Propositional & Predicate Logic

Converting Statements to Logical Expressions

* It is raining and it is Tuesday

can be written as

$R \wedge T$

$\wedge$ → AND operator

where R means "it is raining"

T means "it is Tuesday"

* To express, more complex statements R & T won't be enough

For example,

* It is raining heavily

need to be written as

$H(R)$ → property of R

* It is raining in New York

$N(R)$ → property of R

Here we are emphasising the property of Rain

But if we write

$R(N)$ → property of N

Then we are emphasising the property of New York

* Expressing Negations

"It is not raining in New York"

can be expressed as

$$\underbrace{\neg R(N) \quad \text{or} \quad \neg N(R)}_{\text{Negations}}$$

* Parenthesized Expressions

"It is raining in New York, and I'm either getting sick or just very tired"

$R(N) \longrightarrow$ Raining in New York

$S(I) \longrightarrow$ I'm getting sick

$\longrightarrow$ Property of I $\rightarrow$ (Sick)

$T(I) \longrightarrow$ I'm very tired

$\longrightarrow$ Tired (property of I)

can be expressed as

$$R(N) \wedge (S(I) \vee T(I))$$

* "It is raining in New York and I'm either not well or just very tired"

can be expressed as

$$R(N) \wedge (\neg W(I) \vee T(I)) \checkmark$$

and Not

$$R(N) \wedge \neg(W(I) \vee T(I)) \times$$

$\neg$ is associated with $W(I)$ only not with $T(I)$

* Use of implies ($\rightarrow$)

*For example,

"If it is raining then I will get wet"

can be translated into

$$R \rightarrow W(I)$$

$\left\{ \begin{array}{l} R \rightarrow \text{variable} \\ W \rightarrow \text{Predicate} \end{array} \right.$

can be further converted to $\neg R \vee W(I)$ (Bcoz, $A \rightarrow B \equiv \neg A \vee B$)

* "Whenever he eats sandwiches that have pickles in them, he ends up either asleep at his desk or singing loud songs"

can be translated into

$S(y) \rightarrow y$ is a sandwich

$P(y) \rightarrow y$ (sandwich) has pickles

$E(x, y) \rightarrow x$ (he) eats y (sandwich)

$A(x) \rightarrow x$ ends up asleep at desk

$S_1(x, z) \rightarrow x$ sings z

$L(z) \rightarrow z$ (songs) are loud

Here, S, P, E, A, S_1 & L are called Predicates/variables
(individual terms)

can be written as

$$(S(y) \wedge E(x, y) \wedge P(y)) \rightarrow A(x) \vee (S_1(x, z) \wedge L(z))$$

This expression carries more marks than the other one

This can also be written in terms of variables as

$$(S \rightarrow A \vee L)$$

Don't write this in exam

More accurate than this

$S \rightarrow$ He eats a sandwich that has pickles

$A \rightarrow$ He ends up asleep at his desk

$L \rightarrow$ He sings loud songs

Propositional Logic

* Definition

A proposition is a statement that can either be true or false

For example,

"5 is prime" → This is a proposition as this statement can be either true or false

on the other hand,

"When is this going to end?" → Not a proposition as it can't be true or false

"Read these notes" → Not a proposition as it cannot be evaluated to either T or F

Propositional Calculus

Propositional logic deals with simple propositions such as "I like cheese".

For more complex statements such as "All people who eat cheese like cats", require Predicate logics

Examples on Propositional Logics

1) It is raining today, and it is wet

R → It is raining today

W → It is wet

∴ $\boxed{R \land W}$

2) When I eat apples and pears, I usually like to have a walk.

$$\boxed{(A \land P) \longrightarrow W}$$

A → I eat apples

P → I eat Pears

W → I like to walk.

3) If the sun is out, we have class outside

S → The sun is out

C → We have class outside

i.e $\boxed{S \longrightarrow C}$

4) Given the following propositions, write the English translations.

a: the book has been out for a week

b: I don't have homework

c: I have finished reading the book.

$(a \land b) \rightarrow c$

Sol^n :- I have finished reading the book, if it has been out for a week and I don't have homework

5) If berries are ripe along the trail, hiking is safe if and only if grizzly bears have not been seen in the area.

A → Berries are ripe along the trail.

B → Hiking is safe

C → Grizzly bears have not been seen in the area.

$$A \rightarrow (B \leftrightarrow C)$$

if and only if

if --- then

6) I cannot hear you when you are far away

H → I can hear you

F → You are far away

$$\boxed{F \rightarrow \neg H}$$

7) If I come to Mexico, and I don't know what you like, I won't be able to find you

M → I come to Mexico

L → I ~~don't~~ know what you like

F → I will be able to find you

$$\boxed{M \land \neg L \rightarrow \neg F}$$

Predicate Calculus

In propositional calculus, we can express statements like "I like cheese" by A

But we cannot represent any additional information about the cheese, or me or any other things that I like.

In predicate calculus, we use predicates to express the properties of objects.

∴ The sentence "I like cheese" can be expressed as

$$L(me, cheese)$$

$L \longrightarrow$ predicate that represents the idea of liking.

This statement also expresses a relationship b/w me and cheese

Ex:- $T(A, B)$

could mean, Block A is ontop of Block B.

Ex1:- Use of Universal Quantifier

Everyone likes cheese

Inorder to express the above statement, the symbol $\forall$ (for all) is used.

$\therefore$ $\forall x \longrightarrow$ for every x

$P(x) \longrightarrow$ Property P of person x

This is required to show that we are referring to a person

$L(x, C) \longrightarrow$ Person x likes (L) cheese (C).

Here "$\forall$" is called the "Universal Quantifier"

$\therefore$ The above statement can be expressed as

$$\boxed{\forall x \; P(x) \longrightarrow L(x, c)}$$

Ex2:- Use of Existential Quantifier

Some people like cheese

We use '$\exists$' symbol to express for some

$\therefore$ The above statement can be expressed as

$$\exists x \; P(x) \longrightarrow L(x, c)$$

If we don't write $P(x)$, then it doesn't claim the possible values of 'x'

Examples on Predicate Logics

Some children will eat any food. (V. imp)
No children will eat food that is green
All children like food made by Cadbury's
$\therefore$ No food made by Cadbury's is green. (Conclusion)

Sol^n:- $C(x) \longrightarrow$ x is a child.
$F(x) \longrightarrow$ x is food.
$E(x,y) \longrightarrow$ x eats y
$L(x,y) \longrightarrow$ x likes y
$G(x) \longrightarrow$ x is green
$M(x,y) \longrightarrow$ x makes y
$c \longrightarrow$ Cadbury's

$\therefore$
(i) $\exists x \forall y (C(x) \wedge (F(y) \longrightarrow E(x,y)))$

(ii) $\forall x \forall y (C(x) \longrightarrow (F(y) \wedge G(y) \longrightarrow \neg E(x,y)))$

~~(iii) ∀x(F(x)) ∀x (C(x) → F(x) ∧ M(c,x))~~

(iii) $\forall x \forall y (F(x) \wedge M(c,x) \longrightarrow (C(y) \longrightarrow L(y,x)))$

(iv) $\therefore$ $\boxed{\forall x ((F(x) \wedge M(c,x)) \longrightarrow \neg G(x))}$ $\longrightarrow$ Conclusion

Conclusion needs to be proved if the Question is asked for 10 Marks. Otherwise, this much is sufficient

2) There is a vegetarian that doesn't eat cheese

$$\boxed{\exists x \forall y \left(V(x) \land \left(E(x,y) \rightarrow \neg K(y) \right) \right)}$$

$V(x) \rightarrow x$ is a Vegetarian

$E(x,y) \rightarrow x$ eats y

$K(y) \rightarrow y$ is a cheese.

3) Who doesn't eat meat, is a vegetarian

$V(x) \rightarrow x$ is a vegetarian

$M(x) \rightarrow x$ is a meat

$E(x,y) \rightarrow x$ eats y

$$\boxed{\forall x \forall y \left(\left(M(y) \land \neg E(x,y) \right) \rightarrow V(x) \right)}$$

4) You can fool all of the people some of the time

$P(x) \rightarrow x$ is a person

$T(x) \rightarrow x$ is time

$F(x,y) \rightarrow x$ ~~foots~~ can fool y

$$\boxed{\forall x \exists y \left(P(x) \rightarrow \left(T(x) \land F(x,y) \right) \right)}$$

(V. imp)

5) Translate the following into predicate logic;

(a) "You can fool all of the people some of the time, and you can fool some of the people all of the time but you can't fool all of the people all of the time." Negate your translation. Then translate the negation back ~~into~~ to English.

(b) "All penguins that sing, love a penguin that does not sing. No penguins dance. There is a penguin that loves all ~~penguing~~ penguins. All penguins that dance have a penguin that loves them. There is exists a penguin that loves all penguins who dance and sing"

Resolution in Propositional Logic (V. imp)

Resolution Rule

$$\frac{A \vee B \quad \neg B \vee C}{A \vee C}$$

can also be written as

$$\frac{\neg A \rightarrow B \quad B \rightarrow C}{\neg A \rightarrow C}$$

If a clause that contains literal L and another clause that contains literal ¬L, can be combined together, and L and ¬L can be removed from those clauses.

For example,

$\{(A, B), (\neg B, C)\}$

can be resolved to give

$\{(A, C)\}$

Similarly

$\{(A, B, C), D, (\neg A, D, E), (\neg D, F)\}$

can be resolved to give

$\{(B, C, D, E), D, (\neg D, F)\}$

can be further resolved to

$\{(B, C, D, E), F\}$

Resolution Refutation / Contradiction

Resolve the following clauses

$\{(\neg A, B), (\neg A, \neg B, C), A, \neg C\}$

will lead to

$\{(\neg A, C), A, \neg C\}$

$\{C, \neg C\}$

$\perp$ ⟶ Falsum / Contradiction

Hence Proved or $\boxed{\text{Resolution Refutation}}$

Proof by Refutation / Contradiction *** (V.v.imp)

(i) If it rains and I don't have an umbrella then I will get wet.

(ii) It is raining, and I don't have an umbrella.

(iii) Therefore, I will get wet. ⟶ (Conclusion)

Equivalent propositional ~~logic~~ calculus is

$A \longrightarrow$ It is raining

$B \longrightarrow$ I have an umbrella

$C \longrightarrow$ I will get wet

(i) $(A \wedge B) \longrightarrow C$

(ii) $A \wedge \neg B$

(iii) $\therefore C$ ⟶ Need to prove the conclusion

To prove this by contradiction / refutation, we first negate the conclusion and convert the expressions into clause form

Convert these to CNF form (i.e removing '$\rightarrow$' and '$\leftrightarrow$' by writing equivalent clauses)

equivalent symbol

i.e. $(A \wedge \neg B) \rightarrow C \equiv \neg(A \wedge \neg B) \vee C$

($\because A \rightarrow B \equiv \neg A \vee B$)

i.e. $\neg A \vee B \vee C$

($\because \neg(A \wedge B) \equiv \neg A \vee \neg B$)

To prove that our conclusion is valid, we need to show that

1st clause

Negation of 3rd clause

$\{(\neg A, B, C)\ (A, \neg B), \neg C\} \models \perp$

falsum

2nd clause

Negate the conclusion

When we resolve 1st 2 clauses, we get

$\{C, \neg C\}$

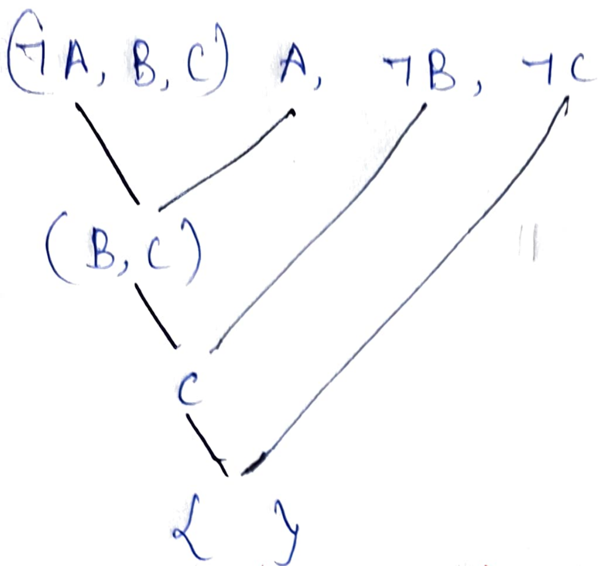
$\perp$

($\because$ A, $\neg$A leads to falsum)

We arrived at a condradiction. Hence, ~~our~~ we have proved that our original conclusion is valid.

i.e. I will get wet, holds

Proof in tree form



Empty or falsum or contradiction

Ex 2: (i) "If it is Sunday, it is not raining and Ram's friends have come, then he will go out to play.
(ii) Today is not Sunday but it is not raining.
(iii) Therefore, Ram will go out to play."
Prove by contradiction, whether the conclusion holds or not.

<u>Propositional Calculus</u>

$S \rightarrow$ It is Sunday.
$R \rightarrow$ It is Raining
$F \rightarrow$ Ram's friends have come
$P \rightarrow$ Ram will go out to play.

(i) $(S \wedge \neg R \wedge F) \longrightarrow P$

CNN form is

$\neg(S \wedge \neg R \wedge F) \vee P \equiv \neg S \vee R \vee \neg F \vee P$

(ii) $\neg S \land \neg R$

(iii) $\therefore P$

The Proof

$\{(\neg S, R, F, P), (\neg S, \neg R), \neg P\}$

Can be resolved to

$\{(\neg S, F, P), \neg P\}$

Can be further resolved to

$\{(\neg S, F)\}$ (No clauses left to resolve further)

Since we could not arrive at an empty clause/contradiction, our original conclusion is not valid

Proof in tree form

$(\neg S, R, F, P)$ $(\neg S, \neg R)$ $\neg P$

$(\neg S, F, P)$

$\{(\neg S, F)\}$ $\longrightarrow$ Not a falsum $(\perp)$

$\therefore$ The conclusion is not valid.