Addition of Angular Momentum

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1 Introduction

suppose we have two operators(J_i^1andJ_i^2) which satisfy the su(2) algebra defined on 2 vector spaces V_1 and V_2 respectively.

We define J_i on $V_1 \otimes V_2$ by :
$J_i = J_i^1 \otimes 1 + 1 \otimes J_i^2$ now this new operator defined on $V_1 \otimes V_2$ also satisfies su(2) algebra (PS. here 1 denotes the identity element not the number 1).

NOTE: If we had some arbitrary coefficiants,then it wouldn't have satisfied the su(2) algebra. check it for yourself. HINT!: write $J_i = aJ_i^1 \otimes 1 + b1 \otimes J_i^2$, now use it the commutator relation $[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$ and you'll that for it to satisfy su(2) algebra, the coefficients must be 1.

2 Spin 1/2

We know that: $S^2|s, m >= \hbar^2 s(s+1)|s, m >$ and
$S_z|s, m >= \hbar m|s, m >$ and we know that these two eigenstates form an orthanormal eigenbase.

here $s = 1/2, m = \pm 1/2$

we define $S_i = S_i^1 \otimes 1 + 1 \otimes S_i^2$ on $V_1 \otimes V_2$ which is spanned by

$\{|1/2, 1/2 >_1 \otimes |1/2, 1/2 >_2, |1/2, 1/2 >_1 \otimes |1/2, -1/2 >_2, |1/2, -1/2 >_1 \otimes |1/2, 1/2 >_2, |-1/2, -1/2 >_1 \otimes |-1/2, -1/2 >_2\}$

which leads to 5 possible multiplets:
1) for $s = 0$ singlets.
2)Two doublets of $s = 1/2$.
3)one doublet of $s = 1/2$ and 2 singlets of $s = 0$.
4)one triplet of $s = 1$ and one singlet of $s = 0$.

5)one $s = 3/2$ multiplet.

only the 4) option is an viable option.Let that sink in and think as to why this is the only correct option, remember s takes half integer and integer 2s+1 values and m ranges between +s and -s.

Thus, we can write: $(s = 1/2) \otimes (s = 1/2) = (s = 1) \oplus (s = 0)$.

now, we can write the basis as:
$|1, 1 >= |1/2, 1/2 >_1 \otimes |1/2, 1/2 >_2$
$|1, 0 >= a|1/2, 1/2 >_1 \otimes |1/2, -1/2 >_2 + b|1/2, -1/2 > \otimes |1/2, 1/2 >$
$|1, -1 >= |1/2, -1/2 >_1 \otimes |1/2, 1/2 >_2$
$|0, 0 >= c|1/2, 1/2 >_1 \otimes |1/2, -1/2 >_2 + d|1/2, -1/2 >_1 \otimes |1/2, 1/2 >_2$

After a careful observation you'll see that m=0 has two states m=0 and s=m=0 and these states are entangled.

We define $S_= S^1_- + S^2_-$ analogous to $J_\pm$ operators.
similarly, we can define $S^2 = (S^1 + S^2)^2 = (S^{(1)})^2 + (S^{(2)})^2 + 2S^1.S^2$

Here, I made use of the following result:

$J(1).J(2) = \Sigma_{i=1}^3 J_i^{(1)} \otimes J_i^{(2)}$
$= 1/2(J^1_+ \otimes J^2_- + J^1_- \otimes J^2_+) + J^1_z \otimes J^2_z$