Unit - 2

Fourier Series

* Periodic functions
* Dirichlet's condition

Q) Obtain the Fourier series for the function x^2 in $(-\pi,\pi)$ & hence deduce that

i) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$

ii) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \quad -(1)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx=\frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$a_0=\frac{1}{3\pi}\left\{\pi^3-(-\pi)^3\right\}=\frac{2\pi^3}{3\pi}=\frac{2\pi^2}{3}$$

$$\boxed{a_0=\frac{2\pi^2}{3}}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx$$

$$a_n=\frac{1}{\pi}\left\{x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right\}_{-\pi}^{\pi}$$

$$=\frac{2}{\pi n^2}\left[x\cos nx\right]_{-\pi}^{\pi} \qquad \sin n\pi=0$$

$$=\frac{2}{\pi n^2}\left[\pi\cos n\pi-(-\pi)\cos n\pi\right] \qquad \cos(-n\pi)=\cos n\pi$$

$$a_n=\frac{2}{\cancel{\pi}n^2}\times 2\cancel{\pi}\cos n\pi=\frac{4(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx\,dx$$

$$b_n=\frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right)-(2x)\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$b_n=\frac{1}{\pi}\left\{-\frac{1}{n}\left[\pi^2\cos n\pi-\pi^2\cos n\pi\right]+0+\frac{2}{n^3}\left[\cos n\pi-\cos n\pi\right]\right\}$$

$$b_n=0$$

Substitute the values a_0, a_n & b_n in ①,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \quad \text{— ②}$$

Putting $x=0$ in ②, we get

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos 0$$

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \qquad \because \cos 0 = f(0) = 1$$

$$-\frac{\pi^2}{3} = 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

Putting $x=\pi$ in ②, we get

$$f(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos n\pi$$

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2} \qquad \because f(\pi) = \pi^2$$

$$\cos n\pi = (-1)^n$$

$$\pi^2 - \frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad (-1)^n \cdot (-1)^n = (-1)^{2n} = 1$$

$$\frac{2\pi^2}{3} = 4 \sum \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

8) $f(x) = e^{-x}$ in $(-\pi, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— ①}$$

$a_0 =$ $2L = 2\pi \Rightarrow L = \pi$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) = \frac{1}{\pi} \left[e^{-x}\right]_{-\pi}^{\pi} = \frac{e^{-\pi} - e^{\pi}}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{-x} \cos nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1^2+n^2}\left(-1\cos nx + n\overset{\sin}{\cancel{\cos}}\,nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-x}(-\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-\pi}(-\cos n\pi + n\cancelto{0}{\sin n\pi}) - e^{\pi}(-\cos n(-\pi) + n\cancelto{0}{\sin n(-\pi)})\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-\pi}(-\cos n\pi) + e^{\pi}\cos n\pi\right]$$

$$= \frac{\cos n\pi}{\pi(n^2+1)}\left[-e^{-\pi} + e^{\pi}\right] \quad \because e^{\pi} - e^{-\pi} = 2\sinh\pi$$

$$= 2\cos\frac{\cancel{2}(-1)^n}{\pi(n^2+1)} \quad \text{OR} \quad = \frac{2(-1)^n}{\pi(n^2+1)}(e^{\pi} - e^{-\pi})$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}\sin nx\,dx \qquad \int e^{ax}\sin nx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin nx - b\cos nx\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-1\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{(1+n^2)\pi}\left[e^{-\pi}(-\cancelto{0}{\sin n\pi} - n\cos n\pi) - e^{\pi}(-\cancelto{0}{\sin n(-\pi)} - n\cos n\pi)\right] \quad \because \sin n\pi = 0$$

$$= \frac{1}{\pi(1+n^2)}\left[-ne^{-\pi}\cos n\pi + ne^{\pi}\cos n\pi\right]$$

$$= \frac{n\cos n\pi}{\pi(1+n^2)}(e^{\pi} - e^{-\pi}) \quad \because \cos n\pi = (-1)^n$$

$$b_n = \frac{n(-1)^n}{\pi(1+n^2)}(2\sinh\pi)$$

sub in (1), we get

$$f(x) = \frac{e^{-\pi} - e^{\pi}}{2} + \sum \frac{2(-1)^n}{\pi(n^2+1)}\sin n\pi\cos nx + \sum \frac{2n(-1)^n}{\pi(1+n^2)}\sinh\pi \cdot \sin nx$$

Q) Obtain the F.S of $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2\pi$. Hence deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

$(0, 2\pi) \Rightarrow 2L = 2\pi \Rightarrow L = \pi$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} \left(\frac{\pi - x}{2}\right) dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\left(2\pi^2 - \frac{4\pi^2}{2}\right) - 0\right] = 0$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right) \cos nx\,dx$$

$$= \frac{1}{2\pi}\left\{(\pi - x)\cancel{\left(\frac{\sin nx}{n}\right)}^{0} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{2\pi}$$

$\because \sin 2n\pi = 0 = \sin 0$

$$= \frac{1}{2\pi}\left\{-\frac{\cos nx}{n^2}\right\}_0^{2\pi}$$

$\because \cos 2n\pi = 1 = \cos 0$

$$a_n = \frac{-1}{2\pi n^2}\left[\cos 2n\pi - \cos 0\right] = 0$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right) \sin nx\,dx$$

$$= \frac{1}{2\pi}\left\{(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\cancel{\left(-\frac{\sin nx}{n^2}\right)}^{0}\right\}_0^{2\pi}$$

$$= \frac{1}{2\pi n}\left[-(\pi - x)\cos nx\right]_0^{2\pi}$$

$$= -\frac{1}{2\pi n}\left[-\pi \cos 2n\pi - (\pi \cos 0)\right] = +\frac{2\pi}{2\pi n} = \frac{1}{n}$$

Sub in (1), we get

$$f(x) = 0 + \sum 0 \cdot \cos nx + \sum \frac{1}{n} \sin nx \quad \text{--- (2)}$$

$$x = \frac{\pi}{2}$$

$$\frac{\pi}{4} = \sum \frac{1}{n} \sin \frac{n\pi}{2}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

Q) Obtain the Fourier series to represent $f(x) = x - x^2$ in $-1 < x < 1$ $(-1, 1)$

$2L = 2 \Rightarrow L = 1$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + \sum b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x \quad \text{--- (1)}$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} (x - x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} = -\frac{2}{3}.$$

$$a_n = \int_{-1}^{1} f(x) \cdot \cos n\pi x$$

$$= \int_{-1}^{1} (x - x^2) \cos n\pi x\,dx$$

$$= \left\{(x - x^2)\cancel{\left(\frac{\sin n\pi x}{n\pi}\right)}^{0} - (1 - 2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\cancel{\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)}^{0}\right\}_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}\left[(1 - 2x)\cos n\pi x\right]_{-1}^{1}$$

$$= \frac{-4\cos n\pi}{n^2\pi^2} = \frac{-4(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{L}\int_{-1}^{1} f(x) \sin n\pi x\,dx$$

$$= \int_{-1}^{1} (x - x^2) \sin n\pi x\,dx.$$

$$= \left\{(x - x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (1 - 2x)\cancel{\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)}^{0} + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right\}_{-1}^{1}$$

$$= \frac{-1}{n\pi}\left\{0 - (-2\cos n\pi)\right\} - \frac{2}{n^3\pi^3}(\cos n\pi - \cos n\pi)$$

$$b_n = \frac{2}{n\pi}(-1)^{n+1}$$

$$f(x) = -\frac{1}{3} + \frac{4}{\pi^2}\sum \frac{(-1)^n}{n^2\pi^2} \cos n\pi x + \frac{2}{n\pi}\sum \frac{(-1)^{n+1}}{n} \sin n\pi x$$

~~Q) Draw the graph of the f^n~~

~~$f(x) = \{ \pi x \text{ in } 0 < x < 1$~~

Q) Obtain the F.S for $f(x) = e^{-x}$ in $(0,2)$

$2L = 2 \Rightarrow L = 1$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x \quad \text{—} (1)$$

$$a_0 = \frac{1}{L}\int_0^2 f(x)\,dx$$

$$= \int_0^2 e^{-x}\,dx = -e^{-x}\Big]_0^2 = -(e^{-2}-1) = 1 - \frac{1}{e^2} = \frac{e^2-1}{e}$$

$$a_n = \frac{1}{L}\int_0^2 f(x)\cos n\pi x\,dx$$

$$= \int_0^2 e^{-x}\cos n\pi x\,dx = \frac{e^{-x}}{1^2+n^2\pi^2}\Big[-\cos n\pi x + \cancel{n\pi \sin n\pi x}^{\to 0}\Big]_0^2$$

$$a_n = \frac{-1}{1+n^2\pi^2}\left\{e^{-2}\cos 2n\pi - 1\right\} = \frac{e^2-1}{e^2(1+n^2\pi^2)}$$

$$b_n = \frac{1}{L}\int_0^2 f(x)\sin n\pi x\,dx$$

$$= \int_0^2 e^{-x}\sin n\pi x\,dx$$

$$= \frac{e^{-x}}{1+n^2\pi^2}\Big[-\sin n\pi x - n\pi\cos n\pi x\Big]_0^2$$

$$= \frac{-n\pi}{1+n^2\pi^2}\Big[e^{-x}\cos n\pi x\Big]_0^2 = \frac{-n\pi}{1+n^2\pi^2}\left(\frac{1}{e^2}-1\right)$$

$$b_n = \frac{n\pi(e^2-1)}{e^2(1+n^2\pi^2)}$$

Sub. in (1), we get

$$f(x) = \frac{e^2-1}{2e} + \sum \frac{e^2-1}{e^2(1+n^2\pi^2)}\cos n\pi x + \sum \frac{n\pi(e^2-1)}{e^2(1+n^2\pi^2)}\sin n\pi x \;//$$

$= \frac{1}{\pi} f$

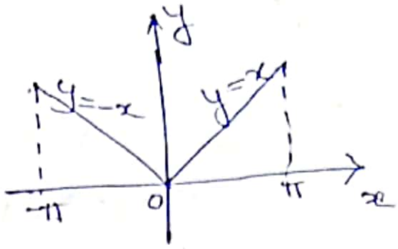
Ex:- An alternating current after passing through a rectifier has the form

$$I = \begin{cases} I_0 \sin\theta & \text{for } 0 < \theta \leq \pi \\ 0 & \text{for } \pi < \theta \leq 2\pi \end{cases}$$

where I_0 is the maximum current. Express I as a Fourier series in $(0, 2\pi)$

<u>Even & odd f^n</u>

Q) Sketch the graph of the function $f(x) = |x|$ in $-\pi \leq x \leq \pi$ & obtain its Fourier series. Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$



$f(x) = |x|$

$f(-x) = |-x| = |x|$ in $(-\pi, \pi)$

$\therefore f(x)$ is an even f^n

$\therefore \boxed{b_n = 0}$

$f(x) = \frac{a_0}{2} + \sum a_n \cos nx$ —— (1)

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx = \frac{2}{\pi} \int_0^{\pi} |x|$

$= \frac{2}{\pi} \int_0^{\pi} x\,dx = \frac{\cancel{2}}{\pi} \left[\frac{x^2}{\cancel{2}}\right]_0^{\pi} = \frac{[\pi^2 - 0]}{\pi} = \pi$

$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$

$= \frac{2}{\pi} \int_0^{\pi} x \cdot \cos nx\,dx$

$= \frac{2}{\pi} \left\{ x \left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right) \right\}_0^{\pi}$

$= \frac{2}{\pi n^2} [\cos n\pi - \cos 0] = \frac{2}{\pi n^2} [(-1)^n - 1]$

Sub in (1), we get

$f(x) = \frac{\pi}{2} + \sum \frac{2}{\pi n^2} [(-1)^n - 1] \cos nx$ —— (2)

put $x = 0$

$0 = \frac{\pi}{2} + \frac{2}{\pi} \sum \frac{(-1)^n - 1}{n^2}$

$1-(-1)^n = 0$, if n is even
$= 2$ if n is odd

$$0 = \frac{\pi}{2} - \frac{2}{\pi}\sum \frac{2}{n^2} \quad \text{if } n \text{ is odd}$$

$$\frac{\pi^2}{4} = 2\sum \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Q) Obtain the Fourier Series expansion of the function $f(x) = \begin{cases} x & \text{in } 0 < x < \pi \\ x - 2\pi & \text{in } \pi < x < 2\pi \end{cases}$

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

$$f(2\pi - x) = 2\pi - x = -(x - 2\pi) = -f(x).$$

$\therefore f(x)$ is odd.

$a_0 = a_n = 0$

$$f(x) = \sum b_n \sin nx \quad \text{—— (1)}$$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx$$

$$= \frac{2}{\pi}\int_0^\pi x \sin nx$$

$$= \frac{2}{\pi}\left\{ x\left(\frac{-\cos nx}{n}\right) - 1\cancel{\left(\frac{-\sin nx}{n^2}\right)}^{\,0} \right\}_0^\pi$$

$$= \frac{-2}{\pi n}\left[\pi\cos n\pi - 0\right] = \frac{-2(-1)^n}{n}$$

$$f(x) = \sum \frac{-2}{n}(-1)^n \sin nx \quad \text{—— (2)}$$

put $x = \frac{\pi}{2}$

$$\frac{\pi}{2} = -2\sum \frac{(-1)^n}{n}\sin\frac{n\pi}{2}$$

$$-\frac{\pi}{4} = \frac{-1}{1} + \frac{1}{3} - \frac{1}{5}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

Q) $f(x) = |x|$ in $(-1, 1)$

Q) $f(x) = \begin{cases} \pi x & \text{in } 0 \le x \le 1 \\ \pi(2-x) & \text{in } 1 \le x \le 2 \end{cases}$

Half Range series

Q) Expand $f(x) = 2x-1$ as a cosine half range Fourier series in $0 < x < 1$

$l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \quad \text{—(1)}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$= 2\int_0^1 (2x-1)\,dx$$

$$= 2\left[\frac{\cancel{2}x^2}{\cancel{2}} - x\right]_0^1 = 0$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= 2\int_0^1 (2x-1)\cos n\pi x\,dx$$

$$= 2\left\{(2x-1)\cancel{\left[\frac{\sin n\pi x}{n\pi}\right]}^{\to 0} - (2)\left[\frac{-\cos n\pi x}{n^2\pi^2}\right]\right\}_0^1$$

$$= \frac{4}{n^2\pi^2}\left[\cos n\pi - \cos 0\right] = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

sub in (1)

$$f(x) = \sum \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\cdot\cos n\pi x$$

Q) Obtain the sine half range series of

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{in } 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \text{in } \frac{1}{2} < x < 1 \end{cases}$$

$L = 1$

$$f(x) = \cancel{\frac{a_0}{2}+} \sum b_n \sin n\pi x \quad \text{—(1)}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin n\pi x\,dx$$

$$= 2\int_0^1 f(x)\sin n\pi x\,dx$$

$$= 2\left\{\int_0^{1/2}\left(\frac{1}{4} - x\right)\sin n\pi x\,dx + \int_{1/2}^1 \left(x - \frac{3}{4}\right)\sin n\pi x\,dx\right\}$$

$$= 2\left\{\left[\left(\frac{1}{4}-x\right)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left[-\frac{\sin n\pi x}{n^2\pi^2}\right]\right]_0^{1/2} + \left[\left(x-\frac{3}{4}\right)\left(-\frac{\cos n\pi x}{n\pi}\right) + 1\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{1/2}^{1}\right\}$$

$$= 2\left\{-\frac{1}{n\pi}\left(-\frac{1}{4}\cos\frac{n\pi}{2} - \frac{1}{4}\right) - \frac{1}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right) - \frac{1}{n\pi}\left(\frac{1}{4}\cos n\pi + \frac{1}{4}\cos\frac{n\pi}{2}\right) + \frac{1}{n^2\pi^2}\left(0 - \sin\frac{n\pi}{2}\right)\right\}$$

$$= 2\left\{\frac{1}{4n\pi}\left(\cos\frac{n\pi}{2} + 1 - \cos n\pi - \cos\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$= 2\left\{\frac{1}{4n\pi}(1-\cos n\pi) - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}\right\}$$

$$b_n = \frac{1}{2n\pi}\{1-(-1)^n\} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{2n\pi}\{1-(-1)^n\} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin n\pi x$$

Q) Find If $f(x) = \begin{cases} x & \text{in } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{in } \frac{\pi}{2} < x < \pi \end{cases}$ S.T

i) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$

ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$

Practical Harmonic Analysis

Q) The turning moment T on the crankshaft of a steam engine for the crank angle θ is given as follows

θ°	0	30	60	90	120	150	180	210	240	270	300	330
T	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2

Expand T as a Fourier series upto to First harmonic

— $0 \le \theta < 2\pi$ $2L = 2\pi \Rightarrow L = \pi$

$N = 12$

θ	T	cosθ	Tcosθ	Sinθ	Tsinθ
0	0	1	0	0	0
30	2.7	0.866	2.3382	0.5	1.35
60	5.2	0.5	2.6	0.866	4.5032
90	7.0	0	0	1	7.0
120	8.1	−0.5	−4.05	0.866	7.0146
150	8.3	−0.866	−7.1878	0.5	4.15
180	7.9	−1	−7.9	0	0
210	6.8	−0.866	−5.8888	−0.5	−3.4
240	5.5	−0.5	−2.75	−0.866	−4.763
270	4.1	0	0	−1	−4.1
300	2.6	0.5	1.3	−0.866	−2.2516
360	1.2	0.866	1.0392	−0.5	−0.6
Total	59.4		−20.4992		8.9032

$$a_0 = \frac{1}{6}\sum T = \frac{1}{6}(59.4) = 9.9$$

$$a_1 = \frac{1}{6}\sum T\cos\theta = \frac{1}{6}(-20.4992) = -3.4165$$

$$b_1 = \frac{1}{6}\sum T\sin\theta = \frac{1}{6}(8.9032) = 1.4839$$

$$\therefore T = \frac{a_0}{2} + (a_1\cos\theta + b_1\sin\theta)$$

$$T = 4.95 - 3.4165\cos\theta + 1.4839\sin\theta$$

8) Express y as a Fourier series upto to the third harmonic given the following values

x	0	1	2	3	4	5
y	4	8	15	7	6	2

$\Rightarrow$ $0 \le x < 6 \Rightarrow [0,6)$

$2L = 6 \Rightarrow L = 3$, $N = 6$

$$f(y) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \left(a_2 \cos\frac{2\pi x}{3} + b_2 \sin\frac{2\pi x}{3}\right) + \left(a_3 \cos\frac{3\pi x}{3} + b_3 \sin\frac{3\pi x}{3}\right)$$

x	y	$y\cos\frac{\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{2\pi x}{3}$	$y\cos\pi x$	$y\sin\pi x$
0	4	4	0	4	0	4	0
1	8	4	6.928	-4	6.928	-8	0
2	15	-7.5	12.99	-7.5	-12.99	15	0
3	7	-7	0	7	0	-7	0
4	6	-3	-5.196	-3	5.196	6	0
5	2	1	-1.732	-1	-1.732	-2	0
	42	-8.5	12.99	-4.5	-2.598	+8	0

$$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(42) = 14$$

$$a_1 = \frac{2}{N}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}(-8.5) = -2.833$$

$$b_1 = \frac{2}{N}\sum y\sin\frac{\pi x}{3} = \frac{2}{6}(12.99) = 4.33$$

$$a_2 = \frac{2}{N}\sum y\cos\frac{2\pi x}{3} = \frac{2}{6}(-4.5) = -1.5$$

$$b_2 = \frac{2}{N}\sum y\sin\frac{2\pi x}{3} = \frac{2}{6}(-2.598) = -0.866$$

$$a_3 = \frac{2}{N}\sum y\cos\pi x = \frac{2}{6}(8) = 2.667, \quad b_3 = 0$$

$$y = 7 + \left(-2.833\cos\frac{\pi x}{3} + 4.33\sin\frac{\pi x}{3}\right) + \left(-1.5\cos\frac{2\pi x}{3} - 0.866\sin\frac{2\pi x}{3}\right) + \left(2.667\cos\pi x\right)$$

Q) Obtain the constant term & the first three co-efficient in the Fourier cosine series for y using the foll. table

x	0	1	2	3	4	5
y	4	8	15	7	6	2.

— $0 \le x < 6 \Rightarrow [0,6) \Rightarrow 2L = 6$ ~~$\Rightarrow L=3$~~

$N = 6$

$$y = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{6} + a_2 \cos\frac{2\pi x}{6} + a_3 \cos\frac{3\pi x}{6} \quad (1)$$

x	y	$y\cos\frac{\pi x}{6}$	$y\cos\frac{\pi x}{3}$	$y\cos\frac{\pi x}{2}$
0	4	4	4	4
1	8	6.928	4	0
2	15	7.5	−7.5	−15
3	7	0	−7	0
4	6	−3	−3	6
5	2	−1.732	1	0
Total	42	13.696	−8.5	−5

$$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(42) = 14$$

$$a_1 = \frac{2}{N}\sum y\cos\frac{\pi x}{6} = \frac{2}{6}(13.696) = 4.565$$

$$a_2 = \frac{2}{N}\sum y\cos\frac{\pi x}{3} = \frac{2}{6}(-8.5) = -2.833$$

$$a_3 = \frac{2}{N}\sum y\cos\frac{\pi x}{2} = \frac{2}{6}(-5) = -1.667$$

$$y = 7 + 4.565\cos\frac{\pi x}{6} + (-2.833)\cos\frac{\pi x}{3} - 1.667\cos\frac{\pi x}{2}$$

Q) The following table give the variation of a periodic current A over a period T

t sec	0	T/6	T/3	T/2	2T/3	5T/6	T
A (amp)	1.98	1.30	1.05	1.30	−0.88	−0.25	1.98

Show that there is a constant part of 0.75 amp in the current A & also obtain the amplitude of the first harmonic

t	A	$A\cos\frac{2\pi t}{T}$	$A\sin\frac{2\pi t}{T}$
0	1.98	1.98	0
T/6	1.3	0.65	1.125
T/3	1.05	−0.52	1.29
T/2	1.3	−1.30	0
2T/3	−0.88	0.44	0.76
5T/6	−0.25	−0.125	0.216
T			
	4.5	1.125	3.391

$\Rightarrow$ $0 \le x \le T$ $\quad N=6$

$2L = T \Rightarrow L = T/2$

$A = \frac{a_0}{2} + a_1\cos\frac{\pi t}{L} + b_1\sin\frac{\pi t}{L}$

$A = \frac{a_0}{2} + a_1\cos\frac{2\pi t}{T} + b_1\sin\frac{2\pi t}{T}$

$a_0 = \frac{2}{N}\sum A = \frac{2}{6}(4.5) = 1.5$

$a_1 = \frac{2}{N}\sum A\cos\frac{2\pi t}{T} = \frac{2}{6}(1.125) = 0.375$

$b_1 = \frac{2}{N}\sum A\sin\frac{2\pi t}{T} = \frac{2}{6}(3.391) = 1.13$

$A = 0.75 + \left(0.375\cos\frac{2\pi t}{T} + 1.13\sin\frac{2\pi t}{T}\right)$

$\therefore$ Direct current $= 0.75$

Amplitude $= \sqrt{a_1^2 + b_1^2} = \sqrt{(0.375)^2 + (1.13)^2} = 1.19$