UNIT - IV

BASIC GRAPH THEORY

GRAPH

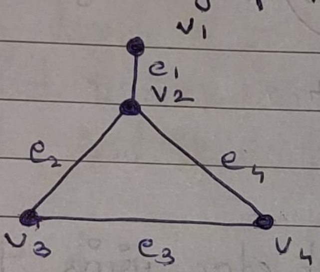
A graph $G = (V, E)$ consists of a non-empty vertex set V (points) and an edge set E (lines).
Each edge has either one or two vertices associated with it. An edge is said to connect or a link of its vertices

FINITE GRAPH

A graph with finite set of vertices is called finite graph. Otherwise an infinite graph.

Examples

G



$V = \{v_1, v_2, v_3, v_4\}$

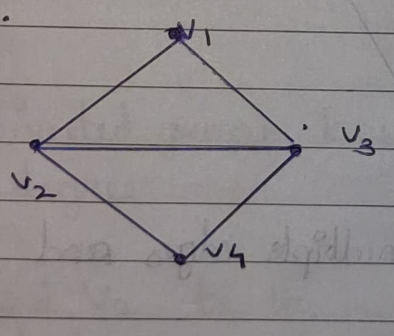
$E = \{e_1\ e_2\ e_3\ e_4\}$

DEGREE OF VERTEX

The degree of the vertex v is denoted by "deg(v)" is the number of edges connected to it.

Example.

G



$deg(v_1) = 2$

$deg(v_2) = 3$

$deg(v_3) = 3$

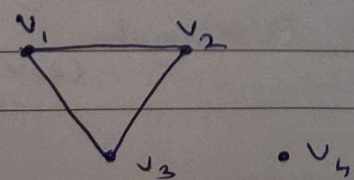
$deg(v_4) = 2$

ISOLATED VERTEX

A vertex v in a graph G is said to be isolated if its degree is 0

Example:

G



v_4 is isolated vertex as $deg(v_4) = 0$

UNDIRECTED GRAPH

An undirected graph is a graph where edges are unordered pairs of distinct vertices

HANDSHAKING LEMMA

Statement: Let $G = (V, E)$ be an undirected graph with e number of edges. Then

$$\sum_{v \in V(G)} \deg(v) = 2e$$

Proof:

While counting degree of an vertex in a graph G, each edge is counted twice [once with one vertex once with other vertex]. And it is true for all the edges in a graph G.

$\therefore$ Sum of degrees of each vertex in a graph G is equal to 2 times the total number of edges in G !

$$\therefore \sum_{v \in VG} \deg(v) = 2e$$

L

LEMMA

An undirected graph has an even number of vertices of odd degree.

Proof

Let V_1 and V_2 be the set of vertices of even degree and odd degree respectively in a undirected graph $G = (V, E)$

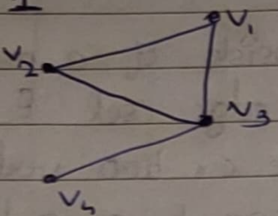
By handshaking Lemma.

$$2e = \sum_{v \in V(G)} \deg(v)$$

PENDENT VERTEX

A vertex v in a graph G is said to be pendent if its degree is 1

Example:

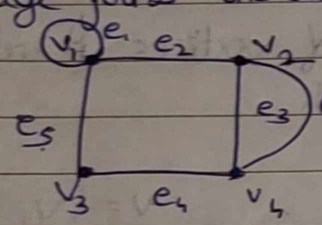


v_4 is pendent vertex as $deg(v_4) = 1$

LOOP.

A an edge joins the same vertex is called loop.

Example:

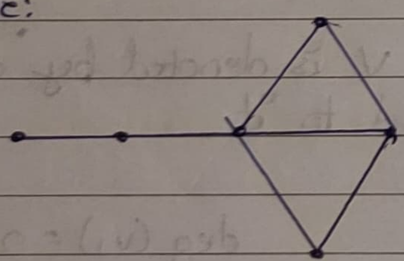


Edge e_1 is a loop

SIMPLE GRAPH.

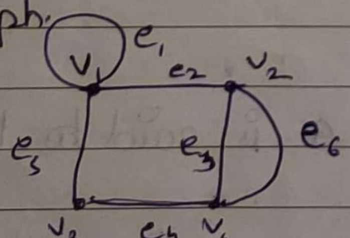
A graph in which each edge joins two different vertices is called a simple graph.

Example:



MULTI GRAPH

A graph which has multiple edges and loops is called multigraph.

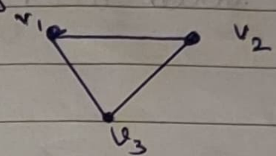


2) Cycle

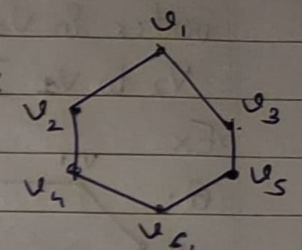
A cycle is a graph in which every vertex has degree 2. A cycle with n vertices is denoted by C(n) where $n \geq 3$

Examples

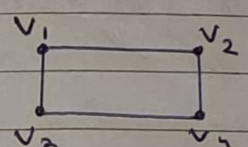
(i) C_3 :



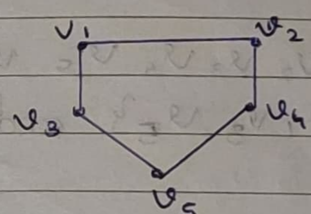
(iv) C_6 :



(ii) C_4 :



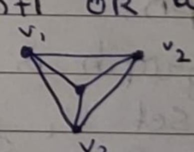
(iii) C_5



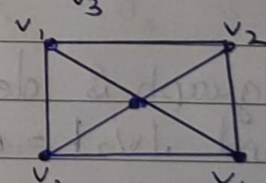
3) Wheel

A wheel is obtained by adding, and joining an additional vertex to all other vertices of a cycle. A wheel can be denoted by W_{n+1} OR $W_{1,n}$

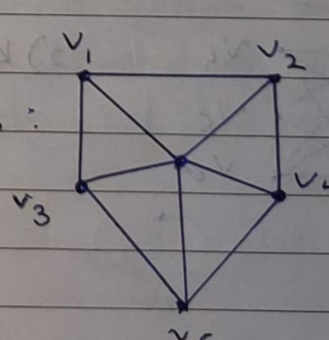
(i) W_{3+1} OR $W_{1,3}$:



(ii) W_{4+1} OR $W_{1,4}$:



(iii) W_{5+1} OR $W_{1,5}$:



$$2e = \underbrace{\sum_{v \in V_1(G)} \deg(v)}_{\text{even}} + \underbrace{\sum_{u \in V_2(G)} \deg(u)}_{\text{even}}$$

Since $2e$ is even in the LHS and 1^{st} term on RHS is also even then 2^{nd} term of RHS must be even.

$\therefore \sum_{v \in V_2(G)} \deg(v) = \text{Even} \Rightarrow$ no of vertices of odd degree must be even.

This complete's the proof.

SOME SPECIAL GRAPHS

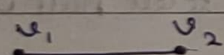
1) Complete Graph.

A simple graph in which every pair of vertices are joined by an edge is called complete graph.

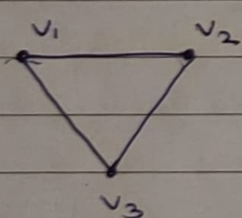
A complete graph with n vertices is denoted by, K_n

Examples

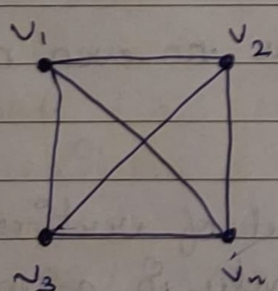
[i] K_2 :



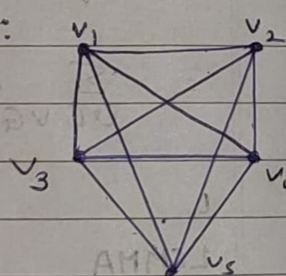
[ii] K_3 :



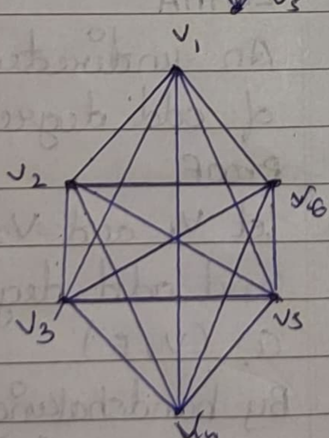
[iii] K_4 :



[iv] K_5 :



[v] K_6

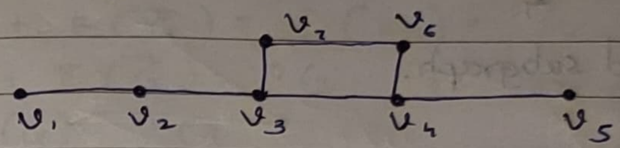


SUBGRAPHS

If $G = (V, E)$ is a graph then $G_1 = (V_1, E_1)$ is called subgraph of G if $V_1 \subseteq V$ and $E_1 \subseteq E$.

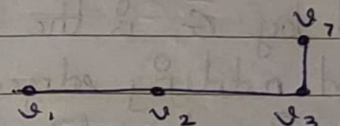
Ex:

G:

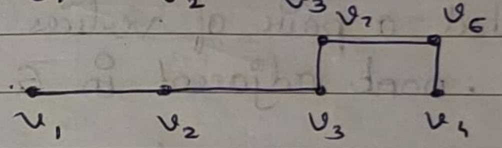


Sub graphs of G are.

G_1 :



G_2 :

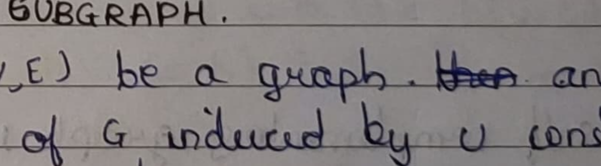


SPANNING SUBGRAPH

If $V_1 = V_2$ then we say that G_1 is the spanning subgraph of G.

Ex: In previous example of graph G spanning subgraph is,

G_3 :

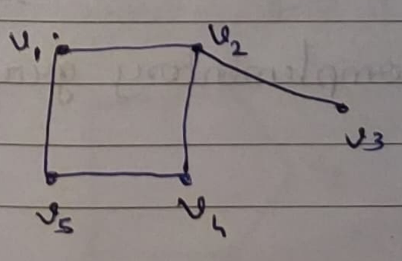


INDUCED SUBGRAPH.

Let $G = (V, E)$ be a graph. ~~then~~ and $U \subseteq V$. Then a subgraph of G induced by U consists of vertex set U and all the edges of U in G

Ex

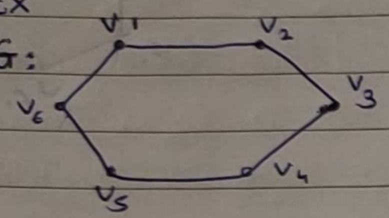
G :



4) Bipertite Graph.
A simple graph G is called bipertite if its vertex set V can be partitioned into two disjoint sets. V_1 & V_2 such that every edge in the graph joins a vertex in V_1 to a vertex in V_2 [no edge joins vertex in V_1 to V_2] or V_2 to V_2]

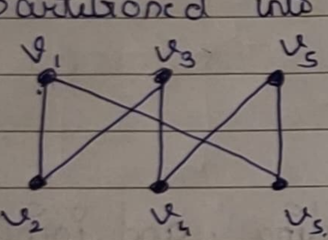
Ex

G:



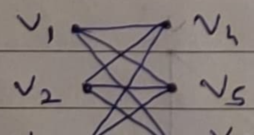
Bipertite graph as $V = \{v_1, v_2\ v_3\ v_4\ v_5\ v_6\}$ is partitioned into $V_1 = \{v_1, v_3\ v_5\}$ and $V_2 = \{v_2\ v_4\ v_6\}$

G:

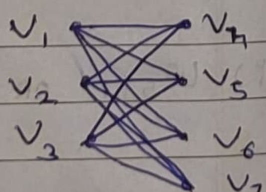


5) Complete Bipertite Graph.
A complete bipertite graph is a bipertite graph in which vertex in one set V_1 joins every vertex in other set V_2.
Complete bipertite graph is denoted by $K_{m,n}$ where $|V_1| = n(V_1) = m$ and $|V_2| = n(V_2) = n$.
Ex

1) $K_{3,3}$:



2) $K_{3,4}$



2) $K_{3,4}$ V.

$\Rightarrow E(G) = E(\bar{G})$ —— (1)

We know that

$E(G) + E(\bar{G}) = E(K_n) = {}^nC_2$

$E(G) + E(\bar{G}) = {}^nC_2$ ($\because$ By eq^n 1)

$2E(G) = {}^nC_2$

$2E(G) = \frac{n!}{(n-2)!\,2!}$

$E(G) = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,2!\,2}$

$\boxed{E(G) = \frac{n(n-1)}{4}}$ —— (2)

Since $E(G)$ is a positive integer, therefore either n should be multiple of 4 or $(n-1)$ should be multiple of 4 i.e $n = 4k$ or $(n-1) = 4k$ for some $k \in \mathbb{Z}^+$

$\Rightarrow \boxed{n = 4k}$ or $\boxed{n = 4k+1}$. for some $k \in \mathbb{Z}^+$.

GRAPH ISOMORPHISM / ISOMORPHIC GRAPHS

let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs a function $f : V_1 \longrightarrow V_2$ is called a graph isomorphism if.

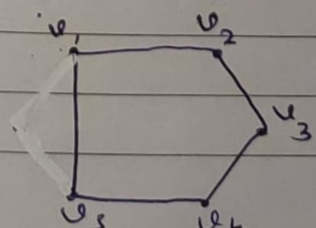
(i) f is one-one and onto (i.e bijective)

(ii) $\forall\, a, b \in V_1$ $(a, b) \in E_1$ if and only if $(f(a), f(b)) \in E_2$

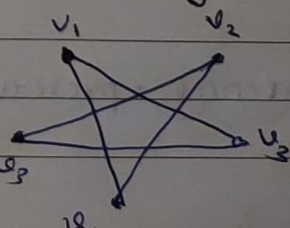
If such function of exists then we say that G_1 and G_2 are isomorphic. And it is denoted by $G_1 \cong G_2$

Ex:

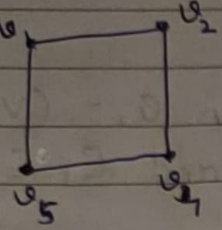
G_1:



G_2:



G_2 :



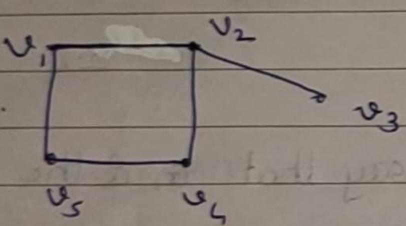
induced subgraph.

COMPLEMENT OF GRAPH.

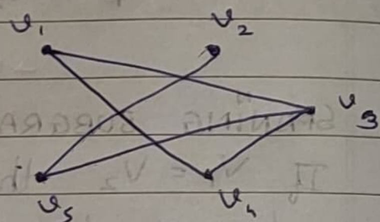
Let G be a loop free undirected graph. with n vertices. The complement of graph G is denoted by $\overline{G}$ is the graph obtained by removing the edges in G and adding edges that are not in G. It means that a pair of vertices in $\overline{G}$ are adjacent if they are not adjacent in G and vise versa

Ex

G :



$\overline{G}$:



Note

1) Clearly we get

$$E(G) + E(\overline{G}) = E(K_n) = {}^nC_2$$

2) A graph G is said to be self complementary is $G \cong \overline{G}$ [isomorphic]

THEOREM

If G is a self complementary graph with n vertices, where $n > 1$, then prove that $n = 4K$ or $n = 4K+1$ for some $K \in \mathbb{Z}^+$

Proof

Given that G is a self complementary graph.

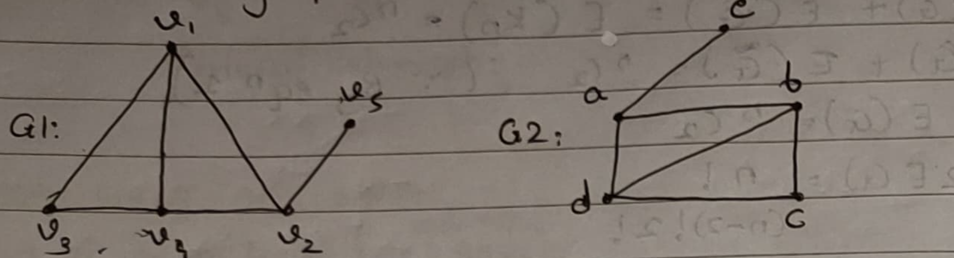
$\therefore \; G \cong \overline{G}$

$\Rightarrow$ no of edges in $G = \overline{G}$

Example

① Show that graphs shown below are isomorphic

(i)



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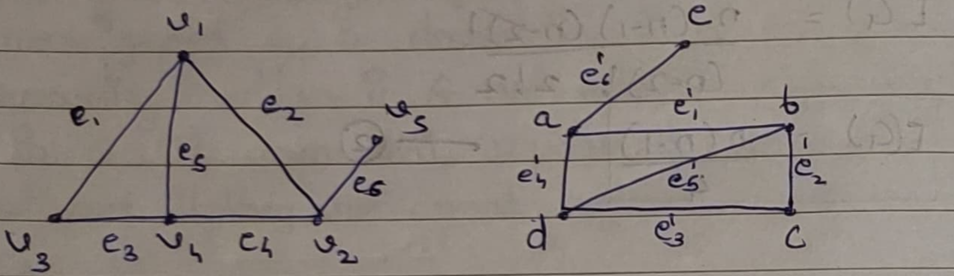


G1 G2

(i) No of vertices in G_1 and G_2

$V_1 = \{v_1, v_2, v_3, v_4, v_5, \cancel{v_6}\} \Rightarrow |V_1| = n(V_1) = 5$

$V_2 = \{a, b, c, d, e\} \Rightarrow |V_2| = n(V_2) = 5$

$\therefore n(V_1) = n(V_2) = 5$

(ii) No of edges in G_1 and G_2

$E_1 = \{e_1, e_2, e_3, e_4, e_5, e_6\} \Rightarrow n(E_1) = 6$

$E_2 = \{e_1', e_2', e_3', e_4', e_5', e_6'\} \Rightarrow n(E_2) = 6$

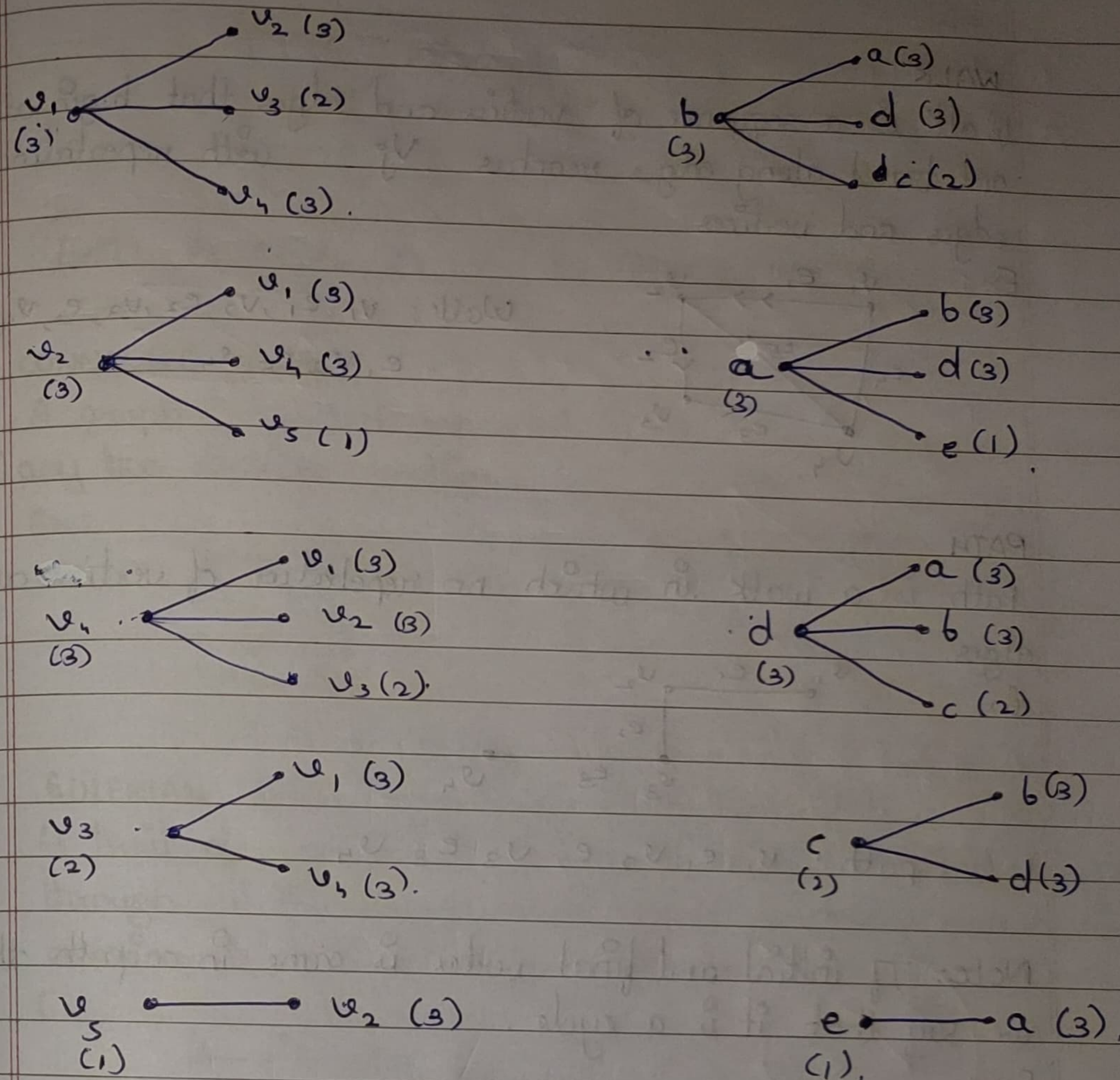
$n(E_1) = n(E_2) = 6$

(iii) Degree of each vertex in G_1 and G_2.

Degree of G_1 : $\{3, 3, 3, 2, 1\}$

$G2$: $\{3, 3, 3, 2, 1\}$

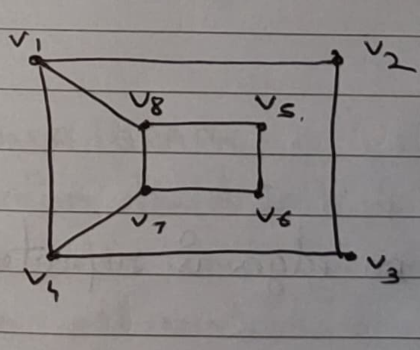
(iv) Adjacency preserving



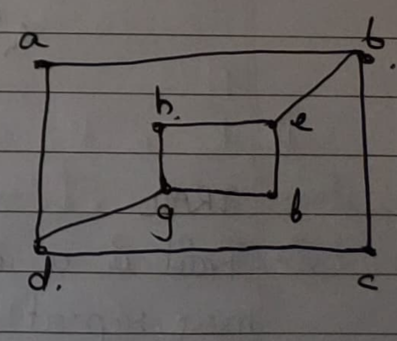
$\therefore G_1 \cong G_2$.

(iii).

G1 :



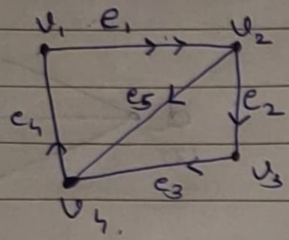
G2 :



WALK

Walk is a sequence of vertices and edges that begin at v_i and travel along edges reaches v_j with repeatation of edges and vertices.

Ex :

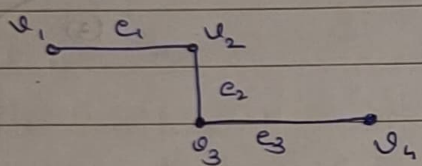


Walk : $v_1, e_1, v_2, e_2, v_3, e_3, v_4, v_1,$ $e_1 v_2 e_5 v_4$

PATH

Path is a walk in which no repeltation of vertices and edges.

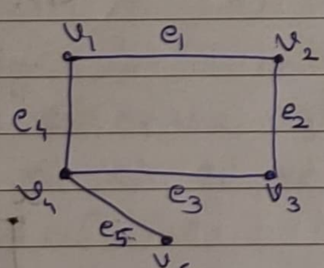
Ex



path : $v_1 e_1 v_2 e_2 v_3 e_3 v_4$

Note: If initial and final vertex is same in a path then we say that it is a cycle.

Ex :

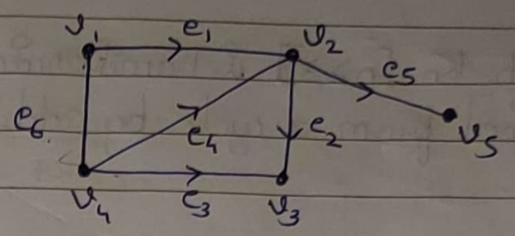


Cycle :- $v_1 e_1 v_2, e_2, v_3, e_3 \cdot v_4\ e_4$

TRAIL

Trail is a walk in which no edges is repeated vertices may repeat

Example.



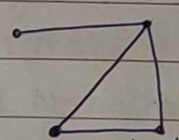
Trail: $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_2, e_5, v_5$

CONNECTED GRAPH

A graph is called connected if their exist a path between [every] any two distinct vertices.

Ex:

G1 :



Connected.

G2 :

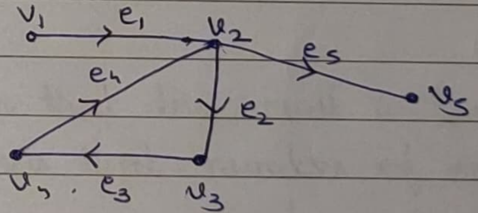
Disconnected.

EULERIAN TRAIL AND EULERIAN GRAPH.

A trail of a graph G is said to be an eulerian trail if it passes through all the edges exactly once, such ~~that~~ graph with eulerian trial is called an eulerian graph.

Ex

G1 :



HAMILTONIA GRAPH

A spanning closed path is known as hamiltonian cycle. It is a path starting and ending at a same point going to all vertices.

A graph with the hamiltonian cycle is called hamiltonian graph.

Smallest complete graph is K_4 which is a planar

Note

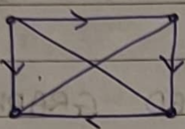
1 Every complete graph K_n $[n \geq 3]$ is hamiltonian graph.
2 Every graph obtained from a cycle by adding edges is called hamiltonian.

[

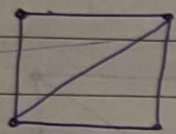
Example.

K_3:

K_4:



G_1:

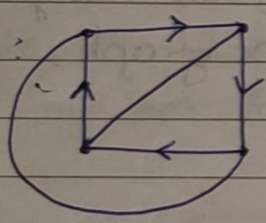


PLANAR GRAPH

A graph is said to be planar if can be drawn in a plane without crossing of edges.

Example:-

K_4:



planar

Note

1) K_5 is the smallest non-planar graph with least number of vertices. [all the graphs upto 4 vertices are planar]
2) $K_{3,3}$ is the smallest non-planar graph with least number of edges. [all the graphs upto 8 edges are planar].

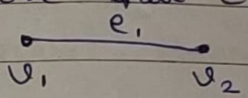
EULER'S FORMULA

Theorem: Let G be a connected planar graph with e edges and 'n' vertices. Let 'r' be the number of region in a planar representation of G. Then Euler formula is,

$\boxed{r = e - n + 2}$ —— (1).

Proof: We will prove the theorem by principle of mathematical induction (PMI).

STEP 1 Prove that $e = 1$.



i.e $e = 1$ (No of edges).
$n = 2$ (No of vertices).
$r = 1$ (No of regions)

$\therefore r = e - n + 2$
$1 = 1 - 2 + 2$
$\boxed{1 = 1.}$

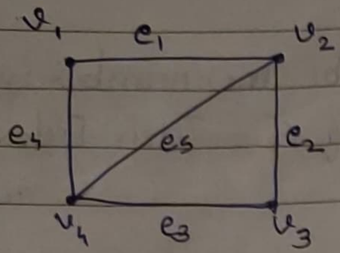
It is true for $e = 1$.

STEP 2 Assume that the result is true for $e = K$.
It means that, number of edges $e = K$
number of vertices $n = n_K$.
number of region $r = r_K$

$\therefore$ Equation (1) becomes.

$\boxed{r_K = K - n_K + 2}$. —— (2).

STEP 3 Prove that the result is true for $e = K+1$
case 1 : Adding one edge within the given vertices



$e = K + 1$

$n = n_K$

$r = r_{K+1} = r_K + 1$

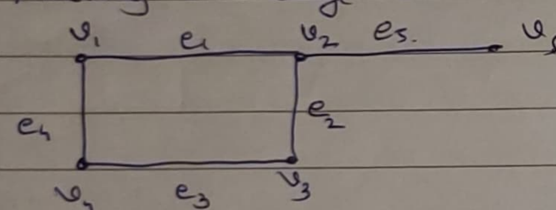
Result ① becomes

$r_{K+1} = (K+1) - n_K + 2$

$r_K + \cancel{1} = K + \cancel{1} - n_K + 2$

$r_K = K - 1$

Case 2: Adding an edge with one extra vertex.



$e = K + 1$

$n = n_{K+1} = n_K + 1$

$r = r_K$

Result ① becomes.

$r_K = (K+1) - n_{K+1} + 2$

$r_K = K + 1 - (n_K + 1) + 2$

$r_K = K + \cancel{1} - n_K - \cancel{1} + 2$

$\boxed{r_K = K + n_K + 2}$

This prove the theorem.

COLOURING

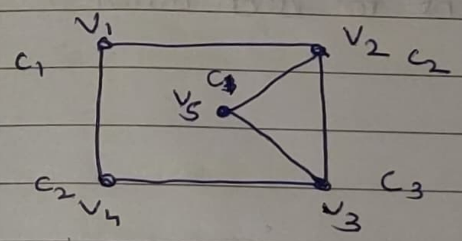
A proper colouring of graph means colouring the vertices of the graph such that adjacent vertices do not have the same colour.

Obiviously for any graph with n vertices a proper colouring with n colour is trivial. Given any colour to one vertex 2^{nd} to the next and so on .....

Chromatic Number

The minimum of colours required to colour a graph G is called chromatic number and it is denoted by $\chi(G)$

Ex



$\chi(G) = 3$

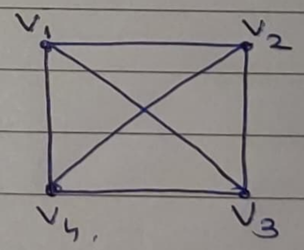
Note

1 A complete graph with n vertices with K_n requires n colours.

$\chi(K_n) = n$

Ex:

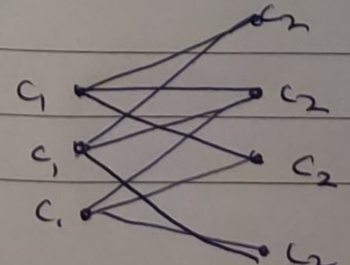
K_4:



$\chi(K_4) = 4$

2 Every bipertite graph B requires 2 colours

$\therefore \chi(B) = 2$



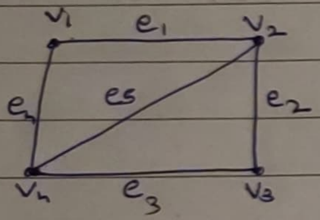
Application of colouring

Storing chemical compounds in a warehouse keeping in mind that some of them are volatile. So condition becomes becomes incompatiable [Reactable], chemical reagents to be store in ~~spec~~ separate compartments. Take the compounds as vertices and draw the edge between them if they are incompatiable. Clearly chromatic number this graph gives the number of compartment.

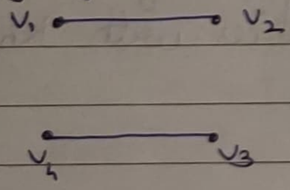
MATCHING

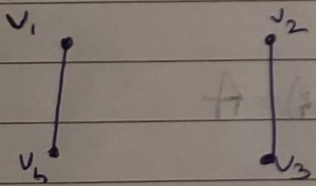
A matching graph is a subgraph of a graph G where their are no edges adjacent to each other.

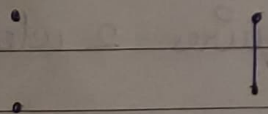
Ex

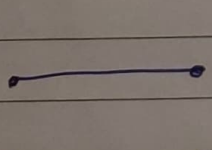
G : 

Matching graphs of G are.

M_1 : 

M_2 : 

M_3 : 

M_4 : 

Maximal Matching

A matching of graph G is said to be a maximal matching if it has maximum number of edges which are not adjacent to each other.

Ex :

M_1 and M_2 are maximal matching of graph G.

Perfect Matching

A matching of a graph G is said to be perfect matching if every vertex of matching is of degree 1.

Ex : M_1 and M_2 are perfect matching of graph G as vertices of both the ~~graphs~~ matchings are of degree 1.