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Theory of airflow over mountains : III – Airstream characteristics

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SUMMARY

The problem of what second boundary condition (the first being the ground profile) should be applied to the differential equation for the variation with height of the wave amplitude is considered and it is thought that solutions corresponding to upward and downward progressing waves are generally irrelevant. In practice we shall be concerned mainly with those waves of length such that their amplitude decreases with increasing height. Only when there is a considerable range of such wavelengths is the motion of the kind supposed in the perturbation theory.

Though it is desirable to characterize an airstream by a single number this is not at present possible for airstreams in general, and it is shown by examples how airstreams may possess the same lee wave length though their l-profiles differ considerably. These examples show that generally the maximum amplitude of the waves is found at or close to levels of maximum values of l, at sharp inversions in particular.

The behaviour of lee (stationary) waves is fairly representative of mountain waves in general.

1. INTRODUCTION

In two previous papers (1949 and 1953) herein to be referred to as I and II, it was shown how lee waves were possible in certain kinds of non-uniform airstream and the idea was developed. One problem, the nature of the flow at great height, was put aside temporarily, and is now to be discussed. This discussion indicates that the chief value of the perturbation theory is in the study of lee waves – those standing waves whose amplitude tends to zero with increasing height in the upper layers, and which occur in the lee of any disturbance if the airstream can contain them.

A general discussion of lee waves is impossible on account of the great number of variables involved, and so ideas must be gleaned from particular cases. Some method of comparing different airstreams is desirable. Uniform streams of water of finite depth can be compared by their Froude number, but for an airstream in which the velocity and stability are variable functions of height – as they must be if lee waves can occur at all – no single number characterizing its properties has been proposed. Any single nûmber must necessarily be derived as a kind of integral, throughout the whole depth of the airstream, of some sort of Froude number for each of an infinite number of infinitely thin superposed layers. Benton (1954) has discussed this question and though some progress has been made practical application to the atmosphere is still far off.

In this event some properties of airstreams are illustrated by particular cases, in the hope that they may be a guide to forecasters presented with similar airstreams.

The notation is the same as that given in II.

2. THE PROBLEM OF THE SECOND BOUNDARY CONDITION

In the solution of the problem of the flow of an airstream over a mountain ridge by the Fourier integral method, or some analogous procedure, it is necessary to apply two boundary conditions to the differential equation

$$w'' + (l^2 - k^2)\, w = 0 \quad . \quad . \quad . \quad . \quad (1)$$

which governs the variation with height of the vertical velocity w, for waves of length $2\pi/k$, in the airstream in which l^2 is given at every height. The first condition is given by the shape of the ground, which determines w at $z = 0$, say. In searching for a second condition various devices have been proposed in order to determine a unique solution to the problem.

If it be assumed that in the uppermost layers of the airstream l^2 is independent of z, and there seems to be no reason why this kind of airstream should behave in an exceptional way, then in those layers

$$w = A \exp \{k^2 - l^2)^{\frac{1}{2}} z\} + B \exp \{-(k^2 - l^2)^{\frac{1}{2}} z\} \quad . \quad . \quad (2)$$

and we have to determine A and B. When $k > l$ we put $A = 0$, because w cannot increase exponentially upwards, and there is no problem because B is then determined by the ground contours.

When $k < l$ the solution proposed in I, and used again in II, is that $A = 0$ because then the wave fronts progress upwards through the air.

All other authors disagree with this conclusion and so the various considerations will now be discussed.

(i) *Convergence of a series*

Zierep (1952), solving the problem for an airstream whose velocity profile is approximately that produced by a constant eddy viscosity, rejected the solution $A = 0$ on the grounds that in the method he used it corresponded to a divergent series. The series is, however, an asymptotic series and it was the first term of this series that was used in I. This asymptotic series is obtained by successive integration by parts of an integral that is convergent. This is a standard method of obtaining the series, but Zierep obtained it in a different way and rejected it as a physically impossible solution on account of its divergence.

(ii) *The effect of friction*

Rayleigh's classical discussion of waves on a stream of water suggests that friction might provide a criterion. A solution, in which a train of lee waves lies on the downstream side of the obstacle only, was then obtained by assuming a small amount of friction. It is equivalent to saying that the stream comes upon the obstacle with no disturbance in it, or that there is no disturbance at a great distance upstream. In this case there is one special wavelength such that its amplitude is always zero on the bed of the stream and so it is excited out of all proportion to the other component wavelengths by an unlevel bed. The solution thrown up by the mathematics, without friction, consists of a train of stationary waves to infinity in the upstream direction from the obstacle which is out of phase with a similar train on the downstream side; but, since it is permissible to add an arbitrary amount of any solution which does not alter the boundary conditions, a train of waves is added which exactly cancels the train extending to infinity upstream and doubles the amplitude of those on the downstream side of the obstacle. The introduction of a small amount of friction produces this same result.

By analogy Lyra (1943) and Queney (1947) argued that the result of friction must be to make the disturbance lie mainly on the downstream side, and when the complete Fourier integral solution is obtained this condition is satisfied by putting $B = 0$.

But there is one important difference, namely that in the solution so obtained the amplitude of the disturbance decreases both upstream and downstream so that friction (or radiation, or any other damping agent) could only be effective in determining the solution if it could produce its effect in the region where the amplitude of the disturbance is not negligible. It was subsequently pointed out by Queney (1952) that friction can only reduce the amplitude of the waves appreciably over a period of days, whereas the air passes through the region in which the disturbance is appreciable in, at most, a few hours.

If we wished to argue by analogy and say that at a great distance upstream the solution with zero amplitude at the ground is absent then we would have to put $A = B$, for $A = -B$ corresponds to $w = 0$ at $z = 0$.

(iii) *A time-dependent solution*

Hoiland (1952) showed that if a stream of water initially at rest everywhere is suddenly made to move with uniform velocity (or, alternatively, an obstacle on the bed of the stream is suddenly started into uniform motion) then the wave train on the lee side, as found by Rayleigh, is soon established. Applying the same method to an airstream, Wurtele (1953) and Palm (1952) have deduced that if started up from rest in this way (or for that matter gradually but not too slowly) the motion would tend to that given by Queney and Lyra, namely the steady-state solution corresponding to $B = 0$.

There is no objection to this as a method of obtaining a solution of a problem, but whether this is how wave motion in an actual airstream is initiated may be questioned. The motion over a hill is of the kind supposed in the perturbation theory only part of the time. There may be standing or moving eddies on the lee side of the hill or important vortical motions up at high levels (usually described in one actual kind of motion as clear-air turbulence), and then, by some change in the nature of the airstream, wave-type motion sets in. There is no reason at present to suppose that it sets in first at the ground; indeed the Sierra Wave in California sometimes seems to behave like a hydraulic jump at low levels and to execute wave motion higher up. There is no proper analogy between the motion gradually spreading downstream to infinity and spreading upwards to infinity from an obstacle in the atmosphere because the whole depth of the atmosphere is linked by gravity and this linkage is instantaneous if sound waves are assumed to travel with infinite speed – as they are in these treatments.

The energy of the wave motion is derived from the energy of the original stream, and unless there is obviously a transport of energy from one level to another there is no reason why the wave motion should appear first at any one level. Hoiland's approach does not give a reason for rejecting one of the solutions in the steady flow case. There may be time-dependent solutions which can tend to a different steady state as time proceeds.

(iv) *The flow of energy*

Arguing that the disturbance owes its origin to the ground contour the solution in which $A = 0$ was chosen in I because waves of the form $\cos(kx - \nu z)$ are as shown in Fig. 1. The air is flowing from left to right and the wave crests are tilted in a downwind direction. This means that the planes containing the wave crests have a component of velocity upstream equal, and opposite, to the velocity of the stream, U, and a component upwards equal to $\nu U/k$. The waves therefore *appear* to travel away from the ground. If $B = 0$ they appear to travel towards the ground.

Over half a wavelength, $X_1 X_2$, the air is flowing upwards and the velocity is greater than in the half, $X_2 X_3$; where it is flowing downwards. More kinetic energy is thus flowing upwards than downwards. In this sense energy is flowing upwards, and a statement to this effect was made in I. This is not strictly correct because the velocity along $X_2 X_3$ is less than along $X_1 X_2$ and so the pressure is higher there. Thus the air below $X_1 X_3$ is having work done on it by the air above.

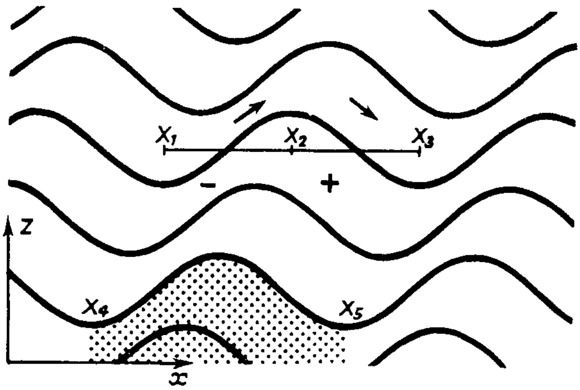

Figure 1. The streamlines of stationary waves represented by cos $(kx \pm \nu z)$. With the positive sign the flow is from right to left, with the negative sign it is from left to right. The wave crests lie in surfaces which have a horizontal component of velocity upstream, relative to the air, and a vertical component downwards and upwards in the respective cases.

In the case of flow from left to right kinetic energy is flowing upwards but work is being done by the upper layers on the lower layers and the net flow of energy is zero.

The pressure on right half of the surface containing the streamline $X_4\ X_5$ is greater than on the left half so that there is a horizontal thrust to the left on it.

When the waves are represented by cos $kz\, e^{-\mu z}$ the crests lie in vertical planes and there is no flow of kinetic energy, no work done by lower layers on upper layers, and no thrust on a stream surface.

The mean upward flow of kinetic energy across a horizontal surface of unit area is equal to

$$\frac{k}{2\pi}\int_0^{\frac{2\pi}{k}} w\,\tfrac{1}{2}\,\rho_0\,(U^2 + 2\,Uu)\,dx + \text{higher-order terms}$$

$$= \frac{k}{2\pi}\int_0^{\frac{2\pi}{k}} \rho_0\,Uu\,w\,dx = E \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

The perturbation of pressure is $p = -\rho_0\, Uu$, so that the mean rate of working on the air above a horizontal surface by the air below is equal to

$$\frac{k}{2\pi}\int_0^{\frac{2\pi}{k}} p\,w\,dx = -E \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

There is therefore no total flow of energy either upwards or downwards in any of these wave motions. We must expect this conclusion because if in any layer of the atmosphere the value of l is less than k the two possible types of wave have their crests in vertical planes. In particular if there is an adiabatic lapse rate and a linear velocity profile in the bottom layer then $l = 0$ and no energy could flow upwards from the ground for any wavelength because the waves are symmetrical and their crests lie in vertical planes.

(v) *The airflow across a plateau*

By integrating the solution for a two-dimensional ridge (II, Eq. (11)) with respect to x we obtain the airflow up on to or down from a semi-infinite plateau. The formulae are

$$U\zeta/U_1 = a^2\left\{\tan^{-1}\frac{x}{b}\cos lz \mp \tfrac{1}{2}\log\frac{b^2}{b^2+x^2}\sin lz\right\} \qquad . \quad . \qquad (5)$$

$$\zeta_1 = a^2\ \tan^{-1}\frac{x}{b} \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

the upper and lower signs in (5) corresponding to $B = 0$ and $A = 0$ respectively.

For large positive or negative x the logarithm term becomes indefinitely large and the only practical solution is to omit it. This is equivalent to putting $A = B$, in accord with the suggestion at the end of (ii) above.

Alternatively we could argue that the perturbation method does not succeed because it leads to infinite displacements if $A \neq B$. But if we are content to reject the positive exponential in (2) when $k > l$ because it is not bounded, it seems legitimate to put $A = B$ in this case.

(vi) *The drag on a mountain*

The last section suggests a solution which is determined entirely by the ground shape. The behaviour at infinite height does not arise. We might insist, for instance, that the airstream should exert a thrust on the mountain in the direction of flow. The mean drag per unit length is seen, by considering the section $X_4\, X_5$, to be

$$\frac{k}{2\pi}\int_0^{\frac{2\pi}{k}} p\,\frac{w}{U}\,dx = -\,U\,E \quad . \quad . \quad . \quad . \quad (7)$$

As we have seen, this is zero, in cases where $k > l$, because the wave crests lie in vertical planes so that, if there were an adiabatic layer at the bottom of the atmosphere, there could be no drag exerted by the ground, even if $k < l$ higher up. Consideration of drag is really irrelevant in perturbation theory because the stream with velocity U is supposed to be maintained by an influence not discussed – the large-scale pressure field.

If the pressure on the lee side of a hill were greater than at the crest we would expect separation of the flow and the formation of standing or moving eddies. This is often observed. If the pressure were less on the lee side we would expect large velocities down the lee slope, and this too is often observed. It seems that both cases are possible. The large velocities down the lee slope have generally been observed when lee waves form : for instance Helm Bars accompany the Helm Wind (Manley 1945); so that to have a pressure minimum on the lee side of a hill may require an airstream capable of containing lee waves.

(vii) *The importance of lee waves*

Lee waves are those which have zero amplitude at the ground, have a maximum at some middle level, and fade away higher up. The neighbouring wavelengths behave in a similar way except that they have finite amplitude at the ground and therefore only appear *over*, and not to the lee of, hills; they too have a maximum amplitude in middle levels and are therefore of great interest, many of their properties are similar to those of lee waves. Waves of much shorter length have a maximum amplitude at the ground and are therefore less interesting in the study of air motion produced in the upper air by flow over mountains.

Waves of much longer length than lee waves present the problem discussed in this section. There is no reason to suppose that the motion is necessarily of the type described by perturbation theory, and the fact that we cannot obtain a solution that dies away upwards is good reason for supposing otherwise.

The behaviour of lee waves is representative of the behaviour of the band of wavelengths that is important, and therefore merits special attention.

(viii) *The possibility of a solution*

It has been supposed that any given airstream can find a way of flowing over an obstacle in the manner prescribed by the linearised equations. Dr. R. R. Long, during a recent conference,* questioned this hypothesis. He pointed out that, in laboratory experiments, if certain fluid streams were blocked by an obstacle a solitary wave like a hydraulic jump travelled upstream in such a way as to modify the whole of the oncoming current. Furthermore, in many cases, a jump occurs on the lee side and the stream passing away downstream is not the same as the one coming from upstream. Professor Queney has recently (privately during the International Conference at Toronto) expressed a similar view – that we are not justified in expecting to find a solution and that the flow is not of the kind supposed.

(ix) *The behaviour of lee waves*

In I and II attention was mainly given to the properties of lee waves. Dr. Long pointed out that waves of finite amplitude could not progress indefinitely without change of form. In the atmosphere it is now known that regions of turbulent motion sometimes occur around the crests of waves and it may be that the dissipation thereby produced prevents any further distortion of the wave as it proceeds through the air.

Even if the flow is not exactly like that supposed in the linearized theory for all wavelengths then the behaviour of the lee waves is probably not very different. This belief was fortified by Förchtgott's observations (described by Scorer 1952) which tended to show that only when conditions were favourable for lee waves was the flow laminar. In other cases the flow separated from the ground and vertical overturning occurred.

For this reason we shall study what kinds of airstream can contain these waves, and compare different airstreams that can contain waves of the same length. In cases where lee waves are not possible we do not at present know what the flow is like for certain, though the observations of Förchtgott and of Long (1953) in connection with the ' Bishop wave ' phenomenon give some enlightenment.

(x) *Mother-of-pearl clouds*

If we accept the last section, and say that only waves that are ' contained by the airstream ' (those for which negative exponential solution in the uppermost layer is possible) can occur we must explain how mother-of-pearl clouds can be produced orographically.

Because these clouds do not always turn into ice clouds their temperature is probably just above $-40°C$ in which case they are probably in the lower part of the ozonosphere, which is known to be lowered in cases of intense cyclonic activity so that such a temperature can reasonably be supposed to occur at heights around 26 km.

It is often argued that the tropopause, because it is under a layer of air of high static stability, acts as a lid to wave motions. This is not true for waves of the length under discussion (where the earth's rotation is unimportant) because l^2 is increased as β is increased and so k exceeds l by a smaller amount in the stratosphere than in the troposphere. A decrease of wind above the tropopause has the same effect. If there were a sufficiently deep layer of zero stability ($\beta = 0$, $l^2 = 0$) then $k > l$ for all k, and $A = 0$ and waves of all lengths would be decreased in amplitude with height. It has been argued by Scorer (1953a) that the upper half of the ozonosphere is such a layer for in it convection is active. Thus, provided that the wind has a component across the mountains at all heights, so that the disturbance due to the mountains can be communicated to those great heights, there is no reason why a disturbance of the kind supposed in the perturbation theory should not occur.

* Symposium on ' The use of models in geophysical fluid dynamics,' held at Johns Hopkins University, September 1953.

(xi) *Flow over an isolated hill*

The above discussion arises because neither of the two possible solutions of (1) decreases as $z \to \infty$. The mountain ridge has been assumed to be two-dimensional. We shall now see that if it is isolated both solutions decrease in amplitude with height. This does not avoid a choice having to be made.

From II we have, for flow across the ridge whose height is

$$\zeta_1 = \frac{a^2\, b}{b^2 + x^2} = a^2 \int_0^\infty e^{-kb+ikx}\, dk \quad \text{(real part)} \qquad (8)$$

the disturbance at height z is

$$\zeta = \frac{U_1}{U}\, a^2 \int_0^\infty e^{-kb+ikx-\mu z}\, dk$$

$$\simeq \frac{U_1}{U}\, a^2\, \frac{b + ix}{b^2 + x^2}\, e^{-ilz} \quad \text{(real part)} \qquad (9)$$

This corresponds to the solution $A = 0$. That for $B = 0$ is obtained by reversing the sign of l. If the airstream makes angle ϕ with the line normal to the crest of the ridge and polar coordinates (r, θ) are used, writing

$$X = r\cos(\theta + \phi), \quad Y = r\sin(\theta + \phi) \qquad (10)$$

we have

$$\zeta_1(\phi) = \frac{a^2\, b}{b^2 + X^2} \qquad (11)$$

$$\zeta(\phi) = \frac{U_1}{U}\, a^2\, \frac{b + iX}{b^2 + X^2}\, e^{-ilz\sec\phi} \qquad (12)$$

$$= \frac{U_1}{U}\, a^2\, (b^2 + X^2)^{-\frac{1}{2}}\, e^{ix} \qquad (13)$$

where $x = T - lz\sec\phi$, and $T = \tan^{-1}\dfrac{X}{b}$. A solitary hill may be represented by the integral

$$\zeta_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Phi(\phi)\, \frac{a^2\, b}{b^2 + X^2}\, d\phi \qquad (14)$$

the hill being imagined as composed of a set of ridges at various angles ϕ all passing through the origin and Φ being the relative intensity of the component at angle ϕ. The corresponding disturbance at height z is therefore

$$\zeta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Phi\, \frac{U_1}{U}\, a^2\, (b^2 + X^2)^{-\frac{1}{2}}\, e^{ix}\, d\phi \qquad (15)$$

Since $x \to -\infty$ as $\phi \to \pm\dfrac{\pi}{2}$ the integral is highly oscillatory at the ends of the range of integration and is therefore suitable for evaluation by the principle of stationary phase. In such case

$$\zeta = \Phi\, \frac{U_1}{U}\, a^2\, (b^2 + X^2)^{-\frac{1}{2}} \left(\frac{2\pi}{|x''|}\right)^{\frac{1}{2}} \cos\left(x \pm \frac{\pi}{4}\right) \qquad (16)$$

evaluated for the value of ϕ which satisfies

$$\partial x/\partial\phi = 0 \qquad (17)$$

where $x'' = \partial^2 x/\partial\phi^2$, and the sign in the argument is the sign of x''. Since x contains z linearly so does x'' and so for large z

$$\zeta \propto z^{-\frac{1}{2}} \qquad . \quad . \quad . \quad . \quad . \quad (18)$$

This result has already been quoted by Scorer (1952). It means that arguments which reject a solution on the grounds that it does not vanish for large z cannot be used for solitary hills.

3. Airstreams with lee waves

In order for lee waves to occur their amplitude must decrease to zero at great heights and be zero at the ground, which is presumed level on the lee side of the hill. Even if the ground is not level, large amplitude vertical motion away from a hill on the leeside is still produced by lee waves whose phase and amplitude may however be modified by other hills.

The waves in the lee of the standard ridge given by (8), and from which any desired profile may theoretically be built up, are (II, Eqs. (26) and (33))

$$2\, i\, a^2\, \pi\, e^{-kb+ikx} \frac{U_1\, w_{z,k}}{U_z\, \partial w_{1,k}/\partial k} \quad \text{(real part)} \qquad . \quad . \quad . \quad (19)$$

At high levels either U_z must increase indefinitely, which is impossible, or $w_{z,k}$ must decrease asymptotically to zero upwards, in order that the waves shall not extend up to infinity. This means that at some level $w_{z,k}$ is increasing downwards with a curvature of the $w_{z,k}$ – profile as indicated in Fig. 2 for $z > 0$. If the slope of $w_{z,k}$ is continuous, which is true if β and U are continuous functions of z, the curvature must change sign at some level if $w_{z,k}$ is zero at the ground. The curvature is given by (1), which $w_{z,k}$ satisfies, and so since w'' must change sign before w, as z decreases, the coefficient $(l^2 - k^2)$ must change sign. This is why the l^2-profile is important.

There are many different l^2-profiles which can achieve very similar results and so, in order to give such guidance as the theory is capable of to those wishing to interpret actual profiles some examples are now presented.

In all examples l_s^2 is assumed to be constant in the uppermost layer, $z \geqslant 0$. In each of the first four cases l_i^2 is constant from the interface down to the ground at $z = -h$, and an inversion (discontinuity) of temperature is supposed to exist at the interface. The value of l_i^2 necessary for waves of a given length to be possible is calculated for four different magnitudes of the inversion. The amplitude of the waves in the four different cases is seen to be not very different so that airstreams with these four l^2-profiles can in some respects be described as nearly 'dynamically equivalent.' The profile of $w_{z,k}$ shows that the more intense the inversion the more pronounced is the maximum of lee wave amplitude at the inversion: this fact was noted by Yates (1949).

In II it was shown that if the airstream is sampled by a balloon which ascends through waves an 'incorrect' temperature and wind profile is obtained. In the example there given a layer of nearly constant lapse rate was replaced in the soundings by two layers of steeper lapse with an inversion in between. One is tempted to hope that the profile obtained would be 'dynamically equivalent' to the original one, but the hope is groundless for the profile obtained depends on the position of release and rate of ascent of the balloon, and the wavelength. It would be possible for the balloon to ascend from the ground through a wave trough, which is warmed, up to levels where the wave amplitude was very small: the only effect then would be a warming of the lower layers. Meanwhile the effect on the wind observations depends not on the vertical displacement of the streamlines, but on the variation of it with height. Inversions and shallow layers of high wind revealed by balloon soundings may, however, often be taken as an indication that wave motion is present.

The calculations proceed as follows: In the upper layer

$$w_{s,k} = e^{-\mu_s z}, \quad \mu_s = +(k^2 - l_s^2)^{\frac{1}{2}} \qquad . \quad . \quad . \quad (20)$$

At the interface, it has been shown (Scorer 1951) Eqs. (20) and (17) that

$$\left.\begin{aligned} w_{i,k}'/w_{i,k} &= -\mu_s \frac{s^2}{1+\epsilon} + \epsilon \frac{g}{U_i^2} \\ \text{and} \qquad w_{i,k} &= w_{s,k}/s \end{aligned}\right\} \qquad . \quad . \quad . \quad (21)$$

where ϵ is the change of density at the interface, namely $(\rho_{0,i} - \rho_{0,s})/\rho_{0,i}$, and $s = U_s/U_i$. When $s \neq 1$ there is a discontinuity of velocity at the interface. In the lower layer

$$w_{i,k} = A \cos \nu_i z + B \sin \nu_i z, \quad \nu_i = +(l_i^2 - k^2)^{\frac{1}{2}} \qquad . \quad . \quad (22)$$

Applying the boundary conditions (21) at the interface, $z = 0$

$$A = 1/s, \quad B = \left(-\frac{\mu_s s^2}{1+\epsilon} + \frac{\epsilon g}{U_i^2}\right)\Big/ s\,\nu_i \qquad . \quad . \quad . \quad (23)$$

whence, at the ground at $z = -h$, using (25) below, we find that

$$\frac{\partial w_{1,k}}{\partial k} = \frac{k \sin \nu_i h}{s\,\nu_i^3}\left\{h\nu_i^2 + \frac{s^2 \nu_i^2}{(1+\epsilon)\mu_s} + \left[h\left(-\frac{\mu_s s^2}{1+\epsilon} + \frac{\epsilon g}{U_i^2}\right) - 1\right]\left[-\frac{\mu_s s^2}{1+\epsilon} + \frac{\epsilon g}{U_i^2}\right]\right\} \quad . \quad (24)$$

This last expression largely determines the amplitude of the lee waves. It is obviously not simply dependent upon any of the variables involved, though changes in s are seen to have least influence since s is almost certain to lie between about 0·8 and 1·2. In any case if $s \neq 1$ the interface is unstable for disturbances of small wavelength, while if $s = 1$ it is not unstable unless $\epsilon < 0$. We shall deal therefore with a stable airstream, and put $s = 1$ henceforth.

ϵ may vary from zero to about ·03 (for an 8°C inversion), ν_i^2 is positive if $l_i > k$ but may vary as far as $-k^2$ for an adiabatic layer ($l_i^2 = 0$). In all these examples we have kept k the same, but this is not the only way in which air streams could be compared. One might, for instance, compare various l^2-profiles which, with a given mountain, give the same maximum vertical component of velocity.

The comparisons most easily made are determined by the form of the lee-wave equation. This states that $w = 0$ at the ground, i.e.,

$$\nu_i \cot \nu_i h = -\frac{\mu_s s^2}{1+\epsilon} + \frac{\epsilon g}{U_i^2} \qquad . \quad . \quad . \quad (25)$$

From this, when k, h, μ_s, ϵ, U_i, and s are given it is easy to calculate ν_i and hence l_i. To calculate k and l_i from the amplitude is incomparably longer because of the complicated way in which k is involved in (19) via (22) and (24).

In all cases we use the kilometre as the unit of length and put $k = 1$ (lee-wavelength $= 2\pi$km), $h = 2$ km, $\mu_s = \frac{1}{2}$ km^{-1}, (i.e. $l_s^2 = \cdot 75$ km^{-2}), $U_i = 14\cdot 1$ m sec^{-1}. It is then found that the relationship between ϵ and l_i is as given in Table 1.

In case 3, since $\nu_i = 0$, (22) is replaced by

$$w_{i,k} = A + Bz. \qquad . \quad . \quad . \quad . \quad . \quad (26)$$

The relative amplitudes of the lee waves produced by the same mountain in these found cases are shown in Fig. 1.

TABLE 1. Characterstics of lee waves in 4 airstreams. The wavelength is the same in all cases. The last column gives their relative excitation by a ridge.

Case No.	ϵ (%)	Approx. inversion (°C)	l_i^2	$(\partial w_{1,k}/\partial k)^{-1}$
1	0	0	2·04	0·125
2	1	2·7	1·79	0·197
3	2	5·4	1	0·255
4	3	8·0	0	0·098

If instead of a sharp inversion ($l^2 = \infty$ at the height of the inversion) there is a shallow middle layer in which l^2 is large, then airstreams equivalent to examples 1-4 in giving the same value of k, can be obtained as follows. The lower layer is divided into two layers denoted by suffixes i (upper) and j (lower), the upper one being of depth h', the lower of depth $h - h'$.

Putting $s = 1$, $\epsilon = 0$, we find, using (21) at both interfaces, that

$$w_{i,k} = \cos \nu_i z - \frac{\mu_s}{\nu_i} \sin \nu_i z \qquad (27)$$

$$w_{j,k} = \frac{1}{\nu_i} (\nu_i \cos \nu_i h' + \mu_s \sin \nu_i h') \cos \nu_j (z + h') + \frac{1}{\nu_j} (\nu_i \sin \nu_i h' - \mu_s \cos \nu_i h') \sin \nu_j (z + h') \qquad (28)$$

and the lee-wave equation, corresponding to (22), is

$$\nu_j \cot \nu_j (h - h') = \nu_i \frac{\nu_i \sin \nu_i h' - \mu_s \cos \nu_i h'}{\nu_i \cos \nu_i h' + \mu_s \sin \nu_i h'} \qquad (29)$$

The left-hand side is replaced by $\mu_j \coth \mu_j (h - h')$ if $k > l_j$, where $\mu_j = + (k^2 - l_j^2)^{\frac{1}{2}} = + i \nu_j$.

TABLE 2. Characteristics of three airstreams which have the same lee-wavelength (the last column gives the relative excitation by a ridge)

Case No.	h' (km)	l_i^2	ν_i	l_j^2	μ_j	$(\partial w_{1,k}/\partial k)^{-1}$
5	0·25	5	2	1·08	·29 i	0·156
6	0·37	5	2	0	1	0·129
7	0·65	3·25	1·5	0	1	0·151

Cases 5-7 in Table 2 show how, with the same lee-wave length, different thicknesses of a very stable middle layer correspond to different stabilities in the lower layer. It must be emphasized that the variety of airstreams all possessing the same lee-wave length is infinite, and that, though in some cases the wavelength is the most easily observed quantity, it may be changed by a factor of two or three without altering very obviously the nature of the flow. Also, it must be emphasized that all the cases presented here are particular. *No* general qualitative conclusion can be drawn without further argument. When the profile of l has been chosen there is an infinity of airstreams with that profile. We are still at liberty to choose either the velocity or temperature profile and the l-profile then determines the other. This means, among other things, that the amplitude of the

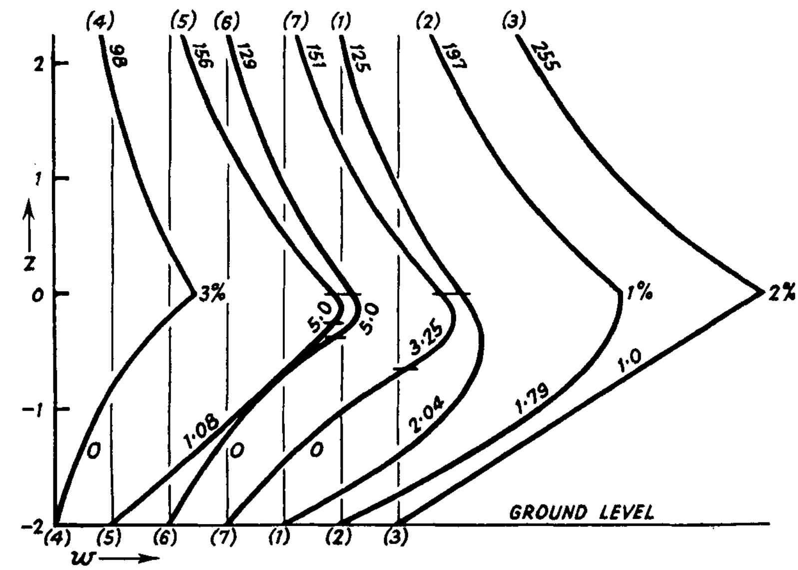

Figure 2. The curves indicate the variation with height of the vertical velocity in waves in the lee of the ridge given by (8), for seven different airstreams which all have the same lee (stationary) wavelength of 2π km. The zero is displaced so as to separate the curves : in all cases $w = 0$ at the ground.

The relative amplitude is indicated against the top of each curve, and the number of the case against the top and bottom in brackets. In all cases $l_s^2 = 0{\cdot}75$ km^{-2} for $z > 0$. For $0 \geqslant z \geqslant -2$ km (the ground is at -2 km) the value of l^2 is indicated against the curves. Where there is a discontinuity of density (cases 2, 3, 4) the percentage discontinuity is indicated : also $\partial w/\partial z$ is discontinuous there, and it is equivalent to a very thin layer in which l is very large. The curvature takes the same sign as $(k - l)$, and in these cases $k = 1$. The maximum of the wave amplitude is at or near to where l is largest.

waves set up by the standard ridge is not uniquely determined by the quantities so far calculated because there is a variety of values of U_1/U_z for a given profile of l. The quantity $(\partial w_{1,k}/\partial k)^{-1}$ only gives one component in the formula (19). In the three-layer case we find that

$$\begin{aligned} \frac{\nu_i^2\,\nu_j^2}{k}\frac{\partial w_{1,k}}{\partial k} = {} & \cos \nu_i h' \Big\{\cos \nu_j (h-h') \Big[-\nu_j^2 - \nu_i^2 - \mu_s \nu_i^2 (h-h') - \nu_j^2 \mu_s h\Big] \\ & - \sin \nu_j (h-h') \Big[-\nu_i^2 \nu_j (h-h') + \mu_s \nu_j - \frac{\nu_i^2 \nu_j}{\mu_s} - h' \nu_j \nu_i^2\Big]\Big\} \\ & + \sin \nu_i h' \Big\{\cos \nu_j (h-h') \Big[-\mu_s \nu_i + \frac{\nu_j^2 \nu_i}{\mu_s} - \nu_i^3 (h-h') + h' \nu_i \nu_j^2\Big] \\ & + \sin \nu_j (h-h') \Big[\nu_i \nu_j (2 + \mu_s h)\Big]\Big\} \qquad . \quad . \quad . \quad (30) \end{aligned}$$

Since there are many ways of producing the same kind of wave motion it is not possible from observations, to conclude that certain features of the velocity or temperature profiles are necessary for waves to occur. Although in a particular locality a certain feature of the airstream, such as a strong inversion, may usually be found when waves occur this may simply be because the actual weather systems that occur do not produce other types of wave-bearing airstream at that place. One might argue that the limitations of the linearized equations make it necessary to seek formulae of prognostic value in an analysis of many observations, but there is no evidence that so long as the flow is laminar the conclusions of perturbation theory are much in error.

4. Conclusion

The unsettled problem of what solution should be chosen for the long wavelengths may be unreal : the perturbation solution may not be a practical one at all. This emphasizes the importance of lee waves and the band of wavelengths around them because of their large amplitude in middle levels. The properties of lee waves are similar to those of neighbouring wavelengths. The complexity of the mathematics prohibits the application of the formulae direct to actual cases, and in particular to forecasting. Some possibilities have been illustrated here, and with an understanding of them it will probably be possible to explain the behaviour of actual airstreams; also it is now clear that the profile of the quantity l^2, or of l, or of l^{-1}, is the feature that should be studied. In particular, lee waves can only exist if there is a layer low down in which l exceeds its value higher up, and the lee wavelength is such that k is equal to an intermediate value of l.

If l increases at higher levels wave motion is probably impossible because almost no waves are ' contained.' The flow is then either not steady, or not wavelike and there is overturning of some sort.

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