Discrete Joint Probability Distribution

JPD1

Example 1. A coin is tossed three times.

$X = \boxed{0}$ or $\underline{1}$ according $\boxed{\text{Tail}}$ or Head occuring on the FIRST toss

$Y \equiv$ Number of Tails.

Determine (i) The Marginal distributions of X and Y.

(ii) Joint PDF of "X and Y".

(iii) Expected values $X+Y$ and XY.

JPD 2

Ans 1 Sample space ≡

≡ {HHH, HHT, HTH, THH, HTT, ~~THH~~ THT, TTH, TTT}

(i) Marginal PDF of X

$X = x$	0 (x_1)	1 (x_2)
$P(X)$	$\frac{4}{8}$	$\frac{4}{8}$
	p_1	p_2

(ii) Marginal PDF of Y

$Y = y$	0 (y_1)	1 (y_2)	2 (y_3)	3 (y_4)
$P(Y)$	$1/8$	$3/8$	$3/8$	$1/8$
	q_1	q_2	q_3	q_4

(ii) Joint PDF of x+y

$X \backslash Y$	0 (y_1)	1 (y_2)	2 (y_3)	3 (y_4)
x_1: 0	0 (p_{11})	$1/8$ (p_{12})	$2/8$ (p_{13})	$1/8$
x_2: 1	$1/8$ (p_{21})	$2/8$ (p_{22})	$1/8$ (p_{23})	0 (p_{24})

(iii) Expectations

$$E(x+y) \equiv \sum (x_i + y_j)\, p_{ij}$$

$$\left.\begin{aligned} &= (0+0)(0) + (0+1)(1/8) \\ &\quad + (0+2)(2/8) + (0+3)(2/8) \\ &\quad + (1+0)(1/8) + (1+1)(2/8) \\ &\quad + (1+2)(1/8) + (1+3)(0) \end{aligned}\right\} = 2$$

$$E(XY) = \sum x_i y_j p_{ij} = 1$$

Eg ② The distributions of two stochastically independent random variables X and Y are

X	0	1
P(X)	0.2	0.8

Y	1	2	3
P(Y)	0.1	0.4	0.5

Find Joint PDF

Ans

X↓ Y→	1	2	3	P(X)
0	0.02	0.08	0.10	0.2
1	0.08	0.32	0.40	0.8
P(Y)→	0.1	0.4	0.5	1

Eg ③ The joint PDF of two random variables X and Y is

$X \backslash Y$	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Determine individual (or marginal) distributions X, Y. Verify whether X & Y are independent

Sol^n

1st step

$j \longrightarrow$, $i \downarrow$

$X \backslash Y$	y_1 2 ($j=1$)	y_2 3 ($j=2$)	y_3 4 ($j=3$)	$P(x) \downarrow$
1 x_1 $i=1$	0.06 p_{11}	0.15 p_{12}	0.09 p_{13}	0.3 p_1
2 x_2 $i=2$	0.14 p_{21}	0.35 p_{22}	0.21 p_{23}	0.7 p_2
$P(y) \longrightarrow$	0.2 q_1	0.5 q_2	0.3 q_3	$\Sigma p_{ij} = 1$

$= \Sigma p_i$
$= \Sigma q_i$

2nd step
Marginal PDF

$X = x_i$	1 x_1	2 x_2
$P(x)$	0.3 p_1	0.7 p_2

Y	2 y_1	3 y_2	4 y_3
$P(y)$	0.2 q_1	0.5 q_2	0.3 q_3

Step 3 X & Y are independent $\because$ for $i = 1, 2$ $j = 1, 2, 3$

$p_1 q_1 = (0.3)(0.2) = p_{11} = 0.06$
$p_1 q_2 = p_{12} = 0.15$ $\quad p_1 q_3 = p_{13} = 0.09$ &

$p_2 q_1 = p_{21}$
$p_2 q_2 = p_{22}$
$p_2 q_3 = p_{23}$

Eg 4 The JPD is

Y \ X	0	1
0	$\boxed{0.1}$ (11)	0.2 (12)
1	0.4 (21)	0.2 (22)
2	0.1 (31)	0 (32)

Q
- Find $P(X+Y>1)$
- Marginal PDF of X
- " " of Y
- Are X & Y independent

Ans
$P(X+Y>1)$
$= P\{(1\ 1), (2\ 0), (2\ 1)\}$
$= 0.2 + 0.1 + 0$
$= 0.3$

Ans cont'd
- Marginal PDF of X

X	0	1	2
P(X)	0.3	0.6	0.1
	p_1	p_2	p_3

Marginal PDF of Y

Y	0	1
P(Y)	0.6	0.4
	q_1	q_2

- Not Independent

Because → $p_1 q_1 = \boxed{0.3 \times 0.6} = 0.18 \neq \boxed{P_{11} = 0.1}$

Definitions

- (1) Joint PDF of two r.v. $X = x_i \; i = 1 \dots m$, $Y = y_j \; j = 1 \dots n$ is $p_{ij} = P(x_i \, y_j)$ s.t. $0 \le p_{ij} \le 1$ & $\sum p_{ij} = 1$

X \ Y	y_1	y_2	y_3	. . .	y_n
x_1	p_{11}	p_{12}	p_{13}	. . . .	
x_2	p_{22}	p_{22}	p_{23}	. . .	
⋮		⋮	⋮	. . . .	
x_m	p_{1m}	p_{2m}	p_{3m}		p_{mn}

JPD Table.

(2) Mean of X $= E(X) = \mu_x = \sum p_i x_i$

Mean of Y $E(Y) = \mu_y = \sum q_j y_j$

(3) Variance X $\sigma_x^2 = E(X^2) - (\mu_x)^2$

$\sigma_y^2 = E(y^2) - (\mu_y)^2$

(3) $Cov(X \, Y) = E(XY) - \mu_x \mu_y$

(4) Correlation $= \dfrac{Cov(X, Y)}{\sigma_x \sigma_y}$

Eg 5 A joint PDF is

$X \backslash Y$	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

Find. Mean of X
Mean of Y
Variance of X
Variance of Y
$Cov(X, Y)$
$Corr^n = \rho(X, Y)$

Sol^n

X	x_1: 1	x_2: 3
$P(X)$	0.5 (p_1)	0.5 (p_2)

Y	y_1: -3	y_2: 2	y_3: 4
$P(Y)$	0.4 (q_1)	0.3 (q_2)	0.3 (q_3)

$$\mu_X = E(X) = \text{Mean } X = \Sigma x_i p_i = x_1 p_1 + x_2 p_2 =$$
$$= 1(0.5) + 3(0.5) = 2$$

$$\mu_y = E(Y) = \Sigma y_j q_j = y_1 q_1 + y_2 q_2 + y_3 q_3$$
$$= (-3)(0.4) + 2(0.3) + 4(0.3) = 0.6$$

$\boxed{E(x^2)} = x_1^2 p_1 + x_2^2 p_2 = (1)^2 0.5 + 3^2(0.5) = \boxed{5}$

$\boxed{E(y^2)} = y_1^2 q_1 + y_2^2 q_2 + y_3^2 q_3 = (-3)^2(0.4) + 2^2(.3) + 4^2(0.3) = \boxed{9.6}$

Variance $x = \boxed{\sigma_x^2} = E(x^2) - (\mu_x)^2 = 5 - (2)^2 = 5 - 4 = 1$

Variance $y = \boxed{\sigma_y^2} = E(y^2) - (\mu_y)^2 = 9.6 - (0.6)^2 = 9.6 - 0.36$

$= \boxed{9.24}$

$E(xy) = \sum x_i y_j p_{ij}$

$= x_1 y_1 p_{11} + \cdots + x_2 y_3 p_{23} = 0$

$\boxed{Cov(x, y)} = E(xy) - E(x) E(y)$

$= E(xy) - \mu_x \mu_y$

$= 0 - 2(0.6)$

$= \boxed{-1.2}$

Correlation betn x & y $= \rho(x, y) = \dfrac{Cov(x, y)}{\sigma_x \sigma_y} =$

$= \dfrac{-1.2}{(1)(3.04)} = -0.3947$

Note:
x & y indpndt
$\Rightarrow$ 1. $E(xy) = E(x) E(y)$
2. $Cov(x, y) = 0$
3. $-1 \le \rho \le 1$