Human Computable Passwords*

Jeremiah Blocki Carnegie Mellon University jblocki@cs.cmu.edu

Anupam Datta Carnegie Mellon University danupam@cmu.edu Manuel Blum Carnegie Mellon University mblum@cs.cmu.edu

Santosh Vempala Georgia Tech vempala@cc.gatech.edu

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Abstract

An interesting challenge for the cryptography community is to design authentication protocols that are so simple that a human can execute them without relying on a fully trusted computer. We propose several candidate authentication protocols for a setting in which the human user can only receive assistance from a semi-trusted computer—a computer that stores information and performs computations correctly but does not provide confidentiality. Our schemes use a semi-trusted computer to store and display public challenges $C_i \in [n]^k$. The human user memorizes a random secret mapping $\sigma : [n] \to \mathbb{Z}_d$ and authenticates by computing responses $f(\sigma(C_i))$ to a sequence of public challenges where $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ is a function that is easy for the human to evaluate. We prove that any statistical adversary needs to sample $m = \tilde{\Omega}(n^{s(f)})$ challenge-response pairs to recover σ , for a security parameter s(f) that depends on two key properties of f. Our lower bound generalizes recent results of Feldman et al. [29] who proved analogous results for the special case d = 2. To obtain our results, we apply the general hypercontractivity theorem [47] to lower bound the *statistical dimension* of the distribution over challenge-response pairs induced by *f* and σ . Our statistical dimension lower bounds apply to arbitrary functions $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ (not just to functions that are easy for a human to evaluate). As an application, we propose a family of human computable password functions f_{k_1,k_2} in which the user needs to perform $2k_1 + 2k_2 + 1$ primitive operations (e.g., adding two digits or looking up a secret value $\sigma(i)$), and we show that $s(f) = \min\{k_1 + 1, (k_2 + 1)/2\}$. For these schemes, we prove that forging passwords is equivalent to recovering the secret mapping. Thus, our human computable password schemes can maintain strong security guarantees even after an adversary has observed the user login to many different accounts.

Keywords: Passwords, Human Authentication, Planted Satisfiability, Statistical Algorithms, Statistical Dimension

1 Introduction

A typical computer user has many different online accounts which require some form of authentication. While passwords are still the dominant form of authentication, users struggle

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to remember their passwords. As a result users often adopt insecure password practices (e.g., reuse, weak passwords) [31, 24, 41, 19] or end up having to frequently reset their passwords. Recent large-scale password breaches highlight the importance of this problem [1, 24, 13, 3, 51, 8, 2, 5, 4, 7, 6, 9]. Blocki et al. [16] initiated the rigorous study of password management schemes — systematic strategies to help users create and remember multiple passwords. Their Shared Cues scheme balances security and usability considerations, but maintains security only when a small (constant) number of plaintext passwords of a user are revealed (e.g., 1 to 4).

The goal of our work is to develop a human computable password management scheme in which security guarantees are maintained after *many* breaches. In a human computable password management scheme the user reconstructs each of his passwords by *computing* the response to a public challenge. The computation may only involve a few very simple operations (e.g., addition modulo 10) over secret values (digits) that the user has memorized. More specifically, in our candidate human computable password schemes the user learns to compute a simple function $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ (we adopt the base d = 10 that is natural for most humans), and memorizes a secret mapping $\sigma: [n] \to \mathbb{Z}_d$. The user authenticates by responding to a sequence of single digit challenges, i.e., a challenge-response pair $(C, f(\sigma(C)))$ is a challenge $C \in X_k \subseteq [n]^k$ and the corresponding response.

One of our candidate human computable password schemes uses the function

$$f(x_0, x_1, x_2, x_3, x_4, x_5, \dots, x_{13}) = x_{13} + x_{12} + x_{(x_{10} + x_{11} \mod 10)} \mod 10$$
.

To evaluate this function a human would only need to perform three addition operations modulo 10. While this function is quite simple we provide strong evidence for our conjecture that a polynomial time attacker will need to see $\tilde{\Omega}(n^{1.5})$ challenge-response pairs before he can forge the user's passwords (accurately predict the responses to randomly selected challenges)¹.

Following Blocki et al. [16] we consider a setting where a user has two types of memory: persistent memory (e.g., a sticky note or a text file on his computer) and associative memory (e.g., his own human memory). We assume that persistent memory is reliable and convenient but not private (i.e., accessible to an adversary). In contrast, a user's associative memory is private but lossy—if the user does not rehearse a memory it may be forgotten. Thus, the user can store a password challenge $C \in X_k$ in persistent memory, but the mapping σ must be stored in associative memory (e.g., memorized and rehearsed). We allow the user to receive assistance from a semi-trusted computer. A semi-trusted computer will perform computations accurately (e.g., it can be trusted to show the user the correct challenge), but it will not ensure privacy of its inputs or outputs. This means that a human computable password management scheme should be based on a function f that the user can compute entirely in his head.

Contributions. We provide precise notions of security and usability for a human computable password management scheme (Section 3). We introduce the notion of UF-RCA security (Unforgeability under Random Challenge Attacks). Informally, a human computable password scheme is UF-RCA secure if an adversary cannot forge passwords after seeing many example challenge-response pairs. We use the usability model of Blocki et al. [16] to quantify the effort that a user must expend rehearsing the secret mapping σ and present a simple model to estimate the time that a user must spend to compute each password in his head.

We present the design of a candidate family of human computable password management schemes f_{k_1,k_2} , and analyze the usability and security of these schemes (Section 4). Our usability

 $^{^{1}}$ We stress that our security guarantees are not information theoretic. A computationally unbounded adversary would need to see at most O(n) challenge-response pairs to break any such human computable password management scheme.

analysis indicates that to compute $f_{k_1,k_2}(\sigma(C))$ the user needs to execute $2k_1 + 2k_2 + 1$ simple operations (e.g., addition of single digits modulo 10). We also propose a mnemonic tool to help users memorize their secret mapping σ . The main technical result of this section (Theorem 1) states that our scheme is UF-RCA secure given a plausible conjecture about the hardness of random planted constraint satisfiability problems (RP-CSP). Our conjecture is that any polynomial time adversary needs to see at least $m = n^{\min\{r(f)/2, g(f) + 1 - \epsilon\}}$ challenge-response pairs (C, $f(\sigma(C))$) to recover the secret mapping σ . Here, $s(f) = \min\{r(f)/2, g(f) + 1\}$ is a composite security parameter involving g(f) (how many inputs to f need to be fixed to make f linear?) and r(f) (what is the largest value of r such that the distribution over challenge-response pairs are (r-1)-wise independent?). We prove that $g(f_{k_1,k_2}) = k_1$ and $r(f_{k_1,k_2}) = (k_2 + 1)/2$.

Next we prove that any statistical adversary needs at least $\tilde{\Omega}(n^{r(f)/2})$ challenge-response pairs $(C, f(\sigma(C)))$ to find a secret mapping σ' that is ϵ -correlated with σ (Section 5). We view this result as strong evidence in favor of the RP-CSP hardness assumption. Almost all known algorithmic techniques have statistical analogues (see discussion in Section 5). While Gaussian Elimination is a notable exception, our composite security parameter accounts for attacks based on Gaussian Elimination—an adversary needs to see $m = \tilde{\Omega}(n^{1+g(f)})$ challenge-response pairs to recover σ using Gaussian Elimination. We also provide experimental evidence for the hardness assumption by using a CSP solver to try to recover $\sigma \in \mathbb{Z}_{10}^n$ given m challenge-response pairs using the functions $f_{1,3}$ and $f_{2,3}$. Our CSP solver failed to find the secret mapping $\sigma \in \mathbb{Z}_{10}^{50}$ given m = 1000 random challenge-response pairs with both functions. Additionally, we constructed public challenges for cryptographers to break our human computable password management schemes under various parameters (e.g., n = 100, m = 1000).

Our lower bound for statistical adversaries is based on the *statistical dimension* of the distribution over challenge-response pairs induced by f and σ . We stress that our analysis of the statistical dimension applies to arbitrary functions $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$, not just to functions that are easy for humans to compute. Our analysis of the statistical dimension generalizes recent results of Feldman et al. [29] for binary predicates and may be of independent interest. Because our function f is not a binary predicate we cannot use the Walsh basis functions to express the Fourier decomposition of f and analyze the statistical dimension of our distribution over challenge-response pairs as Feldman et al. [29] do. Instead, we use a generalized set of Fourier basis functions to take the Fourier decomposition of f, and we apply the general hypercontractivity theorem [47] to obtain our bounds on the statistical dimension.

We complete the proof of Theorem 1 in Section 6 by proving that forging passwords and approximately recovering the secret mapping are equivalent problems for a broad class of human computable password schemes, including f_{k_1,k_2} . This result implies that any adversary who can predict the response f(C) to a random challenge C with better accuracy than random guessing can be used as a blackbox to approximately recover the secret mapping.

We conclude with discussion in Section 7. In particular, we argue that our human computable password schemes could be used to achieve secure cryptography in a panoptic world (the user only has access to a semi-trusted computer) by leveraging recent breakthroughs on code obfuscation[32].

2 Related Work

The literature on passwords has grown rapidly over the past decade. One line of prior work has focused on the effects of password composition rules (e.g., requiring a password to capital letters and numbers) on individual passwords [40, 17]. Another line of prior work has focused on empirical studies of user behavior in password management [31, 24, 41, 19] (e.g., How often

do users reuse the same password? How strong are the passwords that people pick?). These studies consistently paint a grim picture. Many of the passwords that users select have low entropy and users frequently reuse their passwords. Shay et al. [50] empirically studied the usability of system assigned passwords and found that users often had difficulty remembering system assigned passwords. Some researchers have considered replacing text passwords with graphical passwords [22, 12] driven by evidence that humans have a large capacity for visual memories [52] and that cued-recall is easier than pure recall [11]. Fundamentally, both graphical passwords and text passwords rely solely on the user's ability to remember something (e.g., a string, a face or a location on a picture). Many security metrics have been proposed to analyze the security of a dataset of passwords or to estimate the security of an individual password [42, 48, 19, 20]. While these metrics can provide useful feedback about individual passwords (e.g., they rule out some insecure passwords) they do not deal with the complexities of securing multiple accounts against an adversary (e.g., they don't consider correlations between a user's passwords).

Blocki et al. [16] considered the problem of developing usable and secure password management schemes — strategies for creating and remembering *multiple* passwords. We use their usability model to quantify the effort that a user will need to expend to remember his secret mapping in our human computable password schemes. There are two key differences between their work and ours. First, their password management scheme (Shared Cues) only maintains security for a small (constant) number of plaintext password breaches, while our goal is to design protocols that maintain security guarantees even after *many* password breaches. There are scenarios in which it may not be reasonable to assume that the adversary can only compromise a small number of the user's passwords (e.g., if the user's computer is infected with malware for a few days). Second, their Shared Cues scheme only requires users to remember several cue-association pairs to reconstruct their passwords while our schemes require users to perform a few additional computations in their head to reconstruct their passwords.

Hopper and Blum [36] designed a Human Identification Protocol based on noisy parity, a learning problem called that is believed to be hard. Juels and Weis [37] modified the protocol of Hopper and Blum to design HB+ — a lightweight authentication protocol for pervasive devices like smartcards. Subsequent work has explored the security of the HB+ protocol under various threat models (e.g., man-in-the-middle attacks[34, 21], concurrent composition[38]). We emphasize a few fundamental differences between our work and the work of Hopper and Blum. First, they focus on the authentication setting where a human authenticates to a single trusted party with a shared secret. By contrast, we focus on the setting where a human user wishes to authenticate to multiple (possibly untrusted) parties without sharing his secret (e.g., by only sharing the cryptographic hashes of each password he computes). Second, computations in their protocol are randomized (e.g., the human occasionally flips his answer), while the computations in our protocol are deterministic. This is significant because humans are not good at consciously generating random numbers [54, 30, 44] (e.g., noisy parity could be easy to learn when humans are providing source of noise). It also means that their protocol would need to be modified in our setting so that the untrusted third party could validate a noisy response using only a cryptographic hash of the answer — invoking error correcting codes would increase the number of rounds needed to provide an acceptable level of security. Finally, we focus on computations of very simple functions over a constant number of variables so that a human can compute the response to each challenge quickly.

Naor and Pinkas[45] proposed using visual cryptography[46] to address a related problem: how can a human verify that a message he received from a trusted server has not been tampered with by an adversary? Their protocol requires the human to carry a visual transparency (a shared secret between the human and the trusted server in the visual cryptography scheme),

which he will use to verify that messages from the trusted server have not been altered.

A related goal in cryptography, constructing pseudorandom generators in NC^0 , was proposed by Goldreich [35] and by Cryan and Miltersen [26]. In Goldreich's construction we fix $C_1, ..., C_m \in [n]^k$ once and for all, and a binary predicate $P: \{0,1\}^k \to \{0,1\}$. The pseudorandom generator is a function $G: \{0,1\}^n \to \{0,1\}^m$, whose i'th bit G(x)[i] is given by P applied to the bits of x specified by C_i . O'Donnel and Witmer gave evidence that the "Tri-Sum-And" predicate $(TSA(x_1,...,x_5) = x_1 + x_2 + x_3 + x_4x_5 \mod 2)$ provides near-optimal stretch. In particular, they showed that for $m = n^{1.5-\epsilon}$ Goldreich's construction with the TSA predicate is secure against subexponential-time attacks using SDP hierarchies. Our candidate human-computable password schemes use functions $f: \mathbb{Z}_{10}^k \to \mathbb{Z}_{10}$ instead of binary predicates. While our candidate functions are contained in NC^0 , we note that an arbitrary function in NC^0 is not necessarily human computable.

Feldman et al. [29] considered the problem of finding a planted solution in a random binary satisfiability problem. They showed that any statistical algorithm — a class of algorithms that covers almost all known algorithmic techniques — needs to see at least $\tilde{\Omega}(n^{r/2})$ random clauses to efficiently identify the planted solution when the distribution over clauses are (r-1)-wise independent². Feldman et al. [29] also demonstrate that $\tilde{O}(n^{r/2})$ clauses are sufficient. We extend their analysis to handle non-binary planted satisfiability problems, and argue that our candidate human computable password schemes are secure.

3 Definitions

3.1 Notation

Given two strings $\alpha_1, \alpha_2 \in \mathbb{Z}_d^n$ we use $H(\alpha_1, \alpha_2) \doteq |\{i \in [n] \mid \alpha_1[i] \neq \alpha_2[i]\}|$ to denote the Hamming distance between them. We will also use $H(\alpha_1) \doteq H(\alpha_1, \vec{0})$ to denote the Hamming weight of α_1 . We use $\sigma : [n] \to \mathbb{Z}_d$ to denote a secret random mapping that the user will memorize. We will sometimes abuse notation and think of $\sigma \in \mathbb{Z}_d^n$ as a string which encodes the mapping, and we will use $\sigma \sim \mathbb{Z}_d^n$ to denote a random mapping chosen from \mathbb{Z}_d^n uniformly at random.

Definition 1. We say that two mappings $\sigma_1, \sigma_2 \in \mathbb{Z}_d^n$ are ϵ -correlated if $\frac{H(\sigma_1, \sigma_2)}{n} \leq \frac{d-1}{d} - \epsilon$, and we say that a mapping $\sigma \in \mathbb{Z}_d^n$ is δ -balanced if

$$\max_{i \in \{0,\dots,d-1\}} \left| \frac{H(\sigma, \vec{i})}{n} - \frac{d-1}{d} \right| \le \delta.$$

Note that for a random mapping σ_2 we expect σ_1 and σ_2 to differ at $\mathbb{E}_{\sigma_2 \sim \mathbb{Z}_d^n}[H(\sigma_1, \sigma_2)] = n\left(\frac{d-1}{d}\right)$ locations, and for a random mapping σ and $i \sim \{0, \dots, d-1\}$ we expect σ to differ from \vec{i} at $\mathbb{E}_{i \sim \mathbb{Z}_d, \sigma \sim \mathbb{Z}_d^n}[H(\sigma, \vec{i})] = n\left(\frac{d-1}{d}\right)$ locations. Thus, with probability 1 - o(1) a random mapping σ_2 will not be ϵ -correlated with σ_1 , but a random mapping σ will be δ -balanced with probability 1 - o(1).

We let $X_k \subseteq [n]^k$ denote the space of ordered clauses of k variables without repetition. We use $C \sim X_k$ to denote a clause C chosen uniformly at random from X_k and we use $\sigma(C) \in \mathbb{Z}_d^k$ to

²We note that after we have seen $O(n\log n)$ random clauses the planted solution is — with high probability — the only solution which satisfies all of the random clauses. The results of Feldman et al. [29] are evidence that we need $\tilde{\Omega}(n^{r/2})$ examples to find the planted solution efficiently. Feldman et al. [29] also showed that Goldreich's PRG with the TSA predicate is secure against statistical adversaries for $m = n^{r(TSA) - \epsilon}$, where r(TSA) = 1.5.

denote the values of the corresponding variables in *C*. For example, if d = 10, C = (3, 10, 59) and $\sigma(i) = (i + 1 \mod 10)$ then $\sigma(C) = (4, 1, 0)$.

We view each clause $C \in X_k$ as a *single-digit challenge*. The user responds to a challenge C by computing $f(\sigma(C))$, where $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ is a *human computable* function (see discussion below) and $\sigma: [n] \to \mathbb{Z}_d$ is the secret mapping that the user has memorized. For example, if d = 10, C = (3,10,59), $\sigma(i) = (i+1 \mod 10)$ and $f(x,y,z) = (x-y+z \mod 10)$ then $f(\sigma(C)) = (4-1+0 \mod 10) = 3$. A length-t password challenge $\vec{C} = \langle C_1, \dots, C_t \rangle \in (X_k)^t$ is a sequence of t single digit challenges, and $f(\sigma(\vec{C})) = \langle f(\sigma(C_1)), \dots, f(\sigma(C_t)) \rangle \in \mathbb{Z}_d^t$ denotes the corresponding response (e.g., a password).

Let's suppose that the user has m accounts A_1, \ldots, A_m . In a human computable password management scheme we will generate m length-t password challenges $\vec{C}_1, \ldots, \vec{C}_m \in (X_k)^t$. These challenges will be stored in persistent memory so they are always accessible to the user as well as the adversary. When our user needs to authenticate to account A_i he will be shown the length-t password challenge $\vec{C}_i = \left\langle C_1^i, \ldots, C_t^i \right\rangle$. The user will respond by computing his password $p_i = \left\langle f\left(\sigma\left(C_1^i\right)\right), \ldots, f\left(\sigma\left(C_t^i\right)\right)\right\rangle \in \mathbb{Z}_d^t$.

3.2 Requirements for a Human Computable Function

In our setting we require that the composite function $f \circ \sigma : X_k \to \mathbb{Z}_d$ is human computable. Informally, we say that a function f is *human-computable* if a human user can evaluate f *quickly* in his head.

Requirement 1. A function f is \hat{t} -human computable for a human user H if H can reliably evaluate f in his head in \hat{t} seconds.

We believe that a function f will be *human-computable* whenever there is a fast streaming algorithm[10] to compute f. A streaming algorithm is an algorithm for processing a data stream in which the input (e.g., the challenge C) is presented as a sequence of items that can only be examined once. In our context the streaming algorithm must have a very low memory footprint because a typical person can only keep 7 ± 2 'chunks' of information in working memory [43] at any given time. Our streaming algorithm can only involve primitive operations that a person could execute quickly in his head (e.g., adding two digits modulo 10, recalling a value $\sigma(i)$ from memory).

Definition 2. Let P be a set of primitive operations. We say that a function f is (P, \tilde{t}, \hat{m}) -computable if there is a space \hat{m} streaming algorithm \mathcal{A} to compute f using only \tilde{t} operations from P.

In this paper we consider the following primitive operations P: Add, Recall and TableLookup. Add: $\mathbb{Z}_{10} \times \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ takes two digits x_1 and x_2 and returns $x_1 + x_2 \mod 10$. Recall: $[n] \to \mathbb{Z}_{10}$ takes an index i and returns the secret value $\sigma(i)$ that the user has memorized. TableLookup: $\mathbb{Z}_{10} \times [n]^{10} \to [n]$ takes a digit x_1 and finds the x_1 'th value from a table of 10 indices. We take the view that no human computable function should require users to store intermediate values in long-term memory because the memorization process would necessarily slow down computation. Therefore, we do not consider include any operation like **MemorizeValue** in our set of primitive operations.

Example: The function $f \circ \sigma(i_1, ..., i_5) = \sigma(i_1) + ... + \sigma(i_5)$ requires 9 primitive operations (five **Recall** operations and four **Add** operations) and requires space $\hat{m} = 3$ (e.g., we need one slot to store the current total, one slot to store the next value from the data stream and one free slot to execute a primitive operation).

We conjecture that, after training, a human user H will be able to evaluate a $(P,\tilde{t},3)$ -computable function in $\hat{t}=\tilde{t}\gamma_H$ seconds, where γ_H is a small constant that can vary from individual to individual. While we have not conducted a large scale empirical study to test Conjecture 1, the main author of this paper has found that he can evaluate (P,9,3)-computable functions in 7.5-seconds and we believe that similar results $(\gamma_H \leq 1)$ will hold for users with moderate mathematical backgrounds.

Conjecture 1. *Let* $P = \{Add, Recall, TableLookup\}$. *For each human user* H *there is a small constant* $\gamma_H > 0$ *such that any* $(P, \tilde{t}, 3)$ -computable function f *will be* \hat{t} -human computable for H *with* $\hat{t} = \gamma_H \tilde{t}$.

3.3 Password Unforgeability

In the password forgeability game the adversary attempts to guess the user's password for a randomly selected account after he has seen the user's passwords at m/t other randomly selected accounts. We say that a scheme is UF-RCA (Unforgeability against Random Challenge Attacks) secure if any probabilistic polynomial time adversary fails to guess the user's password with high probability. In the password forgeability game we select the secret mapping $\sigma: [n] \to \mathbb{Z}_d$ uniformly at random along with challenges $C_1, \ldots, C_{m+t} \sim X_k$. The adversary is given the function $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ and is shown the challenges C_1, \ldots, C_{m+t} as well as the values $f(\sigma(C_i))$ for $i \in \{1, \ldots, m\}$. The game ends when the adversary \mathcal{F} outputs a guess $\langle q_1, \ldots, q_t \rangle \in \mathbb{Z}_d^t$ for the value of $\langle f(\sigma(C_{m+1})), \ldots, f(\sigma(C_{m+t})) \rangle$. We say that the adversary wins if he correctly guesses the responses to all of the challenges C_{m+1}, \ldots, C_{m+t} , and we use $\mathbf{Wins}(\mathcal{F}, n, m, t)$ to denote the event that the adversary wins the game (e.g., $\forall i \in \{1, \ldots, t\}. q_i = f(\sigma(C_{m+i}))$). We are interested in understanding how many example single digit challenge-response pairs the adversary needs to see before he can start breaking the user's passwords.

Definition 3. (Security) We say that a function $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ is $\mathbf{UF} - \mathbf{RCA}(n, m, t, \delta) - secure$ if for every probabilistic polynomial time (in n, m) adversary \mathcal{A} we have $\Pr[\mathbf{Wins}(\mathcal{A}, n, m, t)] \leq \delta$, where the randomness is taken over the selection of the secret mapping $\sigma \sim \mathbb{Z}_d^n$, the challenges C_1, \ldots, C_{m+t} as well as the adversary's coins.

Discussion Our security model is different from the security model of Blocki et al. [16] in which the adversary gets to adaptively select which accounts to compromise and which account to attack. While our security model may seem weaker at first glance because the adversary does not get to select which account to compromise/attack, we observe that the password management schemes of Blocki et al. [16] are only secure against one to three adaptive breaches. By contrast, our goal is to design human computable password schemes that satisfy **UF-RCA** security for large values of m (e.g. 1000), which means that it is reasonable to believe that the user has at most m/t password protected accounts. If the user has at most m/t accounts then even an adaptive adversary — who gets to compromise all but one account — will not be able to forge the password at any remaining account with probability greater than $m\delta/t$ (typically, $m \ll t/\delta$).

3.4 Security Parameters of *f*

Given a function $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ we define the function $Q^f: \mathbb{Z}_d^{k+1} \to \{\pm 1\}$ s.t. $Q^f(x,i) = 1$ if f(x) = i; otherwise $Q^f(x,i) = -1$. We use Q^f_σ to define a distribution over $X_k \times \mathbb{Z}_d$ (challenge-response pairs) as follows

$$\Pr_{Q^f_\sigma}[C,i] \doteq \frac{Q^f\left(\sigma\left(C\right),i\right) + 1}{2\left|X_k\right|} \;.$$

Intuitively, Q^f_σ is the uniform distribution over challenge response pairs (C,j) s.t. $f(\sigma(C)) = j$. We also use $Q^{f,j}: \mathbb{Z}_d^k \to \{\pm 1\}$ $(Q^{f,j}(x) = Q^f(x,j))$ to define a distribution over X_k . We write the Fourier decomposition of a function $Q: \mathbb{Z}_d^k \to \{\pm 1\}$ as follows

$$Q(x) = \sum_{\alpha \in \mathbb{Z}_d^k} \hat{Q}_{\alpha} \cdot \chi_{\alpha}(x) ,$$

where our basis functions are

$$\chi_{\alpha}(x) \doteq \exp\left(\frac{-2\pi\sqrt{-1}(x \cdot \alpha)}{d}\right).$$

We say that a function Q has degree ℓ if $\ell = \max\{H(\alpha) \mid \alpha \in \mathbb{Z}_d^k \land \hat{Q}_\alpha \neq 0\}$ — equivalently if $Q(x) = \sum_i Q_i(x)$ can be expressed as a sum of functions where each function $Q_i : \mathbb{Z}_d^k \to \mathbb{R}$ depends on at most ℓ variables.

Definition 4. We use $r(Q) \doteq \min \{ H(\alpha) \mid \exists \alpha \in \mathbb{Z}_d^k . \hat{Q}_\alpha \neq 0 \land \alpha \neq \vec{0} \}$ to denote the distributional complexity of Q, and we use $r(f) = \min \{ r(Q^{f,j}) \mid j \in \mathbb{Z}_d \}$ to denote the distributional complexity of f. We use

$$g(f) \doteq \min \left\{ \ell \in \mathbb{N} \cup \{0\} \mid \exists \alpha \in \mathbb{Z}_d^{\ell}, S \subseteq [k], \hat{d} \in \mathbb{Z}_d.s.t \mid S \mid = \ell \text{ and } f_{\mid S, \alpha} \text{ is a linear function} \mod \hat{d} \right\}$$

to denote the minimum number of variables that must be fixed to make f a linear function. Here, $f_{|S,\alpha}: \mathbb{Z}_d^{k-\ell} \to \mathbb{Z}_d$ denotes the function f after fixing the variables at the indices specified by S to α . Finally, we use $s(f) \doteq \min\{r(f)/2, g(f)+1\}$ as our composite security measure.

We conjecture that a polynomial time adversary will need to see $m=n^{s(f)}$ challenge-response pairs before he can approximately recover the secret mapping σ . We call this conjecture about the hardness of random planted constraint satisfiability problems RP-CSP (Conjecture 2). In support of RP-CSP we prove that any statistical algorithm needs to see at least $m=\tilde{\Omega}\left(n^{r(f)/2}\right)$ challenge response pairs to (approximately) recover the secret mapping σ and we observe that a polynomial time adversary would need to see $m=O\left(n^{g(f)+1}\right)$ challenge-response pairs to recover σ using Gaussian Elimination. In Section 6 we show that the human computable password scheme will be UF-RCA secure provided that RP-CSP holds and that f satisfies a few moderate properties (e.g., the output of f is evenly distributed).

Conjecture 2 (RP-CSP). For every probabilistic polynomial time adversary \mathcal{A} and every $\varepsilon, \varepsilon' > 0$ there is an integer N s.t. for all n > N, $m \le n^{\min\{r(f)/2,g(f)+1-\varepsilon'\}}$ we have $\Pr[\mathbf{Success}(\mathcal{A},n,m,\varepsilon)] \le \mu(n)$, where $\mathbf{Success}(\mathcal{A},n,m,\varepsilon)$ denotes the event that \mathcal{A} finds a mapping σ' that is ε -correlated with σ given m randomly selected challenge response pairs $(C_1, f(\sigma(C_1))), \ldots, (C_m, f(\sigma(C_m)))$ and $\mu(n)$ is a negligible function. The probability is over the selection of the random mapping σ , the challenges C_1, \ldots, C_m and the random coins of the adversary.

4 Candidate Secure Human Computable Functions

In this section we present a family of candidate human computable functions. We consider the usability of these human computable password schemes in Section 4.1, and we analyze the security of our schemes in Section 4.2.

We first introduce our family of candidate human computable functions (for all of our candidate human computable functions $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ we fix d = 10 because most humans are

used to performing arithmetic operations on digits). Given integers $k_1 > 0$ and $k_2 > 0$ we define the function $f_{k_1,k_2}: \mathbb{Z}_{10}^{10+k_1+k_2}$ as follows

$$f_{k_1,k_2}(x_0,\ldots,x_{9+k_1+k_2}) = x_j + \sum_{i=10+k_1}^{9+k_1+k_2} x_i \mod 10$$
, where $j = \left(\sum_{i=10}^{9+k_1} x_i\right) \mod 10$.

Authentication Process We briefly overview the authentication process — see Algorithms 2 and 3 in Appendix A for a more formal presentation of the authentication process. We assume that the mapping $\sigma: \{1,...,n\} \to \mathbb{Z}_{10}$ is generated by the user's local computer in secret. The user may be shown mnemonic helpers (see discussion below) to help memorize σ , but these mnemonic helpers are discarded immediately afterward. After the user has memorized σ he can create a password pw_i for an account A_i as follows: the user's local computer generates t random single-digit challenges $C_1^i, \ldots, C_t^i \in X_t$ and the user computes $pw_i = f(\sigma(C_1)), \ldots, f(\sigma(C_t))$. The server for account A_i stores the cryptographic hash of pw_i . The challenges $C_1^i, \ldots, C_t^i \in X_t$ are stored in public memory (e.g., on the server or on the user's local computer), which means that they can be viewed by the adversary as well as the legitimate user. To authenticate the user retrieves the public challenges C_1^i, \ldots, C_t^i for account A_i and computes pw_i . The server for A_i verifies that the cryptographic hash of pw_i matches its records. To protect users from offline attacks in the event of a server breach, the password p_i should be stored using a slow cryptographic hash function H like BCRYPT [49]. Servers could also use GOTCHAs [14] or HOSPs [23] for additional protection.

4.1 Usability

In our discussion of usability we consider (1) the time it would take a human user to compute a password, (2) the challenge of memorizing the secret mapping σ , and (2) the extra effort necessary that a user needs to spend rehearing σ to ensure that he remembers the secret mapping σ over time.

4.1.1 Computation Time

Given a challenge $C = (c_0, ..., c_{9+k_1+k_2}) \in X_{10+k_1+k_2}$ we can compute $f_{k_1,k_2}(\sigma(C))$ we compute $j = \sum_{i=10}^{9+k_1} \sigma(c_i) \mod 10$ using $k_1 - 1$ **Add** operations and k_1 **Recall** operations. We then execute **TableLookup** $(j, c_0, ..., c_9)$ to obtain c_j . Now we need k_2 **Add** operations and $k_2 + 1$ **Recall** operations to compute the final response $\sigma(c_j) + \sigma(c_{10+k_1}) + ... + \sigma(c_{9+k_1+k_2})$.

Fact 1. Let $P = \{ Add, Recall, Table Lookup \}$ then $f_{k_1,k_2} \circ \sigma$ is $(P, 2k_1 + 2k_2 + 1, 3)$ -computable.

Fact 1 and Conjecture 1 would imply that $f_{1,3}$ and $f_{2,2}$ are \hat{t} -human computable with $\hat{t} = 9$ seconds for humans H with $\gamma_H \le 1$. The functions $f_{1,3}$ and $f_{2,2}$ were both \hat{t} -human computable with $\hat{t} = 7.5$ seconds for the main author of this paper.

Discussion To help the user compute the response to a single digit challenge C more quickly the challenge can be displayed to the human in a more helpful manner. For example, if the user memorized a mapping $\sigma \in \mathbb{Z}_{10}^{26}$ from characters to digits then the challenge $C = ('E', 'F', 'G', 'H', 'I', 'J', 'K', 'L', 'M', 'N', 'A', 'B', 'C', 'D') \in X_{14}$ might be displayed to the user as in Table 2. Now to compute $f_{2,2}(\sigma('C'))$ the user would execute the following steps (1) Recall $\sigma('A')$ — the number associated with the letter A, (2) Recall $\sigma('B')$, (3) Compute $i = \sigma('A') + \sigma('B')$ mod 10



Figure 1: Mnemonics to help the user memorize the secret mapping σ

— without loss of generality suppose that i = 8, (4) Find the letter at index i—'M' if i = 8, (5) Recall $\sigma('M')$ (6) Recall $\sigma('C')$ (7) Compute $j = \sigma('M') + \sigma('C') \mod 10$ (8) Recall $\sigma('D')$ (9) Return $j + \sigma('D') \mod 10$.

4.1.2 Memorizing σ

Memorizing the secret mapping might be the most difficult part of our schemes. The primary author of this paper was able to memorize a mapping from n = 100 images to digits in about 2 hours. In practice, we envision that the user memorizes a mapping from n objects (e.g., images) to digits. For example, if n = 26 and d = 10 then the user might memorize a random mapping from characters to digits. We believe that the process could be sped up using mnemonic helpers. For example, to memorize the mapping $\sigma('D') = 2$ we might show a visually inclined user an animation of the letter 'D' transforming into a 2 (see Figure 1a). If instead $\sigma('D') = 9$ then we would show the user a different animation (see Figure 1b). One nice feature of this approach is that we only need to generate nd illustrations to help our users memorize any mapping (e.g., to help users memorize any mapping from characters to digits we would need just 260 such illustrations — 10 for each character).

4.1.3 Rehearsing σ

After the user memorizes σ he may need to rehearse parts of the mapping periodically to ensure that he does not forget it. How much effort does this require? Blocki et al.[16] introduced a usability model to estimate how much extra effort that a user would need to spend rehearsing the mapping σ . We used this model to obtain the predictions in Table 1.

Imagine that we had a program keep track of how often the user rehearsed each association $(i,\sigma(i))$ and predict how much longer the user will safely remember the association $(i,\sigma(i))$ without rehearsing again — updating this prediction after each rehearsal. The user rehearses the association $(i,\sigma(i))$ naturally whenever he needs to recall the value of $\sigma(i)$ while computing the password for any of his password protected accounts. If the user is in danger of forgetting the value $\sigma(i)$ then the program sends the user a reminder to rehearse (Blocki et al.[16] call this an extra rehearsal). Table 1 shows the value of $\mathbb{E}[ER_{365}]$, the expected number of extra rehearsals that the user will be required to do to remember the secret mapping σ during the first year. This value will depend on how often the user rehearses σ naturally. We consider four types of users: Active, Typical, Occasional and Infrequent. An Active user visits his accounts more frequently than a Infrequent user. See Appendix H for specific details about how Table 1 was computed.

Discussion One of the advantages of our human computable passwords schemes is that memorization is essentially a one time cost for our Very Active, Typical and Occasional users. Once the user has memorized the mapping $\sigma : \{1,...,n\} \to \mathbb{Z}_d$ he will get sufficient natural

	Our Scheme $(\sigma \in \mathbb{Z}_{10}^n)$			Shared Cues		
User	n = 100	n = 50	n = 30	SC-0	SC-1	SC-2
Very Active	0.396	0.001	≈ 0	≈ 0	3.93	7.54
Typical	2.14	0.039	≈ 0	≈ 0	10.89	19.89
Occasional	2.50	0.053	≈ 0	≈ 0	22.07	34.23
Infrequent	70.7	22.3	6.1	≈ 2.44	119.77	173.92

A	В	С	D
0	Е	5	J
1	F	6	K
2 3	G	7	L
3	Н	8	M
4	I	9	N

Table 1: $\mathbb{E}[ER_{365}]$: Extra Rehearsals over the first year to remember σ in our scheme with $f_{2,2}$ or $f_{1,3}$. Compared with Shared Cues schemes SC-0,SC-1 and SC-2[16].

Table 2: Single-Digit Challenge Layout in Scheme 1

rehearsal to maintain this memory. In fact, our schemes require the user to expend *less* extra effort rehearsing his secret mapping than the Shared Cues password management scheme of Blocki et al. [16] (with the exception of SC-0 – the least secure Shared Cues scheme). Intuitively, this is because human computable password schemes require to recall $\sigma(i)$ for multiple different values of i to respond to each single-digit challenge C. To compute $f_{2,2}(\sigma(\{0,...,13\}))$ the user would need to recall the values of $\sigma(10)$, $\sigma(11)$, $\sigma(12)$, $\sigma(13)$ and $\sigma(j)$, where $j = \sigma(10) + \sigma(11)$ mod 10. If the user has 10 digit passwords then he will naturally rehearse the value of $\sigma(i)$ for up to fifty different values of i each time he computes one of his passwords. While the user needs to spend extra time computing his password each time he authenticates in our human computable password scheme, this extra computation time gives the user more opportunities to rehearse his secret mapping.

4.2 Security Analysis

Claim 1 demonstrates that $s(f_{k_1,k_2}) = \min\{(k_2+1)/2, k_1+1\}$. Intuitively, the security of our human computable password management scheme will increase with k_1 and k_2 . However, the work that the user needs to do to respond to each single-digit challenge is proportional to $2k_1 + 2k_2 + 1$ (See Fact 1).

Claim 1. Let $0 < k_1 \le 10, 0 < k_2$ be given and let $f = f_{k_1,k_2}$ we have $g(f) = k_1$, $r(f) = k_2 + 1$ and $s(f) = \min\{\frac{k_2+1}{2}, k_1+1\}$.

An intuitive way to see that $r(f_{k_1,k_2}) > k_2$ is to observe that we cannot bias the output of f_{k_1,k_2} by fixing k_2 variables. Fix the value of any k_2 variables and draw the values for the other $k_1 + 10$ variables uniformly at random from \mathbb{Z}_{10} . One of the $k_2 + 1$ variables in the sum $x_j + \sum_{i=10+k_1}^{9+k_1+k_2} x_i$ mod 10 will not be fixed. Thus, the probability that the final output of $f_{k_1,k_2}(x_0,\ldots,x_{9+k_1+k_2})$ will be r is exactly 1/10 for each digit $r \in \mathbb{Z}_{10}$. Similarly, an intuitive way to see that $r(f_{k_1,k_2}) \le k_2 + 1$ is to observe that we can bias the value of $f_{k_1,k_2}(x_0,\ldots,x_{9+k_1+k_2})$ by fixing the value of $k_2 + 1$ variables. In particular if we fix the variables $x_0,x_{10+k_1},\ldots,x_{9+k_1+k_2}$ so that $0 = x_0 + \sum_{i=10+k_1}^{9+k_1+k_2} x_i$ mod 10 then the output of $f_{k_1,k_2}(x_0,\ldots,x_{9+k_1+k_2})$ is more likely to be 0 than any other digit. The full proof of Claim 1 can be found in Appendix F.

Theorem 1 states that our human computable password management scheme is UF-RCA secure as long as RP-CSP (Conjecture 2) holds. In Section 5 we provide strong evidence in support of RP-CSP. In particular, no statistical algorithm can approximately recover the secret mapping given $m = \tilde{O}\left(n^{r(f)/2}\right)$ challenge-response pairs. To prove Theorem 1 we need to show that an adversary that breaks UF-RCA security for f_{k_1,k_2} can be used to approximately recover the secret mapping σ . We prove a more general result in Section 6.

Theorem 1. Let $\epsilon, \epsilon' > 0, t \ge 1$ be given. Under the RP-CSP conjecture (Conjecture 2) the human computable password scheme defined by f_{k_1,k_2} is $\mathbf{UF} - \mathbf{RCA}(n,m,t,\delta) - secure$ for any $m \le n^{\min\{(k_2+1)/2,k_1+1-\epsilon'\}} - t$ and $\delta > \left(\frac{1}{10} + \epsilon\right)^t$.

Remark 1. In the Appendix we demonstrate that our security bounds are asymptotically tight. In particular, there is a statistical algorithm to break our human computable password schemes (f_{k_1,k_2}) which requires $m = \tilde{O}(n^{(k_2+1)/2})$ to 1-MSTAT to recover σ (See Theorem 7 in Section G). We also demonstrate that there is a attack based on Gaussian Elimination that uses $m = \tilde{O}(n^{k_1+1})$ challenge-response pairs to recover σ .

4.2.1 Exact Security Bounds

We used the Constraint Satisfaction Problem solver from the Microsoft Solver Foundations library to attack our human computable password scheme³. In each instance we generated a random mapping $\sigma: [n] \to \mathbb{Z}_{10}$ and m random challenge response pairs $(C, f(\sigma(C)))$ using the functions $f_{2,2}$ and $f_{1,3}$. We gave the CSP solver 2.5 days to find σ on a computer with a 2.83 GHz Intel Core2 Quad CPU and 4 GB of RAM. The full results of our experiments are in Appendix C. Briefly, the solver failed to find the random mapping in the following instances with $f = f_{2,2}$ and $f = f_{1,3}$: (1) n = 30 and m = 100, (2) n = 50 and m = 1,000 and (3) n = 100 and m = 10,000.

Remark 2. While the theoretical security parameter for $f_{1,3}$ ($s(f_{1,3}) = 2$) is slightly better than the security parameter for $f_{2,2}$ ($s(f_{2,2}) = 1.5$), we conjecture that $f_{2,2}$ may be more secure for small values of n (e.g., $n \le 100$) because it is less vulnerable to attacks based on Gaussian Elimination. In particular, there is a polynomial time attack on $f_{1,3}$ based on Gaussian Elimination that requires at most n^2 examples to recover σ , while the same attack would require n^3 examples with $f_{2,2}$. Our CSP solver was not able to crack $\sigma \in \mathbb{Z}_{10}^{100}$ given $10,000 = 100^2$ challenge response pairs with $f_{2,2}$.

Human Computable Password Challenge. We are challenging the security and cryptography community to break our human computable password scheme for instances that our CSP solver failed to crack (see Appendix B for more details about the challenge). Briefly, for each challenge we selected a random secret mapping $\sigma \in \mathbb{Z}_{10}^n$, and published (1) m single digit challenge-response pairs $(C_1, f(\sigma(C_1))), \ldots, (C_m, f(\sigma(C_m)))$, where each clause C_i is chosen uniformly at random from X_k , and (2) 20 length—10 password challenges $\vec{C}_1, \ldots, \vec{C}_{20} \in (X_k)^{10}$. The goal of each challenge is to correctly guess one of the secret passwords $p_i = f(\sigma(\vec{C}_i))$ for some $i \in [20]$. The challenges can be found at http://www.cs.cmu.edu/~jblocki/HumanComputablePasswordsChallenge/challenge.htm. We remark that these challenges remain unsolved even after they were presented during the rump sessions at a cryptography conference and a security conference[15].

5 Statistical Adversaries and Lower Bounds

Our main technical result (Theorem 3) is a lower bound on the number of single digit challengeresponse pairs that a statistical algorithm needs to see to (approximately) recover the secret mapping σ . Our results are quite general and may be of independent interest. Given *any* function $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ we prove that *any* statistical algorithm needs $\tilde{\Omega}\left(n^{r(f)/2}\right)$ examples before it can find a secret mapping $\sigma' \in \mathbb{Z}_d^n$ such that σ' is ϵ -correlated with σ . We first introduce statistical

³Thanks to David Wagner for suggesting the use of SAT solvers.

algorithms in Section 5.1 before stating our main lower bound for statistical algorithms in Section 5.2. We also provide a high level overview of our proof in Section 5.2.

5.1 Statistical Algorithms

A statistical algorithm is an algorithm that solves a distributional search problem \mathcal{Z} . In our case the distributional search problem $\mathcal{Z}_{\epsilon,f}$ is to find a mapping τ that is ϵ -correlated with the secret mapping σ given access to m samples from Q^f_{σ} – the distribution over challenge response pairs induced by σ and f. A statistical algorithm can access the input distribution Q^f_{σ} by querying the 1-MSTAT oracle or by querying the VSTAT oracle (Definition 5).

Definition 5. [29] [1-MSTAT(L) oracle and VSTAT oracle] Let D be the input distribution over the domain X. Given any function $h: X \to \{0,1,\ldots,L-1\}$, 1-MSTAT(L) takes a random sample x from D and returns h(x). For an integer parameter T > 0 and any query function $h: X \to \{0,1\}$, VSTAT(T) returns a value $v \in [p-\tau,p+\tau]$ where $p = \mathbb{E}_{x \sim D}[h(x)]$ and $\tau = \max\left\{\frac{1}{T}, \sqrt{\frac{p(1-p)}{T}}\right\}$.

In our context the domain $X = X_k \times \mathbb{Z}_d$ is the set of all challenge response pairs and the distribution $D = Q_{\sigma}^f$ is the uniform distribution over challenge-response pairs induced by σ and f. Feldman et al. [29] used the notion of statistical dimension (Definition 6) to lower bound the number of oracle queries necessary to solve a distributional search problem (Theorem 2). Before we can present the definition of statistical dimension we need to introduce the *discrimination norm*. Given a set $\mathcal{D}' \subseteq \mathbb{Z}_d^n$ of secret mappings the distributions the discrimination norm of \mathcal{D}' is denoted by $\kappa_2(\mathcal{D}')$ and defined as follows:

$$\kappa_2(\mathcal{D}') \doteq \max_{h, ||h||=1} \{ \mathbb{E}_{\sigma \sim \mathcal{D}'} [|\Delta(h, \sigma)|] \},$$

where
$$h: X_k \times \mathbb{Z}_d \to \mathbb{R}$$
, $||h|| \doteq \sqrt{\mathbb{E}_{(C,i) \sim X_k \times \mathbb{Z}_d} [h^2(C,i)]}$ and $\Delta(h,\sigma) \doteq \mathbb{E}_{C \sim X_k,i \sim \mathbb{Z}_d} [h(C,f(\sigma(C))) - h(C,i)]$.

Definition 6. $[29]^4$. For $\kappa > 0$, $\eta > 0$, $\varepsilon > 0$, let d' be the largest integer such that for any mapping $\sigma \in \mathbb{Z}_d^n$ the set $\mathcal{D}_{\sigma} = \mathbb{Z}_d^n \setminus \left\{ \sigma' \in \mathbb{Z}_d^n \mid \sigma' \text{ is } \varepsilon\text{-correlated with } \sigma \right\}$ has size at least $(1 - \eta) \cdot \left| \mathbb{Z}_d^n \right|$ and for any subset $\mathcal{D}' \subseteq D_{\sigma}$ where $|\mathcal{D}'| \ge |\mathcal{D}_{\sigma}|/d'$, we have $\kappa_2(\mathcal{D}') \le \kappa$. The **statistical dimension** with discrimination norm κ and error parameter η is d' and denoted by $SDN(\mathcal{Z}_{\varepsilon,f},\kappa,\eta)$.

Feldman et al. [29] proved the following lower bound on the number of 1-MSTAT(L) queries needed to solve a distributional search problem. Intuitively, Theorem 2 implies that many queries are needed to solve a distributional search problem with high statistical dimension. In Section 5.2 we argue that the statistical dimension our distributional search problem (finding σ' that is ϵ -correlated with the secret mapping σ given m samples from the distribution Q^f_σ) is high.

Theorem 2. [29, Theorems 10 and 12] For $\kappa > 0$ and $\eta \in (0,1)$ let $d' = \mathrm{SDN}(\mathcal{Z}_{\epsilon,f}, \kappa, \eta)$ be the statistical dimension of the distributional search problem $\mathcal{Z}_{\epsilon,f}$. Any randomized statistical algorithm that, given access to a $VSTAT\left(\frac{1}{3\kappa^2}\right)$ oracle (resp. 1-MSTAT(L)) for the distribution Q^f_{σ} for a secret mapping σ chosen randomly and uniformly from \mathbb{Z}^n_d , succeeds in finding a mapping $\tau \in \mathbb{Z}^n_d$ that is ϵ -correlated with σ with probability $\Lambda > \eta$ over the choice of distribution and internal randomness requires at least $\frac{\Lambda - \eta}{1 - \eta} d'$ (resp. $\Omega\left(\frac{1}{L}\min\left\{\frac{d'(\Lambda - \eta)}{1 - \eta}, \frac{(\Lambda - \eta)^2}{\kappa^2}\right\}\right)$) calls to the oracle.

⁴For the sake of simplicity we define the discrimination norm and the statistical dimension using our particular distributional search problem $\mathcal{Z}_{\epsilon,f}$. Our definitions is equivalent to the definition in [29] if we fix the reference distribution D to be the uniform distribution over all possible challenge response pairs $X_k \times \mathbb{Z}_d$.

As Feldman et al. [29] observe, almost all known algorithmic techniques can be modeled within the statistical query framework. In particular, techniques like Expectation Maximization[28], local search, MCMC optimization[33], first and second order methods for convex optimization, PCA, ICA, k-means can be modeled as a statistical algorithm even with L=2—see [18] and [25] for proofs. One issue is that a statistical simulation might need polynomially more samples. However, for L>2 we can think of our queries to 1-MSTAT(L) as evaluating L disjoint functions on a random sample. Indeed, Feldman et al. [29] demonstrate that there is a statistical algorithm for binary planted satisfiability problems using $\tilde{O}(n^{r(f)/2})$ calls to 1-MSTAT(L).

Remark 3. We can also use the statistical dimension to lower bound the number of queries that an algorithm would need to make to other types of statistical oracles to solve a distributional search problem. For example, we could also consider an oracle MVSTAT(L,T) that takes a query $h: X \to \{0,...,L-1\}$ and a set S of subsets of $\{0,...,L-1\}$ and returns a vector $v \in \mathbb{R}^L$ s.t for every $Z \in S$

$$\left| \sum_{i \in \mathbb{Z}} v[i] - p_{\mathbb{Z}} \right| \le \max \left\{ \frac{1}{T}, \sqrt{\frac{p_{\mathbb{Z}}(1 - p_{\mathbb{Z}})}{T}} \right\},$$

where $p_Z = \Pr_{x \sim D}[h(x) \in Z]$ and the cost of the query is |S|. Feldman et al. [29, Theorem 7] proved lower bounds similar to Theorem 2 for the MVSTAT oracle. In this paper we focus on the 1-MSTAT and VSTAT oracles for simplicity of presentation.

5.2 Statistical Dimension Lower Bounds

We are now ready to state our main technical result⁵.

Theorem 3. Let $\sigma \in \mathbb{Z}_d^n$ denote a secret mapping chosen uniformly at random, let Q_σ^f be the distribution over $X_k \times \mathbb{Z}_d$ induced by a function $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ with distributional complexity r = r(f). Any randomized statistical algorithm that finds an assignment τ such that τ is $\left(\sqrt{\frac{-2\ln(\eta/2)}{n}}\right)$ -correlated with σ with probability at least $\Lambda > \eta$ over the choice of σ and the internal randomness of the algorithm needs at least m calls to the 1-MSTAT(L) oracle (resp. $VSTAT\left(\frac{n^r}{2(\log n)^{2r}}\right)$ oracle) with $m \cdot L \ge c_1\left(\frac{n}{\log n}\right)^r$ (resp. $m \ge n^{c_1 \log n}$) for a constant $c_1 = \Omega_{k,1/(\Lambda - \eta)}(1)$. In particular if we set $L = \left(\frac{n}{\log n}\right)^{r/2}$ then our algorithms needs at least $m \ge c_1\left(\frac{n}{\log n}\right)^{r/2}$ calls to 1-MSTAT(L).

The proof of Theorem 3 follows from Theorems 4 and 2. Theorems 3 and 4 generalize results of Feldman et al. [29] which only apply for binary predicates $f : \{0,1\}^k \to \{0,1\}$. An interested reader can find our proofs in Appendix D. At a high level our proof proceeds as follows: Given any function $h: X_k \times \mathbb{Z}_d \to \mathbb{R}$ we show that $\Delta(\sigma, h)$ can be expressed in the following form:

$$\Delta(\sigma,h) = \sum_{\ell=r(f)}^{k} \frac{1}{|X_{\ell}|} b_{\ell}(\sigma) ,$$

where $|X_{\ell}| = \Theta(n^{\ell})$ and each function b_{ℓ} has degree ℓ (Lemma 2). We then use the general hypercontractivity theorem [47, Theorem 10.23] to obtain the following concentration bound.

⁵We remark that for our particular family of human computable functions f_{k_1,k_2} we could get a theorem similar to Theorem 3 by selecting $\sigma \sim \{0,5\}^n$ and appealing directly to results of Feldman et al. [29]. However, this theorem would be weaker than Theorem 3 as it would only imply that a statistical algorithm cannot find an assignment σ' that is $\frac{1}{2} - \frac{1}{10} + \epsilon$ -correlated with σ for $\epsilon > 0$. In contrast, our theorem implies that we cannot find σ' that is ϵ -correlated.

Lemma 1. Let $b: \mathbb{Z}_d^n \to \mathbb{R}$ be any function with degree at most ℓ , and let $\mathcal{D}' \subseteq \mathbb{Z}_d^n$ be a set of assignments for which $d' = d^n / |\mathcal{D}'| \ge e^{\ell}$. Then $\mathbb{E}_{\sigma \sim \mathcal{D}'}[|b(\sigma)|] \le \frac{2(\ln d'/c_0)^{\ell/2}}{d^{\ell}}||b||_2$, where $c_0 = \ell\left(\frac{1}{2ed}\right)$ and $||b||_2 = \sqrt{\mathbb{E}_{x \sim \mathbb{Z}_d^n}[b(x)^2]}$.

We then use Lemma 1 to bound $\mathbf{E}_{\sigma \sim \mathcal{D}'}[\Delta(\sigma,h)]$ for any set $\mathcal{D}' \subseteq \mathbb{Z}_d^n$ such that $|\mathcal{D}'| = |\mathbb{Z}_d^k|/d'$ (Lemma 5). This leads to the following bound on $\kappa_2(\mathcal{D}') = O_k((\ln d'/n)^{r(f)/2})$.

Theorem 4. There exists a constant $c_O > 0$ such that for any $\epsilon > 1/\sqrt{n}$ and $q \ge n$ we have

$$SDN\left(Z_{\epsilon,f}, \frac{c_Q(\log q)^{r/2}}{n^{r/2}}, 2e^{-n\cdot\epsilon^2/2}\right) \ge q$$
,

where r = r(f) is the distributional complexity of f.

Discussion We view Theorem 3 as strong evidence for RP-CSP (Conjecture 2) because almost all known algorithmic techniques can be modeled within the statistical query framework. In particular, techniques like Expectation Maximization[28], local search, MCMC optimization[33], first and second order methods for convex optimization, PCA, ICA, k-means can be modeled as a statistical algorithm even with L=2— see [18] and [25] for proofs. This implies that many popular heuristic based SAT solvers (e.g., DPLL[27]) will not be able to recover σ in polynomial time. While Gaussian Elimination is a notable exception that cannot be simulated by a statistical algorithm, our composite security parameter $s(f) \ge g(f) + 1$ accounts for attacks based on Gaussian Elimination. In particular, a polynomial time algorithm needs $m = \tilde{O}(n^{g(f)+1})$ examples to extract O(n) linear constraints and solve for σ (see Appendix G.2).

6 Security Analysis

In the last section we presented evidence in support of RP-CSP (Conjecture 2) by showing that any statistical adversary needs $m = \tilde{\Omega}\left(n^{r(f)/2}\right)$ examples to (approximately) recover σ . However, RP-CSP only says that it is hard to (approximately) recover the secret mapping σ , not that it is hard to forge passwords. As an example consider the following NP-hard problem from learning theory: find a 2-term DNF that is consistent with the labels in a given dataset. Just because 2-DNF is hard to learn in the *proper* learning model does not mean that it is NP-hard to learn a good classifier for 2-DNF. Indeed, if we allow our learning algorithm to output a linear classifier instead of a 2-term DNF then 2-DNF is easy to learn [39]. Could an adversary win our password security game without properly learning the secret mapping?

Theorem 5, our main result in this section, implies that the answer is no. Informally, Theorem 5 states that any adversary that breaks UF-RCA security of our human computable password scheme f_{k_1,k_2} can also (approximately) recover the secret mapping σ . This implies that our human computable password scheme is UF-RCA secure as long as RP-CSP holds. Of course, for some functions it is very easy to predict challenge-response pairs without learning σ . For example, if f is the constant function — or any function highly correlated with the constant function — then it is easy to predict the value of $f(\sigma(C))$. However, any function that is highly correlated with a constant function is a poor choice for a human computable passwords scheme. We argue that any adversary that can win the password game can be converted into an adversary that properly learns σ provided that the output of function f is evenly distributed (Definition 7).

Definition 7. We say that the output of a function $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ is evenly distributed if there exists a function $g: \mathbb{Z}_d^{k-1} \to \mathbb{Z}_d$ such that $f(x_1, ..., x_k) = g(x_1, ..., x_{k-1}) + x_k \mod d$.

Clearly, our family f_{k_1,k_2} has evenly distributed output. To see this we simply set $g = f_{k_1,k_2-1}$. We are now ready to state our main result from this section.

Theorem 5. Suppose that f has evenly distributed output, but that f is not $\mathbf{UF}-\mathbf{RCA}(n,m,t,\delta)$ —secure for $\delta > \left(\frac{1}{d} + \epsilon\right)^t$. Then there is a probabilistic polynomial time algorithm (in n, m, t and $1/\epsilon$) that extracts a string $\sigma' \in \mathbb{Z}_d^n$ that is $\epsilon/8$ -correlated with σ with probability at least $\frac{\epsilon^3}{(8d)^2}$ after seeing m+t example challenge response pairs.

The proof of Theorem 5 is in Appendix E. We overview the proof here. The proof of Theorem 5 uses Theorem 6 as a subroutine. Theorem 6 shows that we can, with reasonable probability, find a mapping σ' that is correlated with σ given predictions of $f(\sigma(C))$ for each clause as long as the probability that each prediction is accurate is slightly better than a random guess (e.g., $\frac{1}{d} + \delta$). The proof of Theorem 6 is in Appendix E.

Theorem 6. Let f be a function with evenly distributed output (Definition 7), let $\sigma \sim \mathbb{Z}_d^n$ denote the secret mapping, let $\epsilon > 0$ be any constant and suppose that for every $C \in X_k$ we are given labels $\ell_C \in \mathbb{Z}_d$ s.t.

$$\Pr_{C \sim X_k} [f(\sigma(C)) = \ell_C] \ge \frac{1}{d} + \epsilon.$$

There is a polynomial time algorithm (in n, m, $1/\epsilon$) that finds a mapping $\sigma' \in \mathbb{Z}_d^n$ such that σ' is $\epsilon/2$ -correlated with σ with probability at least $\frac{\epsilon}{2d^2}$

The remaining challenge in the proof of Theorem 5 is to show that there is an efficient algorithm to extract predictions of $f(\sigma(C))$ given blackbox access to an adversary $\mathcal A$ that breaks UF-RCA security. However, just because the adversary $\mathcal A$ gives the correct response to an entire password challenge C_1, \ldots, C_t with probability greater than $\left(\frac{1}{d} + \epsilon\right)^t$ does not mean that the response to each individual challenge C is correct with probability $\frac{1}{d} + \epsilon$. To obtain our predictions for individual clauses C we draw t extra example challenge response pairs $\left(C_1', f\left(\sigma\left(C_1'\right)\right)\right), \ldots, \left(C_t', f\left(\sigma\left(C_t'\right)\right)\right)$, which we use to check the adversary. To obtain the label for a clause C we select a random index $i \in [t]$ and give $\mathcal A$ the password challenge C_1', \ldots, C_t' , replacing C_i' with C. If for some j < i the label for clause C_j' is not correct (e.g., $\neq f\left(\sigma\left(C_j'\right)\right)$) then we discard the label and try again. Claim 2 in Appendix E shows that this process will give us predictions for individual clauses that are accurate with probability at least $\frac{1}{d} + \epsilon$.

7 Discussion

7.1 Improving Response Time

The easiest way to improve response time is to decrease t –the number of single-digit challenges that the user needs to solve. However, if the user wants to ensure that each of his passwords are strong enough to resist offline dictionary attacks then he would need to select a larger value of t (e.g., $t \ge 10$). Fortunately, there is a natural way to circumvent this problem. The user could save time by memorizing a mapping $w : \mathbb{Z}_{10} \to \{x \mid x \text{ is one of } 10,000 \text{ most common english words} \}$ and responding to each challenge C with $w(f(\sigma(C)))$ – the word corresponding to the digit $f(\sigma(C))$. Now the user can create passwords strong enough to resist offline dictionary attacks by responding to just 3–5 challenges. Even if the adversary learns the words in the user's set

he won't be able to mount online attacks. Predicting $w(f(\sigma(C)))$ is at least as hard as predicting $f(\sigma(C))$ even if the adversary knows the exact mapping w^6 .

7.2 One-Time Challenges

Malware Consider the following scenario: the adversary infects the user's computer with a keylogger which is never detected. Every time the user computes a password in response to a challenge the adversary observes the password in plaintext. One way to protect the user in this extreme scenario would be to generate multiple (e.g., 10^6) one-time passwords for each of the user's accounts. When we initially generate the secret mapping $\sigma \sim \mathbb{Z}_d^n$ we could also generate cryptographic hashes for millions of one-time passwords $\mathbf{H}(\vec{C}, f_{k_1,k_2}(\sigma(\vec{C})))$. While usability concerns make this approach infeasible in a traditional password scheme (it would be far too difficult for the user to memorize a million one-time passwords for each of his accounts), it may be feasible to do this using a human computable password scheme. In our human computable password scheme we could select k_1 and k_2 large enough that $s(f_{k_1,k_2}) = \min\{k_1+1,(k_2+1)/2\} \ge 6$. Assuming that the user authenticates fewer than 10^6 times over his lifetime a polynomial time adversary would never obtain enough challenge-response examples to learn σ . The drawback is that f_{k_1,k_2} will take longer for a user to execute in his head.

Secure Cryptography in a Panoptic World Standard cryptographic algorithms could be easily broken in a panoptic world where the user only has access to a semi-trusted computer (e.g., if a user asks a semi-trusted computer to sign an message m using a secret key sk stored on the hard drive then the computer will respond with the correct value Sign(sk, m), but the adversary will learn the values m and sk). Our human computable password schemes could also be used to secure some cryptographic operations in a panoptic world assuming that an initialization phase can be executed in secret. The basic idea is to obfuscate a "password locked" circuit $P_{\sigma,sk,r}$ that can encrypt and sign messages under a secret key sk^{7} . After this secret initialization phase is completed the code for the obfuscated program $O(P_{\sigma,sk,r})$ is published. To sign a message m the user will ask $P_{\sigma,sk,r}$ to sign m and $P_{\sigma,sk,r}$ will respond by generating a one time challenge C_1, \ldots, C_t using random bits from a cryptographic pseudorandom number generator **PRF** (r, m). $P_{\sigma, sk, r}$ will only sign the message if the user can provide the correct response $f_{k_1,k_2}(\sigma(C_1)), \ldots, f_{k_1,k_2}(\sigma(C_1))$. Because $P_{\sigma,sk,r}$ generates a unique password challenge for each request the adversary will not be able to sign messages even if he obtains the obfuscated program $O(P_{\sigma,sk,r})$ — unless the user has already signed that particular message or the adversary has learned σ . However, we could select k_1 and k_2 large enough that a polynomial time adversary would never obtain enough example challenge-response pairs over the user's lifetime to forge passwords. $P_{\sigma,sk,r}$ could also be used to perform other cryptographic operations like decryption for the user. However, in a panoptic world we could only guarantee that the adversary will not be able to obtain *m* from an encrypted message e = Enc(pk, m) before the user provides the password for $P_{\sigma, sk, r}$ to execute Dec(sk,e).

⁶In fact, it is quite likely that it is much harder for the adversary to predict $w(f(\sigma(C)))$ because the adversary will not see which word corresponds with each digit.

⁷Garg et al. recently presented a candidate polynomial time indistinguishability obfuscator [32]. While the obfuscation algorithm of Garg et al. [32] is not yet efficient enough for practical applications this example shows that secure cryptography is theoretically possible in a panoptic world.

7.3 Open Questions

Eliminating the Semi-Trusted Computer Our current scheme relies on a semi-trusted computer to generate and store random public challenges. An adversary with full control over the user's computer might be able to extract the user's secret if he is able to see the user's responses to O(n) adaptively selected password challenges. Can we eliminate the need for a semi-trusted computer? One possibility would be to develop a human computable password scheme that can transform words into passwords. In this case we might be able eliminate the need for a semi-trusted computer by adopting the convention that the name of each site is the challenge (e.g., "Amazon" would be the challenge for the user's Amazon.com).

Cryptography for Humans? We presented a scheme that allows a human to create multiple passwords from a short secret by responding to public challenges. What other cryptographic functions could be performed by humans? Could we develop a human computable protocol for bit commitment or coin flipping? What about a human computable encryption scheme?

Exact Security Bounds While we provided asymptotic security proofs for our human computable password schemes, it is still important to understand how much effort an adversary would need to expend to crack the secret mapping for specific values of n and m. We present some results in Appendix C, but we view these results as preliminary. These results indicate that the value n = 26 is too small to provide UF-RCA security even with small values of m (e.g., m = 50). However, the problem gets harder quickly as n increases (e.g., our CSP solver was not able to find the secret mapping when n = 50 even with m = 1000 examples). We also present a public challenge with specific values of n and m to encourage cryptography and security researchers to attack our scheme.

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Figure 2: Secret Random Mapping from Pictures to Digits







(b) Mnemonic to help the user (c) Mnemonic to help the user remember remember

(a) Original photo (an eagle). $\sigma(eagle) = 2$.

 $\sigma(eagle) = 6.$

Figure 3: Mnemonics to help memorize the secret mapping σ

Authentication Process A

In this section of the appendix we illustrate our human computable password schemes graphically. In our examples, we use the function $f = f_{2,2}$. To compute the response $f(\sigma(C))$ to a challenge $C = \{x_0, ..., x_{13}\}$ the user computes $f(\sigma(C)) = \sigma(x_{\sigma(x_{10}) + \sigma(x_{11}) \mod 10}) + \sigma(x_{12}) + \sigma(x_{13})$ mod 10

Memorizing a Random Mapping. To begin using our human computable password schemes the user begins by memorizing a secret random mapping $\sigma: [n] \to \{0, ..., 9\}$ from *n* objects (e.g., letters, pictures) to digits. See Figure 2 for an example.

The computer can provide the user with mnemonics to help memorize the secret mapping σ — see Figures 3b and 3c. For example, if we wanted to help the user remember that $\sigma(eagle) = 2$ we would show the user Figure 3b. We observe that a $10 \times n$ table of mnemonic images would be sufficient to help the user memorize any random mapping σ . We stress that the computer will only save the original image (e.g., Figure 3a). The mnemonic image (e.g., Figure 3b or 3c) would be discarded after the user memorizes $\sigma(eagle)$.

Single-Digit Challenges. In our scheme the user computes each of his passwords by responding to a sequence of single-digit challenges. For $f = f_{2,2}$ a single-digit challenge is a tuple $C \in [n]^{14}$ of fourteen objects. See Figure 4 for an example. To compute the response $f(\sigma(C))$ to a challenge $C = \{x_0, ..., x_{13}\}$ the user computes $f(\sigma(C)) = \sigma(x_{\sigma(x_{10}) + \sigma(x_{11}) \mod 10}) + \sigma(x_{12}) + \sigma(x_{13})$ mod 10. Observe that this computation involves just three addition operations modulo ten. See Figure 5 for an example. In this example the response to the challenge $C = \{x_0 = \text{burger}, x_1 = \text{burger}, x_1 = \text{burger}, x_2 = \text{burger}, x_3 = \text{burger}, x_4 = \text{b$



Figure 4: A single-digit challenge

eagle,..., x_{10} = lightning, x_{11} = dog, x_{12} = man standing on world, x_{13} = kangaroo} is $f(\sigma(C)) = \sigma(x_{\sigma(x_{10}) + \sigma(x_{11}) \mod 10}) + \sigma(x_{12}) + \sigma(x_{13}) \mod 10$ $= \sigma(x_{\sigma(\text{lightning}) + \sigma(\text{dog}) \mod 10}) + \sigma(\text{man standing on world}) + \sigma(\text{kangaroo}) \mod 10$ $= \sigma(x_{9+3 \mod 10}) + \sigma(\text{man standing on world}) + \sigma(\text{kangaroo}) \mod 10$ $= \sigma(\text{minions}) + \sigma(\text{man standing on world}) + \sigma(\text{kangaroo}) \mod 10$ $= 7 + 4 + 5 \mod 10 = 6.$

We stress that this computation is done entirely in the user's head. It takes the main author 7.5 seconds on average to compute each response.

Creating an Account. To help the user create an account the computer would first pick a sequence of single-digit challenges $C_1, ..., C_t$, where the security parameter is typically t = 10, and would display the first challenge C_1 to the user — see Figure 6 for an example. To compute the first digit of his password the user would compute $f(\sigma(C_1))$. After the user types in the first digit $f(\sigma(C_1))$ of his password the computer will display the second challenge C_2 to the user — see Figure 7. After the user creates his account the computer will store the challenges $C_1, ..., C_{10}$ in public memory. The password $pw = f(\sigma(C_1))...f(\sigma(C_t))$ will not be stored on the user's local computer (the authentication server may store the cryptographic hash of pw).

Authentication. Authenticating is very similar to creating an account. To help the user recompute his password for an account the computer first looks up the challenges C_1, \ldots, C_t which were stored in public memory, and the user authenticates by computing his password $pw = f(\sigma(C_1)) \ldots f(\sigma(C_t))$. We stress that the single-digit challenges the user sees during authentication will be the same single-digit challenges that the user saw when he created the account. The authentication server verifies that the cryptographic hash of pw matches its record.

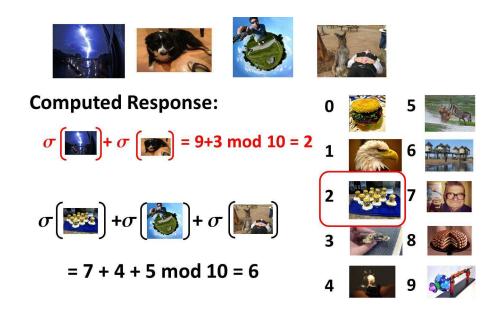


Figure 5: Computing the response $(f(\sigma(C)) = 6)$ to a single-digit challenge



Figure 6: Login Screen

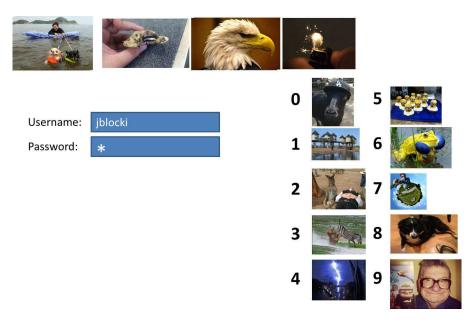


Figure 7: Login Screen after the user responds to the first single-digit challenge

Helping the user remember his secret mapping. The computer keeps track of when the user rehearses each value of his secret mapping (e.g., $(i, \sigma(i))$) for each $i \in [n]$), and reminds the user to rehearse any part of his secret mapping that he hasn't used in a long time. One advantage of our human computable password scheme (compared with the Shared Cues scheme of Blocki et al.[16]) is that most users will use each part of their secret mapping often enough that they will not need to be reminded to rehearse — see discussion in Section 4.1. The disadvantage is that we require the user to spend extra effort computing his passwords each time he authenticates.

A.1 Formal View

We now present a formal overview of the authentication process. Algorithm 1 outlines the initialization process in which the user memorizes a secret random mapping σ generated by the user's computer, and Algorithm 2 outlines the account creation process. In Algorithm 2 the user generates the password for an account i by computing the response to a sequence of random challenges C generated by the user's computer. The sequence of challenges are stored in public memory. We assume that all steps in algorithms 1, 2 and 3 are executed on the user's local computer unless otherwise indicated. We also assume that the initialization phase (Algorithms 1 and 2) is carried out in secret (e.g., we assume that the secret mapping is chosen in secret), but we do not assume that the challenges are kept secret. We use (User) to denote a step completed by the human user and we use (Server) to denote a step completed by a third-party server. Algorithm 3 illustrates the authentication process.

Algorithm 1 MemorizeMapping

```
Input: n, base d, random bits b, images I_1,...,I_n, and mnemonic helpers M_{i,j} for i \in [n], j \in \{0,...,d-1\}. 

ho Generate and Memorize Secret Mapping for i = 1 \rightarrow n do \sigma(i) \sim \{0,...,d-1\} %Using random bits b M_i \leftarrow M_{i,\sigma(i)} (User) Using M_i memorizes the association (I_i,\sigma(i)) for i \in [n].
```

Algorithm 2 CreateChallenge

```
Input: n, i, base d, random bits b and images I_1, ..., I_n.

Generate Password Challenge for account A_i

for j = 1 \rightarrow t do

C_j^i \sim X_k %Using random bits b

\vec{C}_i \leftarrow \left\langle C_1^i, ..., C_t^i \right\rangle

Store record (i, \vec{C}_i)

for j = 1 \rightarrow t do

Load images from C_j^i.

Display C_j^i for the user.

(User) Computes q_j \leftarrow f\left(\sigma\left(C_i^j\right)\right).

Send \langle q_1, ..., q_t \rangle = f\left(\sigma\left(\vec{C}_i\right)\right) to server i.

\Rightarrow H is a strong cryptographic hash function (Server i) Stores h_i = \mathbf{H}\left(\vec{C}_i, \langle q_1, ..., q_t \rangle\right)
```

Algorithm 3 Authenticate

```
Input: Security parameter t. Account i \in [m]. Challenges \vec{C}_1, \ldots, \vec{C}_m. \langle C_1^i, \ldots, C_t^i \rangle \leftarrow \vec{C}_i \Rightarrow Display Single Digit Challenges for j = 1 \rightarrow t do (Semi-Trusted Computer) Load images from C_j^i. (Semi-Trusted Computer) Displays C_j^i to the user. (User) Computes q_j \leftarrow f\left(\sigma\left(C_i^j\right)\right). (Semi-Trusted Computer) Sends \langle q_1, \ldots, q_t \rangle to the server for account i. (Server) Verifies that \mathbf{H}\left(\vec{C}_i, \langle q_1, \ldots, q_t \rangle\right) = h_i
```

B Human Computable Passwords Challenge

While we provided asymptotic security bounds for our human computable password schemes in our context it is particularly important to understand the constant factors. In our context, it may be reasonable to assume that $n \le 100$ (e.g., the user may be unwilling to memorize longer mappings). In this case it would be feasible for the adversary to execute an attack that takes time proportional to $10^{\sqrt{n}} \le 10^{10}$. We conjecture that in practice scheme 2 ($f_{1,3}$) is slightly weaker than scheme 1 ($f_{2,2}$) when $n \le 100$ despite the fact that $s(f_{2,2}) < s(f_{1,3})$ because of the

	Scheme 1 ($f_{2,2}$)		Scheme 2 $(f_{1,3})$	
n	m	Winner	m	Winner
	1000	N/A	500	N/A
100 digits	500	N/A	300	N/A
	300	N/A	200	N/A
50 digits	500	N/A	300	N/A
	300	N/A	150	N/A
	150	N/A	100	N/A
	300	CSP Solver	150	N/A
30 digits	100	N/A	100	N/A
	50	N/A	50	N/A

Table 3: Human Computable Password Challenges n — Secret Length m—# Challenge-Response Pairs

attack described in Remark 5. This attack requires $\tilde{O}(n^{1+g(f)/2})$ examples, and the running time $O\left(10^{\sqrt{n}}n^3\right)$ may be feasible for $n \leq 100$. To better understand the exact security bounds we created several public challenges for researchers to break our human computable password schemes under different parameters (see Table 3). At this time these challenges remain unsolved even after they were presented during the rump sessions at a cryptography conference and a security conference[15]. The challenges can be found at http://www.cs.cmu.edu/~jblocki/HumanComputablePasswordsChallenge/challenge.htm. For each challenge we selected a random secret mapping $\sigma \in \mathbb{Z}_{10}^n$, and published (1) m single digit challenge-response pairs $(C_1, f(\sigma(C_1))), \ldots, (C_m, f(\sigma(C_m)))$, where each clause C_i is chosen uniformly at random from X_k , and (2) 20 length—10 password challenges $\vec{C}_1, \ldots, \vec{C}_{20} \in (X_k)^{10}$. The goal is to guess one of the secret passwords $p_i = f\left(\sigma(\vec{C}_i)\right)$ for some $i \in [20]$.

C CSP Solver Attacks

While Theorems 3 and 5 provide asymptotic security bounds (e.g., an adversary needs to see $m = \tilde{\Omega}(n^{s(f)})$ challenge-response pairs to forge passwords), in our context it is useful to have exact security bounds. To better understand the exact security bounds of our human computable password schemes we (1) used a Constraint Satisfaction Problem (CSP) solver to attack our scheme, and (2) created several public challenges to break our candidate human computable password schemes (see Table 3 in Appendix B).

CSP Solver Our computations were performed on a computer with a 2.83 GHz Intel Core2 Quad CPU and 4 GB of RAM. In each instance we generated a random mapping $\sigma : [n] \to \mathbb{Z}_{10}$ and m random challenge response pairs $(C, f(\sigma(C)))$ using the functions $f_{2,2}$ and $f_{1,3}$. We used the Constraint Satisfaction Problem solver from the Microsoft Solver Foundations library to try to solve for σ^8 . The results of this attack are shown in Tables 4 and 5. Due to limited computational resources we terminated each instance if the solver failed to find the secret mapping within 2.5 days, and if our solver failed to find σ in 2.5 days on and instance (n,m) we did not run the solver on strictly harder instances (e.g., (n',m')) with $n' \ge n$ and $m' \le m$).

⁸http://blogs.msdn.com/b/solverfoundation/ (Retrieved 9/15/2014).

	m = 50	m = 100	m = 300	m = 500	m = 1000	m = 10000
n = 26	23.5 hr	40 min	4.5 hr	29 min	10 min	2 min
n = 30	HARD	UNSOLVED	2.33 hr	35.5 min	10 min	20 s
n = 50	HARD	HARD	HARD	HARD	UNSOLVED	7 hr
n = 100	HARD	HARD	HARD	HARD	HARD	UNSOLVED

Table 4: CSP Solver Attack on $f_{2,2}$

Key: UNSOLVED – Solver failed to find solution in 2.5 days; HARD – Instance is harder than an unsolved instance;

	m = 50	m = 100	m = 300	m = 500	m = 1000	m = 10000
n = 26	8.7 hr	53 min	1.33 hr	13.5 min	6.3min	2 min
n = 30	HARD	UNSOLVED	1 hr	41 min	2 min	15 s
n = 50	HARD	HARD	HARD	HARD	UNSOLVED	6.5 hr
n = 100	HARD	HARD	HARD	HARD	HARD	UNSOLVED

Table 5: CSP Solver Attack on $f_{1,3}$

Key: UNSOLVED – Solver failed to find solution in 2.5 days; HARD – Instance is harder than an unsolved instance;

We remark that the running time of our CSP solver increases exponentially in n (e.g., when we decrease n from 30 to 26 with m = 100 examples the CSP solver can solve for $\sigma \in \mathbb{Z}_{10}^{26}$ in 40 minutes, while the solver failed to find $\sigma \in \mathbb{Z}_{10}^{30}$ in 2.5 days.).

D Statistical Dimension

Our statistical dimension lower bounds closely mirror the lower bounds from [29] for binary predicates. In particular Lemmas 2, 3, 1, 4 and 5 are similar to Lemmas 2, 4, 5, 6 and 7 from [29] respectively. While the high level proof strategy is very similar, our proof does requires new techniques. In particular, because we are working with planted solutions $\sigma \in \mathbb{Z}_d^n$, instead of $\sigma \in \{\pm 1\}^n$, we need to use different Fourier basis functions. We use the basis functions χ_α where for $\alpha \in \mathbb{Z}_d^n$ is

$$\chi_{\alpha}(x) = \exp\left(\frac{-2\pi\sqrt{-1}(x\cdot\alpha)}{d}\right).$$

While the Fourier coefficients \hat{b}_{α} of a function $b: \mathbb{Z}_d^k \to \mathbb{R}$ might include complex numbers, Parseval's identity still applies: $\sum_{\alpha \in \mathbb{Z}_d^k} \left| \hat{b}_{\alpha} \right|^2 = \mathbb{E}_{x \sim \mathbb{Z}_d^k} \left| b(x)^2 \right|$. We first consider the following distributional search problem $\mathcal{Z}_{\epsilon,f,j}$: find σ' that is ϵ -correlated with σ given m randomly chosen challenge clauses from the distribution $Q_{\sigma}^{f,j}$ for $j \in \mathbb{Z}_d$. Remark 4 explains how to generalize our results to the problem $(\mathcal{Z}_{\epsilon,f})$ that we are interested in: find σ' that is ϵ -correlated with σ given m randomly chosen challenge-response pairs from the distribution Q_{σ}^f . In this section we let U_k denote the uniform distribution over X_k .

Definition 8. [29] Given a clause $C \in X_k$ and $S \subseteq [k]$ of size ℓ , we let $C_{|S} \in X_\ell$ denote the clause of variables of C at the positions with indices in S (e.g., if C = (1,...,k) and $S = \{1,5,k-2\}$ then $C_{|S} = (1,5,k-2) \in X_3$). Given a function $h: X_k \to \mathbb{R}$ and a clause $C_\ell \in X_\ell$ we define

$$h_{\ell}(C_{\ell}) = \frac{|X_{\ell}|}{|X_{k}|} \sum_{S \subseteq [k], |S| = \ell, C \in X_{k}, C_{|S} = C_{\ell}} h(C) \ .$$

We first show that $\Delta^j(\sigma,h) = \mathbb{E}_{C \sim Q^{f,j}_\sigma}[h(C)] - \mathbb{E}_{C \sim X_k}[h(C)]$ can be expressed in terms of the Fourier coefficients of \hat{Q} as well as the functions h_ℓ . In particular, we define the degree ℓ function $b_\ell : \mathbb{Z}_d^n \to \mathbb{C}$ as follows

$$b_{\ell}(\sigma) \doteq \sum_{\alpha \in \mathbb{Z}_{d}^{k}: H(\alpha) = \ell} \hat{Q}_{\alpha} \sum_{C_{\ell} \in X_{\ell}} \chi_{\alpha}(\sigma(C_{\ell})) h_{\ell}(C_{\ell}) .$$

Notice that if Q has distributional complexity r and $\ell \le r$ then $b_{\ell}(\sigma) = 0$ because $\hat{Q}_{\alpha} = 0$ for all $\alpha \in \mathbb{Z}_d^k$ s.t. $1 \le H(\alpha) \le r$. This means that first r terms of the sum in Lemma 2 will be zero.

Lemma 2. For every $\sigma \in \mathbb{Z}_{d'}^k$ $j \in \mathbb{Z}_d$ and $h: X_k \to \mathbb{R}$

$$\Delta^{j}(\sigma,h) = \sum_{\ell=1}^{k} \frac{1}{|X_{\ell}|} b_{\ell}(\sigma) .$$

Proof. To simplify notation we set $Q = Q^{f,j}$. We have

$$\mathbb{E}_{Q_{\sigma}}[h] = \sum_{C \in X_{k}} h(C) \cdot Q_{\sigma}(C)$$

$$= \frac{1}{|X_{k}|} \sum_{C \in X_{k}} h(C) \cdot Q(\sigma(C))$$

$$= \frac{1}{|X_{k}|} \sum_{C \in X_{k}} h(C) \sum_{\alpha \in \mathbb{Z}_{d}^{k}} \hat{Q}_{\alpha} \chi_{\alpha}(\sigma(C))$$

$$= \frac{1}{|X_{k}|} \sum_{\alpha \in \mathbb{Z}_{d}^{k}} \hat{Q}_{\alpha} \sum_{C \in X_{k}} h(C) \chi_{\alpha}(\sigma(C))$$

$$= \frac{1}{|X_{k}|} \sum_{\ell=0}^{k} \sum_{\alpha \in \mathbb{Z}_{d}^{k} : H(\alpha) = \ell} \hat{Q}_{\alpha} \sum_{C \in X_{k}} h(C) \chi_{\alpha}(\sigma(C))$$

Observe that whenever $\alpha = \vec{0}$ we have

$$\hat{Q}_{\alpha} = \mathbb{E}_{x \sim \mathbb{Z}_d^k} [Q(x) \chi_{\alpha}(x)] = \mathbb{E}_{x \sim \mathbb{Z}_d^k} [Q(x)] = \sum_{x \in \mathbb{Z}_d^k} \frac{Q(x)}{d^k} = 1.$$

Therefore, for $\ell = 0$ in the above sum we have

$$\frac{1}{|X_k|} \hat{Q}_{\alpha} \sum_{C \in X_k} h(C) \chi_{\alpha}(\sigma(C)) = \frac{1}{|X_k|} \hat{Q}_{\vec{0}} \sum_{C \in X_k} h(C)$$

$$= \frac{1}{|X_k|} \sum_{C \in X_k} h(C)$$

$$= \mathbb{E}_{C \sim X_k} [h(C)]$$

$$= \mathbb{E}_{U_k} [h] .$$

Therefore,

$$\begin{split} \mathbb{E}_{C \sim Q_{\sigma}}[h(C)] - \mathbb{E}_{C \sim X_{k}}[h(C)] &= \frac{1}{|X_{k}|} \sum_{\ell=1}^{k} \sum_{\alpha \in \{0, \dots, d-1\}^{k}: H(\alpha) = \ell} \hat{Q}_{\alpha} \sum_{S \subseteq [k], |S| = \ell} \sum_{C \in X_{k}, C_{|S} = C_{\ell}} h(C) \chi_{\alpha}(\sigma(C)) \\ &= \frac{1}{|X_{k}|} \left(\sum_{\ell=1}^{k} \sum_{\alpha \in \{0, \dots, d-1\}^{k}: H(\alpha) = \ell} \hat{Q}_{\alpha} \sum_{C_{\ell} \in X_{\ell}} \chi_{\alpha}(\sigma(C_{\ell})) \sum_{S \subseteq [k], |S| = \ell} \sum_{C \in X_{k}, C_{|S} = C_{\ell}} h(C) \right) \\ &= \frac{1}{|X_{k}|} \left(\sum_{\ell=1}^{k} \sum_{\alpha \in \mathbb{Z}_{d}^{k}: H(\alpha) = \ell} \hat{Q}_{\alpha} \sum_{C_{\ell} \in X_{\ell}} \chi_{\alpha}(\sigma(C_{\ell})) \cdot h_{\ell}(C_{\ell}) \frac{|X_{k}|}{|X_{\ell}|} \right) \\ &= \left(\sum_{\ell=1}^{k} \frac{1}{|X_{\ell}|} \sum_{\alpha \in \mathbb{Z}_{d}^{k}: H(\alpha) = \ell} \hat{Q}_{\alpha} \sum_{C_{\ell} \in X_{\ell}} \chi_{\alpha}(\sigma(C_{\ell})) h_{\ell}(C_{\ell}) \right) \\ &= \left(\sum_{\ell=1}^{k} \frac{1}{|X_{\ell}|} b_{\ell}(\sigma) \right) \end{split}$$

The following lemma is similar to Lemma 4 from [29]. Lemma 3 is based on the general hypercontractivity theorem [47, Chapter 10] and applies to more general (non-boolean) functions.

Lemma 3. [47, Theorem 10.23] If $b: \mathbb{Z}_d^n \to \mathbb{R}$ has degree at most ℓ then for any $t \ge \left(\sqrt{2e/d}\right)^{\ell}$,

$$\Pr_{x \sim \mathbb{Z}_{d}^{n}}[|b(x)| \ge t||b||_{2}] \le \frac{1}{d^{\ell}} \exp\left(-\frac{\ell}{2ed}t^{2/\ell}\right),$$

where
$$||b||_2 = \sqrt{\mathbb{E}_{x \sim \mathbb{Z}_d^n} \left[b(x)^2 \right]}$$

Lemma 1 and its proof are almost identical to Lemma 5 in [29]. We simply replace their concentration bounds with the concentration bounds in Lemma 3. We include the proof for completeness.

Reminder of Lemma 1. Let $b: \mathbb{Z}_d^n \to \mathbb{R}$ be any function with degree at most ℓ , and let $\mathcal{D}' \subseteq \mathbb{Z}_d^n$ be a set of assignments for which $d' = d^n / |\mathcal{D}'| \ge e^{\ell}$. Then $\mathbb{E}_{\sigma \sim \mathcal{D}'}[|b(\sigma)|] \le \frac{2(\ln d'/c_0)^{\ell/2}}{d^{\ell}}||b||_2$, where $c_0 = \ell\left(\frac{1}{2ed}\right)$ and $||b||_2 = \sqrt{\mathbb{E}_{x \sim \mathbb{Z}_d^n}\Big[b(x)^2\Big]}$.

Proof of Lemma 1. The set \mathcal{D}' contains 1/d' fraction of points in \mathbb{Z}_d^n . Therefore,

$$\Pr_{x \sim \mathcal{D}'}[|b(x)| \ge t||b||_2] \le \frac{d'}{d^{\ell}} \exp\left(-\frac{\ell}{2ed}t^{2/\ell}\right),$$

for any $t \ge (\sqrt{2e/d})^{\ell}$. For any random variable Y and value $a \in \mathbb{R}$,

$$\mathbb{E}[Y] \le a + \int_{a}^{\infty} \Pr[Y \ge t] dt.$$

We set
$$Y = |b(\sigma)|/||b||_2$$
 and $a = \left(\frac{\ln d'}{d^2c_0}\right)^{\ell/2}$ Assuming that $a > \left(\sqrt{2e/d}\right)^{\ell}$ We get

$$\begin{split} \frac{\mathbb{E}_{\sigma \sim \mathcal{D}'}[|b(\sigma)|]}{||b||_{2}} &\leq (\ln d'/c_{0})^{\ell/2} + \int_{(\ln d'/c_{0})^{\ell/2}}^{\infty} \frac{d'}{d^{\ell}} \cdot e^{-c_{0}t^{2/\ell}} dt \\ &= \frac{(\ln d'/c_{0})^{\ell/2}}{d^{\ell}} + \frac{\ell \cdot d'}{2d^{\ell} \cdot c_{0}^{\ell/2}} \cdot \int_{\ln d'}^{\infty} e^{-z} z^{\ell/2 - 1} dz \\ &= \frac{(\ln d'/c_{0})^{\ell/2}}{d^{\ell}} + \frac{\ell \cdot d'}{2d^{\ell} \cdot c_{0}^{\ell/2}} \cdot \left(-e^{-z} z^{\ell/2 - 1} \right) \Big|_{\ln d'}^{\infty} + (\ell/2 - 1) \int_{\ln d'}^{\infty} e^{-z} z^{\ell/2 - 2} dz \\ &= \dots \leq \frac{(\ln d'/c_{0})^{\ell/2}}{d^{\ell}} + \frac{\ell \cdot d'}{2d^{\ell} \cdot c_{0}^{\ell/2}} \sum_{\ell'=1/2}^{\lceil \ell/2 \rceil - 1} \left(-\frac{\lceil \ell/2 \rceil!}{\ell'!} e^{-z} z^{\ell'} \right) \Big|_{\ln d'}^{\infty} \\ &= \frac{(\ln d'/c_{0})^{\ell/2}}{d^{\ell}} + \frac{1}{2d^{\ell} \cdot c_{0}^{\ell/2}} \sum_{\ell'=0}^{\lceil \ell/2 \rceil - 1} \frac{\lceil \ell/2 \rceil!}{\ell'!} (\ln d')^{\ell'} \leq \frac{2(\ln d'/c_{0})^{\ell/2}}{d^{\ell}} , \end{split}$$

where we used the condition $d' \ge e^{\ell}$ to obtain the last inequality.

Lemma 4. Let $\mathcal{D}' \subseteq \{0, ..., d-1\}^n$ be a set of assignments for which $d' = d^n / |\mathcal{D}'|$. Then

$$\mathbb{E}_{\sigma \sim \mathcal{D}'} \left[\left| \frac{1}{|X_{\ell}|} b_{\ell}(\sigma) \right| \right] \leq \frac{4 \left(\ln d' / c_0 \right)^{\ell/2}}{d^{\ell}} \frac{\|h_{\ell}(\sigma)\|_2}{\sqrt{|X_{\ell}|}}$$

Proof. For simplicity of notation we set $b = b_{\ell}$. By Parseval's identity we have

$$\mathbb{E}_{\sigma \sim \mathbb{Z}_{d}^{n}} \left[b(\sigma) \overline{b(\sigma)} \right] = \mathbb{E}_{\sigma \sim \mathbb{Z}_{d}^{n}} \left[|b(\sigma)|^{2} \right]$$

$$= \mathbb{E}_{\sigma \sim \mathbb{Z}_{d}^{n}} \left[\sum_{\substack{\alpha \in \mathbb{Z}_{d}^{k} \\ |H(\alpha) = \ell}} \sum_{C_{\ell} \in X_{\ell}} \hat{Q}_{\alpha} \chi_{\alpha} (\sigma(C_{\ell})) h_{\ell}(C_{\ell}) \right]^{2}$$

$$= \mathbb{E}_{\sigma \sim \mathbb{Z}_{d}^{n}} \left[\sum_{\substack{\alpha \in \mathbb{Z}_{d}^{k} \\ |H(\alpha) = \ell}} \sum_{C_{\ell} \in X_{\ell}} |\hat{Q}_{\alpha}|^{2} h_{\ell}(C_{\ell})^{2} \right]$$

$$= |X_{\ell}| \sum_{\substack{\alpha \in \mathbb{Z}_{d}^{k} \\ |H(\alpha) = \ell}} |\hat{Q}_{\alpha}|^{2} \frac{1}{|X_{\ell}|} \sum_{C_{\ell} \in X_{\ell}} h_{\ell}(C_{\ell})^{2}$$

$$= |X_{\ell}| \sum_{\substack{\alpha \in \mathbb{Z}_{d}^{k} \\ |H(\alpha) = \ell}} |\hat{Q}_{\alpha}|^{2} ||h_{\ell}||_{2}^{2}$$

$$\leq |X_{\ell}| ||h_{\ell}||^{2}$$

Before we can apply Lemma 1 we must address a technicality. The range of $b = b_\ell$ might include complex numbers, but Lemma 1 only applies to functions b with range \mathbb{R} . For $c,d \in \mathbb{R}$ we adopt the notation $Im(c+d\sqrt{-1})=d$ and $Re(c+d\sqrt{-1})=c$. We observe that

$$\mathbb{E}_{\sigma \sim \mathbb{Z}_d^n} \left[b(\sigma) \overline{b(\sigma)} \right] = \mathbb{E}_{\sigma \sim \mathbb{Z}_d^n} \left[Re(b(\sigma))^2 + Im(b(\sigma))^2 \right]$$
$$= ||Re(b)||^2 + ||Im(b)||^2.$$

We first observe that Re(b) and Im(b) are both degree ℓ functions because we can write

$$Re(b(\sigma)) = \frac{1}{|X_{\ell}|} \sum_{\alpha \in \mathbb{Z}_{d}^{k}: H(\alpha) = \ell} \sum_{C_{\ell} \in X_{\ell}} Re(\hat{Q}_{\alpha} \chi_{\alpha}(\sigma(C_{\ell})) h_{\ell}(C_{\ell}))$$

and

$$Im(b(\sigma)) = \frac{1}{|X_{\ell}|} \sum_{\alpha \in \mathbb{Z}_{d}^{k}: H(\alpha) = \ell} \sum_{C_{\ell} \in X_{\ell}} Im(\hat{Q}_{\alpha} \chi_{\alpha}(\sigma(C_{\ell})) h_{\ell}(C_{\ell})).$$

Now we can apply Lemma 1 to get

$$\mathbf{E}_{\sigma \sim \mathcal{D}'} [|Re(b(\sigma))|] \leq \frac{2(\ln d'/c_0)^{\ell/2}}{d^{\ell}} ||Re(b)||_2$$

$$\leq \frac{2(\ln d'/c_0)^{\ell/2}}{d^{\ell}} \sqrt{|X_{\ell}|} ||h_{\ell}(\sigma)||_2.$$

A symmetric argument can be used to bound $\mathbf{E}_{\sigma \sim \mathcal{D}'}[Im(b(\sigma))]$. Now because

$$|b(\sigma)| \le |Re(b(\sigma))| + |Im(b(\sigma))|$$
,

it follows that

$$\begin{split} \mathbf{E}_{\sigma \sim \mathcal{D}'} \left[\left| \frac{1}{|X_{\ell}|} b(\sigma) \right| \right] & \leq & 2 \left(\frac{1}{|X_{\ell}|} \right) \left(\frac{2 \left(\ln d' / c_0 \right)^{\ell/2}}{d^{\ell}} \right) ||h_{\ell}(\sigma)||_2 \sqrt{|X_{\ell}|} \\ & \leq & \frac{4 \left(\ln d' / c_0 \right)^{\ell/2}}{d^{\ell}} \frac{||h_{\ell}(\sigma)||_2}{\sqrt{|X_{\ell}|}} \; . \end{split}$$

We will use Fact 2 to prove Lemma 5. The proof of Fact 2 is found in [29, Lemma 7]. We include it here for completeness.

Fact 2. [29] If $h: X_{\ell} \to \mathbb{R}$ satisfies $||h||_2^2 = 1$ then $||h_{\ell}||_2^2 \le 1$.

Proof. First notice that for any C_{ℓ} ,

$$\left| \{ C \in X_k \mid \exists S \subseteq [k], \text{ s.t. } |S| = \ell \land C_{|S|} = C_\ell \} \right| = \frac{|X_k|}{|X_\ell|}.$$

By applying the definition of h_{ℓ} along with the Cauchy-Schwartz inequality

$$\begin{split} \|h_{\ell}\|_{2}^{2} &= \mathbb{E}_{C_{\ell} \sim U_{\ell}} \left[h_{\ell} (C_{\ell})^{2} \right] \\ &= \left(\frac{|X_{\ell}|}{|X_{k}|} \right)^{2} \mathbb{E}_{C_{\ell} \sim U_{\ell}} \left[\left(\sum_{S \subseteq [k], |S| = \ell, C \in X_{k}, C_{|S} = C_{\ell}} h(C) \right)^{2} \right] \\ &\leq \left(\frac{|X_{\ell}|}{|X_{k}|} \right)^{2} \mathbb{E}_{C_{\ell} \sim U_{\ell}} \left[\frac{|X_{k}|}{|X_{\ell}|} \left(\sum_{S \subseteq [k], |S| = \ell, C \in X_{k}, C_{|S} = C_{\ell}} h(C)^{2} \right) \right] \\ &\leq \left(\frac{|X_{\ell}|}{|X_{k}|} \right) \mathbb{E}_{C_{\ell} \sim U_{\ell}} \left[\left(\sum_{S \subseteq [k], |S| = \ell, C \in X_{k}, C_{|S} = C_{\ell}} h(C)^{2} \right) \right] \\ &= \mathbb{E}_{C \sim U_{k}} \left[h(C)^{2} \right] = ||h||_{2}^{2} = 1 \; . \end{split}$$

Lemma 5. Let r = r(f), let $\mathcal{D}' \subseteq \{0, ..., d-1\}^n$ be a set of secret mappings and let $d' = d^n/|\mathcal{D}'|$. Then $\kappa_2(\mathcal{D}') = O_k((\ln d'/n)^{r/2})$

Proof. Let $h: X_k \to \mathbb{R}$ be any function such that $\mathbb{E}_{U_k}[h^2] = 1$. Using Lemma 2 and the definition of r,

$$|\Delta(\sigma, h)| = \left| \sum_{\ell=r}^{k} \frac{1}{|X_{\ell}|} b_{\ell}(\sigma) \right|$$

$$\leq \sum_{\ell=r}^{k} \left| \frac{1}{|X_{\ell}|} b_{\ell}(\sigma) \right|.$$

We apply Lemma 4 and Fact 2 to get

$$\mathbb{E}_{\sigma \sim \mathcal{D}'} \left[|\Delta(\sigma, h)| \right] \leq \sum_{\ell=r}^{k} \left(\frac{4 (\ln d'/c_0)^{\ell/2}}{d^{\ell}} \frac{||h_{\ell}(\sigma)||_2}{\sqrt{|X_{\ell}|}} \right)$$

$$\leq \sum_{\ell=r}^{k} \left(\frac{4 (\ln d'/c_0)^{\ell/2}}{d^{\ell} \sqrt{|X_{\ell}|}} \right)$$

$$\leq O_k \left(\frac{(\ln d')^{\ell/2}}{d^{\ell} n^{r/2}} \right).$$

Remark 4. Recall that Q^f_{σ} denotes the uniform distribution over pairs $(C,i) \in X_k \times \mathbb{Z}_d$ that satisfy $f(\sigma(C)) = i$. If we let U'_k denote the uniform distribution over $X_k \times \mathbb{Z}_d$ then for any function $h: X_k \times \mathbb{Z}_d \to \mathbb{R}$ s.t. ||h|| = 1 we can apply Lemma 5 to write

$$\begin{split} \kappa_{2}(\mathcal{D}') &= \mathbb{E}_{\sigma \sim \mathcal{D}'} \left[\left| \mathbb{E}_{(C,j) \sim Q_{\sigma}^{f}} \left[h(C,j) \right] - \mathbb{E}_{(C,j) \sim U_{k}'} \left[h(C,j) \right] \right| \right] \\ &= \mathbb{E}_{\sigma \sim \mathcal{D}'} \left[\left| \sum_{i=1}^{d} \Pr_{C \sim U_{k}} \left[f\left(\sigma(C)\right) = i \right] \left(\mathbb{E}_{C \sim Q_{\sigma}^{f,i}} \left[h^{i}(C) \right] - \mathbb{E}_{C \sim U_{k}} \left[h^{i}(C) \right] \right) \right| \right] \\ &\leq \sum_{i=1}^{d} \max_{j} \mathbb{E}_{\sigma \sim \mathcal{D}'} \left[\left| \left\{ \mathbb{E}_{C \sim Q_{\sigma}^{f,j}} \left[h^{j}(C) \right] - \mathbb{E}_{C \sim U_{k}} \left[h^{j}(C) \right] \right\} \right| \right] \\ &\leq O_{k} \left(\frac{(\ln d')^{\ell/2}}{d^{\ell} n^{r/2}} \right), \end{split}$$

where $h^i(C) = h(C, i)$ and $d' = d^n/|\mathcal{D}'|$.

Reminder of Theorem 4. There exists a constant $c_Q > 0$ such that for any $\epsilon > 1/\sqrt{n}$ and $q \ge n$ we have

$$SDN\left(Z_{\epsilon,f}, \frac{c_Q(\log q)^{r/2}}{n^{r/2}}, 2e^{-n\cdot\epsilon^2/2}\right) \ge q$$
,

where r = r(f) is the distributional complexity of f.

Proof of Theorem 4. First note that, by Chernoff bounds, for any solution $\tau \in \mathbb{Z}_d^n$ the fraction of assignments $\sigma \in \mathbb{Z}_d^n$ such that τ and σ are ϵ -correlated (e.g., $H(\sigma, \tau) \leq \frac{n(d-1)}{d} - \epsilon \cdot n$) is at most $e^{-2n \cdot \epsilon^2}$. In other words $|\mathcal{D}_{\sigma}| \geq \left(1 - e^{-2n \cdot \epsilon^2}\right) |\mathbb{Z}_d^n|$, where $\mathcal{D}_{\sigma} = \mathbb{Z}_d^n \setminus \left\{\sigma' \mid H(\sigma, \sigma') \leq \frac{n(d-1)}{d} - \epsilon \cdot n\right\}$ Let

 $\mathcal{D}' \subseteq \mathcal{D}_{\sigma}$ be a set of distributions of size $|\mathcal{D}_{\sigma}|/q$. Then for $d' = d^n/|\mathcal{D}'| = q \cdot d^n/|\mathcal{D}_{\sigma}|$, by Lemma 5 and Remark 4, we get

$$\kappa_2(\mathcal{D}') = O_k\left(\frac{(\ln d')^{r/2}}{n^{r/2}}\right) \tag{1}$$

$$= O_k \left(\frac{(\ln q)^{r/2}}{n^{r/2}} \right), \tag{2}$$

where the last line follows by Sterling's Approximation

$$q = d' |\mathcal{D}_{\sigma}| / d^n = d' |\mathcal{D}_{\sigma}| / d^n \approx d' c' \sqrt{\frac{d}{n}}$$

for a constant c'. The claim now follows from the definition of SDN.

The proof of Theorem 3 follows from Theorem 4 and the following result of Feldman et al. [29].

Reminder of Theorem 2 [29, Theorems 10 and 12]. For $\kappa > 0$ and $\eta \in (0,1)$ let $d' = \mathrm{SDN}(\mathcal{Z}_{\epsilon,f}, \kappa, \eta)$ be the statistical dimension of the distributional search problem $\mathcal{Z}_{\epsilon,f}$. Any randomized statistical algorithm that, given access to a $VSTAT\left(\frac{1}{3\kappa^2}\right)$ oracle (resp. 1-MSTAT(L)) for the distribution Q^f_σ for a secret mapping σ chosen randomly and uniformly from \mathbb{Z}^n_d , succeeds in finding a mapping $\tau \in \mathbb{Z}^n_d$ that is ϵ -correlated with σ with probability $\Lambda > \eta$ over the choice of distribution and internal randomness requires at least $\frac{\Lambda - \eta}{1 - \eta} d'$ (resp. $\Omega\left(\frac{1}{L}\min\left\{\frac{d'(\Lambda - \eta)}{1 - \eta}, \frac{(\Lambda - \eta)^2}{\kappa^2}\right\}\right)$) calls to the oracle.

Reminder of Theorem 3. Let $\sigma \in \mathbb{Z}_d^n$ denote a secret mapping chosen uniformly at random, let Q_σ^f be the distribution over $X_k \times \mathbb{Z}_d$ induced by a function $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ with distributional complexity r = r(f). Any randomized statistical algorithm that finds an assignment τ such that τ is $\left(\sqrt{\frac{-2\ln(\eta/2)}{n}}\right)$ -correlated with σ with probability at least $\Lambda > \eta$ over the choice of σ and the internal randomness of the algorithm needs at least m calls to the 1-MSTAT(L) oracle (resp. $VSTAT\left(\frac{n^r}{2(\log n)^{2r}}\right)$ oracle) with $m \cdot L \ge c_1\left(\frac{n}{\log n}\right)^r$ (resp. $m \ge n^{c_1\log n}$) for a constant $c_1 = \Omega_{k,1/(\Lambda-\eta)}(1)$. In particular if we set $L = \left(\frac{n}{\log n}\right)^{r/2}$ then our algorithms needs at least $m \ge c_1\left(\frac{n}{\log n}\right)^{r/2}$ calls to 1-MSTAT(L).

Proof of Theorem 3. We set $\epsilon = \sqrt{\frac{-2\ln(\eta/2)}{n}}$ and observe that $2e^{-n\epsilon^2/2} = \eta$, and we set $q = n^{\log n}$ in Theorem 4. Now we apply Theorem 2 to get the desired lower bound

$$m = \Omega \left(\min \left\{ \frac{n^{\Omega_k(\log n)} (\Lambda - \eta)}{1 - \eta}, \left(\frac{n^{r/2}}{c_Q \left(\log \left(n^{\log n} \right) \right)^{r/2}} \right)^2 (\Lambda - \eta)^2 \right\} / L \right) = \Omega \left(\left(\frac{n^r}{\log^{2r} n} \right) / L \right),$$

for the 1-MSTAT (*L*) oracle. For the VSTAT $\left(\frac{n^r}{2(\log n)^{2r}}\right)$ oracle we get $m = \frac{n^{\Omega_k(\log n)}(\Lambda - \eta)}{1 - \eta}$.

E Security Proofs

Reminder of Theorem 6. Let f be a function with evenly distributed output (Definition 7), let $\sigma \sim \mathbb{Z}_d^n$ denote the secret mapping, let $\varepsilon > 0$ be any constant and suppose that for every $C \in X_k$ we are

given labels $\ell_C \in \mathbb{Z}_d$ s.t.

$$\Pr_{C \sim X_h} [f(\sigma(C)) = \ell_C] \ge \frac{1}{d} + \epsilon.$$

There is a polynomial time algorithm (in n, m, $1/\epsilon$) that finds a mapping $\sigma' \in \mathbb{Z}_d^n$ such that σ' is $\epsilon/2$ -correlated with σ with probability at least $\frac{\epsilon}{2d^2}$

Proof of Theorem 6. Let $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ be a function with evenly distributed output. We select fix $C^{-1} \sim X_{k-1}$ and $i \sim [n] \setminus C^{-1}$. Given $j \in [n] \setminus C^{-1}$ we let $C_j = (C^{-1}, j) \in X_k$ denote the corresponding clause. Now we generate the mapping σ' by selecting $\sigma'(i)$ at random, and setting $\sigma'(j) = \sigma'(i) + \ell_{C_j} - \ell_{C_i} \mod d$ for $j \in [n] \setminus C_i$. For $j \in C^{-1}$ we select $\sigma'(j)$ at random. We let $\mathbf{GOOD}(C^{-1}, i, \sigma')$ denote the event that

$$\Pr_{j \sim [n] \setminus C^{-1}} \left[\ell_{C_j} = f\left(\sigma(C_j)\right) \right] \ge \frac{1}{d} + \epsilon/2 ,$$

 $\sigma'(i) = \sigma(i)$ and $\ell_{C_i} = f(\sigma(C_i))$. Assume that the event **GOOD** (C^{-1}, i) occurs, in this case for each j s.t. $\ell_{C_j} = f(\sigma(C_j))$ we have

$$\sigma'(j) - \sigma(j) \equiv \left(\sigma'(i) + \ell_{C_j} - \ell_{C_i}\right) - \sigma(j) \mod d$$

$$\equiv \left(\ell_{C_j} - \ell_{C_i}\right) + \sigma(i) - \sigma(j) \mod d$$

$$\equiv g\left(\sigma\left(C^{-1}\right)\right) + \sigma(j) - \left(g\left(\sigma\left(C^{-1}\right)\right) + \sigma(i) - \sigma(j) \mod d$$

$$\equiv 0 \mod d.$$

Therefore, we have

$$\frac{n-H(\sigma,\sigma')}{n} \ge \frac{1}{d} + \epsilon/2 - \frac{k-1}{n} .$$

We note that by Markov's inequality the probability of success is at least

$$\Pr_{C^{-1} \sim X_{k-1}} \left[\mathbf{GOOD} \left(C^{-1} \right) \right] \geq \frac{\epsilon}{2d^2}.$$

Before proving Theorem 5 we introduce some notation and prove an important claim. We use $\mathcal{A}_{C_1,\dots,C_m}:(X_k)^t\to \mathbb{Z}_d^t$ to denote an adversary who sees the challenges $C_1,\dots,C_m\in X_k$ and the corresponding responses $f(\sigma(C_1)),\dots,f(\sigma(C_m))$. $\mathcal{A}_{C_1,\dots,C_m}\left(C_1',\dots,C_t'\right)\in \mathbb{Z}_d^t$ denotes the adversaries prediction of $f\left(\sigma\left(C_1'\right)\right),\dots,f\left(\sigma\left(C_t'\right)\right)$. Given a function $b:(X_k)^t\to\mathbb{Z}_d^t$, challenges $C_1',\dots,C_t'\in X_k$ and responses $f\left(\sigma\left(C_1'\right)\right),\dots,f\left(\sigma\left(C_t'\right)\right)$ we use $\mathcal{P}_{b,i,C_1',\dots,C_m'}:X_k\times[t]\to\mathbb{Z}_d\cup\{\bot\}$ to predict the value of a clause $C\in X_k$

$$\mathcal{P}_{b,C'_1,\dots,C'_t}(C,i) = \begin{cases} b(\hat{C}_1,\dots,\hat{C}_t)[i], & \text{if } f(\sigma(\hat{C}_j)) = b(\hat{C}_1,\dots,\hat{C}_t)[j] \ \forall j < i \\ \bot, & \text{otherwise} \end{cases}$$

where $\hat{C}_i = C$ and $\hat{C}_j = C'_j$ for $j \neq i$. We allow our predictor $\mathcal{P}_{b,C'_1,\dots,C'_t}(C,i)$ to output \bot when it is unsure. Informally, Claim 2 says that for $b = \mathcal{A}_{C_1,\dots,C_m}$ our predictor $\mathcal{P}_{b,i,C'_1,\dots,C'_m}$ is reasonably accurate whenever it is not unsure. Briefly, Claim 2 follows because for $b = \mathcal{A}_{C_1,\dots,C_m}$ we have

$$\Pr\left[\mathbf{Wins}\left(\mathcal{A},n,m,t\right)\right] = \prod_{i=1}^{d} \Pr_{\substack{C \sim X_k \\ C'_1,\dots,C_m \sim X_k \\ C'_1,\dots,C'_t \sim X_k}} \left[\mathcal{P}_{b,C'_1,\dots,C'_t}(C,i) = f\left(\sigma(C)\right) \,\middle|\, \mathcal{P}_{b,C'_1,\dots,C'_t}(C,i) \neq \bot\right].$$

Claim 2. Let \mathcal{A} be an adversary s.t $\Pr[\mathbf{Wins}(\mathcal{A}, n, m, t)] > \left(\frac{1}{d} + \epsilon\right)^t$ and let $b = \mathcal{A}_{C_1, \dots, C_m}$ then

$$\Pr_{\substack{i\sim [t],C\sim X_k\\C_1,\ldots,C_m\sim X_k\\C'_1,\ldots,C'_t\sim X_k}}\left[\mathcal{P}_{b,C'_1,\ldots,C'_t}(C,i)=f(\sigma(C))\;\middle|\mathcal{P}_{b,C'_1,\ldots,C'_t}(C,i)\neq\bot\right]\geq \left(\frac{1}{d}+\epsilon\right).$$

Proof of Claim 2. We draw examples $(C_1, f(\sigma(C_1))), \dots, (C_m, f(\sigma(C_m)))$ to construct $b = \mathcal{A}_{C_1, \dots, C_m}$. Given a random length-t password challenge $(C'_1, \dots, C'_t) \in (X_k)^t$ we let

$$p_{j} = \Pr_{C,C_{1},\dots,C_{m},C_{1}',\dots,C_{t}' \sim X_{k}} \left[\mathcal{P}_{b,j,C_{1}',\dots,C_{t}'}\left(C\right) = f\left(\sigma\left(C\right)\right) \, \middle| \, \mathcal{P}_{b,j,C_{1}',\dots,C_{t}'}\left(C\right) \neq \bot \right]$$

denote the probability that the adversary correctly guesses the response to the j'th challenge conditioned on the event that the adversary correctly guesses all of the earlier challenges. Observe that

$$\Pr_{C,C_1,...,C_m,C'_1,...,C'_{t-1}\sim X_k,i\sim[t]} \left[\mathcal{P}_{b,i,C'_1,...,C'_t}(C,i) = f(\sigma(C)) \right] = \sum_{i=1}^t p_i/t ,$$

so it suffices to show that $\sum_{i=1}^{t} p_i/t \ge \frac{1}{d} + \epsilon$. We obtain the following constraint

$$\prod_{i=1}^{t} p_{i} = \prod_{i=1}^{t} \Pr_{C,C_{1},\dots,C_{m},C'_{1},\dots,C'_{t} \sim X_{k}} \left[\mathcal{P}_{b,j,C'_{1},\dots,C'_{t}}(C) = f(\sigma(C)) \middle| \mathcal{P}_{b,j,C'_{1},\dots,C'_{t}}(C) \neq \bot \right]$$

$$= \prod_{i=1}^{t} \Pr_{C_{1},\dots,C_{m},C'_{1},\dots,C'_{t} \sim X_{k}} \left[\mathcal{A}_{C_{1},\dots,C_{m}} \left(C'_{1},\dots,C'_{t} \right) [i] = f\left(\sigma\left(C'_{i}\right)\right) \middle| \forall j < i. \, \mathcal{A}_{C_{1},\dots,C_{m}} \left(C'_{1},\dots,C'_{t} \right) [j] = f\left(\sigma\left(C'_{j}\right)\right) \right]$$

$$= \Pr_{C_{1},\dots,C_{m},C'_{1},\dots,C'_{t} \sim X_{k}} \left[\mathcal{A}_{C_{1},\dots,C_{m}} \left(C'_{1},\dots,C'_{t} \right) = \left(f\left(\sigma\left(C'_{1}\right)\right),\dots,f\left(\sigma\left(C'_{t}\right)\right) \right) \right]$$

$$\geq \left(\frac{1}{d} + \epsilon \right)^{t}.$$

If we minimize $\sum_{i=1}^{t} p_i/t$ subject to the constraint $\prod_{i=1}^{t} p_i \ge \left(\frac{1}{d} + \epsilon\right)^t$ then we obtain the desired upper bound $\sum_{i=1}^{t} p_i/t \ge \frac{1}{d} + \epsilon$.

Reminder of Theorem 5. Suppose that f has evenly distributed output, but that f is not $\mathbf{UF} - \mathbf{RCA}(n,m,t,\delta) - secure$ for $\delta > \left(\frac{1}{d} + \epsilon\right)^t$. Then there is a probabilistic polynomial time algorithm (in n, m, t and $1/\epsilon$) that extracts a string $\sigma' \in \mathbb{Z}_d^n$ that is $\epsilon/8$ -correlated with σ with probability at least $\frac{\epsilon^3}{(8d)^2}$ after seeing m+t example challenge response pairs.

Proof of Theorem 5. Given random clauses $C_1, \ldots, C_m, C'_1, \ldots, C'_t \sim X_k$ we let **Good** $(C_1, \ldots, C_m, C'_1, \ldots, C'_t)$ denote the event that

$$\Pr_{i \sim [t], C \sim X_k} \left[\mathcal{P}_{b, C_1', \dots, C_t'} \left(C, i \right) = f\left(\sigma\left(C \right) \right) \, \left| \mathcal{P}_{b, C_1', \dots, C_t'} \left(C, i \right) \neq \bot \right] \geq \left(\frac{1}{d} + \frac{\epsilon}{2} \right) \, .$$

By Markov's Inequality and Claim 2 we have $\Pr\left[\mathbf{Good}\left(C_1,\ldots,C_m,C_1',\ldots,C_t'\right)\right] \geq \frac{\epsilon}{2}$. Here, $b = \mathcal{A}_{C_1,\ldots,C_m}$ and

$$\mathcal{P}_{b,C_1',\dots,C_t'}(C,i) = \begin{cases} b\left(\hat{C}_1,\dots,\hat{C}_t\right)[i], & \text{if } f\left(\sigma\left(\hat{C}_j\right)\right) = b\left(\hat{C}_1,\dots,\hat{C}_t\right)[j] \ \forall j < i \\ \bot, & \text{otherwise} \end{cases}$$

Assuming that the event $\mathbf{Good}(C_1, \dots, C_m, C'_1, \dots, C'_t)$ occurs we obtain labels for each clause $C \in X_k$ by selecting a random permutation $\pi : [t] \to [t]$, setting i = 1 and setting $\ell_C = \mathcal{P}_{b,C'_1,\dots,C'_t}(C,\pi(i))$ — if $\ell_C \neq \bot$ then we increment i and repeat. Note that we will always find a label $\ell_C \neq \bot$ within t attempts because $\mathcal{P}_{b,C'_1,\dots,C'_t}(C,1) \neq \bot$. Let $\mathbf{GoodLabels}$ denote the event that

$$\Pr_{C \sim X_k}[G_C] \ge \frac{1}{d} + \frac{\epsilon}{4} ,$$

where G_C is the indicator random variable for the event $\ell_C = f(\sigma(C))$. We have

$$\mathbb{E}\left[\frac{1}{|X_k|}\sum_{C\in X_k}G_C\right]\geq \frac{1}{d}+\frac{\epsilon}{2}\;,$$

so we can invoke Markov's inequality again to argue that $\Pr[\mathbf{GoodLabels} \mid \mathbf{Good}] \ge \frac{\epsilon}{4}$. If the event **GoodLabels** occurs then we can invoke Theorem 6 to obtain σ' that is $\epsilon/8$ -correlated with σ with probability at least $\frac{\epsilon}{8d^2}$. Our overall probability of success is

$$\frac{\epsilon}{8d^2} \times \frac{\epsilon}{4} \times \frac{\epsilon}{2} = \frac{\epsilon^3}{(8d)^2} .$$

F Security Parameters of f_{k_1,k_2}

Reminder of Claim 1. Let $0 < k_1 \le 10, 0 < k_2$ be given and let $f = f_{k_1,k_2}$ we have $g(f) = k_1$, $r(f) = k_2 + 1$ and $s(f) = \min\left\{\frac{k_2+1}{2}, k_1 + 1\right\}$.

Proof of Claim 1. Let $f(x) = f_{k_1,k_2}(x) = x_{\left(\sum_{i=10}^{9+k_1} x_i \mod 10\right)} + \sum_{i=10+k_1}^{9+k_1+k_2} x_i \mod 10$. We first observe that if we fix the values of $x_{10}, \dots, x_{9+k_1} \in \mathbb{Z}_{10}$ and let $i' = \sum_{i=10}^{9+k_1} x_i \mod 10$ then $f'(x_0, \dots, x_9, x_{10+k_1}, \dots, x_{9+k_1+k_2}) = \sum_{i=10}^{9+k_1+k_2} x_i \mod 10$.

we fix the values of $x_{10}, \ldots, x_{9+k_1} \in \mathbb{Z}_{10}$ and let $i' = \sum_{i=10}^{j+k_1} x_i \mod 10$ then $f'(x_0, \ldots, x_9, x_{10+k_1}, \ldots, x_{9+k_1+k_2}, x_{i'} + \sum_{i=10+k_1}^{9+k_1+k_2} x_i \mod 10$ is a linear function. If we don't fix all of the values $x_{10}, \ldots, x_{9+k_1} \in \mathbb{Z}_{10}$ then the resulting function will not be linear. Thus, $g(f) = k_1$. We also note that for any $\alpha \in \mathbb{Z}_{10}^{10+k_1+k_2}$ s.t. $H(\alpha) \le k_2$ and $i, t \in \mathbb{Z}_{10}$ that

$$\Pr_{x \sim \mathbb{Z}_{10}^{10+k_1+k_2}} [f(x) = t \mid \alpha \cdot x \equiv i \mod 10] = \Pr_{x \sim \mathbb{Z}_{10}^{10+k_1+k_2}} [f(x) = t] = \frac{1}{10}.$$

Therefore,

$$\begin{split} \hat{Q}_{\alpha}^{f,t} &= \mathbb{E}_{x \sim \mathbb{Z}_{10}^{10+k_1+k_2}} \left[Q^{f,t}(x) \chi_{\alpha}(x) \right] \\ &= \sum_{i=0}^{9} \Pr[\alpha \cdot x \equiv i \mod 10] \mathbb{E}_{x \sim \mathbb{Z}_{10}^{10+k_1+k_2}} \left[Q^{f,t}(x) \chi_{\alpha}(x) \mid \alpha \cdot x \equiv i \mod 10 \right] \\ &= \sum_{i=0}^{9} \exp\left(\frac{-2\pi i \sqrt{-1}}{10} \right) \Pr[\alpha \cdot x \equiv i \mod 10] \mathbb{E}_{x \sim \mathbb{Z}_{10}^{10+k_1+k_2}} \left[Q^{f,t}(x) \mid \alpha \cdot x \equiv i \mod 10 \right] \\ &= \frac{1}{10} \sum_{i=0}^{9} \exp\left(\frac{-2\pi i \sqrt{-1}}{10} \right) \mathbb{E}_{x \sim \mathbb{Z}_{10}^{10+k_1+k_2}} \left[Q^{f,t}(x) \mid \alpha \cdot x \equiv i \mod 10 \right] \\ &= 0 \,, \end{split}$$

which implies that $r(f) \ge k_2 + 1$.

G Security Upper Bounds

G.1 Statistical Algorithms

Theorem 7 demonstrates that our lower bound for statistical algorithms are asymptotically tight for our human computable password schemes f_{k_1,k_2} . In particular, we demonstrate that $m = \tilde{O}(n^{(k_2+1)/2})$ queries to 1-MSTAT are sufficient for a statistical algorithm to recover σ .

Theorem 7. For $f = f_{k_1,k_2}$ there is a randomized algorithm that makes $O\left(n^{\max\{1,(k_2+1)/2\}}\log^2 n\right)$ calls to the 1-MSTAT $\left(n^{\lceil r(f_i)/2 \rceil}\right)$ oracle and returns σ with probability 1 - o(1).

For binary functions $f': \{0,1\}^k \to \{0,1\}$, Feldman et al. [29] gave a randomized statistical algorithm to find $\sigma' \in \{0,1\}^n$ using just $O\left(n^{r(f)/2}\log^2 n\right)$ calls to the 1-MSTAT $\left(n^{\lceil r(f)/2\rceil}\right)$ oracle. Their main technique is a discrete spectral iteration procedure to find the eigenvector (singular vector) with the largest eigenvalue (singular value) of a matrix M sampled from a distribution $M_{\sigma',p}$ over $\left|X_{\lfloor r(f)/2\rfloor}\right| \times \left|X_{\lceil r(f)/2\rceil}\right|$ matrices. With probability 1-o(1) this eigenvector will encode the value $\sum_{i\in C}\sigma'(i)\mod 2$ for each clause $C\in X_{r(f)/2}$. We show that the discrete spectral iteration algorithm of Feldman et al [29] can be extended to recover $\sigma\in\mathbb{Z}_{10}$ when f_{k_1,k_2} is one of our candidate human computable functions.

Discussion We note that Theorem 7 cannot be extended to arbitrary functions $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$. Consider for example the unique function $f: \mathbb{Z}_{10}^6 \to \mathbb{Z}_{10}$ s.t. $f(x_1, \dots, x_6) \equiv f'(x_1 \mod 2, \dots, x_6 \mod 2)$ mod 2 and $f(x_1, \dots, x_6) \equiv f''(x_1 \mod 5, \dots, x_6 \mod 5)$ mod 5, where $f': \mathbb{Z}_2^6 \to \mathbb{Z}_2$ and $f'': \mathbb{Z}_5^6 \to \mathbb{Z}_5$. By the Chinese Remainder Theorem instead of picking a secret mapping $\sigma \in \mathbb{Z}_{10}^n$ we could equivalently pick the unique secret mappings $\sigma_1 \in \mathbb{Z}_2^n$ and $\sigma_2 \in \mathbb{Z}_5^n$ s.t $\sigma \equiv \sigma_1 \mod 2$ and $\sigma \equiv \sigma_2 \mod 5$. Now drawing challenge response pairs from the distributions $Q_{\sigma_1}^f$ is equivalent to drawing challenge-response pairs from the distributions $Q_{\sigma_1}^f$ and $Q_{\sigma_2}^{f''}$. Suppose that $f'(x_1, \dots, x_6) = x_1 x_2 + x_3 + x_4 + x_5 + x_6 \mod 2$, and $f''(x_1, \dots, x_6) = x_1$. Then we have $r(f) = \min(r(f'), r(f'')) = r(f'') = 1$, but r(f') = 4. We can find σ_2 using $O(n \log^2 n)$ calls to 1-MSTAT(n), but to find σ we must also recover σ_1 . This provably requires at least $\tilde{\Omega}(n^{r(f')/2}) = \tilde{\Omega}(n^2)$ calls to 1-MSTAT (n^2) .

Background The proof of Theorem 7 relies on the discrete spectral iteration algorithm of [29]. We begin by providing a brief overview of their algorithm. In their setting the secret mapping σ is defined over the binary alphabet \mathbb{Z}_2^n . Let $c_1 = \lceil \frac{r(f)}{2} \rceil$, $c_2 = \lfloor \frac{r(f)}{2} \rfloor$ and let $\delta \in [0,2] \setminus \{1\}$. They use σ to define a distribution over $|X_{c_1}| \times |X_{c_2}|$ matrices $M_{\sigma,\delta,p} = \hat{M}(Q_{\sigma,\delta,p}) - Jp$, where J denotes the all ones matrix. For $(C_1) \in X_{c_1}$, $(C_2) \in X_{c_2}$ such that $C_1 \cap C_2 = \emptyset$ we have

$$\hat{M}\big(Q_{\sigma,\delta,p}\big)[(C_1),(C_2)] = \begin{cases} 1, & \text{with probability } (p(2-\delta)) \text{ if } \sum_{j \in C_1 \cup C_2} \sigma(j) \equiv 0 \mod 2 \\ 1, & \text{with probability } (p\delta) \text{ if } \sum_{j \in C_1 \cup C_2} \sigma(j) \not\equiv 0 \mod 2 \\ 0, & \text{otherwise} \end{cases}$$

Given a vector $x \in \{\pm 1\}^{|X_{c_2}|}$ (resp. $y \in \{\pm 1\}^{|X_{c_1}|}$) $M_{\sigma,\delta,p}x$ defines a distribution over vectors in $\mathbb{R}^{|X_{c_1}|}$ (resp. $M_{\sigma,p}^Ty$ defines a distribution over vectors in $\mathbb{R}^{|X_{c_1}|}$).

If r(f) is even then the largest eigenvalue of $\mathbb{E}[M_{\sigma,\delta,p}]$ has a corresponding eigenvector $x^* \in \{\pm 1\}^{X_{r(f)/2}}$, where for $C_i \in X_{r(f)/2}$ we have $x^*[C_i] = 1$ if $\sum_{j \in C_i} \sigma(j) \equiv 1 \mod 2$; otherwise $x^*[C_i] = -1$

(if r(f) is odd then we consider the top singular value instead). Feldman et al [29] use discrete spectral iteration to find x^* (or $-x^*$). Given x^* it is easy to find σ using Gaussian Elimination. The discrete spectral iteration algorithm of Feldman et al [29] starts with a random vector $x^0 \in \{0,1\}^{|X_{c_2}|}$. They then sample $x^{i+1} \sim M_{\sigma,p}x^i$ and execute a normalization step to ensure that $x^{i+1} \in \{0,1\}^{|X_{k_2}|}$. When r(f) is odd, power iteration has two steps: draw a sample $y^i \sim M_{\sigma,\delta,p}x^i$ and sample from the distribution $x^{i+1} = M_{\sigma,\delta,p}^T y^i$. They showed that $O\left(\log \left|X_{r(f)}\right|\right)$ iterations suffice to recover σ whenever $p = \frac{K \log |X_{r(f)}|}{(\delta-1)^2 \sqrt{|X_{r(f)}|}}$, and that for a vector $x \in \{0,1\}^{|X_{k_2}|}$ (resp. $y \in \{\pm 1\}^{|X_{k_1}|}$) it is possible to sample from $M_{\sigma,\delta,p}x$ (resp. $M_{\sigma,\delta,p}^T y$) using O(1/p) queries to 1-MSTAT $\left(\left|X_{c_1}\right|\right)$.

Our Reduction The proof of Theorem 7 uses a reduction to the algorithm of Feldman et al[29]. *Proof of Theorem* 7. (sketch) Given a mapping $\sigma \in \mathbb{Z}_d^n$ and a number $i \in \mathbb{Z}_d$ we define a mapping $\sigma_i \in \mathbb{Z}_2^n$ where

$$\sigma_i(j) = \begin{cases} 1, & \text{if } \sigma(j) = i \\ 0, & \text{otherwise} \end{cases}.$$

Clearly, to recover σ it is sufficient to recover σ_i for each $i \in \mathbb{Z}_d$. Therefore, to prove Theorem 7 it suffices to show that given $x \in \{\pm 1\}^{|X_{k_2}|}$ (resp. $y \in \{\pm 1\}^{|X_{k_1}|}$) we can sample from the distribution $M_{\sigma_i,\delta,p}x$ (resp. $M_{\sigma_i,\delta,p}^Ty$) using O(1/p) queries to 1-MSTAT $(|X_{\lceil r(f)/2\rceil}|)$ for each $i \in \{0,\ldots,d-1\}$, where 1-MSTAT uses the distribution Q_σ^f . In general, this will not possible for arbitrary functions f. However, Lemma 6 shows that for our candidate human computable functions $f_{1,3}$, $f_{2,2}$ we can sample from the distributions $M_{\sigma_i,\delta,p}x$ (resp. $M_{\sigma_i,\delta,p}^Ty$). The proof of Lemma 6 is similar to the proof of [29, Lemma 10].

Lemma 6. Given vectors $\vec{x} \in \{\pm 1\}^{|X_{c_1}|}$, $\vec{y} \in \{\pm 1\}^{|X_{c_2}|}$ we can sample from $M_{\sigma_i,\delta,p}x$ and $M_{\sigma,\delta,p}^Ty$ using $O(n^{(k_2+1)/2}\log^2 n)$ calls to the 1-MSTAT $(n^{\lceil r(f)/2 \rceil})$ oracle for $f = f_{k_1,k_2}$.

The proof of Lemma 6 relies on Fact 3.

Fact 3. For each $j, t \in \mathbb{Z}_{10}$ we have

$$\Pr_{\left(x_0,\dots,x_{9+k_1+k_2}\right)\sim\mathbb{Z}_{10}^{10+k_1+k_2}}\left[x_t+\sum_{i=10+k_1}^{10+k_1+k_2}x_i\equiv j\,\middle|\,f_{k_1,k_2}\left(x_0,\dots,x_{9+k_1+k_2}\right)\equiv j\mod 10\right]=\frac{\left(\frac{9}{10}\left(\frac{1}{10}\right)+\frac{1}{10}\right)\left(\frac{1}{10}\right)}{\left(\frac{1}{10}\right)}=\frac{19}{100}\,,$$

and

$$\Pr_{\left(x_0,\dots,x_{9+k_1+k_2}\right) \sim \mathbb{Z}_{10}^{10+k_1+k_2}} \left[x_t + \sum_{i=10+k_1}^{10+k_1+k_2} x_i \equiv j \,\middle|\, f_{k_1,k_2}\left(\sigma\left(x_0,\dots,x_{9+k_1+k_2}\right)\right) \not\equiv j \mod 10 \right] = \frac{\left(\frac{9}{10}\left(\frac{1}{10}\right) + \frac{1}{10}\left(0\right)\right)\left(\frac{1}{10}\right)}{\left(\frac{1}{10}\right)} = \frac{9}{100} \,.$$

Proof of Lemma 6. Given a value $j \in \mathbb{Z}_{10}$ and a value $i \in \mathbb{Z}_{10}$ we let $x_j^i \in \{0,1\}$ be the indicator variable for the event $x_j = i$. By Fact 3 it follows that

Now for f_{k_1,k_2} we define the function $h^{i,+}: X_{k_1+k_2+10} \times \mathbb{Z}_{10} \to X_{c_1} \cup \{\bot\}$ as follows

$$h^{i,+}\left(x_0,\ldots,x_{9+k_1+k_2},f_{k_1,k_2}\left(\sigma\left(x_0,\ldots,x_{9+k_1+k_2}\right)\right)\right) = \begin{cases} \left(x_0,x_{9+k_1},\ldots,x_{7+k_1+c_1}\right) & \text{if } f_{k_1,k_2}\left(\sigma\left(x_0,\ldots,x_{13}\right)\right) \equiv ic_1 \mod 10 \\ \bot & \text{otherwise.} \end{cases}$$

Intuitively, given a clause $C_1 \in X_{c_1}$ the probability that $h^{i,+}$ returns C_1 is greater if $\sum_{j \in C_1} \sigma_i(j) \equiv 1 \mod 2$.

Given a vector $x \in \{\pm 1\}^{|X_{c_1}|}$ we query our 1-MSTAT $(|X_{c_1}|+1)$ oracle $\lceil 10/p \rceil$ times with the function $h^{i,+}$ to sample from $M_{\sigma_i,\delta,p}x$. Let $q_1,\ldots,q_{\lceil 10/p \rceil} \in X_{c_1}$ denote the responses and let $x \lceil q_j \rceil$ denote the value of the vector x at index q_j . We observe that for some $\delta \neq 1$ we have

$$M_{\sigma_i,\delta,p}x[C] \sim \sum_{\substack{j \in \lceil 10/p \rceil \\ q_i \neq \perp}} x[q_j] - p \sum_{C' \in X_{c_1}} x[C'],$$

for every $C \in X_{c_2}$.

Open Question Can we precisely characterize the functions $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$ for which we can efficiently recover σ after seeing $\tilde{O}\left(n^{r(f)/2}\right)$ challenge-response pairs? Feldman et al. [29] gave a statistical algorithm that recovers the secret mapping whenever d=2 after making $\tilde{O}\left(n^{r(f)/2}\right)$ queries to 1-MSTAT $\left(n^{r(f)/2}\right)$. While we show that the same algorithm can be used to recover σ after making $\tilde{O}\left(n^{r(f)/2}\right)$ queries to 1-MSTAT $\left(n^{r(f)/2}\right)$ in our candidate human computable password schemes with d=10, we also showed that these results do not extend to all functions $f: \mathbb{Z}_d^k \to \mathbb{Z}_d$.

G.2 Gaussian Elimination

Most known algorithmic techniques can be modeled within the statistical query framework. Gaussian Elimination is a notable exception. As an example consider the function $f(x_1,...,x_7) = x_1 + ... + x_7 \mod 10$ (in this example r(f) = 7 and g(f) = 0). Our previous results imply that any statistical algorithm would need to see at least $m = \tilde{\Omega}(n^{7/2})$ challenge response pairs $(C, f(\sigma(C)))$ to recover σ . However, it is trivial to recover σ from O(n) random challenge response pairs using Gaussian Elimination. In general, consider the following attacker shown in algorithm 4, which uses Gaussian Elimination. Algorithm 4 relies on the subroutine $\operatorname{TryExtract}(C, f(\sigma(C)), S, \alpha)$, which attempts to extract a linear constraint from $(C, f(\sigma(C)))$ under the assumption that $\sigma(S) = \alpha$. We assume $\operatorname{TryExtract}(C, f(\sigma(C)), S, \alpha)$ returns \emptyset if it cannot extract a linear constraint. For example, if we assume that $\sigma(1) = 4$ and $\sigma(2) = 8$ and let $C = (i_0, i_1, ..., i_9, 1, 2, i_{10}, i_{11})$ (with $i_j \in [n] \setminus \{1,2\}$) then we have $f_{2,2}(\sigma(C)) = \sigma(i_{4+8 \mod 10}) + \sigma(i_{10}) + \sigma(i_{11}) \mod 10$. In this case, $\operatorname{TryExtract}(C, f(\sigma(C)), \{1,2\}, \{4,8\})$ would return the constraint $f(\sigma(C)) = \sigma(i_2) + \sigma(i_{10}) + \sigma(i_{11})$ mod 10. However, $\operatorname{TryExtract}(C, f(\sigma(C)), \{i_0,2\}, \{4,8\})$ would return \emptyset .

Algorithm 4 Gaussian Attack

```
Input: Clauses C_1, ..., C_m \sim X_k, and labels f(\sigma(C_1)), ..., f(\sigma(C_m)).

for all S \in X_{g(f)}, \alpha \in \mathbb{Z}_d^{g(f)} do

LC \leftarrow \emptyset \blacktriangleright LC is the set of linear constraints extracted for all C \in \{C_1, ..., C_m\} do

LC \leftarrow LC \bigcup TryExtract(C, f(\sigma(C)), S, \alpha)

if |LC| \ge n then

\sigma' \leftarrow LinearSolve(LC)

if \forall i \in [m]. f(\sigma'(C_i)) = f(\sigma(C_i)) \in C then return \sigma'
```

Fact 4 says that an attacker needs at least $m = \tilde{\Omega}\left(n^{1+g(f)}\right)$ challenge-response pairs to recover σ using Gaussian Elimination. This is because the probability that $\mathbf{TryExtract}(C, f(\sigma(C))S, \alpha)$ extracts a linear constraint is at most $O\left(\left(\frac{|S|}{n}\right)^{-g(f)}\right)$, which is $O\left(n^{-g(f)}\right)$ for |S| constant. The adversary needs O(n) linearly independent constraints to run Gaussian Elimination. If the adversary can see at most $\tilde{O}\left(n^{s(f)}\right)$ examples neither approach (Statistical Algorithms or Gaussian Elimination) can be used to recover σ .

Fact 4. Algorithm 4 needs to see at least $m = \tilde{\Omega}(n^{1+g(f)})$ challenge-response pairs to recover σ .

Remark 5 explores the tradeoff between the adversary's running time and the number of challenge-response pairs that an adversary would need to see to recover σ using Gaussian elimination. In particular the adversary can recover σ from $\tilde{O}\left(n^{1+g(f)/2}\right)$ challenge-response pairs if he is willing to increase his running time by a factor of $d^{\sqrt{n}}$. In practice, this attack may be reasonable for $n \le 100$ and d = 10, which means that it may be beneficial to look for candidate human computable functions f that maximize $\min\{r(f)/2, 1+g(f)/2\}$ instead of s(f) whenever $n \le 100$.

Remark 5. If the adversary correctly guesses value of $\sigma(S)$ for $|S| = n^{\epsilon}$ then he may be able to extract a linear constraint from a random example with probability $\Omega(1/n^{(1-\epsilon)g(f)})$. The adversary would only need $\tilde{O}\left(n^{1+(1-\epsilon)g(f)}\right)$ examples to solve for σ , but his running time would be proportional to $d^{\epsilon n}$ — the expected number of guesses before he is correct.

H Rehearsal Model

In this section we review the usability model of Blocki et al. [16]. Their usability model estimates the 'extra effort' that a user needs to expend to memorize and rehearse all of his secrets for a password management scheme. In this section we use (\hat{c}, \hat{a}) to denote a cue-association pair, and we use the variable t to denote time (days). In our context (\hat{c}, \hat{a}) might denote the association between a letter (e.g., 'e') and the secret digit associated with that letter (e.g., $\sigma(e)$). If the user does not rehearse an association (\hat{c}, \hat{a}) frequently enough then the user might forget it. Their are two main components to their usability model: rehearsal requirements and visitation schedules. Rehearsal requirements specify how frequently a cue-association pair must be used for a user to remember the association. Visitation schedules specify how frequently the user authenticates to each of his accounts and rehearses any cue-association pairs that are linked with the account.

H.1 Rehearsal Requirements

Blocki et al. [16] introduce a rehearsal schedule to ensure that the user remembers each cueassociation pair.

Definition 9. [16] A rehearsal schedule for a cue-association pair (\hat{c}, \hat{a}) is a sequence of times $t_0^{\hat{c}} < t_1^{\hat{c}} < \dots$ For each $i \ge 0$ we have a rehearsal requirement, the cue-association pair must be rehearsed at least once during the time window $\left[t_i^{\hat{c}}, t_{i+1}^{\hat{c}}\right] = \{x \in \mathbb{R} \mid t_i^{\hat{c}} \le x < t_{i+1}^{\hat{c}}\}.$

A rehearsal schedule is *sufficient* if a user can maintain the association (\hat{c}, \hat{a}) by following the rehearsal schedule. The length of each interval $\left[t_i^{\hat{c}}, t_{i+1}^{\hat{c}}\right]$ may depend on the strength of the mnemonic techniques used to memorize and rehearse a cue-association pair (\hat{c}, \hat{a}) as well as i—the number of prior rehearsals [55, 53].

Expanding Rehearsal Assumption [16]: The rehearsal schedule given by $t_i^{\hat{c}} = 2^{is}$ is sufficient to maintain the association (\hat{c}, \hat{a}) , where s > 0 is a constant.

H.2 Visitation Schedules

Suppose that the user has m accounts A_1, \ldots, A_m . A visitation schedule for an account A_i is a sequence of real numbers $\tau_0^i < \tau_1^i < \ldots$, which represent the times when the account A_i is visited by the user. Blocki et al. [16] do not assume that the exact visitation schedules are known a priori. Instead they model visitation schedules using a random process with a known parameter λ_i based on $E\left[\tau_{j+1}^i - \tau_j^i\right]$ — the average time between consecutive visits to account A_i .

A rehearsal requirement t_i^c, t_{i+1}^c can be satisfied naturally if the user visits a site A_j that uses the cue \hat{c} ($\hat{c} \in c_j$) during the given time window. Here, c_j denote the set of cue-association pairs that the user must remember when logging into account A_j . Formally,

Definition 10. [16] We say that a rehearsal requirement $\begin{bmatrix} t_i^{\hat{c}}, t_{i+1}^{\hat{c}} \end{bmatrix}$ is naturally satisfied by a visitation schedule $\tau_0^i < \tau_1^i < \dots$ if $\exists j \in [m], k \in \mathbb{N}$ s.t $\hat{c} \in c_j$ and $\tau_k^j \in [t_i^{\hat{c}}, t_{i+1}^{\hat{c}})$. We use

$$ER_{t,\hat{c}} = \left| \left\{ i \mid t_{i+1}^{\hat{c}} \leq t \wedge \forall j, k. \left(\hat{c} \notin c_j \vee \tau_k^j \notin \left[t_i^{\hat{c}}, t_{i+1}^{\hat{c}} \right) \right) \right\} \right| ,$$

to denote the number of rehearsal requirements that are not naturally satisfied by the visitation schedule during the time interval [0,t].

Example: Consider the human computable function $f_{2,2}$ from section 4, and suppose that the user has to compute $f_{2,2}(\sigma(C_i))$ to authenticate at account A_j , where $C_i = (x_0, ..., x_{13})$. When the user computes $f_{2,2}$ he must rehearse the associations $(x_{10}, \sigma(x_{10}))$, $(x_{11}, \sigma(x_{11}))$, $(x_{12}, \sigma(x_{12}))$, $(x_{13}, \sigma(x_{13}))$ and $(x_i, \sigma(x_i))$ where $i = (\sigma(x_{10}) + \sigma(x_{11}) \mod 10)$. Thus $c_j \supset \{x_i, x_{10}, x_{11}, x_{12}, x_{13}\}$. When user authenticates he naturally rehearses each of these associations in c_j .

If a cue-association pair (\hat{c}, \hat{a}) is not rehearsed naturally during the interval $\begin{bmatrix} t_i^{\hat{c}}, t_{i+1}^{\hat{c}} \end{bmatrix}$ then the user needs to perform an extra rehearsal to maintain the association. Intuitively, $ER_{t,\hat{c}}$ denotes the total number of extra rehearsals of the cue-association pair (\hat{c}, \hat{a}) during the time interval [0,t], and $ER_t = \sum_{\hat{c} \in C} ER_{t,\hat{c}}$ denotes the total number of extra rehearsals during the time interval [0,t] to maintain all of the cue-association pairs. Thus, a smaller value of $E[ER_t]$ indicates that the user needs to do less extra work to rehearse his secret mapping.

Schedule λ	$\frac{1}{1}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{31}$	$\frac{1}{365}$
Very Active	10	10	10	10	35
Typical	5	10	10	10	40
Occasional	2	10	20	20	23
Infrequent	0	2	5	10	58

Table 6: Visitation Schedules - number of accounts visited with frequency λ (visits/days)

Poisson Arrival Process The visitation schedule for each account A_j is given by a Poisson arrival process with parameter λ_j , where $1/\lambda_j = E\left[\tau_{j+1}^i - \tau_j^i\right]$ denotes the average time between consecutive visits to account A_j .

Evaluating Usability Blocki et al. [16] prove the following theorem. Given a sufficient rehearsal schedule and a visitation schedule, Theorem 8 predicts the value of ER_t , the total number of extra rehearsals that a user will need to do to remember all of the cue-association pairs required to reconstruct all of his passwords.

Theorem 8. [16] Let $i_{\hat{c}}^* = (\arg \max_x t_x^{\hat{c}} < t) - 1$ then

$$E[ER_t] = \sum_{\hat{c} \in C} \sum_{i=0}^{i_{\hat{c}^*}} \exp \left(-\left(\sum_{\substack{j \text{ s.t.} \\ \hat{c} \in c_j}} \lambda_j \right) \left(t_{i+1}^{\hat{c}} - t_i^{\hat{c}}\right) \right)$$

We use the formula from Theorem 8 to obtain the usability results in Table 1. To evaluate this formula we need to be given the rehearsal requirements, a visitation schedule (λ_i) for each account A_i and a set of public challenges $\vec{C_i} \in (X_{14})^{10}$ for each account A_i . The rehearsal requirements are given by the Expanding Rehearsal Assumption [16] (we use the same association strength parameter $\mathbf{s} = 1$ as Blocki et al. [16]), and the visitation schedules for each user are given in Table 6. We assume that each password is 10 digits long and that the challenges $\vec{C_i} \in (X_{14})^{10}$ are chosen at random by Algorithm 2. Notice that each time the user responds to a single digit challenge he rehearses the secret mapping at five locations (see discussion in Section 4.1). Because the value of $\mathbb{E}[ER_{365}]$ depends on the particular password challenges that we generated for each account, we ran Algorithm 2 and computed the resulting value $\mathbb{E}[ER_{365}]$ one-hundred times. The values in Table 1 represent the mean value of $\mathbb{E}[ER_{365}]$ across all hundred instances.