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UNIT - III

# ADVANCED COUNTING TECHNIQUES

STATING NUMBER OF SECOND KIND

Is the number of ways / partitions of set of size $m$ into $n$ non empty subset.

OR

Is the number of possible ways to assign $m$ object into $n$ identical places / boxes with no place kept / empty.

Denoted by $S(m,n)$

- Properties of SNOSK

(1) $S(m,m) = 1 \quad \forall\, m \geq 1$

(2) $S(m,1) = 1 \quad \forall\, n \geq 1$

~~RESO~~ RECURENCE RELATION OF STERLING NUMBER OF SECOND KIND

If $m$ object we kept in $n$-identical places with $m \geq n$ then sterling number of second kind in recurrence relation form is
$S(m,n) = S(m-1,n-1) + n\,S(m-1,n)$

Proof

let $a_1, a_2, a_3 \cdots\cdots a_n$ are $m$-distinct object. Then $s(m,n)$ is the counts of no of ways in which these $m$-object can be assigned to $n$-identical places (boxes)
Next, keeping $a_m$th object in $n$th place, we have $S(m-1,n-1)$ no of way of assigning $(m-1)$ object in $(n-1)$ places (it includes the $a_m$th object in $n$th boxes)
And assigning $a_1, a_2 \cdots\cdots a_{n-1}$ objects in $n$-places is $S(m-1,n)$ and assigning $a_m$th -object has $n$-possiblites

$\therefore$ $n\, S(m-1, n)$ way are there for this distribution.

$\therefore$ Total no of ways of assigning m-objects in n-places is

$$\boxed{S(m,n) = S(m-1, n-1) + n\, S(m-1, n)}$$

TABLE CONSISTING OF STERLING NUMBER OF SECOND KIND

m = 1 — $S(1,1)$ = 1

m = 2 — $S(2,1)$ = 1, $S(2,2)$ = 1

3

m = 3 — $S(3,1)$ = 1, $S(3,2)$, $S(3,3)$ = 1

7 6

m = 4 — $S(4,1)$ = 1, $S(4,2)$, $S(4,3)$, $S(4,4)$ = 1

15 25 10

m = 5 — $S(5,1)$ = 1, $S(5,2)$, $S(5,3)$, $S(5,4)$, $S(5,5)$ = 1

GENERAL FORMULA TO FIND STERTING NUMBER OF SECOND KIND

If m-objects are kept in n-identical places, we have the following formula to calculate, sterling number of second kind

$$\boxed{S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, {}^nC_k \, (n-k)^m}$$

GENERAL FORMULA TO FIND NUMBER OF ONTO FUNCTIONS

Let A and B are two finite sets with $n(A) = m$ and $n(B) = n$, where $m \geq n$, then number of onto functions from A to B ~~are~~ is denoted by $P(m,n)$ and given by

$$P(m,n) = n!\, S(m,n)$$

OR

$$P(m,n) = \sum_{k=0}^{n} (-1)^k \, {}^nC_k \, (n-k)^m$$

## Examples

① Evaluate

(i) $S(6,4)$

→ ~~$S(6,4) = \frac{1}{}$~~ $m=6$ ; $n=4$

$\therefore$ By sterling no of $2^{nd}$ kind

$$S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \, {}^nC_k \, (n-k)^m$$

$$S(6,4) = \frac{1}{4!} \sum_{k=0}^{4} (-1)^k \, {}^4C_k \, (4-k)^6$$

$$S(6,4) = 65$$

$f(g(x))$ $e^{x^2-3}$

(ii) $S(8,5)$

→ $m=8$ , $n=5$

By sterling no of $2^{nd}$ kind

$$S(8,5) = \frac{1}{5!} \sum_{k=0}^{5} (-1)^k \, {}^5C_k \, (5-k)^8$$

$$S(8,5) = 1050$$

② Find the numbers of ways of assigning 10 students in 4 identical classrooms.

→ No of students = 10

$\therefore m = 10$

No of identical classe = 4

$\therefore n = 4$

$\therefore$ No of ways of assigning 10 students in 4 identical classes is

$$S(10,4) = \frac{1}{4!} \sum_{k=0}^{4} (-1)^k \, {}^4C_k \, (4-k)^{10}$$

$$= 34105$$

(3) A Chemist who has 5 assistants is engaged in a research project that calls for 9. compounds ~~that~~ must be syntesized. How many ways can chemist assign these to the 5 assistants so that each is working on atleast one synthesis?

→ No of Compounds need to syntesized = 9 ∴ $m=9$
No of assistant are 5 ∴ $n=5$

∴ By general formula for onto function

$$P(m,n) = \sum_{K=0}^{n} (-1)^K \, {}^nC_K \, (n-K)^m$$

$$= \sum_{K=0}^{5} (-1)^K \, {}^5C_K \, (5-K)^9$$

$$= 834120$$

(4) Their are 6 programers who can assist 8 executives. In ~~#~~ how many ways can the executive be assisted so that each programmer can assist atleast one executive.

→ $m = 8 \quad n=6$

$$P(8,6) = \sum_{K=0}^{6} (-1)^K \, {}^6C_K \, (6-K)^8$$

$$= 191520$$

(5) If A and B are two finite sets with $n(A)=5$ and $n(B)=3$ then no of onto functions from A to B

→ $n(A) = m = 5$
$n(B) = n = 3$

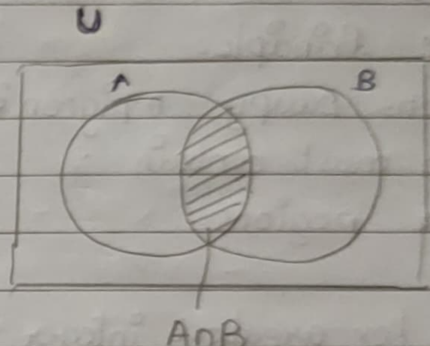
By General formula for no of onto functions,

$$P(m,n) = \sum_{K=0}^{n} (-1)^K \, {}^nC_K \, (n-K)^m)$$

$$P(5,3) = \sum_{k=0}^{3} (-1)^k \, {}^3C_k \, (3-k)^5$$

$$= 150$$

PRINCIPLE OF INCLUSION & EXCLUSION




$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

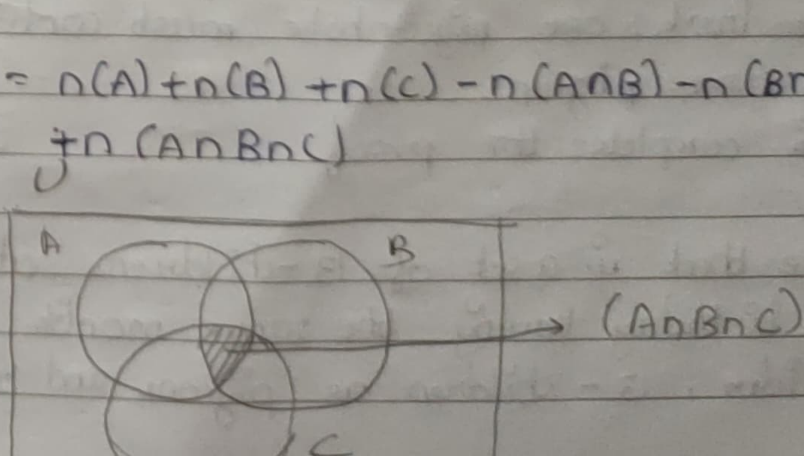
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$

Let A and B be any two sets. Then the number of elements in union of A and B is the sum of the number of elements in A and B minus the number of elements in intersection of A and B.

i.e. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

similary, A, B and C are any 3 sets.
Then the number of elements in union of A, B and C is,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$



PIGEONHOLE PRINCIPLE

Statement : If m- pigeon occupy n-pigeonhole and $m>n$ then 2 or more pigeon occupy the same pigeonhole.

OR

If m-pigeon occupy n-pigeonhole and $m>n$ then atleast one pigeon hole must contain two or more pigeons in it.

Generalized Pigeonhole Principle.

Statement: If m-pigeons occupy n-pigeon's holes then atleast one of the pigeonhole must contain ;

$\left[\frac{m-1}{n}\right] +1$ or more pegions

Where [ ] stands for greatest integer function. (step function)

Proof

We will prove this theorem by the method of contradiction. Suppose each pigeonhole contains at most $\left[\frac{m+1}{n}\right]$ number of pigeons

∴ Total number of pigeons in n- pigeonhole is,

$$n \cdot \left[\frac{m-1}{n}\right] \leq n\left(\frac{m-1}{n}\right) = m-1$$

∴ Total no of piegon in n-pigeonhole are less than or equal to (m-1 piegons

This contradicts the fact that there are m-piegon.

∴ At least one pigeonhole must contain $\left[\frac{m-1}{n}\right]+1$ or more pigeons.

This completes the proof.

① Prove that in a set of 13-childrens atleast two have the birthday having the same month.

→ Consider, 13 - children as pigeons and 12-months. as pageonholes

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$m = 13$ and $n = 12$

By generalized pigeonhole principle, atleast one month has $\left[\frac{m-1}{n}\right]+1$ or more ~~p~~ birthdays of children.

$\therefore \left[\frac{m-1}{n}\right]+1$

$= \left[\frac{13-1}{12}\right]+1$

$= \frac{\cancel{12}^{1}}{\cancel{12}_{1}}+1$

$= 2$

② If 7 cars carry 26 passenger, then prove that atleast one case must have 4 or more passenger.

→ Given that, $m = 26$, $n = 7$

$\therefore$ By generalized pigeohole principle, atleast one car must carry $\left[\frac{m-1}{n}\right]+1$ or more pegion.

$\therefore \left[\frac{m-1}{n}\right]+1$

$\Rightarrow \left[\frac{26-1}{7}\right]+1 = \left[\frac{25}{7}\right]+1 = [3.57]+1 = 3+1 = 4$

③ What should be the minimum number of students, that atleast two students have their last name start with the same english letter.

→ Given that their are 26 english alphabet.

$\therefore n = 26 \quad m = ?$

Also given that atleast 2 students last name starts with same english letter.

$\therefore$ By Generalized pigeonhole principle,

i.e $\left[\frac{m-1}{n}\right]+1 = 2$

$$\left\lceil \frac{m-1}{26} \right\rceil \geq 2-1$$

$m-1 \geq 26$

$m \geq 27$

(4) Find the least number of ways of chosing 3 different numbers from 1 to 10, so that all choices have the same sum.

→ There are ${}^{10}C_3$ ways of chosing 3 different numbers from 1 to 10.

$\therefore {}^{10}C_3 = \frac{10!}{3!\,(10-3)!} = \frac{10 \times 9 \times 8 \times 7!}{3 \times 2 \times 7!} = 120$

$m = 120$

The smallest sum we get by chosing 1, 2, 3 = 6

and largest sum we get by chosing 8, 9, 10 = 27

$\therefore$ The sum of 3 numbers vary from 6 to 27 and they are 22 in number,

$\therefore n = 22$

$\therefore$ By using generalized pigeonhole principle atleast one number [from 6 to 27] is the sum of atleast $\left\lceil \frac{m-1}{n} \right\rceil + 1$ or more choices.

$$\therefore \left\lceil \frac{m-1}{n} \right\rceil + 1 = \left\lceil \frac{120-1}{22} \right\rceil + 1 = \left\lceil \frac{119}{22} \right\rceil + 1 = \lceil 5.4 \rceil + 1 = 5+1 = 6$$

$\therefore$ Atleast 6 choices have the same sum.

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## RECURRENCE RELATION

A recurrence relation for the sequence $\{a_n\}$ (or $a_0, a_1, a_2, a_3 \ldots\ldots a_n \ldots$) is an equation that express $a_n$ in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2 \ldots\ldots a_{n-1}$ for all integer $n \geq 0$

## Solution of Recurrence Relation

A sequence $\{a_n\}$ (or, $a_0, a_1, a_2 \ldots a_n \ldots$) is called solution of a recurrence relation, if its terms satify the recurrence relation.

Example:

The sequence $0, 1, 1, 2, 3, 5 \ldots\ldots$ is the solution of recurrence relation

$$a_n = a_{n-1} + a_{n-2}$$

Note: A recurrence relation

$$a_{n+2} + 2a_{n+1} - 3a_n = 0$$

can also be written as

$$y_{n+2} + 2y_{n+1} - 3y_n = 0$$

OR

$$U_{n+2} + 2U_{n+1} - 3U_n = 0$$

OR

$$f(x+2) + 2f(x+1) - 3f(x) = 0$$

## Order of a recurrence relation

The order of a recurrence relation is the difference between the largest and smallest subscript appearing in the recurrence relation.

Example: Order of a recurrence relation

$a_{n+2} + 2a_{n+1} - 3a_n = 0$ is $n+2-n = 2$

Linear Recurrence Relation with Constant co-efficient.
The linear recurrence relation with constant co-efficient of order K is of the form,

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n) \quad (1)$$

where, $f(n)$ is the function of variable n only
$C_0, C_1, C_2 \cdots C_K$ are the constant.

Homogeneous and Non-Homogeneous linear recurrence relation.
A linear recurrence relation.

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_K a_{n-k} = f(n)$$ is

said to be homogeneous if $f(n) = 0$ Otherwise non-homogenous linear recurrence relation.

GENERATING FUNCTIONS
Let $\{a_n\}$ (or $a_0, a_1, a_2 \ldots$) be a sequence of real numbers. Then generating function of $\{a_n\}$ is denoted by $G(a, z)$ and defined by.

$$G(a,z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{n=0}^{\infty} a_n z^n$$

where z is a variable.

Some Standard Generating Functions.
(i) If $\{a_n\}$ is a sequence and $a_n = c \ \forall \ n \geq 0$ where c is a constant then find generating function of $\{a_n\}$

→ let $\{a_n\} = \{a_0, a_1, a_2 \ldots\}$
given that $a_n = c \ \forall \ n \geq 0$

∴ By Generating function

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$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} c z^n$$

$$= c \sum_{n=0}^{\infty} z^n$$

$$= c\,[1 + z + z^2 + \ldots\ldots]$$

$$\boxed{G(a,z) = \frac{c}{1-z}}$$

$\left[\because \frac{1}{1-x} = 1 + x + x^2 \ldots\right.$

(ii) If $\{a_n\}$ is a sequence and $a_n = b^n \ \forall n \geq 0$, then find generating function of $\{a_n\}$

→ let $\{a_n\} = \{a_0, a_1, a_2 \ldots\ldots\}$

given that $a_n = b^n \ \forall n \geq 0$

By Generating function

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} b^n z^n$$

$$= 1 + bz + b^2 z^2 + b^3 z^3 + \ldots$$

$$= 1 + bz + (bz)^2 + (bz)^3 + \ldots$$

$$\boxed{G(a,z) = \frac{1}{1-bz}}$$

Note: If $\{a_n\}$ is a sequence and $a_n = cb^n \ \forall n \geq 0$ then

$$\boxed{G(a,z) = \frac{c}{1-bz}}$$

(iii) If $\{a_n\}$ is a sequence and $a_n = n \ \forall n \geq 0$, then find generating function of $\{a_n\}$

⇒ Let $\{a_n\} = \{a_0\, a_1, a_2 \ldots\ldots\}$

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Given that $a_n = n \ \forall \ n \geq 0$

By Generating Function,

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n$$

$$= \sum_{n=0}^{\infty} n z^n.$$

$$= 0 + z + 2z^2 + 3z^3 + \cdots$$

$$= z[1 + 2z + 3z^2 + \cdots] \qquad \left[\because \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 \ldots\right]$$

$$\boxed{G(a,z) = \frac{z}{(1-z)^2}}$$

Note If $\{a_n\}$ is a sequence and $a_n = n+1, \ \forall n \geq 0$ then

$$\boxed{G(a,z) = \frac{1}{(1-z)^2}}$$

| S No | Sequence $\{a_n\}$ | Generating Function $G(a,z)$ |
|---|---|---|
| 1 | $a_n = c \quad n \geq 0$ | $\frac{c}{1-z}$ |
| 2 | $a_n = b^n, \quad n \geq 0$ | $\frac{1}{1-bz}$ |
| 3 | $a_n = cb^n \quad n \geq 0$ | $\frac{c}{1-bz}$ |
| 4 | $a_n = n \quad n \geq 0$ | $\frac{z}{(1-z)^2}$ |
| 5 | $a_n = n+1, \ n \geq 0$ | $\frac{1}{(1-z)^2}$ |

SOLUTION OF HOMOGENEOUS LINER RECURRENCE RELATION BY GENERAING FUNCTION

Consider a homogeneous linear recurrence relation.

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0 \quad \forall n \ge k \qquad \text{—— (1)}$$

Solution of (1) involves following steps.

STEP 1: Multiply both side of equation (1) by $z^n$ and take a summation from $k$ to $\infty$

STEP 2: Write each term in the form of $G(a,z)$

STEP 3: Solve $G(a,z)$ by using standard generating functions for the sequence

Example

1 Solve the homogeneous linear recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$ . $\forall n \ge 2$ given that $a_0 = 5$ $a_2 = 3$ by using generating function.

→ let $a_n = 3a_{n-1} + 2a_{n-2}$ $\forall n \ge 2$

$$a_n - 3a_{n-1} + 2a_{n-2} = 0 \qquad \text{—— (1)}$$

Put $n=2$ in (1) and use $a_1 = 1$ $a_2 = 3$.

$$a_2 - 3a_1 + 2a_0 = 0$$

$$2a_0 = 3a_1 - a_2$$

$$2a_0 = 3(5) - 3 \qquad 2a_0 = 12$$

$$\boxed{a_0 = 6}$$

Multiply $z^n$ on both the side of eqn (1) & take summation from $n=2$ to $\infty$

$$\sum_{n=2}^{\infty} (a_n - 3a_{n-1} + 2a_{n-2}) z^n = 0$$

$$\sum_{n=2}^{\infty} a_n z^n - 3\sum_{n=2}^{\infty} a_{n-1} z^n + 2\sum_{n=2}^{\infty} a_{n-2} z^n$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n z^n + a_0 + a_1 z - a_0 - a_1 z$$

$$\Rightarrow -3 \sum_{n=2}^{\infty} a_{n-1} z^{n-1} \cdot z + 2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} z^2 = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n - a_0 - a_1 z - 3z \left[ \sum_{n=2}^{\infty} a_{n-1} z^{n-1} + a_0 - a_0 \right]$$

$$+ 2z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n - 6 - 5z - 3z \left[ \sum_{n=0}^{\infty} a_n z^n - 6 \right] + 2z^2 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\Rightarrow G(a,z) - 6 - 5z - 3z \left[ G(a,z) - 6 \right] + 2z^2 G(a,z) = 0$$

$$\Rightarrow G(a,z) \left[ 1 - 3z + 2z^2 \right] = 6 + 5z - 18z$$

$$\Rightarrow G(a,z) \left[ 2z^2 - 3z + 1 \right] = 6 - 13z$$

$$\Rightarrow G(a,z) \left[ 2z^2 - 2z - z + 1 \right] = 6 - 13z$$

$$\Rightarrow G(a,z) \left[ 2z(z-1) - 1(z-1) \right] = 6 - 13z$$

$$\Rightarrow G(a,z) = \frac{6 - 13z}{(2z-1)(z-1)} \quad \text{—— (2)}$$

∴ By Partial Function.

$$\frac{6 - 13z}{(2z-1)(z-1)} = \frac{A}{z-1} + \frac{B}{2z-1} \quad \text{—— (3)}$$

$$6 - 13z = (2z-1)A + (z-1)B \quad \text{—— (4)}$$

Put $z = 1$ in (4).

$6 - 13 = (2-1)A$

$\boxed{A = -7}$

Put $z = \frac{1}{2}$ in (5)

$6 - \frac{13}{2} = \left(\frac{1}{2} - 1\right) B$

$-\frac{B}{2} = \frac{-1}{2}$

$\boxed{B = 1}$

$\therefore$ Equation (2) becomes

$$G(a,z) = \frac{-7}{z-1} + \frac{1}{2z-1}.$$

$$G(a,z) = \frac{-7}{-(1-z)} + \frac{1}{-(1-2z)}$$

$$= \frac{7}{1-z} - \frac{1}{1-2z}$$

$=$

$\therefore$ By standard generating function

~~G(a, z)~~.

$\boxed{a_n = 7 - 2^n}$

2 Solve homogeneous linear recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ $\forall\, n \geq 2$ given that $a_1 = 1$ & $a_2 = 4$ by generating function

→ let $a_n = 2a_{n-1} - a_{n-2}$ — (1)

$a_n - 2a_{n-1} + a_{n-2} = 0$ — (1)

put $n = 2$ in (1) and $a_1 = 1$ & $a_2 = 4$

$a_2 - 2a_1 + a_0 = 0$

$a_0 = 2a_1 - a_2$

$a_0 = 2 - 4$

$\boxed{a_0 = -2}$

Multiply $z^n$ on both the sides of equation ① and take sumation from $n=2$ to $\infty$

$$\sum_{n=2}^{\infty} (a_n - 2a_{n-1} + a_{n-2}) z^n = 0$$

$$\sum_{n=2}^{\infty} a_n z^n - \sum_{n=2}^{\infty} 2a_{n-1} z^n + \sum_{n=2}^{\infty} a_{n-2} z^n = 0$$

$$\sum_{n=2}^{\infty} a_n z^n - 2\sum_{n=2}^{\infty} a_{n-1} z^{n-1} z + \sum_{n=2}^{\infty} a_{n-2} z^{n-2} z^2 = 0$$

$$\sum_{n=2}^{\infty} a_n z^n - 2z \sum_{n=2}^{\infty} a_{n-1} z^{n-1} + z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} = 0$$

$$\underline{\sum_{n=2}^{\infty} a_n z^n + a_0 + a_1 z} - a_0 - a_1 z - 2z\left[\underline{\sum_{n=2}^{\infty} a_{n-1} z^{n-1} + a_0} - a_0\right] + z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} = 0$$

$$\sum_{n=0}^{\infty} a_n z^n - a_0 - a_1 z - 2z\left[\sum_{n=0}^{\infty} a_n z^n - a_0\right] + z^2 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} a_n z^n + 2 - z - 2z\left[\sum_{n=0}^{\infty} a_n z^n + 2\right] + z^2 \sum_{n=0}^{\infty} a_n z^n = 0$$

$$\sum_{n=0}^{\infty} a_n z^n \left[1 - 2z + z^2\right] + 2 - z - 4z = 0$$

$$\sum_{n=0}^{\infty} a_n z^n \left[1 - 2z + z^2\right] = 5z - 2$$

$$\sum_{n=0}^{\infty} a_n z^n \left[(z-1)^2\right] = 5z - 2$$

$$\sum_{n=0}^{\infty} a_n z^n = \frac{5z-2}{(z-1)^2} \quad \text{—— ②} \quad \equiv \Sigma$$

∴ By Partial Fraction

$$= \frac{5z-2}{(1-z)^2} = \frac{A}{(1-z)} + \frac{B}{(1-z)^2} \quad \text{—— ③}$$

$$= 5z - 2 = (1-z)A + B \quad \text{—— ④}$$

put $z=1$ in (4).
$5-2=B$
$\boxed{B=3}$

put $z=0$ in (4).
$-2=A+B$
$A=-2-3$
$\boxed{A=-5}$

$\therefore$ Equ$^n$ (2) becomes.

$$\sum_{n=0}^{\infty} a_n z^n = \frac{-5}{(1-z)} + \frac{3}{(1-z)^2}$$

By standard generating function
$a_n = -5 + \cancel{3A}\ 3(n+1)$
$a_n = -5 + 3n + 3$
$\boxed{a_n = 3n-2}$

SOLUTION OF NON-HOMOGENEOUS LINEAR RECURRENCE RELATION BY USING GENERATING FUNCTIONS

(1) Solve non-homogeneous linear recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n \ \forall n \geq 0$ given that $a_0 = 2$ and, $a_1 = 1$ by generating function
→ Let $a_{n+2} - 2a_{n+1} + a_n = 2^n \quad \forall n \geq 0$ —— (1)

Multiply $z^n$ on both side of eq$^n$ (1) & take summation from $n=0$ to $\infty$

$$\sum_{n=0}^{\infty} (a_{n+2} - 2a_{n+1} + a_n) z^n = \sum_{n=0}^{\infty} 2^n z^n$$

$$\sum_{n=0}^{\infty} a_{n+2} z^n - 2\sum_{n=0}^{\infty} a_{n+1} z^n + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}$$

$$\sum_{n=0}^{\infty} a_{n+2} z^{n+2} z^{-2} - 2\sum_{n=0}^{\infty} a_{n+1} z^{n+1} z^{-1} + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}$$

$$\Rightarrow \frac{1}{z^2}\left[\sum_{n=0}^{\infty} a_{n+2} z^{n+2} + a_0 + a_1 z - a_0 - a_1 z\right]$$

$$-\frac{2}{z}\left[\sum_{n=0}^{\infty} a_{n+1} z^{n+1} + a_0 - a_0\right] + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}$$

$$\Rightarrow \frac{1}{z^2}\left[\sum_{n=0}^{\infty} a_n z^n - a_0 - a_1 z\right] - \frac{2}{z}\left[\sum_{n=0}^{\infty} a_n z^n - a_0\right] + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}$$

$$\Rightarrow \frac{1}{z^2}\left[\sum_{n=0}^{\infty} a_n z^n - 2 - z\right] - \frac{2}{z}\left[\sum_{n=0}^{\infty} a_n z^n - 2\right] + \sum_{n=0}^{\infty} a_n z^n = \frac{1}{1-2z}$$

$$\Rightarrow \left[\sum_{n=0}^{\infty} a_n z^n - 2 - z\right] - 2z\left[\sum_{n=0}^{\infty} a_n z^n - 2\right] + z^2 \sum_{n=0}^{\infty} a_n z^n = \frac{z^2}{1-2z}$$

$$\Rightarrow G(a,z) - 2 - z - 2z\,G(a,z) + 4z + z^2 G(a,z) = \frac{z^2}{1-2z}$$

$$\Rightarrow G(a,z)\left[1 - 2z + z^2\right] - 2 + 3z = \frac{z^2}{1-2z}$$

$$\Rightarrow G(a,z)\left[1 - 2z + z^2\right] = \frac{z^2}{1-2z} + 2 - 3z$$

$$\Rightarrow G(a,z)\ (z-1)^2 = \frac{z^2 + (1-2z)(2-3z)}{1-2z}$$

$$\Rightarrow G(a,z)\ (z-1)^2 = \frac{z^2 + 2 - 3z - 4z + 6z^2}{1-2z}$$

$$\Rightarrow G(a,z)\ (z-1)^2 = \frac{7z^2 - 7z + 2}{1-2z}$$

$$\Rightarrow G(a,z) = \frac{7z^2 - 7z + 2}{(z-1)^2(1-2z)} \quad \text{--- (2)}$$

By Partial Fraction

$$\frac{7z^2 - 7z + 2}{(1-2z)(1-z)^2} = \frac{A}{(1-2z)} + \frac{B}{(1-z)} + \frac{C}{(1-z)^2} \quad \text{--- (3)}$$

~~put~~

~~$7z^2-7z+2 = (1-z)^2A + (1-z)B + C$ — (4)~~

$7z^2-7z+2 = (1-z)^2A + (1-2z)(1-z)B + (1-2z)C$ — (4)

~~put z =~~

put $z = \frac{1}{2}$

$\frac{7}{4} - \frac{7}{2} + 2 = \frac{A}{4}$

$A = 7-14+8$

$\boxed{A=1}$

put $z=1$

$7-7+2 = (-1)C$

$\boxed{C=-2}$

put $z=0$ in (4)

$2 = A+B+C$

$B = 2-A-C$

$B = 2-1+2$

$\boxed{B=-3}$

∴ Equation (2) becomes

$\frac{1}{1-2z} + \frac{3}{1-z} - \frac{2}{(1-z)^2} = \sum_{n=0}^{\infty} a_n z^n$

By standard generating function

$a_n = 2^n + 3 - 2(n+1)$

$a_n = 2^n + 3 - 2n - 2$

$\boxed{a_n = 2^n - 2n + 1}$

$r^{[-(n-k)]}$

SOLUTION OF HOMOGENEOUS LINEAR OCURENCE RELATIONS BY CHARACTERISTIC ROOT METHOD

Consider a homogeneous linear recurrence relation of order k

$c_0a_n + c_1a_{n-1} + c_2a_{n-2} + \dots + c_k a_{n-k} = 0$ — (1)

Characteristic equation or auxillary equations of degree k is obtained by replacing $a_n$ by $r^n$, $a_{n-1}$ by $r^{n-1}$

$$c_0 r^n + c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_K r^{n-K} = 0$$

$$r^{n-1}\{c_0 r^K + c_1 r^{K-1} + c_2 r^{K-2} + \dots + c_K\} = 0.$$

$$c_0 r^K + c_1 r^{K-1} + c_2 r^{K-2} + \dots + c_K = 0 \quad \text{—— (2)}$$

Roots of equation (2) are $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_K$.

Solution of relation (1) has following four cases :

case 1: If roots are real and unequal the general solution of (1) is

$$\boxed{a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + A_3 \alpha_3^n + \dots + A_K \alpha_K^n}$$

case 2: If two roots are real and equal ($\alpha_1 = \alpha_2$). The general solution of (1) is

$$\boxed{a_n = (A_1 + nA_2)\alpha_1^n + A_3 \alpha_3^n + \dots + A_K \alpha_K^n}$$

Case 3: If roots are complex. The general solution of (1) is

$$\boxed{a_n = A_1(\alpha_1 + i\beta_1)^n + A_2(\alpha_2 - i\beta_2)^n + \dots}$$

If two complex roots are equal i.e $\alpha_1 + i\beta_1 = \alpha_2 + i\beta_2$ The general solution of (1) is

$$\boxed{a_2 = (A_1 + nA_2)(\alpha_1 + i\beta_1)^n + (A_3 + nA_4)(\alpha_2 + i\beta_2)^n + \dots}$$

where $A_1, A_2, A_3$ are auxilary constant.

(1) Solve $a_{n+2} - 3a_{n+1} + 2a_n = 0$ given that $a_0 = 1$ $a_1 = 3$ by characteristic root method.

→ let $a_{n+2} - 3a_{n+1} + 2a_n = 0$ —— (1)

Characteristic equation of (1) can be written by replacing $a_{n+2}$ by $r^{n+2}$, $a_{n+1}$ by $r^{n+1}$ and $a_n$ by $r^n$

$$r^{n+2} - 3r^{n+1} + 2r^n = 0$$

$$r^n [r^2 - 3r + 2] = 0$$
$$\Rightarrow r^2 - 3r + 2 = 0 \quad \text{—— (2)}$$
$$\Rightarrow r^2 - 2r - r + 2 = 0$$
$$\Rightarrow r(r-2) - 1(r-2) = 0$$
$$\Rightarrow (r-1)(r-2)$$
$$\Rightarrow r - 1 = 0 \text{ and } r - 2 = 0$$
$$r = 1 \text{ and } r = 2$$

∴ Roots are $\alpha_1 = 1$ and $\alpha_2 = 2$

∴ General solution of (1) is

$$a_n = A_1\alpha_1^n + A_2\alpha_2^n.$$
$$a_n = A_1(1)^n + A_2(2)^n$$
$$a_n = A_1 + A_2 2^n \quad \text{—— (3)}$$

Put $n = 0$ in (3) and use $a_0 = 1$.

$$\boxed{1 = A_1 + A_2} \quad \text{—— (4)}$$

Put $n = 1$ in (3) and $a_1 = 3$

$$3 = A_1 + A_2 2^1$$
$$\boxed{3 = A_1 + 2A_2} \quad \text{—— (4)}$$

Solving equation (4) and (5) we get

$\boxed{A_1 = -1}$ and $\boxed{A_2 = 2}$

$\therefore$ Particular solution of (1) is

$a_n = -1 + 2 \cdot 2^n$

$\boxed{a_n = 2^{n+1} - 1}$

(2) Solve $a_n - 8a_{n-1} + 16a_{n-2} = 0$ given that $a_2 = 6$; $a_3 = 80$ by characteristic root method.

$\longrightarrow$ Let $a_n - 8a_{n-1} + 16a_{n-2} = 0$ —— (1)

Characteristic equation of (1) can be written by replacing $a_{n-2}$ by $r^{n-2}$, $a_{n-1}$ by $r^{n-1}$ and $a_n$ by $r^n$

$\therefore \; r^n - 8r^{n-1} + 16r^{n-2} = 0$

$\Rightarrow r^{n-2} \cancel{[1 - 8r + 16r^2}\; [r^2 - 8r + 16] = 0$

$\Rightarrow r^2 - 8r + 16 = 0$

$\Rightarrow (r-4)^2 = 0$

$r = 4$ and $r = -4$.

$\therefore$ Roots are $\alpha_1 = \alpha_2 = 4$

General solution of (1) is

$a_n = (A_1 + nA_2)\alpha_1^n$

$a_n = (A_1 + nA_2)4^n$ —— (3)

Put $n = 2$ in (3) and use $a_2 = 6$

$6 = (A_1 + 2A_2)4^2$

$A_1 + 2A_2 = \frac{3}{8}$ —— (4)

Put $n = 3$ in (3) and use $a_3 = 80$

$$80 \; ~~t~~ \doteq (A_1 + 3A_2)\, 4^3$$

$$A_1 + 3A_2 = \frac{5}{4} \quad \text{——} (5)$$

Solving equation (4) and (5) we get

$$A_1 = \frac{-11}{8} \qquad A_2 = \frac{7}{8}.$$

Particular solution of (1) is

$$a_n = \left(\frac{-11}{8} + n\frac{7}{8}\right) 4^n.$$

$$a_n = (7n-11)\, \frac{4^n}{8}.$$

$$\boxed{a_n = (7n-11)\, \frac{4^{n-1}}{2}}$$

SOLUTION OF NON-HOMOGENEOUS LINEAR RECURRENCE RELATION BY CHARACTERISTIC ROOT METHOD.

Consider a non-homogeneous linear recurrence relation of order $K$.

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots\cdots + c_K a_{n-K} = f(n) \quad \text{——} (1)$$

General solution of recurrence relation (1) is

$$a_n = a_n^{(h)} + a_n^{(P)} \quad \text{——} (2)$$

where $a_n^{(h)}$ is the general solution of homogeneous part of recurrence relation (1) and $a_n^{(P)}$ is particular solution of relation (1)

The following are the cases ~~of~~ to find $a_n^{(P)}$

Case 1: If $f(n)$ is a polynomial of degree $q$ and 1 is not a root of characteristic equation of homogeneous part of ① then, $a_n^{(P)}$ is as follows:

$$\boxed{a_n^{(P)} = B_0 + B_1 n + B_2 n^2 + \cdots + B_q n^q}$$

case 2: If $f(n)$ is a polynomial of degree $q$ and 1 is the root of multiplicity $m$ of characteristic equation of homogeneous part of ① then, $a_n^{(P)}$ as follows

$$a_n^{(P)} = n^m (B_0 + B_1 n + B_2 n^2 + \cdots + B_q n^q)$$

case 3: If $f(n) = \alpha b^n$, where $\alpha$ and $b$ are constants, and $b$ is not a root of characteristic equation of homogeneous part of ①, then $a_n^{(P)}$ is as follows.

$$a_n^{(P)} = B_0 b^n$$

case 4: If $f(n) = \alpha b^n$, where $\alpha$ and $b$ are constant and $b$ is the root of multiplicity $m$ of characteristic equation of homogeneous part of ① then $a_n^{(P)}$ is as follows

$$a_n^{(P)} = B_0 n^m b^n$$

where $B_0, B_1, B_2, \ldots, B_q$ are constants to be evaluated by the fact that $a_n = a_n^{(P)}$ satisfy recurrence relation ①

① Solve the recurrence relation $a_n + 4a_{n-1} + 4a_{n-2} = 8 \ \forall\, n \geq 2$ and given that $a_0 = 1$ & $a_2 = 2$

→ let $a_n + 4a_{n-1} + 4a_{n-2} = 8$ —— ①

Characteristic equation of homogeneous part of ① is

$$r^n + 4r^{n-1} + 4r^{n-2} = 0$$

$r^{n-2}[r^2+4r+4]=0$

$r^2+4r+4=0$ —— (2)

$(r+2)^2=0$

$r=-2$ or $r=-2$

Roots are $\alpha_1=\alpha_2=-2$

$\therefore$ Solution of homogeneous part of (1) is,

$a_n^{(h)} = (A_1+nA_2)\alpha_1^n$

$\boxed{a_n^{(h)} = (A_1+nA_2)(-2)^n}$ —— (3)

Next, $f(n)=8$ is a polynomial of degree zero and 1 is not a root of equation (2), then particular solution of (1) is,

$\boxed{a_n^{(p)} = B_0}$ —— (4)

Considering the fact that $a_n = a_n^{(p)} = B_0$ satisfying relation (1)

$\therefore B_0+4B_0+4B_0=8$

$9B_0=8$

$\boxed{B_0=\frac{8}{9}}$

$\therefore \boxed{a_n^{(p)}=\frac{8}{9}}$ —— (5)

$\therefore$ General solution of (1) is

$a_n=a_n^{(h)}+a_n^{(p)}$

$a_n = (A_1+nA_2)\cdot(-2)^n+\frac{8}{9}$ —— (6)

Put $n=0$ in (6) and use $a_0=1$

$$a_n = \left(-\frac{4}{5}\right)(-2)^n + \left(\frac{2}{4}\right)(-1)^n + \frac{3n}{10}$$

$$1 = A_1(-2)^0 + \frac{8}{9}$$

$$A_1 = 1 - \frac{8}{9}$$

$$A_1 = \frac{1}{9}$$

Put $n = 1$ in (6) and use $a_1 = 2$ and $1/9 = A_1$

$$2 = \left(\frac{1}{9} + 2A_2\right)(-2)^2 + \frac{8}{9}$$

$$2 = \left(\frac{1}{9} + 2A_2\right)(4) + \frac{8}{9}$$

$$2 = \frac{4}{9} + 8A_2 + \frac{8}{9}$$

$$4 = \frac{12}{9} + 8A_2$$

$$8A_2 = 4 - \frac{4}{3}$$

$$A_2 = \frac{8}{3 \times 8}$$

$$A_2 = \frac{1}{3}$$

$$2 = \left(\frac{1}{9} + A_2\right)(-2) + \frac{8}{9}$$

$$2 = \frac{-2}{9} + -2A_2 + \frac{8}{9}$$

$$2 = -2A_2 + \frac{6}{9}$$

$$-2A_2 = \frac{6}{9} - 2$$

$$A_2 = -\frac{12}{9 \times 2}$$

$$A_2 = -\frac{2}{3}$$

Solution (5) becomes

$$a_n = \left(\frac{1}{9} - \frac{2n}{3}\right)(-2)^n + \frac{8}{9}$$

(2) Solve the recurrence relation $a_{n+2} + 3a_{n+1} + 2a_n = 3^n \ \forall n \geq 0$ and $a_0 = 0$ and $a_1 = 1$

Let $a_{n+2} + 3a_{n+1} + 2a_n = 3^n$ —— (1)

Charateristic eqⁿ of homogeneous part of (1) is

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$$r^{n+2} + 3r^{n+1} + 2r^n = 0$$

$$r^n\,[r^2 + 3r + 2] = 0$$

$$r^2 + 3r + 2 = 0 \quad \text{—— } (2)$$

$$r^2 + r + 2r + 2 = 0$$

$$r(r+1) + 2(r+1) = 0$$

$$(r+1)(r+2) = 0$$

$r = -1$ and $r = -2$

Solution for part (1)

$$a_n^{(h)} = A_1\alpha_1^n + A_2\alpha_2^n$$

$$a_n^{(h)} = A_1(-1)^n + A_2(-2)^n \quad \text{—— } (3)$$

Next $f(n) = 3^n$ and 3 is not a root of eq$^n$ (2) the particular solution of (1) is

$$a_n^{(p)} = B_0\, 3^n \quad \text{—— } (4)$$

Consider the fact that $a_n = a_n^{(p)} = B_0 3^n$ satisfying relation (1)

$\therefore\ B_0\, 3^{n+2} + 3\, B_0\, 3^{n+1} + 2^{B_0} 3^n = 3^n$

$$3^n B_0\,[3^2 + 3\,3^1 + 2] = 3^n$$

$$B_0\,[20] = 1$$

$$B_0 = \frac{1}{20}$$

$$a_n^{(p)} = \frac{1}{20}\,3^n \quad \text{—— } (5)$$

$\therefore$ General solution (1) is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$= A_1(-1)^n + A_2(-2)^n + \frac{3^n}{20} \quad \text{—— } (6)$$

Put $n=0$ in (6) and use $a_0 = 0$ Put $n=1$ in (6) and use $a_1 = 1$

$$a_0 = A_1(-1)^0 + A_2(-2)^0 + \frac{3^0}{20}$$

$$0 = A_1 + A_2 + \frac{1}{20}$$

$$A_1 + A_2 = \frac{-1}{20} \quad \text{— (7)}$$

$$a_1 = A_1(-1)^1 + A_2(-2)^1 + \frac{3^1}{20}$$

$$1 = -A_1 - 2A_2 + \frac{3}{20}$$

$$A_1 + 2A_2 = \frac{3}{20} - 1$$

$$A_1 + 2A_2 = \frac{-17}{20} \quad \text{— (8)}$$

By solving (7) and 8

$A_1 = \frac{3}{4}$ $\quad$ $A_2 = -\frac{34}{105}$

Solution (6) becomes

$$a_n = \frac{3}{4}(-1)^n - \frac{34}{50}(-2)^n + \frac{3^n}{20}$$

DIVIDE AND CONQUER ALGORITHM

It works recursivly breaking down a given problem P into two or more sub problems [P1, P2, etc] of the same or related type, until this becomes simple enough to be solved directly. The solutions to this sub problems then combined to get the solution for orignal problem P.

A Typical divide and conquer algorithm solves a problem using following steps

Step 1 : Divide

- Breaks or divide a given problem P into sub problem ($P_1$, $P_2$, $P_3$.....etc) of the same type

Step 2 : Conquer

→ Recursively solve the subproblem ($P_1$, $P_2$, $P_3$ ..... etc) to get a solutions (S1, S2, S3 .... etc) step 3 - combine

→ combine the solution of sub-problem ($S1, S2$ ... etc) to get solution of orignal problem P.

Idea behind Divide and conquer Algorithm

Given a problem P of size $n = 2^k$.

Algorithm DAC (P)

If 'n' is small solve it.

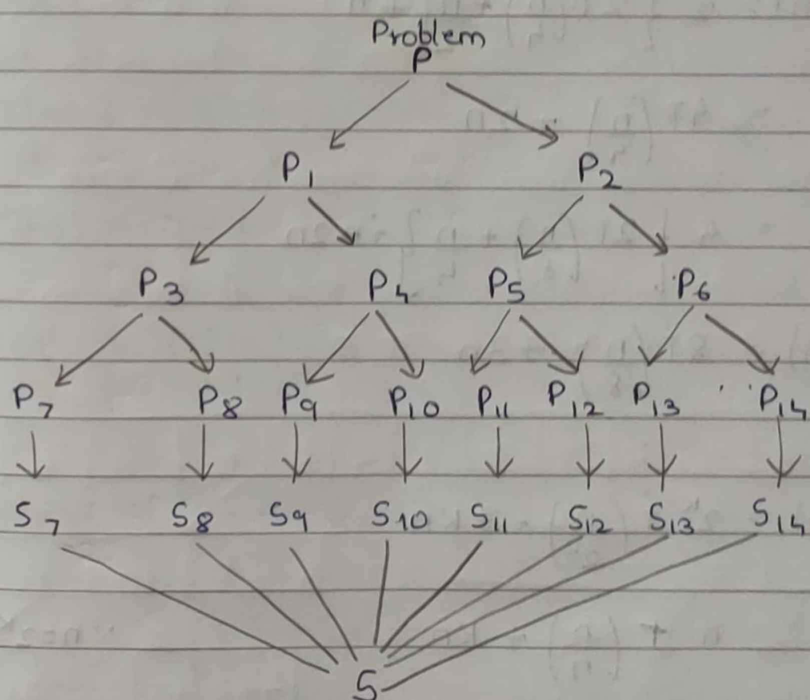
else

divide a problem P into subproblems. P1 P2 ... etc.

DAC (P1)

DAC (P2)

Combine the solution of sub-problems P1, P2, ....etc.




Time required to solve problem P.

$$\underset{P}{T(n)} = \underset{P_1}{T\left(\frac{n}{2}\right)} + \underset{P_2}{T\left(\frac{n}{2}\right)} + f(n)$$

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where $f(n) =$ is called additional cost of combining the solutions.

$$T(n) = 2T\left(\frac{n}{2}\right) + f(n)$$

In general,

$$T(n) = b\,T\left(\frac{n}{a}\right) + f(n)$$

It is the recurrence relation for divide and conquer Algorithm.

① Solve $T(n) = 2T\left(\frac{n}{2}\right) + n$, using iteration or substitution method taking $n = 2^k$

→ Let $T(n) = 2T\left(\frac{n}{2}\right) + n$ —— ①

$$T(n) = 2\left\{2T\left(\frac{n}{4}\right) + \frac{n}{2}\right\} + n$$

$$= 4T\left(\frac{n}{4}\right) + 2n$$

$$= 4\left\{2T\left(\frac{n}{8}\right) + \frac{n}{4}\right\} + 2n$$

$$T(n) = 8T\left(\frac{n}{8}\right) + 3n$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$T(n) = 2^k\,T\left(\frac{n}{2^k}\right) + kn$$

$$= n\,T\left(\frac{n}{n}\right) + kn \qquad \because n = 2^k$$

$$= n.T(1) + kn$$

$T(1)$ is negligibly small $T(1) \simeq 0$

$T(n) = kn$

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$T(n) =$

Since $n = 2^k$

$\log n = 2k \log 2$

$k = \frac{\log n}{\log 2}$

$\boxed{k = \log_2 n}$

$\therefore \boxed{T(n) = n \log n}$

$\therefore$ Complexity is $O(n \log n)$

② In an algorithm we divide large problems into 3 equal parts and discard two of them, in a constant time. What is the complexity of this algorithm for the size $n = 3^k$.

→ Given that

$T(n) = T\left(\frac{n}{3}\right) + c$ —— ①

$T(n) = T\left(\frac{n}{9}\right) + 2c$

$T(n) = T\left(\frac{n}{27}\right) + 3c$

$\vdots \quad \vdots \quad \vdots$

$T(n) = T\left(\frac{n}{3^k}\right) + kc$

$= T\left(\frac{n}{n}\right) + kc \qquad n = 3^k$

$T(n) = T(1) + kc$

$T(1)$ is negligibly $\therefore$

$T(n) = kc$

Since $n = 3^k$

$\log n = k \log 3$

$\boxed{k = \log_3 n}$

$\boxed{T(n) = \log_3 n \cdot c}$

Complexity is $O(c \log_3 n)$

MERGE SORT ALGORITHM

Merge sort keeps on dividing given list of numbers into equal halves until it can ~~not~~ no more be divided, then merges sort combine smaller sorted list keeping the new list sorted.

Algorithm

Step 1: If list has only one element then it is sorted, ~~it~~ returns

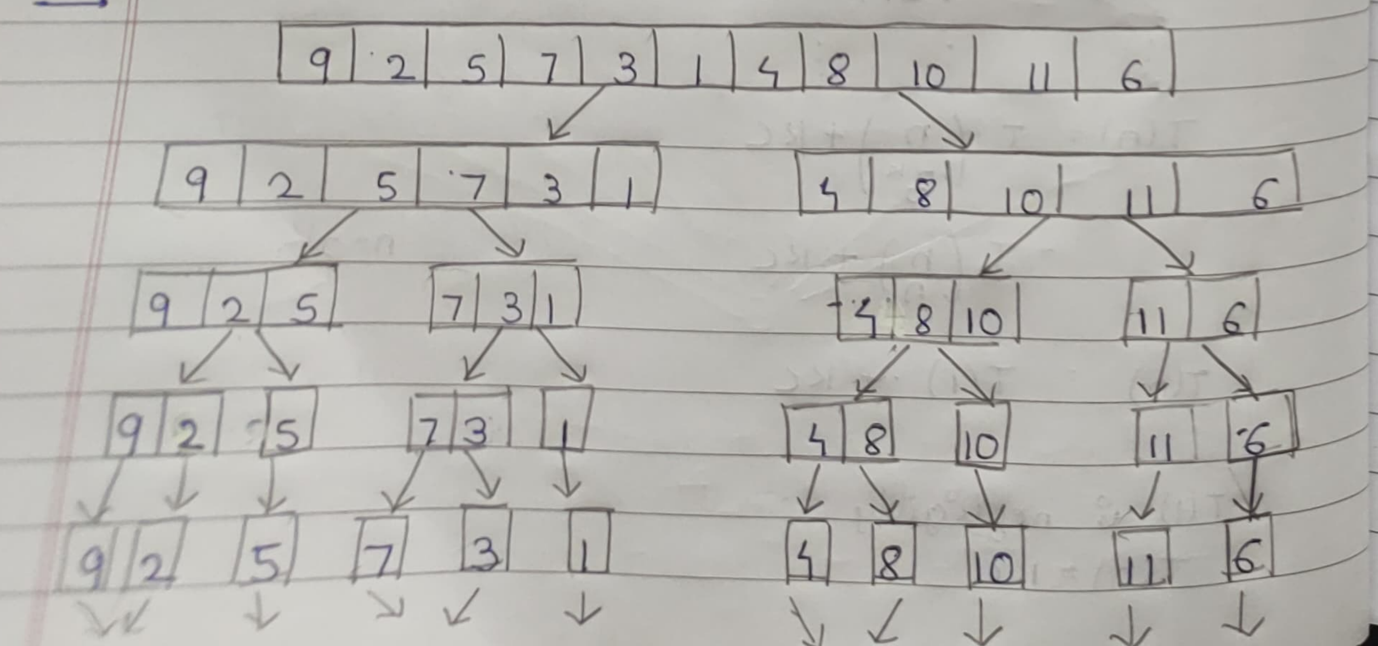
Step 2: Divide the list recursively into two halves until it can no more ~~of~~ be divided.

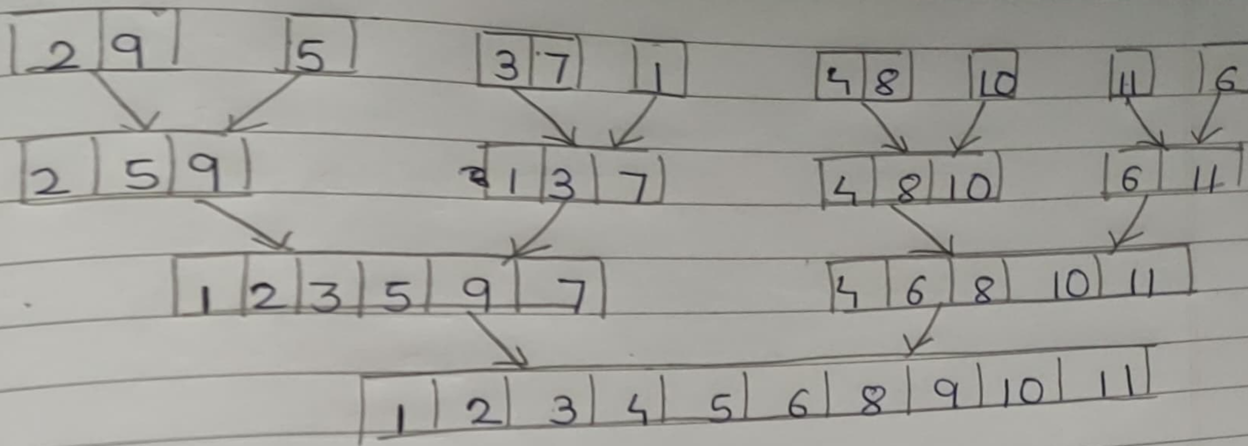
Step 3: Merge the smaller lists into new list in sorted order

① Consider the following list and sort it by merge sort algorithm

9, 2, 5, 7, 3, 1, 4, 8, 10, 11, 6

→




2 9
5
3 7
1
4 8
10
11
6
2 5 9
1 3 7
4 8 10
6 11
1 2 3 5 9 7
4 6 8 10 11
1 2 3 4 5 6 8 9 10 11