Path integrals, spontaneous localization, and the classical limit

Bhavya Bhatt *[1], Manish Ram Chander†[2], Raj Patil ‡[3], Ruchira Mishra §[4], Shlok Nahar ¶[5], and Tejinder P. Singh ‖[6]

[1]*Indian Institute of Technology Mandi, Mandi 175005, India*
[2]*Indian Institute of Technology Madras, Chennai 600036, India*
[3]*Indian Institute of Science Education and Research Pune, Pune 411008, India*
[4]*Indian Institute of Science Education and Research Mohali, Mohali 140306, India*
[5]*Indian Institute of Technology Bombay, Mumbai 400076, India*
[6]*Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India*

November 15, 2018

Abstract

We recall that in order to obtain the classical limit of quantum mechanics one needs to take the $\hbar \to 0$ limit. In addition, one also needs an explanation for the absence of macroscopic quantum superposition of position states. One possible explanation for the latter is the Ghirardi-Rimini-Weber (GRW) model of spontaneous localisation. Here we describe how spontaneous localisation modifies the path integral formulation of density matrix evolution in quantum mechanics. (Such a formulation has been derived earlier by Pearle and Soucek; we provide two new derivations of their result). We then show how the von Neumann equation and the Liouville equation for the density matrix arise in the quantum and classical limit, respectively, from the GRW path integral. Thus we provide a rigorous demonstration of the quantum to classical transition.

*coder1704@gmail.com
†physicophillic@gmail.com
‡patil.raj@students.iiserpune.ac.in
§ruchiramishra98@gmail.com
¶naharshlok@gmail.com
‖tpsingh@tifr.res.in

Contents

1 Introduction

The limit $\hbar \to 0$ is often said to yield the classical limit of quantum mechanics, in the sense of recovering particles traveling on trajectories, among other things. This is the same as the limit $S \gg \hbar$, where S is the classical action. Here we recall how this is a necessary but insufficient criterion, by using the simple example of a free wave packet, and thus motivate sponataneous collapse as one possible resolution. The Schrödinger equation for a single particle of mass m, describing a pure state[1]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{1}$$

with $\psi = \sqrt{P}e^{i\chi/\hbar}$ and P, χ real functions, reduces to

$$\frac{1}{2m}|\boldsymbol{\nabla}\chi|^2 + V(\mathbf{x}, t) + \frac{\partial\chi}{\partial t} = 0 \tag{2}$$

$$\frac{\partial P}{\partial t} + \frac{1}{m}\boldsymbol{\nabla}\cdot(P\boldsymbol{\nabla}\chi) = 0 \tag{3}$$

when we take[2] $\hbar \to 0$. From the transformation, we see that

$$P = \psi\psi^* \tag{4}$$

is the probability distribution of finding the particle in space upon measurement and $\chi = \hbar\varphi$, φ being the phase of the wavefunction. The solution to these coupled equations must describe what quantum mechanics behaves like in the said limit.

Now let us consider a free particle in 1-D, with initial probability distribution and phase being

$$P(x,0) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-x^2/(2\sigma^2)} \qquad \chi(x,0) = px \tag{5}$$

This is a Gaussian wave packet of initial spread σ with average momentum p (non-zero). We are interested in its time evolution. Writing equations (2) and (3) in 1-D, we get

$$\frac{1}{2m}\left(\frac{\partial\chi}{\partial x}\right)^2 = -\frac{\partial\chi}{\partial t} \tag{6}$$

$$\frac{\partial P}{\partial t} + \frac{1}{m}\frac{\partial P}{\partial x}\frac{\partial\chi}{\partial x} + \frac{P}{m}\frac{\partial^2\chi}{\partial x^2} = 0 \tag{7}$$

Equation (6) is not separable - its solution for given initial condition is

$$\chi(x,t) = px - \frac{p^2 t}{2m} \tag{8}$$

as can be verified by explicitly plugging form $ax + bt$. Now, (7) becomes

$$\frac{\partial P}{\partial t} + \frac{p}{m}\frac{\partial P}{\partial x} = 0 \tag{9}$$

[1] More precisely, position representation of state $|\psi\rangle \in \mathcal{H}$ in the abstract Hilbert space.
[2] Refer J. J. Sakurai for details.

for constant $v = p/m$ with velocity dimensions. The solution to this equation with given initial condition is

$$P(x,t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-vt)^2/2\sigma^2} \tag{10}$$

which is again a Gaussian of same spread, centred at vt. We wish to emphasise that *only* $\hbar \to 0$ limit is taken to arrive here, without further assumptions.

Interpretation

We have established that spread of the wavepacket is unchanged for $\hbar \to 0$. By taking $\sigma \to 0$ therefore, we obtain a *trajectory* - a delta-localized particle moves along the line at constant speed, remaining delta-localized forever. This is signature of a free particle in classical mechanics.

However, finite σ values are allowed by quantum mecanics, which are not seen for classical particles. Furthermore, quantum mechanics allows the initial state to be a superpsotion

$$\psi(x,0) = \frac{1}{\sqrt[4]{8\pi\sigma^2}} e^{-(x-a)^2/(4\sigma^2)} e^{ipx/\hbar} + \frac{1}{\sqrt[4]{8\pi\sigma^2}} e^{-(x+a)^2/(4\sigma^2)} e^{ipx/\hbar} \tag{11}$$

where $a \gg \sigma$, so that

$$P(x,0) = |\psi(x,0)|^2 \simeq \frac{1}{\sqrt{8\pi\sigma^2}} e^{-(x-a)^2/(2\sigma^2)} + \frac{1}{\sqrt{8\pi\sigma^2}} e^{-(x+a)^2/(2\sigma^2)} \qquad \chi(x,0) = px \tag{12}$$

as the cross term is suppressed heavily. Quantum mechanics does not prevent us from doing so. As before $\chi(x,t) = px - pt^2/(2m)$ since initial condition is same. The probability density becomes

$$P(x,t) \simeq \frac{1}{\sqrt{8\pi\sigma^2}} e^{-(x-a-vt)^2/(2\sigma^2)} + \frac{1}{\sqrt{8\pi\sigma^2}} e^{-(x+a-vt)^2/(2\sigma^2)} \tag{13}$$

Thus, a "macroscopic superposition" in intial conditions is possible to prescribe, and superposition persists even as time evolves. Taking the limit of σ doesn't help: a trajectory simply is not guaranteed by just $\hbar \to 0$.

A similar situation arises while defining the classical limit in the Feynman path integral formulation of quantum mechanics. It is indeed the case that $S \gg \hbar$ ensures that in the sum over paths, destructive interference takes place between all paths except those in the vicinity of the classical path. Near the classical path, since variation of the action vanishes, there is constructive interference, and the dominant contribution to the path integral comes from the path satisfying the classical equations of motion. However, in order to conclude from here that the particle in question evolves along a classical trajectory, it is essential that the initial state be highly localised. This is not dictated by the requirement that $S \gg \hbar$, and we are back to the same problem as above.

In this article, we follow the viewpoint that the $\hbar \to 0$ limit is insufficient to rectify the situation. As one possible solution, we take resort to a phenomenological modification of quantum mechanics to deal with this, i.e. the idea of spontaneous localisation. The idea of spontaneous localization, and collapse models in general, has been extensively studied in recent years, as a possible approach to solve the quantum measurement problem, and explain the absence of macroscopic position superpositions. This was first proposed by Pearle in the 1970s [**?**] and subsequently by other authors in [**?**] and generalised to the case of identical particles in the CSL model [**?**]. The work of Ghirardi-Rimini-Weber is often referred to as the GRW model. The proposal is that every quantum object in nature undergoes spontaneous localisation to a region of size r_c, at random times given by a Poisson process with a mean collapse rate λ. Between every two collapses, the wave function obeys Schrödinger evolution. The collapse rate can be shown to be proportional to the number N of nucleons in the object, and we write $\lambda = N\lambda_{GRW}$, where λ_{GRW} is the collapse rate for a nucleon. Thus, λ_{GRW} and r_C are two new constants of nature, whose values must be fixed by experiment. Formally, the two postulates of the GRW model are stated as follows [**?**]:

Postulate 1. Given the wave function $\psi(\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_N})$ of an N particle quantum system in Hilbert space, the n-th particle undergoes spontaneous localization to a random position $\mathbf{x}$ as described by the following jump operator:

$$\psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_N) \quad \longrightarrow \quad \frac{L_n(\mathbf{x})\psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_N)}{\|L_n(\mathbf{x})\psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_N)\|} \tag{14}$$

The jump operator $L_n(\mathbf{x})$ is a linear operator which is defined to be the normalised Gaussian:

$$L_n(\mathbf{x}) = \frac{1}{(\pi r_C^2)^{3/4}} e^{-(\hat{\mathbf{q}}_n - \mathbf{x})^2/2r_C^2} \tag{15}$$

Here, $\hat{\mathbf{q}}_n$ is the position operator for the n-th particle of the system, and the random variable $\mathbf{x}$ is the spatial position to which the jump occurs. r_C, which is the width of the Gaussian, is a new constant of nature.

The probability density for the n-th particle to jump to the position $\mathbf{x}$ is assumed to be given by:

$$p_n(\mathbf{x}) \quad \equiv \quad \|L_n(\mathbf{x})\psi_t(\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_N)\|^2 \tag{16}$$

Also, it is assumed that the jumps are distributed in time as a Poissonian process with frequency λ_{GRW}. This is the second new constant of nature, in the model.

Postulate 2. In between any two successive jumps, the wave function evolves according to the Schrödinger equation.

Here, we use the path integral formulation for the evolution of the density matrix, to show that in addition to the limit $\hbar \to 0$, one needs the limit $\lambda T \gg 1$, to properly obtain the classical limit of quantum theory. Here, T is the time integral over which the path integral is evaluated. The process of spontaneous localisation serves to provide an exponential damping of the exponential oscillations in the path integral amplitude. Inevitably, the damping is important for macroscopic systems, but insignificant for microscopic ones.

2 The GRW path integral and its derivation

The path integral formulation of quantum mechanics is a description of quantum theory that generalizes the action principle of classical mechanics. It replaces the classical notion of a single, unique classical trajectory for a system with a sum, or functional integral, over an infinity of quantum-mechanically possible trajectories to compute a quantum amplitude. The GRW path integral has been previously derived by Pearle and Soucek [?]; here we give two alternative derivations of their result, and then discuss the classical and quantum limits of the GRW path integral. [For further applications of path integrals to collapse models, see also [? ? ?]].

2.1 Method-1

2.1.1 Introduction

Standard techniques [?] can be used to derive the propagator starting from the Schrödinger equation. However, these techniques cannot directly be used for mixed states represented by density matrices. Hence, we first purify the state-vector [?] so that it obeys Schrödinger-like evolution with an effective Hamiltonian. The methods followed in [?] can then be directly applied to this pure state ket.

2.1.2 Getting the Hamiltonian Form

The GRW master equation [? ?] is

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}(H\rho - \rho H) - \lambda\left(\rho - \int d^3r L_r \rho L_r\right) \tag{17}$$

where H is the Hamiltonian for Schrödinger evolution of the system and

$$L_r = \frac{1}{\mathcal{N}}\exp\left(-\frac{(\hat{q}-r)^2}{2r_C^2}\right) \tag{18}$$

is the collapse operator for the particle to localize around r. λ is the collapse rate, and r_C is the length scale to which localization takes place, as defined in the introduction. This master equation was first derived for the CSL model [?] where the authors noted that for the one particle case this equation is the same as for the GRW model, although this is not true in general.

In order to convert Eq. (17) into an equation of the form

$$\frac{d|\psi\rangle}{dt} = -\frac{i}{\hbar}\tilde{H}|\psi\rangle \tag{19}$$

we define $|\psi\rangle$ as

$$|\psi\rangle = \sum_{m,n} \rho_{mn} |m\rangle \otimes |n\rangle \tag{20}$$

where $\rho_{mn} = \langle m|\rho|n\rangle$ are elements of the density matrix ρ from Eq. (17). We note that there is an isomorphism between $|\psi\rangle$ as defined here, and ρ. Thus, knowing the evolution of $|\psi\rangle$ would give us all the information about how ρ whould evolve. Using Einstein's summation convention, we rewrite Eq. (17) as,

$$\frac{d\rho_{mn}}{dt} = -\frac{i}{\hbar}(H_{ma}\rho_{an} - \rho_{ma}H_{an}) - \lambda\left(\rho_{mn} - \int d^3r L_{rma}\rho_{ab}L_{rbn}\right) \tag{21}$$

From Eq. (19) and Eq. (20), it follows that the equation

$$\frac{d\rho_{mn}}{dt} = -\frac{i}{\hbar}\tilde{H}_{mabn}\ \rho_{ab} \tag{22}$$

must also hold. Comparing Eq. (21) and Eq. (22) we get

$$\tilde{H} = (H \otimes \mathbb{I} - \mathbb{I} \otimes H^{\mathsf{T}}) - (i\hbar\lambda \mathbb{I} \otimes \mathbb{I}) + i\hbar\lambda \int d^3r L_r \otimes L_r^{\mathsf{T}} \tag{23}$$

So, $|\psi(t)\rangle$ evolves as

$$|\psi(t)\rangle = \exp\Big(-i\tilde{H}t/\hbar\Big)\, |\psi(0)\rangle \tag{24}$$

This gives us the evolution of $\rho(t)$ via Eq. (20), and the above equation can be used to derive the propagator and the path integral.

2.1.3 Derivation of the Path Integral

The total time $t = T$ can be divided into N intervals such that $\epsilon = T/N$ and the finite time propagator in Eq. (24) can be written as

$$U = \left[\exp\left(\frac{-i\epsilon}{\hbar}\left(H \otimes \mathbb{I} - \mathbb{I} \otimes H^{\mathsf{T}}\right) - \lambda\epsilon\left(\mathbb{I} \otimes \mathbb{I} - \int d^3r L_r \otimes L_r^{\mathsf{T}}\right)\right)\right]^N \tag{25}$$

As $N \to \infty$ and $\epsilon \to 0$, we can make the approximation

$$U \approx \left[\exp\left(\frac{-i\epsilon}{\hbar}(H \otimes \mathbb{I} - \mathbb{I} \otimes H^{\mathsf{T}})\right) \times \exp\left(-\lambda\epsilon(\mathbb{I} \otimes \mathbb{I} - \int d^3r L_r \otimes L_r^{\mathsf{T}})\right)\right]^N \tag{26}$$

Introducing resolution of the identity

$$\int_{-\infty}^{\infty} dx_{k\epsilon} dy_{k\epsilon}\, |x_{k\epsilon}\rangle\, |y_{k\epsilon}\rangle\, \langle x_{k\epsilon}|\, \langle y_{k\epsilon}| \tag{27}$$

between every time step we get N terms, each of the form

$$\langle x_{k\epsilon}, y_{k\epsilon}| \exp\left[\frac{-i\epsilon}{\hbar}(H \otimes \mathbb{I} - \mathbb{I} \otimes H^{\mathsf{T}})\right] \times \exp\left[-\lambda\epsilon\left(\mathbb{I} \otimes \mathbb{I} - \int d^3r L_r \otimes L_r^{\mathsf{T}}\right)\right] \left|x_{(k-1)\epsilon}, y_{(k-1)\epsilon}\right\rangle \tag{28}$$

Evaluating one of these terms

$$\begin{aligned}&\langle x_{k\epsilon}, y_{k\epsilon}| \exp\left[\frac{-i\epsilon}{\hbar}\left(H \otimes \mathbb{I} - \mathbb{I} \otimes H^{\mathsf{T}}\right)\right] \times \exp\left[-\lambda\epsilon\left(\mathbb{I} \otimes \mathbb{I} - \int d^3r L_r \otimes L_r^{\mathsf{T}}\right)\right] \left|x_{(k-1)\epsilon}, y_{(k-1)\epsilon}\right\rangle \\ &= \langle x_{k\epsilon}, y_{k\epsilon}| \exp\left[\frac{-i\epsilon}{\hbar}\left(H \otimes \mathbb{I} - \mathbb{I} \otimes H^{\mathsf{T}}\right)\right] \left|x_{(k-1)\epsilon}, y_{(k-1)\epsilon}\right\rangle \exp\left[-\lambda\epsilon\left(1 - \exp\frac{-(x_{(k-1)\epsilon} - y_{(k-1)\epsilon})^2}{4r_C^2}\right)\right]\end{aligned} \tag{29}$$

The first exponent is simply the Feynman propogator[3] for Schrödinger evolution.[4] Thus, taking all N terms we get

$$U(x_{N\epsilon}, y_{N\epsilon}, x_0, y_0) = \frac{m}{2\pi\epsilon\hbar} \times \int_{-\infty}^{\infty} \prod_{n=1}^{N-1} \frac{m}{2\pi\epsilon\hbar} dx_{n\epsilon} dy_{n\epsilon} \times \exp\Big[\sum_{n=1}^{N} \Big(\frac{im}{2\hbar\epsilon}((x_{n\epsilon} - x_{(n-1)\epsilon})^2 - (y_{n\epsilon} - y_{(n-1)\epsilon})^2)\Big) - \frac{i\epsilon}{\hbar}\big(V(x_{(n-1)\epsilon}) - V(y_{(n-1)\epsilon})\big)\Big] \times \exp\left[\sum_{k=1}^{N} -\lambda\epsilon\left(1 - \exp\frac{-(x_{(k-1)\epsilon} - y_{(k-1)\epsilon})^2}{4r_C^2}\right)\right] \tag{30}$$

$$= \frac{m}{2\pi\epsilon\hbar} \int_{-\infty}^{\infty} \prod_{n=1}^{N-1} \frac{m}{2\pi\epsilon\hbar} dx_{n\epsilon} dy_{n\epsilon} \times \exp\left[\sum_{n=1}^{N} \frac{i}{\hbar}(S[x_{n\epsilon}, x_{(n-1)\epsilon}] - S[y_{n\epsilon}, y_{(n-1)\epsilon}])\right] \times \exp\left[\sum_{k=1}^{N} -\lambda\epsilon\left(1 - \exp\frac{-(x_{(k-1)\epsilon} - y_{(k-1)\epsilon})^2}{4r_C^2}\right)\right] \tag{31}$$

In the continuum limit with $N \to \infty$ while still keeping $N\epsilon = T$, the evolution of the density matrix element thus becomes,

$$\rho(x_T, y_T, T) = \int_{\text{all paths}} [Dx_t][Dy_t] \times \exp\left(\frac{i}{\hbar}(S[x_t, T, t = 0] - S[y_t, T, t = 0])\right) \times \exp\left[-\lambda \int_0^T dt \left(1 - \exp\frac{-(x_t - y_t)^2}{4r_C^2}\right)\right] \rho(x_0, y_0) dx_0 dy_0 \tag{32}$$

where

$$\int [Dx_t] = \lim_{N\to\infty} \left(\frac{m}{2\pi\hbar\epsilon}\right)^{1/2} \int \prod_{n=1}^{N-1} \left(\frac{m}{2\pi\hbar\epsilon}\right)^{1/2} dx_n \tag{33}$$

This is the same result as derived in [**?**]. Here, x_t and y_t can be understood as individual paths that might be traversed. Thus, $\int_{\text{all paths}}[Dx_t][Dy_t]$ can be understood as an integral over all such paths. The exponential in the second line of the above equation serves as the GRW induced regulator of the Feynman path integral, and improves the understanding of the classical limit, as we will see in the next section.

2.2 Method-2

2.2.1 Introduction

In this case, we use a more physically motivated approach. We use the fact that after every time interval ϵ the wave function has a probability $\lambda\epsilon$ to collapse. Thus, by taking discrete time steps and using the above fact, we can derive the propagator.

[3]The propagator is a function that specifies the probability amplitude for a particle to travel from one place to another in a given time, or to travel with a certain energy and momentum.

[4]Refer R. Shankar [**?**]; Eq. (21.1.15)

2.2.2 Derivation of the Path Integral

Consider $\rho(x_0, y_0, t=0)$ to be a density matrix at initial time $t=0$. We intend to find $\rho(x_T, y_T, T)$ at final time $t=T$. We divide the total time into smaller intervals such that $\epsilon = \frac{T}{N}$. So, we have

$$\rho(x_\epsilon, y_\epsilon, \epsilon) = A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}\left(\frac{m}{2}(\frac{x_\epsilon - x_0}{\epsilon})^2 - V(\frac{x_\epsilon + x_0}{2})\right)\epsilon\right] \exp\left[-\frac{i}{\hbar}\left(\frac{m}{2}(\frac{y_\epsilon - y_0}{\epsilon})^2 - V(\frac{y_\epsilon + y_0}{2})\right)\epsilon\right]\rho(x_0, y_0, t=0)dx_0dy_0 \tag{34}$$

$$= A \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} prop_1 \rho(x_0, y_0, t=0)dx_0dy_0 \tag{35}$$

representing standard Schrödinger evolution[5] where A is the appropriate normalization constant to recover Von-Neumann Equation. Now from Eq. (17) we know that at a given instant say, $t=\epsilon$ the probability of collapse is $\lambda\epsilon$ while that of it evolving according to Schrödinger's equation is $1-\lambda\epsilon$. Thus, the new density matrix after ϵ time becomes,

$$\rho_{new}(x_\epsilon, y_\epsilon, \epsilon) = (1-\lambda\epsilon)\rho_1 + \lambda\epsilon \int_{-\infty}^{\infty} L_r(x_\epsilon)\rho_1 L_r(y_\epsilon)dr \tag{36}$$

where $\rho_1 = \rho(x_\epsilon, y_\epsilon, \epsilon)$ and $L_r(x_\epsilon) = \langle x_\epsilon| L_r |x_\epsilon\rangle$ are as defined in Eq. (18). Here, since ρ_1 does not depend on r (it is a function of x_ϵ, y_ϵ, x_0 and y_0), we can evaluate the above integral by taking ρ_1 outside the integration. We get,

$$\int_{-\infty}^{\infty} L_r(x_\epsilon)\rho_1 L_r(y_\epsilon)dr = \left(\int_{-\infty}^{\infty} L_r(x_\epsilon)L_r(y_\epsilon)dr\right)\rho_1 \tag{37}$$

$$= \left[\int_{-\infty}^{\infty} \exp\left(-\frac{(x_\epsilon - r)^2}{2r_C^2}\right)\exp\left(-\frac{(y_\epsilon - r)^2}{2r_C^2}\right)dr\right]\rho_1 \tag{38}$$

$$= \exp\left[-\frac{(x_\epsilon - y_\epsilon)^2}{4r_C^2}\right]\rho_1 \tag{39}$$

Thus, we can write

$$\rho_{new}(x_\epsilon, y_\epsilon, \epsilon) = (1-\lambda\epsilon)\rho_1 + \lambda\epsilon\left[\exp\left(-\frac{(x_\epsilon - y_\epsilon)^2}{4r_C^2}\right)\right]\rho_1 \tag{40}$$

For simplicity we write

$$G_i = \exp\left[-\frac{(x_{i\epsilon} - y_{i\epsilon})^2}{4r_C^2}\right]$$

and thus

$$\rho_{new}(x_\epsilon, y_\epsilon, \epsilon) = [(1-\lambda\epsilon) + \lambda\epsilon G_1]\rho_1 \tag{41}$$

[5] Refer R. Shankar [**?**] Eq. (8.5.4)

We propagate again according to Schrödinger's equation from time $t = \epsilon$ to time $t = 2\epsilon$,

$$\rho(x_{2\epsilon}, y_{2\epsilon}, 2\epsilon) = A \int_{-\infty}^{\infty} prop_2 \left[(1-\lambda\epsilon) + \lambda\epsilon G_1\right] \rho_1 dx_\epsilon dy_\epsilon \tag{42}$$

Substituting ρ_1 according to the Eq. (34) and writing new $\rho_{new}(x_{2\epsilon}, y_{2\epsilon}, 2\epsilon)$, we get

$$\begin{aligned}\rho_{new}(x_{2\epsilon}, y_{2\epsilon}, 2\epsilon) =& A^2((1-\lambda\epsilon) + \lambda\epsilon G_2) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} prop_2 \left((1-\lambda\epsilon) + \lambda\epsilon G_1\right) \\ & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} prop_1 \rho(x_0, y_0, t=0) dx_0 dy_0 dx_\epsilon dy_\epsilon \end{aligned} \tag{43}$$

Further rearranging the terms gives,

$$\begin{aligned}\rho_{new}(x_{2\epsilon}, y_{2\epsilon}, 2\epsilon) =& A^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} prop_2 prop_1 \left((1-\lambda\epsilon) + \lambda\epsilon G_2\right)\left((1-\lambda\epsilon) + \lambda\epsilon G_1\right) \\ & \rho(x_0, y_0, t=0) dx_0 dy_0 dx_\epsilon dy_\epsilon \end{aligned} \tag{44}$$

We repeat the above procedure $N-1$ times. Taking continuum limit $N \to \infty$ gives us the final density matrix as

$$\begin{aligned}\rho(x_T, y_T, T) = \lim_{N\to\infty} A^{N-1} \int \cdots \int \prod_{i=0}^{N-1} prop_i \prod_{i=0}^{N-1} \left((1-\lambda\epsilon) + \lambda\epsilon G_i\right) \\ \rho(x_0, y_0, t=0) dx_0 dy_0 \cdots dx_{(N-1)\epsilon} dy_{(N-1)\epsilon} \end{aligned} \tag{45}$$

We know that

$$\begin{aligned}\lim_{N\to\infty} \prod_{i=1}^{N} prop_i = \lim_{N\to\infty} \prod_{i=1}^{N} \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{x_{i\epsilon} - x_{(i-1)\epsilon}}{\epsilon}\right)^2 - V\left(\frac{x_{i\epsilon} + x_{(i-1)\epsilon}}{2}\right)\right]\epsilon\right\} \\ \exp\left\{-\frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{y_{i\epsilon} - y_{(i-1)\epsilon}}{\epsilon}\right)^2 - V\left(\frac{y_{i\epsilon} + y_{(i-1)\epsilon}}{2}\right)\right]\epsilon\right\} \end{aligned} \tag{46}$$

$$\begin{aligned} = \exp\left\{\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{x_{i\epsilon} - x_{(i-1)\epsilon}}{\epsilon}\right)^2 - V\left(\frac{x_{i\epsilon} + x_{(i-1)\epsilon}}{2}\right)\right]\epsilon\right\} \\ \exp\left\{\lim_{N\to\infty} \sum_{i=1}^{N} -\frac{i}{\hbar}\left[\frac{m}{2}\left(\frac{y_{i\epsilon} - y_{(i-1)\epsilon}}{\epsilon}\right)^2 - V\left(\frac{y_{i\epsilon} + y_{(i-1)\epsilon}}{2}\right)\right]\epsilon\right\} \end{aligned} \tag{47}$$

$$= \exp\left\{\frac{i}{\hbar}\int_0^T L\left(x_t\right) dt\right\} \exp\left\{-\frac{i}{\hbar}\int_0^T L\left(y_t\right) dt\right\} \tag{48}$$

$$= \exp\left\{\frac{i}{\hbar}(S(x_t, T, t=0) - S(y_t, T, t=0))\right\} \tag{49}$$

where $L\,(x_t, T, t = 0)$ is the Lagrangian and $S(x_t, T, t = 0)$ the action thus obtained. Expanding the second product term gives us,

$$\lim_{N\to\infty}\prod_{i=1}^{N}((1-\lambda\epsilon)+\lambda\epsilon G_i) = \lim_{N\to\infty}(1-\lambda\epsilon)^N(1+\lambda\epsilon\frac{(G_1+\cdots G_N)}{(1-\lambda\epsilon)}+ \lambda^2\epsilon^2\frac{(G_1G_2+\cdots)}{(1-\lambda\epsilon)^2}+\cdots\infty) \quad (50)$$

$$= \lim_{N\to\infty}(1-\lambda\epsilon)^N(1+\lambda\epsilon\frac{(G_1+\cdots G_N)}{(1-\lambda\epsilon)}+ \lambda^2\epsilon^2\frac{(G_1+\cdots G_N)^2}{2!(1-\lambda\epsilon)^2}+\cdots\infty) \quad (51)$$

$$= \exp(-\lambda T)\exp\left(\lim_{N\to\infty}\sum_{i=1}^{N}\lambda G_i\epsilon\right) \quad (52)$$

$$= \exp(-\lambda T)\exp\left(\lambda\int_0^T G(t)dt\right) \quad (53)$$

$$= \exp(-\lambda T)\exp\left[\lambda\int_0^T \exp\Big(-\frac{(x_t-y_t)^2}{4r_C^2}\Big)dt\right] \quad (54)$$

Substituting these two terms back in Eq. (45) we get an integral form solution of Eq. (17)

$$\rho(x_T, y_T, T) = \int_{\text{all paths}} \exp\left(\frac{i}{\hbar}(S[x_t,T,t=0]-S[y_t,T,t=0])\right)\exp\left(-\lambda\int_0^T(1-\exp\left\{-\frac{(x_t-y_t)^2}{4r_C^2}\right\})dt\right) [Dx_t][Dy_t]\rho(x_0,y_0,t=0)dx_0dy_0 \quad (55)$$

which is the same as what we got using the previous method.

3 Classical and Quantum Limits of GRW path integral

3.1 Quantum Limit

From equations (32) or (55), the path integral for the GRW model is written as

$$\rho(x_T, y_T, T) = \int_{\text{all paths}} \exp\left[-\lambda\int_0^T\left(1-e^{\frac{-(x_t-y_t)^2}{4r_C^2}}\right)dt\right]\exp\left[\frac{i}{\hbar}(\ S(x_t,T,t=0)-S(y_t,T,t=0)\ )\right] \rho(x_0,y_0,t=0)[Dx_t][Dy_t]dx_0dy_0 \quad (56)$$

If we consider the limit $\lambda T \to 0$, i.e. we look at the system at timescales $(t = T)$ much smaller than the time period of collapse $(\tau = 1/\lambda)$, then the non-oscillating part of the above given propagator could be approximated as,

$$\exp\left[-\lambda\int_0^T(1-e^{\frac{-(x_t-y_t)^2}{4r_C^2}})dt\right]\approx 1 \quad (57)$$

This makes the propagator of GRW look exactly like that for normal quantum mechanics,

$$\rho(x_T, y_T, T) = \int_{\text{all paths}} \exp\left[\frac{i}{\hbar}\left(\ S(x_t, T, t=0) - S(y_t, T, t=0)\ \right)\right]\rho(x_0, y_0, t=0)[Dx_t][Dy_t]dx_0 dy_0 \tag{58}$$

From here the standard quantum mechanical result follows easily - we recall the calculation here, for sake of completeness. We can write the above equation for infinitesimal time interval ϵ as,

$$\rho(x_\epsilon, y_\epsilon, \epsilon) = A\iint \exp\left[\frac{i}{\hbar}\int_0^\epsilon\left(\frac{m\dot{x}^2}{2} - V(x)\right)dt\right]\exp\left[-\frac{i}{\hbar}\int_0^\epsilon\left(\frac{m\dot{y}^2}{2} - V(y)\right)dt\right]\rho(x_0, y_0, t=0)dx_0 dy_0 \tag{59}$$

where A is as defined in the previous section. Using the following finite difference substitution

$$\dot{x} \rightarrow \frac{x_\epsilon - x_0}{\epsilon}$$

$$x \rightarrow \frac{x_0 + x_\epsilon}{2}$$

and using the standard substitution of $\eta_x = x_0 - x_\epsilon$ and $\eta_y = y_0 - y_\epsilon$ and rearranging the terms we have

$$\rho(x_\epsilon, y_\epsilon, \epsilon) = A\iint e^{\frac{i}{\hbar}\frac{m\eta_x^2}{2\epsilon}} e^{\frac{-i}{\hbar}\frac{m\eta_y^2}{2\epsilon}} \exp\left[\frac{i}{\hbar}\left(-V(x) + V(y)\right)\epsilon\right]\rho(x_\epsilon + \eta_x, y_\epsilon + \eta_y, t=0)d\eta_x d\eta_y \tag{60}$$

The exponentials oscillate very rapidly as ϵ could be made arbitrarily small. When such a rapidly oscillating function multiplies a smooth function, the integral vanishes for the most part due to the random phase of the exponential. Just as in the case of the path integration, the only substantial contribution comes from the region where the phase is stationary. The region of constructive interference is,

$$\frac{m\eta^2}{2\hbar\epsilon} \leq \pi \tag{61}$$

Now, Taylor expanding the terms in equation (60) upto first order in ϵ i.e. upto order η^2 we get

$$\begin{aligned}
\rho(x_\epsilon, y_\epsilon, \epsilon) &= A \iint e^{\frac{i}{\hbar}\frac{m\eta_x^2}{2\epsilon}} e^{\frac{-i}{\hbar}\frac{m\eta_y^2}{2\epsilon}} \left(1 - \frac{i}{\hbar}V(x)\epsilon + \frac{i}{\hbar}V(y)\epsilon\right) \rho(x_\epsilon + \eta_x, y_\epsilon + \eta_y, t=0) d\eta_x d\eta_y \\
&= A \iint e^{\frac{i}{\hbar}\frac{m\eta_x^2}{2\epsilon}} e^{\frac{-i}{\hbar}\frac{m\eta_y^2}{2\epsilon}} \left(1 - \frac{i}{\hbar}V(x)\epsilon + \frac{i}{\hbar}V(y)\epsilon\right) \big(\rho(x_\epsilon, y_\epsilon, t=0) \\
&\qquad + \frac{\partial \rho}{\partial y}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \eta_y + \frac{\partial \rho}{\partial x}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \eta_x \\
&\qquad + \frac{\partial^2 \rho}{2\partial y^2}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \eta_y^2 + \frac{\partial^2 \rho}{2\partial x^2}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \eta_x^2\big) d\eta_x d\eta_y \\
&= A \iint e^{\frac{i}{\hbar}\frac{m\eta_x^2}{2\epsilon}} e^{\frac{-i}{\hbar}\frac{m\eta_y^2}{2\epsilon}} \big(\rho(x_\epsilon, y_\epsilon, t=0) - \frac{i}{\hbar}V(x)\rho(x_\epsilon, y_\epsilon, t=0)\epsilon \\
&\qquad + \frac{i}{\hbar}V(y)\rho(x_\epsilon, y_\epsilon, t=0)\epsilon + \frac{\partial^2 \rho}{2\partial y^2}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \eta_y^2 + \frac{\partial^2 \rho}{2\partial x^2}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \eta_x^2 \\
&\qquad + \frac{\partial \rho}{\partial y}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \eta_y + \frac{\partial \rho}{\partial x}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \eta_x\big) d\eta_x d\eta_y
\end{aligned} \tag{62}$$

Evaluating the Gaussian integral and using $A = \sqrt{\frac{-2\epsilon\hbar\pi i}{m}}\sqrt{\frac{2\epsilon\hbar\pi i}{m}}$ we get

$$\begin{aligned}
\rho(x_\epsilon, y_\epsilon, \epsilon) &= \rho(x_\epsilon, y_\epsilon, t=0) - \frac{i}{\hbar}V(x)\rho(x_\epsilon, y_\epsilon, t=0)\epsilon + \frac{i}{\hbar}V(y)\rho(x_\epsilon, y_\epsilon, t=0)\epsilon \\
&\qquad + \frac{-i\hbar}{2m}\frac{\partial^2 \rho}{\partial y^2}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \epsilon + \frac{i\hbar}{2m}\frac{\partial^2 \rho}{\partial x^2}\Big|_{(x_\epsilon, y_\epsilon, t=0)} \epsilon \\
&= \rho(x_\epsilon, y_\epsilon, t=0) - \frac{i}{\hbar}[H, \rho]\epsilon
\end{aligned} \tag{63}$$

which describes how a density operator evolves in time:

$$\frac{d\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H, \rho] \tag{64}$$

The above equation is the von Neumann equation and it describes the statistical state of a system in quantum mechanics. We refer to the above equation as the statistical quantum limit of GRW model.

3.2 Classical Limit

The following analysis is previously done by Ajanapon [?] for the propagator of the density matrix in standard quantum mechanics. We here make use of the same analysis for the propagator of the GRW model. From equations (32) or (55), the path integral for GRW model could be written as,

$$\begin{aligned}
\rho(x_T, y_T, T) = \int_{\text{all paths}} \exp\left[-\lambda \int_0^T (1 - e^{\frac{-(x_t - y_t)^2}{4r_C^2}}) dt\right] \exp\left[\frac{i}{\hbar}\left(\ S(x_t, T, t=0) - S(y_t, T, t=0)\ \right)\right] \\
\rho(x_0, y_0, t=0)[Dx_t][Dy_t]dx_0 dy_0
\end{aligned} \tag{65}$$

Now we consider the limit $\lambda T \gg 1$ which could be interpreted as waiting for a sufficiently long time, or the collapse rate λ for the system is sufficiently large. Large λ implies large mass since the collapse rate is directly proportional to number of entangled particles in the system. As a result large λ implies large action. On the other hand, large time also results in large action. As a result, large masses and large times are both representatives of classical limit which causes S to be large, and thus implies the limit $S \gg \hbar$.

When a rapidly oscillating function is multiplied with a smooth function then the integral of their product could be approximated by the smooth function at the stationary point of the rapidly oscillating function. This is commonly called the stationary phase approximation. Here x_t^{cl} and y_t^{cl} are the stationary paths for $S(x_t, T, t=0)$ and $S(y_t, T, t=0)$ respectively in the limit $S \gg \hbar$. Thus the stationary phase approximation leads us to the following equation,

$$\rho(x_T, y_T, T) = \int \exp\left[-\lambda \int_0^T (1 - e^{\frac{-(x_t^{cl} - y_t^{cl})^2}{4r_C^2}})dt\right] \exp\left[\frac{i}{\hbar}\Big(\ S(x_t^{cl}, T, t=0) - S(y_t^{cl}, T, t=0)\ \Big)\right] \rho(x_0, y_0, t=0) dx_0 dy_0 \tag{66}$$

For brevity, we here drop the notation for stationary paths and use $x_t^{cl} = x_t$ and $y_t^{cl} = y_t$. The $\rho(x_T, y_T, T)$ in the above expression represents diagonal as well as off-diagonal terms in position basis [**?**]. Now we look for the off-diagonal terms of the final ρ, which are specified by large $(x_t - y_t)$. In the limit $(x_t - y_t) \gg r_C$, the non-oscillating part of the propagator could be approximated as,

$$\exp\left[-\lambda \int_0^T (1 - e^{\frac{-(x_t - y_t)^2}{4r_C^2}})dt\right] \approx \exp\left[-\lambda T\right] \tag{67}$$

This leads to damping of the off-diagonal terms of the density matrix. Thus, in the limit $\lambda T \gg 1$, the integral can be considered to be vanish. This could also be interpreted as destruction of interference in the system as the off-diagonal terms are the primary representatives of interference. Now let us consider the diagonal terms of the final ρ, specified by $(x_t - y_t) \approx 0$. In the limit $(x_t - y_t) \ll r_C$, the non-oscillating part of the propagator could be approximated as,

$$\exp\left[-\lambda \int_0^T (1 - e^{\frac{-(x_t - y_t)^2}{4r_C^2}})dt\right] \approx 1 \tag{68}$$

Now, we consider an infinitesimal time step ϵ.

$$\begin{aligned} S(x_t, \epsilon, t=0) - &S(y_t, \epsilon, t=0) \\ &= \frac{m}{2\epsilon^2}(x_\epsilon - x_0)^2\epsilon - \frac{1}{2}\left[V(x_\epsilon) + V(x_0)\right]\epsilon - \frac{m}{2\epsilon^2}(y_\epsilon - y_0)^2\epsilon + \frac{1}{2}\left[V(y_\epsilon) + V(y_0)\right]\epsilon \qquad (69) \\ &= \frac{m}{\epsilon}\left[\frac{1}{2}(x_\epsilon + y_\epsilon) - \frac{1}{2}(x_0 + y_0)\right]\ \left[(x_\epsilon - y_\epsilon) - (x_0 - y_0)\right] \\ &\qquad - \frac{\epsilon}{2}\left[V(x_\epsilon) - V(y_\epsilon)\right] - \frac{\epsilon}{2}\left[V(x_0) - V(y_0)\right] \qquad (70) \end{aligned}$$

Motivated by the above expression, we implement the following change of variables,

$$\bar{q}_t = \frac{1}{2}(x_t + y_t) \tag{71}$$

$$\Delta_t = (x_t - y_t) \tag{72}$$

$$U(\bar{q}_t, \Delta_t) = V(\bar{q}_t + \frac{1}{2}\Delta_t) - V(\bar{q}_t - \frac{1}{2}\Delta_t) \tag{73}$$

Thus the equation (66) could be written as,

$$\rho(\bar{q}_\epsilon, \Delta_\epsilon, \epsilon) = A \int \exp\left[\frac{i}{\hbar}\left(\frac{m}{\epsilon}(\bar{q}_\epsilon - \bar{q}_0)(\Delta_\epsilon - \Delta_0) - \frac{\epsilon}{2}U(\bar{q}_\epsilon, \Delta_\epsilon) - \frac{\epsilon}{2}U(\bar{q}_0, \Delta_0)\right)\right] \rho(\bar{q}_0, \Delta_0, t = 0)dx_0dy_0 \tag{74}$$

As the state of a system is specified by position and momentum in classical mechanics, we take the Fourier transform of Δ as given by,

$$\rho(\bar{q}_t, p_t, t) = A \int e^{(-ip_t\Delta_t)}\rho(\bar{q}_t, \Delta_t, t)d\Delta_t \tag{75}$$

Thus the equation (74) in terms of p_t could be written as,

$$\rho(\bar{q}_\epsilon, p_\epsilon, \epsilon) = A \int \exp\left[\frac{i}{\hbar}\left(\Delta_0 p_0 - \Delta_\epsilon p_\epsilon + \frac{m}{\epsilon}(\bar{q}_\epsilon - \bar{q}_0)(\Delta_\epsilon - \Delta_0) - \frac{\epsilon}{2}U(\bar{q}_\epsilon, \Delta_\epsilon) - \frac{\epsilon}{2}U(\bar{q}_0, \Delta_0)\right)\right] \rho(\bar{q}_0, p_0, t = 0)d\Delta_0 d\Delta_\epsilon dx_0dy_0 \tag{76}$$

The $\rho(\bar{q}_t, p_t, t)$ could be interpreted as the phase space representation of the diagonal terms of the density matrix in the limit $S \gg \hbar$. As the $\Delta_\epsilon \ll r_C$, $U(\bar{q}_t, \Delta_t)$ could be approximated by Taylor expanding and ignoring Δ_t^2 and its higher orders

$$U(\bar{q}_t, \Delta_t) \approx \Delta_t \frac{\partial V}{\partial q}(\bar{q}_t) \tag{77}$$

The equation (76) could be further simplified by using the above approximation,

$$\rho(\bar{q}_\epsilon, p_\epsilon, \epsilon) = \frac{1}{N'} \int \exp\left[\frac{i\Delta_0}{\hbar}\left(p_0 - \frac{m}{\epsilon}(\bar{q}_\epsilon - \bar{q}_0) - \frac{\epsilon}{2}\frac{\partial V}{\partial q}(\bar{q}_0)\right)\right] \exp\left[\frac{-i\Delta_\epsilon m}{\hbar\epsilon}\left(\bar{q}_0 - \bar{q}_\epsilon + \frac{\epsilon}{m}p_\epsilon + \frac{\epsilon^2}{2m}\frac{\partial V}{\partial q}(\bar{q}_\epsilon)\right)\right] \rho(\bar{q}_0, p_0, t = 0)d\Delta_0 d\Delta_\epsilon dx_0dy_0 \tag{78}$$

$$= \frac{1}{N''} \int \delta\left(p_0 - \frac{m}{\epsilon}(\bar{q}_\epsilon - \bar{q}_0) - \frac{\epsilon}{2}\frac{\partial V}{\partial q}(\bar{q}_0)\right) \delta\left(\bar{q}_0 - \bar{q}_\epsilon + \frac{\epsilon}{m}p_\epsilon + \frac{\epsilon^2}{2m}\frac{\partial V}{\partial q}(\bar{q}_\epsilon)\right) \rho(\bar{q}_0, p_0, t = 0)dx_0dy_0 \tag{79}$$

$$= \frac{1}{N'''}\rho(\bar{q}_\epsilon - \frac{\epsilon}{m}p_\epsilon, p_\epsilon + \epsilon\frac{\partial V}{\partial q}(\bar{q}_\epsilon), t = 0) \tag{80}$$

The above equation could also be written as follows by changing the variables of ρ,

$$\rho\left(\bar{q}_0 + \frac{\epsilon}{m}p_0, p_0 - \epsilon\frac{\partial V}{\partial q}(\bar{q}_0), t = \epsilon\right) = \frac{1}{N'''}\,\rho\left(\bar{q}_0, p_0, t = 0\right) \tag{81}$$

Now Taylor expanding the left hand side around the point (q_0, p_0, t=0) and equating orders of ϵ, we get, at zeroth order,

$$N''' = 1 \tag{82}$$

at first order,

$$\frac{\partial \rho}{\partial t}\Big|_{(\bar{q}_0, p_0, t=0)} = -\frac{p_0}{m}\,\frac{\partial \rho}{\partial \bar{q}}\Big|_{(\bar{q}_0, p_0, t=0)} + \frac{\partial V}{\partial \bar{q}}\Big|_{(\bar{q}_0)}\frac{\partial \rho}{\partial p}\Big|_{(\bar{q}_0, p_0, t=0)} \tag{83}$$

and dropping the subscript,

$$\frac{\partial \rho}{\partial t} = -\{\rho, H\} \tag{84}$$

where $H = \frac{1}{2m}p^2 + V(\bar{q})$. We refer to this equation (84) as being the statistical classical limit of GRW. The above limit does not depend on a specific form of the initial density matrix, and hence is a phase space representation of a general density matrix following GRW evolution.

3.3 Absence of macroscopic position superpositions

To summarise the discussion this far, we first developed a path integral formulation of the GRW model. We then showed that this gives us the correct quantum and classical limits. We shall now illustrate some important features of the classical limit through some examples. Since we are taking the classical limit, we would consider large action and large number of nucleons (which implies large λ). Hence, the stationary phase approximation shown in Eq. (66) would be valid. If we consider the case of a free particle, the stationary paths would be straight lines with $\dot{x}(t) = constant$.

Let us consider an initial condition that is formed by the superposition of two Gaussians separated by a macroscopic distance $|a_1 - a_2| >\gg r_C$. The resulting density matrix would be

$$\rho(x_0, y_0, t = 0) = \sum_{i,j=1}^{2} A_{ij} e^{-\frac{(x_0 - a_i)^2}{r^2}} e^{-\frac{(y_0 - a_j)^2}{r^2}}$$

with $r \ll r_C$. Here, the coefficients A_{ij} can be chosen such that the density matrix is a valid one (i.e. it has unit trace, it is positive semi-definite, and it is Hermitian). Putting this into Eq. (66), we get

$$\rho(x_t, y_t, T) = \int \exp\left[-\lambda \int_0^T (1 - e^{\frac{-(x_t^{cl} - y_t^{cl})^2}{4r_C^2}})dt\right] \exp\left[\frac{i}{\hbar}\left(\ S(x_t^{cl}, T, = 0) - S(y_t^{cl}, T, t = 0)\ \right)\right]$$
$$\sum_{i,j=1}^{2} A_{ij} \exp\left[-\frac{(x_0 - a_i)^2}{r^2}\right] \exp\left[-\frac{(y_0 - a_j)^2}{r^2}\right] dx_0 dy_0 \tag{85}$$

We can see that the terms of the initial density matrix

$$A_{12}e^{-\frac{(x_0-a_1)^2}{r^2}}e^{-\frac{(y_0-a_2)^2}{r^2}}+A_{21}e^{-\frac{(x_0-a_2)^2}{r^2}}e^{-\frac{(y_0-a_1)^2}{r^2}}$$

would have $|x_t^{cl}-y_t^{cl}|\gg r_C$ for a large time. Hence, the final density matrix would have these terms damped exponentially as

$$\exp\left(-\lambda\int_0^T(1-e^{\frac{-(x_t^{cl}-y_t^{cl})^2}{4r_C^2}})dt\right)\approx e^{-\lambda T}$$

Additionally, in the remaining terms where both paths start in the same Gaussian, the paths must finally also remain within a distance which is of the order r_C. Thus, the so-called off-diagonal terms are destroyed, while the approximately diagonal terms are preserved. Note that the system transforms from a state with the superposition of two Gaussians to a statistical ensemble of the two Gaussians with probabilities A_{11} and A_{22} respectively. Note also that this statistical ensemble is different from a superposition as this represents classical probabilities which do not interfere. In this way, GRW destroys macroscopic superpositions.

4 Discussion and Conclusion

We have presented the Feynman Path integral approach to GRW model and the associated transition from GRW to standard quantum mechanics and classical mechanics. We note that in this approach, the transition from GRW to classical and quantum mechanics is quite naturally obtained.

In order to see the transition to standard quantum mechanics, we took the limit $\lambda T\ll 1$ of the path integral for the GRW model and were left with quantum mechanics for a density matrix i.e. the Von Neumann equation. We understand this by noting that this limit corresponds to looking at the system for time-scales smaller than those necessary for spontaneous collapse. Without spontaneous collapse, GRW is identical to standard quantum mechanics and all paths of the propagator contribute to the path integral with equal amplitudes. The limit $\hbar\to 0$ is often taken as the classical limit of quantum mechanics. This limit can be understood in the following context. Liouville's equation implies that the density of points in phase space always remains constant. Since each point has a deterministic evolution, the density of points never changes. However, in quantum mechanics the theory in phase space is similar to that of a stochastic process and each point does not undergo deterministic evolution.Thus, the density of points in phase space diffuses and is not constant [**?**]. This is a direct consequence of the uncertainty principle which depends on $\hbar$.

Following [**?**] we have highlighted how simply taking $\hbar\to 0$ is not sufficient to give classical mechanics. In addition to this limit, the initial state must not be in a superposition of position states, if we are to obtain classical mechanics in the limit. GRW does not suffer from this limitation. The initial state is naturally kept localised for a macroscopic object, by the GRW localisation mechanism, provided the initial instant is understood as having being coarse grained over a time interval larger than $1/\lambda$. This coarse graining ensures that continual collapse keeps the object localised. The GRW modification of the propagator can be interpreted as a term that damps paths that are far from each other. These far-off paths are directly related to the off-diagonal terms of the density matrix. This could be interpreted as destruction of superposition in the system as the off-diagonal terms represent superposition. We have additionally shown how this leads us directly to Liouville's equation following the analysis in [**?**].