Lecture 0: Probability and Markov Chains

EL2805 - Reinforcement Learning

Alexandre Proutiere

KTH, The Royal Institute of Technology

Objectives of this lecture

- Introduce probability theory
- Define Markov chains and provide their basic properties
- Illustrate concepts through simple examples

Probability Theory

- The goal is to formally model "random" phenomena or experiments
- Samples: all information you need in understanding an experiment is contained in a sample randomly selected by nature
- Set of samples: Ω, a sample ω
  - **Example 1:** throwing a die, $\Omega = \{1, 2, 3, 4, 5, 6\}$
  - **Example 2:** select a real number uniformly at random between 0 and 1, $\Omega = [0, 1]$

σ-algebra

- A σ-algebra is a subset $\mathcal{F}$ of sets of the sample set such that:
  1. $\Omega \in \mathcal{F}$
  2. $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$
  3. If $F_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, then $\cup_{n\in\mathbb{N}} F_n \in \mathcal{F}$
- σ-algebra generated by a set G of subsets is the smallest σ-algebra containing the subsets of G
- **Example 1:** throwing a die, σ-algebra $=$ the set of all subsets of $\{1, 2, 3, 4, 5, 6\}$
- **Example 2:** select a real number uniformly at random between 0 and 1, the natural algebra is that generated by the open sets of $[0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$

Probability Measures

- Measurable space: $(\Omega, \mathcal{F})$
- A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is such that:
    1. $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\emptyset) = 0$
    2. If $F_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, and if $F_n \cap F_m = \emptyset$ when $n \neq m$, then

$$\mathbb{P}(\cup_{n \in \mathbb{N}} F_n) = \sum_{n \in \mathbb{N}} \mathbb{P}(F_n)$$

- Terminology: $(\Omega, \mathcal{F}, \mathbb{P})$ is a *probability space*, $F \in \mathcal{F}$ is an *event*
- **Example 1:** throwing a die, $\mathbb{P}(\omega) = 1/6$, for all $\omega \in \Omega$
- **Example 2:** select a real number uniformly at random between 0 and 1, $\mathbb{P}([0, x)) = x$, for all $x \leq 1$

Random Variables

- A random variable X is a measurable function $X : \Omega \to \mathbb{R}$, i.e.,

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathcal{F}$$

- **Example 1:** throw a die

$$X(\omega) = \begin{cases} 0 & \text{if } \omega \text{ is even} \\ 1 & \text{if } \omega \text{ is odd} \end{cases}$$

- Interpretation: we run an experiment, and observe the value of a random variable. It provides partial information about the sample selected by nature.
- Distribution of X defined by $\forall B \in \mathcal{B}(\mathbb{R})$, $\mathbb{P}[X \in B]$

Expectation

- Restrict attention to countable sample sets
- Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
- The expectation of the r.v. X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is (if it exists):

$$\mathbb{E}[X] = \sum_{a \in A} a \mathbb{P}[X = a],$$

where $A = \{X(\omega), \omega \in \Omega\}$

Conditional Expectation

- Restrict attention to countable sample sets
- Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
- Conditional probability: for $F, G \in \mathcal{F}$,

$$\mathbb{P}(F|G) = \frac{\mathbb{P}(F \cap G)}{\mathbb{P}(G)}$$

- Let X and Y two r.v. defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $A = X(\Omega)$ and $B = Y(\Omega)$. The conditional expectation of X given $Y = b$, $b \in B$, is:

$$\mathbb{E}[X|Y = b] = \sum_{a \in A} a\mathbb{P}(X = a|Y = b)$$

Conditional Expectation

- The r.v. $Z = \mathbb{E}[X|Y]$ is defined by:

$$Z(\omega) = \mathbb{E}[X|Y = b], \quad \text{if } Y(\omega) = b$$

- Interpretation: Z is the expectation of X given that we know the value of Y
- **Example 1:** See slide 6. Define $Y(\omega) = \omega$. Then:

$$\mathbb{E}[Y|X] = \begin{cases} 4 & \text{if } \omega \text{ is even} \\ 3 & \text{if } \omega \text{ is odd} \end{cases}$$

- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|Y]]$

σ-algebra Generated by r.v.

- Family of random variables on $(\Omega, \mathcal{F})$: $(X_i, i \in I)$
- The σ-algebra generated by $(X_i, i \in I)$ is the smallest σ-algebra $\mathcal{G}$ containing $X_i^{-1}(B)$. $\mathcal{G} = \sigma(X_i, i \in I)$
- Interpretation: We run an experiment. Nature selects a sample ω. $\mathcal{G}$ consists of those events F for which for all sample, you are able to decide whether F occurred or not by observing $(X_i(\omega), i \in I)$.
- **Example 1:** See Slide 6. $\sigma(X) = \{\emptyset, \Omega, \{1, 3, 5\}, \{2, 4, 6\}\}$
- $\mathbb{E}[X|Y]$ is $\sigma(Y)$-measurable, i.e., for all $B \in \mathcal{B}(\mathbb{R})$, $\mathbb{E}[X|Y]^{-1}(B) \in \sigma(Y)$

Independent Events

- Two events A and B are *independent* if,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

  or equivalently when $\mathbb{P}(A) > 0$, if $\mathbb{P}(B|A) = \mathbb{P}(B)$

Independent Events

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  or equivalently when $\mathbb{P}(A) > 0$, if $\mathbb{P}(B|A) = \mathbb{P}(B)$
- **Interpretation:** The information about the outcome of A does not help us to predict the outcome of B.

Independent Events

- **Example:** Consider throwing a die and flipping a coin simultaneously. Note that

$$\Omega = \{1, 2, \ldots, 6\} \times \{H, T\}.$$

Define

$$A = \text{die's outcome is even}$$
$$B = \text{coin's flip is tail}$$

We have $\mathbb{P}(B|A) = \mathbb{P}(A)$, so A and B are independent.

Independent Events

- **Example.** Consider an urn containing 5 white balls and 5 black balls. Pick two balls sequentially *with replacement*. Define

$$A = \text{the first ball is white}$$
$$B = \text{the second ball is white}$$

We have $\mathbb{P}(B|A) = \mathbb{P}(B)$, so A and B are independent events.

Independent Events

- **Example.** Consider an urn containing 5 white balls and 5 black balls. Pick two balls sequentially *with replacement*. Define

$$A = \text{the first ball is white}$$
$$B = \text{the second ball is white}$$

  We have $\mathbb{P}(B|A) = \mathbb{P}(B)$, so A and B are independent events.

- **Example.** Now pick two balls *without replacement*. Now $\mathbb{P}(B|A) = \frac{4}{4+5}$ but $\mathbb{P}(B) = \frac{1}{2}$. So A and B are dependent.

Independent Events

- A collection of events $A_1, \ldots, A_n$ are independent if for any subset $I \in \{1, \ldots, n\}$,
$$\mathbb{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathbb{P}(A_i).$$

Independent Events

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- Events can be pairwise independent but not independent!
- **Example:** Flip two coins. Let

$$\begin{aligned} A &= \text{1st flip is tail,} \\ B &= \text{2nd flip is tail} \\ C &= \text{both flips are the same} \end{aligned}$$

Show that these events are pairwise independent but not jointly independent.

Independent Random Variables

- Two r.v. X and Y are *independent* if $\{X \in A\}$ and $\{Y \in B\}$ are independent events for all Borel sets A and B.
- If X and Y are independent, then for each possible pair of values a and b,

$$\mathbb{P}(X = a,\ Y = b) = \mathbb{P}(X = a)\mathbb{P}(Y = b).$$

Independent Random Variables

- Random variables $X_1, \ldots, X_n$ are mutually independent if for all $x_1, x_2, \ldots, x_n$,

$$\mathbb{P}\left(\bigcap_{i=1}^{n} X_i = x_i\right) = \prod_{i=1}^{n} P(X_i = x_i).$$

Moreover

$$\mathbb{E}\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} \mathbb{E}(X_i).$$

A Useful Property for $\mathbb{E}(X)$

- If r.v. X only takes non-negative values, then

$$\mathbb{E}(X) = \sum_{i=0}^{\infty} \mathbb{P}(X \geq i).$$

Markov Chains

- A stochastic dynamical system
- Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
- Finite state space: $S = \{1, \ldots, |S|\}$
- A sequence of r.v.'s with values in S is a Markov chain iff for all $n \geq 1$ and all $j, i_1, \ldots, i_n \in S$

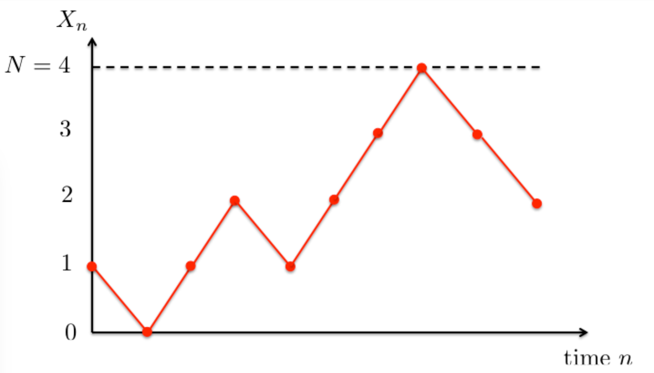
$$\mathbb{P}(X_{n+1} = j | X_1 = i_1, \ldots, X_n = i_n) = \mathbb{P}(X_{n+1} = j | X_n = i_n)$$

- Transition matrix for homogenous Markov chains: P

$$P_{i,j} = \mathbb{P}(X_{n+1} = j | X_n = i), \quad \forall i, j \in S$$

Example: Reflected random walk

- $S = \{0, 1, \ldots, N\}$
- $P_{0,1} = P_{N,N-1} = 1$,
  $\forall i \neq 0, N$, $P_{i,i+1} = P_{i,i-1} = 1/2$



Kolmogorov Equations

- The distribution of the state at time n is described by a row vector $\mu_n \in [0,1]^{|S|}$
- Kolmogorov equation: $\mu_{n+1} = \mu_n P$
- m-steps transition: $\mu_{n+m} = \mu_n P^m$

$$(P^m)_{i,j} := p^m(i,j) = \mathbb{P}(X_{n+m} = j | X_n = i)$$

- Accessibility, Communication:

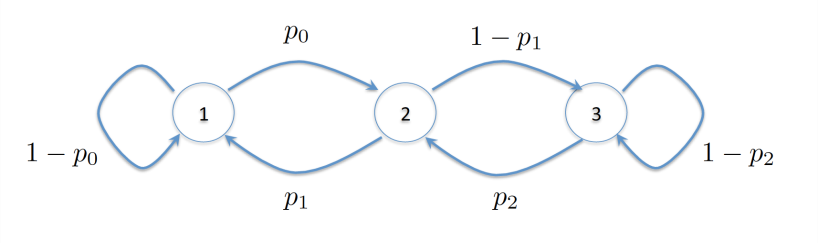
$$(i \to j) \iff (\exists m : p^m(i,j) > 0)$$

$$(i \longleftrightarrow j) \iff (i \to j \text{ and } j \to i)$$

Communication Classes, Irreducibility

- By definition: each state communicates with itself
- Communication is an equivalence class
- A communicating class is a maximal set of states C such that every pair of states in C communicates with each other
- A finite Markov chain is irreducible iff there is a unique communication class
- If a communication class consists of a single state, the latter is called absorbing

Transition Graph



State Classification

- Time to reach i: $\tau_i = \inf(n \geq 1 : X_n = i)$
- Recurrent state: $\mathbb{P}_i(\tau_i < \infty) = 1$ where $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot | X_0 = i)$
- Positive recurrent state: $\mathbb{E}_i(\tau_i) < \infty$
- Transient state: $\mathbb{P}_i(\tau_i < \infty) < 1$
- Recurrence is a class property:

$$i \leftrightarrow j \implies (i, j \text{ are both recurrent or transient})$$

- Number of visits: $N_i = \sum_{n \geq 1} 1_{X_n = i}$

$$\mathbb{P}_i(\tau_i < \infty) = 1 \iff \mathbb{P}_i(N_i = \infty) = 1$$

Irreducibility and Recurrence

- In an irreducible finite Markov chain, all states are positive recurrent

Periodicity

- The period of state i is the defined by: gcd$\{n > 0 : p^n(i,i) > 0\}$
- A state is aperiodic if its period is equal to 1
- In an irreducible Markov, all states have the same period
- An irreducible Markov chain with period d has a cyclic structure:

$$\exists S_0, \ldots, S_{d-1} : \cup_k S_k = S, S_d = S_0$$

$$\forall k, \forall i \in S_k, \quad \sum_{j \in S_{k+1}} p(i,j) = 1$$

Periodicity

- An irreducible Markov chain with period d has a cyclic structure: for instance, order the states so that we get in order $S = S_0, \ldots, S_3$, then
$$P = \begin{pmatrix} 0 & A_0 & 0 & 0 \\ 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \\ A_3 & 0 & 0 & 0 \end{pmatrix}$$

Stationary Distribution

- A distribution π is stationary if: $\pi = \pi P$
- Global balance equations: π is stationary iff:

$$\forall i \in S, \quad \pi_i = \sum_j P_{j,i} \pi_j$$

- A finite irreducible Markov chain has a unique stationary distribution

$$\forall i \in S, \quad \pi(i) = \frac{\mathbb{E}_0[\sum_{n \geq 1} 1_{X_n = i} 1_{n \leq \tau_0}]}{\mathbb{E}_0[\tau_0]}$$

$$\pi_i = \frac{1}{\mathbb{E}_i[\tau_i]}$$

Ergodicity

- For a finite irreducible Markov chain:

$$\forall f : S \to \mathbb{R} : \sum_{i \in S} |f(i)| \pi_i < \infty$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) = \sum_{i \in S} f(i) \pi_i$$

Example: Reflected random walk

- What are the communication classes?
- Compute $\mathbb{E}_i[\tau_i]$
- Compute the stationary distribution

Summary

- Set of samples Ω. A sample $\omega \in \Omega$ contains all the information about an experiment
- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$
  - $\mathcal{F}$ is a σ-algebra, $F \in \mathcal{F}$ is an event
  - the probability measure $\mathbb{P}$ is such that $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(F \cup G) = \mathbb{P}(F) + \mathbb{P}(G)$ if $F \cap G = \emptyset$
  - $F, G \in \mathcal{F}$ are independent if $\mathbb{P}(F \cap G) = \mathbb{P}(F)\mathbb{P}(G)$
  - Conditional probability: $F, G \in \mathcal{F}$, $\mathbb{P}(F|G) = \mathbb{P}(F \cap G)/\mathbb{P}(G)$
- Random variable $X : \Omega \to A$ (partial information about the experiment)
  - Distribution of X: $(\mathbb{P}(X = a))_{a \in A}$
  - Expectation of X: $\mathbb{E}[X] = \sum_{a \in A} a\mathbb{P}(X = a)$
  - Conditional expectation: $\mathbb{E}[X|Y = b] = \sum_{a \in A} a\mathbb{P}(X = a|Y = b)$
    $Z = \mathbb{E}[X|Y]$ is a r.v. such that if $Y(\omega) = b$, $Z(\omega) = \mathbb{E}[X|Y = b]$
    $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$ and $\mathbb{E}[f(X)|X] = f(X)$

Summary

- Markov chains: $(X_n)_{n\geq 1}$ is a MC in S if for all $n \geq 1$ and all $j, i_1, \dots, i_n \in S$

$$\mathbb{P}(X_{n+1} = j | X_1 = i_1, \dots, X_n = i_n) = \mathbb{P}(X_{n+1} = j | X_n = i_n)$$

  - $j \in S$ is a state
  - Homogenous MC: transition matrix $P_{i,j} = \mathbb{P}(X_{n+1} = j | X_n = i)$ (stochastic matrix)
  - Kolmogorov equations: if $\mu_n = (\mathbb{P}(X_n = i))_{i=1,\dots,S}$, then $\mu_{n+1} = \mu_n P$
    $\forall m \geq 0, \mu_{n+m} = \mu_n P^m$

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More in the exercice session ...