blog post

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1 Introduction

Cohomology is used in QFT to check for non-contractible surfaces on a manifold i.e. if $H^2(C) \neq 0$ where C is our configuration on which we define our path integrals, we get that there are non-contractible two surfaces in our configuration space. So, today we'll be focusing are attention on specifically on cohomology of lie groups. They come up

2 Basic Set-Up

Suppose $g \in G$ where G is our lie-group and the representation of that element g can be given by $g = exp(i\theta^a t^a)$. Now, we can construct lie-algebra valued one form as follows:
$\omega = g^{-1}dg = -it^a E_i^a d\theta^i = -it^a E^a$
where, $E^a = E_i^a(\theta)d\theta^i$ are the frame fields. For my more information on frame bundles and frame fields refer to my blog post on Principal G-Bundles.

3 Constructing Traces of forms

we can build $Tr(\omega^k)$ and these will vanish if k is even because of cyclicity of trace.
$Tr(\omega^2) = Tr(t^a t^b)E^a \wedge E^b = -Tr(t^a t^b)E^b \wedge E^a = -Tr(\omega^2)$
For odd values we define $\Omega^{(k)} = Tr(\omega^k)$ these are non-zero generally and:
$d\omega = d(g^{-1}dg) = -g^{-1}dgg^{-1}dg = -\omega^2$
$d\Omega^{(k)} = -Tr(\omega^2\omega^{k-1} - \omega\omega^2\omega^{k-2}...) = 0$
Thus $\Omega^{(k)}$ are closed but not exact and hence will be an element of the cohomology group.
Example: $U(1)$ here $g = exp(i\theta)$ and $\Omega^{(1)} = id\theta$, integrating the form over a circle with constraints $\theta \in [0, 2\pi]$
$-i \oint \Omega^{(1)} = 2\pi$,
Thus $\Omega^{(1)}$ is closed but not exact if it were exact it could be written in the form

like $\Omega^{(1)} = d\alpha$ for some one-form α which is periodic function of θ. This shows that $\Omega^{(1)}$ is an element of $H^1(U(1), R)$