PG-bundles

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1 Prelimnaries

A) If $(G, \circ)$ is a lie group and P is a smooth manifold and there exists a smooth map satisfying:

$$L : G \times P \to P$$

such that:
1) $\forall x \in P$: $eLx = x$
2)$\forall$ $g_1, g_2 \in G, \forall x \in P$ such that $(g_1 \circ g_2)L(x) = g_1 L(g_2 Lx)$
Then L is called the left G-action.sly, one can define right G-action.

B)Let L be a left G-action. We can define orbit for each $x \in P$ such that:
$G_x \equiv \{q \in P : \exists g \in G : q = gLx\}$

We can define an equivalence relation such that x y iff $G \in G$: $y = g(L)x$

Then, $M/G \equiv M/\ = \{G_x | x \in P\}$. This is called the orbit space of p.

We define the stablizer of $x \in P$ such that:
$S_x \equiv \{g \in G | g(L)p = p\}$
S_x is a subgroup of G

Left G-action is said to be free if:
$\forall x \in P, S_x = \{e\}$

If a left G-action is free,then:
$\forall y \in G$ then $G_y \cong_{diff} G$

2 Principle Fibre Bundles

Definition: If G is a lie group and (E, π, M) is a smooth bundle and E is equipped with free right action of G and $E - \pi \to M$ is isomorphic to $E - \rho \to E/G$

where E/G is the orbit space and ρ is the inclusion map.

Careful observation: We know that E is equipped with FREE right action of G, this means $Preim_{\rho}(G_p) = G_p \cong_{diff} G$ where $p \in E$.

Thus we can contemplate that Principle bundle is bundle which is isomorphic to a bundle whose fibres are the orbits under the free right action of G, which again is isomorphic to G since, the action is free.More intuitively,Principle G-bundle is a bundle whose fibres at each point is a lie group.

Definition: Let (P, π, M) and (Q, π, N) be two principle G-bundles.A principle bundle morphism $(P, \pi, M) - -- > (Q, \Pi, N)$ is set of smooth maps (u,f) such that the following commutes: $(u : P \to Q)$ and $f : M \to N$
$(f \circ \pi)(p) = (\Pi \circ u)(p)$
$u(P(R)g) = u(p)(R')g$

Definition: A Principle bundle morphism is said to be an isomorphism if it is also bundle isomorphism.

Important Theorem: A principle G-bundle (P, π, M) is trivial iff there exists a smooth section $\sigma \in \Gamma(P)(\sigma : M - - > P)$ such that $\pi \circ \sigma = id_M$

3 Associated Fibre Bundles

Definition: If (P, π, M) be a principle G-bundle and F be a manifold equipped with left action of G (L): 1)$P_F \equiv (P \times F)/ - G$, $-G$ is the quivalence relation $(p, f)\ _G(p', f')$ iff $\exists g \in G : p' = p(R)g$ and $f' = g^-1(L)f$ and we use a bit of notation from now on P_F as $[p, f]$

2) $\pi_F : P_F - -- > M$, $[p, f] - -- > \pi(p)$

Then, (P_F, π_F, M) is called the associated bundle to (P, π, M)

Definition: (P_F, π_F, M) and (Q_F, π_F, N) be the associated bundles of two principle G-bundles (P, π, M) and (Q, π, N).An associated bundle map between the associated bundles is bundle map (u', v), the pair (u,v) is a principle bundle map between underlying Principle bundles and :
$u'([p, f]) \equiv [u(p), f]$

Definition: An associated map (u',v) is said to be isomorphsim if u' and v are invertible and (u'^{-1}, v^{-1}) is also an associated bundle map.

An associated bundle (P_F, π_F, M) is trivial if the underlying principle G-bundle (P, π, M) is trivial.

4 An Example(Frame Bundles)

If M is a smooth manifold. Consider this space of it's tangent space's basis vectors:
$L_pM \equiv \{(e_1, ..., e_{\dim(M)}) \mid e_1, ..., e_{\dim(M)}$ is basis of $T_pM\} \cong_{vec} GL(dim(M), R)$

Frame Bundle is defined as:
$LM \equiv \cup_{p\in M} L_pM$, with the projection map $\pi : LM - -- > M$, where $\cup$ denotes the disjoint union.

Then, (LM, π, M) is called the frame-bundle of M

$dim(LM) = dim(M) + dim(T_pM)$

Definition: Let (LM, π, M) be frame bundle of the smooth bundle of M, With right $GL(d, R)$-action, $F \equiv (R^d)^{\times p} \times (R^{d*})^{\times q}$ and define left $GL(d, R)$-action on F by,:
$(g(L)f)^{a_1,...,a_p}_{b_1,...,b_q} \equiv (\det g^{-1})^\omega g^{a_1}_{a'_1} ... g^{a_p}_{a'_p} (g^{-1})^{b'_1}_{b_1} .... (g^{-1})^{b'_q}_{b_q} f^{a'_1,...,a'_p}_{b'_1,..,b'_q}$, $\omega \in Z$

Then, associated bundle (LM_F, π_F, M) is called the (p,q)-tensor ω-density bundle on M, and its sections are (p,q)-tensor densities of weight ω.

1. If $\omega = 0$, we recover the (p,q)-tensor bundle on M.
2. If $F = R(p = q = 0)$, the left action is $(g(L)f) = (\det g^{-1})^\omega f$
3. If $GL(d, R)$ is restricted such that $\det g^{-1} = 1$, then tensor densities can't be differentiated from ordinary tensor fields. This is the why we don't encounter tensor densities in special relativity.