Unit-02

"Relations & Function"

Zero-One matrix:- Consider the sets $A = \{a_1, a_2, a_3 --- a_m\}$ & $B = \{b_1, b_2, b_3 --- b_n\}$ of orders m & n respectively. Then $A \times B$ consists of all ordered pairs (a_i, b_j) $1 \le i \le m$, $1 \le j \le n$, which are 'mn' in no. Let R be a relation from A to B. then R is subset of $A \times B$.

$$\text{Let } m_{ij} = (a_i, b_j) = \begin{cases} 1, & (a_i, b_j) \in R \\ 0, & (a_i, b_j) \notin R. \end{cases}$$

The $m \times n$ matrix formed by these m_{ij}'s is called Matrix of Relation R or Relation matrix of R, denoted by M_R or $M(R)$. Since the elements of M_R are only 0's & 1's it is also called as Zero-One matrix.

Note: The rows of $M(R)$ corresponds to A & columns correspond to B.

Ex:- Let $A = \{1, 2, 3\}$ & $B = \{a, b\}$ Let $R = \{(1,a), (2,b), (3,a)\}$

$m_{11} = 1$, $m_{12} = 0$, $m_{21} = 0$, $m_{22} = 1$, $m_{31} = 1$, $m_{32} = 0$

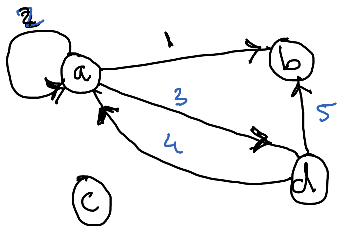
$$M_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Directed graph or Digraph:- Let R be a relation on a finit set A. Then the pictorial representation of R is discribed as below,

For every element of 'A' draw a circle or a dot & label them accordingly. These are called Nodes or Vertices. Draw an arrow from a to b for every $(a,b) \in R$. which is called Edge. The resulting picture is called Directed graph or Digraph.

In a digraph, the vertex from which an edge is starting is called Origin or Source of the edge. & the vertex where the edge is ending is called Terminus of the edge. The edge whose origin & terminus are same is called a Loop. The vertex which is neither source nor terminus is called Isolated vertex. The no. of edges starting from a vertex is called Out-degree of the vertex. & the no. of edges terminating in a vertex is called In-degree of the vertex.

Ex:- Let $A = \{a, b, c, d\}$ & $R = \{(a,b), (a,a), (a,d), (d,b), (d,a)\}$



	a	b	c	d
In-degree	2	2	0	1
Out-degree	3	0	0	2

Examples / Problems :-

1> Let $A = \{1, 2\}$, $B = \{p, q, r, s\}$ & let the relation from A to B be given by - $R = \{(1,q), (1,r), (2,p), (2,q), (2,s)\}$. Write the matrix of R.

$\longrightarrow$ $m_{11} = 0$, $m_{12} = 1$, $m_{13} = 1$, $m_{14} = 0$

$m_{21} = 1$, $m_{22} = 1$, $m_{23} = 0$, $m_{24} = 1$

$$\therefore M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

2> Let $A = \{1, 2, 3, 4\}$ & R be a relⁿ of A defined by $(a,b) \in R$ iff $a < b$. Write down R as a set of ordered pairs. Also write the zero-one matrix of it.

$\longrightarrow$ $R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

$$\therefore M_R = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

* Determine the relation R from a set A to set B as represented by the following matrix. Also draw the digraph.

$$M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

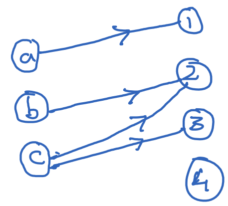
matrix. Also draw the digraph.

$$M_R = \begin{array}{c} a \\ b \\ c \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

→ Let $A = \{a, b, c\}$ & $B = \{1, 2, 3, 4\}$

$\therefore R = \{(a,1), (b,2), (c,2), (c,3)\}$

& digraph:



* Let $A = \{1,2,3,4\}$ & let R be the relation on A defined by xRy iff x divides y written as $x|y$.

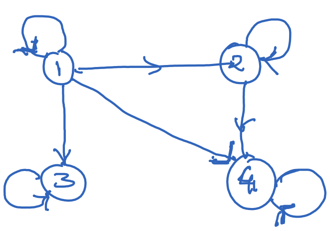
i) Write down R as a set of ordered pairs

ii) Draw the digraph of R.

iii) Determine the in-degree & out-degree of the vertices.

→ i) $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

ii) digraph:

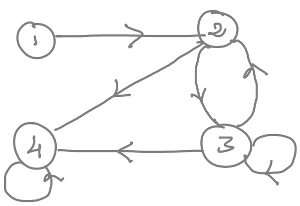


iii)

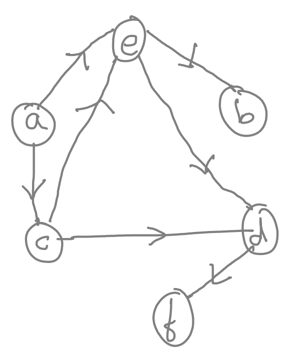
	1	2	3	4
In-degree	1	2	2	3
Out-degree	4	2	1	1

* Find the relation R determined by each of the digraph given below. Also write the matrix of the relation.

i)



ii)



→ $R_1 = \{(1,2), (2,3), (2,4), (3,3), (3,2), (3,4), (4,4)\}$

$\longrightarrow$ $R_1 = \{(1,2),(2,3),(2,4),(3,3),(3,2), \ldots$

$$M_{R_1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

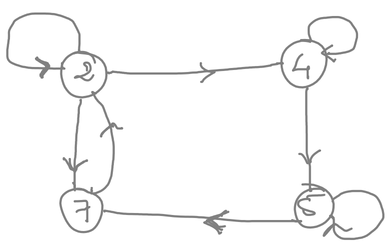
$R_2 = \{(a,c),(a,e),(c,e),(d,f),(e,d),(e,b),(c,d)\}$

$$M_{R_2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

* Let $A = \{2,4,5,7\}$ & R be the relation on A having the matrix,

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$ construct the digraph of R.

$\longrightarrow$



Properties of relations:-

Let A be any non-empty set & let R be a relation on A.

i> Reflexive relⁿ :- If $\forall a \in A$, $(a,a) \in R$. then we say that R is reflexive.

& if for any $a \in A$, if $(a,a) \notin R$ then R is non-reflexive.

Ex:- $A = \{1,2,3,4\}$ $R_1 = \{(1,1),(2,2),(3,3),(4,4)\}$ — reflexive, T

$R_2 = \{(1,1),(2,2),(3,3),(1,4)\}$ — non-reflexive, T

$R_3 = \{(1,1)\}$ — non-reflexive, T

$R_4 = \{(1,1),(2,2),(1,2),(3,3),(4,4),(3,2)\}$. — reflexive, T ii> Symmetric relⁿ :- A relation R on the set A is said to be symmetric if $(b,a) \in R$ whenever $(a,b) \in R$.

Note:- A relⁿ which is not symmetric is called Assymetric relⁿ.

iii> Transitive relⁿ : A relⁿ R defined on 'A' is said to be Transitive if $(a,c) \in R$ whenever $(a, b) \in R$ & $(b,c) \in R$.

Ex:- $R_5 = \{(1,1), (1,3), (3,4)\}$ — non-Transitive

iv> Equivalence relⁿ :- A relation R defined on a set A is said to be Equivalence if its is reflexive, symmetric & transitive.

Examples / Problems :-

1> Let $A = \{1,2,3,4\}$ & $R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}$ be a relation on A. S.T. R is equivalence relation.

→ Reflexive:- We can see that $(1,1), (2,2), (3,3), (4,4) \in R$. ∴ R is reflexive.

Symmetric:- $(1,2), (2,1) \in R$ also $(3,4), (4,3) \in R$ ∴ R is symmetric.

Transitivity:- For $(1,2)$ & $(2,1) \in R$, $(1,1)$ & $(2,2) \in R$; $(3,4)$ & $(4,3) \in R$, $(3,3)$ & $(4,4) \in R$. } ∴ R is transitive.

∴ Hence R is Equivalence relation.

* Let $A = \{1,2,3 ---- 12\}$. On this set define the relation R by $(x,y) \in R$ iff $x-y$ is a multiple of 5. Verify that R is equivalence.

→ Reflexivity:- $\forall x \in A$, wkt, $x-x=0$ which is multiple of 5

∴ $(x,x) \in R$ $\forall x \in A$.

∴ R is reflexive.

Symmetric:- Let $(x,y) \in R \Rightarrow x-y = 5k$ (where $k \in Z$)

$\Rightarrow y-x = 5(-k)$ [here $-k \in Z$]

$\Rightarrow (y,x) \in R$

∴ R is symmetric.

Transitivity :- Let $(x,y) \in R$ & $(y,z) \in R$

$\Rightarrow \quad x - y = 5k_1$ & $y - z = 5k_2$

$\Rightarrow \quad x - y + y - z = 5k_1 + 5k_2 = 5(k_1 + k_2)$

$\Rightarrow \quad x - z = 5k_3$

$\Rightarrow \quad (x,z) \in R \quad \therefore R$ is transitive.

$\therefore$ Hence R is equivalence relation //

* A relation R on a set $A = \{a, b, c\}$ is represented by the following matrix,

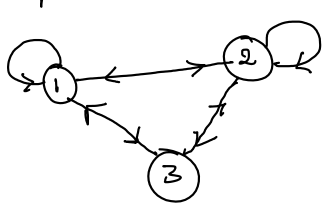
$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Determine whether R is equivalence.

→ From the matrix we can write $R = \{(a,a), (a,c), (b,b), (c,c)\}$. We note that $(a,c) \in R$ but $(c,a) \notin R$. $\therefore$ R is not symmetric.

Also, just by observing the matrix, given matrix of relation is not - symmetric. Hence R is non-symmetric.

$\therefore$ R is not equivalence. ~~re~~ relation.

* The digraph of a relation R on a set $A = \{1, 2, 3\}$ is as given below. Determine whether R is an equivalence relation.



→ From the digraph we see that the vertex 3 doesn't have a loop. Hence R is not reflexive. & $\therefore$ R is not equivalence relation.

* Let S be the set of all non-zero integers & $A = S \times S$. On A, define R by $(a,b) R (c,d)$ iff $ad = bc$. S.T. R is an equivalence relation.

→ We note that $\forall a \in S$, $(a,a) \in A$ also, $aa = aa$

$\Rightarrow (a,a) R (a,a)$

$\therefore$ R is reflexive.

Now, consider, $(a,b) R (c,d) \iff ad = bc$

$\iff bc = ad$

$\iff cb = da$

$\iff (c,d) R (a,b)$ $\therefore$ R is symmetric

Let us consider, $(a,b) R (c,d)$ & $(c,d) R (p,q)$

$\iff ad = bc$ & $cq = pd$

$\iff \frac{a}{b} = \frac{c}{d}$ & $\frac{c}{d} = \frac{p}{q}$

$\iff \frac{a}{b} = \frac{p}{q} \iff aq = bp \iff (a,b) R (p,q)$

$\therefore$ R is transitive.

$\therefore$ R is Equivalence rel^n //

* For a fixed integer $n > 1$, prove that the relation 'congruent modulo n', is an equivalence relation.

→ $a \equiv b \pmod{n} \implies a-b$ is divisible by n.

Reflexivity :- For any 'a' $\in R$, $a-a$ is divisible by $n \implies a \equiv a \pmod{n}$

$\therefore$ R is reflexive.

Symmetric :- Let $(a,b) \in R \implies a \equiv b \pmod{n}$

$\implies a-b$ is divisible by n

$\implies b-a$ is divisible by n

$\implies b \equiv a \pmod{n}$

$\implies (b,a) \in R$. $\therefore$ R is symmetric.

Transitivity :- Let $(a,b) \in R$ & $(b,c) \in R$.

$\implies a \equiv b \pmod{n}$ & $b \equiv c \pmod{n}$.

$\implies a-b = nk_1$ & $b-c = nk_2$

$\implies a-b+b-c = nk_1 + nk_2 = n(k_1+k_2)$

$\implies a-c = nk_3 \implies a \equiv c \pmod{n}$. $\therefore (a,c) \in R$.

$\therefore$ R is transitive

$\therefore$ R is equivalent.

Equivalence Class:- Let R be an equivalence relation on 'A' & $a \in A$. Then the set of all $x \in A$ which are related to 'a' (i.e. xRa) or (i.e. $(x,a) \in R$) is called an Equivalence relation. It is represented by $R(a)$, $[a]$, or $\bar{a}$.

i.e. $[a] = \{x \in A \mid (x,a) \in R\}$

Ex:- $A = \{1,2,3,4\}$ $\quad R = \{(1,1), (1,2), (2,1), (3,4), (2,2), (4,3), (3,3), (4,4)\}$.

$[1] = \{1,2\}$, $[2] = \{1,2\}$, $[3] = \{3,4\}$, $[4] = \{3,4\}$

Note: ① If R is an equivalence relation on A & $a \in A$ then $a \in [a]$.

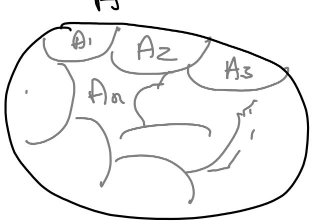
② For $a,b \in A$ & if aRb then $[a] = [b]$

③ If $[a] \cap [b] \neq \phi$ then $[a] = [b]$

④ If $b \in [a]$ then $[a] = [b]$.

Partition of a Set:- Let 'A' be a non-empty set. Suppose there exist non-empty subsets A_1, A_2, A_3 --- A_n of A such that (i) $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = A$ & (ii) $A_i \cap A_j = \phi$ when $i \neq j$, then $P = \{A_1, A_2, A_3, \ldots, A_n\}$ is called a Partition of A. & A_1, A_2, A_3 --- A_n are called Blocks / Cells of the the partition.

A



$A_1 \cup A_2 \cup A_3 \cdots \cup A_n = A$.

$A_i \cap A_j = \phi$.

Ex:- $A = \{1,2,3,4,5,6\}$

$A_1 = \{1,2\}$ $\quad A_2 = \{3,4\}$ $\quad A_3 = \{5,6\}$

$A_1 \cup A_2 \cup A_3 = A$ $\qquad A_1 \cap A_2 = \phi = A_2 \cap A_3 = A_3 \cap A_1$

$\Rightarrow$ ✓ $P_1 = \{A_1, A_2, A_3\}$

$A_1 = \{1,3,5\}$ $\quad A_2 = \{2,4,6\}$

$A_1 \cup A_2 = A \qquad A_1 \cap A_2 = \emptyset$

$P_2 = \{ A_1, A_2 \}$

Thm: If A is a non-empty set, then

i) Any equivalence relation R on A, induces a partition of A.

ii) Any partition of A gives rise to an equivalence relation R on A.

→ (i) Suppose R is an equivalence relation on A & let P be the set of all the distinct equivalence classes of the elements of A w.r.t. R.

i.e. $P = \{ [a] \mid a \in A \}$

Then we note that every element a of A belongs to an equivalence class in P. Therefore, A is the union of the equivalence classes in P. Also every two equivalence classes in P are mutually disjoint. Therefore P is a partition of A.

Thus every equivalence relation on A induces a partition of A.

(ii) Let $P = \{A_1, A_2, A_3 - - -, A_n\}$ be a partition of A. Define a relation R on A by aRb iff a & b both belong to the same block of the partition.

Take any $a \in A$, then $a \in A_i$ for some i, hence $(a,a) \in R$ because a & a will be in A_i. $\therefore$ R is reflexive.

For any $a, b \in A$, let $(a,b) \in R$ then a & b should belong to A_i for some i. that means b & a belongs to the same A_i. $\Rightarrow (b,a) \in R$. Hence R is Symmetric.

For any a, b & $c \in A$ let $(a,b) \in R$ & $(b,c) \in R$ then,

$\rightarrow$ a & b belongs to the same block A_i & b & c also belong to A_i

$\Rightarrow$ a & c belong to same block A_i

$\Rightarrow$ aRc $\Rightarrow \therefore$ R is transitive.

$\therefore$ R is equivalent. & hence the proof//

Ex:-

* Let $A=\{1,2,3,4,5,6,7\}$ & R be the equivalence relation on A that induces the partition $A=\underbrace{\{1,2\}}_{A_1}\cup\underbrace{\{3\}}_{A_2}\cup\underbrace{\{4,5,7\}}_{A_3}\cup\underbrace{\{6\}}_{A_4}$. Find R.

$\longrightarrow$

$\therefore\ R=\{(1,1),(1,2),(2,2),(2,1),(3,3),(4,4),(4,5),(4,7),(5,5),(5,4),(5,7),(7,7),$
$(7,4),(7,5),(6,6)\}$ //

* On the set Z, a relation R is defined by aRb iff $a^2=b^2$. Verify that R is an equivalence relation. Determine the partition induced by this relation.

$\longrightarrow$ Here aRb iff $a^2=b^2$

Now we note that, $\forall a\in Z$, $a^2=a^2 \Rightarrow aRa$ $\therefore R$ is reflexive.

For any $a,b\in Z$, let $aRb \Rightarrow a^2=b^2$
$\Rightarrow b^2=a^2 \Rightarrow bRa$ $\therefore R$ is symmetric.

For any $a,b,c\in Z$, let aRb & bRc
$\Rightarrow a^2=b^2$ & $b^2=c^2 \Rightarrow a^2=c^2 \Rightarrow aRc$ $\therefore R$ is transitive.

$\therefore R$ is equivalence relation.

For any $a\in Z$, $[a]=\{x\in Z \mid (x,a)\in R\}$
$=\{x\in Z \mid x^2=a^2\}$
$=\{x\in Z \mid x=\pm a\} = \{x\in Z \mid a=\pm x\}$

Here x can be either '0' or non-zero integer.

$\therefore$ There exist only two equivalence classes $[0]$, $[n]$
$\Rightarrow [0]=\{0\}$ & $[n]=\{n,-n\}$ $\forall n\in Z^+$

$\therefore$ required partition is $P=\{[0],[n]\}$ // where $n\in Z^+$

Anti-Symmetric Relation:- A relation 'R' defined on a set 'A' is said to be — Anti-Symmetric if $(a,b)\in R$ & $(b,a)\in R$ then $a=b$.

Ex:- i) The relation 'less than or equal to' defined on $\mathbb{R}$.

ii) The relation 'divisibility' on $\mathbb{R}$.

Note:- * Anti-symmetric relation & assymmetric relation are not same.

* A relation can be Anti-symmetric & Symmetric at-a-time.

Partial Order:- A relation which is reflexive, anti-symmetric & transitive is called Partial Order.

Note:- The set A with the partial order R defined on it is called a Partially Ordered Set or Poset. & it is denoted by (A, R).

Ex:- i) $(\mathbb{R}, \leq)$

ii) $(P(A), \subseteq)$

Total Order:- Let 'R' be a partial order on A, then R is called Total Order if $\forall x, y \in A$ either xRy or yRx.

Note:- Every total order is a partial order but every partial order need not be total order.

Hasse Diagram:- A digraph drawn for a Partial order is called as Hasse diagram. or Poset diagram.

Note:- i) All the elements (vertices) are represented by 'dots'

ii) We avoid the loops at every vertex in Hasse diagram because the Partial order is by default reflexive (by convention).

iii) Whenever $(a,b) \in R$ & $(b,c) \in R$ we draw an arrow from a to b & from b to c & if $(a,c) \in R$ we draw an arrow from $a \to c$. Here in Hasse diagram we avoid the arrow from a to c. because partial order is by default transitive (by convention)

iv) All the edges of Hasse diagram are always pointed upwards.

(a, b) a b

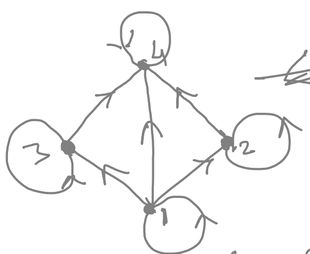
(a,b) a b

Example:-

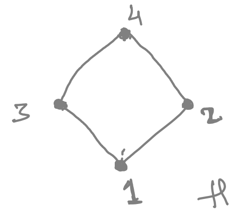
1) Let $A = \{1,2,3,4\}$. & $R = \{(1,1),(1,2),(2,2),(2,4),(1,3),(3,3),(3,4),(1,4),(4,4)\}$. Verify that R is a partial order on A. Write down the Hasse diagram for R.

→ Here we observe that $\forall a \in A$, $(a,a) \in R$ ∴ R is reflexive. Also we can observe that R is transitive. Further R doesnot contain the ordered pairs (a,b) & (b,a) with $a \neq b$. ∴ R is anti-symmetric. ∴ R is a partial order on A.

Hasse diagram:-



digraph



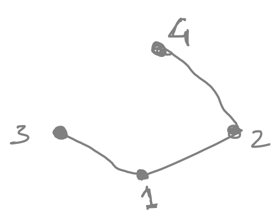
Hasse diagram

* Let R be a relation on the set $A = \{1,2,3,4\}$ defined by xRy iff x divides y. Prove that (A,R) is a poset & hence draw the Hasse diagram.

→ $R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$

Since $\forall a \in A$, $(a,a) \in R$ ∴ R is reflexive. & whenever $(a,b) \in R$ & $a \neq b$, then $(b,a) \notin R$. ∴ R is anti-symmetric. Also we can observe that is R is transitive. ∴ (A,R) is a Poset.

Hasse diagram:-

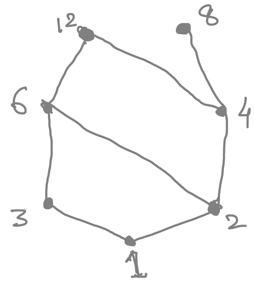


* Let $A = \{1,2,3,4,6,8,12\}$. On A, define R as xRy iff $x|y$. Draw the Hasse diagram.

→ $R = \{(1,1),(1,2),(1,3),(1,4),(1,6),(1,8),(1,12),(2,2),(2,4),(2,6),(2,8),(2,12),$
$(3,3),(3,6),(3,12),(4,4),(4,8),(4,12),(6,6),(6,12),(8,8),(12,12)\}$

Observing the relation we can see that R is reflexive & transitive. Also for all $(a,b) \in R$ & $a \neq b$, $(b,a) \notin R$. $\therefore R$ is antisymmetric.
$\therefore R$ is a partial order.

Hasse diagram:-



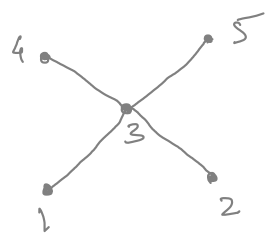
* Draw the Hasse diagram of the relation R on $A = \{1,2,3,4,5\}$ whose matrix is as given below.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\longrightarrow$ Here $R = \{(1,1),(1,3),(1,4),(1,5),(2,2),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5),(4,4),(5,5)\}$

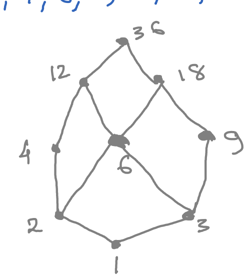
$\therefore$ By observing we can say that R is reflexive, anti-symmetric & transitive. $\therefore R$ is a partial order.

Hasse diagram:-

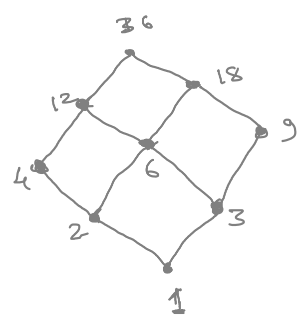


* Draw the Hasse diagram representing the positive divisors of 36.

$\longrightarrow$ $D_{36} = \{1,2,3,4,6,9,12,18,36\}$



or



Extremal elements in a poset:-

i) **Maximal element:** An element $a \in A$ is called maximal element of A if there exists no element $x \in A$ & $x \neq a$ such that $a \preceq x$.
In other words an element $a \in A$ is called maximal element of A

if no edges comes out of a in Hasse diagram.

ii) <u>Minimal element</u>:- An element $a \in A$ is called minimal element of A if there exists no element $x \in A$ & $x \neq a$ such that $x \preceq a$

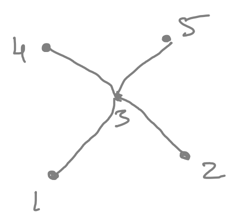
In other words an element $a \in A$ is called minimal element of A if no edge comes towards "a" in hasse diagrame.

<u>Note</u>:- Maximal element & minimal elements together are called as Exteremal element.

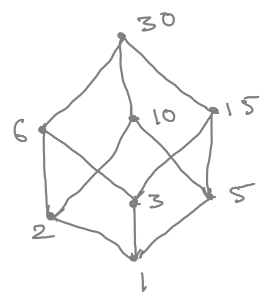
* <u>Greatest element</u>:- An element $a \in A$ is called greatest element of A if $\forall x \in A$, $x \preceq a$.

* <u>Least element</u>:- An element $a \in A$ is called least element of A if $\forall x \in A$, $a \preceq x$.

<u>Lattice</u>:- A hasse diagram is called as Lattice iff there exist both - greatest element & least element in the hasse diagram.



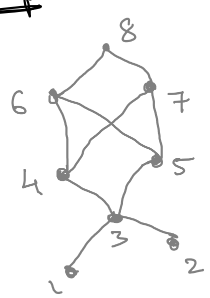
Non-lattice



<u>Lattice</u>

<u>Example</u>:-

(1)

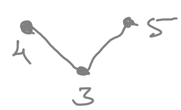


It's not a lattice because least element is not there.

(2)

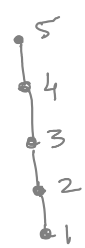
Not a hasse diagram
∴ non-lattice.

(3)



It's not a lattice because greatest element is not there.

(4)



It is a lattice because greatest element is 5 & least element is 1.

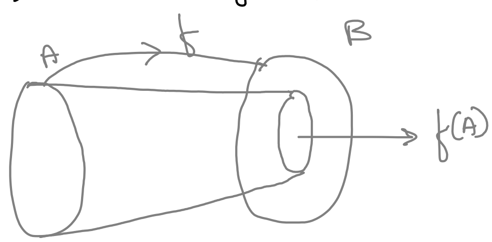
$f: A \to B$

<u>Functions</u>:- A relation defined from A to B is called a function iff.

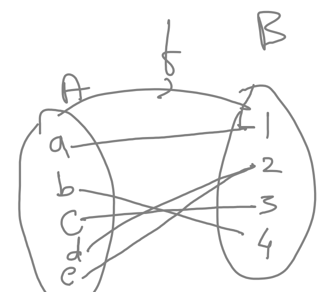
Functions:- A relation defined from A to B is called a function iff every element of A has a unique image in B.

Here A is called as domain & B is called Co-domain.

& $f(A) \subseteq B$ is called Range of 'f'.



Types of functions:-



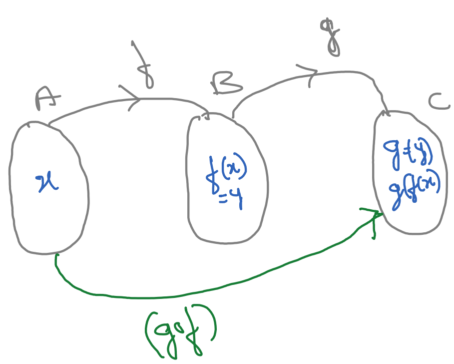
Injection / One - One : Every image should have a unique pre-image in the domain, then that function is called as One-One.

Surjection / Onto : If every element of co-domain has the preimage in the domain then it is called Onto function.

Bijection:- A function which is both one-one & onto is called Bijection.

Inverse of 'f' :- If $f: A \to B$ is a bijection then $f^{-1}: B \to A$ is called - inverse of 'f'.

Composition of functions:-



Let x be an arbitrary element of A then f(x) will be an element of B. Again since $g: B \to C$ is a function then $g(f(x))$ will be in C. [i.e. $g(f(x)) = (g \circ f)(x)$] this '$g \circ f$' is called a - Composite function from A to C i.e. $g \circ f : A \to C$

Note:- Let $f: A \to B$ & $g: B \to A$ be the two given functions then, if $f \circ g = g \circ f = I$ then f & g are inverse of each other.

Properties of Functions:-

Th^m 01] "Let $f: X \to Y$ be a function & A and B be arbitrary non-empty subsets of X. Then,

i) If $A \subseteq B$ then $f(A) \subseteq f(B)$

ii) $f(A \cup B) = f(A) \cup f(B)$

iii) $f(A \cap B) \subseteq f(A) \cap f(B)$ & equality holds if f is one-one."

$\longrightarrow$ Let $y \in Y$ be an arbitrary element.

(i) $y \in f(A) \implies y = f(x)$ for some $x \in A$.

$\implies y = f(x)$ for some $x \in B$, $\because A \subseteq B$

$\implies y \in f(B)$

$\therefore f(A) \subseteq f(B)$

ii) $y \in f(A \cup B) \implies y = f(x)$ for some $x \in A \cup B$

$\implies y = f(x)$ for $x \in A$ or $x \in B$

$\implies y \in f(A)$ or $y \in f(B)$

$\implies y \in [f(A) \cup f(B)]$

$\therefore f(A \cup B) \subseteq f(A) \cup f(B)$ —— (1)

Since $A \subseteq A \cup B$ & $B \subseteq A \cup B$ it follows from (i) that $f(A) \subseteq f(A \cup B)$ & $f(B) \subseteq f(A \cup B)$. $\therefore f(A) \cup f(B) \subseteq f(A \cup B)$ —— (2)

$\therefore$ from (1) & (2)

$f(A \cup B) = f(A) \cup f(B)$

(iii) $y \in f(A \cap B) \implies y = f(x)$ for some $x \in A \cap B$.

$\implies y = f(x)$ for $x \in A$ & $x \in B$

$\Rightarrow y \in f(A)$ & $y \in f(B)$

$\Rightarrow y \in f(A) \cap f(B)$

$\therefore f(A \cap B) \subseteq f(A) \cap f(B)$ —— (1)

Let $y \in [f(A) \cap f(B)] \Rightarrow y \in f(A)$ & $y \in f(B)$

$\Rightarrow y = f(x_1)$ for some $x_1 \in A$ & $y = f(x_2)$ for some $x_2 \in B$.

$\Rightarrow y = f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$ if f is one-one.

$\Rightarrow y = f(x_1)$ for some $x_1 \in A$ & $x_1 \in B$.

$\Rightarrow y = f(x_1)$ for some $x_1 \in A \cap B$.

$\Rightarrow y \in f(A \cap B)$.

$\therefore$ if f is one-one, $[f(A) \cap f(B)] \subseteq f(A \cap B)$ —— (2)

$\therefore$ from (1) & (2)

$f(A) \cap f(B) = f(A \cap B)$ if f is one-one

Thm 02] Let A & B be finite sets. & f be a function from A to B. Then the following are true.

i) If f is one-to-one, then $n(A) \leq n(B)$

ii) If f is onto, then $n(B) \leq n(A)$

iii) If f is a one-to-one correspondance, then $n(A) = n(B)$

iv) If $n(A) > n(B)$ then at least two different elements of A must have same image in B under f

proof: Here A & B are finite sets with $|A| = n$ & $|B| = m$

i) Suppose f is one-to-one. Then the images of the elements of A, namely $f(a_1), f(a_2), f(a_3) \ldots f(a_n)$ are all different & so their no. is 'n'. All these images belong to B. Therefore, B must have atleast 'n' elements.

i.e. $|B| \geq n = |A|$ $\therefore |A| \leq |B|$

ii> Suppose 'f' is onto. Then with each 'b' in B there is an 'a' in A such that $f(a) = b$. Since 'f' is a function, no two different b's can correspond to same 'a'. Therefore, the number 'k' of a's which are pre images of b's can't be less than the no. of b's. Thus, we should have $k \geq m$. On other hand every 'a' is a pre-image of some 'b'. Therefore $k = n$. Thus $m \leq k$ & $k = n$. As such, $m \leq n$, i.e. $|B| \leq |A|$.

iii> Suppose f is one-to-one correspondance. Then f is both one-to-one & onto. From results proved in the above two paragraphs, we have $|A| \leq |B|$ & $|A| \geq |B|$. Therefore $|A| = |B|$

iv> The contrapositive of the result ① reads: "If $|A| > |B|$ then f is not one-one". This means that if $|A| > |B|$ then at least two different elements of A have the same image under 'f'. This result is true because the result ① is true.

Th^m 03] Suppose A & B are finite sets having the same no. of elements & f is a function from A to B, then f is one-to-one iff f is onto.

proof → Let $f: A \to B$ be a function such that $|A| = |B| = n$ (say).

i.e. $A = \{a_1, a_2, a_3 \text{----} a_n\}$.

Suppose f is one-to-one function, then images of elements of A must be different. i.e. $f(a_1)$, $f(a_2)$, $f(a_3)$ ---- $f(a_n)$ all must be different.

$\therefore$ $f(A)$ has 'n' no. of elements.

$$\Rightarrow \quad |f(A)| = n = |B|$$

$$\therefore \quad |f(A)| = |B| \Rightarrow \text{Range} = \text{Co-domain}$$

$\therefore$ f is onto.//

Conversely, suppose f is onto. then $B = f(A) = \{f(a_1), f(a_2), f(a_3) \text{----} f(a_n)\}$.

& $|B| = n = |A|$. (given).

Also [from theorem ②] we know that if $n(A) \leq n(B)$ then f is one-one function. But here $n(A) \nless n(B)$ because $n(A) = n(B)$.

Therefore f is one-one.//