SPONTANEOUS SYMMETRY BREAKING

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1. Motivation

We've all heard of the famous Higgs boson and Higgs mechanism. But, to really understand it at a deeper scientific level, we must take the path of least crankery which means (most of the time) maximum mathematics.In order to understand Higgs mechanism, we first need to understand spontaneous symmetry breaking.I'll do the math in the following section. First, let me paint a brief (vague) picture in your mind of this concept.It can be understood in a way which means that "change" in some parameters to which the particle Lagrangian should be invariant to, results in the field acquiring a non-zero vacuum expectation value. This means our symmetry is broken! Which may seem like a disaster but is indeed a bliss because it lead to the discovery of something known as Goldstone's theorem which I'll cover in this article. Eventually, this lead to Peter Higgs formulating the idea of massive gauge bosons (which he generalized to the non-abelian case). I'll try to provide a "rooftop" view of "Higgs interaction" for the people who don't feel well acquainted with math that I've used here, in probably the last post on this series.

Contents

2. Spontaneous Symmetry Breaking(scalar field example)

We start with writing the Lagrangian for a scalar particle:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\,.$$

Now, we perform a sort of reparametrization and replace $m^2 \to -\mu^2$. The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4\,.$$

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Now, we expect our Lagrangian to be invariant under the discrete symmetry $\phi \to -\phi$. The Hamiltonian is

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4\right].$$

We write our potential:

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 .$$

We choose ϕ_0 to minimize our potential $V(\phi)$:

$$\phi_0 = \pm V = \pm\sqrt{\frac{6}{\lambda}}\mu .$$

V is the vacuum expectation value of ϕ. Suppose we choose a "place" near one of the minima, we can write ϕ as

$$\phi(X) = V + \sigma(X) .$$

Thus, we get the Lagrangian of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(2\mu)^2\sigma^2 - \sqrt{\frac{\lambda}{6}}\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4 .$$

The Lagrangian gives us mass of the scalar field as $\sqrt{2}\mu$ with σ^3 and σ^4 interactions. The $\phi \to -\phi$ is now broken.

3. Linear Sigma Model

We start by writing a Lagrangian of n scalar particles:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}(\mu\phi^i)^2 - \frac{\lambda}{4}[(\phi^i)^2]^2 .$$

The Lagrangian is invariant under the symmetry

$$\phi^i \to R^{ij}\phi^j ,$$

which basically means $O(N)$ symmetry. We can write our potential as

$$V(\phi^i) = -\frac{1}{2}(\mu\phi^i)^2 + \frac{\lambda}{4}[(\phi^i)^2]^2 .$$

$V(\phi)$ can be minimized for an ϕ_0^i if

$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda_0} .$$

Now, technically we can choose any direction for ϕ_0^i but it is conventional to choose the coordinates such that

$$\phi_0^i = (0, 0, ..., V), V = \frac{\mu}{\lambda} .$$

Now, we define a set of "shifted fields":

$$\phi^i(X) = (\pi^m(X), V + \sigma(X)), K = 1, ..., N-1 .$$

We write the Lagrangian in terms of shifted parameters as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\pi^m)^2 + \frac{1}{2}(\partial_\mu\sigma^m)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\pi^m)^2\sigma\frac{\lambda}{4}\sigma^4$$
$$- \frac{\lambda}{2}(\pi^m)^2\sigma^2 - \frac{\lambda}{4}[(\pi^m)^2]^2 .$$

We get a massive σ field with $(k-1)$ massless π fields.Now, we are left with $O(N-1)$ symmetry for π fields.

4. Goldstone's Theorem

It states that for each spontaneously broken symmetry there exists a massless particle. As we saw the example of linear sigma model above. The massless particles are called Goldstone bosons.

5. A Much Simpler Example

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2] - V(\sigma^2 + \pi^2)\,.$$

and

$$V(\sigma^2 + \pi^2) = -\frac{\mu^2}{2}(\sigma^2 + \pi^2) + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2\,.$$

Potential has a global symmetry $O(2)$, i.e. σ and π transform by our good ol' 2×2 rotation matrix. Now, to get the minimum of the potential, we impose

$$\partial_\sigma V = [-\mu^2 + \lambda(\sigma^2 + \pi^2)]\sigma = 0\,,$$
$$\partial_\pi V = [-\mu^2 + \lambda(\sigma^2 + \pi^2)]\pi = 0\,,$$

giving us

$$\sigma^2 + \pi^2 = \frac{\mu^2}{\lambda} = v^2\,,$$

i.e. a circle in $\sigma - \pi$ plane.

Now, we choose classical vacuum field configurations which are $\sigma = v$, $\pi = 0$, giving us new quantum fields

$$\sigma_n = \sigma - v\,, \quad \pi_n = \pi\,.$$

Now we observe that $\langle \sigma \rangle = v$ which doesn't equal zero, breaking our symmetry leaving us with a new Lagrangian:

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma_n)^2 + (\partial_\mu \pi)^2] - \mu^2 \sigma_n^2 - \lambda v \sigma_n(\sigma_n^2 + \pi_n^2) - \frac{\lambda}{4}(\sigma_n^2 + \pi_n^2)^2\,.$$

Here the Goldstone boson is π_n.

6. What's Next?

Well, our main goal will be understanding of Higgs mechanism. But, the concept of spontaneous symmetry breaking is also used superconductors, which I shall briefly consider next time. [1]

Check out this graph for a relatively better intuition of spontaneous symmetry breaking:

[1]Special thanks to Christian Ferko for typesetting the code.