Advanced Counting Techniques

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Unit - 03

"Advanced Counting Techniques"

Sterling no. of Second Kind:-

Let A & B be two finite sets with $|A| = m$ & $|B| = n$ where $m \geq n$. then the no. of onto functions from A to B is given by,

$$P(m,n) = \sum_{k=0}^{n} (-1)^k \cdot {}^nC_{n-k} \cdot (n-k)^m$$

So, the Sterling number is given by,

$$S(m,n) = \frac{P(m,n)}{n!} = \frac{1}{n!} \cdot \sum_{k=0}^{n} (-1)^k \cdot {}^nC_{n-k} \cdot (n-k)^m$$

The Sterling no. represents the no. of possible ways of assigning 'm' objects into 'n' identical places with no place is left empty.

Note:- i) $S(m,1) = 1$ ii) $S(m,m) = 1$ $\forall\ m \geq 1$

iii) If empty places are allowed,

$$P(m) = \sum_{i=1}^{n} S(m,i) \quad \text{for } m \geq n$$

Problems:-

i) Let $A = \{1,2,3,4,5,6,7\}$ & $B = \{w,x,y,z\}$ Find the no. of onto functions from A to B.

$\longrightarrow$ $m = n(A) = 7$, $n = n(B) = 4$

$\therefore$ No. of onto functions from A to B $= P(7,4)$

W.K.T. $P(m,n) = \sum_{k=0}^{n} (-1)^k \cdot {}^nC_{n-k} \cdot (n-k)^m$

$$P(7,4) = \sum_{k=0}^{4} (-1)^k \, {}^4C_{4-k} \, (4-k)^7$$

$$= {}^4C_4 \cdot 4^7 - {}^4C_3 \cdot 3^7 + {}^4C_2 \cdot 2^7 - {}^4C_1 \cdot 1^7 + 0$$

$$= 4^7 - 4 \cdot 3^7 + 6 \cdot 2^7 - 4 \cdot = \underline{8,400}$$

* Evaluate i) $S(5,4)$ ii) $S(8,6)$.

$\longrightarrow$ (i) $S(5,4)$ $\quad \sum^{n} (-1)^k \, {}^nC \ldots (n-k)^m$

→ (i) $S(5,4)$

WKT, $S(m,n) = \frac{P(m,n)}{n!} = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k \cdot {}^nC_{n-k}\,(n-k)^m$

$$\therefore\ S(5,4) = \frac{1}{4!}\sum_{k=0}^{4}(-1)^k \cdot {}^4C_{4-k} \cdot (4-k)^5$$

$$= \frac{1}{4!}\left[{}^4C_4 \cdot 4^5 - {}^4C_3 \cdot 3^5 + {}^4C_2 \cdot 2^5 - {}^4C_1 \cdot\right]$$

$$= \frac{1}{4!}\left[4^5 - 4\cdot 3^5 + 6\cdot 2^5 - 4\right] = \underline{10}$$

* There are six programmers who can assist eight executives. In how many ways can the executives be assisted so that each programmer assists atleast one executive?

→ Let the set of executives = A & the set of programmers = B.

$\therefore\ |A| = 8 \qquad |B| = 6$

Here no. of ways of executives being assisted by the programmers $\Big\}$ $P(m,n) = P(8,6) = \sum_{k=0}^{6}(-1)^k \cdot {}^6C_{6-k}\,(6-k)^8$

$$= 6^8 - {}^6C_5 \cdot 5^8 + {}^6C_4 \cdot 4^8 - {}^6C_3 \cdot 3^8 + {}^6C_2 \cdot 2^8 - {}^6C_1 \cdot 1^8 + 0$$

$$= 6^8 - 6\cdot 5^8 + 15\cdot 4^8 - 20\cdot 3^8 + 15\cdot 2^8 - 6\cdot(1)$$

$$= \underline{\underline{1{,}91{,}520}}$$

Properties of Sterling no. of Second Kind :-

i) $S(m,1) = 1, \quad m \geq 1$

ii) $S(m,m) = 1, \quad m \geq 1$

iii) If 'm' objects are kept in 'n' identical places with $m \geq n$, then Sterling no. of second kind can be written as,

$$\boxed{S(m,n) = S(m-1,n-1) + n\cdot S(m-1,n)}$$

$S(3,2)$
$= S(2,1) + 2\cdot S(2,2)$.
$= 1 + 2(1) = \underline{3}$

Table consisting of possible values of Sterling no.s.

S.N.

Table consisting of possib...

'm'	S.N.
1	$S(1,1) = 1$
2	$S(2,1) = 1$, $S(2,2) = 1$
3	$S(3,1) = 1$, $S(3,2) = 3$, $S(3,3) = 1$
4	$S(4,1) = 1$, $S(4,2) = 7$, $S(4,3) = 6$, $S(4,4) = 1$
⋮	⋮

$S(m,1) = 1$

$S(m,m) = 1$

* Find $S(5,4)$ using the recurring formula.

→ WKT, $S(m,n) = S(m-1,n-1) + n\cdot S(m-1,n)$

$$S(5,4) = S(4,3) + 4\cdot S(4,4)$$
$$= S(3,2) + 3\cdot S(3,3) + 4\cdot S(4,4)$$
$$= S(2,1) + 2\cdot S(2,2) + 3\cdot S(3,3) + 4\cdot S(4,4)$$
$$= 1 + 2\cdot(1) + 3(1) + 4(1) = \underline{10}$$

* If A & B are two finite sets with $n(A) = 5$ & $n(B) = 3$ then find the no. of onto functions from A to B.

→ No. of onto functions from A to B $= P(m,n)$.

Here $m = 5$, $n = 3$

$$\therefore P(5,3) = \sum_{k=0}^{3} (-1)^k\, {}^3C_{3-k} \cdot (3-k)^5$$

$$= {}^3C_3 \cdot 3^5 - {}^3C_2 \cdot 2^5 + {}^3C_1 \cdot 1^5 - 0$$

$$= 3^5 - 3\cdot 2^5 + 1^5 = \underline{150}$$

* Evaluate $S(8,7)$ given that $S(7,6) = 21$

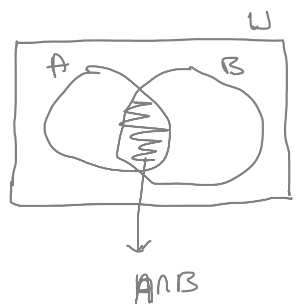
→ WKT, $S(m,n) = S(m-1,n-1) + n\cdot S(m-1,n)$

$$S(8,7) = S(7,6) + 7\cdot S(7,7)$$
$$= 21 + 7\cdot(1) = \underline{28}$$

Principle of Inclusion & Exclusion:-

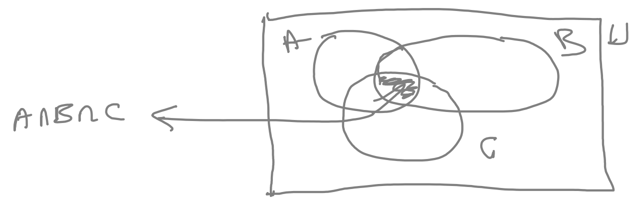
Statement: Let A & B be any two sets. then the number of elements in the Union of A & B is equal to the sum of nos of elements in A & B & difference with the intersection of A & B.

i.e. $$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$



Similarly:- If A, B & C are any three sets then the no. of elements in the union of A, B & C is given by

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$



Pigeonhole principle:-

Statement: If 'm' Pigeons occupy 'n' pigeonholes with $m > n$ then - two or more pigeons occupy the same pigeonhole.

In other words, If 'm' pigeons occupy 'n' pigeonholes & $m > n$ then atleast one pigeonhole must contain two or more pigeons in it.

Generalized Pigeonhole principle:

Statement:- "If 'm' pigeons occupy 'n' pigeonholes & $m > n$ then atleast one pigeonhole must contain atleast $(p+1)$ pigeons in it. where $P = \left[\frac{m-1}{n}\right]$"

proof → We prove by the method of contradiction.

Suppose every pigeonhole contains not more than 'p' pigeons.

$\Rightarrow$ Every pigeonhole contains not more than $\left(\frac{m-1}{n}\right)$ pigeons.

Therefore 'n' pigeonholes contain $\leq \cancel{n} \cdot \left(\frac{m-1}{\cancel{n}}\right)$ pigeons

'n' pigeonholes contain $\leq$ $m-1$ pigeons

It is contradicting our fact that there are 'm' pigeon. Therefore our assumption is wrong. Hence the proof//

$\therefore$ Atleast one pigeonhole contains atleast $\left(\frac{m-1}{n}\right)+1$ pigeons//

Problems:-

* P.T. in a set of 13 children at two have their birthdays in the same month.

$\rightarrow$ Let us take children as pigeons & months as pigeonholes.

Then we have 13 pigeons & 12 pigeonholes. $\Rightarrow m=13, n=12$

By generalised pigeonhole principle, we have atleast one month which has $\left[\frac{(m-1)}{n}\right]+1$ births in it.

$$\therefore \left[\frac{13-1}{12}\right]+1 = \left[\frac{12}{12}\right]+1 = 2$$

$\therefore$ Atleast 2 children have their births in the same month.

If seven cars carry 26 passengers, prove that atleast one car must have 4 or more passengers.

$\rightarrow$ Here 26 passengers = 26 pigeons & 7 cars = 7 pigeonholes.

$\therefore m=26$ & $n=7$.

$\therefore$ By generalized pigeonhole principle, atleast one car must have $\left(\frac{m-1}{n}\right)+1$ passengers.

$$= \left[\frac{26-1}{7}\right]+1 = \left[\frac{25}{7}\right]+1 = [3.5]+1 = 4.$$

$\therefore$ atleast one car must have 4 or more passengers//

* What should be the minimum no. of students, so that atleast two students have their last name beginning with the same english letter.

$\rightarrow$ Here m no. of students = m. pigeons & 26 english alphabets = 26 pigeonholes.

have their last name beginning with the same english letter.

→ Here m no. of students = m. pigeons & 26 english alphabets = 26 pigeonholes.

Given that, atleast 2 students have their name beginning with same letter.

$$\therefore \left\lceil \frac{m-1}{n} \right\rceil + 1 = 2$$

$$\frac{m-1}{26} + 1 \geq 2$$

$$\frac{m-1}{26} \geq 1 \implies m-1 \geq 26$$

$$m \geq 27$$

∴ Minimum no. of students is 27. so that atleast 2 students have their last name beginning with same letter.

* Find the least no. of ways of choosing three different no.s from 1 to 10 so that all choices have the same sum.

→ No. of ways of choosing 3 different no.s from 1 to 10 = $^{10}C_3$ = 120 = m

The least sum we can have = 1+2+3 = 6

The greatest sum we can have = 8+9+10 = 27

∴ No. of different sums we can have = 22 = n

Let us take 120 = m & 22 = n

∴ the least no. of choices of 3 no.s to have the same sum is given by,

$$= \left\lceil \frac{m-1}{n} \right\rceil + 1$$

$$= \left\lceil \frac{120-1}{22} \right\rceil + 1 = \left\lceil \frac{119}{22} \right\rceil + 1 = \lceil 5.409 \rceil + 1 = 6.$$

∴ least no. of choices of 3 no.s to have the sam sum = 6

Recurrance Relations :-

A recurrance relation for the sequence $\{a_n\}$ is an equation that expresses 'a_n' in terms of one or more of its previous terms of the sequence, namely $a_0, a_1, a_2, \ldots, a_{n-1}$ for all $n \geq 1$.

A sequence $\{a_n\} = \{a_0, a_1, a_2, \ldots\}$ is called a solution ... recurrence relation.

A sequence $\{a_n\} = \{a_0, a_1, a_2, ----\}$ is called a solution of a reccurrence relation if its terms satisfy the reccurrence relation.

Example:-

i) A reccurrence relation $a_n = 2 \cdot a_{n-1} - a_{n-2}$, $\forall\, n = 2,3,4,---$ Satisfy the sequence $\{a_n\}$ where $a_n = 3n$.

$\longrightarrow$
$$\begin{aligned} RHS &= 2a_{n-1} - a_{n-2} \\ &= 2(3(n-1)) - 3(n-2) \\ &= 6n - \cancel{6} - 3n + \cancel{6} \\ &= 3n = a_n = LHS \end{aligned}$$

Note:- A reccurrance relation can also be written as

i) $4y_{n+3} - y_{n+2} + 11y_{n+1} - 6y_n = 0$

ii) $4a_{n+3} - a_{n+2} + 11a_{n+1} - 6a_n = 0$

iii) $4u_{n+3} - u_{n+2} + 11u_{n+1} - 6u_n = 0$

iv) $4f(x+3) - f(x+2) + 11 \cdot f(x+1) - 6 \cdot f(x) = 0$.

Order of reccurrance relation:-

The order of reccurrance relation is the difference between the largest and the smallest subscript appearing in the relation.

Ex:- $a_{n+2} = a_{n-1} + 2a_n + 3a_{n+3}$ $\qquad (n+3) - (n-1) = 4$. is the order.

Degree of a reccurrance relation:-

The degree of a reccurrance relation is the degree of the function, 'f' in the equation, $a_n = f(a_{n-1}, a_{n-2}, --- a_{n-k})$ where $a_{n-1}, a_{n-2}, --- a_{n-k}$ are all variables & 'n' is a constant.

The function 'f' needs to be a polynomial, otherwise no degree is assigned to the given reccurrance relation.

Linear reccurrance relation with constant co-efficients:-

The linear reccurrance relation with constant co-efficient of order 'k' is written as,

$$C_0 a_n + C_1 a_{n-1} + C_2 \cdot a_{n-2} + C_3 a_{n-3} + ---- + C_k \cdot a_{n-k} = f(n) \quad — (1)$$

... ...ction of

is ... as,

$$C_0 a_n + C_1 a_{n-1} + C_2 \cdot a_{n-2} + C_3 a_{n-3} + \cdots + C_k \cdot a_{n-k} = f(n) \quad \text{--- (1)}$$

where $C_0, C_1, C_2 \cdots C_k$ are constant co-efficients & $f(n)$ is a function of variable 'n'.

Note:- If $f(n) = 0$ then (1) is called Homogeneous linear reccurrance relation.
& if $f(n) \neq 0$ then (1) is called Non-homogeneous Linear reccurrance relation.

Solution of homogeneous linear reccurrance relation :

i) Characteristic Roots method:

Consider a linear homogeneous reccurrance relation

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k \cdot a_{n-k} = 0 \quad \text{--- (1)}$$

Characteristic/Auxiluary equation of degree 'k' of (1) is given by putting

$a_n = \alpha^k$, $a_{n-1} = \alpha^{k-1}$, $a_{n-2} = \alpha^{k-2}$, $\cdots$, $a_{n-k} = \alpha^0$

$\therefore$ from (1).

$$C_0 \cdot \alpha^k + C_1 \alpha^{k-1} + C_2 \alpha^{k-2} + \cdots + C_k \alpha^0 = 0 \quad \text{--- (2)}$$

$\therefore$ Roots of (2) are (say) $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_k$

The solution of (1) depends on the nature of roots of (2)

case i] When the roots are real & distinct.

then, Solution of (1) is of the form,

$$\boxed{a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + A_3 \alpha_3^n + \cdots + A_k \cdot \alpha_k^n}$$

where $A_1, A_2, A_3 \cdots A_k$ are arbitrary constants.

case ii] When the roots are real & repeated,

If $\alpha_1 = \alpha_2$ & other roots are different then we have,

$$a_n = (A_1 + A_2 n)\alpha_1^n + A_3 \alpha_3^n + A_4 \alpha_4^n + \cdots + A_k \cdot \alpha_k^n$$

... roots are different then we have,

If $\alpha_1 = \alpha_2 = \alpha_3$ & other roots are different then we have,

$$a_n = (A_1 + A_2 n + A_3 n^2)\alpha_1^n + A_4 \alpha_4^n + \cdots + A_k \cdot \alpha_k^n$$

If all roots are equal, then we have,

$$a_n = \left[A_1 + A_2 n + A_3 n^2 + A_4 n^3 + \cdots + A_k \cdot n^{k-1}\right] \alpha_1^n.$$

case iii] When roots are complex.

If $\alpha_1 \pm i\beta_1$ be the complex roots, then the general solution will be,

$$a_n = A_1(\alpha_1 + i\beta_1)^n + A_2(\alpha_1 - i\beta_1)^n.$$

case iv] When complex roots are equal.

If $\alpha_1 \pm i\beta_1 = \alpha_2 \pm i\beta_2$ then the general solution will be,

$$\begin{cases} \alpha_1 + i\beta_1 = \alpha_2 + i\beta_2 \\ \alpha_1 - i\beta_1 = \alpha_2 - i\beta_2 \end{cases}$$

$$a_n = (A_1 + A_2 n)(\alpha_1 + i\beta_1)^n + (A_3 + A_4 n)(\alpha_1 - i\beta_1)^n$$

Examples/ Problems:-

* Solve the linear homogeneous reccurrance relation $a_{n+2} - 3 \cdot a_{n+1} + 2a_n = 0$ by Characteristic roots method.

$\rightarrow$ Let $a_{n+2} - 3a_{n+1} + 2a_n = 0$ — (1)

Put $a_{n+2} = \alpha^2$, $a_{n+1} = \alpha$, $a_n = \alpha^0 = 1$

from (1)

A.E. $\alpha^2 - 3\alpha + 2(1) = 0$

$\alpha^2 - 3\alpha + 2 = 0 \Rightarrow \alpha = 1, 2$

$\therefore \alpha_1 = 1, \alpha_2 = 2$ roots are real & unequal.

$\therefore$ G.S. $a_n = A_1\alpha_1^n + A_2\alpha_2^n = A_1(1)^n + A_2(2)^n$

$$\boxed{a_n = A_1 + A_2 \cdot 2^n}$$

* Solve the linear homogeneous reccurrance relation $a_{n+2} - 2a_{n+1} + 4a_n = 0$ by Characteristic root method.

Sol^n $\rightarrow$ Given, $a_{n+2} - 2a_{n+1} + 4a_n = 0.$ — (1)

put $a_{n+2} = \alpha^2$, $a_{n+1} = \alpha$, $a_n = 1$

$\therefore$ from (1).

put ...

$\therefore$ from (1).

$$\alpha^2 - 2\alpha + 4 = 0$$

$$\Rightarrow \alpha = 1 \pm i\sqrt{3}$$

$\therefore$ G.S. $a_n = A_1(\alpha_1 + i\beta_1)^n + A_2(\alpha_1 - i\beta_1)^n$

$$\therefore a_n = A_1(1+i\sqrt{3})^n + A_2(1-i\sqrt{3})^n //$$

* Solve the linear homogeneous reccurrance relation $a_n - 8a_{n-1} + 16a_{n-2} = 0$ with initial conditions $a_2 = 6$, $a_3 = 80$ by characteristic root method.

$\longrightarrow$ Given, $a_n - 8 \cdot a_{n-1} + 16a_{n-2} = 0$ — (1)

put $a_n = \alpha^2$, $a_{n-1} = \alpha$, $a_{n-2} = 1$

from (1).

$$\alpha^2 - 8\alpha + 16 = 0 \Rightarrow (\alpha - 4)^2 = 0$$

$$\alpha = 4, 4 \Rightarrow \alpha_1 = 4 \ \& \ \alpha_2 = 4$$

$\therefore$ G.S. $a_n = (A_1 + nA_2)4^n$ — (2)

put $n = 2$, $a_2 = (A_1 + 2A_2)4^2$

$$6 = (A_1 + 2A_2)16$$

$$A_1 + 2A_2 = 3/8 \Rightarrow 8A_1 + 16A_2 = 3 \ — (3)$$

put $n=3$ in (2). $a_3 = (A_1 + 3A_2)4^3$

$$80 = (A_1 + 3A_2)64$$

$$5 = (A_1 + 3A_2)4 \Rightarrow 4A_1 + 12A_2 = 5 \ — (4)$$

Solving (3) & (4)

$$8A_1 + 16A_2 = 3$$
$$4A_1 + 12A_2 = 5$$

$$\cancel{8A_1} + 16A_2 = 3$$
$$-\cancel{8A_1} + 24A_2 = 10$$
$$-8A_2 = -7$$

$$\boxed{A_2 = 7/8}$$

111^{ly} $\boxed{A_1 = -11/8}$

$\therefore$ from (2)

$$a_n = \left(\frac{-11}{8} + n \cdot \frac{7}{8}\right)4^n = \frac{4^n}{4}\left(\frac{7n-11}{2}\right)$$

$$\boxed{a_n = 4^{n-1}\left(\frac{7n-11}{2}\right)} //$$

$$\boxed{a_n = 4^{n-1}\left(\frac{7n-1}{2}\right)}$$

* Solve the homogeneous linear reccurrance relation $a_n - 7a_{n-2} + 6a_{n-3} = 0$ with the conditions $a_0 = 8$, $a_1 = 6$ & $a_2 = 22$.

→ Here, $a_n - 7a_{n-2} + 6a_{n-3} = 0$ — (1)

put $a_n = \alpha^3$, $a_{n-1} = \alpha^2$, $a_{n-2} = \alpha$, $a_{n-3} = 1$

from (1) A.E. $\alpha^3 - 7\alpha + 6 = 0 \Rightarrow \alpha^3 + 0\cdot\alpha^2 - 7\alpha + 6 = 0$

$$\begin{array}{r|rrrr} & 1 & 0 & -7 & 6 \\ \alpha=1 & 0 & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & \boxed{0} \end{array}$$

$\therefore \alpha^2 + \alpha - 6 = 0$

$\alpha = -3, 2$

$\therefore \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = -3$

$\therefore$ G.S. $a_n = A_1(1)^n + A_2(2)^n + A_3(-3)^n$ — (2)

put $n=0$, $a_0 = A_1 + A_2 + A_3$

$A_1 + A_2 + A_3 = 8$ — (3)

put $n=1$, $a_1 = A_1 + A_2 2 + A_3(-3)$

$A_1 + 2A_2 - 3A_3 = 6$ — (4)

put $n=2$, $a_2 = A_1 + A_2 4 + A_3 9$

$A_1 + 4A_2 + 9A_3 = 22$ — (5)

from (3) & (4)

$$\begin{array}{rcl} \cancel{A_1} + A_2 + A_3 &=& 8 \\ \cancel{A_1} + 2A_2 - 3A_3 &=& 6 \\ - \quad - \quad + && - \\ \hline -A_2 + 4A_3 &=& 2 \end{array}$$ — (6)

from (3) & (5)

$$\begin{array}{rcl} \cancel{A_1} + A_2 + A_3 &=& 8 \\ \cancel{A_1} + 4A_2 + 9A_3 &=& 22 \\ - \quad - \quad - && - \\ \hline -3A_2 - 8A_3 &=& -14 \end{array}$$ — (7)

from (6) & (7)

$-A_2 + 4A_3 = 2$

$2A_2 - \cancel{8A_3} = -4$

$-3A_2 - \cancel{8A_3} = -14$

from ⑥ & ⑦

$$-A_2 + 4A_3 = 2$$
$$-3A_2 - 8A_3 = -14$$

$$2A_2 - \cancel{8A_3} = -4$$
$$\underline{-3A_2 - \cancel{8A_3} = -14}$$
$$5A_2 = 10 \Rightarrow \boxed{A_2 = 2}$$

$\therefore$ from ⑥ $\quad -2 + 4A_3 = 2 \Rightarrow \boxed{A_3 = 1}$

from ③ $\quad A_1 + 2 + 1 = 8 \Rightarrow \boxed{A_1 = 5}$

$\therefore$ from eqn ②

$$a_n = 5(1)^n + 2(2)^n + 1(-3)^n$$

$$\boxed{a_n = 5 + 2^{n+1} + (-3)^n}$$

Generating functions :-

Defn :- Let $\{a_n\}$ or $a_0, a_1, a_2, ----, a_n$ be a sequence of real numbers. Then the Generating function denoted by & defined by,

$$G(a,z) = a_0 + a_1 z + a_2 z^2 + ---- + a_n z^n + --- = \sum_{n=0}^{\infty} a_n z^n$$

where 'z' is a variable.

Generating function of some standard sequences :-

① If $\{a_n\}$ is a sequence and $a_n = c$ (a constant) then,

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ----$$

$$= c + cz + cz^2 + cz^3 + ----$$

$$= c[1 + z + z^2 + ----]$$

$$= c \cdot \left(\frac{1}{1-z}\right)$$

$$\therefore \boxed{G(a,z) = \frac{c}{1-z}}$$

② If $\{a_n\}$ is a sequence and $a_n = b^n$, then

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b^n \cdot z^n = \sum_{n=0}^{\infty} (b \cdot z)^n$$

$= 1 + bz + (bz)^2 + (bz)^3 + \cdots$

$\boxed{G(a,z) = \frac{1}{1-bz}}$ where $a_n = b^n$

③ If $\{a_n\}$ is a sequence & $a_n = c \cdot b^n$, then, $\boxed{G(a,z) = \frac{c}{1-bz}}$

④ If $\{a_n\}$ is a sequence & $a_n = n \quad \forall n \geq 0$ then,

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} n \cdot z^n = z + 2z^2 + 3z^3 + 4z^4 + \cdots$$

$$= z\left[1 + 2z + 3z^2 + 4z^3 + \cdots\right]$$

$$= z(1-z)^{-2} = z \cdot \left[\frac{1}{(1-z)^2}\right]$$

$$\therefore \boxed{G(a,z) = \frac{z}{(1-z)^2}}$$

i.e. Sequence	Generating function
1) $a_n = c$	$G(a,z) = \frac{c}{1-z}$
2) $a_n = b^n$	$G(a,z) = \frac{1}{1-bz}$
3) $a_n = c \cdot b^n$	$G(a,z) = \frac{c}{1-bz}$
4) $a_n = n$	$G(a,z) = \frac{z}{(1-z)^2}$

II Solution of Homogeneous linear reccurrance relation by Generating functions:-

Consider a homogeneous linear reccurrance relation,

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k \cdot a_{n-k} = 0 \quad \text{—①} \quad \forall n \geq k.$$

Step 1] Multiply both sides of eqⁿ ① by z^n & take summation from $n=k$ to ∞

Step 2] Write each term in the form of $G(a,z)$.

Step 3] Solve $G(a, z)$ by using standard generating functions for $\{a_n\}$.

Problems :-

1> Solve the homogeneous linear reccurrance relation $U_n = U_{n-1} + U_{n-2}$ $\forall n \geq 2$ given that $U_1 = 1$, $U_2 = 3$ by generating functions.

Sol^n $\rightarrow$ Let $U_n = U_{n-1} + U_{n-2}$ $\forall n \geq 2$

put $n = 2$ $\quad U_2 = U_1 + U_0 \Rightarrow 3 = 1 + U_0 \Rightarrow \boxed{U_0 = 2}$

$$U_n - U_{n-1} - U_{n-2} = 0 \quad \text{--- (1)} \quad \forall n \geq 2$$

multiplying B.S. of (1) by z^n & taking summation from $n = 2$ to ∞.

$$\sum_{n=2}^{\infty} U_n z^n - \sum_{n=2}^{\infty} U_{n-1} \cdot z^n - \sum_{n=2}^{\infty} U_{n-2} \cdot z^n = 0$$

$$[U_2 z^2 + U_3 z^3 + U_4 z^4 + \cdots] + U_0 + U_1 z - U_0 - U_1 z - \sum_{n=2}^{\infty} U_{n-1} z^n - \sum_{n=2}^{\infty} U_{n-2} \cdot z^n = 0$$

$$\sum_{n=0}^{\infty} U_n z^n - U_0 - U_1 z - [U_1 z^2 + U_2 z^3 + U_3 z^4 + \cdots] - \sum_{n=2}^{\infty} U_{n-2} \cdot z^n = 0$$

$$\sum_{n=0}^{\infty} U_n z^n - U_0 - U_1 z - z[U_1 z + U_2 z^2 + U_3 z^3 + \cdots + U_0 - U_0] - \sum_{n=2}^{\infty} U_{n-2} \cdot z^n = 0$$

$$\sum_{n=0}^{\infty} U_n z^n - U_0 - U_1 z - z\left[\sum_{n=0}^{\infty} U_n z^n - U_0\right] - \sum_{n=2}^{\infty} U_{n-2} \cdot z^n = 0$$

$$\sum_{n=0}^{\infty} U_n z^n - U_0 - U_1 z - z \cdot \sum_{n=0}^{\infty} U_n z^n + U_0 z - [U_0 z^2 + U_1 z^3 + U_2 z^4 + \cdots] = 0$$

$$U_n z^n - U_0 - U_1 z - z \quad U_n z^n + U_0 z - z^2 [U_0 + U_1 z + U_2 z^2 + \cdots] = 0$$

$$\sum U_n z^n - U_0 - U_1 z - z \sum_{n=0}^{\infty} U_n z^n + U_0 z - z^2 \sum_{n=0}^{\infty} U_n z^n = 0$$

$$G(U, z) - 2 - z - z \cdot G(U, z) + 2z - z^2 \cdot G(U, z) = 0$$

$$G(U, z)[1 - z - z^2] = 2 + z - 2z$$

$$G(U, z) = \frac{2 - z}{} = \frac{z - 2}{}$$

$G(u,z)\left[1-z-z^2\right] \ldots$

$$G(u,z) = \frac{2-z}{1-z-z^2} = \frac{z-2}{z^2+z-1}$$

$$G(u,z) = \frac{z-2}{z^2+z-1} \quad \text{—— (2)}$$

$$G(u,z) = \frac{z-2}{\left(z^2+z+\frac{1}{4}\right)-\frac{1}{4}-1} = \frac{z-2}{\left(z+\frac{1}{2}\right)^2-\frac{5}{4}} = \frac{z-2}{\left(z+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2} = \frac{z-2}{\left(z+\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\cdot\left(z+\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}$$

$$G(u,z) = \frac{z-2}{\left(z+\frac{1+\sqrt{5}}{2}\right) \ldots} \qquad \frac{A}{\left(z+\frac{1+\sqrt{5}}{2}\right)} + \frac{B}{\left(z+\frac{1-\sqrt{5}}{2}\right)} \quad \text{—— (3)}$$

$$z-2 = A\left(z+\frac{1-\sqrt{5}}{2}\right) + B\left(z+\frac{1+\sqrt{5}}{2}\right)$$

put $z = -\left(\frac{1+\sqrt{5}}{2}\right)$

$$-\left(\frac{1+\sqrt{5}}{2}\right)-2 = A\left(-\left(\frac{1+\sqrt{5}}{2}\right)+\frac{1-\sqrt{5}}{2}\right)$$

$$\frac{-1-\sqrt{5}-4}{2} = A\left[\frac{-\cancel{1}-\sqrt{5}+\cancel{1}-\sqrt{5}}{2}\right]$$

$$\frac{-5-\sqrt{5}}{\cancel{2}} = A\left(\frac{-2\sqrt{5}}{\cancel{2}}\right) \Rightarrow A = \frac{-5-\sqrt{5}}{-2\sqrt{5}} = \frac{5+\sqrt{5}}{2\sqrt{5}}$$

$$\boxed{A = \frac{1+\sqrt{5}}{2}}$$

put $z = -\left(\frac{1-\sqrt{5}}{2}\right)$ $\qquad \boxed{B = \frac{1-\sqrt{5}}{2}}$

$\therefore$ from (3)

$$G(u,z) = \frac{\left[\left(1+\sqrt{5}\right)/2\right]}{\left(z+\frac{1+\sqrt{5}}{2}\right)} + \frac{\left(1-\sqrt{5}\right)/2}{\left(z+\frac{1-\sqrt{5}}{2}\right)}$$

$$= \frac{1}{\left(z+\frac{1+\sqrt{5}}{2}\right)\Big/\left(\frac{1+\sqrt{5}}{2}\right)} + \frac{1}{\left(z+\frac{1-\sqrt{5}}{2}\right)\Big/\left(\frac{1-\sqrt{5}}{2}\right)}$$

$$= \frac{1}{\frac{z}{\left(\frac{1+\sqrt{5}}{2}\right)}+1} + \frac{1}{\frac{z}{\left(\frac{1-\sqrt{5}}{2}\right)}+1}$$

$$= \frac{1}{\ldots} + \frac{1}{1+z\left(\frac{2}{\ldots}\right)}$$

$$= \frac{1}{1+z\left(\frac{2}{1+\sqrt{5}}\right)} + \frac{1}{1+z\left(\frac{2}{1-\sqrt{5}}\right)}$$

$$G(u,z) = \frac{1}{1-z\left(\frac{-2}{1+\sqrt{5}}\right)} + \frac{1}{1-z\left(\frac{-2}{1-\sqrt{5}}\right)}$$

WKT, $G(u,z) = \frac{1}{1-bz} \Rightarrow u_n = b^n$

$$\therefore \quad G(u,z) = \left(\frac{-2}{1+\sqrt{5}}\right)^n + \left(\frac{-2}{1-\sqrt{5}}\right)^n \quad \text{—— } ③$$

$$= (-2)^n \left[\frac{1}{(1+\sqrt{5})^n} + \frac{1}{(1-\sqrt{5})^n}\right]$$

$$= (-2)^n \left[\frac{(1-\sqrt{5})^n + (1+\sqrt{5})^n}{(1+\sqrt{5})^n(1-\sqrt{5})^n}\right]$$

$$\left[(1+\sqrt{5})(1-\sqrt{5})\right]^n$$

$$\left[1^2 - (\sqrt{5})^2\right]^n$$

$$(1-5)^n$$

$$= (-2)^n \left[\frac{(1-\sqrt{5})^n + (1+\sqrt{5})^n}{(1-5)^n}\right]$$

$$= (-2)^n \left[\frac{(1-\sqrt{5})^n + (1+\sqrt{5})^n}{(-4)^n}\right]$$

$$(-4)^n = 2^n(-2)^n$$

$$= \frac{(1-\sqrt{5})^n + (1+\sqrt{5})^n}{2^n}$$

$$\boxed{u_n = \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^n}$$

* Solve homogeneous linear reccurrance relation $C_n = 3C_{n-1} - 2C_{n-2}$ $\forall\, n \geq 2$, given initial conditions $c_1 = 5$ & $C_2 = 3$ by generating function.

soln →

$$C_n = 3C_{n-1} - 2C_{n-2} \quad \forall\, n \geq 2 \quad \text{—— } ①$$

put $n=2$, $C_2 = 3C_1 - 2C_0$

$3 = 3(5) - 2C_0 \Rightarrow 2C_0 = 15 - 3 = 12$

$$\boxed{C_0 = 6}$$

from ①.

$$C_n - 3C_{n-1} + 2C_{n-2} = 0$$

multiply z^n on B.S. & take the summation from $n=2$ to ∞

$$\sum_{n=2}^{\infty} C_n z^n - 3\sum_{n=2}^{\infty} C_{n-1} \cdot z^n + 2\sum_{n=2}^{\infty} C_{n-2} z^n = 0$$

$$[C_2 z^2 + C_3 z^3 + C_4 z^4 + \cdots] + C_0 + C_1 z - C_0 - C_1 z - 3\sum_{n=2}^{\infty} C_{n-1} z^n + 2\sum_{n=2}^{\infty} C_{n-2} z^n = 0$$

$$\sum_{n=0}^{\infty} C_n z^n - C_0 - C_1 z - 3[C_1 z^2 + C_2 z^3 + C_3 z^4 + \cdots] + 2\sum_{n=2}^{\infty} C_{n-2} z^n = 0$$

$$\sum_{n=0}^{\infty} C_n z^n - C_0 - C_1 z - 3z \; C_1 z + C_2 z^2 + C_3 z^3 + \cdots + C_0 - C_0] + 2\sum_{n=2}^{\infty} C_{n-2} z^n = 0$$

$$\sum_{n=0}^{\infty} C_n z^n - C_0 - C_1 z - 3z \; \sum_{n=0}^{\infty} C_n z^n - C_0] + 2[C_0 z^2 + C_1 z^3 + C_2 z^4 \cdots] = 0$$

$$\sum_{n=0}^{\infty} C_n z^n - C_0 - C_1 z - 3z\sum_{n=0} C_n z^n + 3zC_0 + 2z^2[C_0 + C_1 z + C_2 z^2 + \cdots] = 0$$

$$\sum^{\infty} C_n z^n - C_0 - C_1 z - 3z\sum_{n=0}^{\infty} C_n z^n + 3zC_0 + 2z^2\sum_{n=0}^{\infty} C_n z^n = 0$$

WKT, $\sum_{n=0} a_n z^n = G(a,z)$ & $C_0 = 6,\; C_1 = 5,\; C_2 = 3$

$$G(c,z) - 6 - 5z - 3z\,G(c,z) + 18z + 2z^2 G(c,z) = 0$$

$$G(c,z)[1 - 3z + 2z^2] = 6 + 5z - 18z$$

$$G(c,z) = \frac{6-13z}{2z^2 - 3z + 1} = \frac{6-13z}{(2z-1)(z-1)} = \frac{A}{} \quad \frac{}{z-1} \quad \text{—— (2)}$$

$$6 - 13z = A(z-1) + B(2z-1)$$

$z=1$, $\boxed{-7 = B}$ when $z = 1/2$ $\quad -1/2 = A(-1/2) \Rightarrow \boxed{A = 1}$

$\therefore$ from (2) $\quad G(c,z) = \dfrac{1}{1-2z} + 7 \cdot \dfrac{1}{1-z}$

$$G(c,z) = (-1)\frac{1}{1+2z} + 7 \cdot \frac{1}{1-z} \quad \text{—— (3)}$$

WKT, $G(a,z) = \dfrac{1}{1-bz}$ then $a_n = b^n$

$\therefore$ from (3)

$$C_n = (-1)2^n + 7 \cdot (1)$$

$\therefore \quad C_n = (-1)2^n + 7$

$$\boxed{C_n = 7 - 2^n}$$

Solution of Non-homogeneous linear reccurrance relation by Generating function:-

* ~~Solve non-homogeneous linear reccurrance relation~~ $a_{n+2} - 2a_{n+1} + a_n = 2^n \ \forall n \geq 0$. given initial conditions are $a_0 = 2$, $a_1 = 1$ by generating function.

Sol^n >

$$a_{n+2} - 2a_{n+1} + a_n = 2^n \ \forall n \geq 0 \quad \text{— (i)}$$

put $n=0$,

$$a_2 - 2a_1 + a_0 = 1$$

$$a_2 - \cancel{2} + \cancel{2} = 1 \Rightarrow \boxed{a_2 = 1}$$

Given,

$$a_{n+2} - 2a_{n+1} + a_n = 2^n$$

multiply z^n on B.S. & taking summation from $n=0$ to ∞

$$\sum_{n=0}^{\infty} a_{n+2} \cdot z^n - 2\sum_{n=0}^{\infty} a_{n+1} \cdot z^n + \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} 2^n \cdot z^n$$

$$\sum_{n=0}^{\infty} a_{n+2} \cdot z^n - 2\sum_{n=0}^{\infty} a_{n+1} z^n + G(a,z) = \frac{1}{1-2z}$$

$$a_2 + \quad - \quad \Big] - 2\sum^{\infty} a_{n+1} z^n + G(a,z) = \frac{1}{1-2z}$$

$$\frac{1}{z^2}\Big[a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \Big] - 2\sum_{n=0}^{\infty} a_{n+1} z^n + G(a,z) = \frac{1}{1-2z}$$

$$\frac{1}{z^2}\Big[a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots + a_0 + a_1 z - a_0 - a_1 z\Big] - 2\sum^{\infty} a_{n+1} z^n + G(a,z) = \frac{1}{1-2z}$$

$$\frac{1}{z^2}\Big[\sum^{\infty} a_n z^n - a_0 - a_1 z\Big] - 2\Big[a_1 + a_2 z + a_3 z^2 + \cdots \quad + G(a,z) = \frac{1}{1-2z}$$

$$\frac{1}{z^2}\sum_{n=0} a_n z^n - \frac{a_0}{z^2} - \frac{a_1}{z} - 2 \cdot \frac{1}{z}\Big[a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_0 - a_0\Big] + G(a,z) = \frac{1}{1-2z}$$

$$\frac{1}{z^2} G(a,z) - \frac{a_0}{\quad} \qquad \frac{\quad}{z}\Big[G(a,z) - a_0\Big] + G(a,z) = \frac{1}{1-2z}$$

$$\frac{1}{z^2} \cdot G(a,z) - \frac{a_0}{z^2} - \frac{a_1}{z} - \frac{2}{z} G(a,z) + \frac{2a_0}{z} + G(a,z) = \frac{1}{1-2z}$$

$$G(a,z)\Big[\frac{1}{z^2} - \frac{2}{\quad} \quad \Big] = \frac{1}{1-2z} + \frac{a_0}{z^2} + \frac{a_1}{z} - \frac{2a_0}{z}$$

$$G(a,z)\left[\frac{1}{z^2} - \frac{2}{ }\right] = \frac{1}{1-2z} + \frac{a_0}{z^2} + \frac{a_1}{z} - \frac{2a_0}{z}$$

$$G(a,z)\left[\frac{1-2z+z^2}{ }\right] = \frac{1}{1-2z} + \frac{a_0 + a_1 z - 2a_0 z}{z^2}$$

Here $a_0 = 2, a_1 = 1$

$$G(a,z)\left[\frac{z^2-2z+1}{z^2}\right] = \frac{1}{1-2z} + \frac{2+z-4z}{z^2} = \frac{1}{1-2z} + \frac{2-3z}{z^2}$$

$$G(a,z)\left[\frac{z^2-2z+1}{\cancel{z^2}}\right] = \frac{z^2 + (1-2z)(2-3z)}{\cancel{z^2}(1-2z)} = \frac{z^2+2-3z-4z+6z^2}{(1-2z)}$$

$$G(a,z)\left(z^2-2z+1\right) = \frac{7z^2-7z+2}{(1-2z)}$$

$$G(a,z) = \frac{(7z^2-7z+2)}{(1-2z)(1-z)^2} = \frac{A}{1-2z} + \frac{B}{1-z} + \frac{C}{(1-z)^2} \quad \text{—— (2)}$$

$$7z^2-7z+2 = A(1-z)^2 + B(1-z)(1-2z) + C(1-2z)$$

put $z=1$ $\quad 2 = -C \Rightarrow \boxed{C=-2}$

put $z=1/2$ $\quad \frac{7}{4} - \frac{7}{2} + 2 = \frac{A}{4} \Rightarrow \frac{1}{4} = \frac{A}{4} \Rightarrow \boxed{A=1}$

comparing co-efficients of z^2, $\quad 7 = A + 2B \Rightarrow \boxed{B=3}$

$\therefore$ from (2) $\quad G(a,z) = \frac{1}{ } \qquad \frac{ }{(1-z)^2}$

$\therefore$ By standard generating functions,

$$a_n = 2^n + 3 - 2(n+1)$$

$$a_n = 2^n + 3 - 2n - 2$$

$$\boxed{2^n - 2n + 1}$$

Divide & Conqure Rule / Algorithm :-

The divide & conquer algorithm works reccursively breaking down a problem into two or more sub-problems of the same or related type, until these problems become simple enough to solve. The solutions to the subproblems are then combined to give a solution to the original problem.

The typical divide & conquer algorithm solves a problem using the following steps:

1) Divide: Break or divide the given problem into sub-problems of same type.

2) Conquer: Reccursively solve the sub-problems.

3) Combine: Combine the solutions of sub-problems to get the solution for the given problem.

The idea behind the algorithm:-

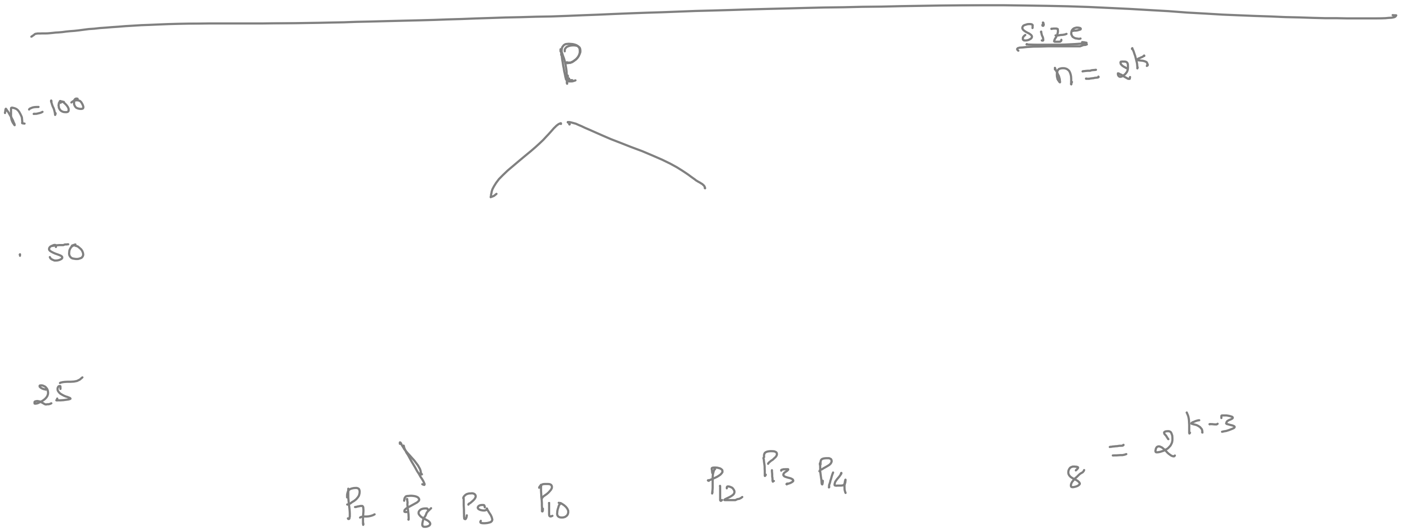
* Given problem P of size $n = 2^k$

→ If 'n' is small, directly solve it.

else,

→ If 'n' is larger, then divide 'P' into two sub-problems P_1 & P_2 of size $\frac{n}{2} = 2^{k-1}$.

→ Solve P_1 & P_2 & combine the solutions of P_1 & P_2 to get the solution of 'P'.



$\therefore$ Time to solve problem 'P',

$$\underset{P}{T(n)} = \underset{P_1}{T(n/2)} + \underset{P_2}{T(n/2)} + f(n)$$

where $f(n)$ is called additional constant of combining.

$$\boxed{T(n) = 2 \cdot T(n/2) + f(n)}$$

In general,

$$\boxed{T(n) = a \cdot T(n/b) + f(n)} \quad \text{—— (1)}$$

This is the Recurrance relation for Divide & conquer algorithm.

Problems:-

* Solve $T(n) = 2 \cdot T(n/2) + n$ recurrance relation using iteration method. (take $n = 2^k$)

sol^n →

$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n$ —— (1) $\quad n \to n/2$ in (1)

$T(n) = 2\left[2 \cdot T\left(\frac{n}{4}\right) + \frac{n}{2}\right] + n$

$T(n/2)$

$T(n) = 4 \cdot T(n/4) + 2n = 2^2 \cdot T(n/2^2) + 2n$

$T(n/4)$ $\quad n \to n/4$ in (1)

$T(n) = 4\left[2 \cdot T\left(\frac{n}{8}\right) + \frac{n}{4}\right] + 2n$

$T(n) = 8 \cdot T(n/8) + 3n = 2^3 \cdot T(n/2^3) + 3n$ $\quad n \to n/8$ in (1)

$T(n) = 8\left[2 \cdot T(n/16) + \frac{n}{8}\right] + 3n$

2^k

$T(n) = 16 \cdot T\left(\frac{n}{16}\right) + 4n = 2^4 \cdot T(n/2^4) + 4n$

after k-iterations

$T(n) = 2^k \cdot T(n/2^k) + kn.$ $\quad$ but $n = 2^k$

$T(n) = 2^k \cdot T(1) + kn$ $\quad$ [Here $T(1)$ is very small]

$$\boxed{T(n) = kn} \text{ —— (1)}$$

$n = 2^k \Rightarrow \log n = k \cdot \log 2 \Rightarrow k = \frac{\log n}{\log 2} = \log_2 n$

$$\boxed{T(n) = kn} + ①$$

but given that $n = 2^k \Rightarrow \log n = k \cdot \log 2 \Rightarrow k = \frac{\log n}{\log 2} = \log_2 n$

$\Rightarrow k = \log_2 n$

from ① $\boxed{T(n) = n \cdot \log_2 n}$

* In an algorithm, we divide large problems into 3 equal parts & discard two of them, in the constant time. What is the complexity of this algorithm for size $n = 3^k$.

sol^n $\rightarrow$ Given,

$$T(n) = T(n/3) + c \quad — ①$$

$$T(n) = \{T(n/9) + c\} + c$$

$$T(n) = T(n/9) + 2c = T(n/3^2) + 2c$$

$$T(n) = \{T(n/27) + c\} + 2c$$

$$T(n) = T\left(\frac{n}{27}\right) + 3c = T(n/3^3) + 3c$$

$$T(n) = \left\{T\left(\frac{n}{81}\right) + c\right\} + 3c$$

$$T(n) = T\left(\frac{n}{81}\right) + 4c = T(n/3^4) + 4c$$

$$\vdots$$

$$T(n) = T\left(\frac{n}{3^k}\right) + k \cdot c = T(n/n) + kc$$

$$T(n) = T(1) + kc$$

Since $T(1)$ is very small.

$$T(n) = kc \quad — ①$$

Given, $n = 3^k \Rightarrow \log n = k \cdot \log 3 \Rightarrow \boxed{\log_3 n = k}$

$\therefore$ from ①, $\boxed{T(n) = c \cdot \log_3 n}$

$\therefore$ Complexity is $T(n) = c \cdot \log_3 n$

Merge-Sort algorithm :-

Merge sort keeps on dividing list into equal halves until it can no more be divided. Then merge sort combines the smaller sorted lists keeping the new list sorted.

Algorithm:

Step 1: If it is only one element in the list, then it is already sorted. Return.

Step 2: Divide the list recursively into two halves until it can no more be divided.

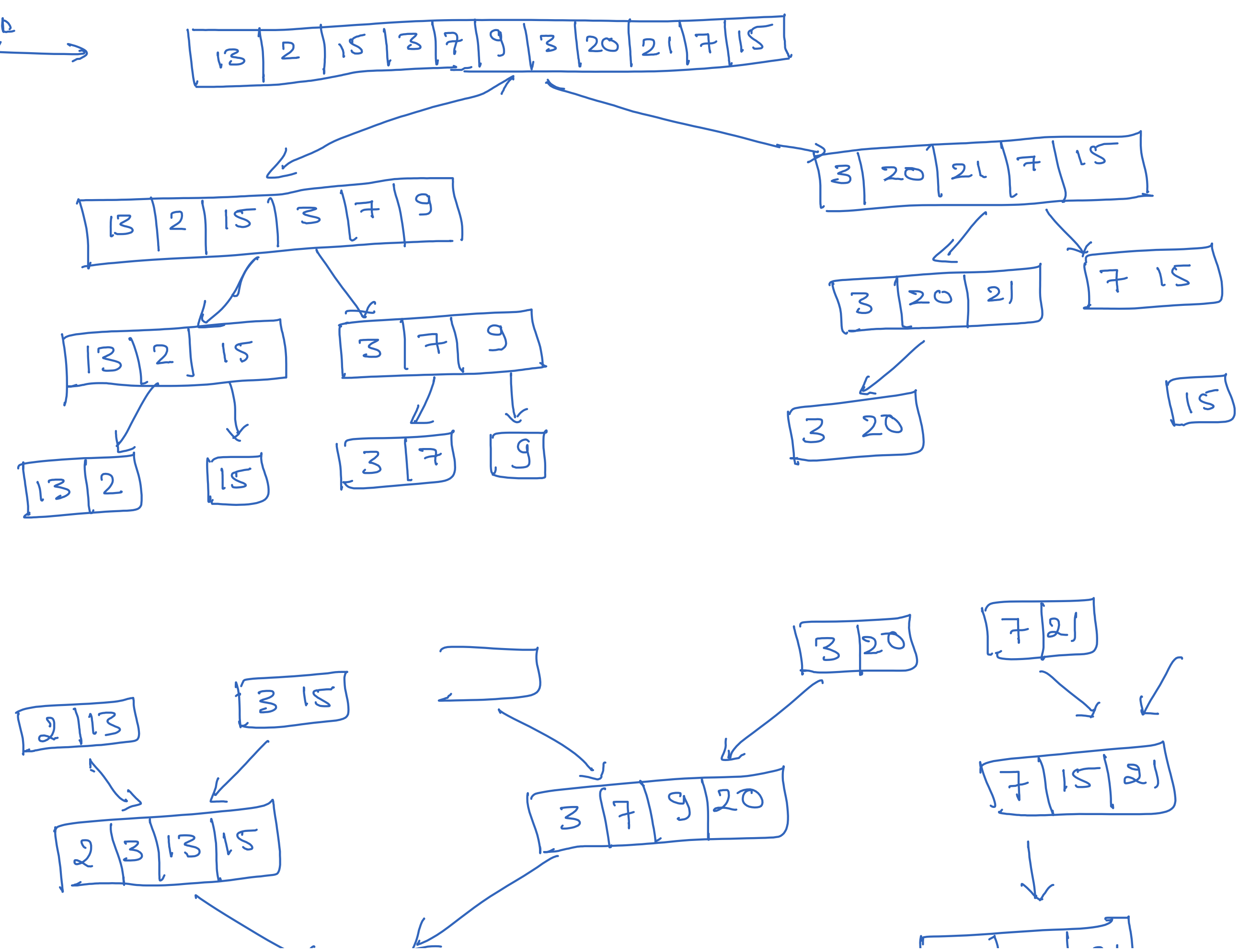
Step 3: Merge the smaller lists into new list in the sorted order.

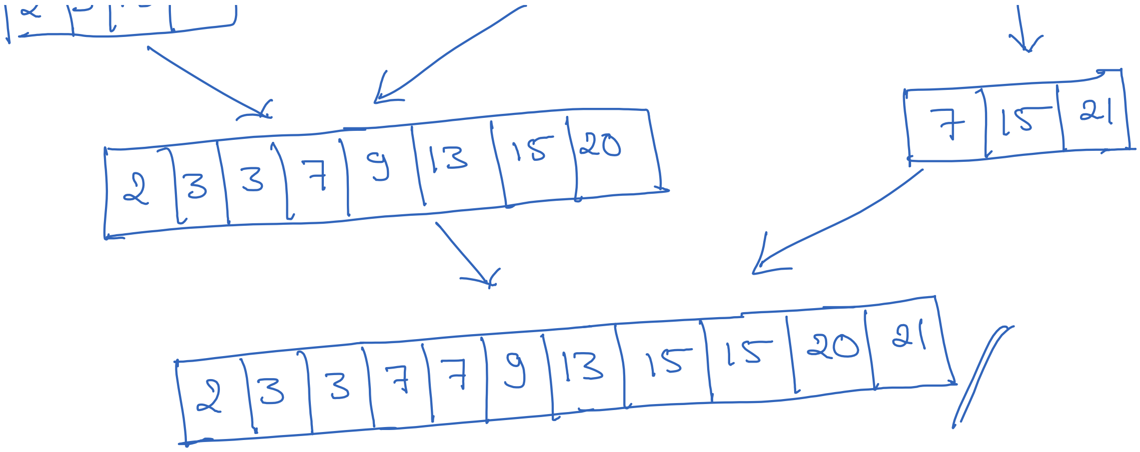
Example:-

1) Sort the following data in the increasing order, using merge sort.

13, 2, 15, 3, 7, 9, 3, 20, 21, 7, 15.

sol^n →




7 15 21
2 3 3 7 9 13 15 20
2 3 3 7 7 9 13 15 15 20 21