Newton- Raphson Method.

1. Using Newton-Raphson iterative Method find the real root of $x\log_{10} x = 1.2$ correct to four decimal places.

Sol^n: Let $f(x) = x\log_{10} x - 1.2 = 0$

On rewriting $f(x) = x\left[\dfrac{\log_e x}{\log_e 10}\right] - 1.2 = 0$

$$\Rightarrow f(x) = \frac{1}{\log_e 10}\left[x\log_e x\right] - 1.2 = 0$$

$$\therefore f'(x) = \frac{1}{\log_e 10}\left[x\cdot\frac{1}{x} + \log_e x \; 1\right] = 0$$

$$\therefore f'(x) = \frac{1}{\log_e 10}\left[1 + \log_e x\right] = 0.4343\left[1+\log_e x\right]$$

Now to find the interval

$f(2.7) = -0.035$, $f(2.8) = 0.052$

$\therefore f(2.7)\cdot f(2.8) < 0$

$\therefore$ Root lies in $(2.7, 2.8)$

Let $x_0 = 2.7$; By N-R method.

$$x_1 = x - \frac{f(x)}{f'(x)} = x - \frac{x\log_{10} x - 1.2}{0.4343\left[1+\log_e x\right]}$$

$$x_1 = 2.7 - \frac{2.7\log_{10}(2.7) - 1.2}{0.4343\left[1+\log_e 2.7\right]} = \boxed{2.74108}$$

$x_2 = \boxed{2.74064}$, $x_3 = \boxed{2.74064}$

$\therefore$ $\boxed{x_3 = 2.74064}$ is the required root

(2) Using N-R method find the root of $x\sin x + \cos x = 0$ which is near to $x = \pi$ correct to 3 decimal places

Sol^n: Let $f(x) = x\sin x + \cos x$

$$f'(x) = x\cos x + \sin x - \sin x$$

$$f'(x) = x\cos x$$

Let $\boxed{x_0 = \pi}$

By N-R method we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{\pi\sin(\pi) + \cos(\pi)}{\pi\cos\pi} = \boxed{2.8232}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8232 - \frac{2.8232\sin(2.8232) + \cos(2.8232)}{2.8232\cos(2.8232)}$$

$$\boxed{x_2 = 2.7985}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \boxed{2.7983}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = \boxed{2.7983}$$

$\therefore$ x_3 & x_4 are identical $\boxed{x_4 = 2.7983}$ is the approximate root

Fixed Point Iteration Method

Let $f(x) = 0$ be the given equation.

Let us write this equation in the form

$$x = \phi(x) \quad \text{——— (1)}$$

Let x_0 be the initial approximation value to the actual root 'α' and substituting $x = x_0$ in RHS of (1) we get

$$x_1 = \phi(x_0) \quad \text{——— (2)}$$

Again put $x = x_1$ in (2)

$$x_2 = \phi(x_1)$$

$$\cdots\cdots\cdots$$

$$x_n = \phi(x_{n-1})$$

The sequnce of approximate roots $x_1, x_2, x_3 \ldots x_n$ if it converges to 'α' is taken as the root of the equation $f(x) = 0$

Note: 1) The smaller the value of $\phi'(x)$, the more rapid will be the convergence

2) The sufficient condition for the convergence is $\boxed{|\phi'(x)| < 1}$ for all x in the interval 'I' containing the root $x = \alpha$

* [$f(x) = 0$ can be algebraically expressed as $x = \phi(x)$.
A point say 'α' is fixed point if it satisfies $x = \phi(x)$]

① Find the root of the equation $x^2+x-1=0$ using fixed point iteration.

Sol^n: Let $f(x) = x^2+x-1 = 0$

$\Rightarrow x^2+x = 1$

$x(1+x) = 1 \Rightarrow x = \frac{1}{1+x} = \phi(x)$

Now $\phi'(x) = -\frac{1}{(1+x)^2}$

Consider the function $f(x) = x^2+x-1 = 0$

Root lies in the interval $(0.6, 0.7)$

$f(0.6) = -0.04$, $f(0.7) = 0.19$

Let $x_0 = 0.6$

Also $|\phi'(0.6)| = 0.3906 < 1$; $|\phi'(0.7)| = 0.346 < 1$

$x_0 = 0.6$ be the initial approximation.

$x_1 = \phi(x_0) = \frac{1}{1+x_0} = \frac{1}{1+0.6} = 0.625$

$x_2 = \phi(x_1) = \frac{1}{1+0.625} = 0.61538$

$x_3 = \phi(x_2) = \frac{1}{1+0.61538} = 0.61904$

$x_4 = 0.61764$, $x_5 = 0.61818$, $x_6 = 0.61797$

$x_7 = 0.61805$, $x_8 = 0.61802$

$\boxed{x_8 = 0.61802}$ is the required root.

(2) Find the root of $x = \frac{1}{2} + \sin x$ by fixed point iteration method.

Sol^n: Let $f(x) = \frac{1}{2} + \sin x - x = 0$

Root lies in $(1.4, 1.5)$

$f(1.4) = 0.0854$, $f(1.5) = -2.163$

Consider $x = \frac{1}{2} + \sin x = \phi(x)$

Also $\phi'(x) = \cos x$

$|\phi'(x)| = |\phi'(1.4)| = |\cos(1.4)| = 0.1699 < 1$

Let $x_0 = 1.4$

$x_1 = \phi(x_0) = \frac{1}{2} + \sin(x_0) = \frac{1}{2} + \sin(1.4) = 1.48544$

$x_2 = \phi(x_1) = \frac{1}{2} + \sin(1.48544) = 1.49635$

$x_3 = \phi(x_2) = \frac{1}{2} + \sin(1.49635) = 1.49723$

$x_4 = 1.497295$ $\quad x_5 = 1.497296$

$\boxed{x_5 = 1.49729}$ is the required root