UNIT-1

FUNDAMENTALS OF LOGIC

PROPOSITION / STATEMENT

A proposition or statement is a sentence which can be either true or false but not both.

Ex:

(i) Mumbai is the capital of Maharastra. (true)

(ii) 2 is an odd number (false)

LOGICAL CONNECTIVES

Compound statements obtained by the use of words or phrases like "not", "and", "or", "if, then" and "if and only if" are called logical connectives.

1 Negation (~ or not) ($\sim p$)

If p is any proposition then negation of p is denoted by $\sim p$ and define by "not p"

Truth Table

p	$\sim p$
0	1
1	0

Ex: p

- Banglore is IT capital of India
- Banglore is not a IT capital of India

2 Conjection ("and", "$\wedge$", $p \wedge q$)

A compound proposition obtained by combining two given statements (proposition) p and q by inserting "and" in between is called conjection and it is denoted by "p and q" or $p \wedge q$

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

~~Ex~~: Remark: Conjection is true only when both the preposition p and q are true, ~~&~~ in all other cases it is false.

3) disjenction ("or", "v", $p \vee q$)

A compound preposition obtained by combining two given preposition p and q by inserting "or" in between is called disjenction. And it is obtained by "p or q" or $p \vee q$

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Remark: Disjenction is false only when both the preposition p and q are false, in all other cases it is true.

4) Conditional ($p \rightarrow q$ or "if p, then q")

A compound preposition obtained by ~~this~~ combining two given preposition p and q by the use of words "if" and "then" at appropriate place is called a condition and it is denoted by $p \rightarrow q$ or "if p, then q".

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Remark: The conditional $p \rightarrow q$ is false only when p is true and q is false, in all other cases it is true.

5) Biconditional ($p \leftrightarrow q$)

let p and q are any two preposition, then the conjuction of the conditional of $p \rightarrow q$ and $p \rightarrow p$ is called biconditional of p and q. And it is denoted by $p \leftrightarrow q$

$$(p \longrightarrow q) \wedge \cancel{p}(q \rightarrow p) = p \longleftrightarrow q$$

p	q	$p \rightarrow q$	$q \rightarrow \cancel{q}p$	$p \leftrightarrow q$
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

TAUTOLOGY

A compound preposition which is true for all the posible truth values of its components is called tautol. Generally Tautology is denoted by. T_0.

CONTRADICTION

A compound preposition which is false for all its possible truth values of its components is called contradiction.

Generally contradiction is denoted by F_0

LAW'S OF LOGICS

1) Law of double negation

If p is any preposition, then ~~pe~~.

$\sim(\sim p) \equiv p$

2) ~~Addex~~ Idempotent law

If p is any preposition, then

(i) $p \vee p = p$

(ii) $p \wedge p = p$

3) Identity law

If p be any proposition and T_0 and ~~for~~ F_0 are Tautology and Contradiction respectively then

(i) $p \wedge T_0 \equiv p$ (ii) $p \vee F_0 \equiv p$

4) Inverse law

If p be any proposition and T_0 and F_0 are Tautology and Contradiction respectively then,

(i) $p \vee (\sim p) \equiv T_0$ (ii) $p \wedge (\sim p) \equiv F_0$

5) Domination Law

If p be any proposition and T_0 and F_0 are Tautology contradiction respectively then,

(i) $p \vee T_0 \equiv T_0$ (ii) $p \wedge F_0 = F_0$

6) Commutative law

If p and q are any two proposition then

(i) $p \wedge q \equiv q \wedge p$ (ii) $p \vee q = q \vee p$

7) Absorption Law

If p and q are any two proposition then

(i) $p \wedge (p \vee q) \equiv p$ (ii) $p \vee (p \wedge q) \equiv p$

8) De morgan's law

(i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

(ii) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

9 Assosiative law

If p, q and r are any three propositional then,

(i) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

(ii) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

10 Distributed Law

If p, q and r are any three proposition then,

(i) $p \wedge (q \vee r) \equiv \cancel{p} \; (p \wedge q) \vee (p \wedge r)$

(ii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

11 Law for negation of a conditional

If p and q are any two propositions, then for the given conditional $p \rightarrow q$ its negation is obtained by,

$\sim(p \rightarrow q) \equiv p \wedge (\sim q)$

Truth Table.

p	q	$\sim q$	$p \rightarrow q$	$p \wedge (\sim q)$	$\sim(p \rightarrow q)$
1	1	0	1	0	0
1	0	1	0	1	1
0	1	0	1	0	0
0	0	1	1	0	0

Remark:

$p \rightarrow q \equiv \sim[\sim(p \rightarrow q)]$ $\quad$ [∵ Double negation

$\equiv \sim[p \wedge (\sim q)]$ $\quad$ ∵ negation of a conditio

$\equiv (\sim p) \vee [\sim(\sim q)]$ $\quad$ ∵ By De-Morgan law

$\equiv (\sim p) \vee q$ $\quad$ ∵ By double negation

12 Transitive law

If p, q and r are any 3 preposition such that $p \equiv q$ and $q \equiv r$, then $p \equiv r$ is called transitive law.

13 Substitution law

Suppose U is a compound statement which is tautology and p is a component of U. If p replace each operands of p by by any other preposition q, then resulting compound preposition. V is also tautology.

Examples:

(1) Simplify the following compound prepositions by using laws of logic.

1 $\sim[\sim\{(p \vee q) \wedge r\} \vee (\sim q)]$

$\rightarrow [\sim[\sim\{(p \vee q) \wedge r\}]] \wedge [\sim(\sim q)]$ ∵ By De-Morgan's law

$\{(p \vee q) \wedge r\} \wedge q$ ∵ Law of double negation

$(p \vee q) \wedge (r \wedge q)$ ∵ By Associative law

$(p \vee q) \wedge (q \wedge r)$ ∵ By Commutative law

$\{(p \vee q) \wedge q\} \wedge r$ ∵ By Associative law

$\{q \wedge (p \wedge q)\} \wedge r$ ∵ By Commutative law

$q \wedge r$ ∵ By absorption law

$p \wedge (p \vee q) \equiv p$

OR

$\sim[\sim\{(p \vee q) \wedge r\} \wedge (\sim q)]$

$\sim[\sim(p \vee q) \vee \sim r]$

2 $[p \vee q \vee \{(\sim p) \wedge (\sim q) \wedge r\}]$

$\rightarrow [p \vee q \vee \{\sim (p \vee q) \wedge r\}]$ $\qquad \because$ By De-Morgan law

$(p \vee q) \vee \{\sim (p \vee q) \wedge r\}$

$[(p \vee q) \vee \sim (p \vee q)] \wedge [(p \vee q) \vee r]$ $\qquad \because$ By Distributive law

$=$

$= T_0 \wedge [(p \vee q) \vee r]$ $\qquad \because$ Inverse law.

$= (p \vee q) \vee r$ $\qquad \because$ Identity law.

$= p \vee q \vee r$

3 $p \vee [p \wedge (p \vee q)]$

$p \vee [p]$ $\qquad \because$ By absorption law,

p $\qquad \because$ By idempotent law.

4 $(p \vee q) \wedge [\sim \{(\sim p) \wedge q]$

$(p \vee q) \wedge (\sim(\sim p) \vee \sim q]$ $\qquad \because$ By De-Morgan law

$(p \vee q) \wedge (p \vee \sim q)$ $\qquad \because$ By double negation

$\cancel{((p \vee q) \wedge p)\ (}$

$(p \vee q \wedge p) \vee (p \vee q \wedge \sim q)$ $\qquad \because$ By distributive law

$(p \wedge p \vee q) \vee (p \vee F_0)$ $\quad \wedge$ $\qquad \because$ By Inverse law

$(p \vee q) \vee (T_0)$

$p \vee q$

$p \vee (q \wedge (\sim q))$ $\qquad \because$

$p \vee F_0$ $\qquad \because$ By Inverse law

p $\qquad \because$ Identity law.

(2) Prove the following logical equivalences without using truth table.

1 $(p \vee q) \wedge [\sim \{(\sim p) \wedge q\}] \equiv p$

$\rightarrow (p \vee q) \wedge [(\sim(\sim p)) \vee \sim q]$ $\qquad$ by De-Morgan Law

$(p \wedge \sim q) \vee (q \wedge \sim q)$

$(p \wedge \sim q) \vee F_0$

$p \vee (p \wedge \sim q)$

$(p \vee q) \wedge (p \vee \sim q)$ ∵ by double negation

$((p \vee q) \wedge p) \vee ((p \vee q) \wedge \sim q)$ ∵ By distributive law

~~$((p \vee q) \wedge p) \vee (p \vee (q \wedge \sim q))$~~ ∵ By

$(p \wedge (p \vee q)) \vee ((p \vee q) \wedge \sim q)$ ∵ By Comutative law

$p \vee ((p \vee q) \wedge \sim q)$ ∴ By absorption law

$p \vee ($

$p \vee [q \wedge \sim q]$ ∵ Distributive law

$p \vee F_0$ ∴ By Inverse law

p ∴ By identity law

2. $[(\sim p) \vee (\sim q)] \rightarrow (p \wedge q \wedge r) \equiv p \wedge q$

→ $LHS \equiv [(\sim p) \vee (\sim q)] \rightarrow (p \wedge q \wedge r)$

$[\because p \rightarrow q = (\sim p) \vee q]$

$LHS = [\sim (p \wedge q)] \rightarrow [p \wedge q \wedge r]$

$= [\sim (\sim (p \wedge q)] \vee [p \wedge q \wedge r]$ ∵ By De-Morgan's law

$= (p \wedge q) \vee [p \wedge q \wedge r)$ ∴ By Doble negation

$= (p \wedge q) \vee [(p \wedge q) \wedge r]$ ∴ By Associative law

$= p \wedge q$ ∴ By Absorption law

$= RHS$

DUALITY

Suppose u is a compound preposition that contains connectives $\vee$, and $\wedge$ and also contains T_0 and F_0 as components. Suppose we replace each occurance of $\wedge$, $\vee$, T_0 and F_0 By $\vee$, $\wedge$, F_0 and T_0 respectively. Then the resulting compound preposition is called dual of u. and it is denoted by u^d.

Ex:

- If $u : p \wedge (q \vee \sim r) \wedge (r \vee T_0)$
  then $u^d : p \vee (q \wedge \sim r) \vee (r \wedge F_0)$

Principle of Duality.
For any two compound prepositions u and v such that $u \equiv v$ then $u^d \equiv v^d$ is called principle of duality.

THE CONNECTIVE'S NAND and NOR

(i) NAND
The compound preposition $\sim(p \wedge q)$ is read as "Not p and q" and it is denoted by $\overline{p \wedge q}$ or $p \uparrow q$ is called connective NAND

$$p \uparrow q \equiv \overline{p \wedge q} \equiv \sim(p \wedge q)$$

(ii) NOR
The compound preposition $\sim(p \vee q)$ is read as "Not p or q" and it is denoted by $p \downarrow q$ or $\underline{p \vee q}$ is called connective NOR

$$p \downarrow q \equiv \sim(p \vee q)$$

REMARK

(i) The connective NAND is the combination of connectives NOT and AND
(ii) The connective NOR is the combination of connectives NOT and OR
(iii) Evidently, $(p \uparrow q)$ and $(p \downarrow q)$ are duals to each other.

CONVERSE , INVERSE & CONTRAPOSITIVE OF A CONDITIONAL

If p and q are two proposition and $p \to q$ is a conditional, then

(i) Converse of $p \to q$

$q \to p$ is called converse of $p \to q$

(ii) Inverse of $p \to q$

$(\sim p) \to (\sim q)$ is called inverse of $p \to q$

(iii) Contrapositive of $p \to q$

$(\sim q) \to (\sim p)$ is called contrapositive of $p \to q$

Truth Table

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$(\sim p) \to (\sim q)$	$(\sim q) \to (\sim p)$
1	1	0	0	1	1	1	1
1	0	0	1	0	1	1	0
0	1	1	0	1	0	0	1
0	0	1	1	1	1	1	1

REMARK

(i) From the above truth table we have following observations

(a) The conditional and its contrapositive are logically equivalent, i.e $p \to q \equiv (\sim q) \to (\sim p)$

(b) The converse and inverse of a conditional are logically equivalent,

i.e. $q \to p \equiv (\sim p) \to (\sim q)$

- Example:

let us consider two preposition p and q.

p: 2 is a prime number

q: 5 is an odd number.

Then.

$p \rightarrow q$: If 2 is a prime number then 5 is an odd number.

(i) Converse $p \rightarrow q$

$q \rightarrow p$: If 5 is an odd number then 2 is a prime a number.

(ii) Inverse of $p \rightarrow q$

$(\sim p) \rightarrow (\sim q)$: If 2 is not a prime number then 5 is not an odd number.

(iii) Contraposibive of $p \rightarrow q$

$(\sim q) \rightarrow (\sim p)$: If 5 is not an odd number then 2 is not a ~~even~~ prime number.

HYPOTHETICAL STATEMENT / IMPLICATIVE STATEMENT

The conditional $p \rightarrow q$ where p and q are any two prepositions which are related in one or other way, so that the truth value of q depends on truth value of p. or vise-versa.

Such conditions are called hypothetical statements or Implicative statement.

Logical Implication

In an implicative statement $p \rightarrow q$, q is true whenever p is true, then we say that "p logically implies q" or "p implies q".

It is symbolically written as, $p \Rightarrow q$.

In an implicative statement $p \rightarrow q$, q is not necessarily true whenever p is true then we say that "p does not implies q"

It is symbolically written as, $p \not\Rightarrow q$.

(1) Write down duals of the following prepositions

1. $\sim(p \vee q) \wedge [p \wedge \sim\{q \wedge (\sim r)\}]$

$\rightarrow \sim(p \wedge q) \vee [p \vee \sim\{q \vee (\sim r)\}]$

Given that,

$u: \sim(p \vee q) \wedge [p \wedge \sim\{q \wedge (\sim r)\}]$

$\therefore$ Dual of u is,

$u^d: \sim(p \wedge q) \vee [p \vee \sim\{q \vee (\sim r)\}]$

2. $p \rightarrow q$

Given that,

$u = p \rightarrow q$

$= (\sim p) \vee q$

$\therefore u^d = (\sim p) \wedge q$

3. $u: (p \rightarrow q) \rightarrow r$

$\rightarrow (\sim p \vee q) \longrightarrow r$	$u: (p \rightarrow q) \longrightarrow r$
$\equiv \sim(\sim p \vee q) \vee r$	$\equiv \sim(p \rightarrow q) \vee r$
	$\equiv [p \wedge \sim q] \vee r$
$u^d = \sim(\sim p \wedge q) \wedge r$	
$\equiv (p \vee \sim q) \wedge r$	$u^d \equiv p \vee (\sim q) \wedge r$

4 $p \rightarrow (q \rightarrow r)$

$\rightarrow$ $u: p \rightarrow (q \rightarrow r) \equiv p \rightarrow (\sim q \vee r)$

$\sim p \vee ((\sim q) \vee r)$

$u^d: \sim p \wedge ((\sim q) \wedge r)$

$\equiv \sim p \wedge \sim q \wedge r$

(2) Verify principle of duality for the following logical equivalence

$(p \wedge q) \vee [(\sim p) \vee \{(\sim p) \vee q\}] \equiv (\sim p) \vee q$

$\rightarrow$ ~~$(p \wedge q) \vee [\{(\sim p) \vee (\sim p)\} \vee q]$~~ $\because$ by ~~associative law~~

$(p \wedge o$

Given that,

$u: (p \wedge q) \vee [(\sim p) \vee \{(\sim p) \vee q\}]$

$v: (\sim p) \vee q$

Dual of u and v are

$u^d: (p \vee q) \wedge [(\sim p) \wedge \{(\sim p) \wedge q\}]$

$v^d: (\sim p) \wedge q$

Consider,

$u^d: (p \vee q) \wedge [(\sim p) \wedge \{\sim p \wedge q\}]$

$\equiv (p \vee q) \wedge [\{(\sim p) \wedge (\sim p)\} \wedge q]$ $\because$ by associative law

$\equiv (p \vee q) \wedge [\sim p \wedge q]$ $\because$ by idempotent law

$\equiv (p \vee q) \wedge (q \wedge \sim p)$ $\because$ by commutative

$\equiv ((p \vee q) \wedge q) \wedge \sim p$ $\because$ by associative law

$\equiv [q \wedge (p \vee q)] \wedge \sim p$ $\because$ by commutative

$= q \wedge \sim p$ $\because$ by absorption

$= (\sim p) \wedge q$ $\because$ by commutative

$u^d \equiv v^d$

$\therefore$ principle of duality is verified.

③ Express following prepositions in terms of only NAND and only NOR

(i) $\sim p$

$\rightarrow$ By idempotent law, we have

$p \equiv p \wedge p$ and $p \equiv p \vee p$

$\sim p \equiv \sim(p \wedge p)$ and $\sim p \equiv \sim(p \vee p)$

$\sim p \equiv p \uparrow p$ and $\sim p \equiv p \downarrow p$

(ii) $p \wedge q$

$\rightarrow$ By De-Morgan law

$\sim(\sim p \vee \sim q)$

$\sim p \downarrow \sim q$

by using (i) ~~b~~ we get

$\cancel{(p \uparrow p) \downarrow (q \uparrow q)}$

OR

$(p \downarrow p) \downarrow (q \downarrow q)$

$p \wedge q \equiv \sim[\sim(p \wedge q)]$ ∵ by double

$\equiv \sim(p \uparrow q)$

By using (i)

$p \wedge q \equiv (p \uparrow q) \uparrow (p \uparrow q)$

(iii) $p \vee q$

$\rightarrow$ ~~By~~ $p \vee q \equiv \sim[\sim p \wedge \sim q]$ ∵ by De-Morgan law

$\equiv \sim p \uparrow \sim q$

$\equiv (p \uparrow p) \uparrow (q \uparrow q)$ ∵ by (i)

④ Prove that

(i) $p \uparrow q \equiv q \uparrow p$

$\rightarrow$ Consider,

$p \uparrow q \equiv \sim(p \wedge q)$

$\equiv \sim(q \wedge p)$ ∵ By commutative law

$\equiv q \uparrow p$ ∵ By defn of NAND

$\equiv$ RHS

classmate

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(ii) $p \downarrow q \equiv q \downarrow p$

$\rightarrow$ $p \downarrow q \equiv \sim (p \vee q)$

$\equiv \sim (q \vee p)$ $\therefore$ by commutative law

$\equiv q \downarrow p$

$\boxed{p \downarrow q = q \downarrow p}$

(iii) $p \uparrow (q \uparrow r) \equiv (\sim p) \vee (q \wedge r)$

$\rightarrow$ $p \uparrow (q \uparrow r) \equiv \sim (p \wedge (q \uparrow r))$ $\therefore$ by def NAND

$\equiv \sim (p \wedge \{\sim (q \wedge r)\})$ $\therefore$ by def NAND

$\equiv \sim p \vee \sim (\sim (q \wedge r))$ $\therefore$ by De Morgan

$\equiv (\sim p) \vee (q \wedge r)$ $\therefore$ by double negati

$\therefore \boxed{p \uparrow (q \uparrow r) \equiv (\sim p) \vee (q \wedge r)}$

(iv) $(p \downarrow q) \downarrow r \equiv (p \vee q) \wedge (\sim r)$

$\rightarrow$ $(p \downarrow q) \downarrow r \equiv \sim ((p \downarrow q) \vee r)$ $\therefore$ by def NOR

$\equiv \sim (\sim (p \vee q) \vee r)$ "

$= \sim (\sim (p \vee q)) \wedge \sim r$ $\therefore$ by De Morgan

$= (p \vee q) \wedge \sim r$ $\therefore$ by double negati

$\therefore \boxed{(p \downarrow q) \downarrow r \equiv (p \vee q) \wedge (\sim r)}$

(5) State the converse, inverse and contrapositive of the following condition

(i) If a real number x^2 is greater than zero then x is equal to zero.

$\rightarrow$ " " " " " " "

p : A real number x^2 is greater than zero

q : x is equal to zero.

$\sim$p : A real number x^2 is not greater than zero

$\sim$q : x is not equal to zero

Given statement can be written as, $p \to q$.

(i) Converse of $p \to q$

$q \to p$: If a real number x is equal to zero then x^2 is greater than zero.

(ii) Inverse of $p \to q$

$\sim p \to \sim q$: If a real number x^2 is not greater than zero then x is not equal to zero.

(iii) Contrapositive of $p \to q$

$\sim q \to \sim p$: If a real number x is not equal to zero then x^2 is not greater than zero.

(ii) If x is a natural number, then x is positive

→ p: x is a natural number

q: x is positive

$\sim p$: x is not a natural number

$\sim q$: x is not positive

(i) Converse

$q \to p$: If x is positive then x is a natural number

(ii) Inverse

$\sim p \to \sim q$: If x is not a natural number then x is not positive.

(iii) Contrapositive

$\sim q \to \sim p$: If x is not positive then x is not a natural number.

(iii) If I study Mathematics, then I will not fail in examination

p : I study Mathematics
q : I will not fail in examination
$\sim p$: I will not study Mathematics
$\sim q$: I will fail in examination

(i) converse
$q \to p$: If I will not fail in examination then I will study mathematics

(ii) Inverse
$\sim p \to \sim q$: If I will not study Mathematics then I will fail in examination

(iii) Contrapositive
$\sim q \to \sim p$: If I will fail in examination then I will not study Mathematics

(6) Prove the following logical implications.

[i] $p \Rightarrow (p \vee q)$
→ Truth Table

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

By truth table (col : 1) if p is true then $(p \vee q)$ (col : 4) is also true.
Therefore,
$p \Rightarrow (p \vee q)$

(ii) $p \wedge q \Rightarrow p \vee q$

→

p	q	$p \wedge q$	$p \vee q$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

By the truth table if $p \wedge q$ (col : 3) is true, then $p \vee q$ (col : 4) is also true

$\therefore$ $p \wedge q \Rightarrow p \vee q$.

(iii) $q \Rightarrow (p \rightarrow q)$

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

By the truth table if q (col : 2) is true, then $p \rightarrow q$ (col : 3) is also true

$\therefore$ $q \Rightarrow (p \rightarrow q)$

(iv) $[p \vee (q \vee r)] \wedge (\sim q) \Rightarrow (p \vee r)$

p	q	r	$\sim q$	$q \vee r$	$p \vee (q \vee r)$	$p \vee r$	$p \vee (q \vee r) \wedge \sim q$
1	1	1	0	1	1	1	0
1	1	0	0	1	1	1	0
1	0	1	1	1	1	1	1
1	0	0	1	0	1	1	1
0	1	1	0	1	1	1	0
0	1	0	0	1	1	0	0
0	0	1	1	1	1	1	1
0	0	0	1	0	0	0	0

By truth table if $[p \vee (q \vee r)] \wedge (\sim q)]$ (col : 8) is true then $p \vee r$ (col ; 7) is true.

$\therefore [p \vee (q \vee r)] \wedge (\sim q) \Longrightarrow (p \vee r)$

(v) $(p \to q) \wedge (q \to r) \Longrightarrow (p \wedge q) \to r$

p	q	r	$p \to q$	$q \to r$	$p \wedge q$	$p \wedge q \to r$	$(p \to q) \wedge (q \to r)$
1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	0
1	0	1	0	1	0	1	0
1	0	0	0	1	0	1	0
0	1	1	1	1	0	1	1
0	1	0	1	0	0	1	0
0	0	1	1	1	0	1	1
0	0	0	1	1	0	1	1

By truth table if $(p \to q) \wedge (q \to r)$ (col : 8) is true then $[(p \wedge q) \to r]$ (col : 7) is true.

$\therefore (p \to q) \wedge (q \to r) \Longrightarrow (p \wedge q) \to r$

ARGUMENTS

let us consider a set of prepositions $\{p_1, p_2, p_3, p_4 \dots p_n\}$ and another preposition q. Then, a compound preposition of the form

$$p_1 \wedge p_2 \wedge p_3 \wedge \dots \wedge p_n) \to q$$

is called an argument.

Here $p_1, p_2 ; p_3 \dots p_n$ are called premises of an argument and q is called conclusion of an argument.

In practise we write above argument in the following form.

$$\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \underline{p_n} \\ \therefore q \end{array}$$

The argument is said to be valid if each premises $p_1, p_2, p_3 \ldots p_n$ are true, then conclusion q is also true.
In other words the argument $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \to q$ is valid, when

$$(p_1 \wedge p_2 \wedge p_3 \wedge \ldots \wedge p_n) \Rightarrow q$$

RULE OF INFERANCES

Their are 7 rule of inferences :

i Rule of conjuctive simplification

For any two preposition p and q if $p \wedge q$ is true, then p is true.

i.e $p \wedge q \Longrightarrow p$

OR

$$\begin{array}{l} p \\ \underline{q} \\ \therefore p \end{array}$$

Prof : Truth Table

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

By the truth table if $p \wedge q$ is true (col:3) then p is also true (col:1) therefore, $p \wedge q \Rightarrow p$.

2 Rule of disjuctive Amplification

For any two preposition p and q. if p is true, then $p \vee q$ is true. ie,

$$p \Rightarrow p \vee q$$

OR

$$\frac{p}{\therefore p \vee q}$$

3 Rule of ~~sillo~~ syllogism

For any 3 preposition p ~~and~~ ; q and r if $p \rightarrow q$ is true and $q \rightarrow r$ is true, then $p \rightarrow r$ is true

$$(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$$

OR

$$\frac{\begin{array}{c} p \rightarrow q \\ q \rightarrow r \end{array}}{\therefore p \rightarrow r}$$

Truth table

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$
1	1	1	1	1	1	1
1	1	0	1	0	0	0
1	0	1	0	1	0	1
1	0	0	0	1	0	0
0	1	1	1	1	1	1
0	1	0	1	0	0	1
0	0	1	1	1	1	1
0	0	0	1	1	1	1

$\therefore$ By truth table $(p \rightarrow q) \wedge (q \rightarrow r)$ is true (col: 6) then $p \rightarrow r$ is true (col: 7) i.e $(p \rightarrow q) \wedge (q \rightarrow r) \Longrightarrow (p \rightarrow r)$

4 MODUS PONES [RULE OF DETACHMENT]

For any 2 preposition, p and q if p and $p \rightarrow q$ is true then

i.e $p \wedge (p \rightarrow q) \Rightarrow q$

OR

$$\frac{\begin{array}{c} p \\ (p \rightarrow q) \end{array}}{\therefore q}$$

prof : Truth Table

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$
1	1	1	1
1	0	0	0
0	1	1	0
0	0	1	0

∴ By truth Table, If $p \wedge (p \rightarrow q)$ is true (col: 4) then q is also true (col: 2).

$\therefore \quad p \wedge (p \rightarrow q) \Longrightarrow q$

5 Modus Tollens

For any two proposition p and q, if $p \rightarrow q$ is true then q is false then p is false

$\therefore \quad (p \rightarrow q) \wedge (\sim q) \Rightarrow (\sim p)$

OR

$$\frac{\begin{array}{c} p \rightarrow q \\ \sim q \end{array}}{\therefore \sim p}$$

6 Rule of disjunctive syllogism

For any 2 proposition p and q if $p \vee q$ is true and p is false then q is true

$$(p \vee q) \wedge (\sim p) \Rightarrow q$$

7 Rule of contradiction

For any proposition p and contradiction F_0, if $\sim p \rightarrow F_0$ is true then p is ~~false~~ true

$$(\sim p) \rightarrow F_0 \Rightarrow p.$$

OR

$$\frac{(\sim p) \rightarrow F_0}{\therefore p.}$$

Examples:

(1) Test the validity of the following arguments.

(i) If Sachin hits a century, then he gets a free car

Sachin hits a century

Sachin gets a free car

→ p: Sachin hits a century

q: (He) (sachin) gets a free car.

Above argument in mathematical form

$$\begin{array}{c} p \rightarrow q \\ \underline{\quad p} \\ \therefore q \end{array} \Rightarrow \begin{array}{c} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

∴ By the rule Modus Pones we said that above argument is valid.

ii] If I study, I will not fail in examination.
If I do not watch TV in the evening then I will study.

If failed examination

I must have watched TV in the evening

→ p : I study

q : I will not fail in examination

$\sim q$: I failed examination

r : I do not ~~a~~ watch TV in the evening

$\sim r$: I watch TV in the evening.

Given argument in Mathematical form,

$$\begin{array}{l} p \to q \\ r \to p \\ \underline{\sim q} \\ \sim r \end{array} \implies \begin{array}{l} \left.\begin{array}{l} r \to p \\ p \to q \end{array}\right\} \\ \underline{\sim q} \\ \sim r \end{array}$$

$$\implies \begin{array}{l} r \to q \\ \underline{\sim q} \\ \sim r \end{array} \quad (\because \text{By Rule of syllogism})$$

∴ By the rule Modus Tollens we conclude that above argument is valid.

iii] • I will get grade A in this course or I will not graduate.

• If I do not graduate then I will join army.

• I got grade A

∴ I will not join army.

→ p : I will get grade A

q : I will ~~a~~ graduate $\quad \sim q$: I will not graduate

r : I will join army

$\sim r$: I will not join army

Given argument in Mathematical form

$$\begin{array}{l} p \vee (\sim q) \\ \sim q \rightarrow r \\ \underline{p} \\ \sim r \end{array} \Longrightarrow \begin{array}{l} (\sim q) \vee p \\ (\sim q) \rightarrow r \\ \underline{p} \\ \sim r \end{array}$$ (By Commutative law)

$$\Longrightarrow \begin{array}{l} q \rightarrow p \\ (\sim q) \rightarrow r \\ \underline{p} \\ \therefore \sim r \end{array}$$ ($\because$ By $p \rightarrow q = \sim p \vee q$)

$$\Longrightarrow \begin{array}{l} (\sim p) \rightarrow (\sim q) \\ (\sim q) \rightarrow r \\ \underline{p} \\ \therefore \sim r \end{array}$$ [$\because p \rightarrow q \equiv (\sim q) \rightarrow (\sim p)$ contrapositive]

$$\Longrightarrow \begin{array}{l} (\sim p) \rightarrow r \\ \underline{p} \\ \therefore \sim r \end{array}$$ [$\because$ Rule of syllogism]

$$\Longrightarrow \begin{array}{l} (\sim p) \rightarrow r \\ \underline{\sim (\sim p)} \\ \therefore \sim r \end{array}$$ [By ~~Rule of~~ double negation]

~~$\therefore$ By the rule of Modus Tollen we conclude that above argument is valid~~

Truth table

p	r	~p	~p → r	(~p → r) ∧ p	~r
1	1	0	1	1	0
1	0	0	1	1	1
0	1	1	1	0	0
0	0	1	0	0	1

$\therefore$ By above truth table $(\sim r)$ is not true when $(\sim p \to r) \wedge p$ is true.

$\therefore \quad (\sim p \to r) \wedge p \not\Longrightarrow \sim r$

$\therefore$ Argument is invalid.

(2) Test the validity of the following arguments

(i) $p \to r$

$\underline{q \to r}$

$\therefore (p \vee q) \to r$

$\rightarrow$ Given that

$p \to r \quad \Longrightarrow \quad \sim p \vee r \qquad [\because p \to q \equiv \sim p \vee r]$

$\underline{q \to r} \qquad \underline{\sim q \vee r}$

$\therefore (p \vee q) \to r \qquad \therefore (p \vee q) \to r$

$(p \vee q) \wedge (p \vee r)$
$= p \wedge (q \vee r)$

$\Longrightarrow \dfrac{[(\sim p) \wedge (\sim q)] \vee r}{(p \vee q) \to r} \qquad [\because$ Distributive law$]$

$\Longrightarrow \dfrac{\sim (p \vee q) \vee r}{(p \vee q) \to r}$

$\therefore$ Given argument is valid

(ii) $p \wedge q$

$\underline{p \to (q \to r)}$

$\therefore r$

$\rightarrow$ Given that

$p \wedge q$

$\underline{p \to (q \to r)}$

$\therefore r$

$\Rightarrow$ p

q

$\underline{p \rightarrow (q \rightarrow r)}$

$\therefore r$

$\Rightarrow$ p $\quad\quad$ $\therefore$ By Commutative law

$p \rightarrow (q \rightarrow r)$

$\underline{q}$

r

$\Rightarrow$ $q \rightarrow r$ $\quad\quad$ $\therefore$ By Modus Pones

$\underline{q}$

r

By Modus Pones the above argument is valid.

(iii) $p \rightarrow q$

$r \rightarrow s$

$\underline{p \vee r}$

$q \vee s$

$\rightarrow$ Given that

$$\begin{array}{lcl} p \rightarrow q & \Rightarrow & p \rightarrow q \\ r \rightarrow s & & p \vee r \\ \underline{p \vee r} & & \underline{r \rightarrow s} \\ q \vee s & & q \vee s \end{array}$$

[$\because$ By Commutative law]

$\Rightarrow$ $p \rightarrow q$ $\quad\quad$ [$\because$ By ~~to~~ double negation]

$\sim(\sim p) \vee r$

$\underline{r \rightarrow s}$

$q \vee s$

$$\Rightarrow \begin{array}{l} p \to q \\ \sim p \to r \\ \underline{r \to s} \\ q \vee s \end{array}$$

$$\Rightarrow \begin{array}{l} \sim q \to \sim p \\ \sim p \to r \\ \underline{r \to s} \\ q \vee s \end{array}$$ $[\because p \to q \equiv \sim q \to \sim p]$

$$\Rightarrow \begin{array}{l} \sim q \to r \\ \underline{r \to s} \\ q \vee s \end{array}$$ $[\because$ By Rule of syllogism$]$

$$\Rightarrow \begin{array}{l} \underline{\sim q \to s} \\ q \vee s \end{array}$$ $[\because$ By Rule of syllogism$]$

$\therefore$ Above argument is valid because $\sim q \to s = \sim(\sim q) \vee s$
$= q \vee s$
$[\because p \to q \equiv \sim p \vee q]$

QUANTIFIERS

The words or phrases like "for all", "for each", "for every", "for any", "their exist", "for some" in the propositions are associated with the idea of the quantity. Such words are called quantifiers.

For example:- • For every x in integer set, x^2 is a +ve number
• Their exists an integer x, such that x is equal to x^2

Note:

1) The symbol $\forall$ is used to denote the quantifier "for all".

"for each", "for any", "for every".

2) The symbol $\exists$ is used to denote the quantifier "there exists", "for some".

Types of Quantifiers

(i) Universal Quantifiers
The words "for all", "for every", "for any", "for each" are called universal quantifiers
$\therefore$ $\forall$ symbol is used for universal quantifier
Ex: For every integer x, $(x+1)$ is also an integer
Symbolically it can be written as
$\forall x \in Z, (x+1) \in Z$

(ii) Existential Quantifier
The word "there exist", "for some" are called existential Quantifiers.
$\therefore$ $\exists$ symbol is used for existential quantifiers.
Ex: There exist an even prime number.
Symbollically,
$\exists x \in N, \exists x$ is even prime

QUANTIFIED STATEMENT
A proposition involving universal or existential quantifiers are called quantified statement.

Truth value of a Quantified statement.
The following rules are employed for determining truth value of quantified statement
Rule - 1:

The statement "$\forall x \in S, p(x)$", is true only when $p(x)$ is true for each value of x chosen from universal set S

Rule 2 :
The statement "$\exists x \in S, q(x)$". is false only when $q(x)$ is false for every value of x chosen from universal set S

Rule 3 : Rule of Universal Specification
If the statement $p(x)$ is known to be true for all value of x choosen from universal set S and if a belongs to S then $p(a)$ is true

Rule 4: Rule of Universal Generalisation
If the statement $p(x)$ is proved to be true for any arbitary x chosen from universal set S, then quantified statement "$\forall x \in S, p(x)$" is true

- Logical Equivalence of Quantified statement
For any two proposition $p(x)$ and $q(x)$ and universal set S, we have following results.

1) $\forall x \in S\ [p(x) \wedge q(x)] = [\forall x \in S, p(x)] \wedge [\forall x \in S, q(x)]$

2) $\exists x \in S\ [p(x) \vee q(x)] = [\exists x \in S, p(x)] \vee [\exists x \in S, q(x)]$

3) $\exists x \in S, [p(x) \rightarrow q(x)] = \exists x \in S, \sim p(x) \vee q(x)$

4) $\forall x \in S . \sim p(x) =$ For no value of x, $p(x)$ is true.

Rule for a negation of a Quantified statement
To form a negation of a quantified statement, change

universal quantifer to existential quantifier and vise-versa, and replace the proposition by its negation.
i.e,

1) $\sim[\forall x \in S, p(x)] = \exists x \in S, \sim p(x)$

2) $\sim[\exists x \in S, q(x)] = \forall x \in S, \sim q(x)$

Examples

(1) Consider the following open statements defined on all set of real ~~nats~~ numbers.
i.e $p(x): x \geq 0$ and $q(x): x^2 \geq 0$
Determine the truth value of the following statement
[i] $\exists x \in R, p(x) \wedge q(x)$
[ii] $\forall x \in R, p(x) \rightarrow q(x)$
$\rightarrow$
Give that
$p(x): x \geq 0$
$q(x): x^2 \geq 0$

[i] Given statement is,
$\exists x \in R, p(x) \wedge q(x)$
OR
$\exists x \in R \ (x \geq 0) \wedge (x^2 \geq 0)$

Consider. $x = 1$
$\therefore p(1): 1 \geq 0$ is true
$q(1): 1^2 \geq 0$ is true.

$\Rightarrow p(1) \wedge q(1)$ is true
$\therefore$ Given ~~the~~ statement is true

(ii) Given statement is,

$\forall x \in R,\ p(x) \rightarrow q(x)$

OR

$\forall x \in R,\ (x \geq 0) \rightarrow (x^2 \geq 0)$

We know that,
$p \rightarrow q$ is false only when p is true and q is false.
Let us consider $x = 1$ [$\because p(x)$ is true for $\forall x \in [0, \infty)$]
$\Rightarrow p(1) : 1 \geq 0$ is true.
& $q(1) : 1^2 \geq 0$ is true.

$\therefore p(x) \rightarrow q(x)$ is true for all $x \in R$.
$\therefore$ Given statement is true.

(2) Write down the following prepositions in symbolic for

[i] All real numbers are complex number
→ Given statement,
All real numbers are complex number
Symbollically
$\forall x \in R,\ x \in \mathbb{C}$

(ii) There exists a matrix whose transpose is itself.
→ $P(A) : A = A'$
Symbolically,
$\exists A \in M \ni A = A'$ where, M is set of all Matrices

OR

$\exists A \in M \ni P(A)$

(iii) For all real number x, y, $x^2 > y^2$ then $x > y$
$p : x^2 > y^2$ $q : x > y$

Symbolically

$\forall x, y \in \mathbb{R}, \; p \to q.$

OR

$\forall x, y \in \mathbb{R}, \; (x^2 > y^2) \to (x > y)$

(3) Write the converse, inverse and contrapositive for the following statements, for which set of real number in universe. Also indicate their truth value

i] $\forall x \in \mathbb{R}, \; [(x > 3) \to (x^2 > 9)]$

→ Given that,

$\forall x \in \mathbb{R}, \; [(x > 3) \to (x^2 > 9)]$

where,

$p(x) : x > 3$, $\sim p(x) : x \leq 3$

$q(x) : x^2 > 9$, $\sim q(x) : x^2 \leq 9$

$\forall x \in \mathbb{R}, \; [p(x) \to q(x)]$

Consider, $x = 4$ ($\because p(x) : x > 3$ is true in $(3, \infty)$)

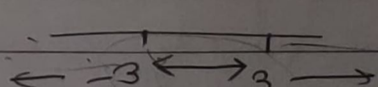
$\Rightarrow p(4) : 4 > 3$ is true

$q(4) : 4^2 > 9$ is true

$\therefore p(x) \to q(x)$ is true for all $x \in \mathbb{R}$.

$\therefore$ Give condition is true.

- Converse of $p(x) \to q(x)$



$\forall x \in \mathbb{R}, \; q(x) \to p(x)$

OR

$\forall x \in \mathbb{R}, \; (x^2 > 9) \to (x > 3)$

$x^2 > 9$

$\pm x > 3$

Let us consider, $x = -4$ ($\because q(x)$ is true in $(-\infty, 3)$)

$\Rightarrow q(-4) : (-4)^2 > 9$ is true

& $p(-4) : -4 > 3$ is false

$\therefore q(x) \rightarrow p(x)$ is false

$\therefore$ Given converse of a conditional is false

- Inverse of $p(x) \rightarrow q(x)$

$\forall x \in R, \sim(p(x)) \rightarrow \sim(q(x))$

OR

$\forall x \in R \quad (x \leq 3) \rightarrow (x^2 \leq 9)$

Since, converse and inverse of a conditional are logically equivalent.

$\therefore$ Inverse is also false.

- Contrapositive of $p(x) \rightarrow q(x)$

$\forall x \in R, \quad \sim q(x) \rightarrow \sim p(x)$

OR

$\forall x \in R \quad (x^2 \leq 9) \rightarrow (x \leq 3)$

Since, conditional and its contrapositive are logically equivalent

$\therefore$ Contrapositive is also true.

(ii) $\forall x \in R, \{[x^2 + 4x - 21 > 0] \rightarrow [(x > 3) \vee (x < -7)]\}$

$x^2 + 4x + 2^2 - 25$

$(x+2)^2 - 5^2$

$(x+2-5)(x+2+5)$

$(x-3)(x+7) > 0$

$x - 3 = 0 \quad x = -7$

$x > 3 \quad x \not< -7$

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(4) Write the negation of the following.

(i) $\{\forall x \in S, p(x)\} \vee \{\forall x \in S, \sim q(x)\}$

$\longrightarrow$ $\sim [\{\forall x \in S, p(x)\} \vee \{\forall x \in S : \sim q(x)\}$

$\sim \{\forall x \in S, p(x)\} \wedge \sim \{\forall x \in S, \sim q(x)\}$ by De Morgan

$\{\exists x \in S, \sim p(x)\} \wedge \{\exists x \in S, \sim(\sim q(x))\}$ $\because$ by negation of QS

$\{\exists x \in S, \sim p(x)\} \wedge \{\exists x \in S\ q(x)\}$ $\because$ by double negat

(ii) $\forall x \in S, p(x) \rightarrow q(x)$

$\longrightarrow$ $\sim [\forall x \in S, p(x) \rightarrow q(x)]$

$\sim [\forall x \in S, \sim p(x) \vee q(x)]$

$\exists x \in S, \sim [\sim p(x) \vee q(x)]$ $\because$ By negation of Q

$\exists x \in S\ \sim(\sim p(x) \wedge \sim q(x))$ $\because$ By De-Morgan law

$\exists x \in S\ p(x) \wedge \sim q(x)$ $\because$ By double negati

(iii) $\{\exists x \in S \sim p(x)\} \longrightarrow q(x)\}$

$\longrightarrow$ $\sim \{\exists x \in S \sim p(x) \rightarrow q(x)\}$

$\forall x \in S\ \sim [p(x) \vee q(x)]$

$\forall x \in S\ \sim p(x) \wedge \sim q(x)$

(iv) $\{\exists x \in S, p(x)\} \rightarrow \{\exists x \in S, q(x)\}$

$\longrightarrow$

UNIT-II

RELATIONS AND FUNCTIONS

RELATIONS

~~Relation~~ Consider a set $A = \{a_1, a_2 \dots a_n\}$ [or $A = \{a_i\} \forall i = 1, 2, 3 \dots n$] and $B = \{b_1, b_2, \dots b_m\}$ [$B = \{b_i\} \forall i = 1, 2 \dots m$] of order n and m respectively. Then a relation R defines on set A to B denoted by ~~A related to B~~ or [$(a_i, b_j) \in R$] is the subset of cartesian product $A \times B$ i.e, $R \subseteq A \times B$.

ZERO-ONE MATRIX

A zero-one matrix of a relation R defined on set A to set B is denoted by M_R or $M(R)$ and defined by,

$$M_R = M(R) = [m_{ij}]_{n \times m}$$

where m_{ij} is a function defined as

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

In particular of $A = B$ then zero-one matrix M_R is $n \times m$ matrix and whose elements are,

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in R \\ 0 & \text{if } (a_i, a_j) \notin R \end{cases}$$

DIA GRAPH / ~~DIRECT~~

A relation R defined on set $A = \{a, b, c\}$ to set $B = \{1, 2\}$ as,

$$R = \{(a,1), (b,1), (b,2), (c,1), (c,2)\}$$

$\therefore$ Zero-one matrix of this relation R is,

$$M_R = M(R) = \begin{array}{c|c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

DIAGRAPH / DIRECTED GRAPH

let R be the relation on a finite set A. Then R can be pictorially represented as small circles for each element of set A called vertices or nodes and draw an arrow from one vertex a to another vertex b, if $(a, b) \in R$ or a R b is called an edge.

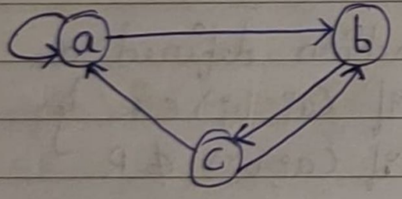
This resulting pictorial representation of relation R is called diagraph of R

For Example:

Consider a set $A = \{a, b, c\}$ and a relation R defined on set A,

$R = \{(a,a), (a,b), (b,c), (c,a), (c,b)\}$

Therefore diagraph of R is



Origin / Source and Terminus of an edge

In a diagraph a vertex from which an edge leaves is called origin of an edge and a vertex where an edge ends is called terminus of an edge.

loop

An edge for which the source and terminus is same vertex called loop.

Isolated vertex

A vertex in a diagraph which is neither a source nor a terminus for any edge is called isolated vertex

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Indegree and Outdegree

- The number of edges terminating at the vertex is called the indegree of that vertex.
- The number of edges leaving a vertex is called outdegree of that vertex

Examples

① let $A = \{a, b, c\}$ and $B = \{0, 1\}$ and $R = \{(a,0), (b,0), (c,1)\}$ be a relation on set A to B. Write down the matrix of the relation

→ let ; $A = \{a, b, c\}$ & $B = \{0, 1\}$

$R = \{(a,0), (b,0), (c,1)\}$

Zero-one matrix of R is.

$M_R = M(R) = [m_{ij}]_{3\times 2}$

$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix}$$

where, $m_{11} = 1$ $[\because (a_1, b_1) = (a, 0) \in R]$

$m_{12} = 0$ $(\because (a_1, b_2) = (a, 1) \notin R]$

$m_{21} = 1$ $[\because (a_2, b_1) = (b, 0) \in R]$

$m_{22} = 0$ $[\because (a_2, b_2) = (b, 1) \notin R]$

$m_{31} = 0$ $[\because (a_3, b_1) = (c, 0) \notin R]$

$m_{32} = 1$ $[\because (a_3, b_2) = (c, 1) \in R]$

$$M_R = M(R) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

OR

$$M_R = M(R) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2 Determine the relation R on a set A to set B as represented by the following matrix.

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

→ Let us take elements of set A and set B as follows,

$A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 0 & 0 \\ c & 1 & 1 & 1 & 0 \\ d & 0 & 1 & 0 & 1 \end{array}$$

∴ Relation, R defined by above matrix is,

$R = \{(a,1), (b,2), (c,1), (c,2), (c,3), (d,2), (d,4)\}$

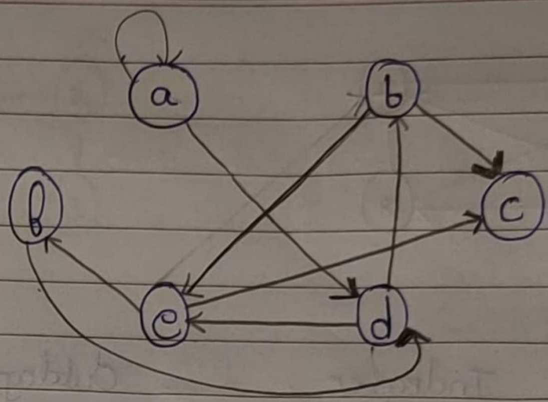
3 Let $A = \{a, b, c, d, e, f\}$ and R be the relation on set A defined by $R = \{(a,a), (a,d), (b,c), (b,e), (d,b), (d,e), (e,c), (e,f), (f,d)\}$.

Draw the diagraph of R. Also determine indegree and outdegree of each vertices of diagraph.

→ Let $A = \{a, b, c, d, e, f\}$

$R = \{(a,a), (a,d), (b,c), (b,e), (d,b), (d,e), (e,c), (e,f), (f,d)\}$

Diagraph of this relation R is



Vertex	Indegree	Outdegree
a	1	2
b	1	2
c	2	0
d	2	2
e	2	2
f	1	1

4 If $R = \{(x, y) : x > y\}$ is relation defined on set $A = \{1, 2, 3, 4\}$ write down the matrix and diagraph of R. Also find indegree and outdegree of each vertices of diagraph.

→ let $A = \{1, 2, 3, 4\}$

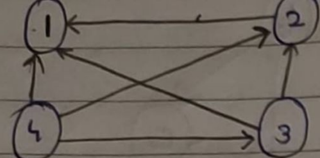
$R = \{(x, y) : x > y\}$

$R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

Zero-one matrix of relation R is

$$M_R = M(R) = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

Diagraph of this Relation is



Vertex	Indegree	Outdegree
1	3	0
2	2	1
3	1	2
4	0	3

(5) Find the relation R on the set A and write down its diagraph given that $A = \{a, b, c, d, e\}$ and the matrix of R is,

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

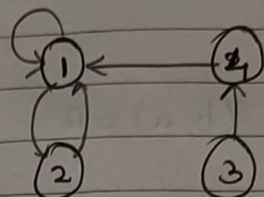
→ let $A = \{a, b, c, d, e\}$

$$\text{let } M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Relation R defined by the matrix is

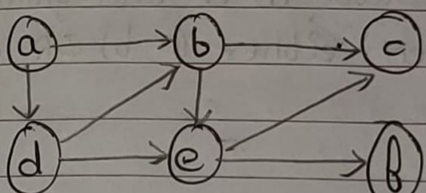
$R = \{(a,a), (a,b), (b,c), (b,d), (c,d), (d,b), (d,c), (e,a)\}$

(6) Find the relation R and matrix of R determined by the following diagraph

(i) 

→ $R = \{$

$M_R =$ $\left[\quad \right.$

(ii) 

→ $R = \{$

$$M_R = \begin{array}{c|c} & a \\ a & \\ b & \\ c & \\ d & \\ e & \\ f & \end{array}$$

TYPES OF RELATION [PROPERTIES OF RELATION]

If R be the relation defined on non-empty set A, then

(i) Reflexive Relation

R is reflexive if $(a, a) \in R \quad \forall a \in A$

OR $a R a, \forall a \in R$

(ii) Symmetric Relation

R is symmetric if $(a, b) \in R$ then $(b, a) \in R$
(OR if $a R b$ then $b R a$)

(iii) Transitive Relation

R is transitive if $(a,b) \in R$ and $(b, c) \in R$ then $(a,c) \in R$

OR

if aRb and bRc, then aRc

(iv) Anti-symmetric Relation

A Relation R on set A is said to be antisymmetric if whenever $a \neq b$, then either $(a, b) \notin R$ or $(b, a) \notin R$

(v) Equivalence Relation

A Relation R on a set A is said to be equivalence relation if R satisfies reflexive, symmetric and transitive relation

CLOSURE OF A RELATION

Defination

If R is a relation set A that does not posses a particular property [like, Reflexive, symmetric and Transitive], then we may wish to add related order pairs to R until we get a relation. that does have the required property is called closure of a relation.

(i) Reflexive Closure

The Reflexive closure of a relation R on set A is the smallest Reflexive relation containing R.

(ii) Symmetric Closure

The Symmetric closure of a relation R on a set A is the smallest symmetric relation containing R.

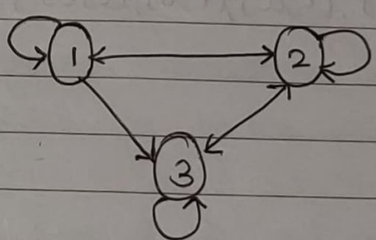
(iii) Transitive Closure

The transitive closure of a relation R on a set A is the smallest transitive relation containing R.

Example

1 The digraph of a relation are on set $A = \{1, 2, 3\}$ is given below. Determine whether R is equivalence relation or not

(i)



→ let $A = \{1, 2, 3\}$

Relation, R defined above digraph is,

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,2), (3,3)\}$

(i) Reflexive relation

since $(1,1), (2,2), (3,3) \in R$

∴ R is reflexive Relation

(ii) Symmetric Relation

since $(1,3) \in R$ but $(3,1) \notin R$

∴ R is not a symmetric Relation.

∴ R is not a equivalence Relation.

② A relation R on a set $A = \{a, b, c\}$ is represented by the following matrix. Determine whether R is equivalence relation or not

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

→ Let, $A = \{a, b, c\}$

Given that

$$M_R = \begin{array}{c|c} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

∴ Relation R defined by above matrix is,

$R = \{(a,a), (a,c), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

(i) Reflexive relation

Since $(a,a), (b,b), (c,c) \in R$

∴ R is Reflexive relation,

(ii) Symmetric relation

Since $(a,c) \in R$ and $(c,a) \in R$

$(b,c) \in R$ and $(c,b) \in R$

∴ R is symmetric relation

(iii) Transitive Relation

Since $(a,c) \in R$ and $(c,b) \in R$ but $(a,b) \notin R$ $(a,a) \rightarrow (a,c) \Rightarrow (a,c)$

∴ R is not Transitive relation

$(c,a) \Rightarrow (a,a) \in R$?

$(a,c) \rightarrow (c,b) \Rightarrow (a,b) \notin R$

$(c,c) \Rightarrow (a,c) \in R$

∴ R is not a equivalence Relation

3 Find the reflexive, symmetric and transitive closure of a relation $R = \{(a,a), (b,c), (a,c)\}$ defined on set $A = \{a, b, c\}$

→ Let $A = \{a, b, c\}$

$R = \{(a,a), (b,c), (a,c)\}$

(i) Reflexive Closure

To make R as a reflexive relation then we have to add following order pairs.

$R' = \{(b,b), (c,c)\}$

$\therefore R_1 = R \cup R' = \{(a,a) (b,c), (a,c)\} \cup \{(b,b), (c,c)\}$
$= \{(a,a), (b,c), (a,c), (b,b), (c,c)\}$

(ii) Symmetric Closure

To make R as a symmetric relation we need to add following order pairs.

$R'' = \{(b,c), (c,a)\}$

$\therefore R_2 = R \cup R'' = \{(a,a) (b,c) (a,c)\} \cup \{(b,c) (c,a)\}$
$= \{(a,a) (b,c) (a,c) (b,c) (c,a)\}$

(iii) Transitive closure.

Since R is transitive relation,

$\therefore$ R is transitive closure of itself.

EQUIVALENCE CLASS

Let R be the equivalence relation on set A. Then the set of all those elements, x of A which are related to a by R. is called an equivalence class of a (i.e. $x R a$ or $(x,a) \in R$). with respect to relation R and it is denoted by $[a]$ or $R(a)$

i.e $[a] = R(a) = \{x : x R a\}$

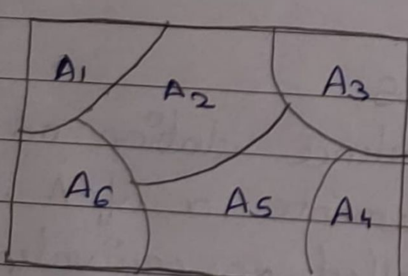
OR $[a] = R(a) = \{ x : (x, a) \in R \}$

For example:
Consider an equivalence relation R defined on set $A = \{a, b, c\}$ as follows
$R = \{ (a,a) (a,b) (b,a) (b,b) (c,c) \}$
$\therefore$ Equivalence classes of each element of set A w.r.t R
$[a] = R(a) = \{ x : x R a \}$
i.e $[a] = R(a) = \{a, b\}$
$[b] = R(b) = \{a, b\}$
$[c] = R(c) = \{c\}$

Theorem: Any two equivalence classes of R are either identical or disjoint.

PARTITION OF A SET
let A be a non-empty set. Suppose there exists a non-empty subset $A_1, A_2, A_3 \dots A_4$ of A called partition of set A if following conditions holds,
(i) $A = A_1 \cup A_2 \cup A_3 \dots \cup A_n = \bigcup_{i=1}^{n} A_i$

(ii) Any two subsets $A_1, A_2, A_3 \dots A_n$ are disjoint
i.e $A_i \cap A_j = \emptyset$ for $i \neq j$



$\therefore A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$
& $A_i \cap A_j = \emptyset \quad i \neq j$

Theorem:
let R be an equivalence relation on set A, Then,

(i) $\forall\, a \in A$, $[a] \neq \phi$
(ii) if $b \in [a]$, then $[a] = [b]$, where $a, b \in A$
(iii) $\forall\, a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \phi$
(iv) A is the union of all equivalence classes wrt R.
i.e $A = \bigcup_{a \in A} [a]$

Proof: Given that R is an equivalence relation.
$\Rightarrow$ R is reflexive, symmetric, transitive.

(i) Consider arbitary $a \in A$
$\Rightarrow aRa$ ($\because$ R is reflexive)
OR $(a, a) \in R$
$\Rightarrow a \in [a]$ ($\because$ Defn of equivalence class)
$\Rightarrow [a] \neq \phi$, $\forall\, a \in A$

(ii) Given that $b \in [a]$
$\Rightarrow bRa$ or $(b, a) \in R$
$\Rightarrow aRb$ or $(a, b) \in R$ ($\because$ R is symmetric)
$\Rightarrow bRa$ and aRb [OR $(b, a) \in R$ and $(a, b) \in R$]
Now, consider $x \in [a]$ —— ①
$\Rightarrow xRa$ or $(x, a) \in R$
$\Rightarrow xRa$ and aRb
$\Rightarrow xRb$ ($\because$ R is transitive relation)
$\Rightarrow x \in [b]$ —— ②
From ① and ② we get
$[a] \subseteq [b] \longrightarrow$ ③

Next, consider $y \in [b]$ —— ④
$\Rightarrow yRb$ OR $(y, b) \in R$
$\Rightarrow yRb$ and bRa OR $[(y, b) \in R$ and $(b, a) \in R]$

$\Rightarrow yRa$ ($\because R$ is transitive)

$\Rightarrow y \in [a]$ — 5

~~$\Rightarrow [b] \subseteq [a]$ — 5~~

From 4 and 5 we get

$[b] \subseteq [a]$ — 6

[iii] Consider $a, b \in A$

Suppose $[a] \cap [b] \neq \phi$

$\Rightarrow \exists x \in A, \ni x \in [a] \cap [b]$

$\Rightarrow x \in [a]$ and $x \in [b]$

By proof of result [ii] , we can write

$\Rightarrow [a] = [x]$ and $[b] = [x]$

$\Rightarrow [a] = [b] = [x]$

$\Rightarrow [a] = [b]$

[iv] Consider $a \in A$

$\Rightarrow aRa$

$\Rightarrow a \in [a]$

$\Rightarrow a \in \bigcup_{a \in A} [a]$

$\Rightarrow A \subseteq \bigcup_{a \in A} [a]$ — 7

Next, we know that

$[a] \subseteq A$

$\Rightarrow \bigcup_{a \in A} [a] \subseteq A$ — 8

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From (7) and (8) we get

$$A = \bigcup_{a \in A} [a]$$

FUNDAMENTAL THEOREM ON EQUIVALENCE RELATION

Statement: If A be any non-empty set, then

[i] Any equivalence relation R on A induces a partition of A

[ii] Any partition of A gives rise to equivalence relation R on A.

Proof

(1) Given that R is an equivalence relation

$\Rightarrow$ R is reflexive, symmetric and transitive.

Let us consider a set P of all distinct equivalance classes of A.

i.e $P = \{[a] : a \in A\}$

where $[a] = \{x : xRa\}$

We need to prove that P is the partition of A

It means that P has to satisfy two conditions of a partition of set.

[i] Since R is reflexive relation.

$\Rightarrow aRa$ or $(a,a) \in R, \forall a \in A$.

$\Rightarrow a \in [a], \forall a \in A$

$\Rightarrow A = \bigcup_{a \in A} [a]$

[ii] We know that any two equivalence classes of A are either identical or disjoint.

It means that intersection of two distinct equivalence cl

$\therefore$ R is transitive Relation

$\therefore$ R is an equivalence Relation.

Examples

① For the equivalence relation $R = \{(1,1)(1,2)(2,1),(2,2)(3,4)(4,3)$ $(3,3)(4,4)\}$. Defined on set $A = \{1,2,3,4\}$

$\rightarrow$ let $A = \{1,2,3,4\}$

$R = \{(1,1)(1,2)(2,1)(2,2)(3,4)(4,3)(3,3)(4,4)\}$

Equivalence class of R is,

$[a] = \{x : xRa\}$

OR $[a] = \{x : (a,x) \in R\}$

$[1] = \{1,2\}$

$[2] = \{1,2\}$

$[3] = \{3,4\}$

$[4] = \{3,4\}$

$\therefore$ Partition of set A is

$P = \{[1], [3]\}$

$= \{\{1,2\}, \{3,4\}\}$

2 For the set $A = \{1,2,3 \ldots\ldots\}$ Consider the Relation R defined on A as aRb if and only if a-b is divisible by 5 find the partition of A induced by R

$\rightarrow$ let $A = \{1,2,3 \ldots\ldots\}$

$R = \{(a,b) :$ a-b is divisible by 5$\}$

$R = \{(a,b) : 5/(a-b)\}$

is empty.
i.e $[a] \cap [b] = \phi$ if $[a] \neq [b]$

$\therefore$ P satisfies both the condition of Partition.
$\therefore$ P is the Partition of set A.

(2) Suppose $P = \{A_1, A_2, A_3, \ldots A_n\}$ be the partition of A
Define a relation R on set A as aRb if and only if a, b $\in$ belongs to the same block of P (same subset A_i)
i.e $R = \{(a,b) : a, b \in A_i, \forall i = 1, 2, 3, \ldots n\}$

We need to prove that R is an equivalence relation
(i) Reflexive relation
aRa, since a belongs to same block.
$\Rightarrow$ aRa $\forall$ a $\in$
$\therefore$ R is reflexive relation.

(ii) Symmetric relation
if aRb
$\Rightarrow$ a, b belongs to same block
$\Rightarrow$ b, a also belongs to same block
$\Rightarrow$ bRa
R is symmetric relation

(iii) Transitive relation
if aRb and bRc
$\Rightarrow$ a, b belongs to same block and b, c belongs to same block
$\Rightarrow$ a, b, c belongs to the same block.
$\Rightarrow$ a, c belongs to same block
$\Rightarrow$ aRc.

(i) Reflexive relation

aRa as $5 / (a-a)$ $\forall a \in A$

$\therefore$ R is reflexive

(ii) Symmetric relation

If aRb $\Rightarrow 5/(a-b)$

$\Rightarrow 5 / -(b-a)$

$\Rightarrow 5/(b-a)$

$\Rightarrow$ bRa $\forall a,b \in A$

$\therefore$ R is symmetric relation

(iii) Transitive relation

If aRb and bRc

$\Rightarrow 5/(a-b)$ and $5/(b-c)$

$\Rightarrow 5 / \{(a-b)+(b-c)\}$

$\Rightarrow 5/(a-b+b-c)$

$\Rightarrow 5/(a-c)$

$\Rightarrow$ aRc

$\therefore$ R is transitive relation

$\therefore$ R is equivalence relation.

Equivalence classes of A are

$[a] = \{x : xRa\}$

OR

$[a] = \{x : (x,a) \in R$

$[a] = \{x : 5/(x-a)\}$

$[a] = \{x : x-a = 5n\}$

$[a] = \{x : x = 5n+a\}$

$[a] = \{5n+a\}$

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$[1] = \{5n+1\} = \{1, 6, 11, 16, 21, \ldots\}$
$[2] = \{5n+2\} = \{2, 7, 12, 17, 22, \ldots\}$
$[3] = \{5n+3\} = \{3, 8, 13, 18, 23, \ldots\}$
$[4] = \{5n+4\} = \{4, 9, 14, 19, 24, \ldots\}$
$[5] = \{5n+5\} = \{5, 10, 15, 20, 25, \ldots\}$
$[6] = \{5n+6\}$
$= \{5n+5+1\}$
$= \{5(n+1)+1\}$
$= \{5N+1\}$ $\quad N = n+1$
$[6] = \{5N+1\} = \{$

Partition of set A is,
$P = \{[1], [2], [3], [4,] [5]\}$
OR
$P = \{[5n+1], [5n+2], [5n+3], [5n+4], [5n+5]\}$ $n = 0, 1, 2, 3 \ldots$

(3) If $A = \{1, 2, 3, 4, 5\}$ and R be the relation on A that induces the partition $A = \{1,2\} \cup \{3,4\} \cup \{5\}$, then find R

→ let $A = \{1, 2, 3, 4, 5\}$
Partition of A is,
$P = \{\{1,2\}, \{3,4\}, \{5\}\}$

Relation R on set A is defined as,
aRb if and only if a, b belongs to same block,
i.e $R = \{(a,b) : a, b \in A_i$ for some $i\}$
$R = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,3)\ (3,4)\ (4,3)\ (4,4)\ (5,5)\}$

(4) On the set of all integers $\mathbb{Z}$, the relation R is defined by aRb if and only if $a^2 - b^2$ is an even integer show that R is an equivalence relation. Find the partition of $\mathbb{Z}$.

induced by R

→ let $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\ldots\}$

OR $\mathbb{Z} = \{\ldots\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\ldots\}$

Relation R on Z is,

$R = \{(a, b) : a^2 - b^2 \text{ is an even integer}\}$

OR $R = \{(a, b) : a^2 - b^2 \text{ divisible by } 2\}$

OR $R = \{(a, b) : 2 \mid (a^2 - b^2)\}$

(i) Reflexive Relation

aRa as $2 \mid (a^2 - a^2)$ $\forall a \in \mathbb{Z}$

∴ R is a reflexive relation

(ii) Symmetric Relation.

if $aRb \Rightarrow 2 \mid (a^2 - b^2)$

$\Rightarrow 2 \mid -(b^2 - a^2)$

$\Rightarrow 2 \mid (b^2 - a^2)$

$\Rightarrow bRa$

∴ R is symmetric relation

(iii) Transitive Relation

If aRb and bRc

$\Rightarrow 2 \mid (a^2 - b^2)$ and $2 \mid (b^2 - c^2)$

$\Rightarrow 2 \mid \{(a^2 - b^2) + (b^2 - c^2)\}$

$\Rightarrow 2 \mid [a^2 - b^2 + b^2 - c^2]$

$\Rightarrow 2 \mid (a^2 - c^2)$

$\Rightarrow aRc$

∴ R is transitive relation

∴ R is equivalence Relation

Equivalence classes of $\mathbb{Z}$ are

$[a] = \{x : xRa\} = \{x : (x,a) \in R\}$

$[a] = \{x : 2 / (x^2 - a^2)\}$ $\quad$ a/b

$[a] = \{x : x^2 - a^2 = 2n\}$ $\quad \forall n \in \mathbb{Z}$ $\quad$ $b = 2n$

$[a] = \{x : x^2 = 2n + a^2\}$ $\quad \forall n \in \mathbb{Z}$

$[a] = \{x : x = \pm\sqrt{2n+a^2}\}$ $\quad \forall n \in \mathbb{Z}$

$[a] = \{-\sqrt{2n+a^2}, \sqrt{2n+a^2}\}$ $\quad \forall n \in \mathbb{Z}$

provided $\sqrt{2n+a^2} \in \mathbb{Z}$

$[1] = \{-\sqrt{2n+1}, \sqrt{2n+1}\}$; $\forall n \in \mathbb{Z}$, provided $\sqrt{2n+1} \in \mathbb{Z}$

$[1] = \{-1, 1, -3, 3, -5, 5, -7, 7 \ldots\ldots\}$

$[1] = \{\ldots\ldots -7, -5, -3, -1, 1, 3, 5, 7 \ldots\}$

$[2] = \{-\sqrt{2n+4} ; \sqrt{2n+4}\}$ $\quad \forall n \in \mathbb{Z}$ provided $\sqrt{2n+1} \in \mathbb{Z}$

$[2] = \{0, -2, 2, -4, 4, -6, 6 \ldots\ldots\}$

$[2] = \{\ldots\ldots -6, -4, -2, 0, 2, 4, 6 \ldots\}$

$\therefore$ [1] and [2] are only two distinct equivalence classes of Z

$\therefore$ Partition of $\mathbb{Z}$ is,

$P = \{[1], [2]\}$

OR

$P = \{\{\ldots -7, -5, -3, -1, 1, 3, 5, 7 \ldots\}, \{\ldots -6, -4, -2, 0, 2, 4, 6 \ldots\}\}$

PARTIAL ORDERING RELATION

Defination

A Relation R on a set A is said to partial ordering relation (partial order) on A if R is reflexive, antisymmetric and transitive.

Partial ordered set (Poset)

A set A with partial ordering relation R on it is called Partial order set of A (poset of A). And it is denoted by, (A, R).

Examples

(1) A relation "less than or equals" ($\leq$) defined on a set $\mathbb{Z}$ of all integer is a poset of $\mathbb{Z}$ $(\mathbb{Z}, \leq)$

(2) A relation "Greater than or equals" ($\geq$) defined on a set $\mathbb{Z}$ of all integer is a poset of $\mathbb{Z}$ $(\mathbb{Z}, \geq)$

HASSE DIAGRAM

The Hasse diagram of a finite partial order set A [poset A] is the diagram whose vertices represent the elements of set A. If $a, b \in A$ and b is immediate successor of a [$a < b$], then an edge is drawn directed from a to b by placing element b at the higher level than a.

To draw Hasse diagram of a poset A a diagraph of the given partial ordering relation R is drawn and remove the following.

[i] All the loops.

[ii] All the edges implied by transitive relation.

Examples

~~(1) The diagraph of relation R on set A = {1, 2, 3} is given below~~

(1) Let A = {1, 2, 3, 5, 6 10, 15, 30} and R is the relation on A defined by 'x Ry if and only if x divides y. Verify R is partial ordering relation and Draw its

Hasse diagram.

→ let $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$R = \{(x, y) : x \text{ divides } y\}$

OR

$R = \{(x, y) : x/y\}$

(i) Reflexive Relation

xRx as x/x $\forall x \in A$

$\therefore R$ is reflexive Relation

(ii) Anti Symmetric Relation.

whenever $x \neq y$ and if xRy, then $y \not R x$ and:

whenever $x \neq y$ and $x \nmid y$ and $y \nmid x$; the $x \not R y$ and $y \not R x$.

$\therefore R$ is antisymmetric relation.

(iii) Transitive Relation.

If xRy and yRz

$\Rightarrow x/y$ and y/z

$\Rightarrow x/z$

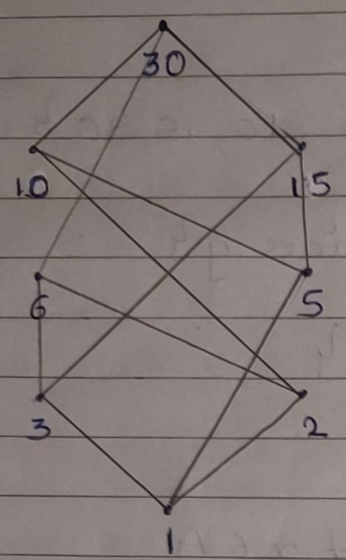
$\Rightarrow xRz$

$\therefore R$ is transitive relation.

$\therefore R$ is partial ordering relation. on A

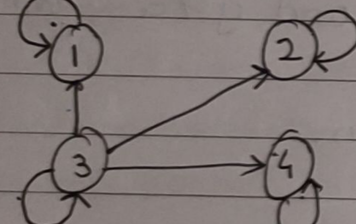
$\therefore (A, R)$ is a poset.

Hasse diagram of (A, R) is



② The diagraph of a relation R defined on set A = {1, 2, 3, 4} is as shown below. Verify (A, R) is a poset and draw the corresponding Hasse diagram.

(i)



→ (i) Relation R on set A = {1, 2, 3, 4} defined by the given diagraph is

R = {(1, 1) (2, 2) (3, 3) (3, 1) (3, 4) (4, 4) (3, 2)}

(i) Reflexive Relation

Since (1, 1), (2, 2), (3, 3), (4, 4) ∈ R.

∴ R is reflexive relation

(ii) Antisymmetric Relation

Since (3, 1) ∈ R but (1, 3) ∉ R

(3, 4) ∈ R but (4, 3) ∉ R

(3, 2) ∈ R but (2, 3) ∉ R

∴ R is Antisymmetric Relation.

(iii) Transitive Relation

since $(3,1) \in R$ & $(1,1) \in R \Rightarrow (3,1) \in R$

$(3,3) \in R$ & $(3,2) \in R \Rightarrow (3,2) \in R$

$(3,3) \in R$ & $(3,4) \in R \Rightarrow (3,4) \in R$

$(3,3) \in R$ & $(3,1) \in R \Rightarrow (3,1) \in R$.

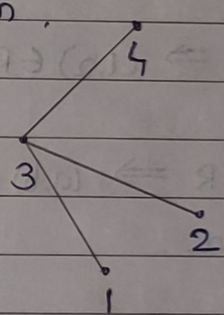
$(3,4) \in R$ & $(4,4) \in R \Rightarrow (4,4) \in R$.

$\therefore$ R is Transitive Relation.

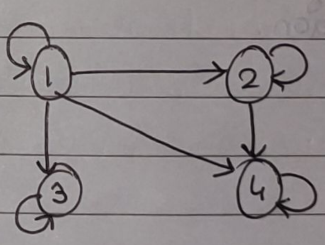
$\therefore$ R is partial ordering relation.

$\therefore (A, R)$ is poset

Hasse diagram.



(ii)



(3) Draw the hasse diagram of the relation on set $A = \{1,2,3,4,5\}$, whose matrix is given below and,

$$M(R) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

→ The relation R on the set $A = \{1,2,3,4,5\}$ defined by

$R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (2,2)\ (3,3)\ (4,4)\ (5,5)\}$

(i) Reflexive relation

$(1,1), (2,2), (3,3), (4,4)\ (5,5) \in R$

$\therefore$ R is reflexive relation

(ii) Anti symmetric relation.

since $(1,2) \in R$ but $(2,1) \notin R$

$(1,3) \in R$ but $(3,1) \notin R$

$(1,4) \in R$ but. $(4,1) \notin R$

$(1,5) \in R$ but $(5,1) \notin R$

$\therefore$ R is anti-symmetric relation

[iii] Transitive Relation.

Since $(1,1) \in R$ $(1,2) \in R \Rightarrow (1,2) \in R$

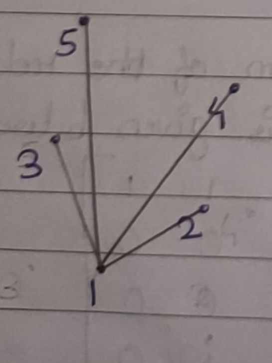
Since $(a,a) \in R$ and $(a,b) \in R \Longrightarrow (a,b) \in R$.

$\therefore$ R is transitive relation.

$\therefore$ R is ~~be~~ partial ordering relation.

$\therefore (A,R)$ is a poset

$\therefore$ Hasse diagram of (A,R) is



Lower Bound

Def^n Let A be a poset and B is a subset of A. Then an element $a \in A$ is called lower bound of B. if $a \leq b$, $\forall\, b \in B$.

Upper Bound.

Let A be a poset and B is a subset of A. Then an element $a \in A$ is called upper bound of B, if $a \geq b$ $\forall\, b \in B$

Example

Consider a set $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{3, 4\}$ be the subset of A

Lower Bounds of B are : 3, 2, 1

Greatest lower Bound of B : 3

Upper Bounds of B are : 4, 5, 6

Least Upper Bound of B are : 4

Greatest lower Bound [GLB]

An element a is called Greatest lower bound or GLB of subset B of poset A, if a is ~~an~~ lower bound that is greater than anyother lowers bounds of B.

Least Upper Bound [LUB]

An element a is called least Upper Bound or LUB of subset B of poset A, if a is upper bound that is lesser than any other upper bounds of B.

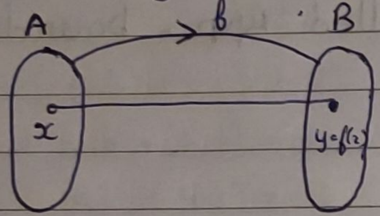
LATTICE

A partial order set (A, R) in which every pair of eleme-nts have both least upper bound [LUB] and greatest lower bound [GLB] is called Lattice

FUNCTIONS

A relation defined from non-empty set A to non-empty set B is said to be function (mapping) from A to B, if each element of A has unique image in B.

It is denoted by $f : A \longrightarrow B$



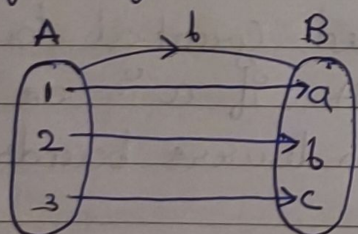
TYPES OF FUNCTIONS

(i) One-One function (One-to-One function / Injective

A function, $f : A \rightarrow B$ is said to be one-one if distinct elements of A should have distinct images in B

i.e if $a, b \in A$ and $a \neq b$, then $f(a) \neq f(b)$

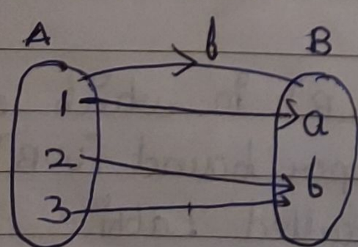
OR if $f(a) = f(b)$, then $a = b$.



(ii) Onto Function / Surjective

A function, $f : A \rightarrow B$ is said to be onto if every element of B is the image of atleast one element of A.

i.e $y \in B$, then $\exists\, x \in A \ni f(x) = y$.



PROPERTIES OF FUNCTIONS

Theorem :- 1

Let $f : X \longrightarrow Y$ be a function and A and B are subset of X. Then prove that

(i) if $A \subseteq B$, then $f(A) \subseteq f(B)$

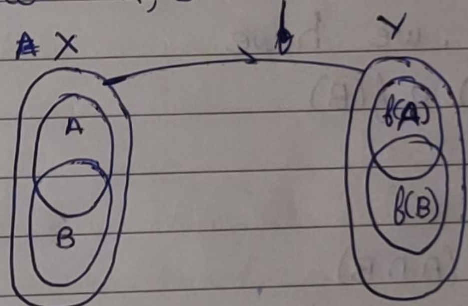
(ii) $f(A \cup B) = f(A) \cup f(B)$

(iii) $f(A \cap B) = f(A) \cap f(B)$.

equality holds if f is one-one

Let $f : X \longrightarrow Y$

Given that $A, B \subseteq X$



(i) Given that ; suppose $A \subseteq B$

let us consider an arbitrary element $y \in Y$

if $y \in f(A) \Rightarrow y = f(x)$, for some $x \in A$

$\Rightarrow y = f(x)$, for some $x \in B$ ($\because A \subseteq B$)

$\Rightarrow y \in f(B)$

$\Rightarrow$ Any arbitrary element $y \in f(A)$. is element of $f(B)$.

$f(A) \subseteq f(B)$

(ii) Suppose $y \in f(A \cup B)$

$\Rightarrow y = f(x)$, for some $x \in A \cup B$.

$\Rightarrow y = f(x)$ for some $x \in A$ or $x \in B$

$\Longrightarrow (y=f(x),$ for some $x\in A)$ OR $(y=f(x)$ for some $x\in B)$
$\Rightarrow \quad y\in f(A)$ OR $y\in f(B)$
$\Rightarrow \quad y\in f(A)\cup f(B)$

$\therefore \; f(A\cup B)\subseteq f(A)\cup f(B)$ —— (1)

$\therefore$ We know that,
$A\subseteq A\cup B$ and $B\subseteq A\cup B$
By the result (i) we have,
$f(A)\subseteq f(A\cup B)$ and $f(B)\subseteq f(A\cup B)$
$\Rightarrow f(A)\cup f(B)\subseteq f(A\cup B)\cup f(A\cup B)$
$\Rightarrow f(A)\cup f(B)\subseteq f(A\cup B)$ —— (2)

From (1) and (2), we have
$f(A\cup B) = f(A)\cup f(B)$

(iii) Suppose; $y\in f(A\cap B)$
$\Longrightarrow \quad y=f(x)$, for some $x\in A\cap B$
$\Longrightarrow \quad y=f(x)$ for some $x\subseteq A$ and $x\subseteq B$
$\Rightarrow (y=f(x)$, for some $x\in A)$ ~~OR~~ AND
$(y=f(x)$, for some $x\in B)$
$\Rightarrow \quad y\in f(A)$ and $y\in f(B)$
$\Rightarrow \quad y\in f(A)\cap f(B)$

$\therefore \; f(A\cap B)\subseteq f(A)\cap f(B) \longrightarrow$ (3)

Next, given ~~we know~~ that f is one-one function
~~$A\subseteq A$.~~

Suppose, $y\in f(A)\cap f(B)$
$\Rightarrow \quad y\in f(A)$ and $y\in f(B)$

$\Rightarrow [y = f(x_1)$ for some $x_1 \in A]$
AND $[y = f(x_2)$, for some $x_2 \in B]$

$\Rightarrow y = f(x_1) = f(x_2)$
$\Rightarrow x_1 = x_2$ $\quad [\because f$ is one-one$]$
$\Rightarrow y = f(x_1)$; for some $x_1 \in A$ and $x_2 \in B$
$\Rightarrow y = f(x_1)$ for some $x_1 \in A \cap B$.
$\Rightarrow y \in f(A \cap B)$

$f(A) \cap f(B) \subseteq f(A \cap B)$ —— (4)

$\therefore$ From (3) and (4), we get
$f(A \cap B) = f(A) \cap f(B)$

Theorem - 2
let A and B be finite sets and $f: A \to B$ be a function. Then prove the following.
(i) If f is one-one ; then $n(A) \leq n(B)$
(ii) If f is onto, then $n(B) \leq n(A)$
(iii) If f is bijective, then $n(A) = n(B)$
(iv) If $n(A) > n(B)$, then at least two different elements of A have the same image under f.

Proof : let $f: A \longrightarrow B$.
let $A = \{a_1, a_2, a_3 \ldots a_n\}$ and $B = \{b_1, b_2, b_3, \ldots b_m$
are the two sets.
$\therefore n(A) = n$; $n(B) = m$

(i) Suppose f is one-one
$\Rightarrow$ Each element of f has distinct images in set B

$\Rightarrow a_1, a_2, a_3 \cdots\cdots a_n \in A$; then images

$f(a_1), f(a_2), f(a_3), \cdots\cdots f(a_n) \in B$

$\Rightarrow$ B must have atleast n elements.

$\Rightarrow n(B) \geq n$

$\Rightarrow n(B) \geq n(A) \quad (\because n(A) = n)$

$\Rightarrow n(A) \leq n(B)$ —— (1)

(ii) Suppose f is onto

$\Rightarrow$ Each element of B is the image of atleast one element of

$\Rightarrow$ A must have atleast m elements

$\Rightarrow n(A) \geq m$

$\Rightarrow n(A) \geq n(B) \quad (n(B) = m)$

$\Rightarrow n(B) \leq n(A)$ —— (2)

(iii) Suppose f is bijective.

$\Rightarrow$ f is one-one and onto

$\Rightarrow$ From result (1) and (2) we get

$n(A) = n(B)$

[iv] Suppose $n(A) > n(B)$

By the contrapositive of result [i] we get

if $n(A) > n(B)$, then f is not one-one

$\Rightarrow$ Atleast two elements $x_1, x_2 \in A$ has same image in $y \in B$

$\Rightarrow y = f(x_1)$ and $y = f(x_2)$ for $x_1 \neq x_2 \in A$

$\therefore$ It means that two distinct elements $x_1, x_2 \in A$ have the same image y in B.

THEOREM-3

Suppose A and B are finite set having the same number of elements and $f: A \longrightarrow B$ is a function.

Then f is ...

Proof

let $f: A \rightarrow B$.

Given that number of elements in A and B are same

then, $n(A) = n(B) = n$

let $A = \{a_1, a_2, a_3 \dots a_n\}$ and $n(A) = n$.

Suppose f is one-one

$\Rightarrow$ Images of each element of A are distinct and they ar $f(a_1), f(a_2), f(a_3) \dots f(a_n)$

$\Rightarrow$ set of distinct images of A constitute the range of function f and it is denoted by $f(A)$

$\Rightarrow n(f(A)) = n$

$\Rightarrow n(f(A)) = n(B) \quad [\because n = n(B)]$

$\therefore$ It means that f is onto.

Conversly, suppose f is onto, then

$\Rightarrow B = f(A) = \{f(a_1), f(a_2), \dots f(a_n)\}$

$\Rightarrow f(a_1), f(a_2) \dots f(a_n)$ are distinct images of set A

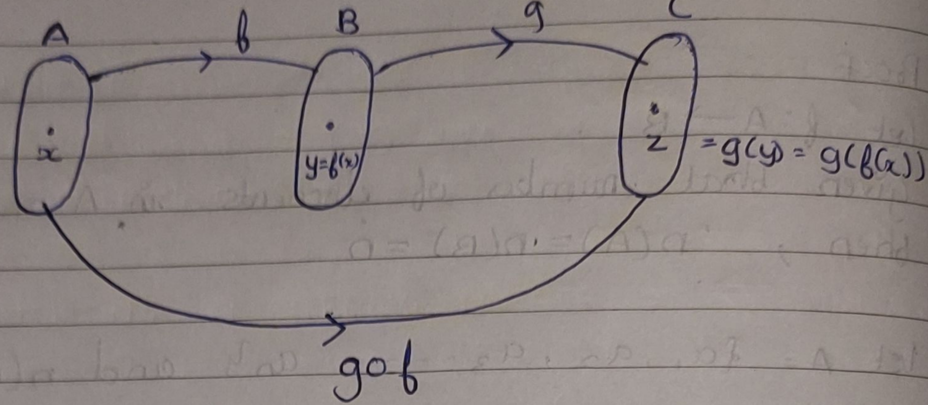
$\Rightarrow f$ is one-one.

COMPOSITION OF FUNCTIONS

let $f: A \rightarrow B$ and $g: B \rightarrow C$ be two functions then the compositions of two functions f and g is denoted by $(g \circ f)$ is defined by the function:

$(g \circ f): A \longrightarrow C$ such that

$$(g \circ f)(x) = g(f(x)), \forall x \in A$$



INVERTIBLE FUNCTIONS

A function $f : A \longrightarrow B$ is said to be invertible, if there exist a function $g : B \longrightarrow A$ such that $(g o f)(x) = x$ and $(f o g)(y) = y$. Then the function g is called inverse of f and it is denoted by $g = f^{-1}$

Note

A function f a invertible if and only if f is bijective