UNIT 1

Numerical Analysis

In this chapter, we study

i) **Numerical Methods to solve Algebraic and Transcendental equations**

ii) **Numerical Methods to solve Ordinary Differential Equations**

Why Numerical Methods:

Most of the problems in Engg , Physical and Economical Sciences can be formulated in terms of systems of Linear or Non-Linear equations, ordinary or Partial Differential Equations.

In majority of cases, the solutions to these problems in analytic form are non-existent or difficult or not amenable for direct interpretation

Why Numerical Methods:

Numerical Analysis provides approximate solutions, practical and amenable for analysis.

Algebraic and Transcedental Equations:

An algebraic equation of degree n is

$$p(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$$

$$a_0 \neq 0, \text{ here } \quad n \geq 1$$

Transcendental equation are non algebraic equations involving transcendental functions such as exponential, logarthmic, trigonometric or hyperbolic functions.

A general form of an algebraic or transcendental equation is $f(x)=0$

where $f(x)$

is defined and continuous on an interval

$$a \,\square\, x \,\square\, b$$

a

Root:

Any value ‘a’ for which f(a)=0 is known as root or solution of the equation f(x)=0 or and ‘a’ is called the zero of the function f(x)

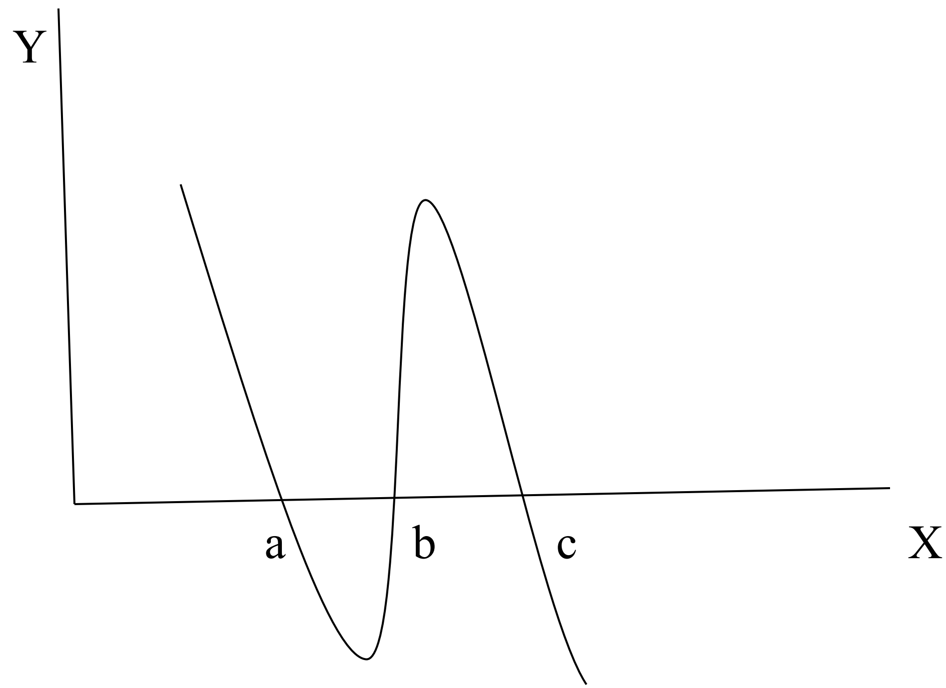
Note:

The approximate solutions of an algebraic or transcendental equation are to be found by numerical methods consisting of

i) Isolating the roots and

ii) Then improving the value of the approximate roots.

Geometrically, the root of the equation f(x)=0 is a point where the graph (or curve) of y=f(x) crosses the x- axis.



Graphical Method

Disadvantage:

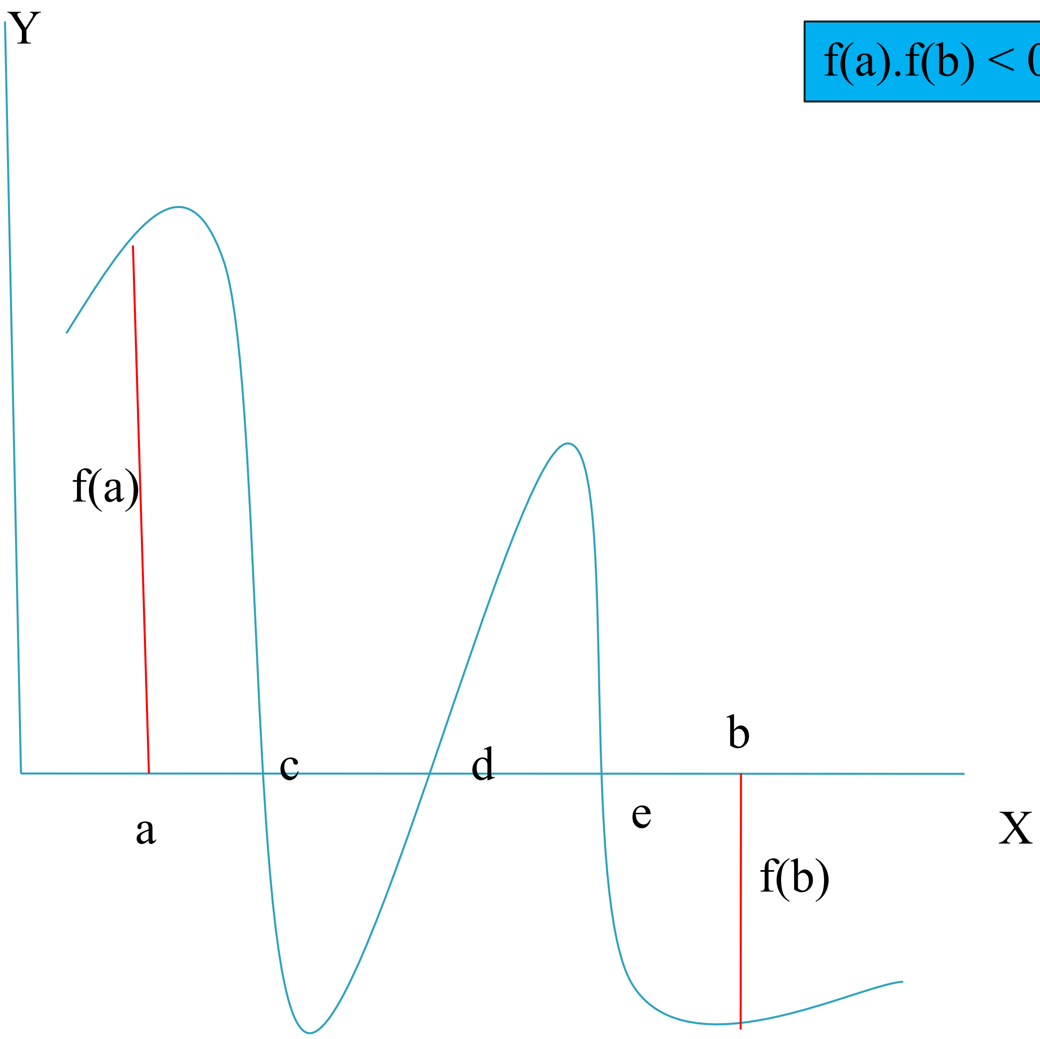
Though the roots can be isolated by drawing the graphs of the curve, the graphical methods are cumbersome

Hence the next Theorem…….

Theorem:

If a continuous function f(x) assumes values of opposite signs at the end points of an interval [a,b] that is., f(a).f(b)<0, then the interval will contain atleast one root of the equation f(x)=0

That is
There exists c belongs to (a,b) such that f(c)=0



f(a).f(b) < 0

Descarte’s rule of sign:

The number of positive roots of f(x)=0 cannot exceed the number of changes of sign in f(x).

Also, the number of negative roots of f(x) cannot exceed the number of changes of sign in f(-x).

Example:

$$f(x) = x^5 - 6x^2 - 4x + 5 = 0$$

f(x): + - - + : 2 changes of sign; no more than 2 positive roots

f(-x): - - + + : 1 change of sign; no more than one negative root

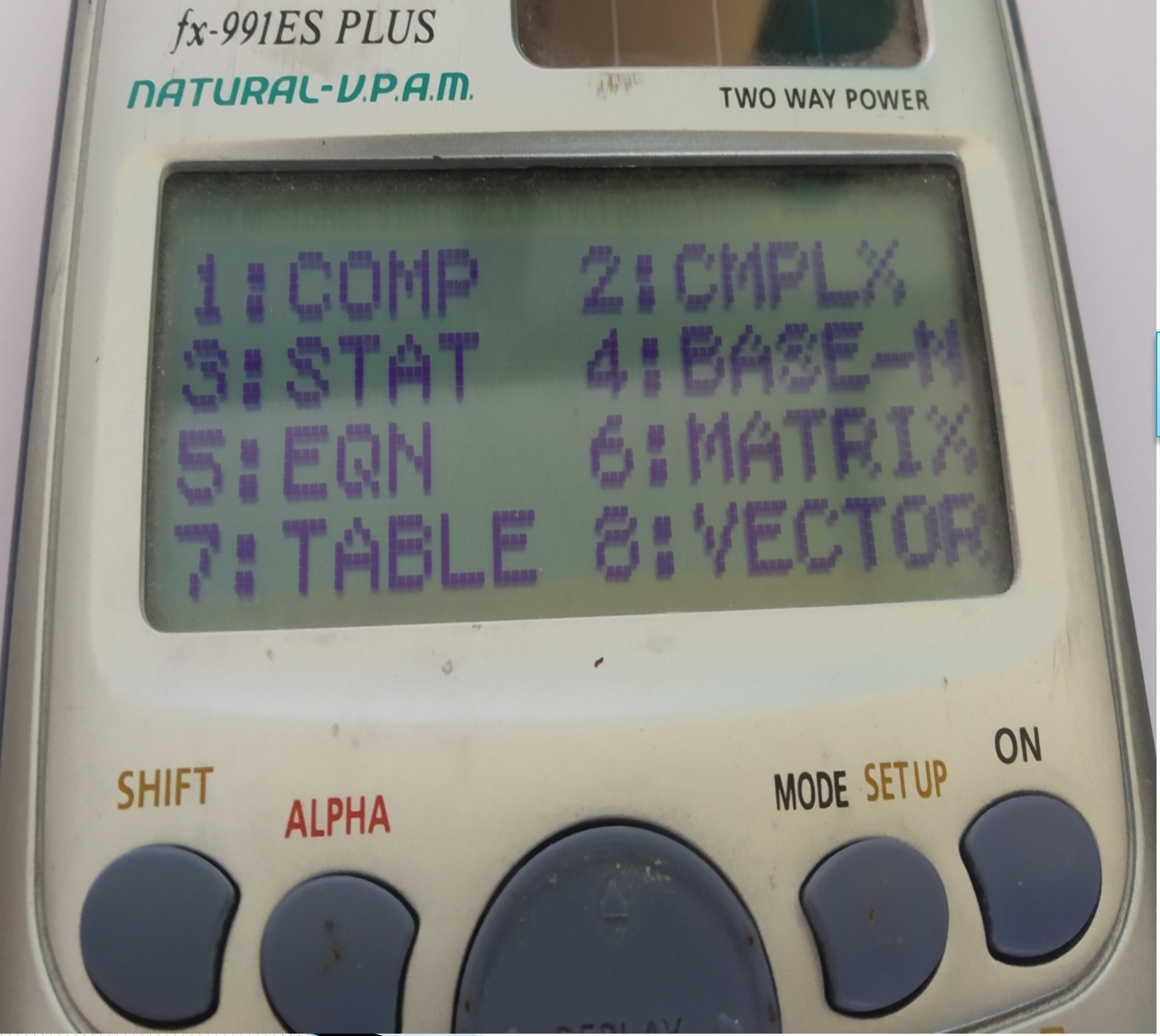
Regula - Falsi Method or
Method Of False Position or
Method of Chords or
Method of Propotional Part

This is a geometrical method to find an approximate root of an equation.

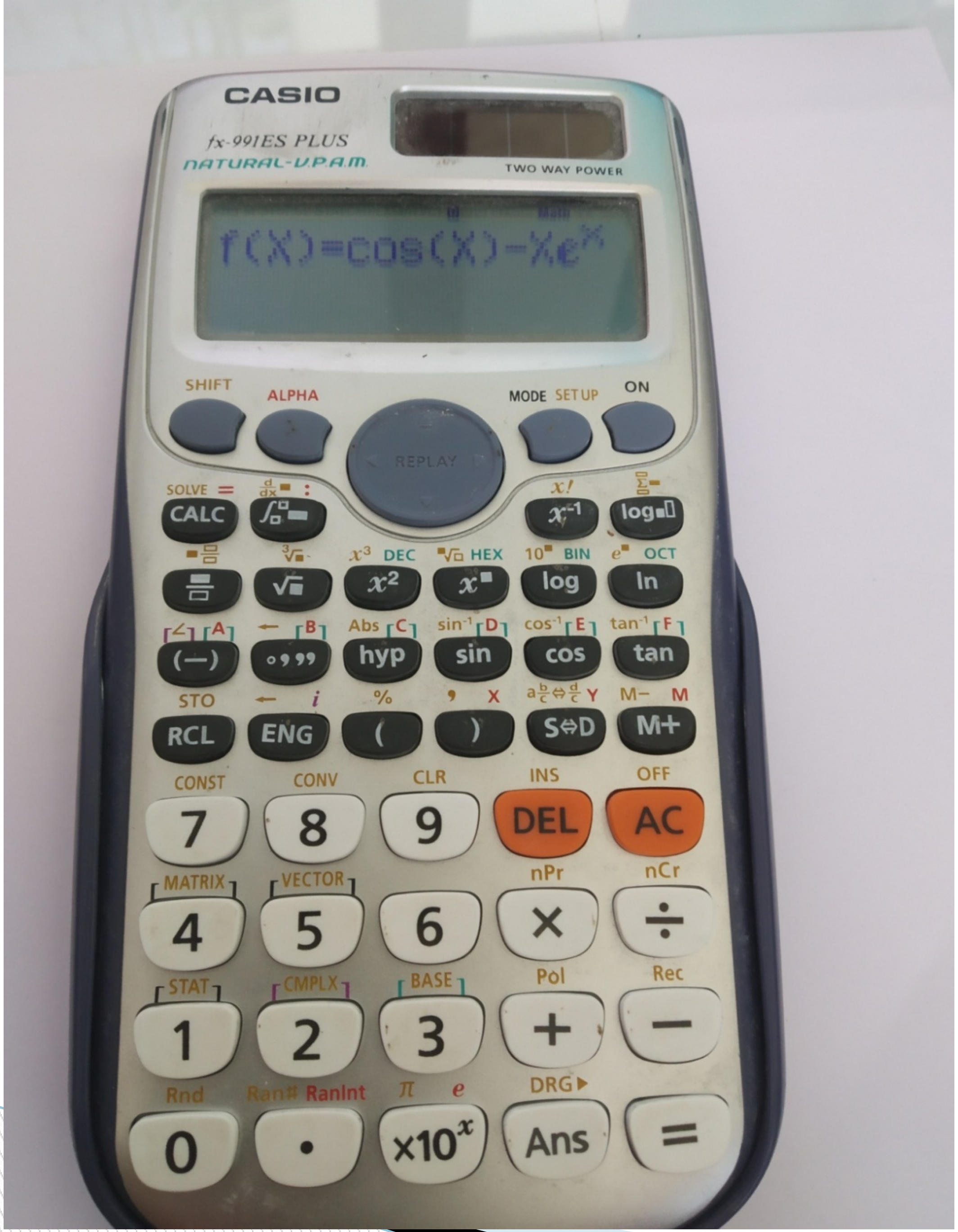
This method is equivalent to replacing the curve y=f(x) by a chord that passes through the points A(a, f(a)) and B(b, f(b))

Formula:

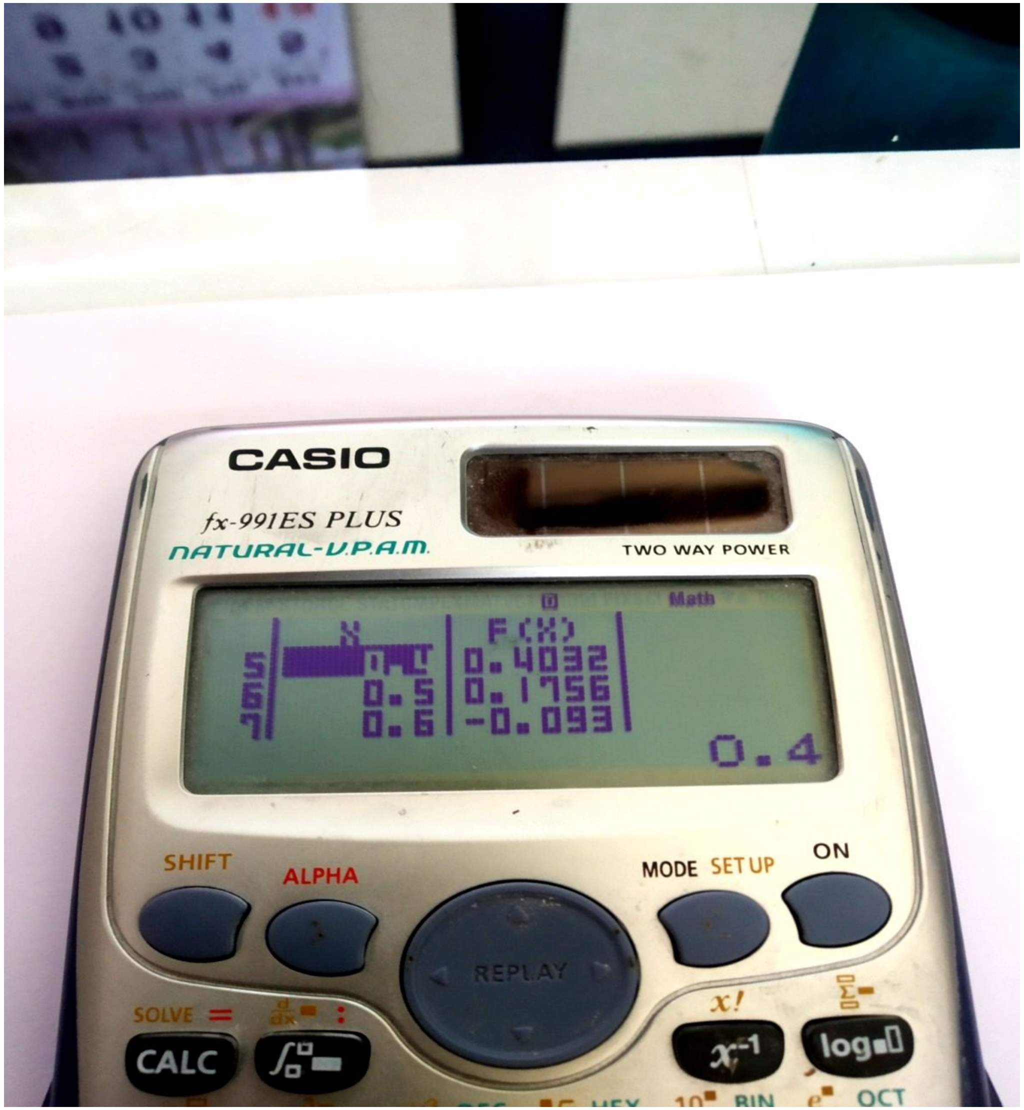
$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$



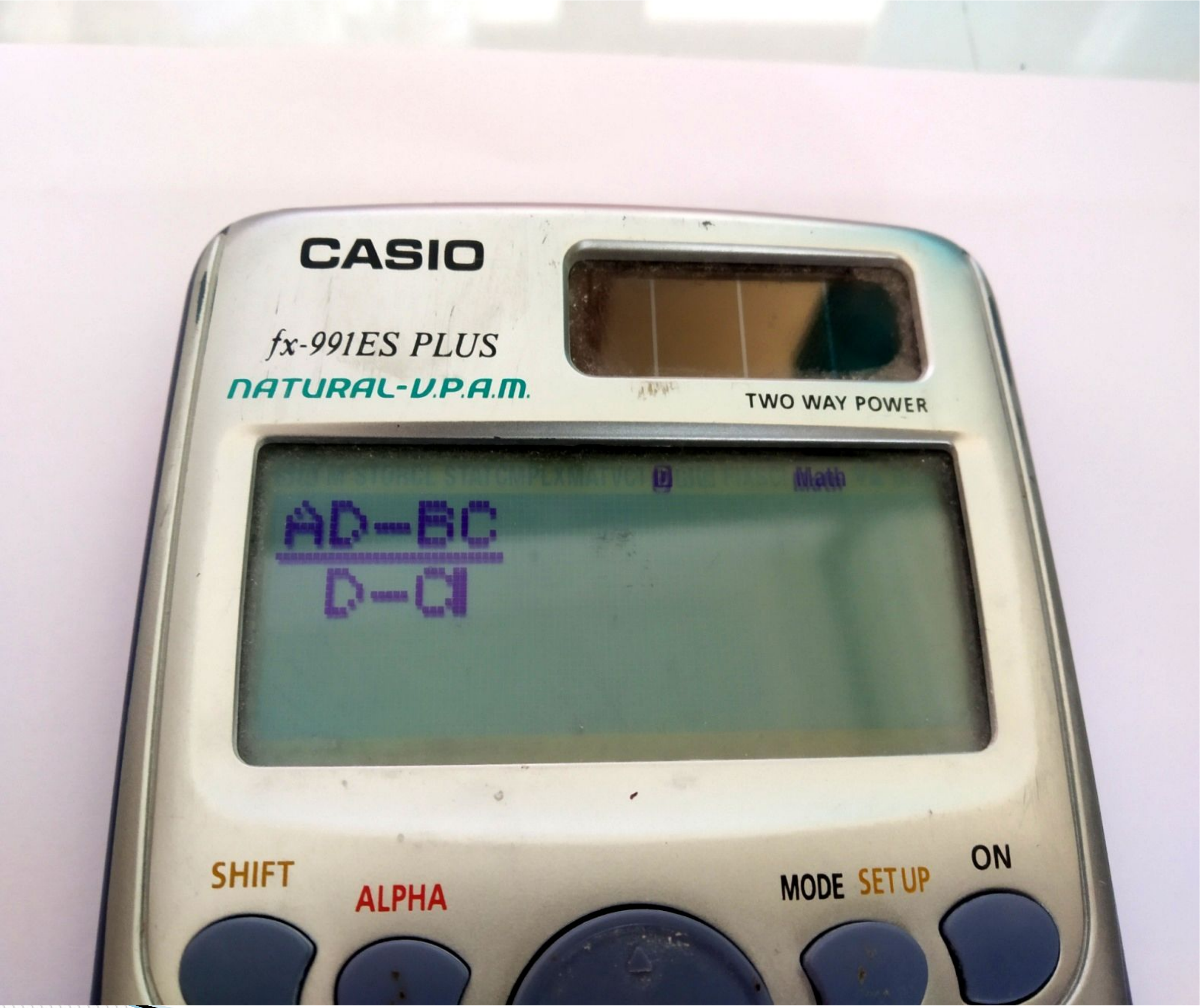
Go to: MODE
Press 7: table



Enter the function f(x)



Enter :Start
End
Step



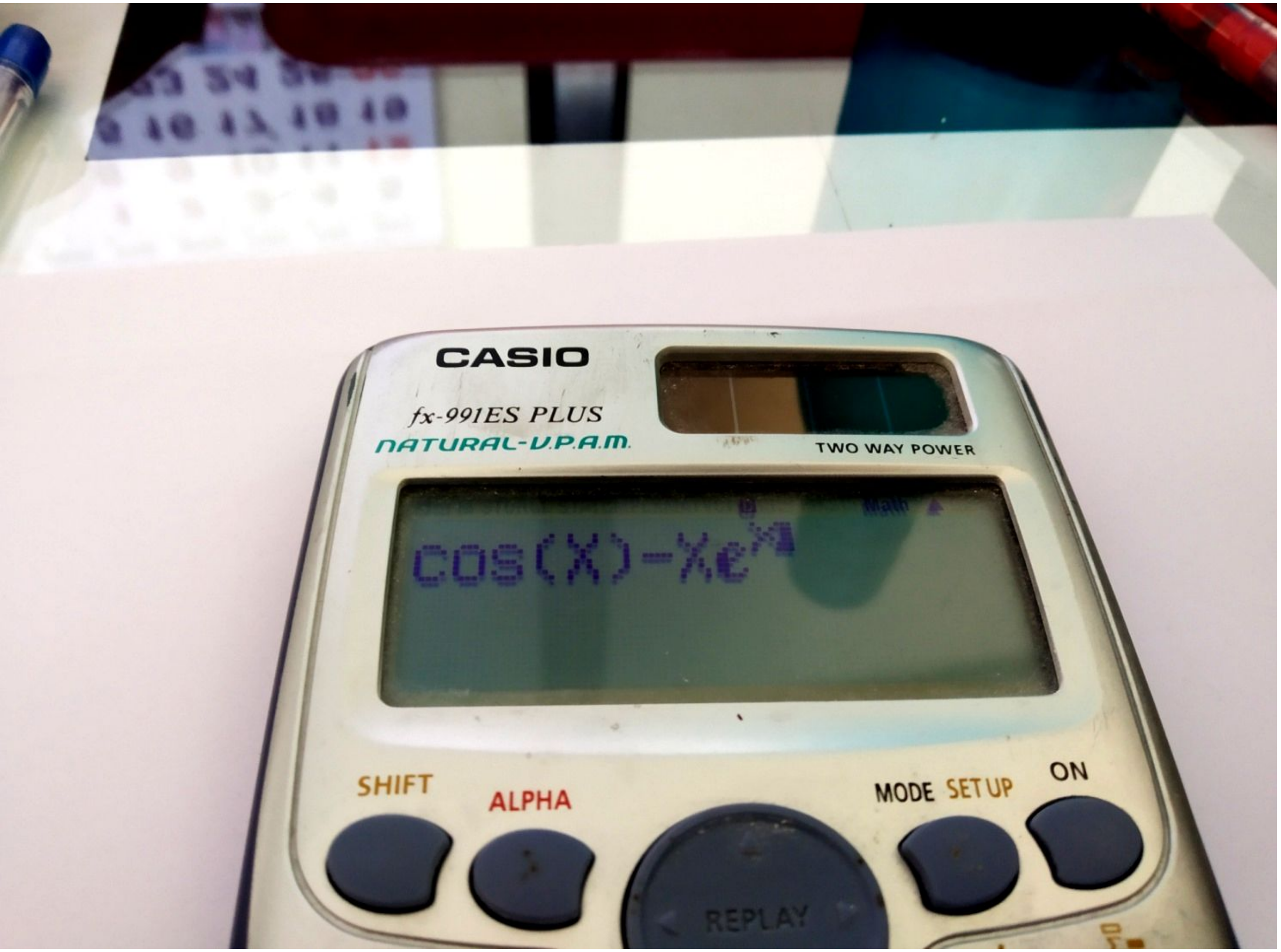
Write Regula Falsi Formula as shown:

a : A

f(b) : D

b: B

f(a) :C



Enter given f(x)

1) Find the root of the eq^n $\cos x = xe^x$ using the regula-falsi method correct to four decimal places.

Solution:

let $f(x) = \cos x - xe^x = 0$

$f(0) = \cos 0 - 0 \cdot e^0 = 1$

$f(1) = \cos 1 - 1e^1 = -2.17798$

the root lies betn 0 & 1 ,

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$a = 0 \quad b = 1, \; f(a) = 1 \; , \; f(b) = -2.17798$

$$\boxed{x_1 = \frac{0f(1) - 1f(0)}{f(1) - f(0)} = 0.31467}$$

$f(0.31467) = 0.51987$

$\therefore$ the root lies betn (0.31467) & 1

Since $f(0.31467) > 0$ & $f(1) < 0$

$a = 0.31467 \quad b = 1$

$$x_2 = \frac{0.31467\, f(1) - 1\, f(0.31467)}{f(1) - f(0.31467)}$$

$$\boxed{x_2 = 0.44673}$$

$f(0.44673) = 0.20356$

$\therefore$ the root lies betn 0.44673 & 1

∴ the root lies bet ...

$$x_3 = \frac{0.44673\,f(1) - 1\,f(0.44673)}{f(1) - f(0.44673)} = 0.49407$$

Continuing in this way we get

$x_4 = 0.50995$, $x_5 = 0.51520$, $x_6 = 0.51692$

$x_7 = 0.51748$, $x_8 = 0.51767$ $x_9 = 0.51775$ & so on

∴ Hence the required root is $\boxed{0.5177}$ correct to 4 decimal places.

2) Use the method of falsi position, to find the fourth root of 32 correct to three decimal places

Solution:

Let $x = (32)^{1/4} \Rightarrow x^4 - 32 = 0$

$\therefore f(x) = x^4 - 32$

$f(2) = -16$ & $f(3) = 49$.

the root lies beth 2 & 3

$a = 2, \quad b = 3$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = 2.2462.$$

$f(2.2462) = -6.5438 < 0.$

$\therefore f(3) = 49 > 0$

$\therefore$ root lies betn 2.2462 & 3

$$\boxed{x_2 = \frac{2.2462\, f(3) - 3 f(2.2462)}{f(3) - f(2.2462)} = 2.335}$$

$f(2.335) = -2.2732 < 0$

$f(3) = 49 > 0$

$\therefore$ root lies betn 2.335 & 3.

$$x_3 = \frac{2.335 f(3) - 3 f(2.335)}{f(3) - f(2.335)} = 2.3645$$

Continuing in this way, we get

$x_4 = 2.\underline{3770}$, $x_5 = 2.\underline{3779}$ x_6.

$\therefore x_4 \approx x_3$

$\therefore$ the required root is $\boxed{2.377}$ correct to three decimal places

Home Work

3) Find a real root of the eqn $x \log_{10} x = 1.2$ by regula-falsi method correct to four decimal places

Answer : 2.74064

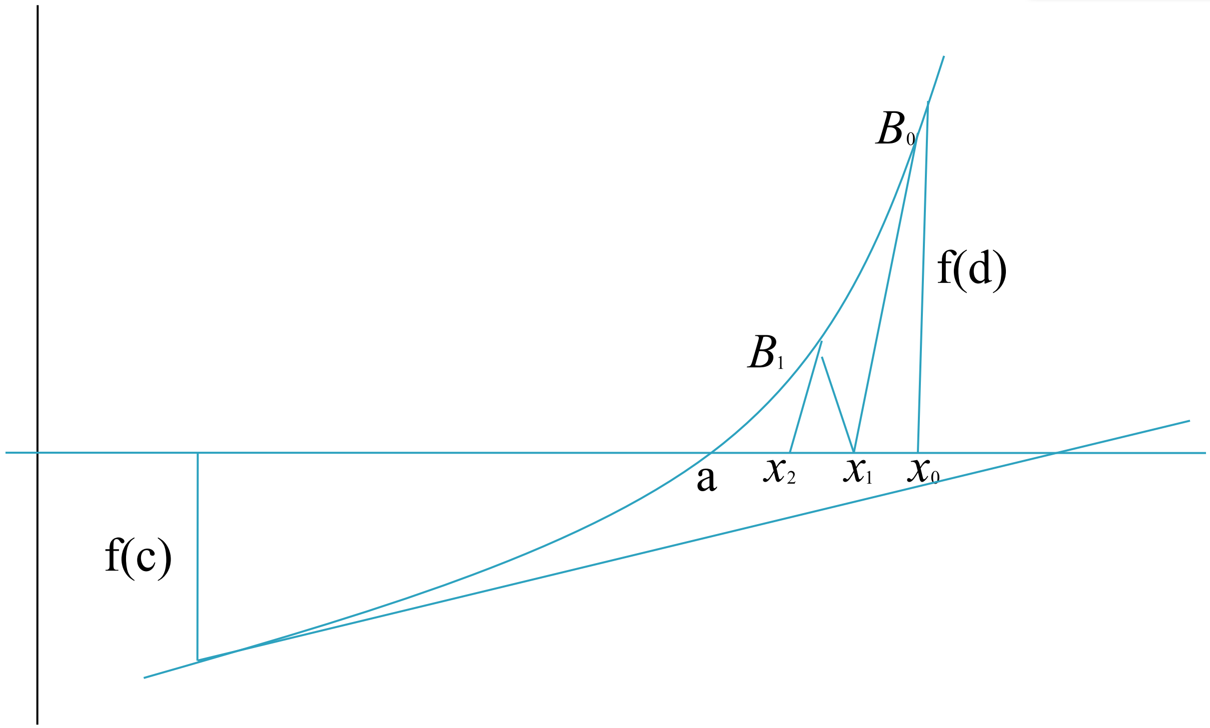
NEWTON- RAPHSON METHOD
OR
METHOD OF TANGENTS

Geometrically, Newton - Raphson method is equivalent to replacing a small arc of the curve y=f(x) by a tangent line drawn at a point of the curve.

Draw a tangent to the curve at B_0
which meets x-axis at x_1. Then draw a tangent at B_1
which meets x-axis at x_2

Continuing this process, the root ‘a’ is obtained .

f(c).f(d)<0



Suppose a=x+h where h is a small quantity,
Then applying Taylor's formula

$$0 = f(x+h) \approx f(x) + hf'(x)$$

or $$h = -\frac{f(x)}{f'(x)}$$

Thus, $$a = x + h = x - \frac{f(x)}{f'(x)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

n= 0,1 2,3……

Formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Problems:

2) using N-R method find a root of the eqn $x\sin x + \cos x = 0$ which is near $x = \pi$ correct to 3 decimal places

Solution:

$$f(x) = x\sin x + \cos x$$

$$f'(x) = x\cos x + \cancel{\sin x} - \cancel{\sin x}$$

$$f'(x) = x\cos x$$

$x_0 = \pi$ ($\because$ it is near $x = \pi$)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \pi - \frac{f(\pi)}{f'(\pi)} = \boxed{2.8232}$$

$$x_2 = 2.8232 - \frac{f(2.8232)}{f'(2.8232)} = \boxed{2.7985}$$

$$x_3 = 2.7985 - \frac{f(2.7985)}{f'(2.7985)} = \boxed{2.7983}$$

$$x_4 = 2.7983 - \frac{f(2.7983)}{f'(2.7983)} = \boxed{2.7983}$$

∴ the required root is $\boxed{2.7983}$

1) Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to four decimal places

Solution:

$f(x) = x \log_{10} x - 1.2$ —— (1)

$f(x) = x \frac{\log_e x}{\log_e 10} - 1.2$ (by change of base theorem)

$= x \log_{10} e \cdot \log_e x - 1.2$

$= 0.4343\, x \log_e x - 1.2$

$f'(x) = 0.4343 \left[x \cdot \frac{1}{x} + \log x \right] = 0.4343 (1 + \log x)$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

~~$f(1) = -1.2 < 0$~~ $f(2) = -0.59794 < 0$

$f(3) = 0.23136 > 0$

Root lies betn 2 & 3. $x_0 = 3 \quad \because |f(3)| < |f(2)|$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 2.7461}$$

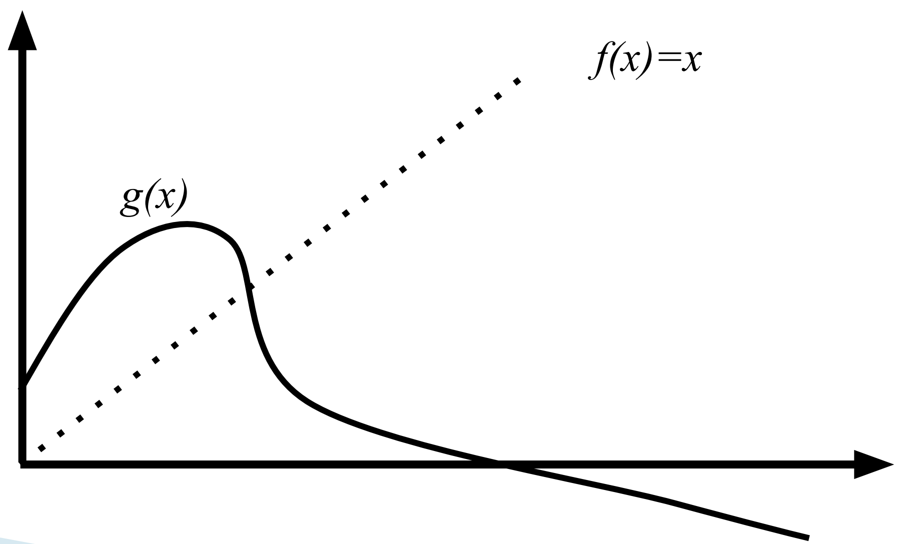
$$\boxed{x_2 = 2.7461 - \frac{f(2.7461)}{f'(2.7461)} = 2.7406}$$

$$\boxed{x_3 = 2.7406 - \frac{f(2.7406)}{f'(2.7406)} = 2.7406}$$

Thus the required root is 2.7406.//

Fixed-Point Iteration----
Successive Approximation

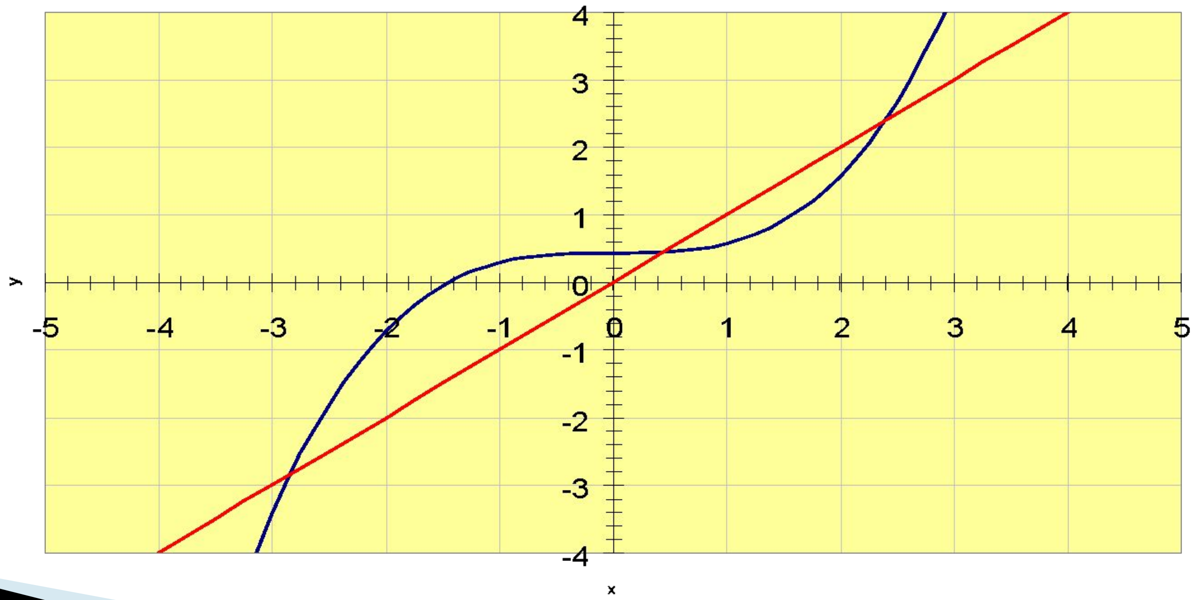
Many problems also take on the specialized form: $g(\boldsymbol{x})=\boldsymbol{x}$, where we seek, $\boldsymbol{x}$, that satisfies this equation.



Fixed Point Iteration

The equation f(x) = 0, where f(x) = $x^3 - 7x + 3$, may be re-arranged to give $x = (x^3 + 3)/7$.

Intersection of the graphs of $y = x$ and $y = (x^3 + 3)/7$ represent roots of the original equation $x^3 - 7x + 3 = 0$.



- *Fixed point* of given function $g: \mathbb{R} \to \mathbb{R}$ is value x such that

$$x = g(x)$$

- Many iterative methods for solving nonlinear equations use *fixed-point iteration* scheme of form

$$x_{k+1} = g(x_k)$$

where fixed points for g are solutions for $f(x) = 0$

- Also called *functional iteration*, since function g is applied repeatedly to initial starting value x_0
- For given equation $f(x) = 0$, there may be many equivalent fixed-point problems $x = g(x)$ with different choices for g

Fixed Point Iteration Method

Let $f(x) = 0$ be the given equation.
Let us write this equation in the form

$$x = \phi(x) \quad \text{——— (1)}$$

Let x_0 be the initial approximation value to the actual root 'α' and substituting $x = x_0$ in RHS of (1) we get

$$x_1 = \phi(x_0) \quad \text{——— (2)}$$

Again put $x = x_1$ in (2)

$$x_2 = \phi(x_1)$$

$$\vdots$$

$$x_n = \phi(x_{n-1})$$

The sequnce of approximate roots $x_1, x_2, x_3 \ldots x_n$ if it converges to 'α' is taken as the root of the equation $f(x) = 0$

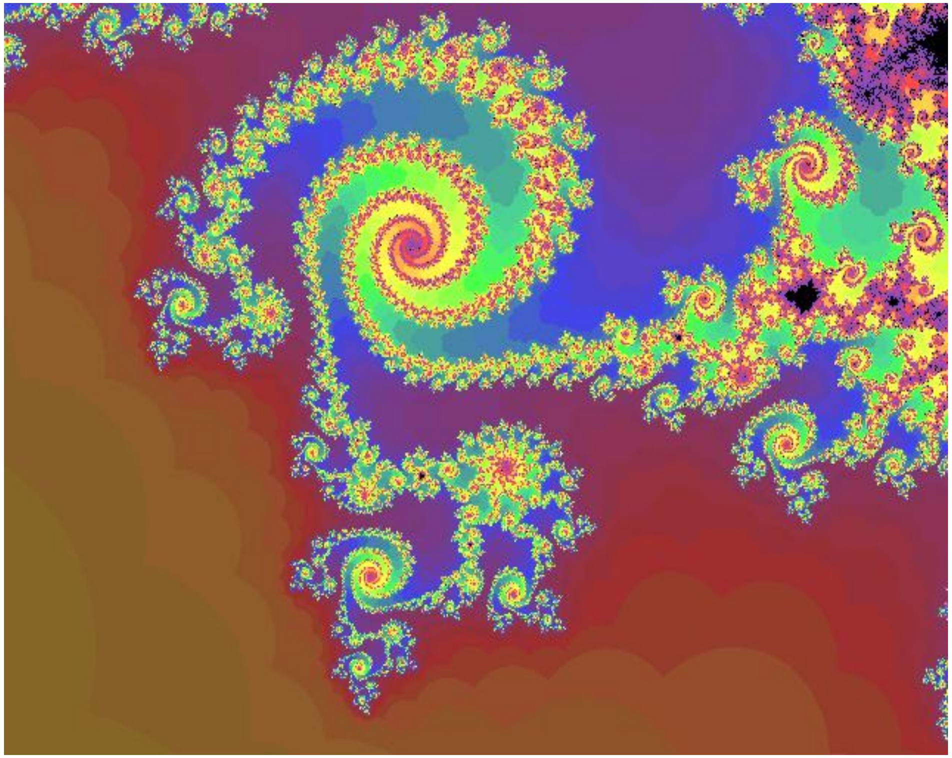
Note : 1) The smaller the value of $\phi'(x)$, the more rapid will be the convergence

2) The sufficient condition for the convergence is $\boxed{|\phi'(x)| < 1}$ for all 'x' in the interval 'I' containing the root $x = \alpha$

★ [$f(x) = 0$ can be algebraically expressed as $x = \phi(x)$
A point say 'α' is fixed point if it satisfies $x = \phi(x)$]

Fractals

Images result when we deal with 2-dimensions. Such as complex numbers.
Color indicates how quickly it converges or diverges.

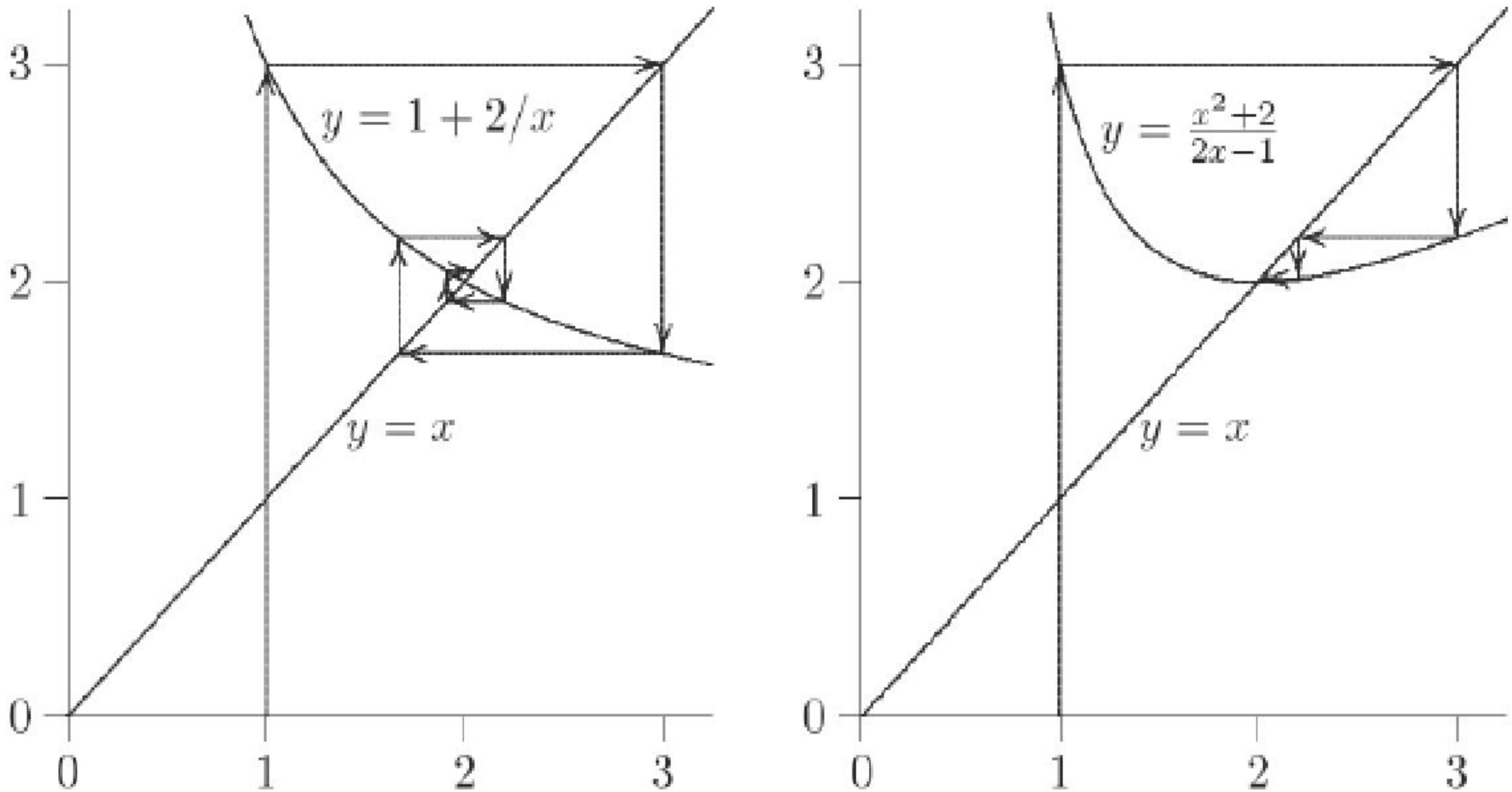


Examples

If $f(x) = x^2 - x - 2$, then fixed points of each of functions

- $g(x) = x^2 - 2$
- $g(x) = \sqrt{x+2}$
- $g(x) = 1 + 2/x$
- $g(x) = \dfrac{x^2+2}{2x-1}$

are solutions to equation $f(x) = 0$



$y = 1 + 2/x$
$y = x$
$y = \frac{x^2+2}{2x-1}$
$y = x$
0
1
2
3