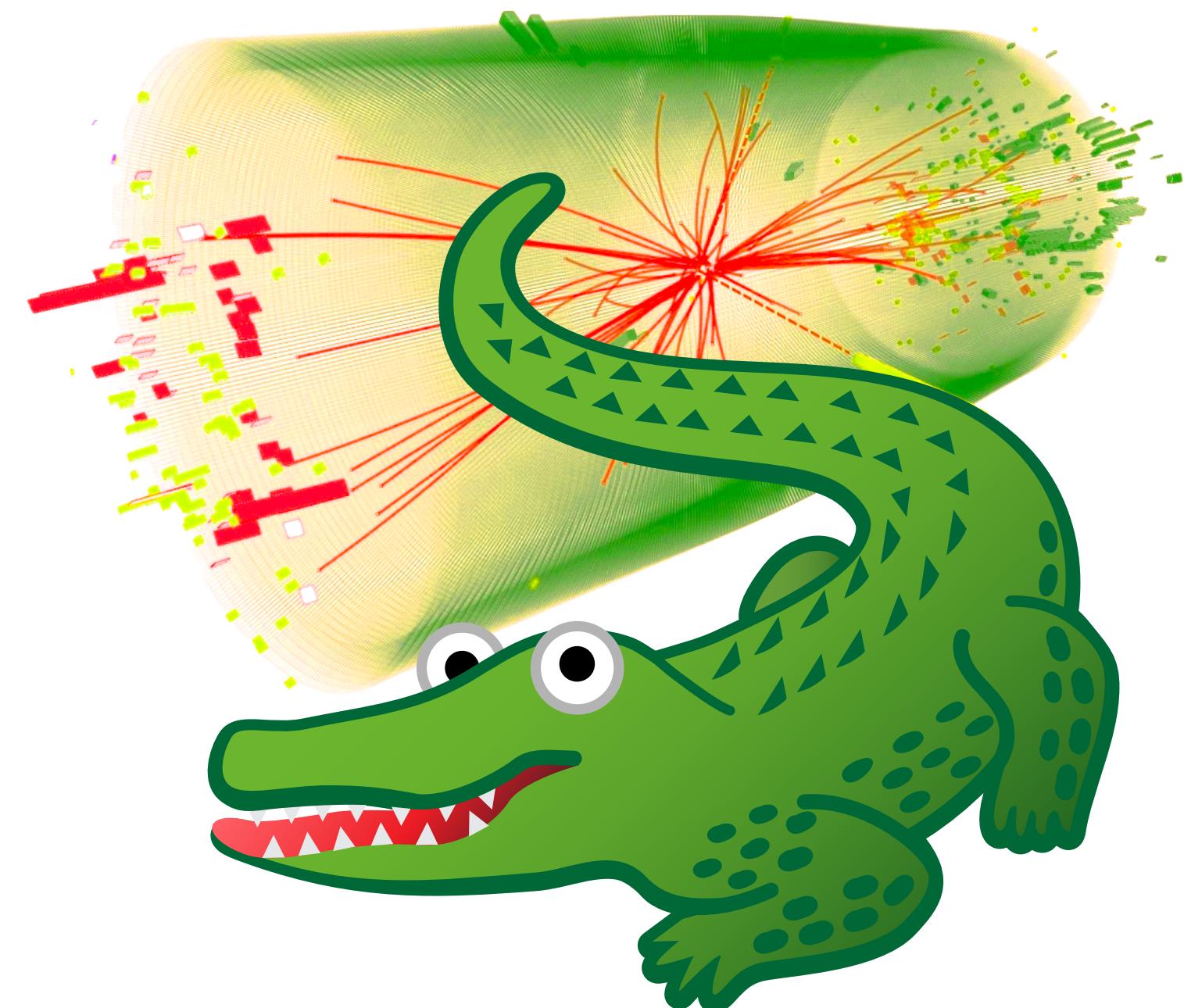


# L-GATr

Lorentz-Equivariant  
Geometric Algebra Transformer  
for High-Energy Physics

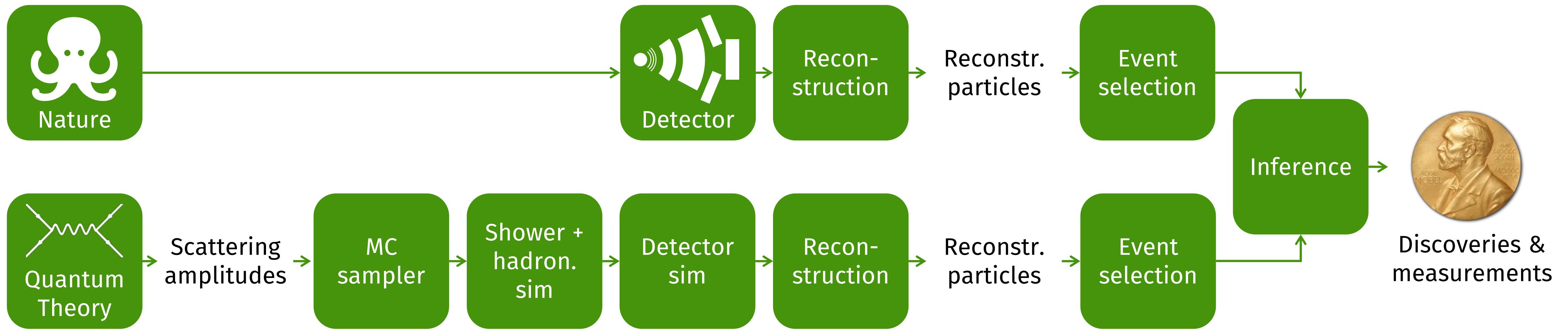


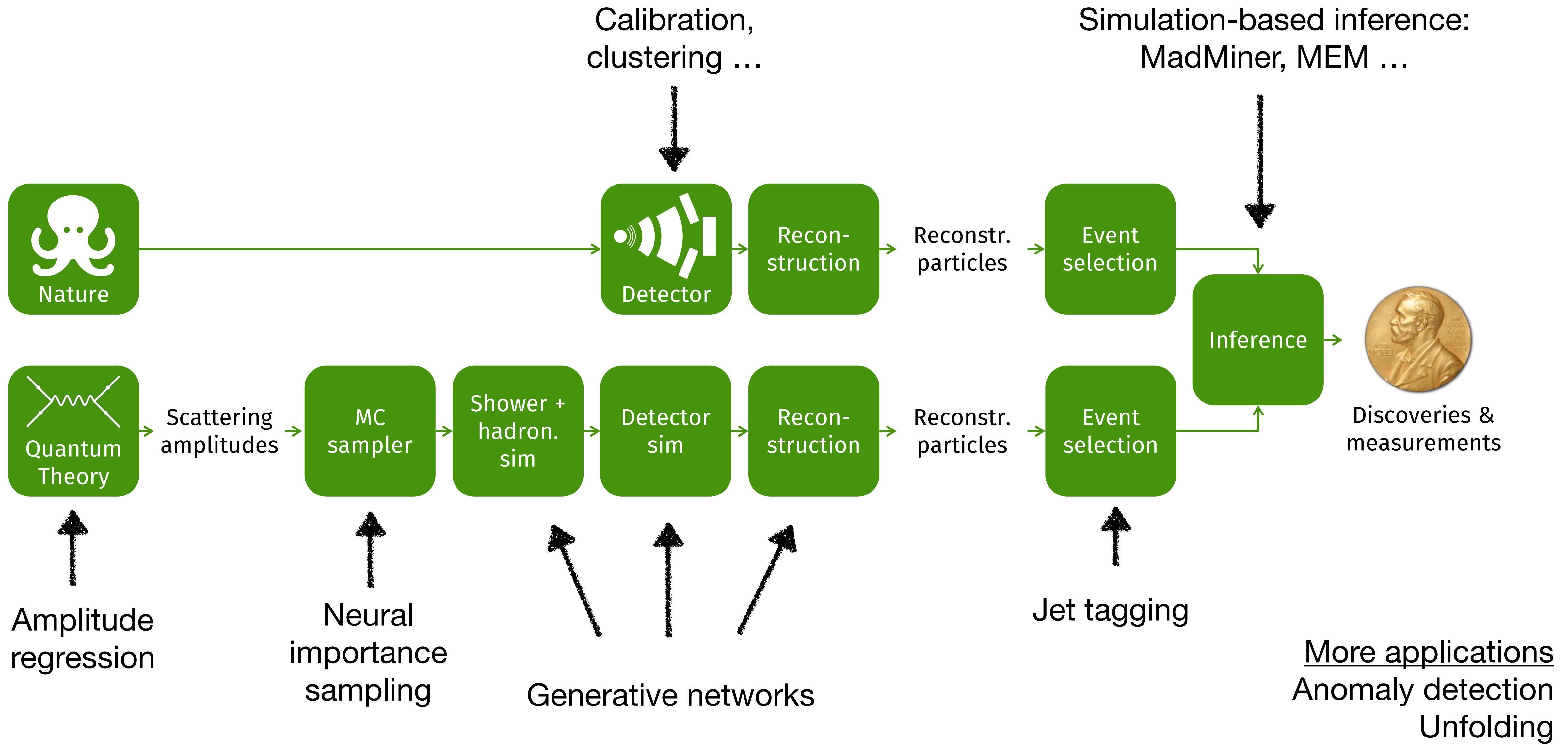
Johann Brehmer, Víctor Bresó,  
Pim de Haan, Tilman Plehn, Huilin Qu,  
Jonas Spinner, Jesse Thaler  
arXiv:2405.14806, arXiv:2411.00446

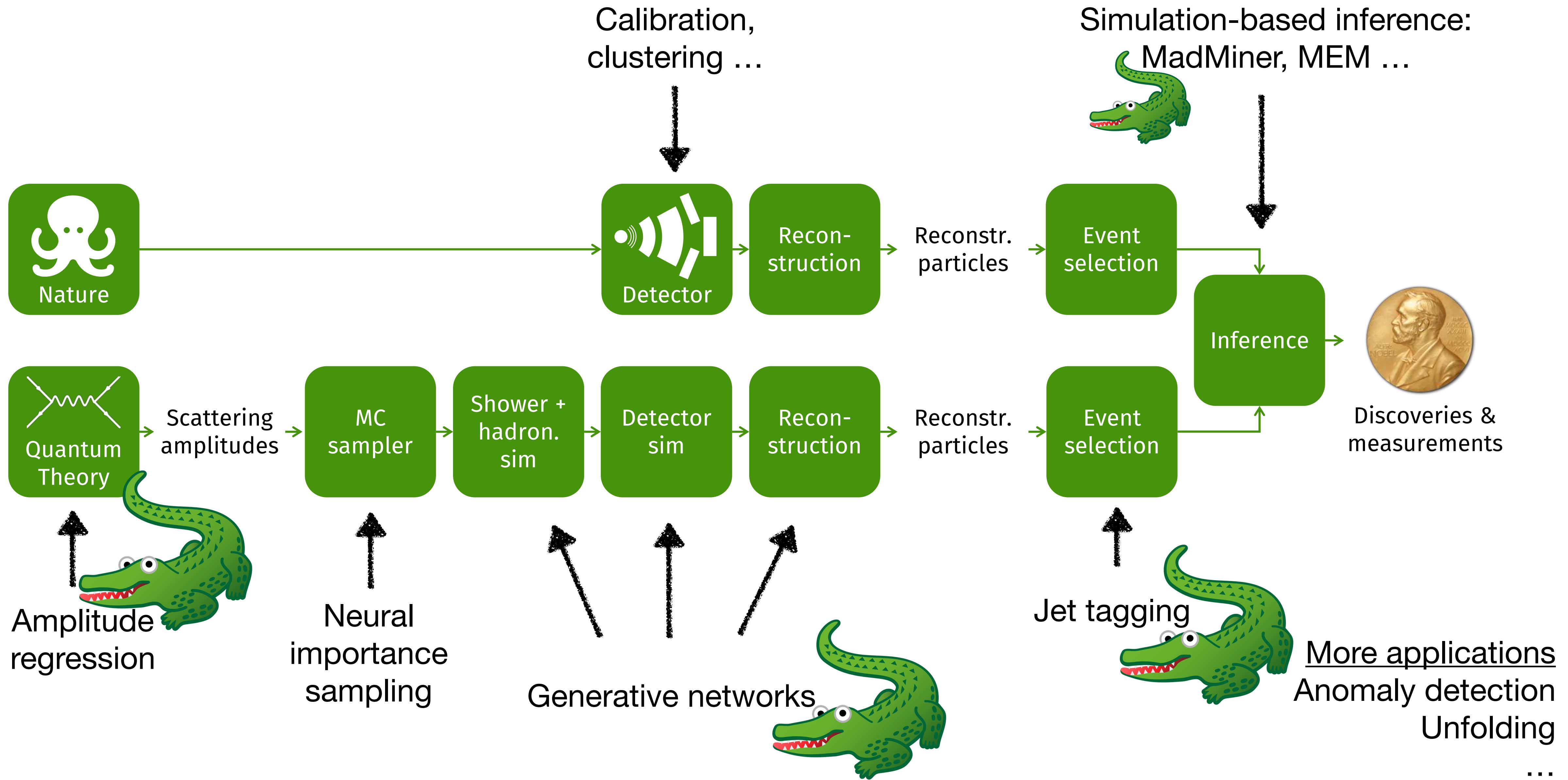
IRN Terascale @ IPHC Strasbourg  
20.05.2025

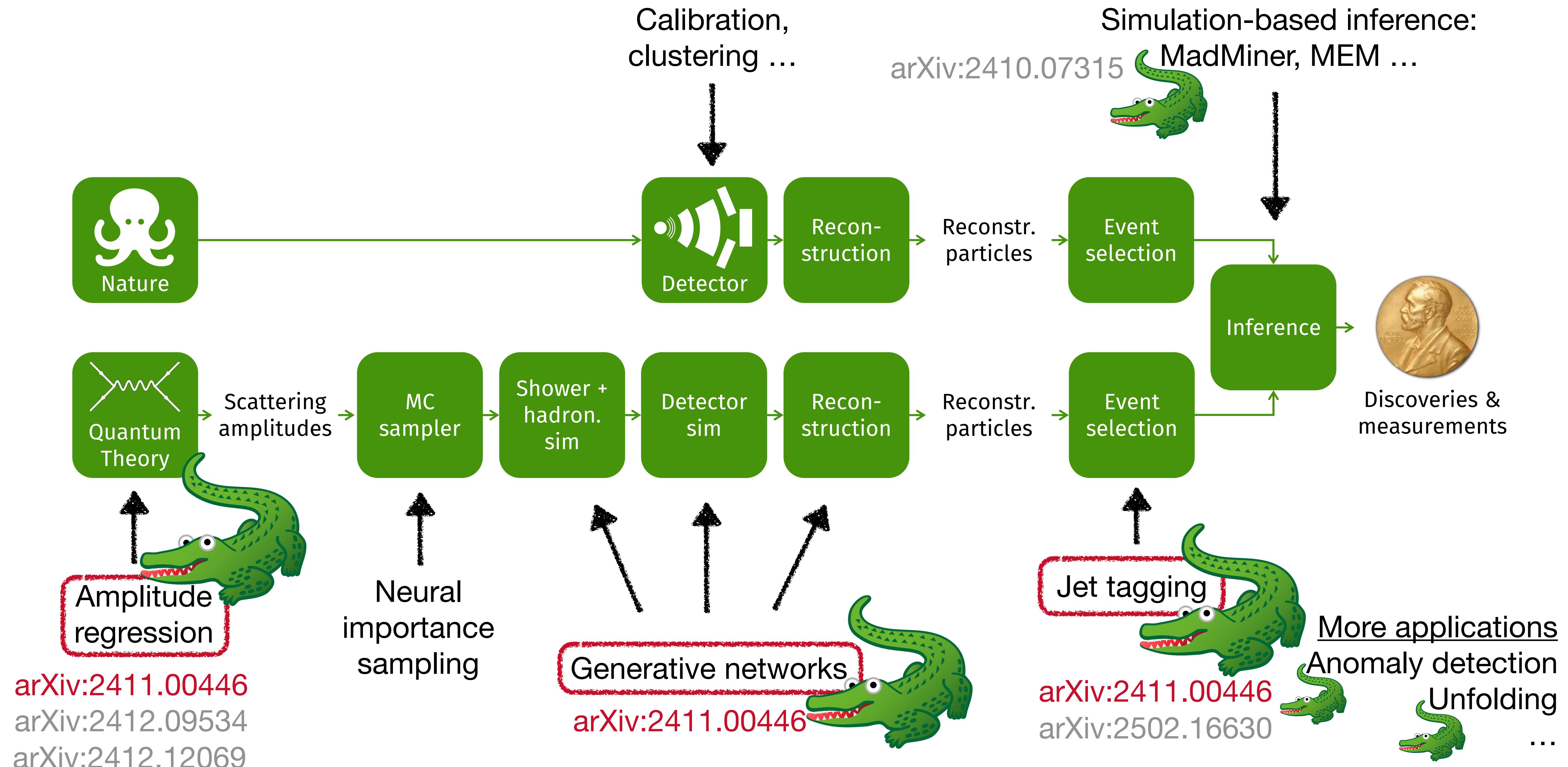


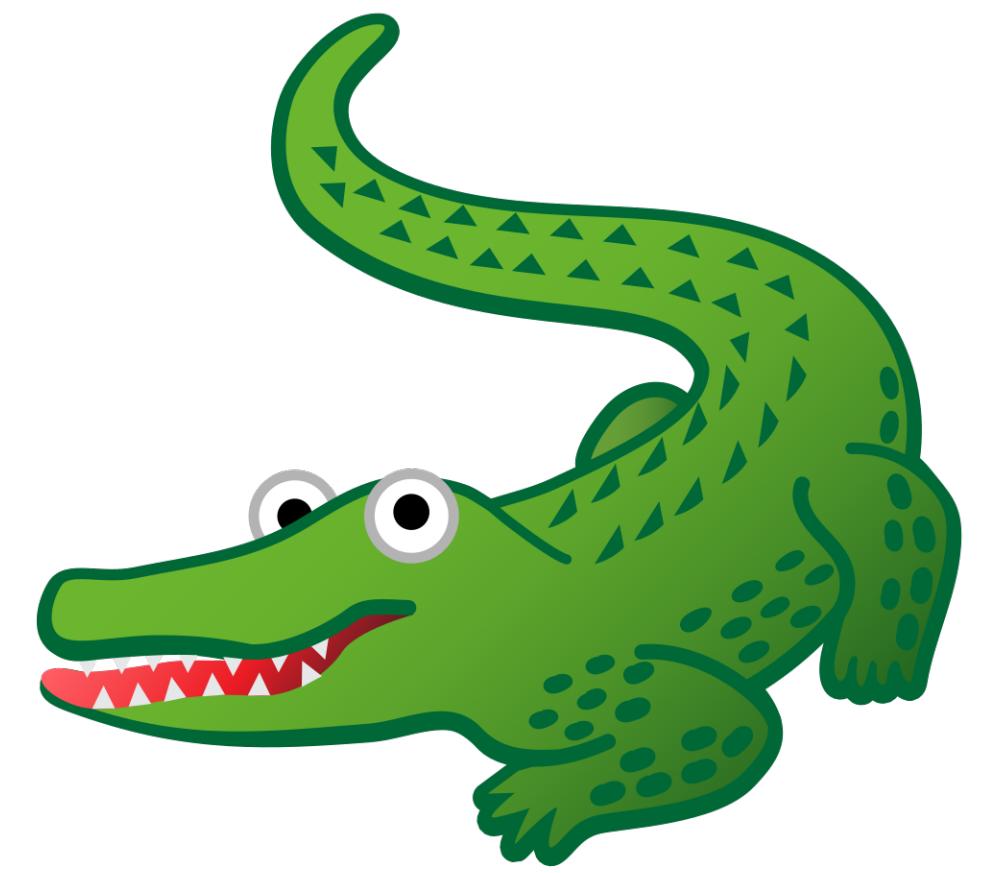
UNIVERSITÄT  
HEIDELBERG  
ZUKUNFT  
SEIT 1386



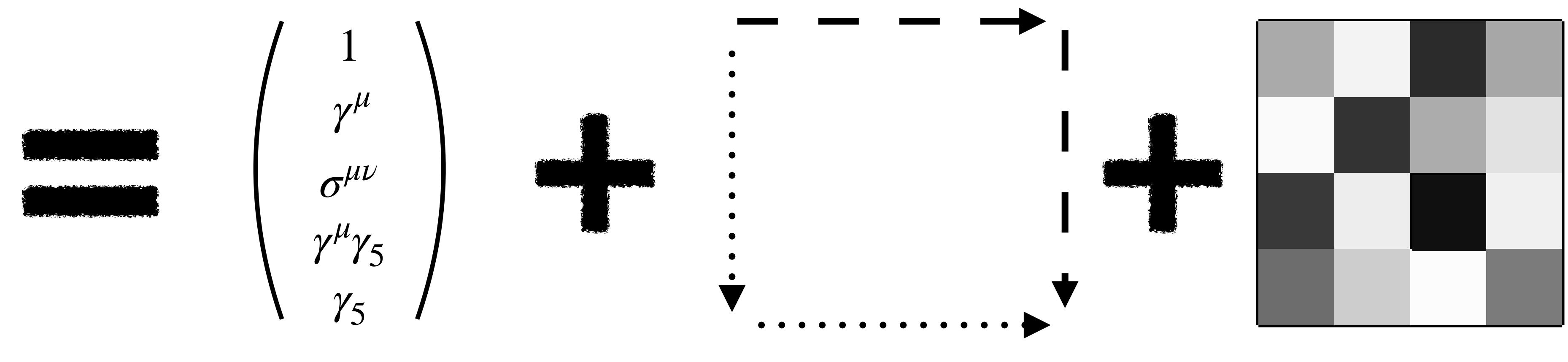








Lorentz-Equivariant  
Geometric **A**lgebra  
Transformer



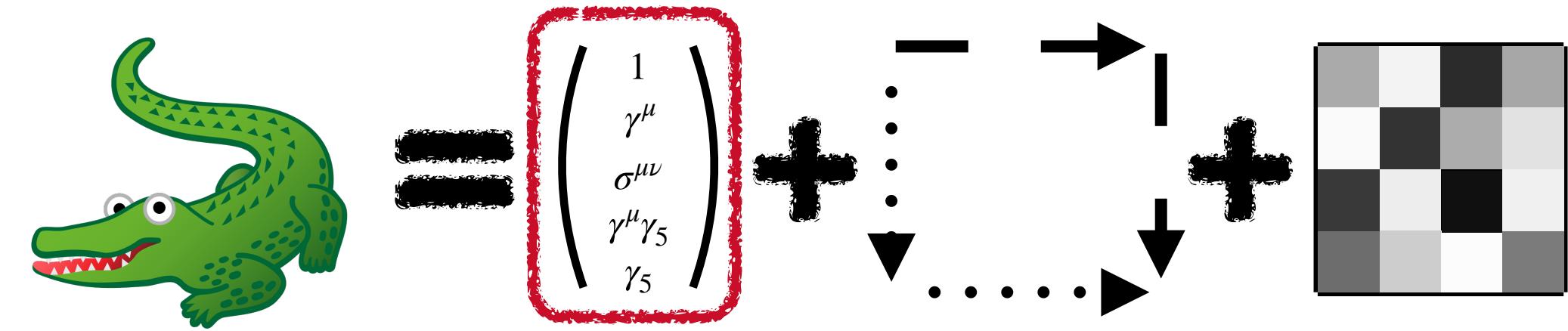
**Geometric algebra**  
representations

**Lorentz-Equivariant**  
layers

**Transformer**  
architecture

GATr was originally  
developed for E(3)  
arXiv:2305.18415

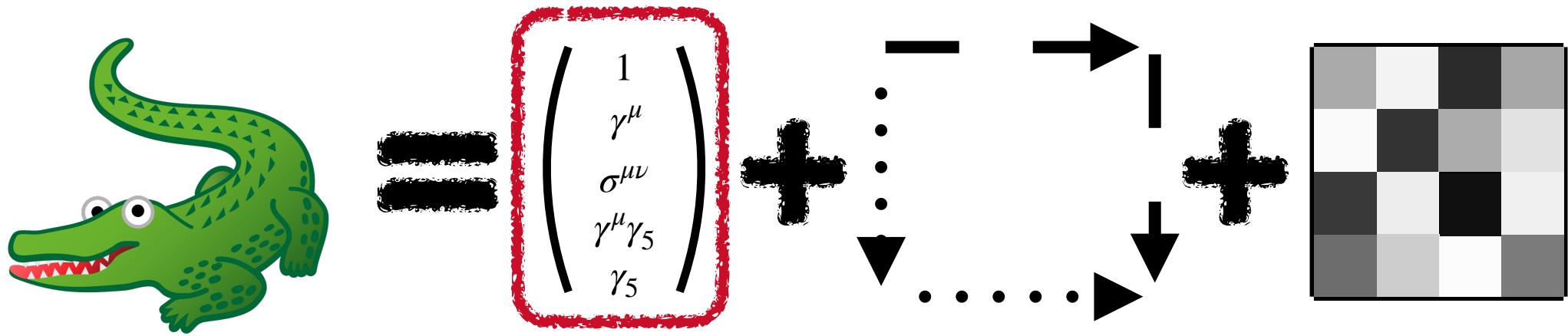
# L-GATr



**Geometric algebra = Clifford algebra**

Geometric algebra = Vector space + geometric product  $xy = \frac{\{x, y\}}{2} + \frac{[x, y]}{2}$

# L-GATr



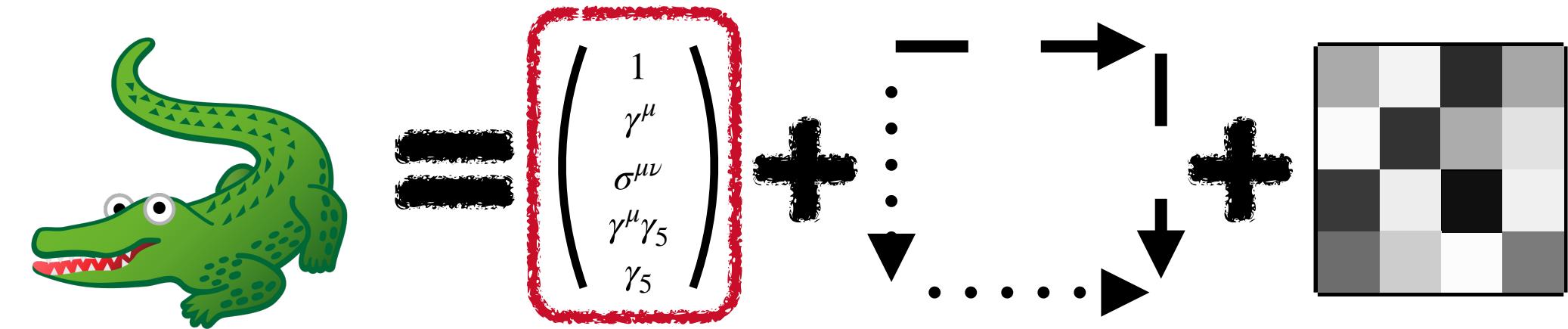
**Geometric algebra = Clifford algebra**

Geometric algebra = Vector space + geometric product  $xy = \frac{\{x, y\}}{2} + \frac{[x, y]}{2}$

Spacetime geometric algebra: Geometric algebra over vector space  $\mathbb{R}^4$  with Minkowski metric  $g = \text{diag}(1, -1, -1, -1)$

- Basis elements  $\gamma^\mu$  are orthonormal:  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$
- Dirac algebra is the same up to  $\mathbb{R} \rightarrow \mathbb{C}$

# L-GATr



## Building multivectors

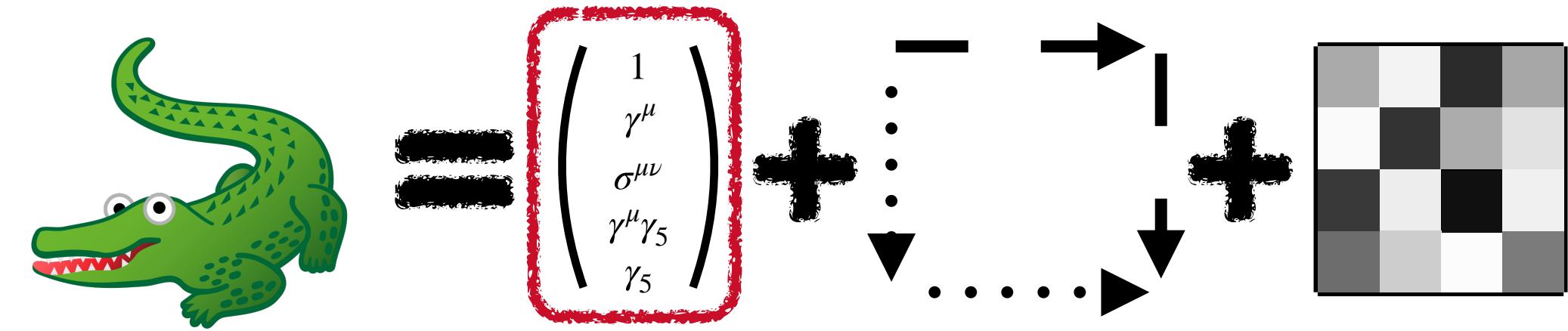
- Scalar and vectors  $1, \gamma^\mu$  (1+4 objects)
- Product of two vectors:  $\gamma^\mu \gamma^\nu = \frac{\{\gamma^\mu, \gamma^\nu\}}{2} + \frac{[\gamma^\mu, \gamma^\nu]}{2} = g^{\mu\nu} + \sigma^{\mu\nu}$  (6 new objects)
- Axial vector:  $\epsilon_{\mu\nu\rho\sigma} \gamma^\nu \gamma^\rho \gamma^\sigma \propto \gamma_\mu \gamma^5$  (4 new objects)
- Pseudoscalar:  $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \frac{1}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$  (1 new object)

Multivector:  $x = x^S 1 + x_\mu^V \gamma^\mu + x_{\mu\nu}^B \sigma^{\mu\nu} + x_\mu^A \gamma^\mu \gamma^5 + x^P \gamma^5$  with  $(x^S, x_\mu^V, x_{\mu\nu}^B, x_\mu^A, x^P) \in \mathbb{R}^{16}$

Particle:  $x^V = (E, p_x, p_y, p_z), \quad x^S = \text{PID}$

# L-GATr

## Geometric algebra representations



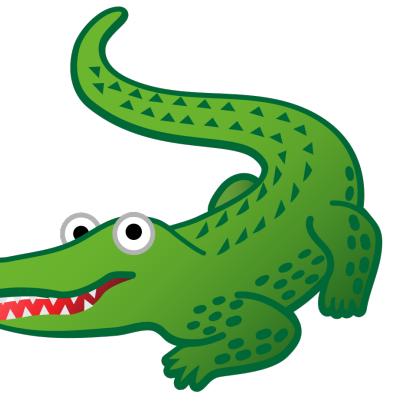
$$x^S \in \mathbb{R} \rightarrow \begin{pmatrix} x^S \\ x_\mu^V \\ x_{\mu\nu}^B \\ x_\mu^A \\ x_\mu^P \end{pmatrix} \in \mathbb{R}^{16}$$

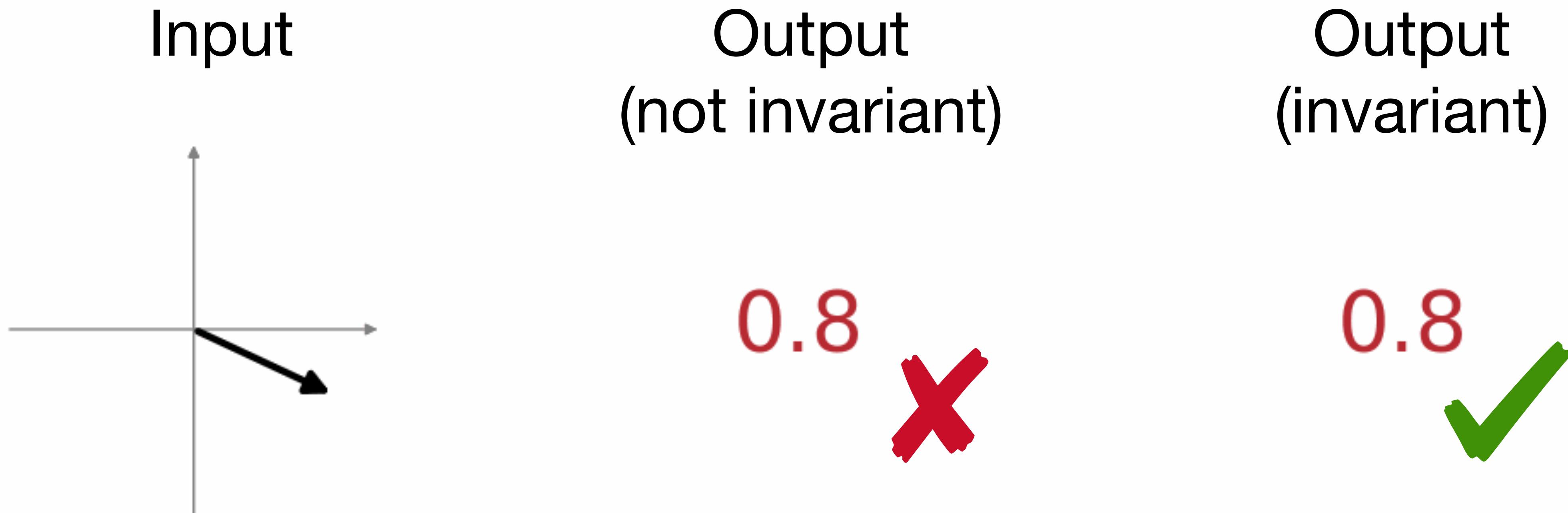
scalar channels

multivector channels

# L-GATr

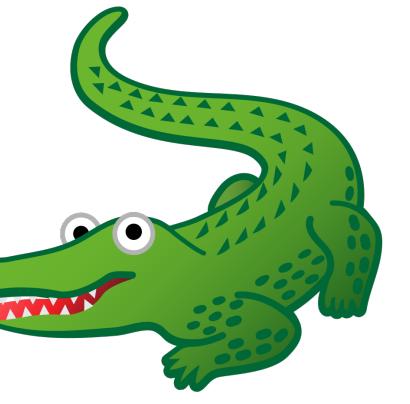
## Invariance

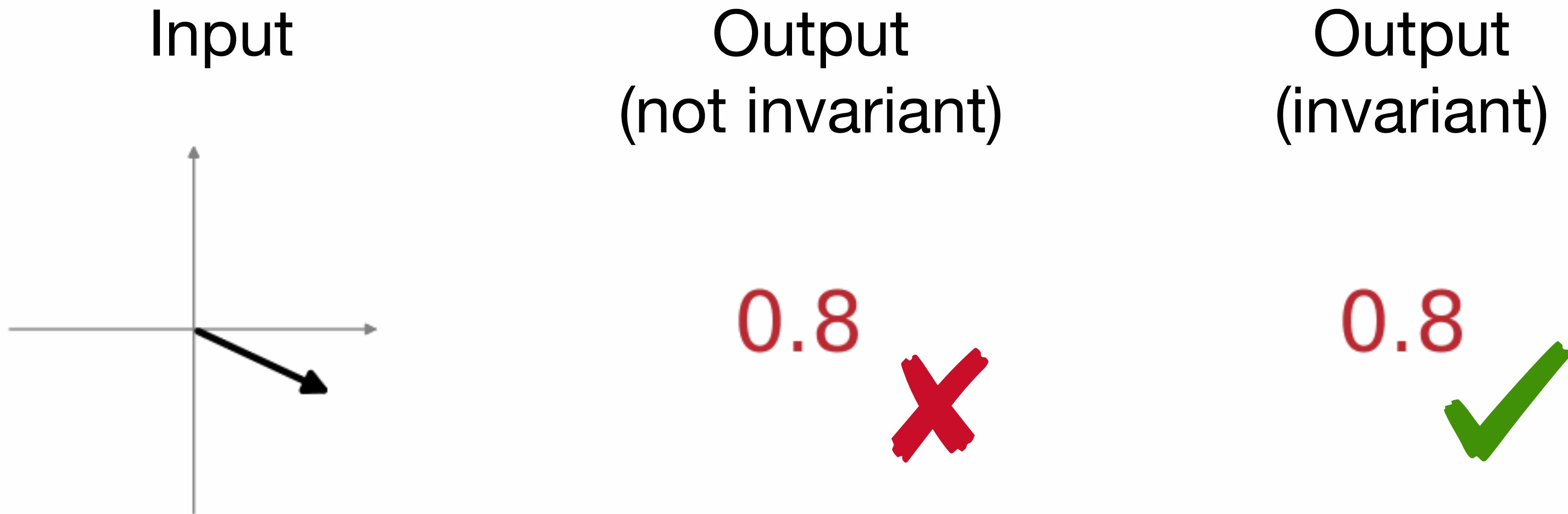

$$= \begin{pmatrix} 1 \\ \gamma^\mu \\ \sigma^{\mu\nu} \\ \gamma^\mu \gamma_5 \\ \gamma_5 \end{pmatrix} + \boxed{\dots} + \boxed{\dots} + \boxed{\text{checkered image}}$$



# L-GATr

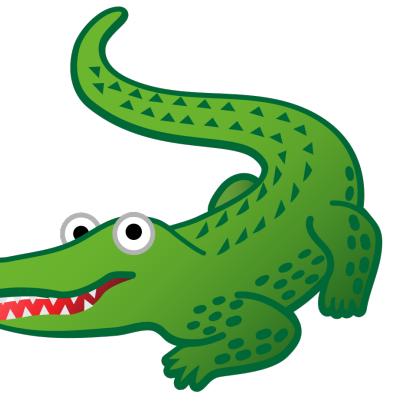
## Invariance


$$= \begin{pmatrix} 1 \\ \gamma^\mu \\ \sigma^{\mu\nu} \\ \gamma^\mu \gamma_5 \\ \gamma_5 \end{pmatrix} + \boxed{\dots} + \boxed{\dots} + \boxed{\text{checkered image}}$$

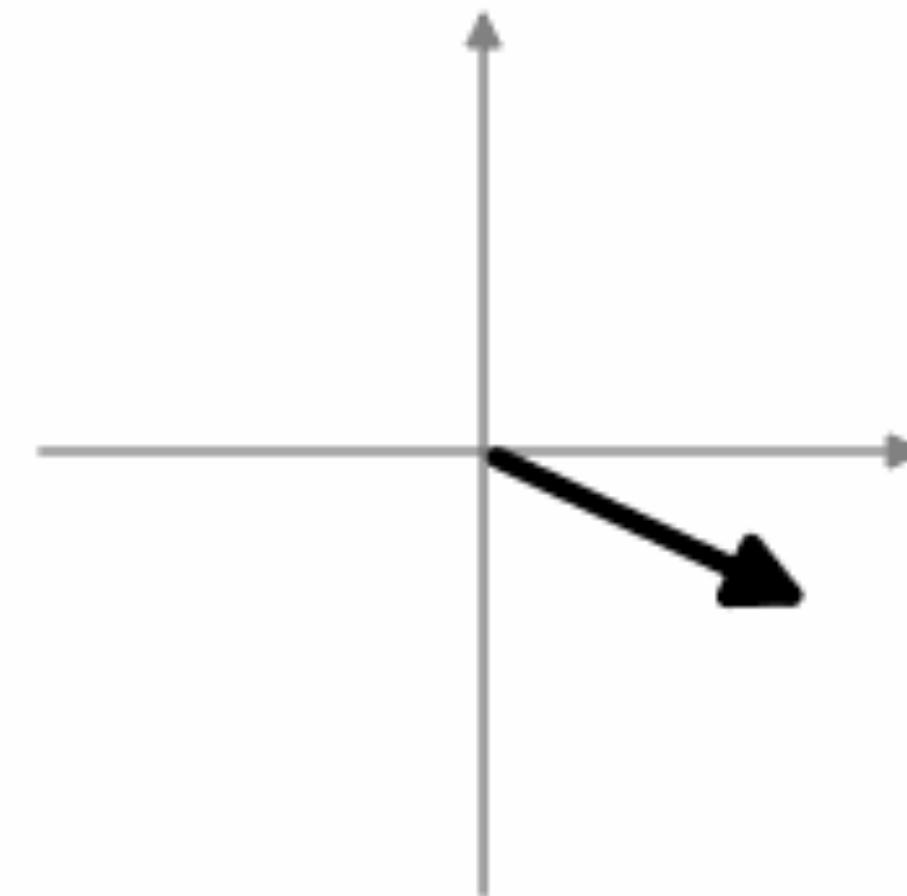


# L-GATr

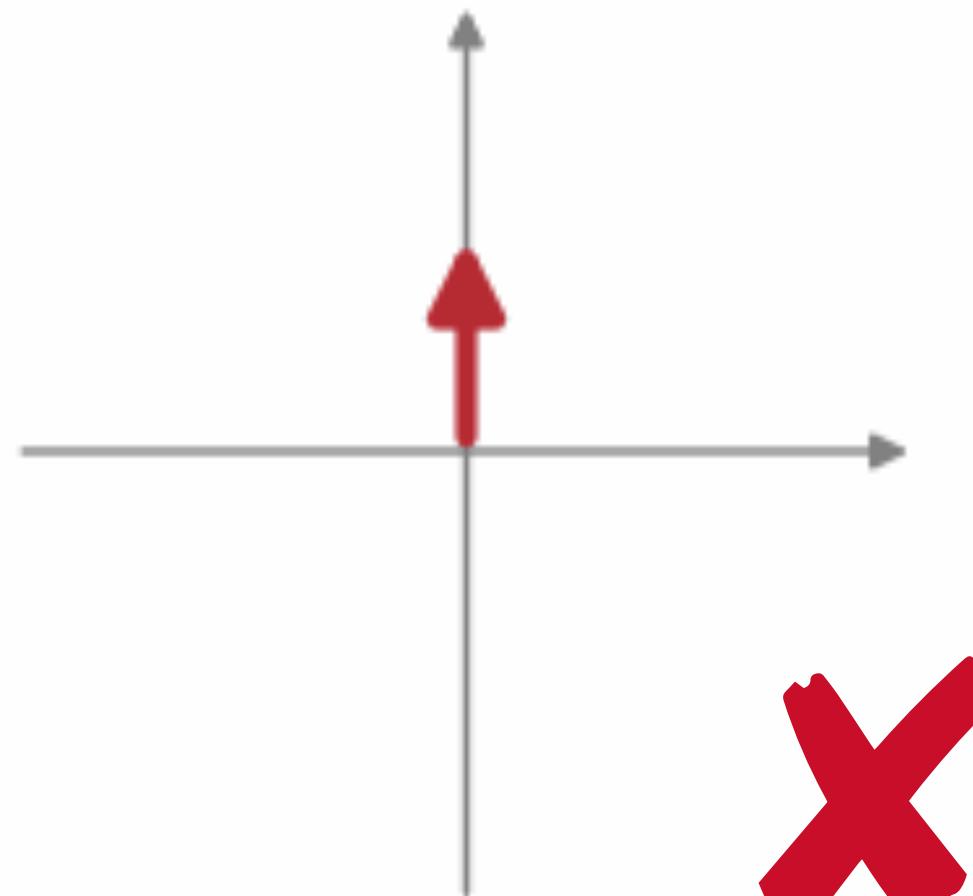
**Equivariance = Covariance**


$$= \begin{pmatrix} 1 \\ \gamma^\mu \\ \sigma^{\mu\nu} \\ \gamma^\mu \gamma_5 \\ \gamma_5 \end{pmatrix} + \boxed{\dots} + \boxed{+} + \boxed{\text{checkered image}}$$

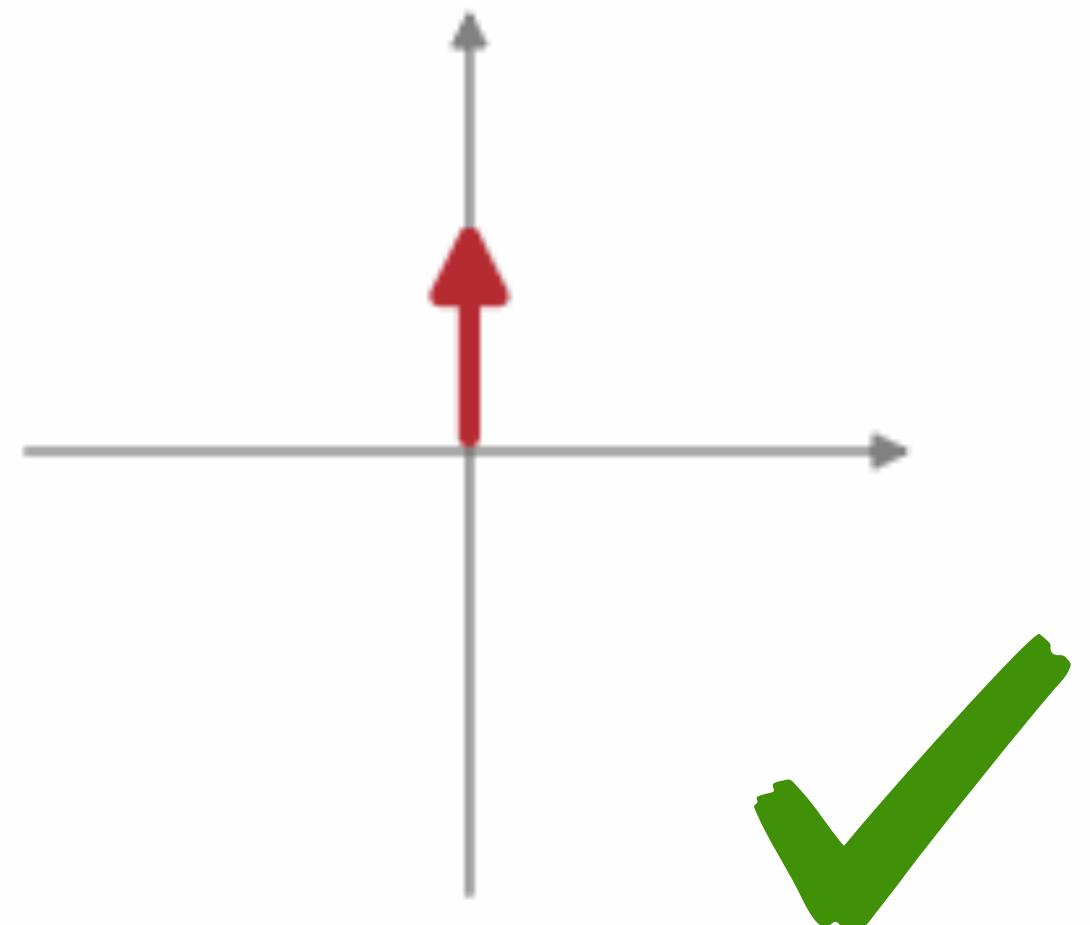
Input



Output  
(not equivariant)



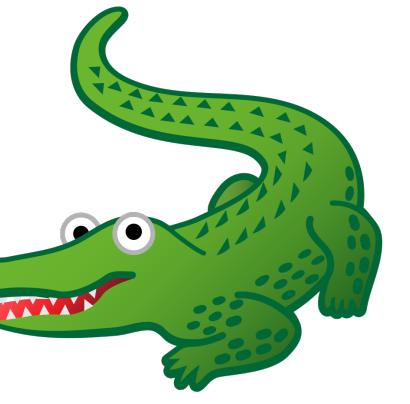
Output  
(equivariant)



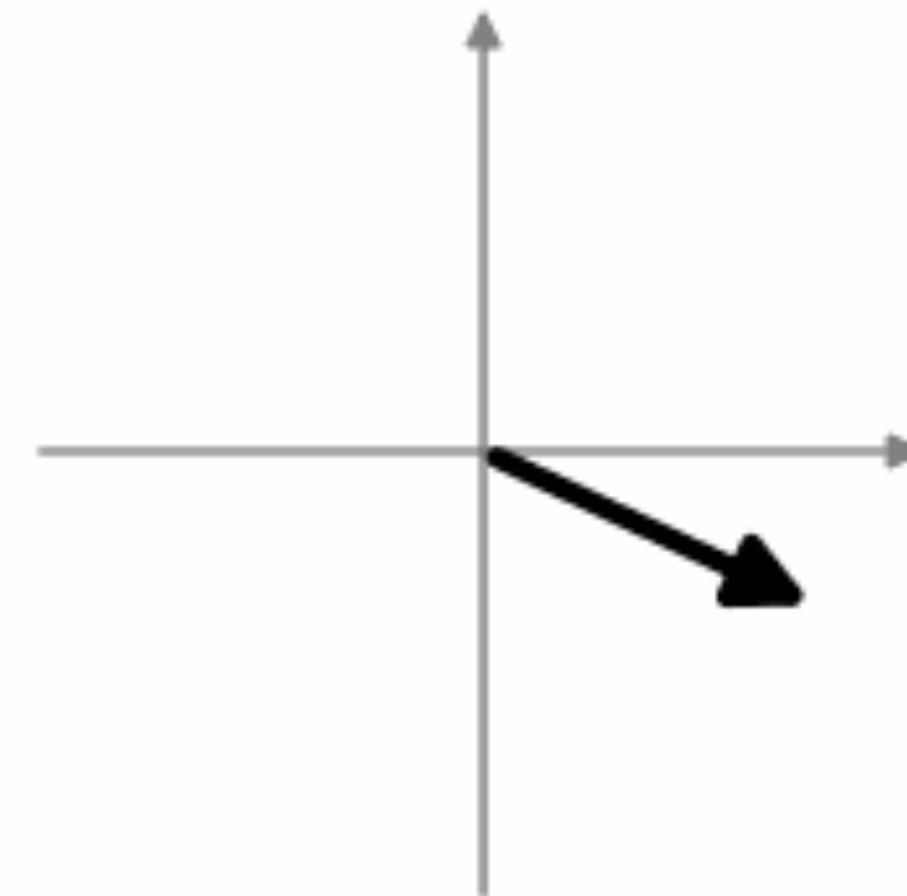
Invariance = Equivariance with scalar output

# L-GATr

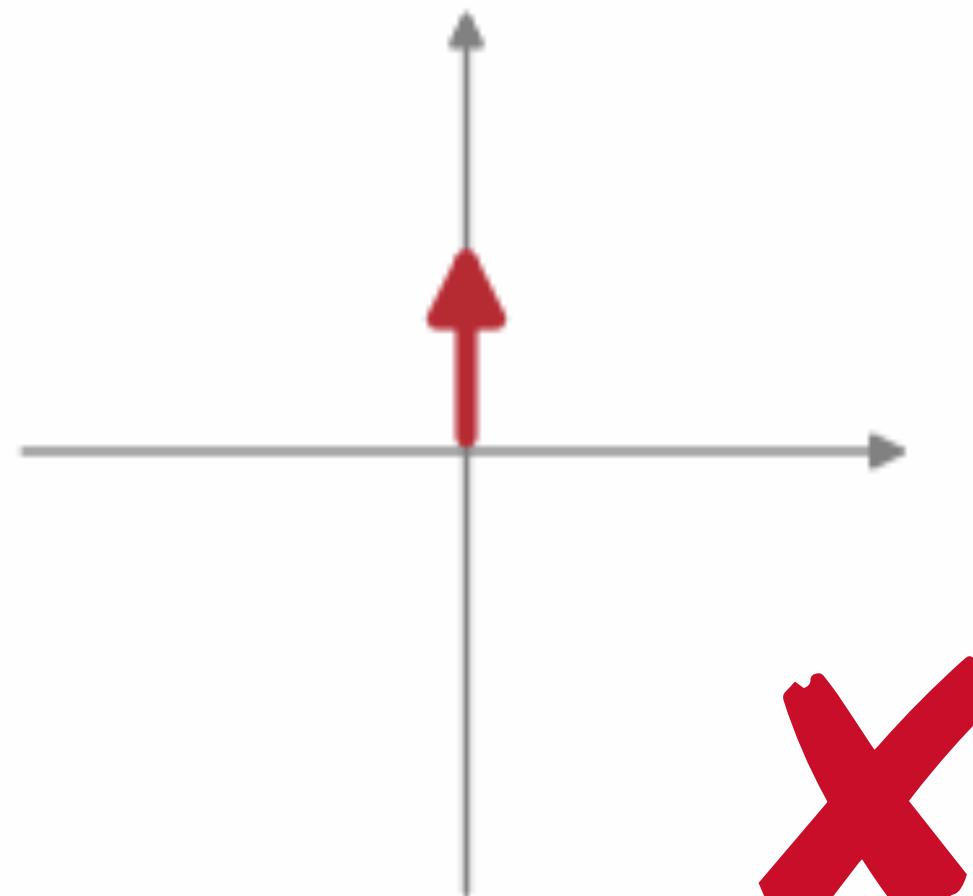
**Equivariance = Covariance**


$$= \begin{pmatrix} 1 \\ \gamma^\mu \\ \sigma^{\mu\nu} \\ \gamma^\mu \gamma_5 \\ \gamma_5 \end{pmatrix} + \boxed{\dots} + \boxed{+} + \boxed{\text{checkered image}}$$

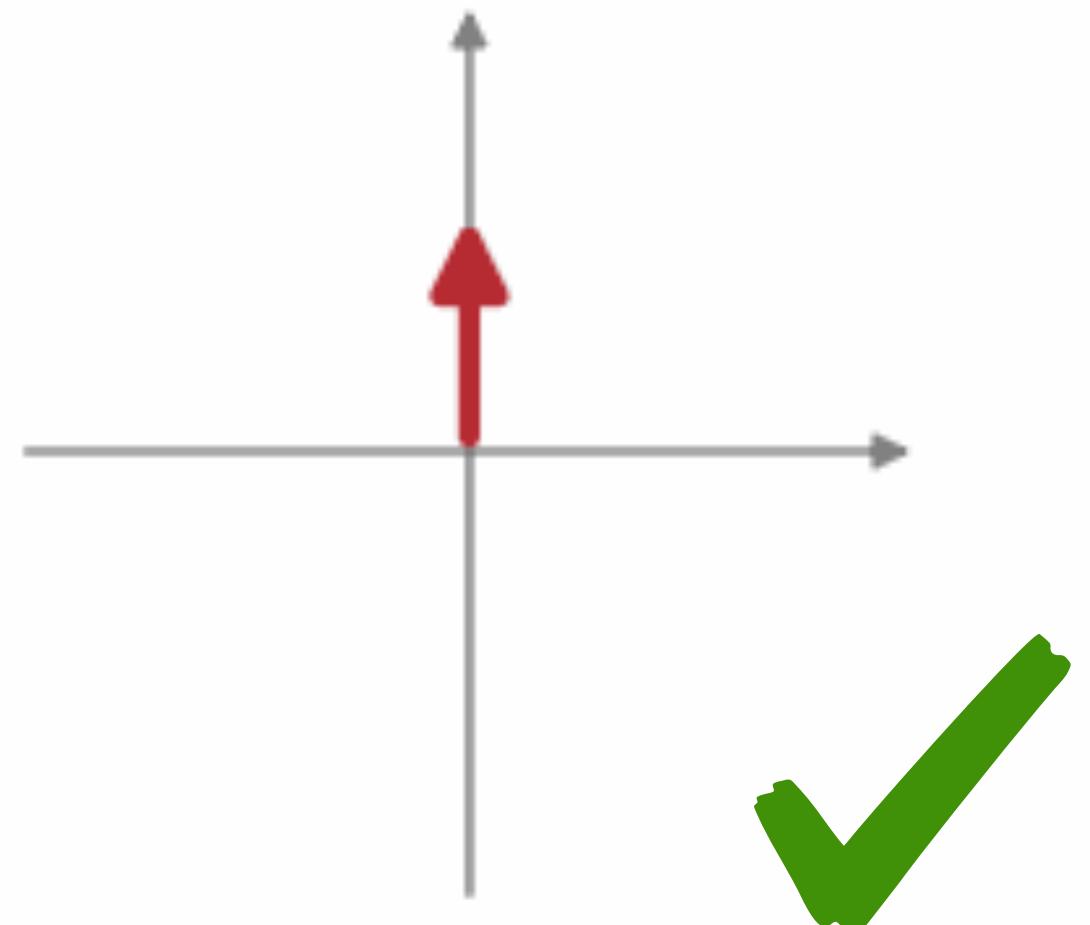
Input



Output  
(not equivariant)



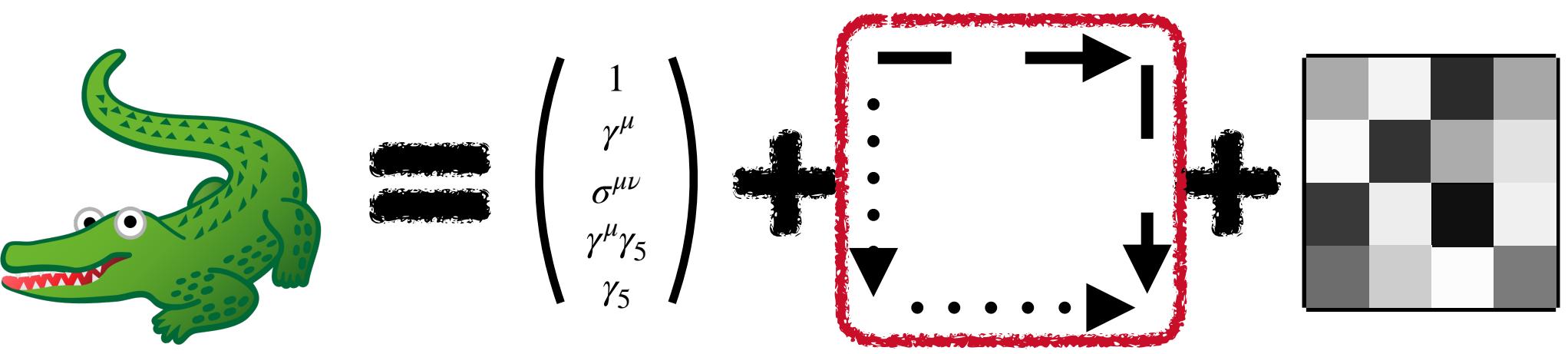
Output  
(equivariant)



Invariance = Equivariance with scalar output

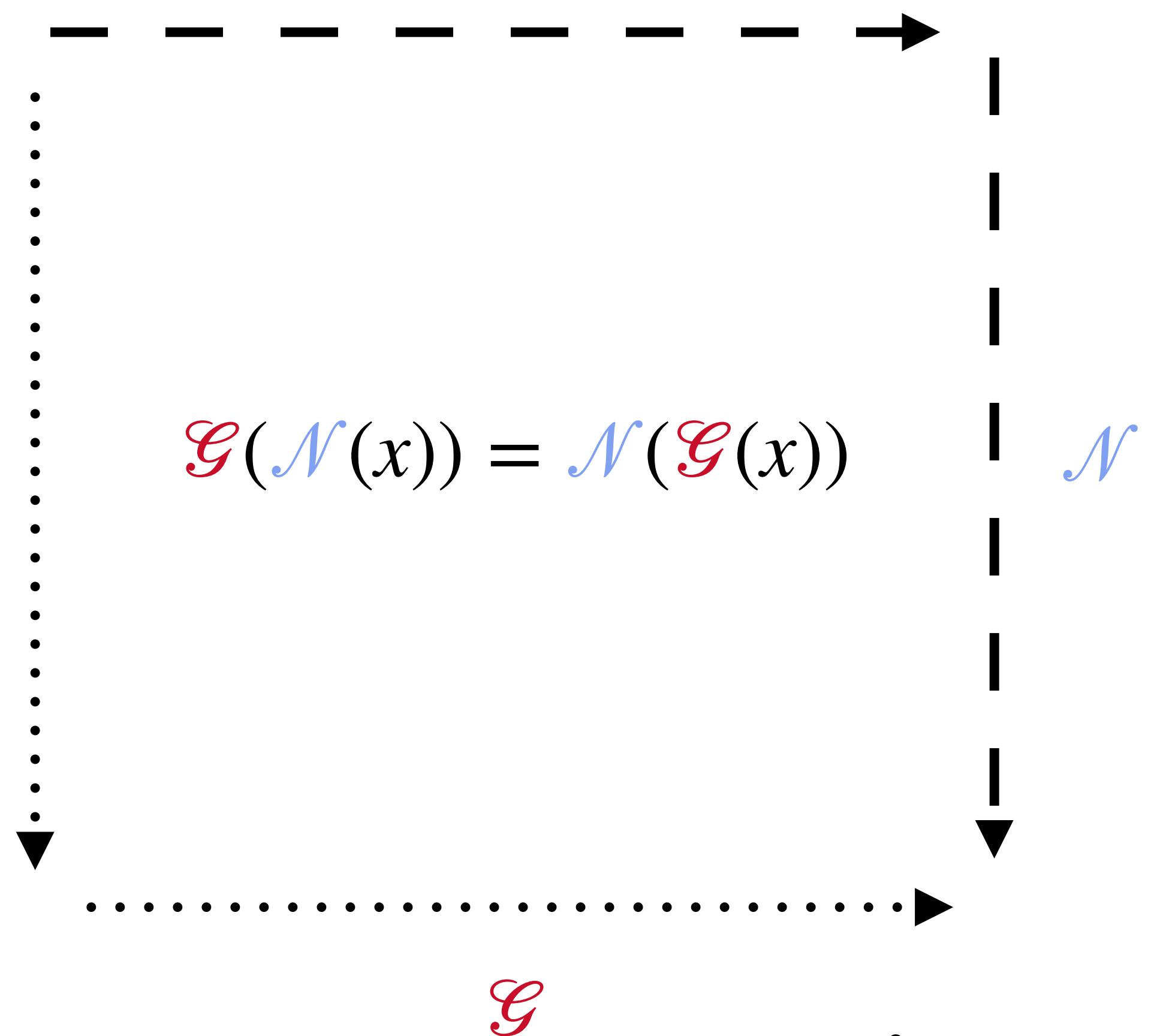
# L-GATr

## Equivariance - formalised

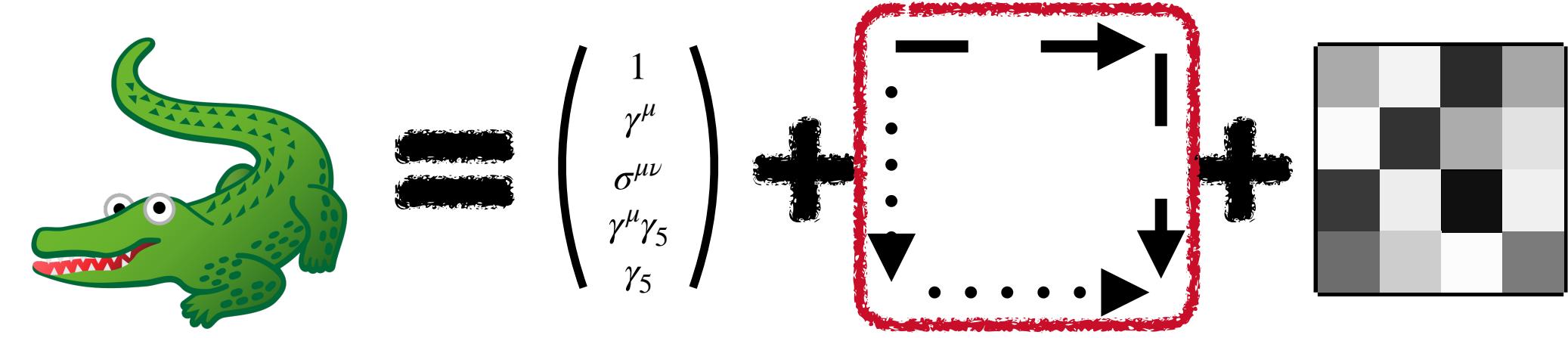


neural network  
transformation  $\mathcal{N}$

Lorentz group  
transformation  $\mathcal{G}$



# L-GATr



# Equivariance - formalised

The diagram illustrates the commutativity of the composition of neural network and Lorentz group transformations. It features two parallel horizontal axes. The top axis, labeled "Lorentz group transformation  $\mathcal{G}$ ", is represented by a dashed line with a right-pointing arrow at its end. The bottom axis, labeled "neural network transformation  $\mathcal{N}$ ", is represented by a dotted line with a right-pointing arrow at its end. A vertical dotted line connects the two axes. On the left side, there is a vertical stack of dots between the two axes, with a downward-pointing arrow below it. On the right side, there is a vertical stack of bars between the two axes, with a downward-pointing arrow below it. The equation  $\mathcal{G}(\mathcal{N}(x)) = \mathcal{N}(\mathcal{G}(x))$  is centered between the two axes.

Lorentz group transformation  $\mathcal{G}$

neural network transformation  $\mathcal{N}$

$$\mathcal{G}(\mathcal{N}(x)) = \mathcal{N}(\mathcal{G}(x))$$

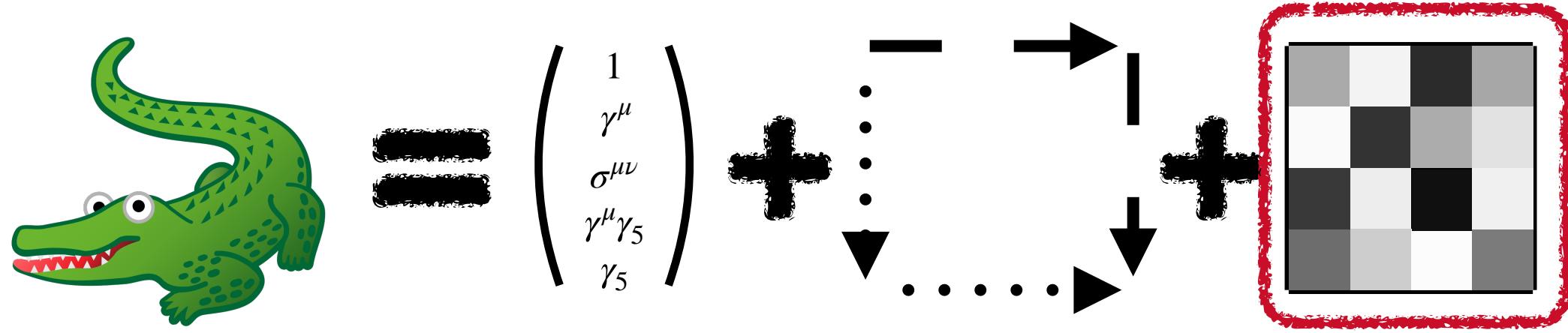
Each L-GATr operation  $\mathcal{N}$   
transforms all components  
within a given grade equally

The diagram illustrates a neural network layer. On the left, a vertical vector labeled  $\mathcal{N}$  contains elements  $x^S$ ,  $x_\mu^V$ ,  $x_{\mu\nu}^B$ ,  $x_\mu^A$ , and  $x^\mu_P$ . A large bracket groups these elements. In the center, a horizontal dashed arrow labeled  $\mathcal{N}$  indicates the transformation process. On the right, another vertical vector contains the transformed elements  $w^S x^S$ ,  $w^V x_\mu^V$ ,  $w^B x_{\mu\nu}^B$ ,  $w^A x_\mu^A$ , and  $w^P x^\mu_P$ . A large bracket groups these transformed elements.

This turns L-GATr equivariant,  
because grades form  
sub-representations of  $\mathcal{G}$

# L-GATr

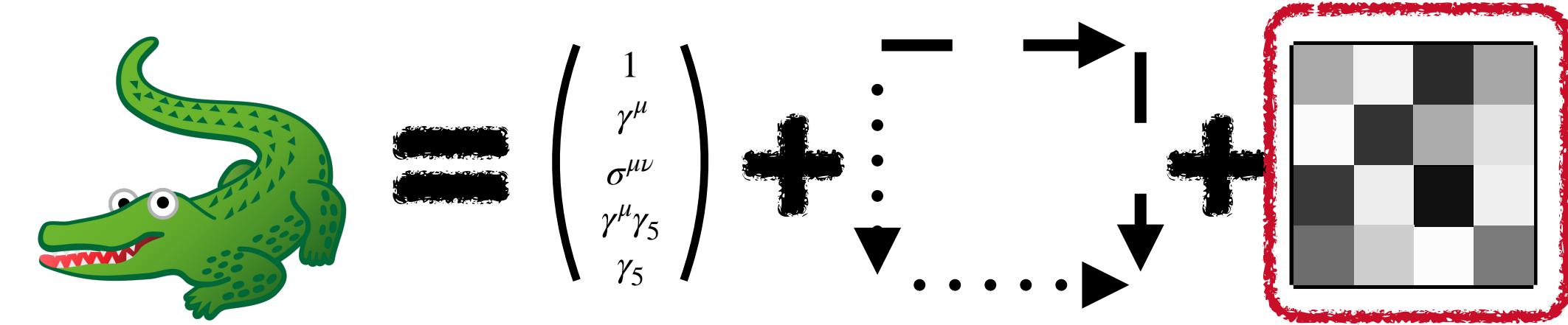
## GATr-ing all transformer layers



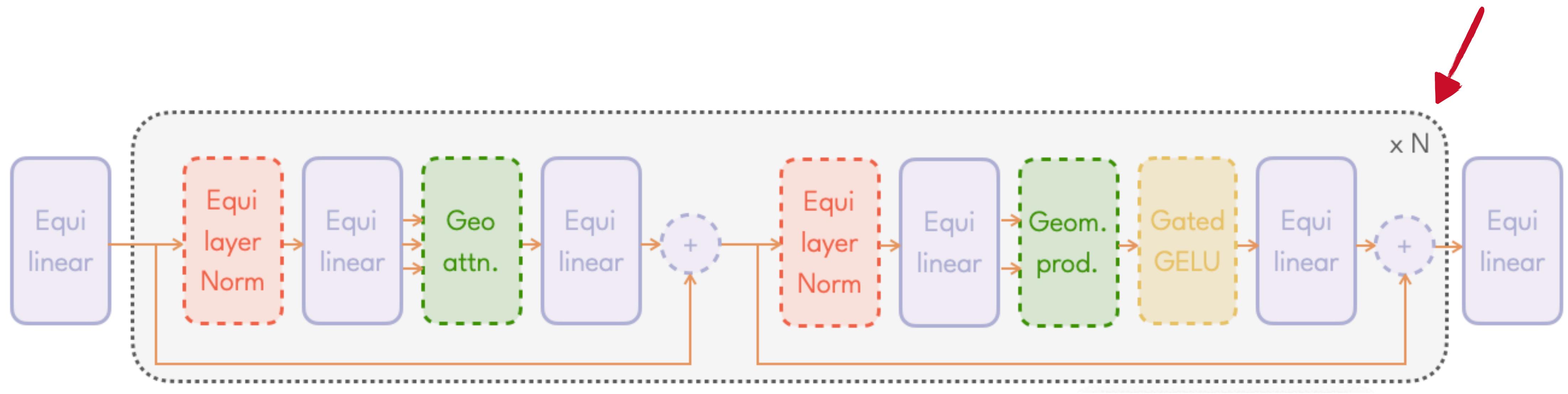
Layer type	Transformer	L-GATr
Linear( $x$ )	$vx + w$	$\sum_{k=0}^4 v_k \langle x \rangle_k \left( + \sum_{k=0}^4 w_k \gamma^5 \langle x \rangle_k \right)$
Attention( $q, k, v$ ) <sub>ic</sub>	$\sum_{j=1}^{n_t} \text{Softmax}_j \left( \sum_{c'=1}^{n_c} \frac{q_{ic'} k_{jc'}}{\sqrt{n_c}} \right) v_{jc}$	$\sum_{j=1}^{n_t} \text{Softmax}_j \left( \sum_{c'=1}^{n_c} \frac{\langle q_{ic'}, k_{jc'} \rangle}{\sqrt{16n_c}} \right) v_{jc}$
LayerNorm( $x$ )	$x \left[ \frac{1}{n_c} \sum_{c=1}^{n_c} x_c^2 + \epsilon \right]^{-1/2}$	$x \left[ \frac{1}{n_c} \sum_{c=1}^{n_c} \sum_{k=0}^4 \left  \langle \langle x_c \rangle_k, \langle x_c \rangle_k \rangle \right  + \epsilon \right]^{-1/2}$
Activation( $x$ )	GELU( $x$ )	GELU( $\langle x \rangle_0$ ) $x$
GP( $x, y$ )	—	$xy$

# L-GATr

## Full architecture

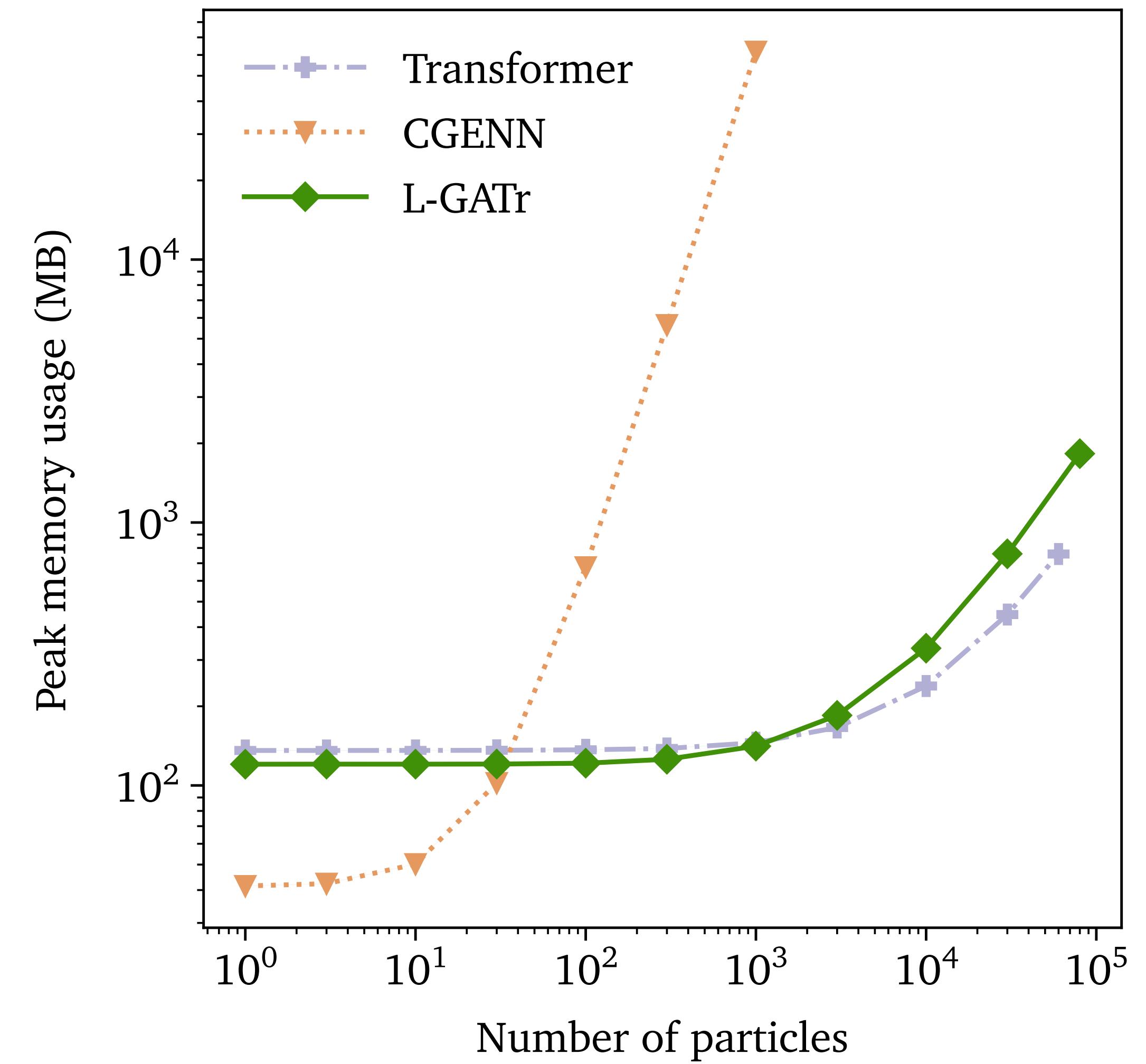
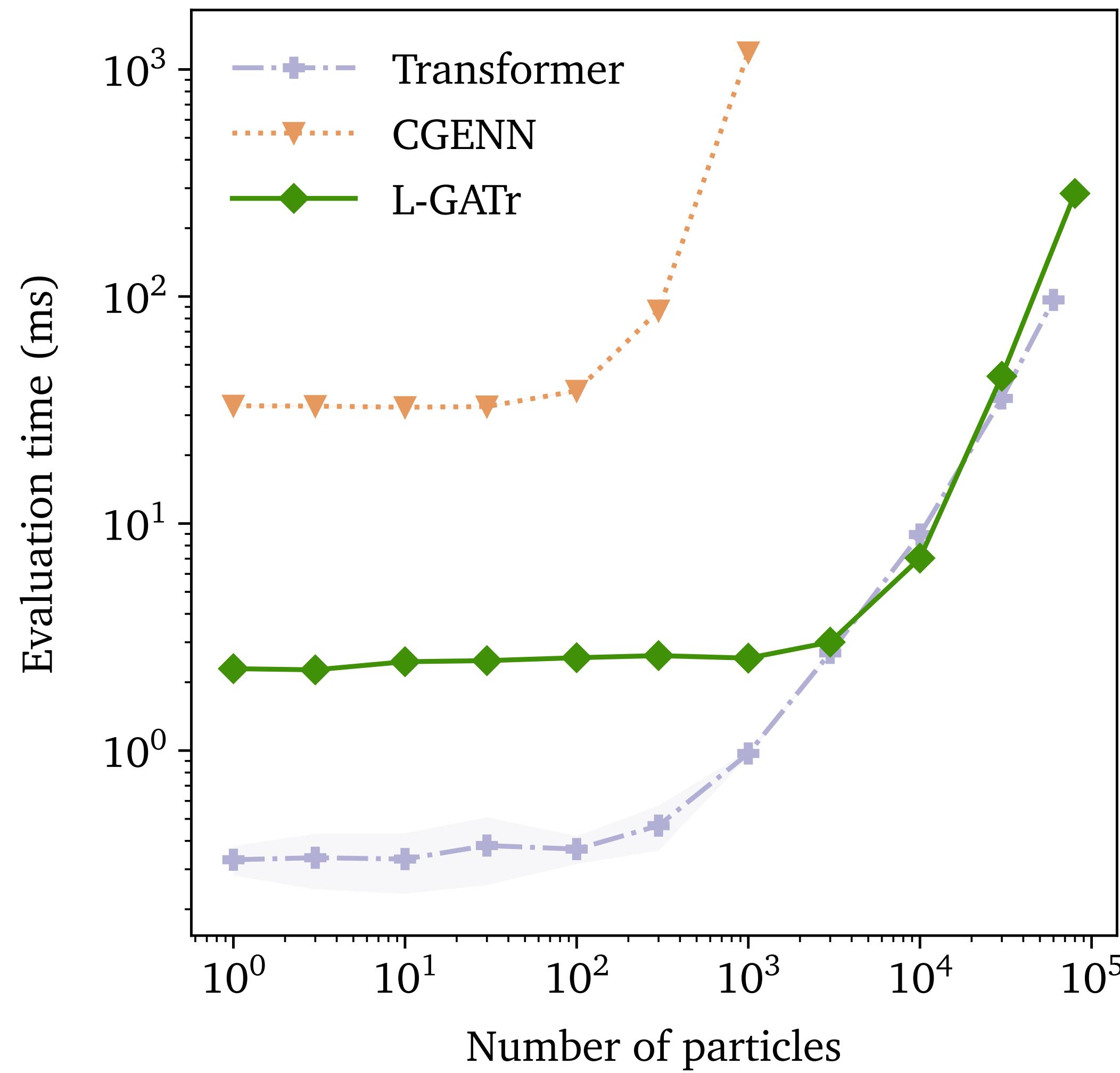
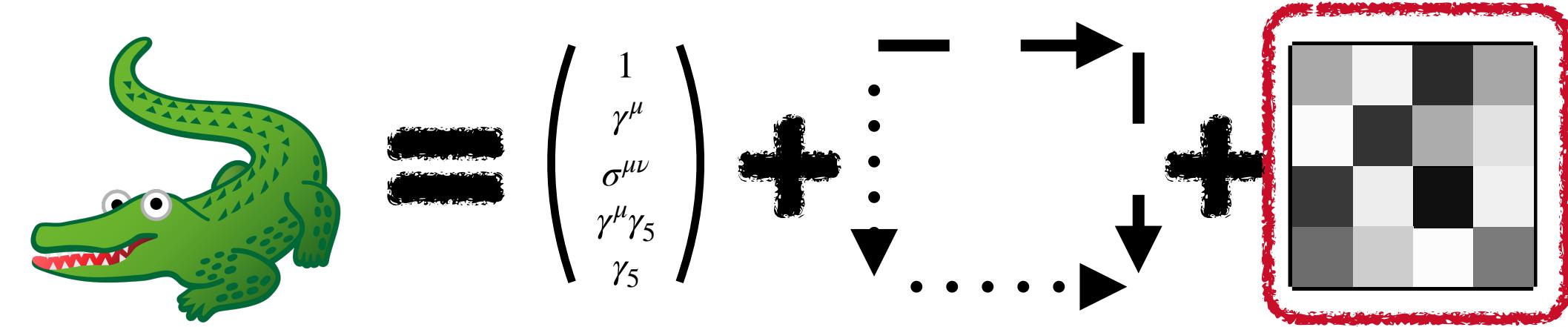


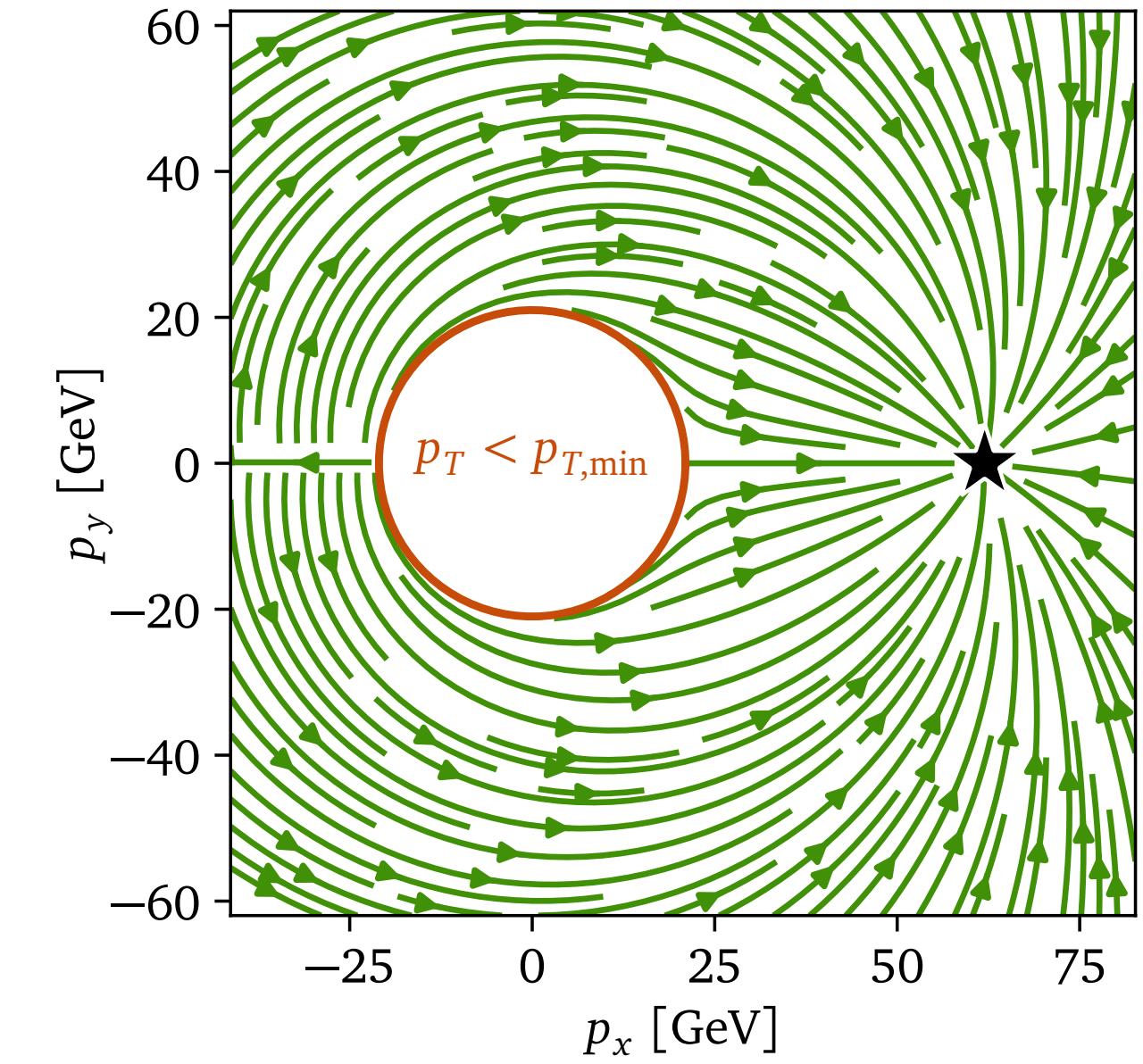
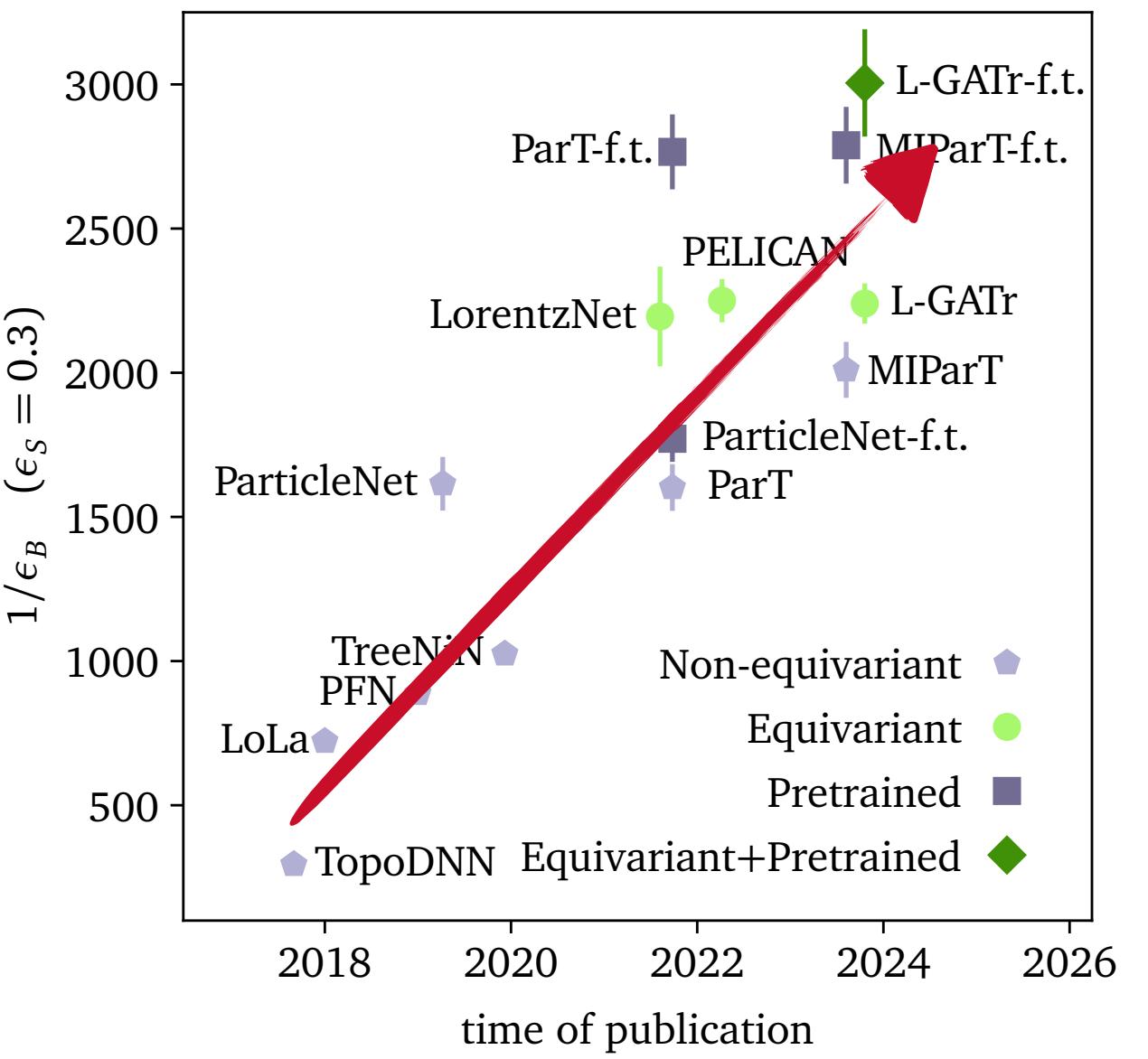
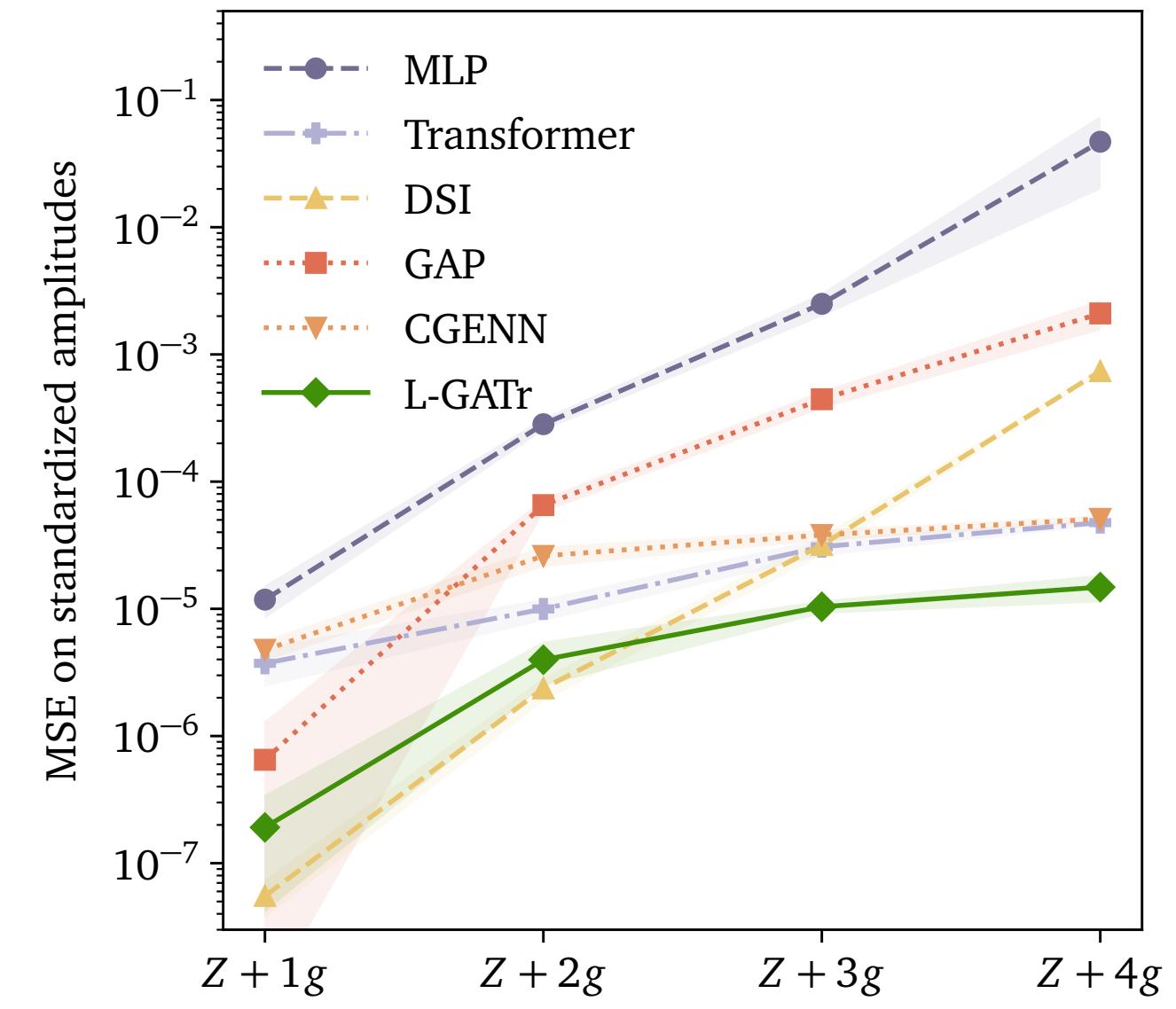
**GATr blocks**  
can be stacked  
to large depth



# L-GATr

Transformers scale better than graphs





# Regression

QFT amplitudes

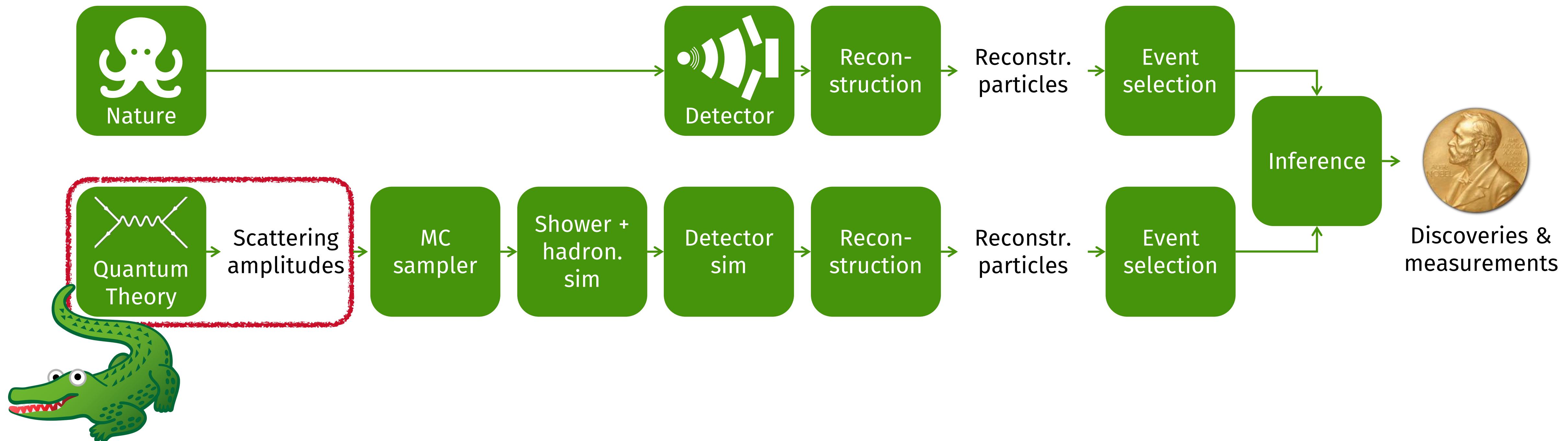
# Classification

Jet tagging

# Generation

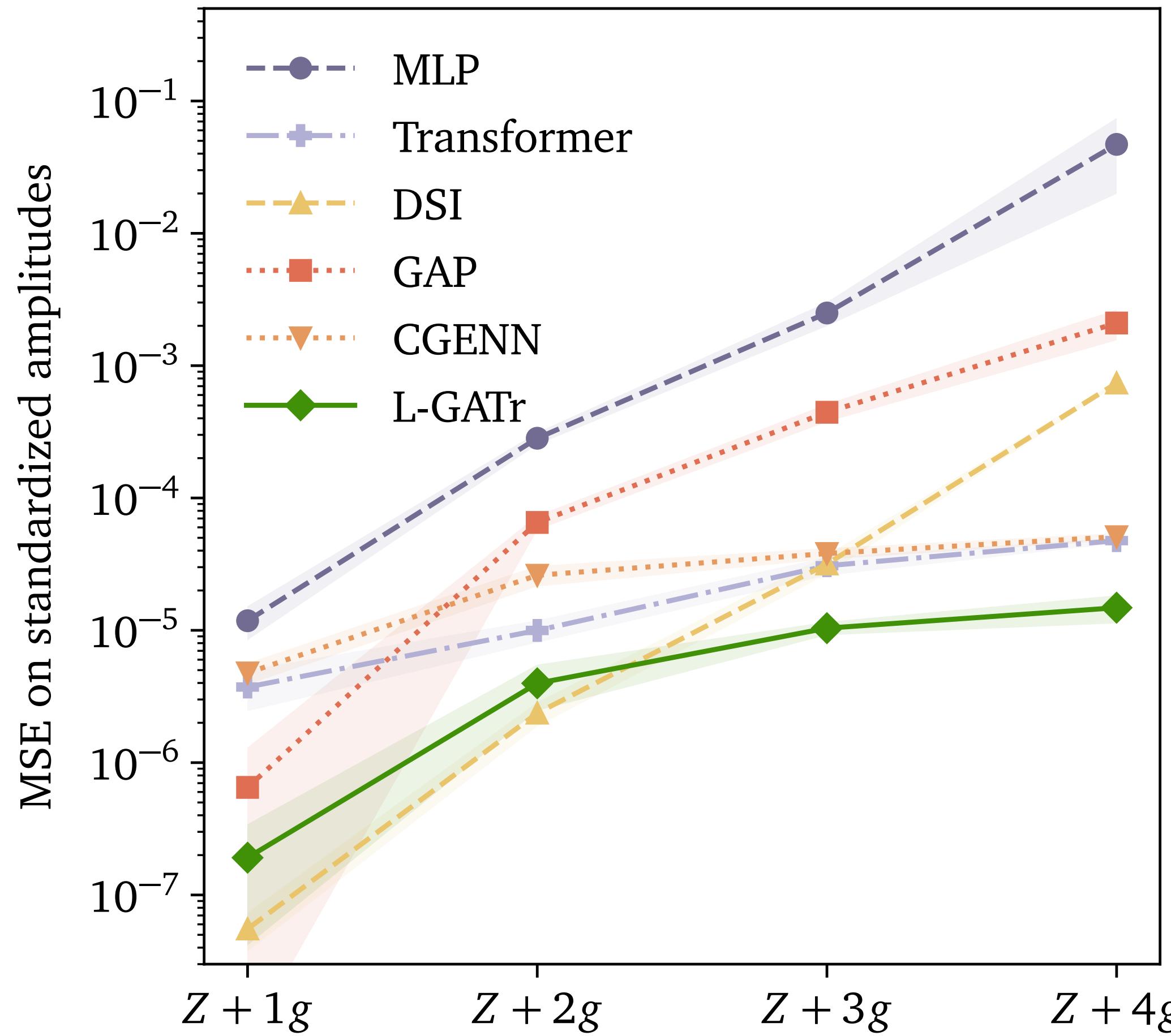
Reconstructed events

# QFT amplitude regression



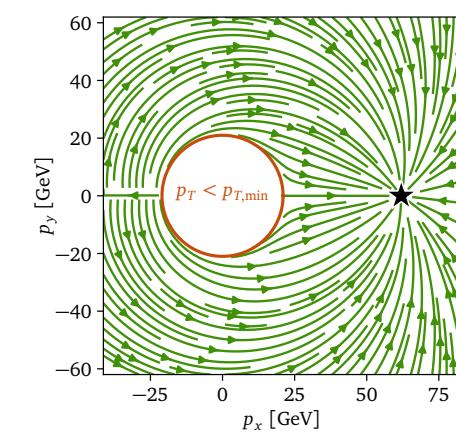
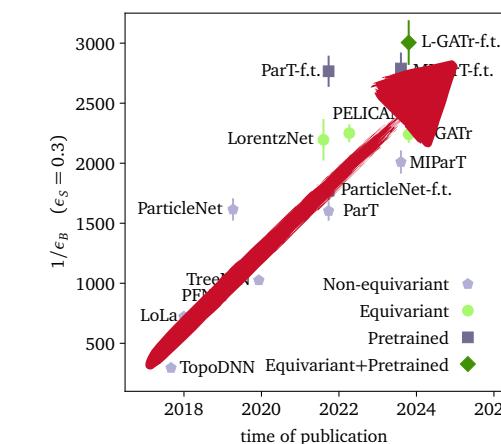
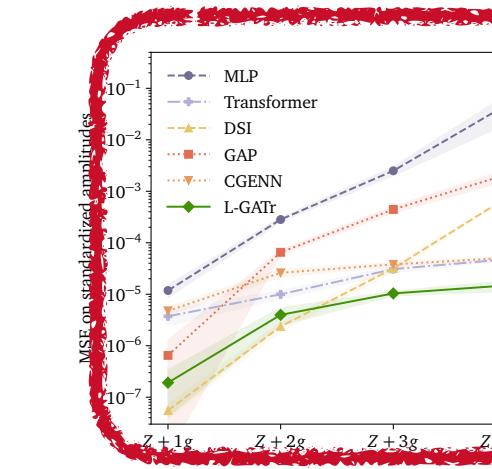
# QFT amplitude regression

## Permutation + Lorentz equivariance

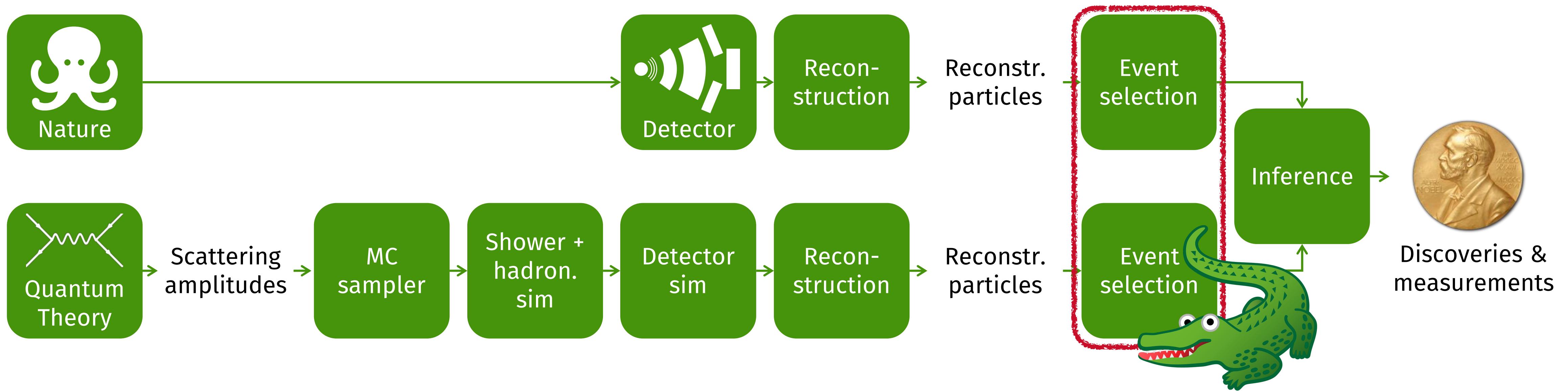


Permutation equivariance improves the scaling to high multiplicity

Lorentz equivariance gives a constant improvement across multiplicities



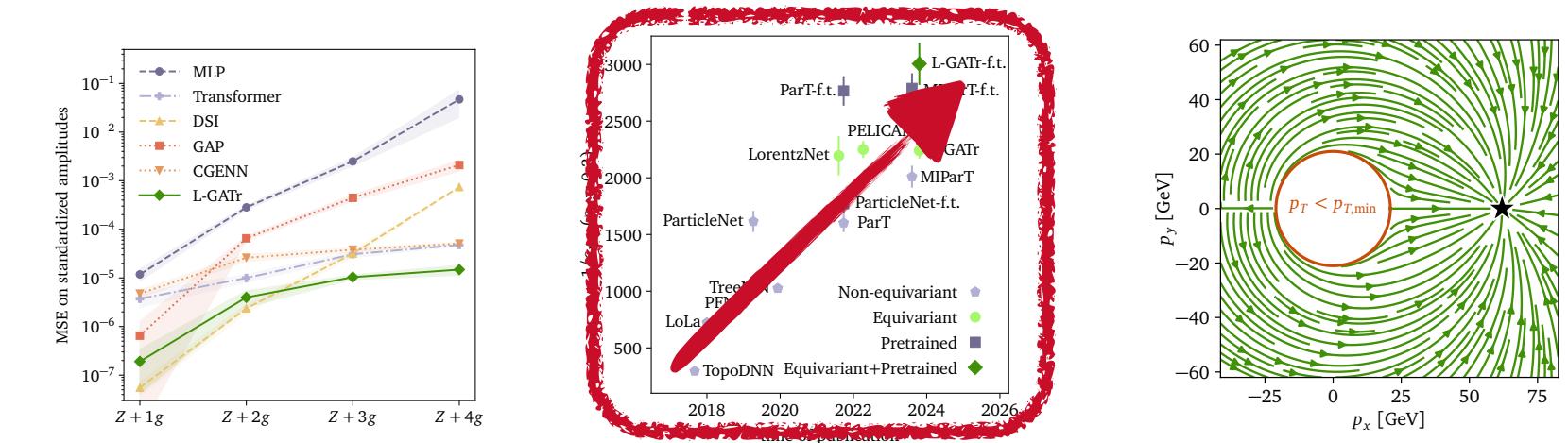
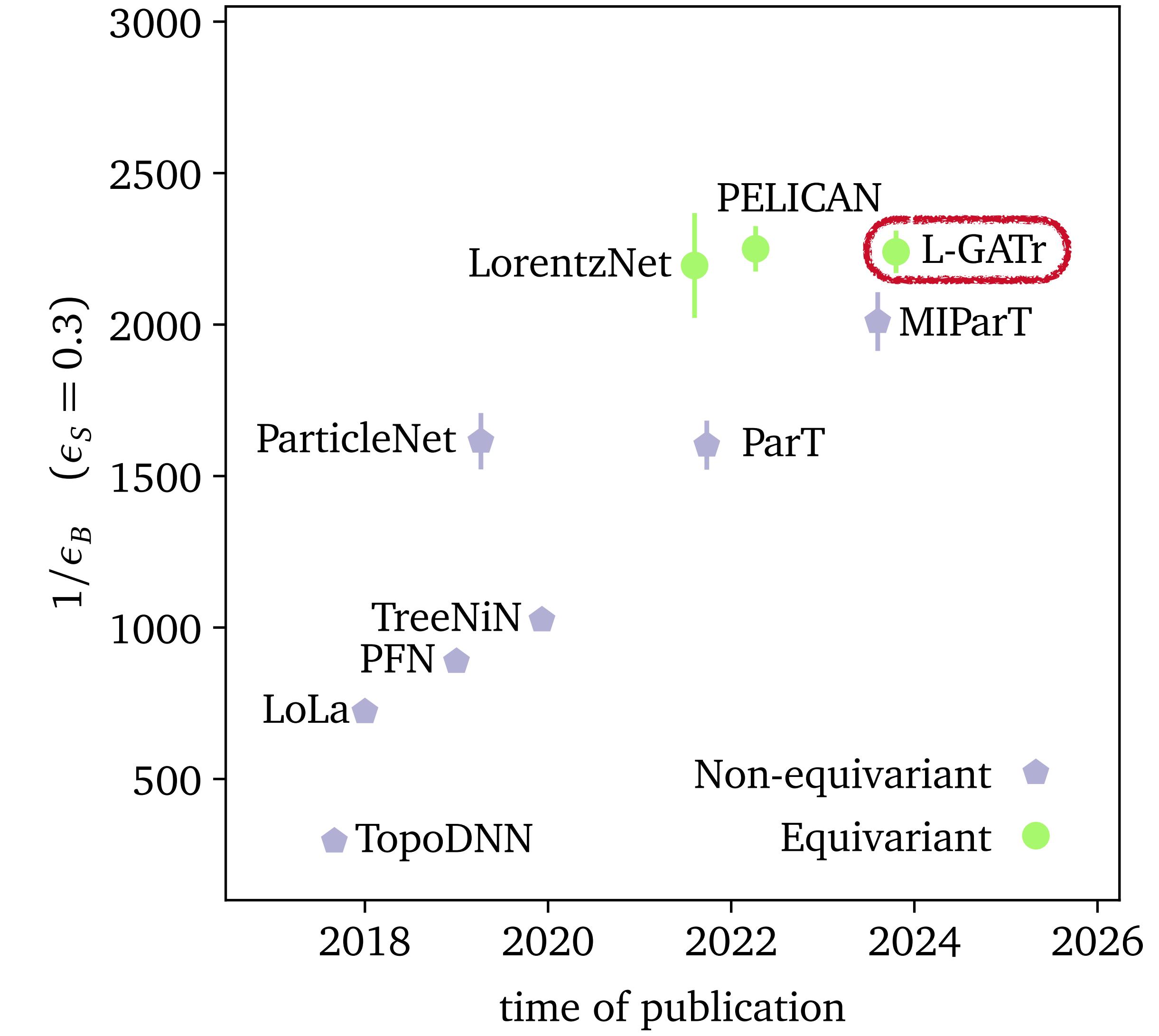
# Jet Tagging



# Jet Tagging

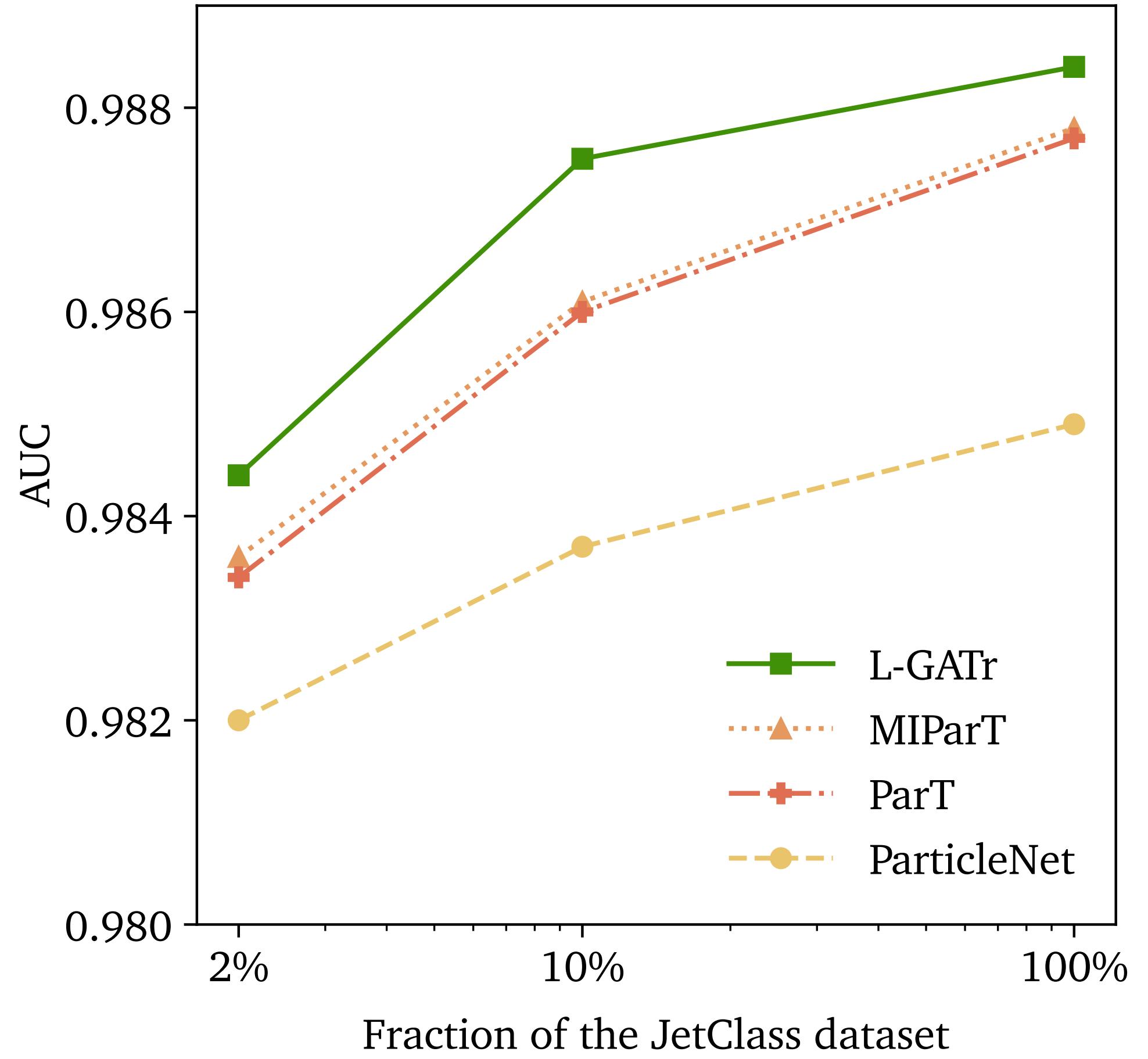
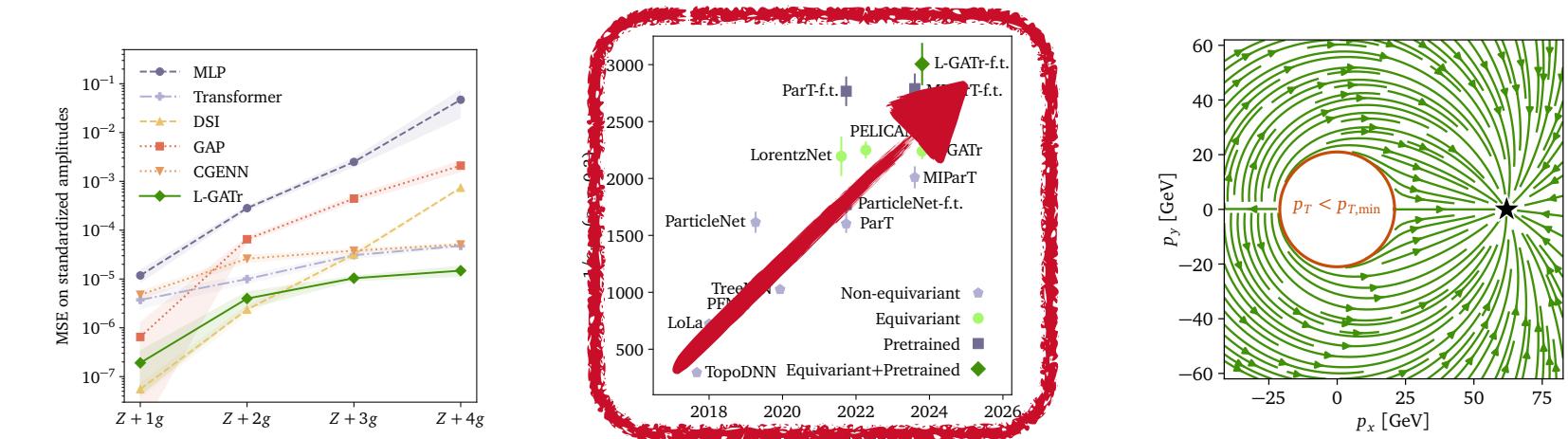
## The history of top tagging

All top-performing taggers  
are Lorentz-equivariant



# Jet Tagging

## Training on 100M jets

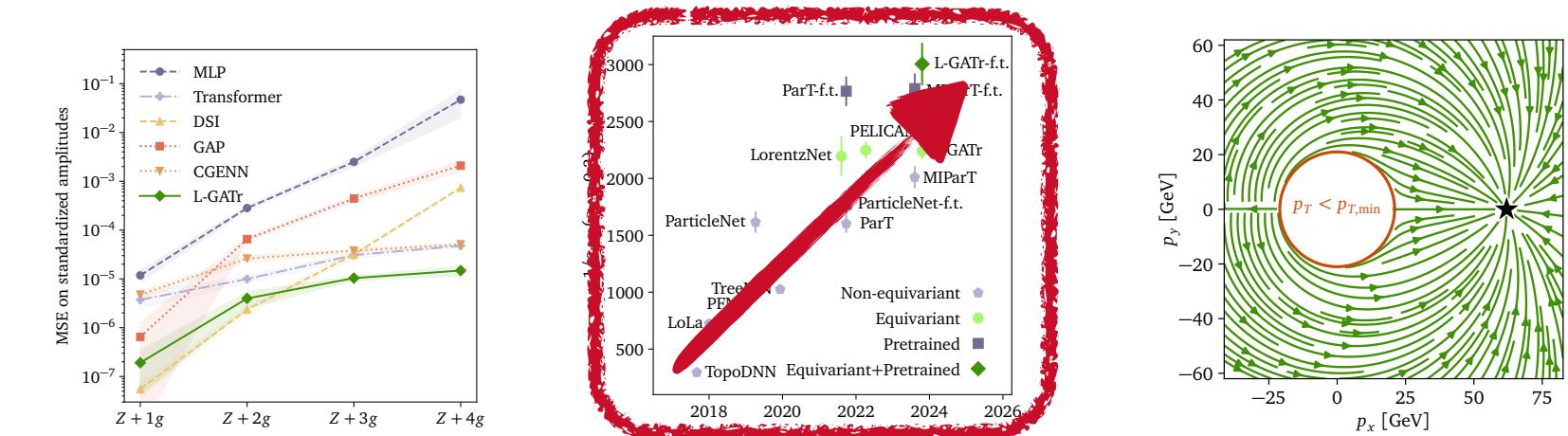
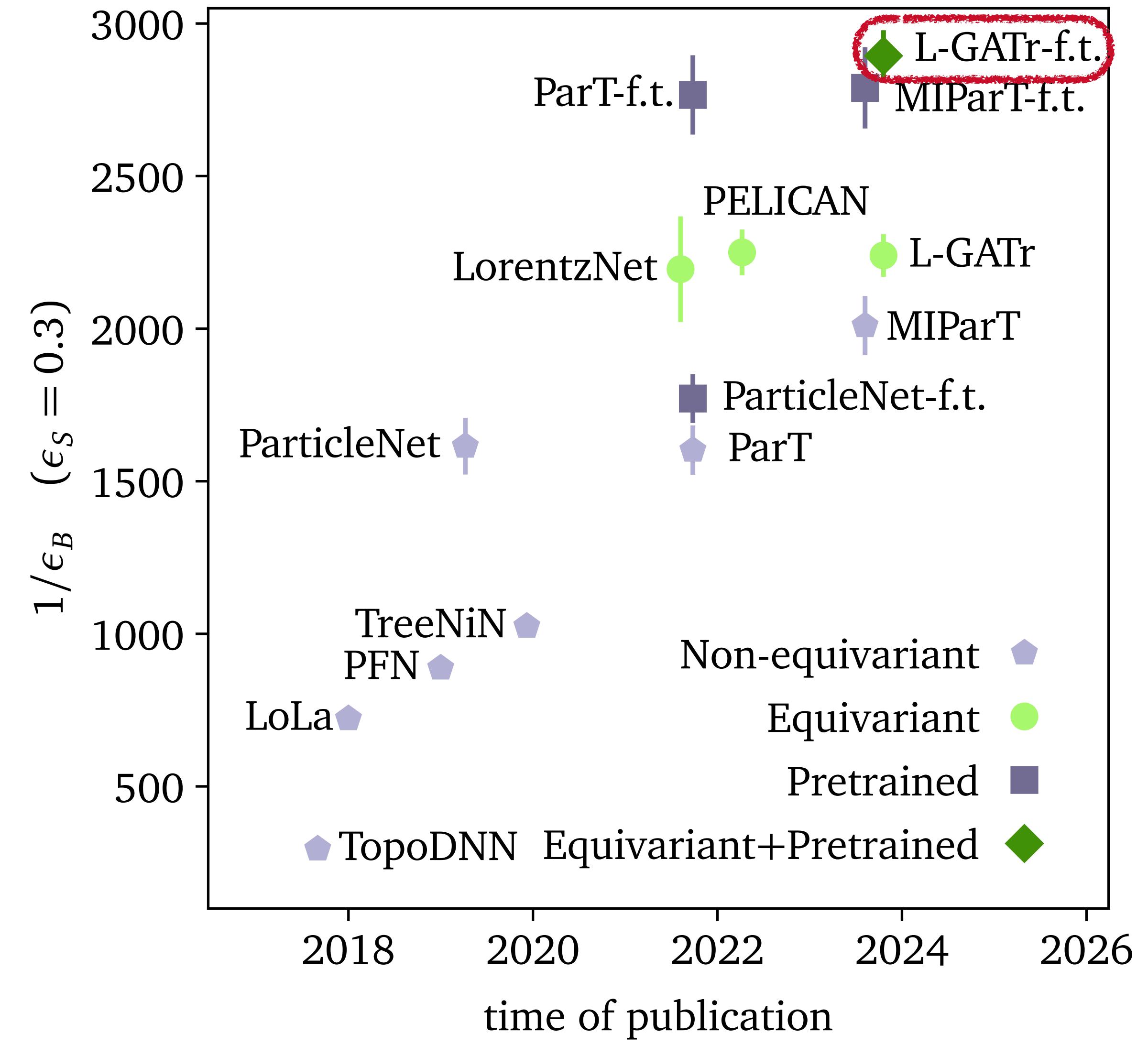


Transformers outperform  
graph networks  
on large datasets

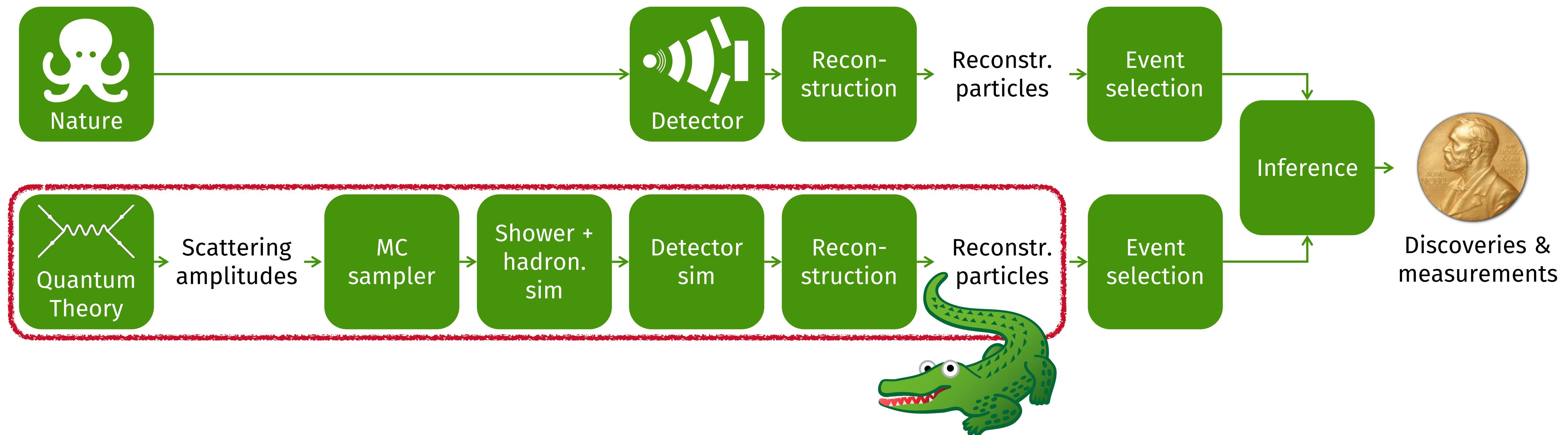
# Jet Tagging

## Transfer learning for top tagging

L-GATr combines the benefits of pretraining and Lorentz equivariance



# Event generation

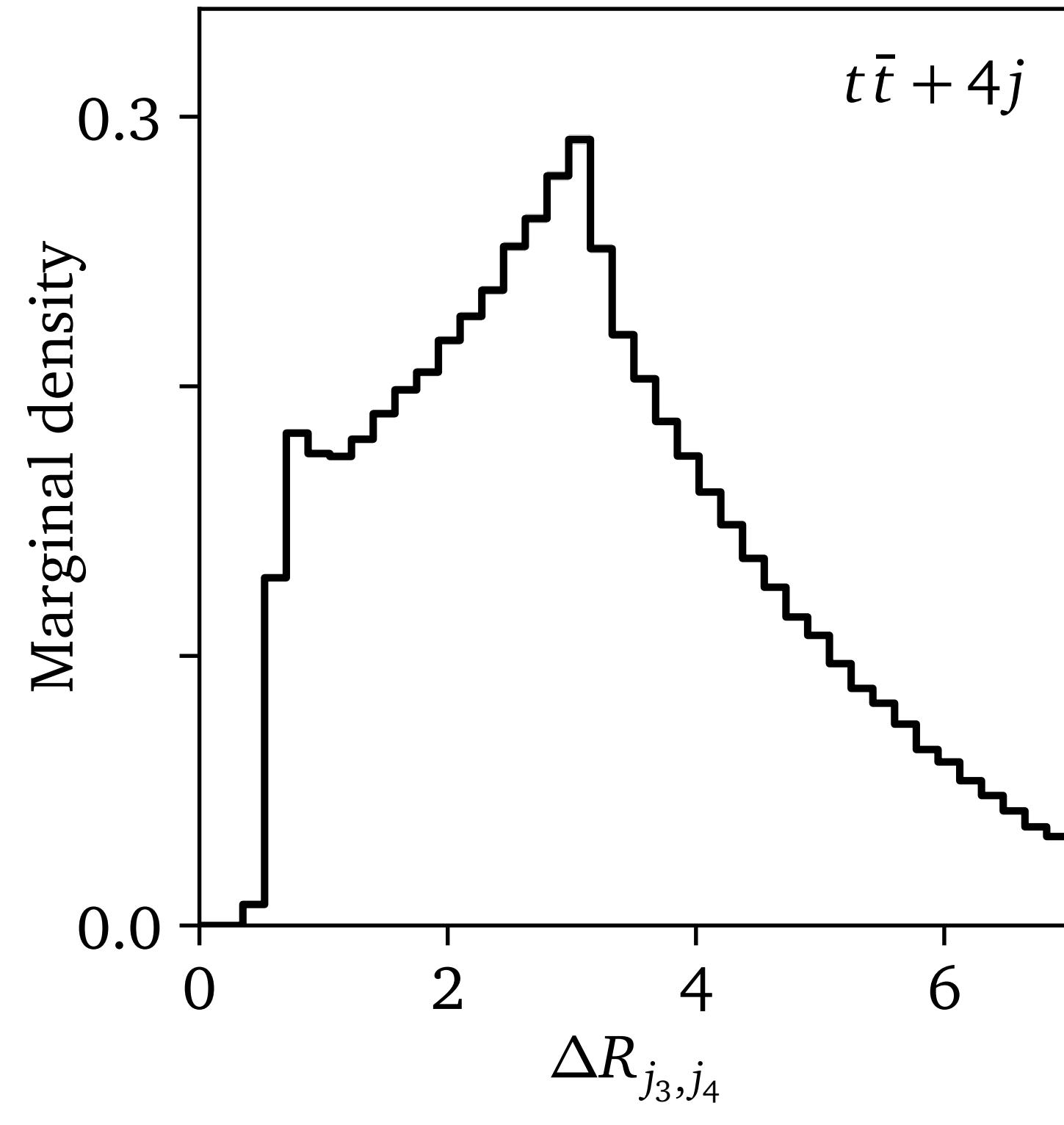
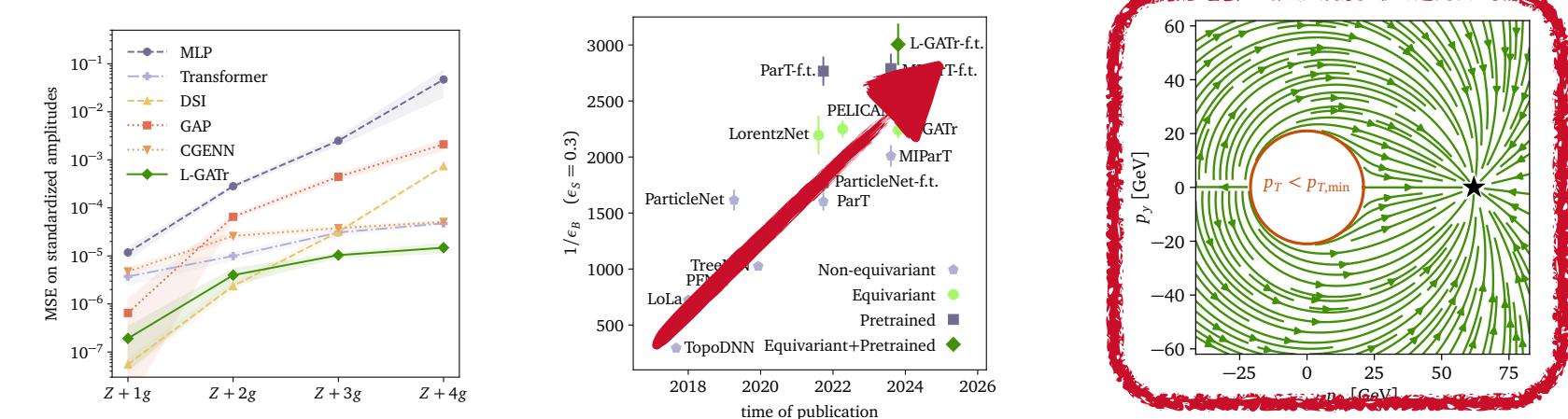
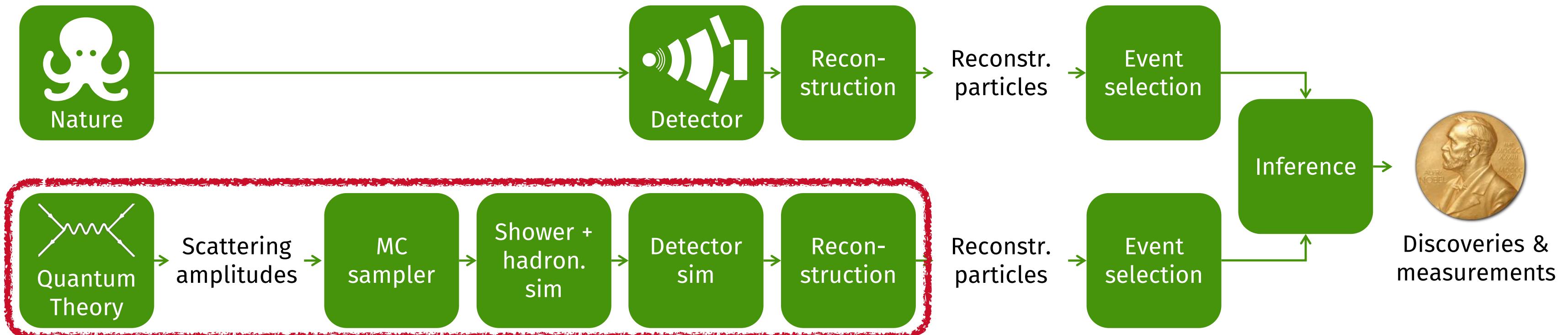


# Event generation

## Task

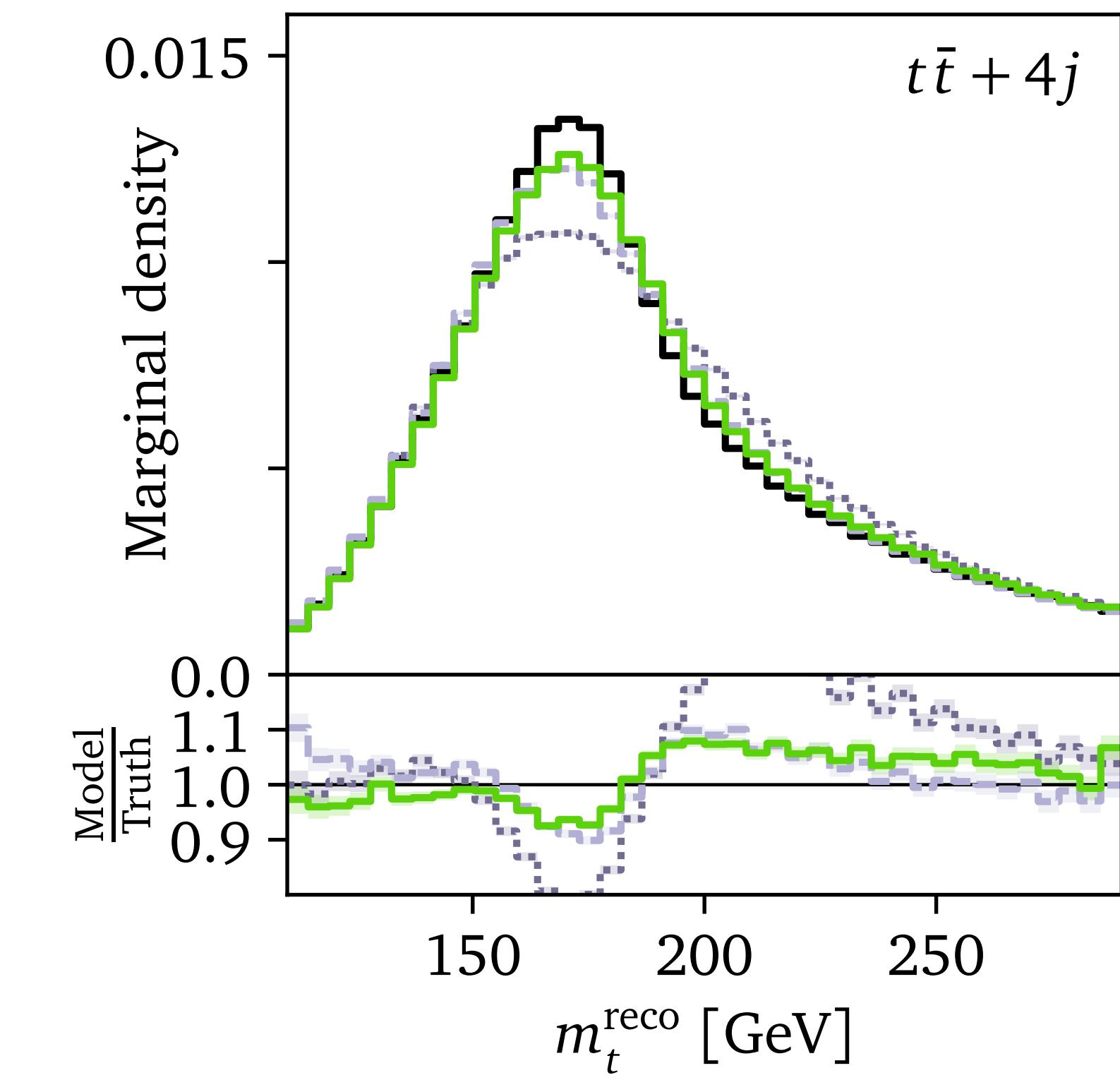
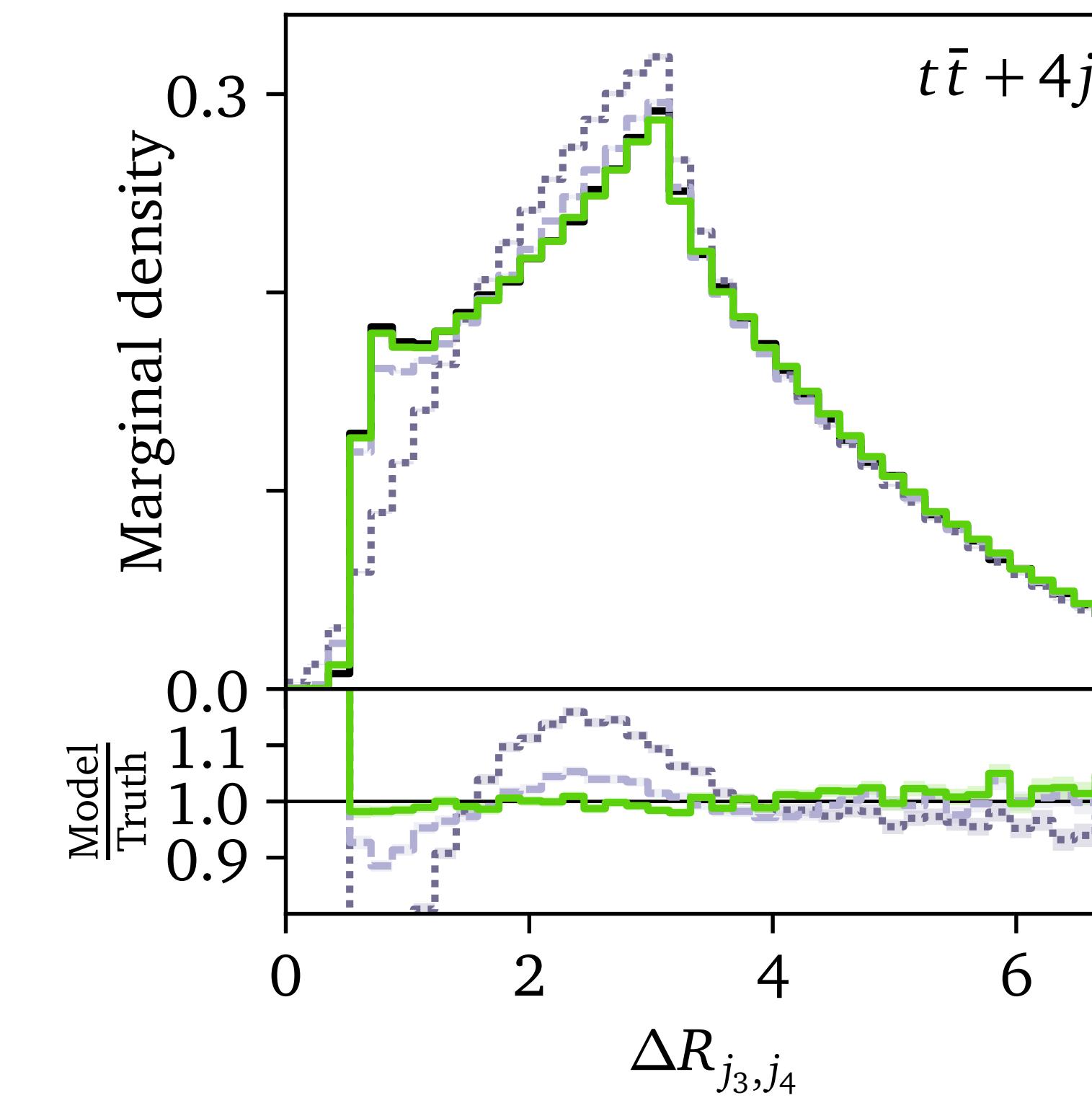
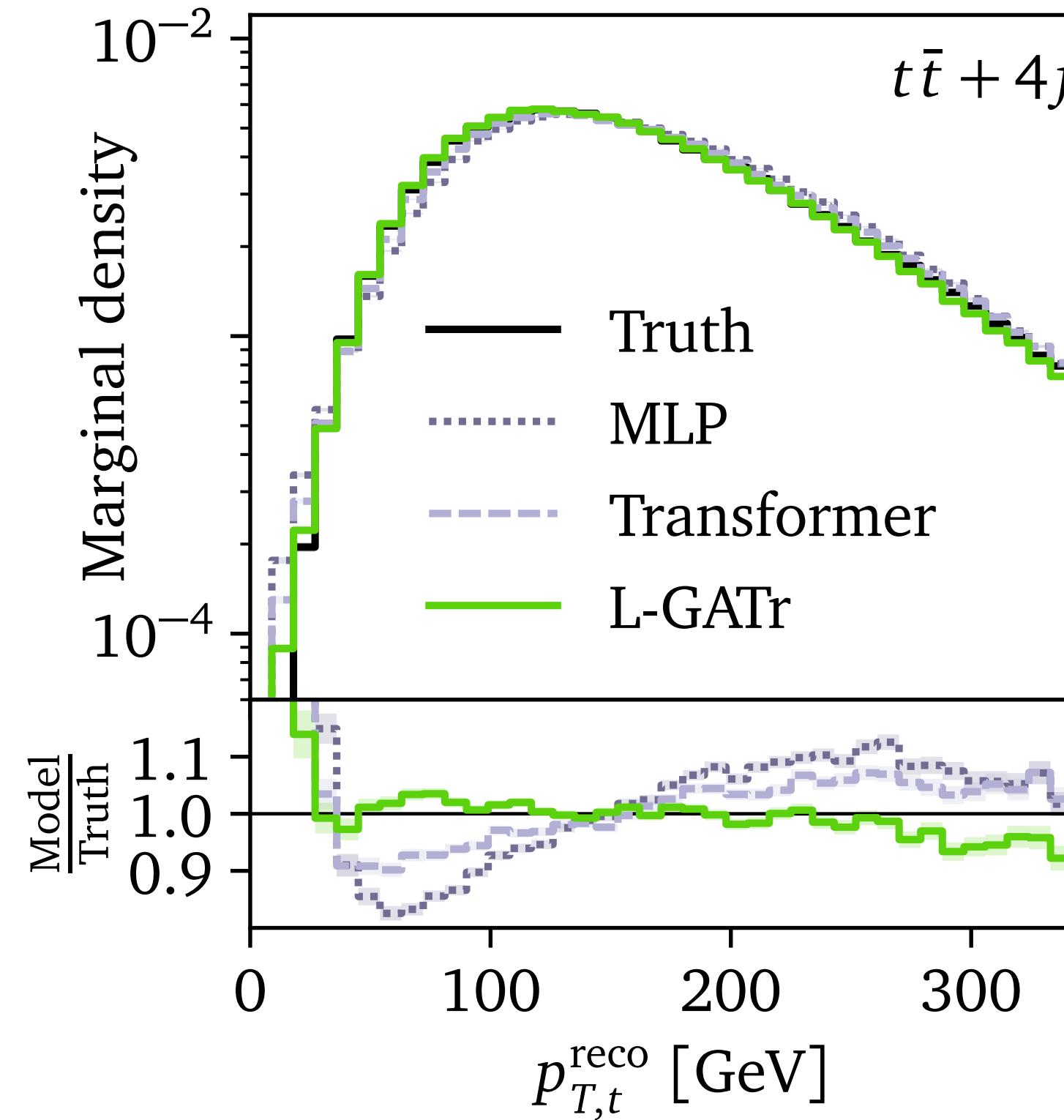
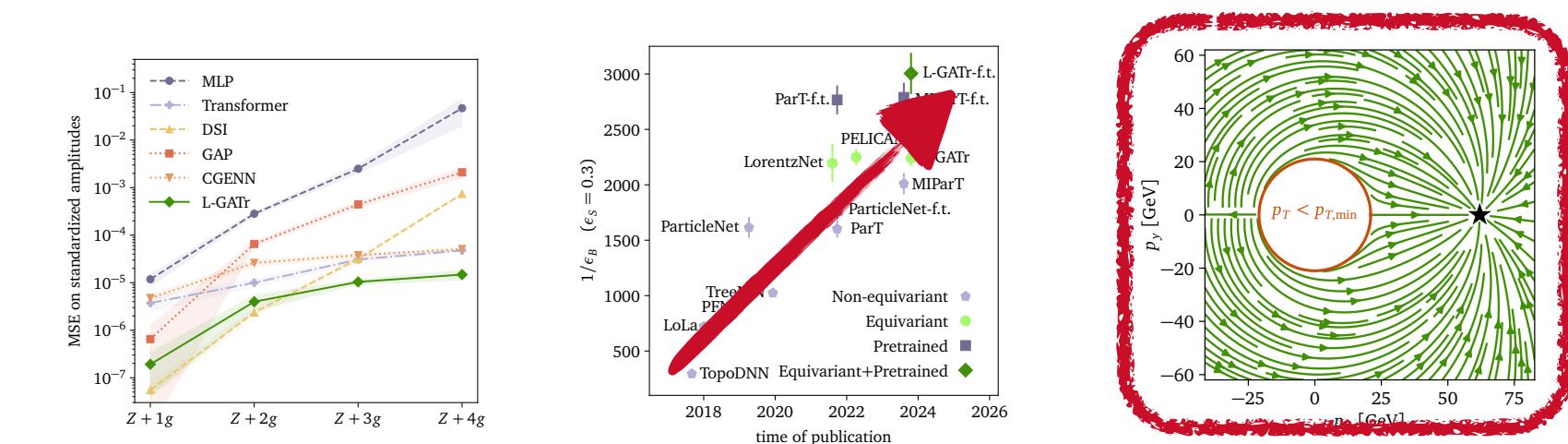
Dataset:  $pp \rightarrow t_h\bar{t}_h + nj, n = 0\dots4$

- MadGraph + Pythia + Delphes + Reconstruction
- Challenging features:  $m_t, m_W, \Delta R_{jj} > 0.5$



# Event generation

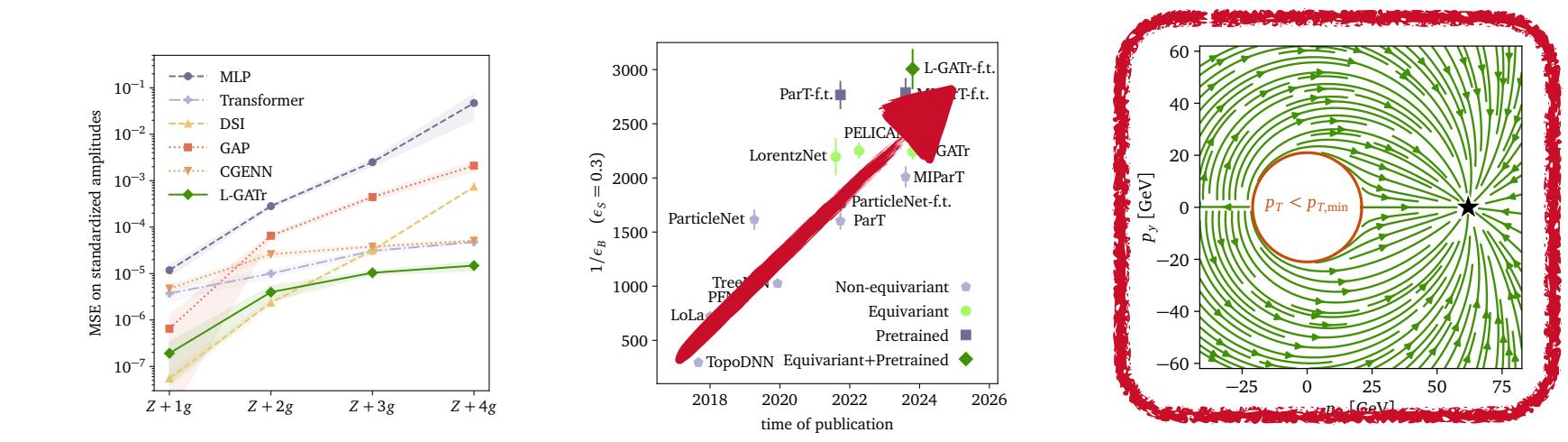
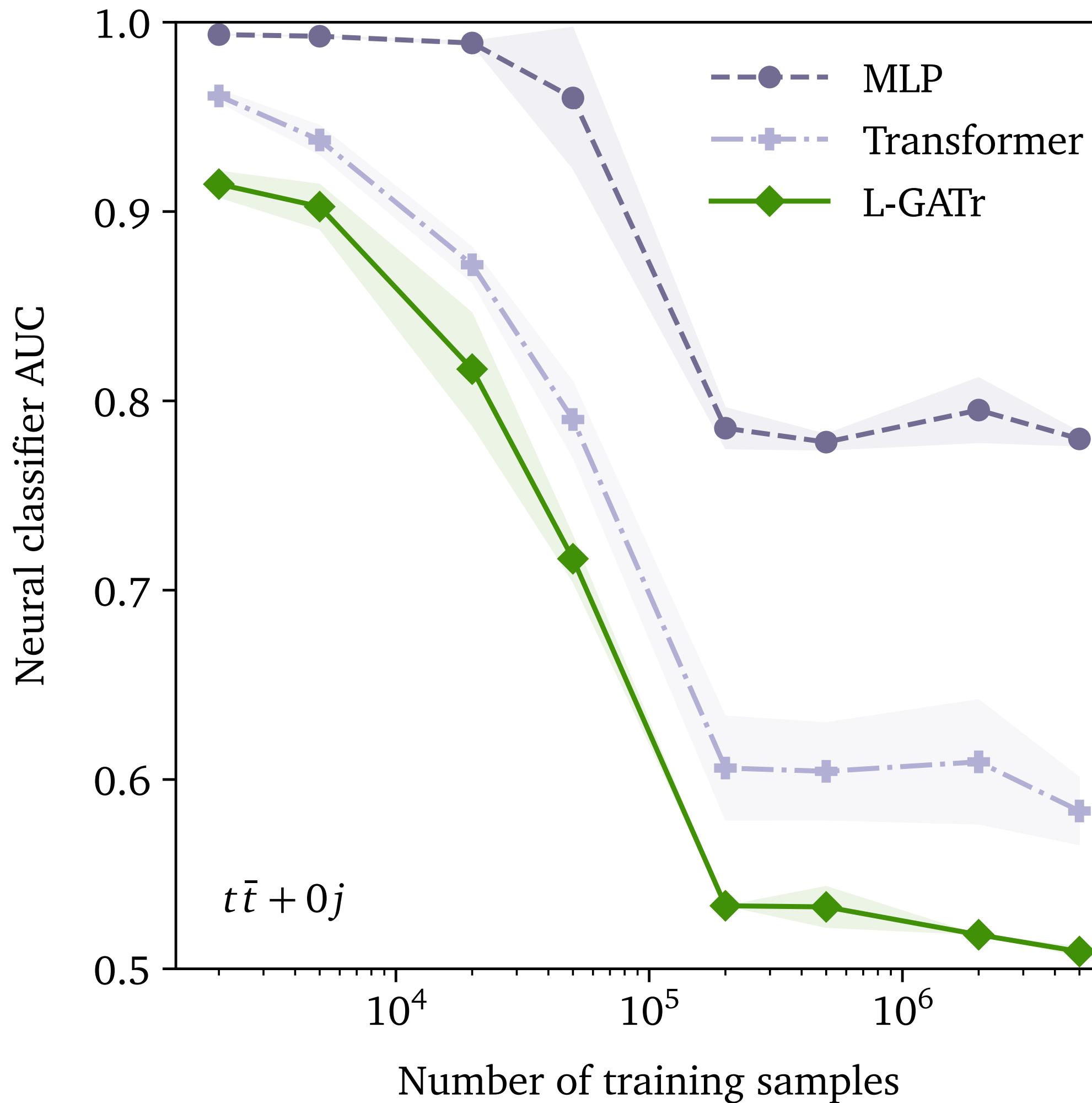
## Kinematic distributions



Equivariance helps,  
especially for angular correlations

# Event generation

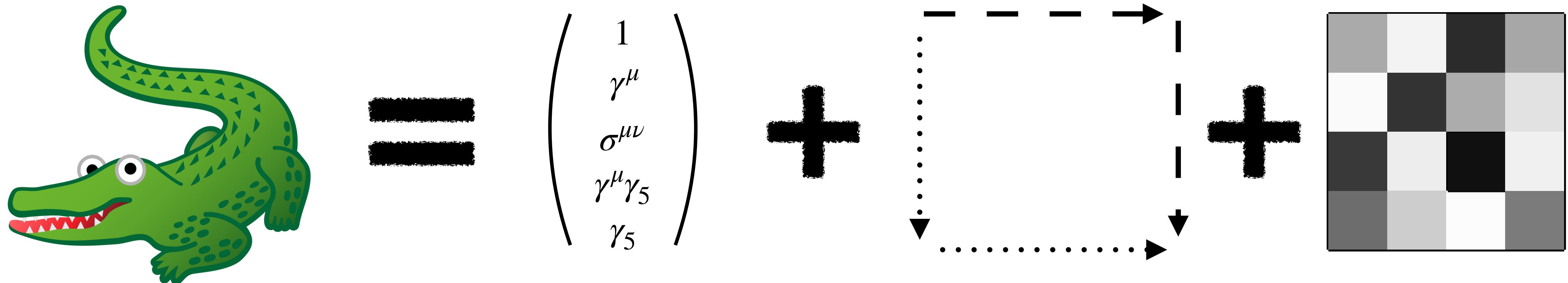
## Neural classifier metric



L-GATr generates samples that a classifier can barely distinguish from the ground truth

# Messages

- L-GATr is a multi-purpose architecture for LHC physics
- In jet tagging, L-GATr combines the benefits from Lorentz equivariance and pretraining
- Lorentz-equivariant generators are tricky but great



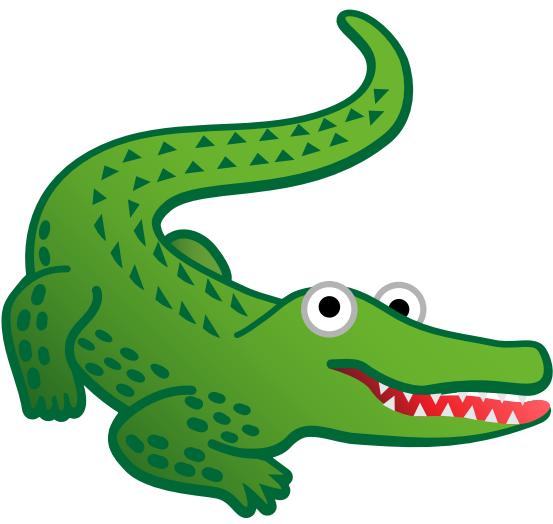
Lorentz-Equivariant  
Geometric Algebra  
Transformer

Geometric algebra  
representations

Lorentz-Equivariant  
layers

Transformer  
architecture

# L-GATr is easy to use



```
from gatr import GATr, SelfAttentionConfig, MLPConfig
from gatr.interface import embed_vector, extract_scalar, embed_spurions
import torch

class ExampleWrapper(torch.nn.Module):
    """Example wrapper around a L-GATr model.

    Parameters
    -----
    num_blocks : int
        Number of transformer blocks
    hidden_mv_channels : int
        Number of hidden multivector channels
    hidden_s_channels : int
        Number of hidden scalar channels
    """

    def __init__(self, blocks=6, hidden_mv_channels=16, hidden_s_channels=32):
        super().__init__()
        self.gatr = GATr(
            in_mv_channels=3,
            out_mv_channels=1,
            hidden_mv_channels=hidden_mv_channels,
            in_s_channels=None,
            out_s_channels=None,
            hidden_s_channels=hidden_s_channels,
            num_blocks=num_blocks,
            attention=SelfAttentionConfig(), # Use default parameters for attention
            mlp=MLPConfig(), # Use default parameters for MLP
        )
```

```
def forward(self, fourmomenta):
    """Forward pass.

    Parameters
    -----
    fourmomenta : torch.Tensor with shape (batchsize, num_points, 4)
        fourmomentum point cloud input data

    Returns
    -----
    outputs : torch.Tensor with shape (batchsize, 1)
        Model prediction: a single scalar for the whole point cloud.
    """
    batchsize, num_points, _ = fourmomenta.shape

    # Embed fourmomentum point cloud inputs in GA
    multivectors = embed_vector(fourmomenta).unsqueeze(-2) # (batchsize, num_points, 1, 1)

    # Append spurions for symmetry breaking (optional)
    spurions = embed_spurions(beam_reference="xyplane", add_time_reference=True, device=fourmomenta.device)
    spurions = spurions[None, None, ...].repeat(batchsize, num_points, 1, 1) # (batchsize, num_points, 1, 1)
    multivectors = torch.cat((multivectors, spurions), dim=-2) # (batchsize, num_points, 1, 2)

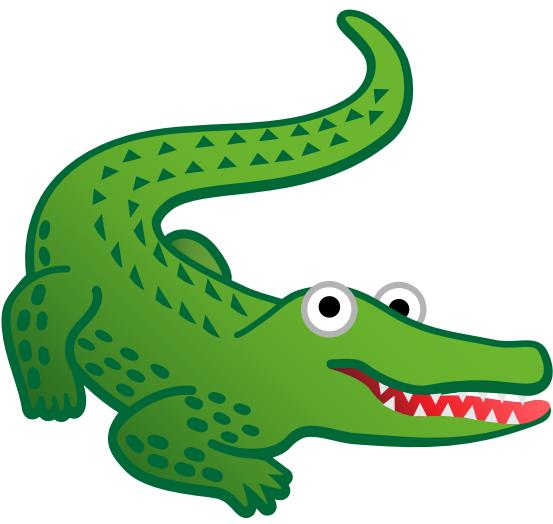
    # Pass data through GATr
    multivector_outputs, _ = self.gatr(multivectors, scalars=None) # (batchsize, num_points, 1, 1)

    # Extract scalar outputs
    outputs = extract_scalar(multivector) # (batchsize, num_points, 1)

    # Mean aggregation to extract a single scalar for the whole point cloud
    score = outputs.mean(dim=1)

    return score
```

# L-GATr is easy to use



[github.com/heidelberg-hepml/lgatr](https://github.com/heidelberg-hepml/lgatr)

```
from gatr import GATr, SelfAttentionConfig, MLPConfig
from gatr.interface import embed_vector, extract_scalar, embed_spurions
import torch

class ExampleWrapper(torch.nn.Module):
    """Example wrapper around a L-GATr model.

    Parameters
    -----
    num_blocks : int
        Number of transformer blocks
    hidden_mv_channels : int
        Number of hidden multivector channels
    hidden_s_channels : int
        Number of hidden scalar channels
    """

    def __init__(self, blocks=6, hidden_mv_channels=16, hidden_s_channels=32):
        super().__init__()
        self.gatr = GATr(
            in_mv_channels=3,
            out_mv_channels=1,
            hidden_mv_channels=hidden_mv_channels,
            in_s_channels=None,
            out_s_channels=None,
            hidden_s_channels=hidden_s_channels,
            num_blocks=num_blocks,
            attention=SelfAttentionConfig(), # Use default parameters for attention
            mlp=MLPConfig(), # Use default parameters for MLP
        )
```

```
def forward(self, fourmomenta):
    """Forward pass.

    Parameters
    -----
    fourmomenta : torch.Tensor with shape (batchsize, num_points, 4)
        fourmomentum point cloud input data

    Returns
    -----
    outputs : torch.Tensor with shape (batchsize, 1)
        Model prediction: a single scalar for the whole point cloud.
    """
    batchsize, num_points, _ = fourmomenta.shape

    # Embed fourmomentum point cloud inputs in GA
    multivectors = embed_vector(fourmomenta).unsqueeze(-2) # (batchsize, num_points, 1, 1)

    # Append spurions for symmetry breaking (optional)
    spurions = embed_spurions(beam_reference="xyplane", add_time_reference=True, device=fourmomenta.device)
    spurions = spurions[None, None, ...].repeat(batchsize, num_points, 1, 1) # (batchsize, num_points, 1, 1)
    multivectors = torch.cat((multivectors, spurions), dim=-2) # (batchsize, num_points, 1, 2)

    # Pass data through GATr
    multivector_outputs, _ = self.gatr(multivectors, scalars=None) # (batchsize, num_points, 1, 1)

    # Extract scalar outputs
    outputs = extract_scalar(multivector) # (batchsize, num_points, 1)

    # Mean aggregation to extract a single scalar for the whole point cloud
    score = outputs.mean(dim=1)

    return score
```



Victor Bresó



Pim de Haan



Tilman Plehn



Huilin Qu



Jesse Thaler



Johann Brehmer

## Lorentz-Equivariant Geometric Algebra Transformer for High-Energy Physics

Jonas Spinner\*, Víctor Bresó\*, Pim de Haan,  
Tilman Plehn, Jesse Thaler, Johann Brehmer  
NeurIPS 2024, arXiv:2405.14806

## A Lorentz-Equivariant Transformer for all of the LHC

Johann Brehmer, Víctor Bresó,  
Pim de Haan, Tilman Plehn,  
Huilin Qu, Jonas Spinner, Jesse Thaler  
arXiv:2411.00446



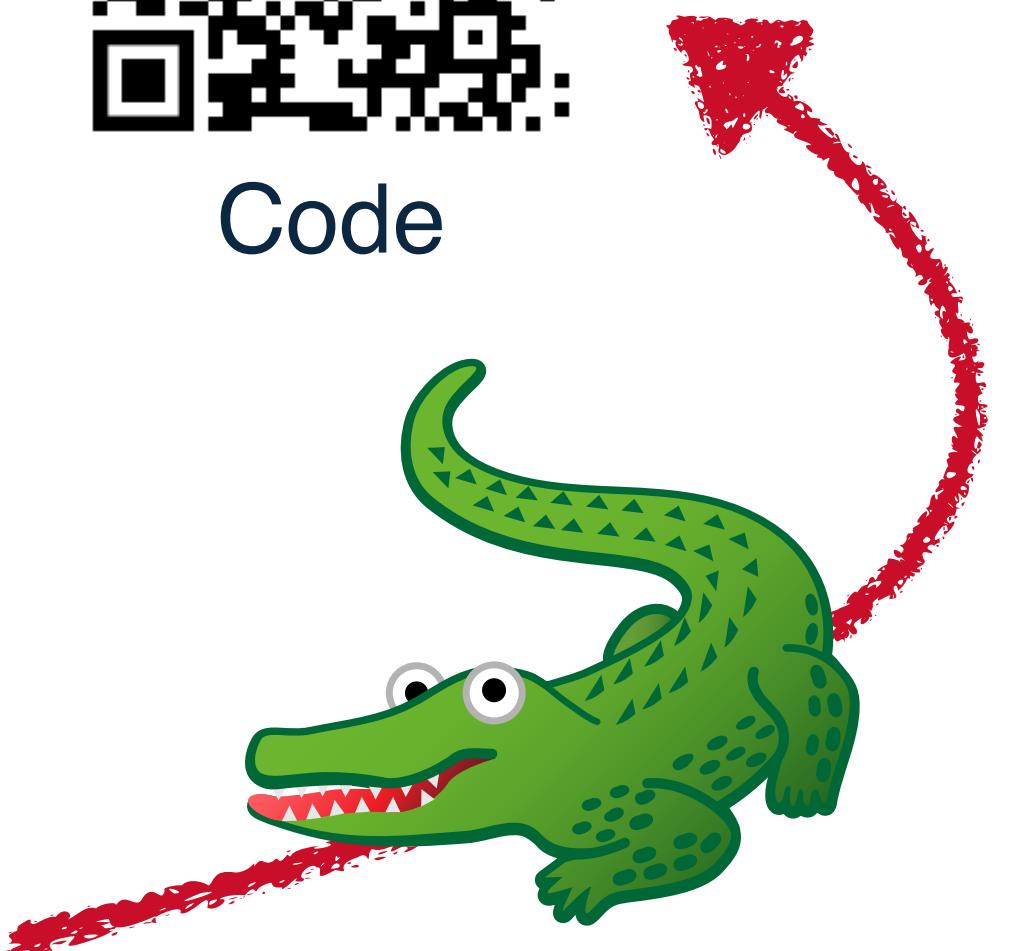
CS paper



HEP paper



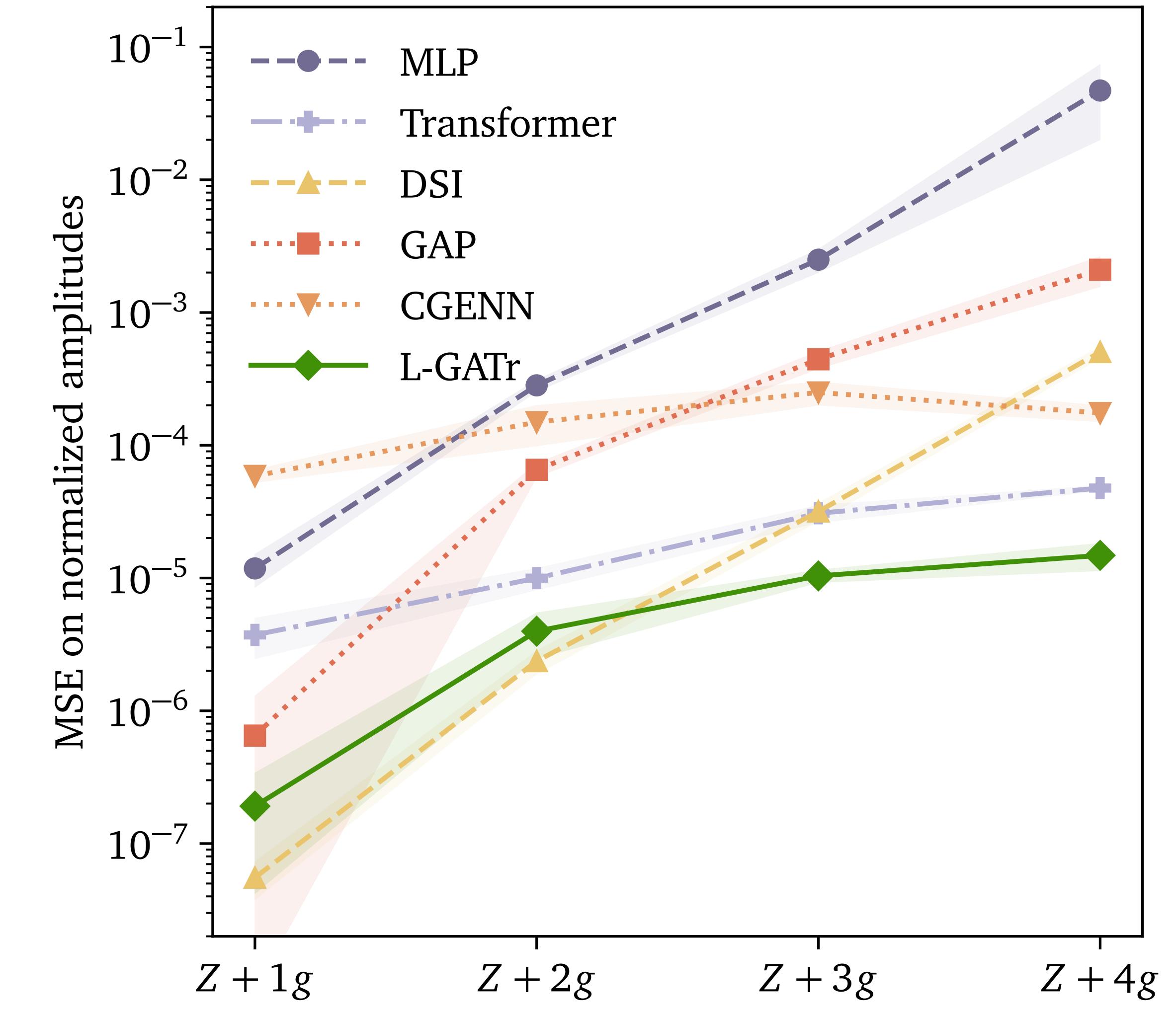
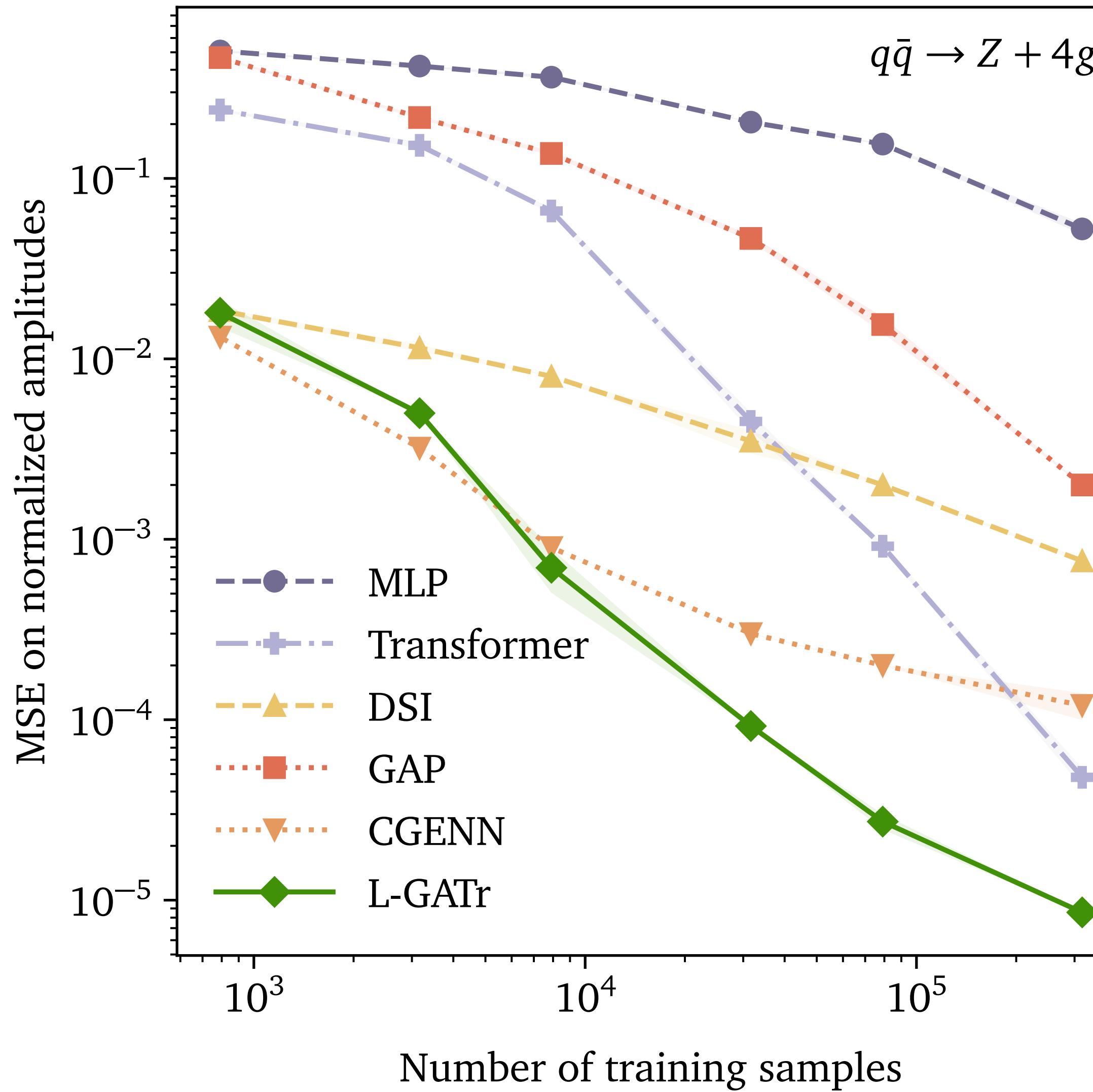
Code



For what will **you** use L-GATr?

# Bonus material

# Amplitude regression

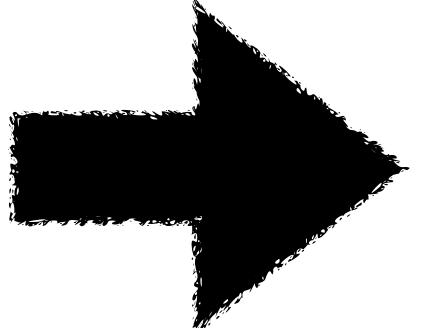


# Jet Tagging

## Symmetry breaking with multivectors

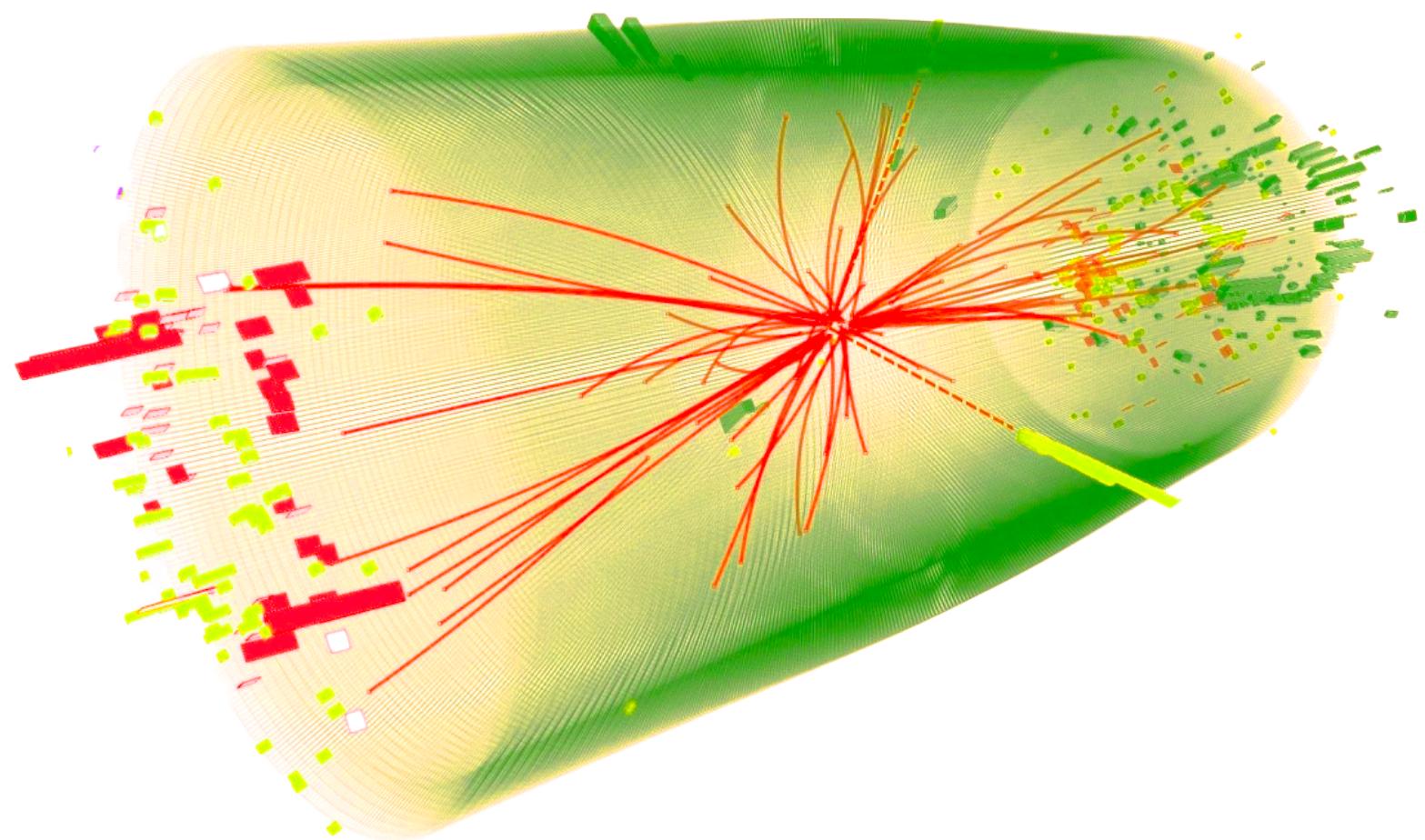
The jet tagging score is not Lorentz-equivariant

- Beam direction breaks invariance under rotations around the x- and y-axis
- Detector breaks boost invariance

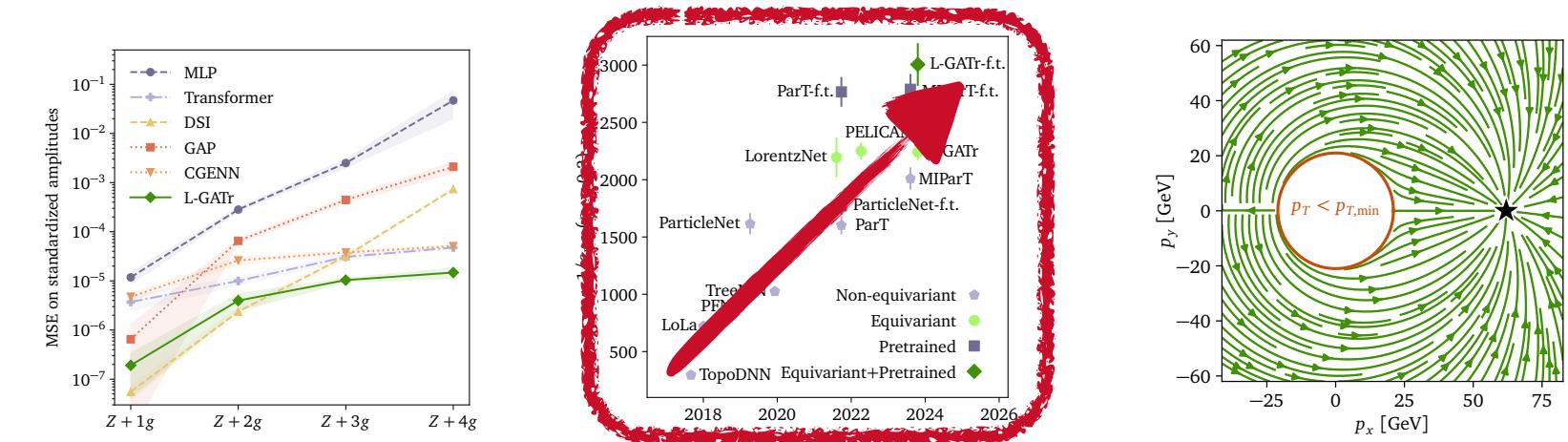


Add reference multivectors as extra particles to break Lorentz symmetry

- Beam reference multivector:  $x^V = (0,0,0, \pm 1)$  or  $x_{12}^B = 1$
- Time reference multivector:  $x^V = (1,0,0,0)$



All Lorentz-equivariant taggers use symmetry-breaking inputs



Network	Accuracy	AUC	$1/\epsilon_B$ ( $\epsilon_S = 0.5$ )	$1/\epsilon_B$ ( $\epsilon_S = 0.3$ )
TopoDNN [54]	0.916	0.972	–	$295 \pm 5$
LoLa [9]	0.929	0.980	–	$722 \pm 17$
<i>N</i> -subjettiness [55]	0.929	0.981	–	$867 \pm 15$
PFN [56]	0.932	0.9819	$247 \pm 3$	$888 \pm 17$
TreeNiN [57]	0.933	0.982	–	$1025 \pm 11$
ParticleNet [58]	0.940	0.9858	$397 \pm 7$	$1615 \pm 93$
ParT [59]	0.940	0.9858	$413 \pm 16$	$1602 \pm 81$
MIParT [60]	0.942	0.9868	$505 \pm 8$	$2010 \pm 97$
LorentzNet* [10]	0.942	0.9868	$498 \pm 18$	$2195 \pm 173$
CGENN* [14]	0.942	0.9869	500	2172
PELICAN* [42]	$0.9426 \pm 0.0002$	$0.9870 \pm 0.0001$	–	$2250 \pm 75$
L-GATr* [35]	$0.9423 \pm 0.0002$	$0.9870 \pm 0.0001$	$540 \pm 20$	$2240 \pm 70$
ParticleNet-f.t. [60]	0.942	0.9866	$487 \pm 9$	$1771 \pm 80$
ParT-f.t. [60]	0.944	0.9877	$691 \pm 15$	$2766 \pm 130$
MIParT-f.t. [60]	0.944	0.9878	$640 \pm 10$	$2789 \pm 133$
L-GATr-f.t.* (new)	$0.9442 \pm 0.0002$	$0.98792 \pm 0.00004$	$661 \pm 24$	$3005 \pm 186$

	All classes	$H \rightarrow b\bar{b}$	$H \rightarrow c\bar{c}$	$H \rightarrow gg$	$H \rightarrow 4q$	$H \rightarrow l\nu q\bar{q}'$	$t \rightarrow bq\bar{q}'$	$t \rightarrow bl\nu$	$W \rightarrow q\bar{q}'$	$Z \rightarrow q\bar{q}$	
	Accuracy	AUC	Rej <sub>50%</sub>	Rej <sub>50%</sub>	Rej <sub>50%</sub>	Rej <sub>99%</sub>	Rej <sub>50%</sub>	Rej <sub>99.5%</sub>	Rej <sub>50%</sub>	Rej <sub>50%</sub>	
ParticleNet [58]	0.844	0.9849	7634	2475	104	954	3339	10526	11173	347	283
ParT [59]	0.861	0.9877	10638	4149	123	1864	5479	32787	15873	543	402
MIParT [60]	0.861	0.9878	10753	4202	123	1927	5450	31250	16807	542	402
L-GATr	0.865	0.9884	12195	4819	128	2304	5764	37736	19231	580	427

# Event generation

## Conditional Flow Matching

Continuous normalising flow (CNF)

connect a simple base density  
to a complex target density  
through a neural differential equation

$$\frac{d}{dt}x = v_t(x)$$

arXiv:1806.07366

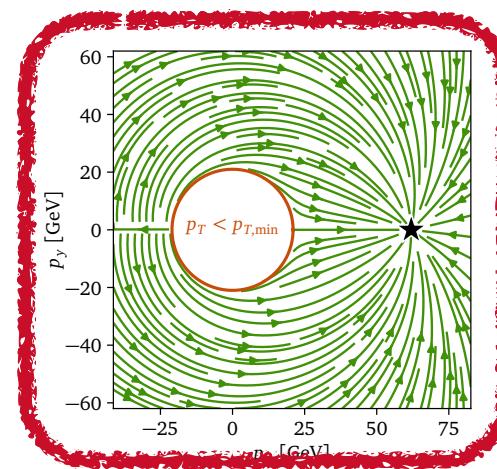
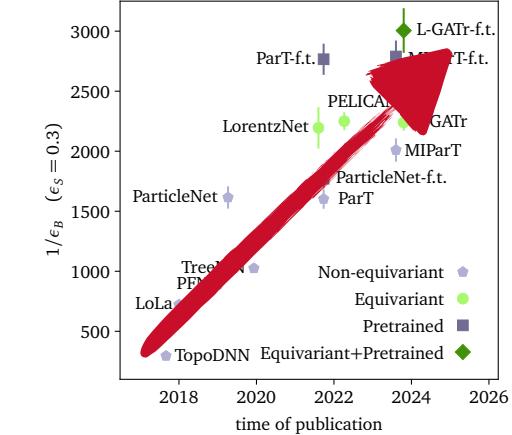
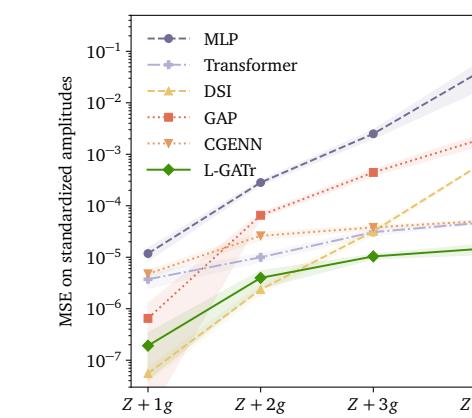
Conditional flow matching (CFM)

is a simple way to train CNFs  
by comparing the learned velocity  $v_t(x)$   
to a conditional target velocity  $u_t(x | x_1)$

$$\mathcal{L} = \left\langle (v_t(x) - u_t(x | x_1))^2 \right\rangle$$

arXiv:2210.02747

How to pick the target velocity  $u_t(x | x_1)$ ?

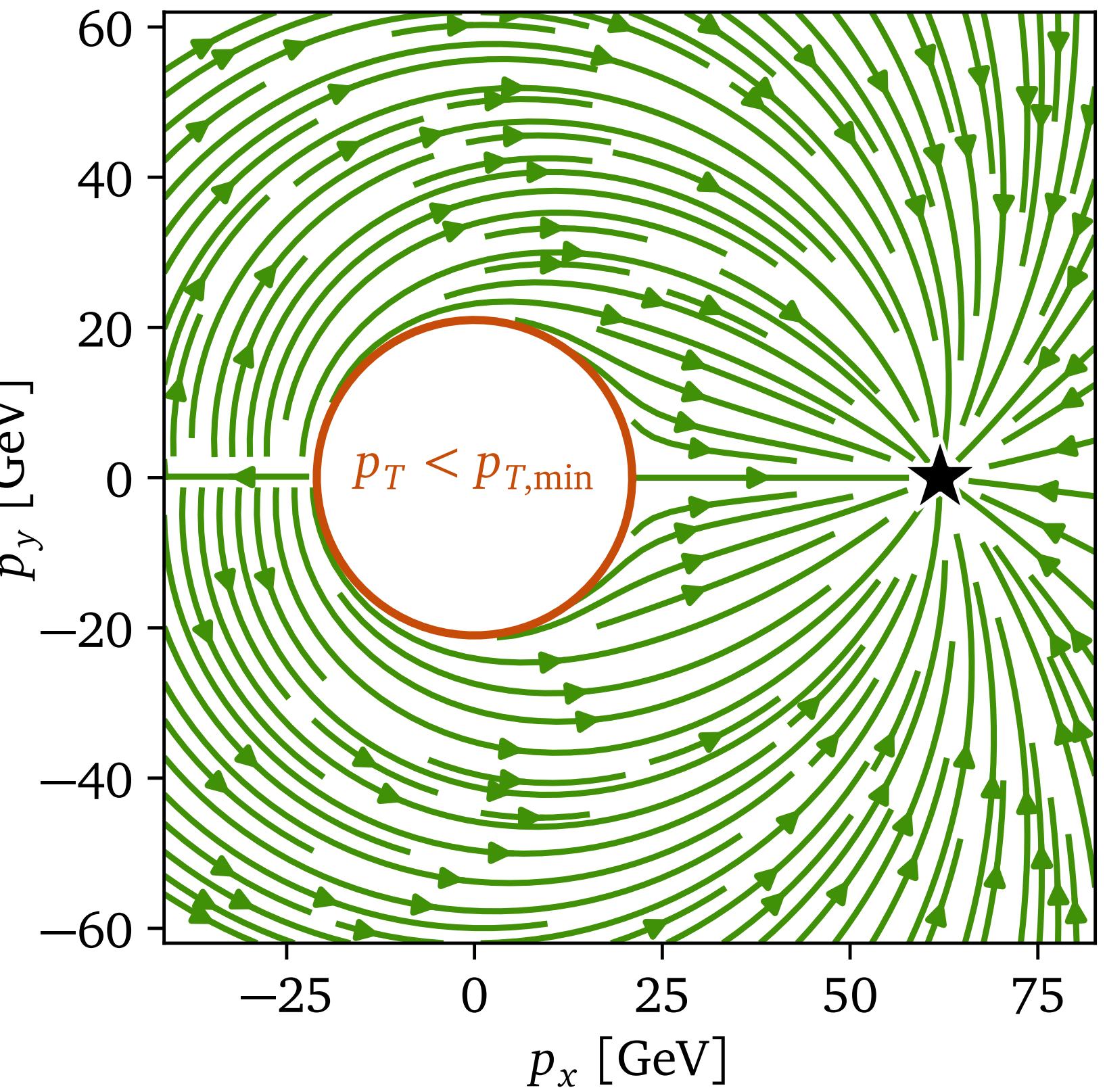
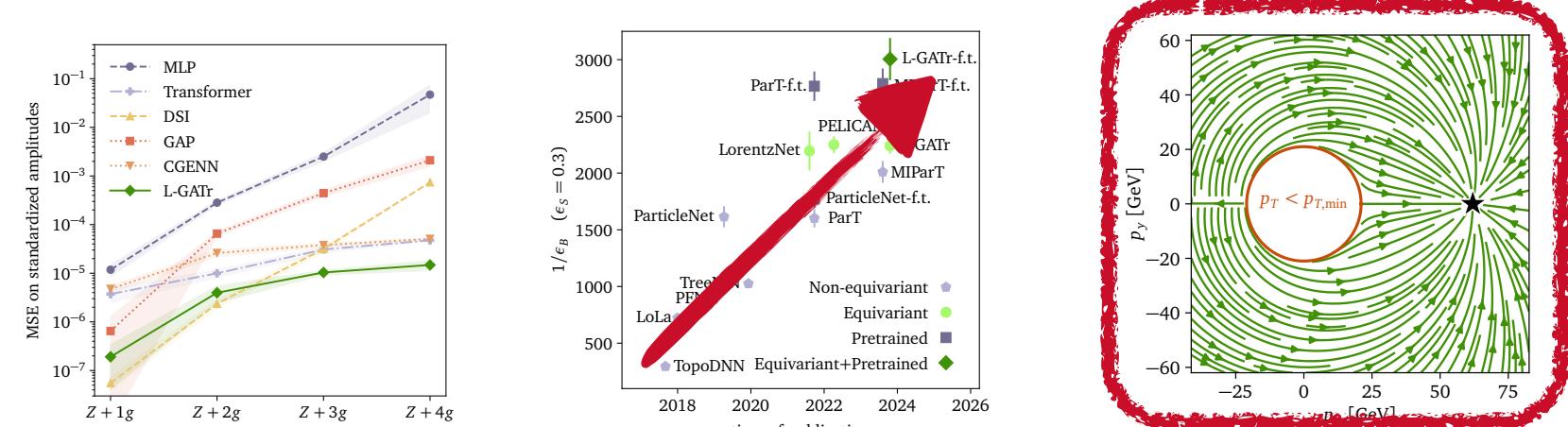


# Event generation

## Physics-inspired target trajectories

Straight trajectories in ‘modified jet momenta’  $x$ :

$$p = \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} \rightarrow f^{-1}(p) = x = \begin{pmatrix} x_p \\ x_m \\ x_\eta \\ x_\phi \end{pmatrix} = \begin{pmatrix} \log(p_T - p_{T,\min}) \\ \log m^2 \\ \eta \\ \phi \end{pmatrix}$$



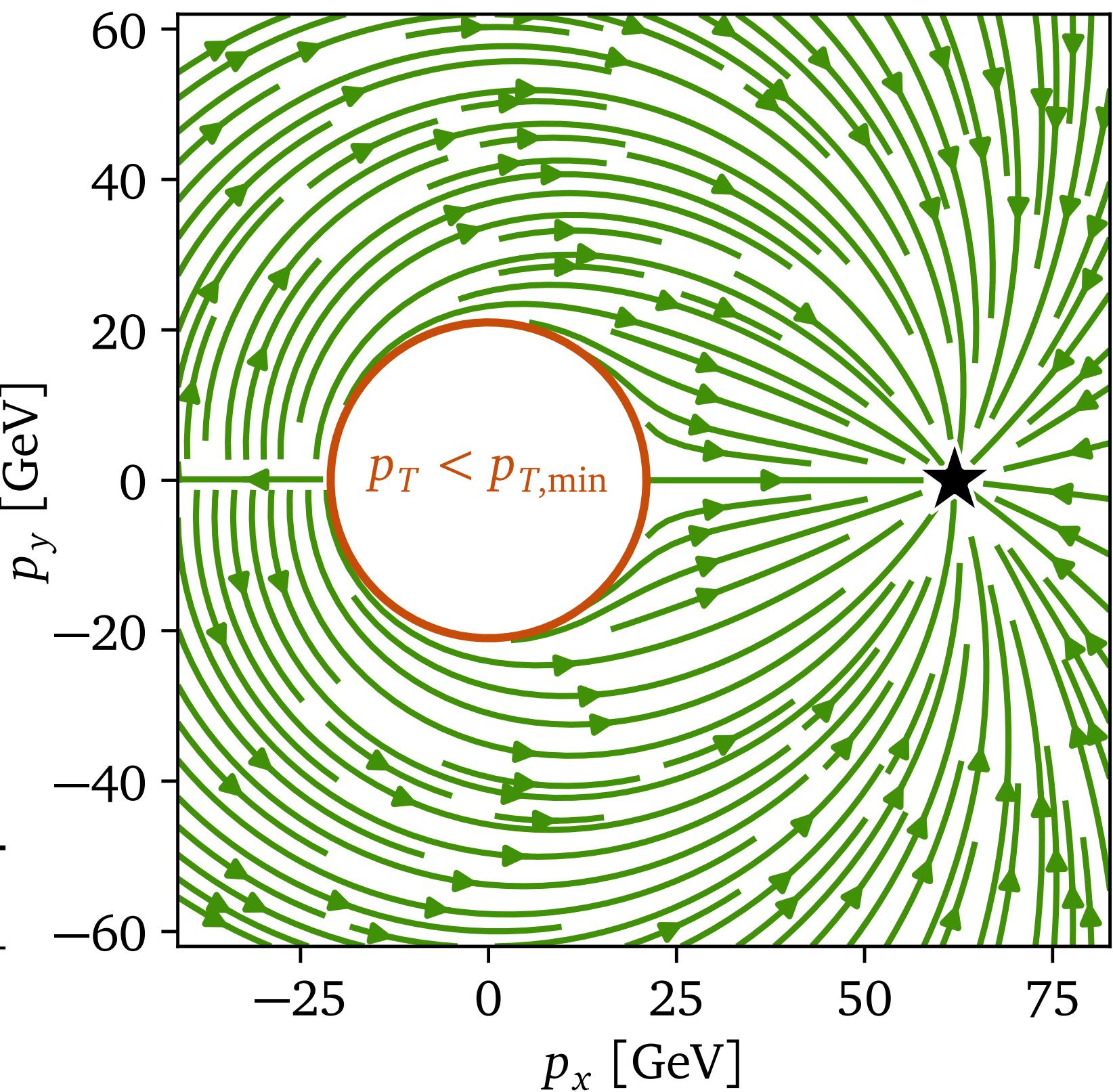
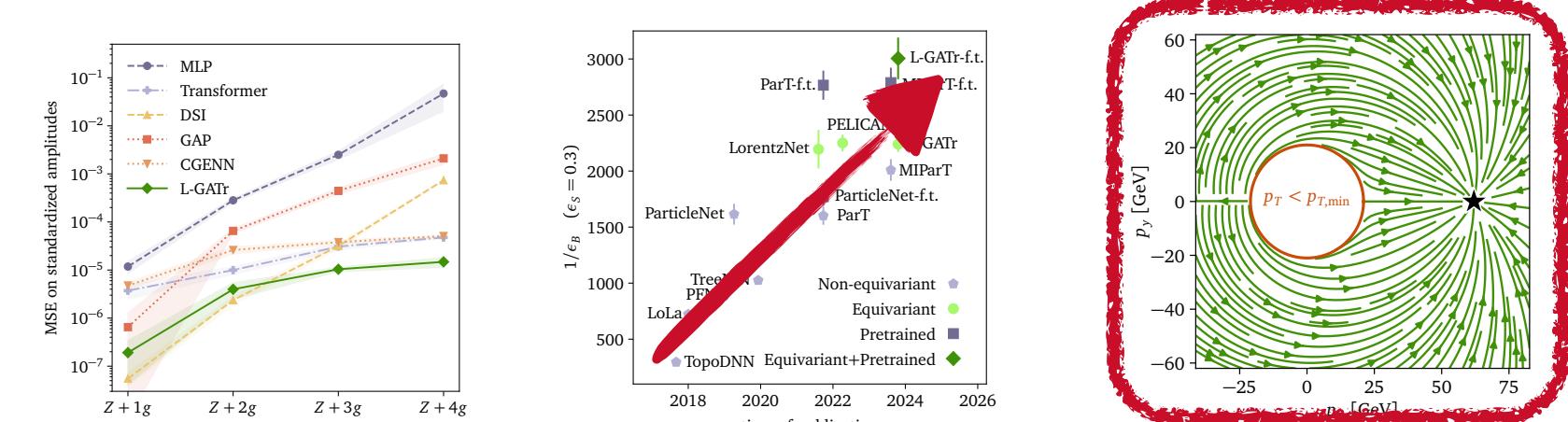
# Event generation

## Physics-inspired target trajectories

Straight trajectories in ‘modified jet momenta’  $x$ :

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Data	Architecture	Base distribution	Periodic	Neg. log-likelihood	AUC
$p$	L-GATr	rejection sampling	✓	$-30.80 \pm 0.17$	$0.945 \pm 0.004$
$x$	MLP	rejection sampling	✓	$-32.13 \pm 0.05$	$0.780 \pm 0.003$
$x$	L-GATr	rejection sampling	✗	$-32.57 \pm 0.05$	$0.530 \pm 0.017$
$x$	L-GATr	no rejection sampling	✓	$-32.58 \pm 0.04$	$0.523 \pm 0.014$
$x$	L-GATr	rejection sampling	✓	$-32.65 \pm 0.04$	$0.515 \pm 0.009$



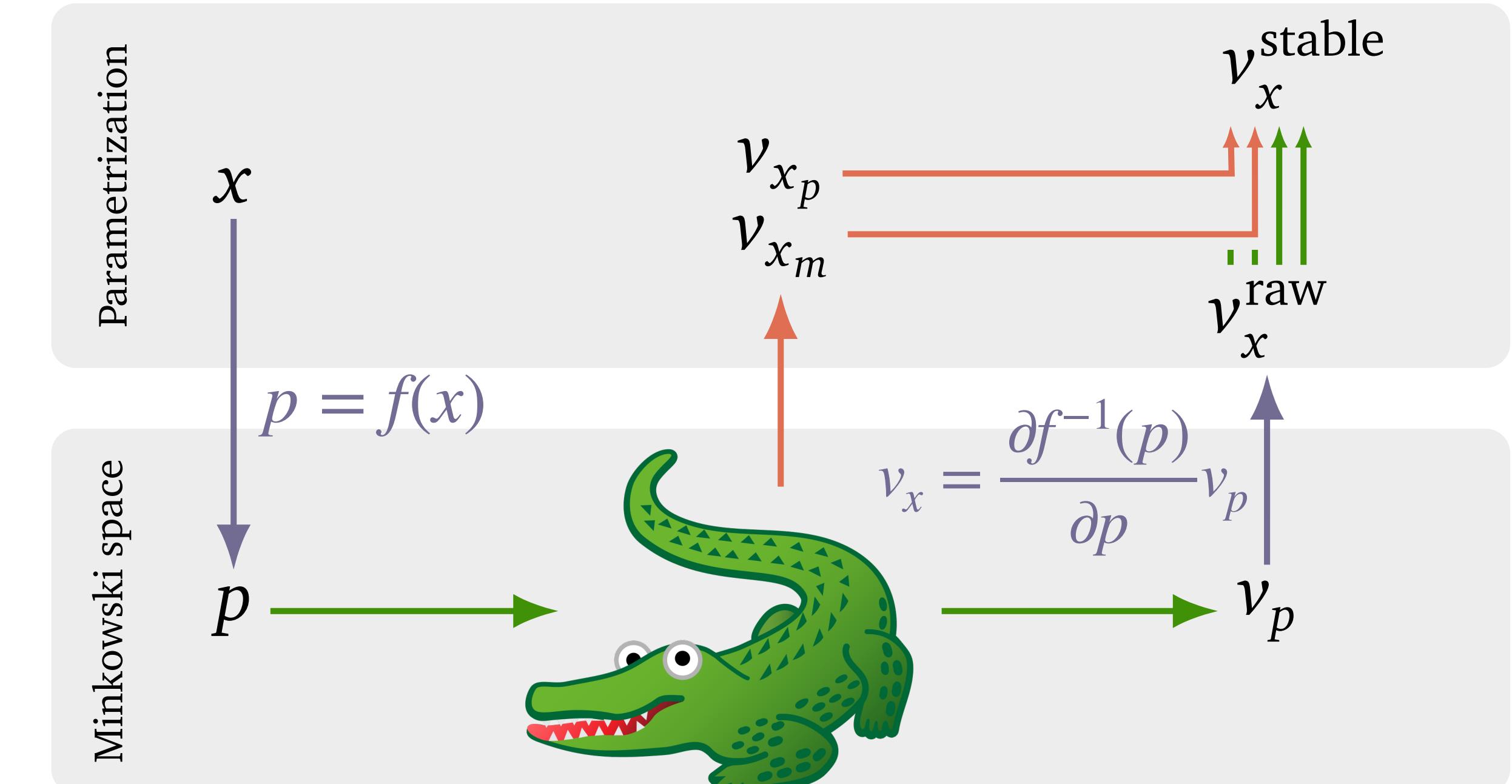
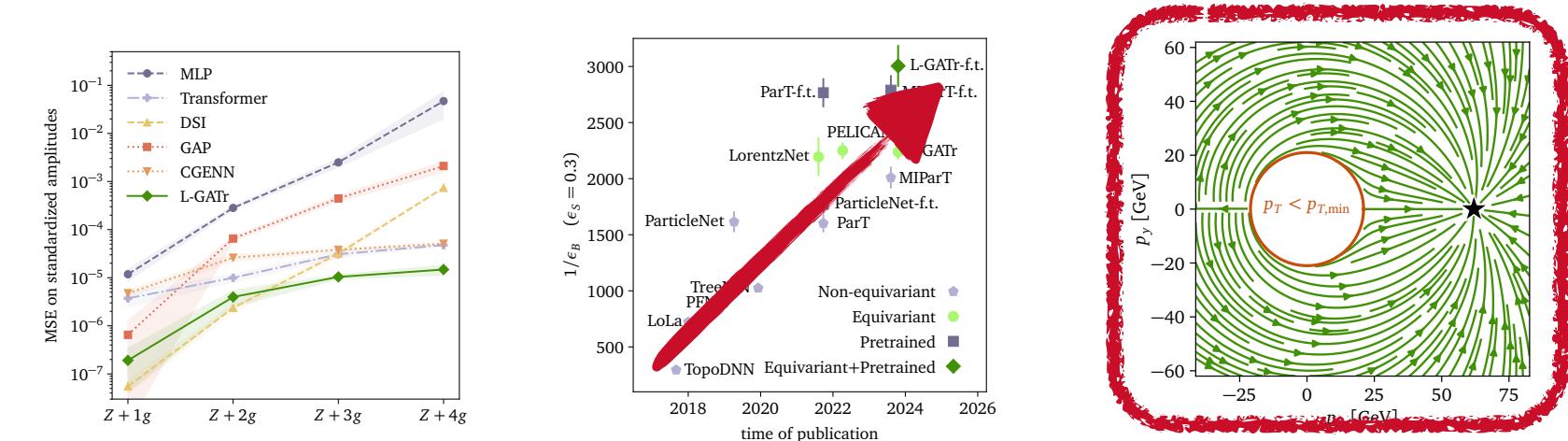
→ default

# Event generation

How to build an equivariant CFM field  $v_x(x)$ ?

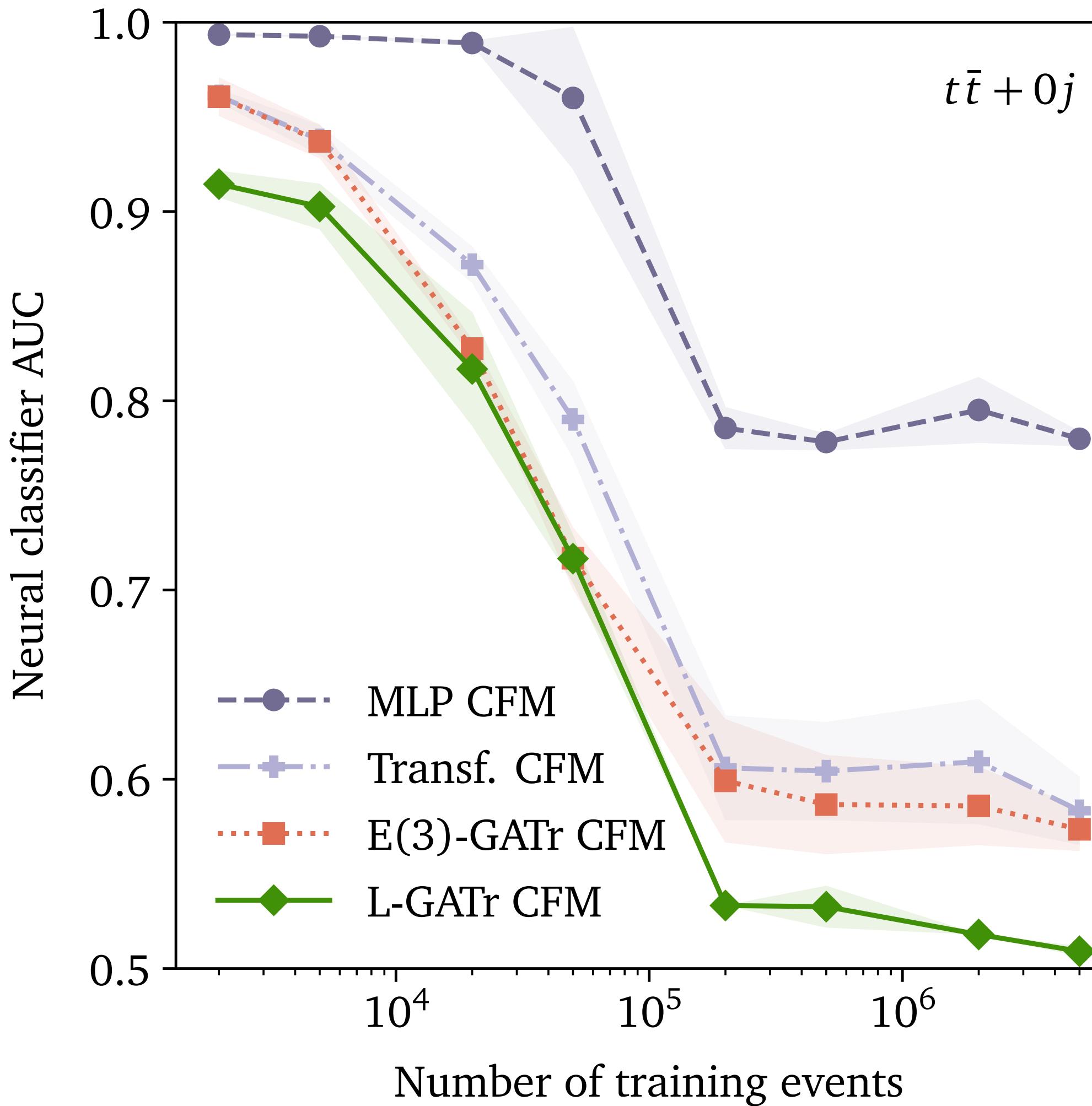
Extend standard CFM workflow  
with L-GATr:

- Transformations  $f(x)$   
between Minkowski space  $p$   
and the parametrization  $x$
- Equivariant L-GATr operations  
using multivectors
- Symmetry-breaking operations  
using scalars  
(required for numerical stability)



# Event generation

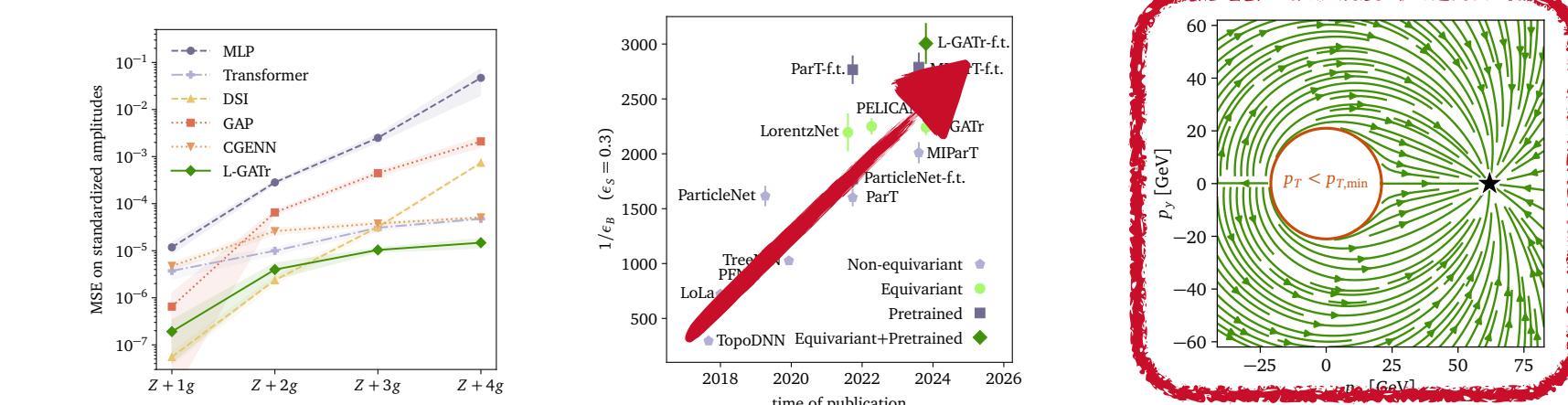
## Boost equivariance lets L-GATr shine



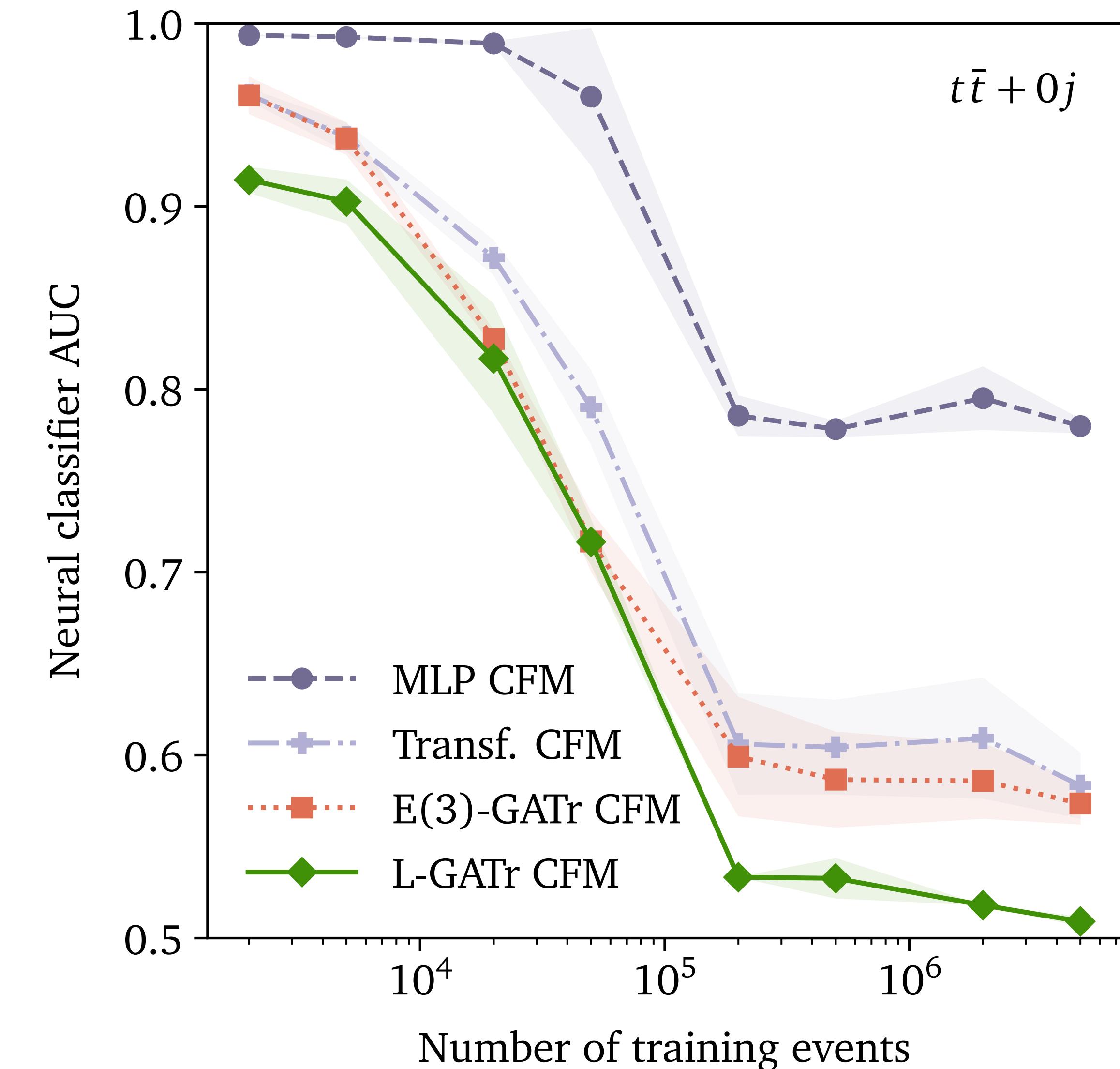
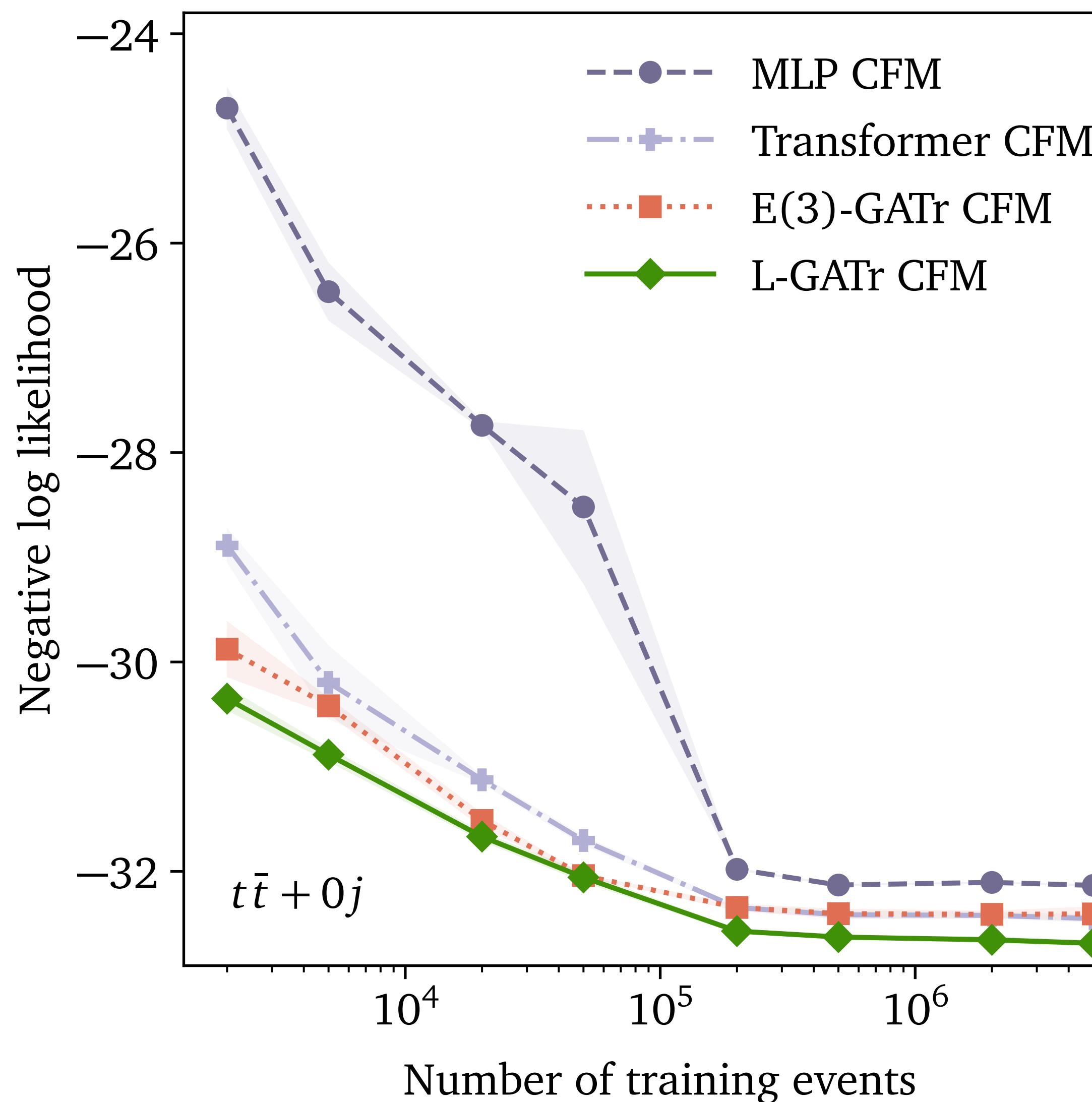
L-GATr and E(3)-GATr are both not boost-equivariant:

- E(3)-GATr: No boost-equivariance at all
- L-GATr: Architecture is boost-equivariant, but network inputs are not

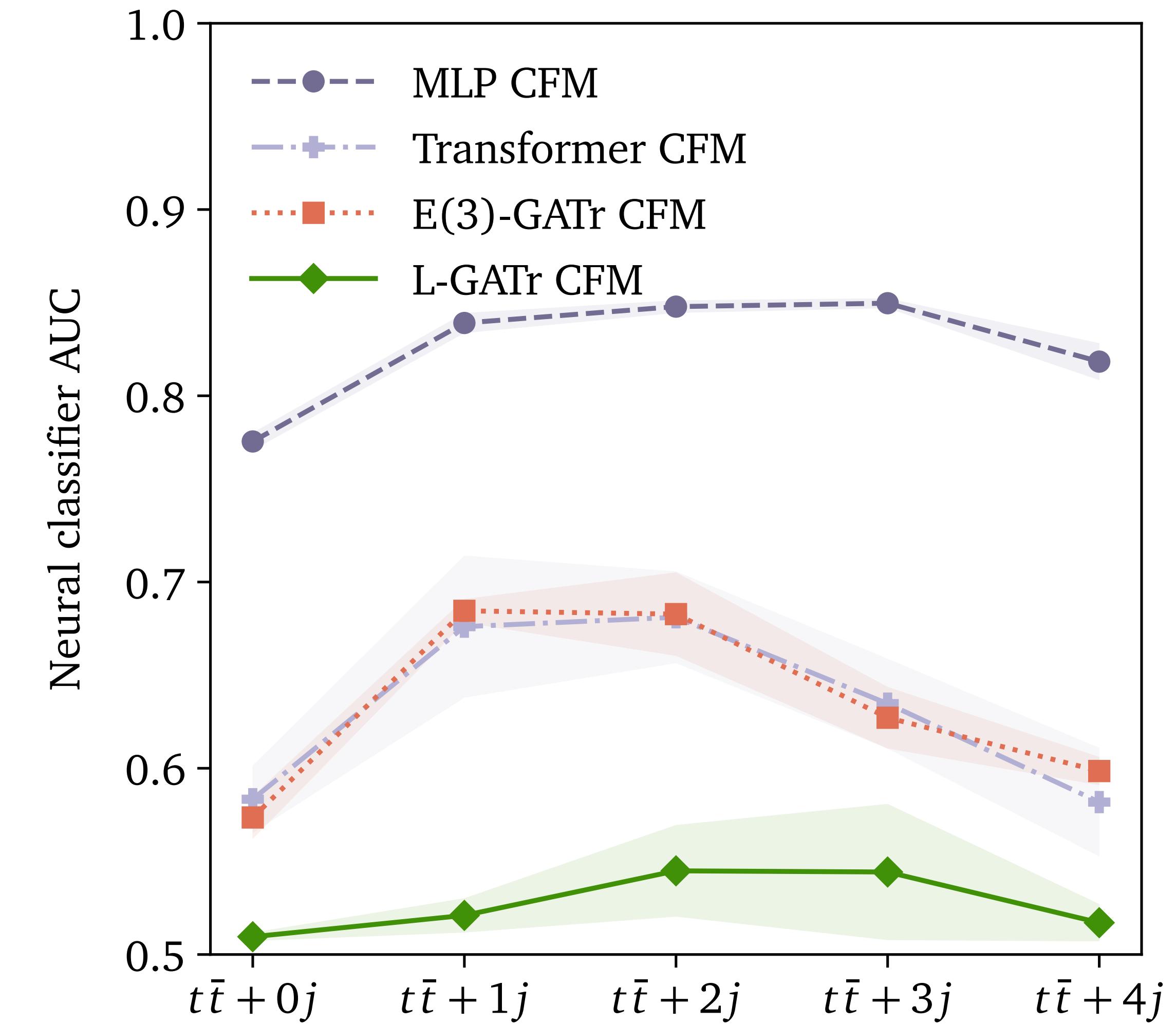
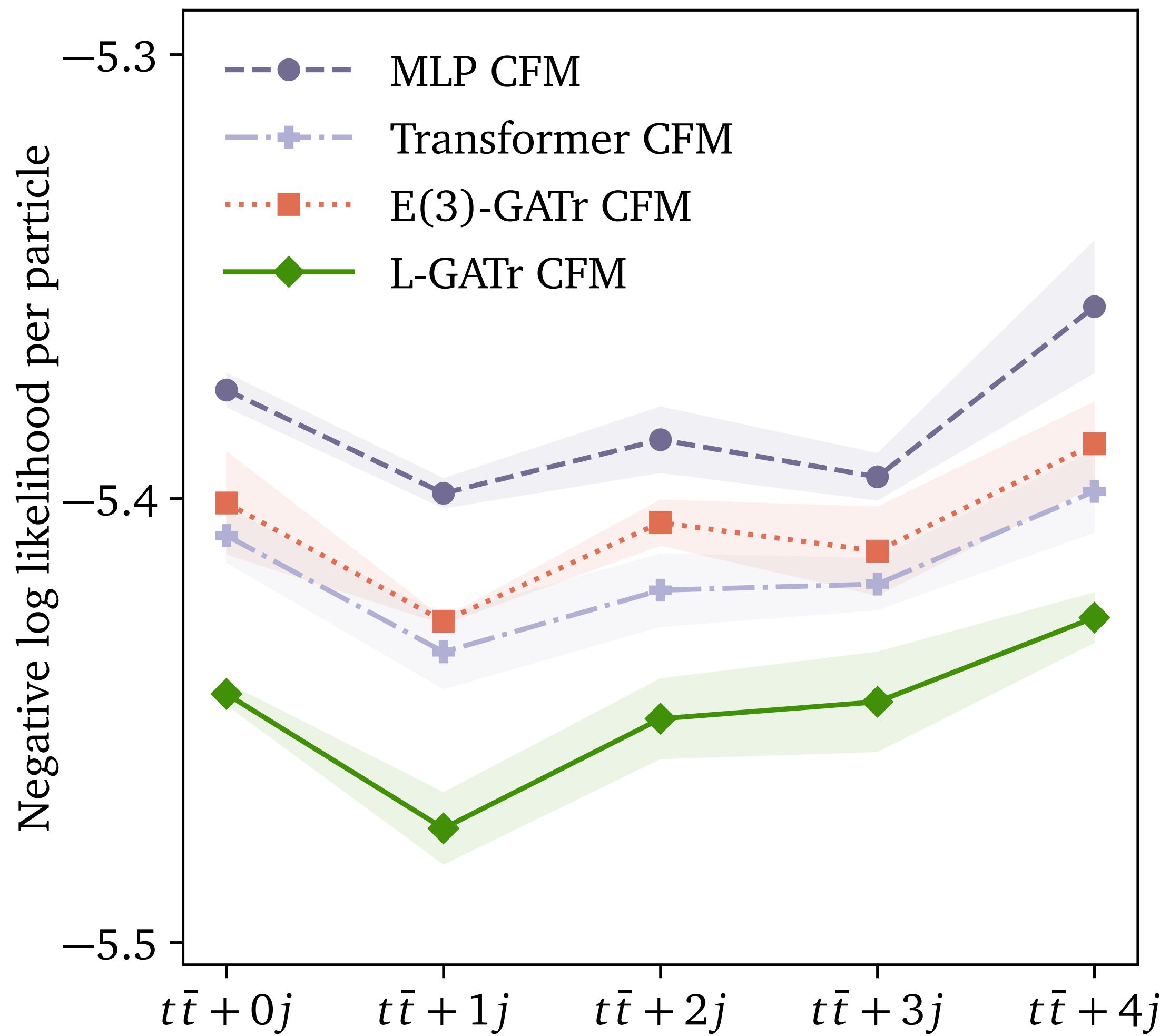
Its all about how we break the symmetry!



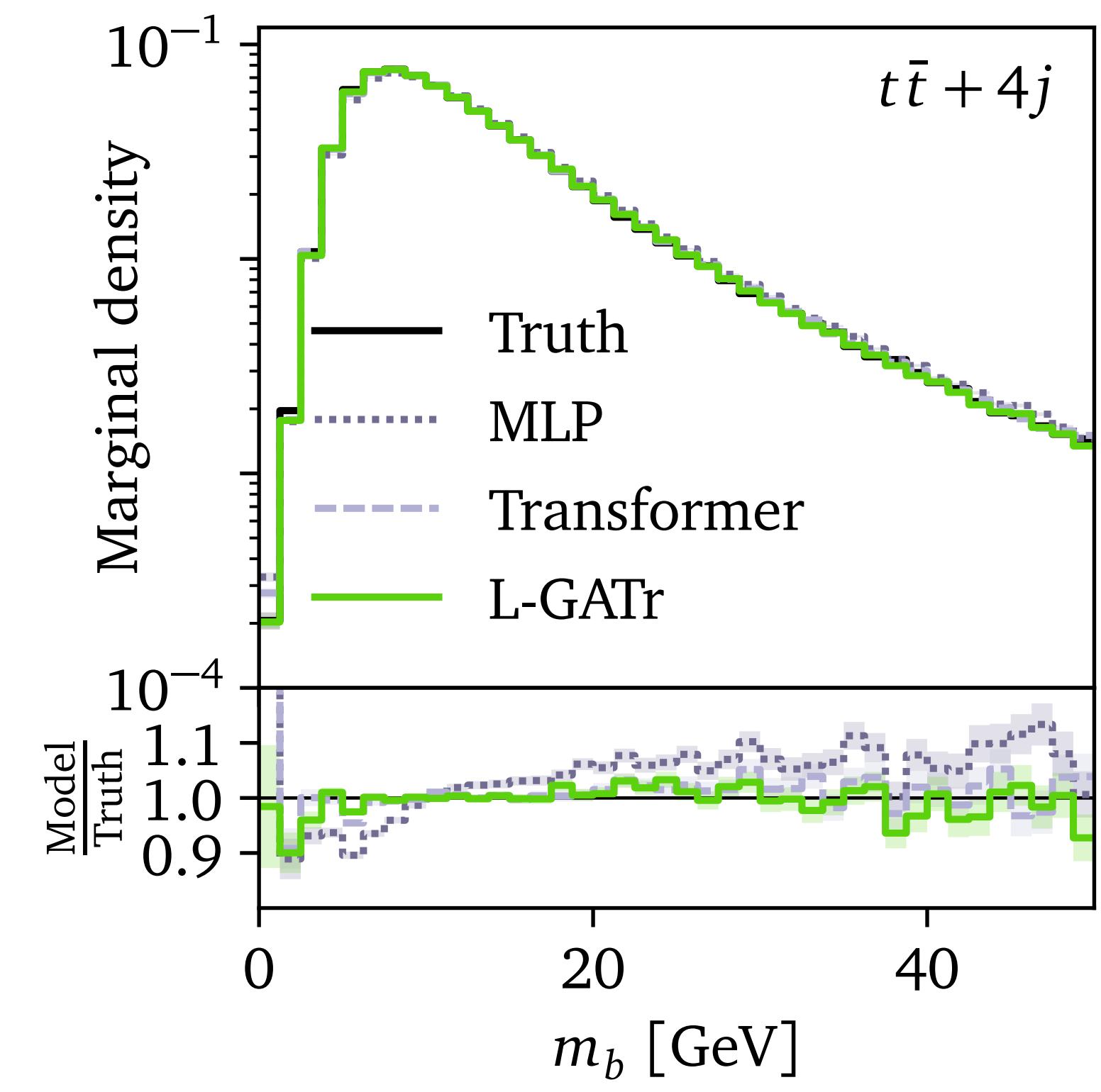
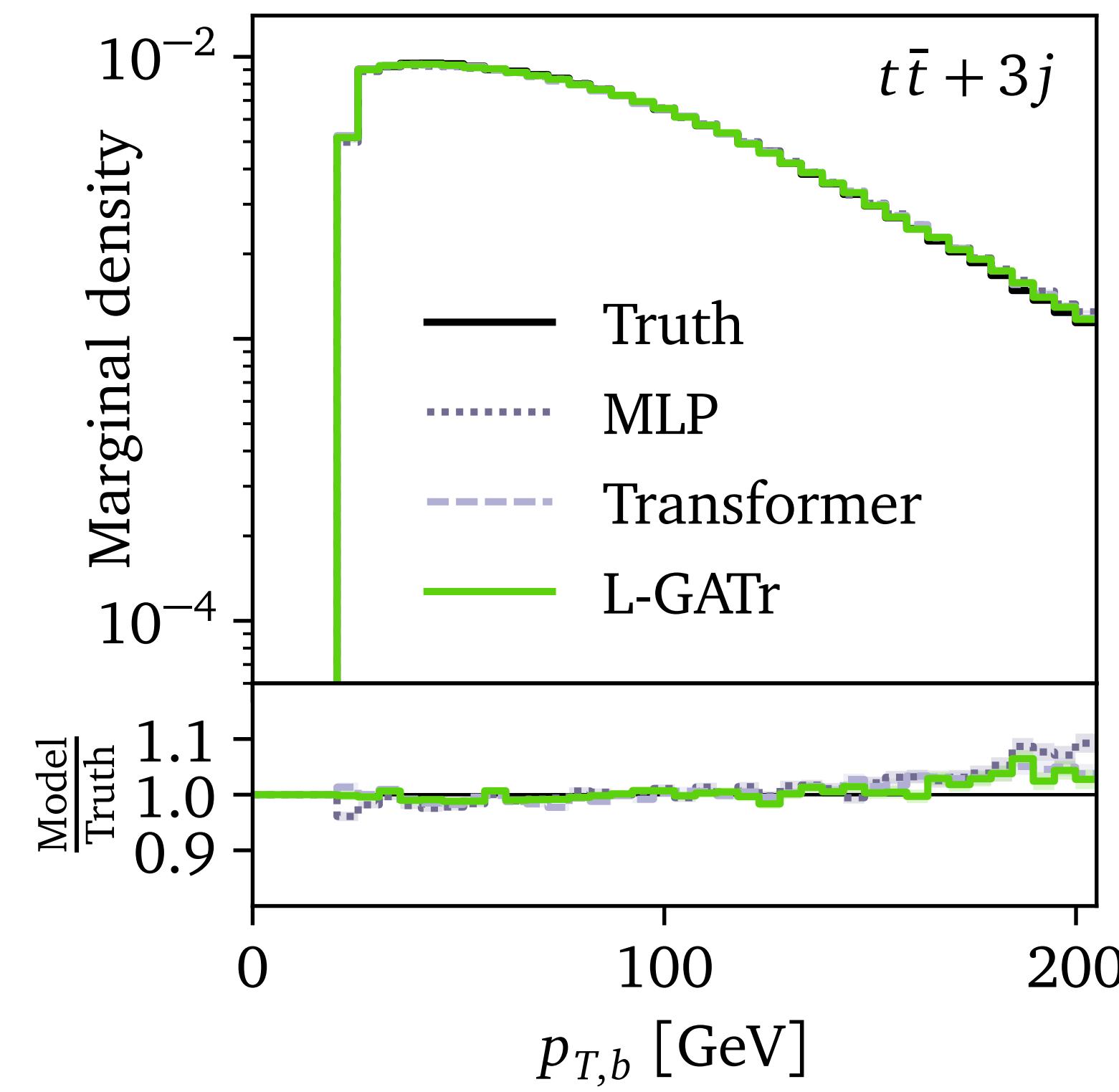
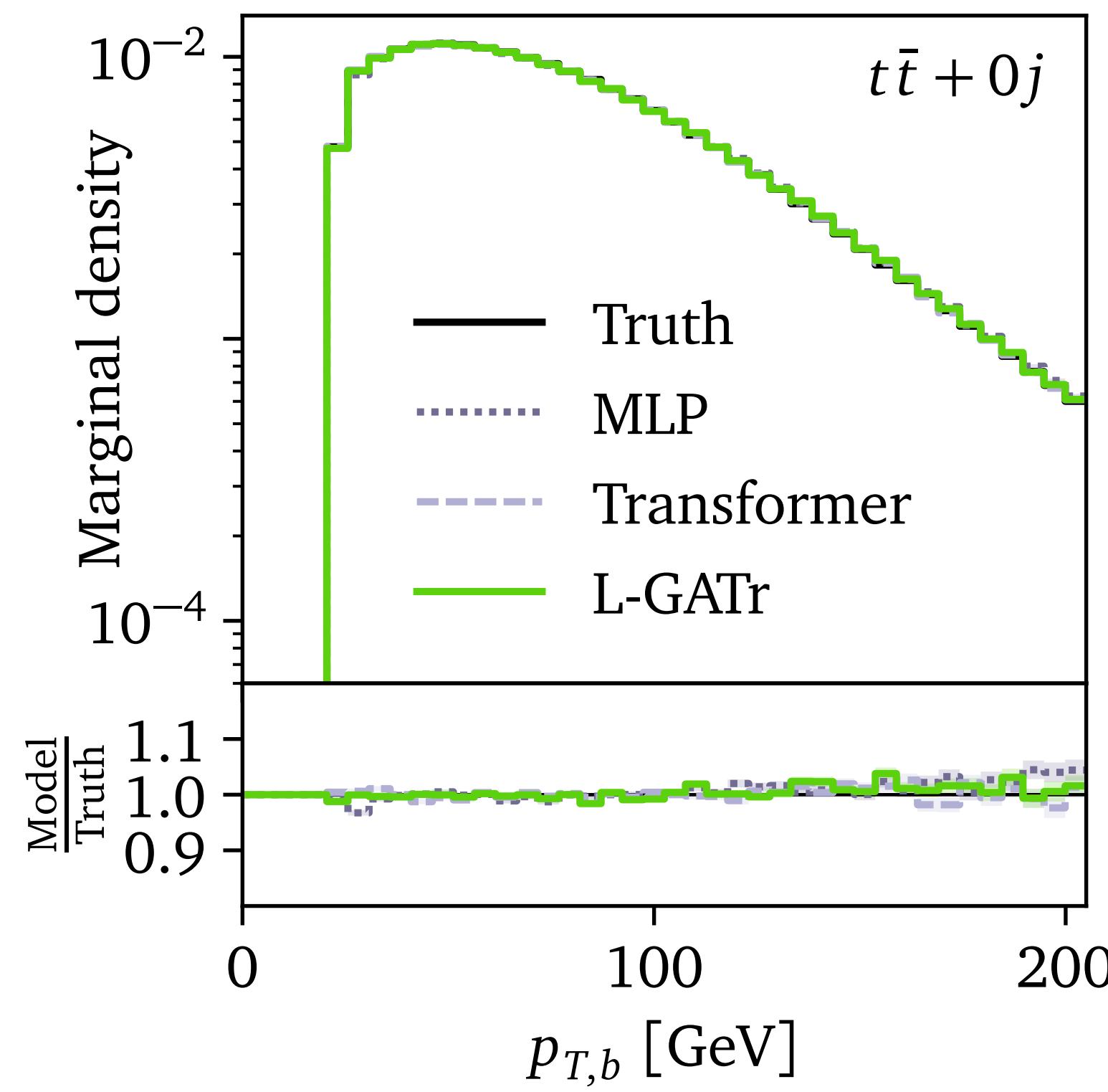
# Event generation



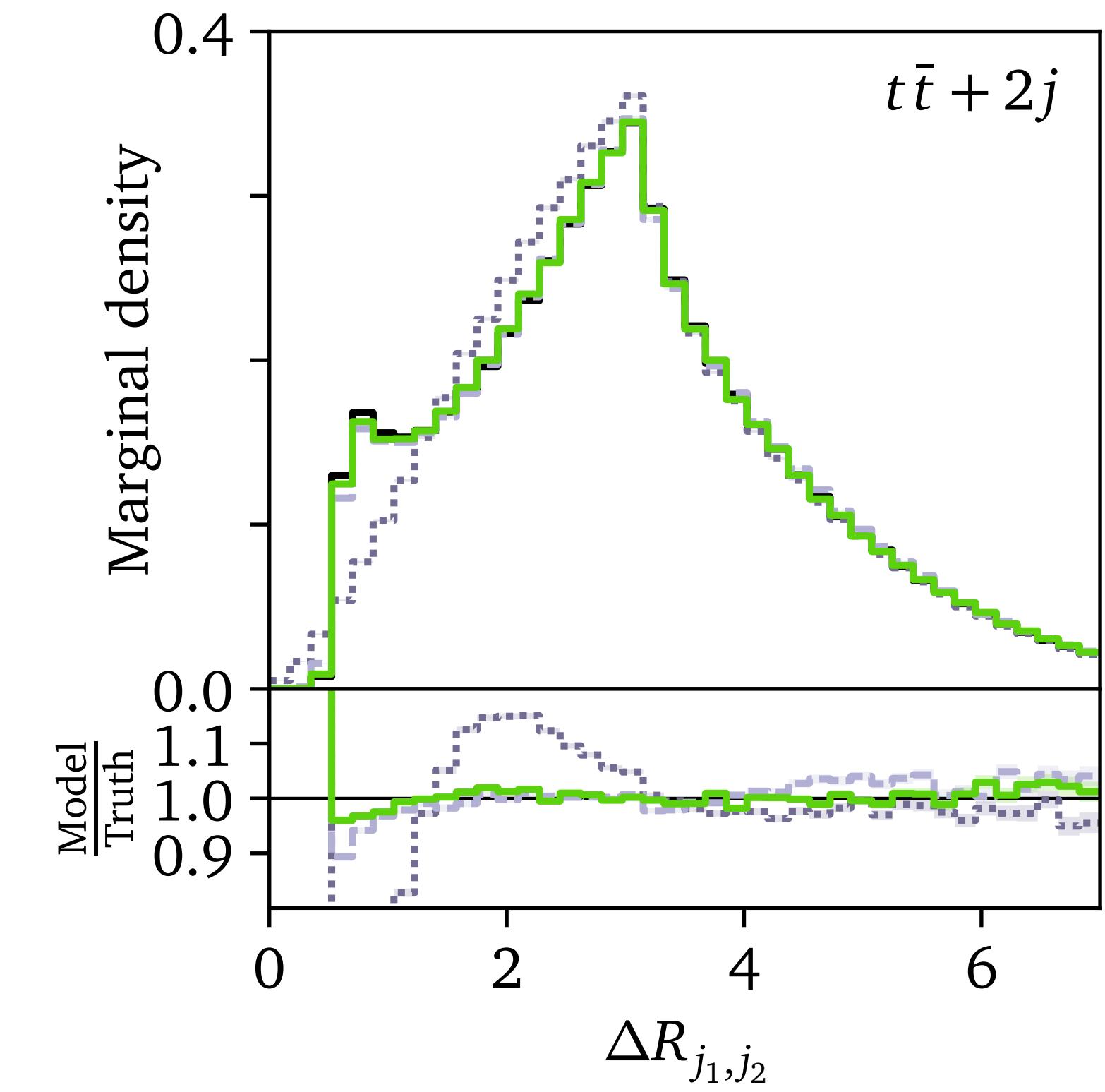
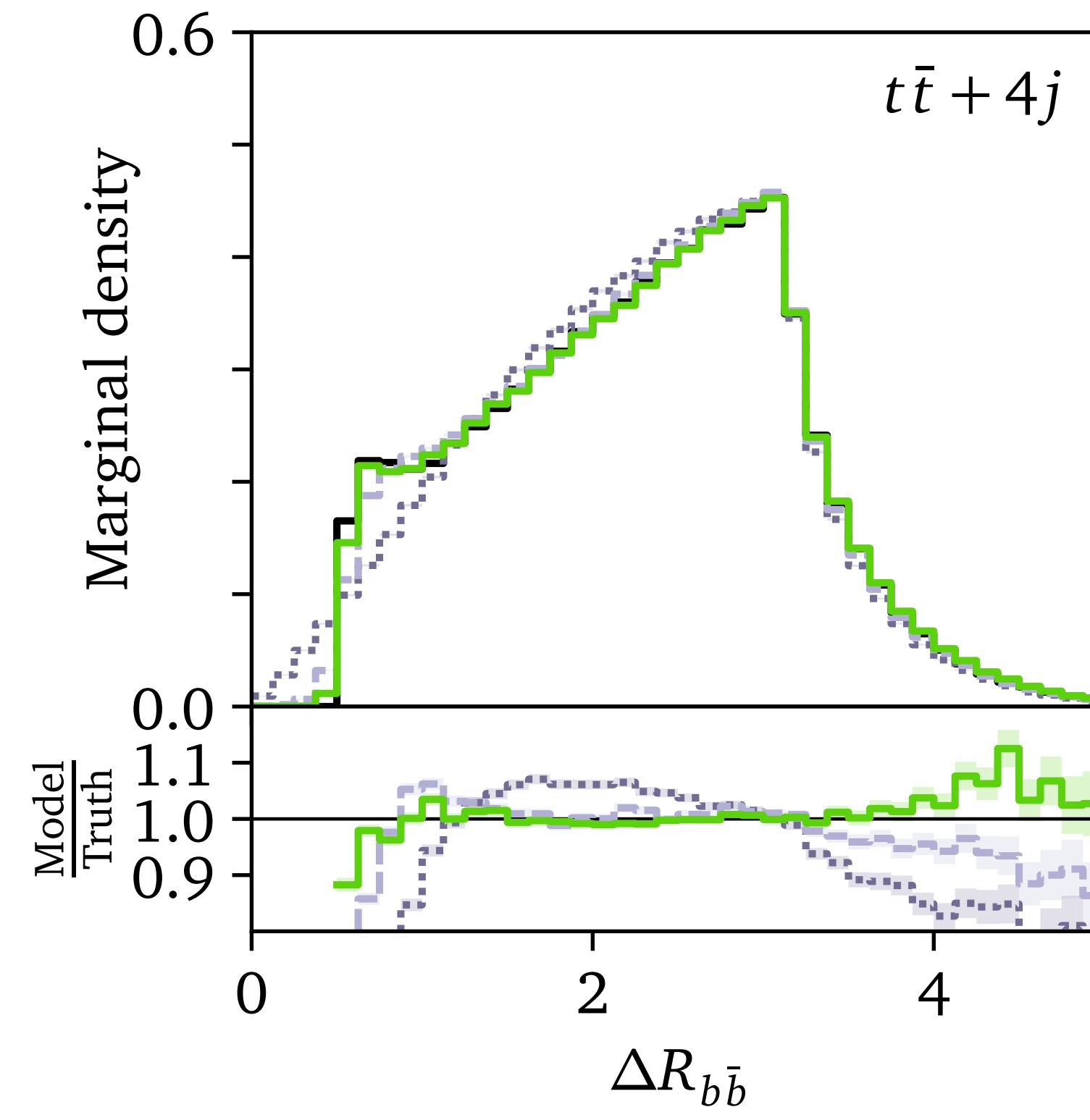
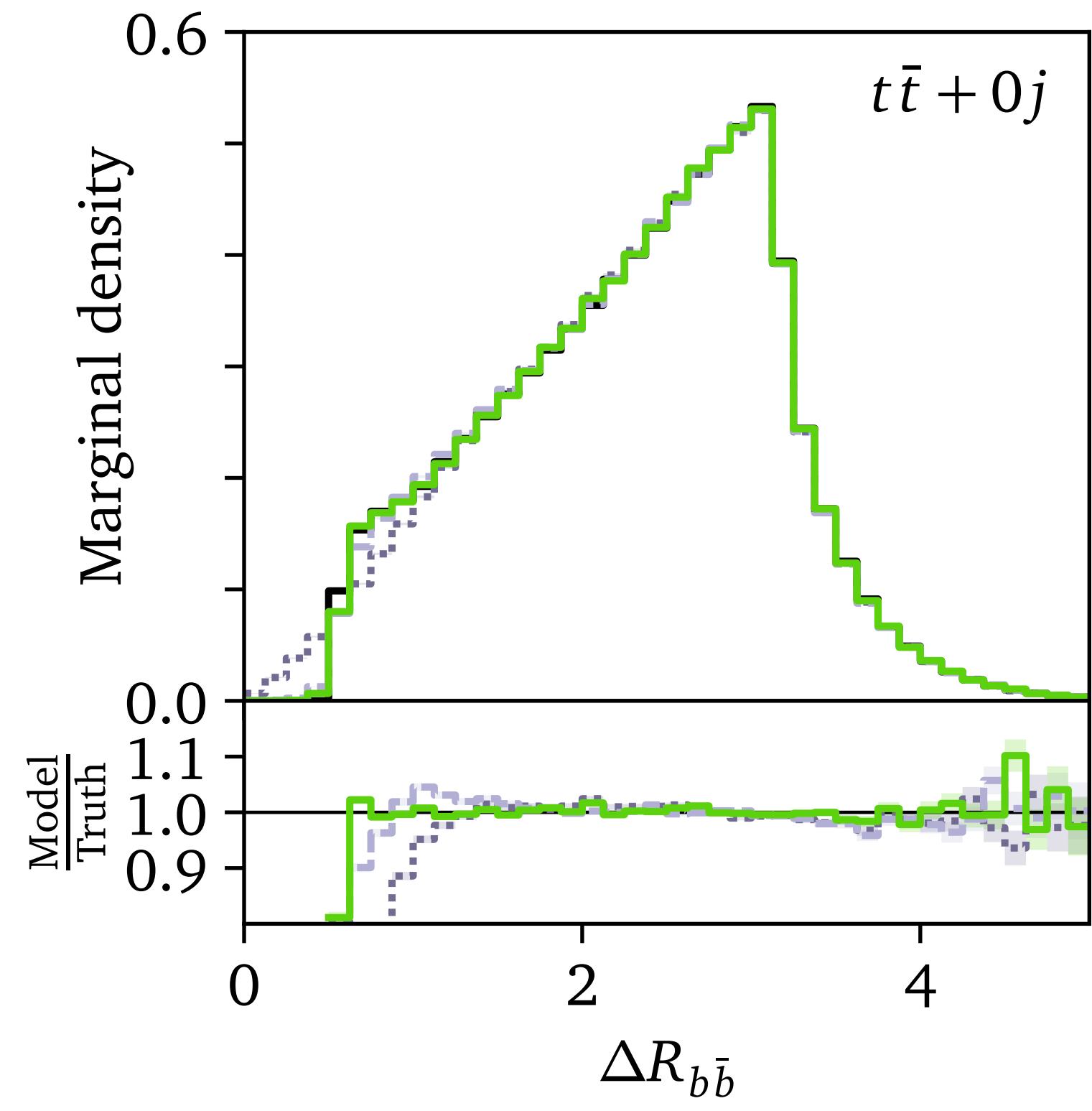
# Event generation



# Event generation



# Event generation



# Event generation

