Topological Aspects of Gauge Theories

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Dirac Magnetic Monopoles

Let's keep our magnetic monopole at the origin of our coordinate system.
The magnetic field will be given by
$B_i = g\frac{x^i}{|x|^3}$
Now, this is singular at $x = 0$. $M = \mathbb{R}^3 - \{0\}$.
For this M we've non-contractible two-spheres.
$F = \frac{1}{2}\epsilon_{ijk} g \frac{x^k}{|x|^3} dx^i \wedge dx^j$
$dF = 0$ iff $M = \mathbb{R}^3 - \{0\}$ otherwise we'll get a delta function at $\delta^{(3)}(x)$
Integrating F over a sphere around the origin.
$\int_{S^2} F = \int_{S^2} \frac{1}{2}\epsilon_{ijk} g \frac{x^k}{|x|^3} dx^i \wedge dx^j = g\int_{S^2} \frac{r^2 sin^2\theta}{r^2} d\theta d\phi = 4\pi g$
F is closed but not exact hence it's an element of $H^2(\mathbb{R}^3 - \{0\}) \neq 0$

The spheres around the origin can be described by two coordinates, northern hemisphere $x^3 > 0 - \epsilon$ and the southern hemisphere $x^3 < 0 + \epsilon$ with small overlap region of 2ϵ near equator. For each patch we should be able to write F as dA for some arbitrary A

$A_N = g\frac{\epsilon_{abc} n_a x_b}{r(r+n.x)} dx^c$

$A_S = -g\frac{\epsilon_{abc} n_a x_b}{r(r-n.x)} dx^c$

where, $n_1 = n_2 = 0, n_3 = 1$ and $r^2 = x.x$

A_N has singularity at south pole $(n.x = -r)$ and similarly A_S has singularity at the north pole $(n.x = r)$

Singularity is a reflection of the fact that F is closed but not exact. The singularity is at a single point so if we consider the whole M we've a line of singularities, till infinity. That line is called dirac string. Using potentials defined on patches like we did, we can avoid singularities over all M

The patches overlap at equator where $n.x = 0$

$A_N - A_S = 2g\frac{x_1 dx_2 - x_2 dx_1}{(x_1^2 + x_2^2)} = 2gd\phi$

where ϕ is the azimuthal angle. Difference in the overlap is a gauge transformation with gauge parameter $\Lambda = 2g\phi$
Physical $\vec{B}$ is same in the overlap region. So one can either use A_N or A_S

Deep Into Configuration Space

Let A denote the set of gauge potentials(set of all lie algebra value one-form on R^4)
$G_* = \{g(x)|g(x) \to 1 as \sqrt{x_\mu x^\mu} \to \infty\}$
Because of boundary conditions transformations are topologically equivalent to transformation on S^4.
The gauge invariant set of configuration is $C = A/G_*$
There exists a natural splitting $A = C \times G_*$ for neighbhorhood of C
A is fibre bundle over C with G_* as the fibre.
Bundle need to be trivial so the splitting might not be globally true.
NOTE:The non-triviality shall not be discussed in this talk. It's related to what's know as Gribov Ambiguity but sadly we don't have the time to cover it in this talk.

Topology of A

A is an affine space which means any potential can be written as $A_i^{(0)} + h_i$. $A_i^{(0)}$ is a fixed potential while h_i is a lie-algebra valued vector field.Any two point in A can be connected by a straight line.
$A_i(x, \tau) = A_i^{(1)}(1 - \tau) + \tau A_i^{(2)}$
where $\tau \in [0, 1]$...provides straight line interpolation between $A_i^{(1)}$ and $A_i^{(2)}$
A appears as an infinite dimensional Eucledian space. Toplogicially it's rather trivial $\Pi_0(A) = 0$ and $\Pi_1(A) = 1$.

Topology of G_*

G_* is made up of maps $g(x) : \mathbb{R}^3 \to SU(2)$ with the condition$g(x) \to 1$ as $|x| \to \infty$. We can parametrize the elements of $SU(2)$ as:
$g = \phi_0 + i\phi_i\sigma_i$, with $g^+ = -g$ and $\phi_0, \phi_i \in \mathbb{R}$
$det(g) = 1$. $SU(2)$ is topologically as three-sphere.ϕ_μ gives mapping $\mathbb{R}^3 \to S^3$. Furthermore $g(x) \to 1$ as $|x| \to \infty$. Hence we can think of $\mathbb{R}^3$ being a three-sphere itself.
More concretely, we have a mapping:
$y_0 = \frac{x^2-1}{x^2+1}$ and $y_i = \frac{2x_i}{x_i^2+1}$ now:
$y_0^2 + \Sigma_i y_i^2 = 1$ defines a three-sphere.
spatial infinity corrosponds to $|x| \to \infty$ being mapped to the pole $y_0 = 1, y_i = 0$. Given a $SU(2)$ valued function on S^3 we use the sum of the square condition to write it as a function on $\mathbb{R}^3$ with $g \to g_\infty$. g_∞ doesn't depends upon angles as $|x| \to \infty$ and choosing $g_\infty = 1$ requires $g(1,0,0,0) = 1$.

Now, we've the map $S^3_{(1)} \to S^3_{(2)}$ where $S^3_{(1)}$ is our $\mathbb{R}^3$ and $S^3_{(2)}$ is $SU(2)$.
we know that, $\Pi_3(S^3) = \mathbb{Z}$.
Now, consider the map $\phi_\mu : S^3 \to S^3$. The volume element for the image generated by ϕ_μ is:
$dS_\mu \epsilon^{\mu\nu\alpha\beta} = \partial_i \phi^\nu \partial_j \phi^\alpha \partial_k \phi^\beta \epsilon^{ijk} d^3x$
$= \frac{1}{3!} \epsilon^{\mu\nu\alpha\beta} \epsilon^{ijk} \partial_i \phi^\nu \partial_j \phi^\alpha \partial_k \phi^\beta$
$dS_\mu = \phi^\mu d\Omega^{(3)}$
$d\Omega^{(3)} = \frac{1}{3!} \epsilon^{\mu\nu\alpha\beta} \epsilon^{ijk} \partial_i \phi^\nu \partial_j \phi^\alpha \partial_k \phi^\beta$
$vol(S^3) = 2\pi^2$, $Q = \frac{1}{2\pi^2}$[volume traced out by $\phi^\mu(x)$]
$Q = \frac{1}{12\pi^2} \int d^3x \epsilon_{\mu\nu\alpha\beta} \epsilon^{ijk} \phi^\mu \partial_i \phi^\nu \partial_j \phi^\partial_k \phi^\beta$
$Q[g] = \frac{-1}{24\pi^2} \int Tr(g^{-1} \partial_i g g^{-1} \partial_j g g^{-1} \partial_k g) \epsilon^{ijk}$

$$Q[g] = \frac{-1}{24\pi^2} \int Tr(g^{-1}dg)^3 \tag{1}$$

Properties of $Q[g]$:

- $Q[\phi + \delta\phi] - Q[\phi] = 0$.... invariant under small deformations ϕ
- $Q[g_1 g_2] = Q[g_1] + Q[g_2]$

Consider a configuration for which $Q = 1$. Which can be given by:
$g_1(x) = \frac{x^2-1}{x^2+1} + i\frac{2x^i}{x^2+1}\sigma_i$
equivalent to $\phi^\mu = y^\mu$. The ϕ's form a three-sphere, so do the y's and give one covering of S^3, which is $SU(2)$. The configuration $g(x) = 1$ has $Q = 0$(no covering). Thus $Q = 0$ for $g = 1$ and $Q = 1$ for $g = g_1(x)$ they are indeed invariant under smooth deformations but $g_1(x)$ can't be smoothly deformed into identity everywhere.
$g = g_1(x)g_2(x)$, which has $Q = 2$ also of $g = g^+(x)$ we have $Q = -1$.
We can write G_* as the direct sum of different components each of which is connected and characterized by Q:
$G_* = \sum_{Q=-\infty}^{+\infty} \bigoplus G_{*Q}$

Q's add up upon taking the product of two maps and hence the structure is isomorphic to Z(additive group of integers). Structure of A/G_*take a

look at line given in A as:
$A_i(x, \tau) = A_i(x)(1-\tau) + A_i^g \tau$, generally
$A_i(x, \tau)$ with $A_i(x, 0) = A_i(x)$ and $A_i(x, 1) = A_i^{g_1}(x)$
where $A_i^{g_1}(x)$ is the gauge transformed version of $A_i(x)$ by $g_1(x)$. Since $A_i^{g_1}(x)$ is the gauge transform of $A_i(x)$, both configurations represent the same point in $C = A/G_*$. Thus $A_i(x, \tau)$ is a closed loop in C and we'd like to question it's "contractibility" if it is then we should be able to deform the curve along gauge flow directions which would then connect $g = 1$ to $g = g_1(x)$ i.e. $g_1(x)$ is smoothly deformable to identity but we remember that it's impossible from our previous discussion on G_* which means $\Pi_1(C) \neq 0$ this argument can be extended to $g = g_1 g_2$,etc. Thus giving us:
$\Pi_1(C) = \Pi_1(A/G_*) = \mathbb{Z}$

Basic Path Integral Construction

First we'll see how to construct a configuration space for path integrals and how to write path integrals over manifolds with no non-trivial topology.

We are going to denote a specific field configuration as $\phi(x)$(at a given time instance)
hence we can write the configuration space as $C = \{\phi(x) : R^3 \longrightarrow R^3\}$
The wave function $\psi[\phi]$ is a complex no. at every point in C. As the time progresses we obtain a path in $C(\phi(x, t))$. Thus the usual integral representation of path integral can be interpreted as follows:
$\psi[\phi, t] = \Sigma_p e^{is_P[\phi]}\psi[\phi', t']$ (sum over all paths P)

where $S_p[\phi]$ is the action for a path in C connecting $\phi'(x)$ at t' to $\phi(x)$ at x. It is basically the result from Huygens' principle from wave optics.
Getting back to the matter at hand, the path integral on a configuration space with no non-trivial topology can be written as:
$\psi[Q,t] = \int [dQ] e^{iS[Q,t,Q',t']} \psi_i(Q',t')$,
where Q is a point in configuration space and we're integrating over all the paths from Q to Q'.
Our main focus today is to talk about config spaces with non trivial topology. Specifically, today we're concerned about configuration spaces with $\Pi_1(C) \neq 0$ and $H^2(C) \neq 0$.

$\Pi_1(C) \neq 0$

Here $C = \mathbb{R}^3/\{0\}$ and we know that $\Pi_1(\mathbb{R}^3 - \{0\}) = \mathbb{Z}$.
Whenever there's a talk about non trivial fist homotopy group, you instantly get thoughts about non triviality of closed loops in that space(in our case C). And to talk about non triviality of those close loops, we talk about their winding no.("to measure their non-triviality"). Let's say that closed loop is C then,

$\nu(C) = \frac{-1}{2\pi} \oint_C \frac{\epsilon_{ij} x^j}{x^2} dx^i = \oint_c \alpha$

Here, α is a one-form and is closed. for homotopic paths P and P': $\int_P \alpha = \int_{P'} \alpha$
given structure of the path, the path integral could be generalized as:
$\psi(x,t) = \int [dx] e^{is(x,t,x',t')} e^{i\theta \int_{x'}^{x} \alpha} \psi(x',t')$ θ is an arbitrary parameter.
Any path can be written as $P + C_\nu$ where P is a path with no non contractible loops around the origin and C_ν is a non contractible loop around origin with winding no. ν
$\int_{P+C_\nu} = \nu + \int_P \alpha$

Thus our path integral can be simplified as:
$\psi(x,t) = [\Sigma_{P\simeq P'} e^{iS_{P'}+i\theta\int_P \alpha} + \Sigma_{P'\simeq P+C_1} e^{iS_{P'}+i\theta\int_p \alpha + i\theta} + ...]\psi(x',t')$
$= \Sigma_\nu \Sigma_{P'\simeq P+C_\nu} e^{iS_{P'}+i\theta\int_P \alpha + i\nu\theta}\psi(x',t')$
We've summed over all Paths P' which are homtopic to $P' + C_\nu$.Overall phase of $i\int_P \alpha$ doesn't matter for matrix elements.
$\psi(x',t') = \Sigma_\nu \Sigma_{P'\simeq P+C_\nu} e^{iS_{P'}+i\nu\theta}\psi(x',t')$
ψ unchanged under $\theta \to \theta + 2\pi$, $\theta \in [0, 2\pi]$
$\theta\nu$ term has no classical effect. Thus we regard θ as an extra parameter which arises when we quantize theories with $\Pi_1(C) \neq 0$. To mimic the affect $\theta\nu$ for writing actions:
$S = \int dtL - \frac{\hbar\theta}{2\pi}\int dt \frac{\epsilon_{ij}x^i x^j}{x^2}$

$\Pi_1(C)$ can be smth other than Z depending on C. Thus in general in addition to e^{iS} we add a K(C) such that :

- $K(C+\delta C) = K(C)$.....conveys that K(C) should be a topological invariant for the path.
- $K(C_1)K(C_2) = K(C_1 + C_2)$.....just the basic composition law.
- $K(-C) = K^*(C)$.....tells us the K(C) changes sign under change of orientation.

The quantization rule can be stated as:
$\psi(Q,t) = \int [dQ] e^{iS(Q,t,Q',t')} K(C) \psi(Q',t')$
NOTE!: $K(C)$ will possibly include new parameters, which are to be extra coupling constants in the theory.

Examples

Examples of theories with $\Pi_1(C) \neq 0$

- QCD with $\Pi_1(C) = Z$.
- Charged particle dynamics in the presence of magnetic vortex.
- Particles with fractional spin in 2d. $\theta \in [-\pi, \pi]$ with $\theta = 0$ giving bosons and if $theta = \pi$ or $\theta = -\pi$ giving us fermions anything in between five anyons.

$H^2(C) \neq 0$

Now, we shift our focus to a quantization problem which could occur when there are non contractible two-surfaces. Very basic and intuitive example is motion of particle on a two-sphere.

$\psi(x,t) = \int [dx] W(P,x,x')\psi(x',t')$

weight factor$(W(P,x,x')) = e^{is_P}$, we can think of e^{is_P} as, for every path P between x' and x, it gives a complex no. of unit modulus:

$W(P,x,x') = e^{is_P}$: set of paths $\rightarrow$U(1)

path integral gives summation over all paths with e^{iS} as a weight factor for each path.

Idea to resolve our problem:

- we associate our weight factor with a surface instead of a path.
- chose a standard path between x and x' as P_0 and path of our interest as P
- Now, $P - P_0$ gives a closed loop and we consider a surface σ whose boundary is $P - P_0$ i.e. $\partial\Sigma = P - P_0$

If $P - P_0$ isn't a boundary of a surface then we've to modify our construction too.

let's say coordinates $x^\mu(t)$ gives our path we can generalize this idea to $x^\mu(\sigma, t) = x^\mu(\xi)$ and $\xi^i = (\sigma, t)$ parametrize Σ. The area element of Σ can be written as:
$\frac{1}{2}\partial_i x^\mu \partial_j x^\nu \epsilon^{ij}$. The weight factor can now be written as:
$W(P, x, x') = e^{iS_P(x,x')} e^{i\Gamma(\Sigma)}$
$\Gamma(\Sigma) = k \int_\Sigma d^2\xi \frac{1}{2} B_{\mu\nu i} x^\mu \partial_j x^\nu \epsilon^{ij}$
Where, $B_{\mu\nu}$ is a function of x^μ and is anti-symmetric.
There's no physical meaning of σ thus our choice of shouldn't affect the physics.
Choice of different Σ corrosponds to a different functional hence $\Gamma(\Sigma)$ must be invariant under small deformations $(\delta\Gamma(\Sigma))$ of σ or x^μ

$\delta\Gamma =$
$\frac{k}{2}\int_\Sigma d^2\xi[\partial_\alpha B_{\mu\nu}\delta x^\alpha \partial_i x^\mu \partial_j x^\nu \epsilon^{ij} + B_{\mu\nu}\partial_i(\delta x^\mu)\partial_j x^\nu \epsilon^{ij} + B_{\mu\nu}\partial_i x^\mu_j(\delta x^\nu)\epsilon^{ij}]$
$= \frac{k}{2}\int_\Sigma d^2\xi[\partial_\alpha B_{\mu\nu} + \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\mu\alpha}]\partial_i x^\mu \partial_j x^\nu \epsilon^{ij}\delta x^\alpha$
$+k\oint_{\partial\Sigma} d\xi_i B_{\mu\nu}\partial_j x^\nu \epsilon^{ij}\delta x^\mu$
Thus, we require $[\partial_\alpha B_{\mu\nu} + \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\mu\alpha}] = 0 \rightarrow 1)$
second tern depends on $\partial\Sigma = P - P_0$ and δx^μ is 0 for P_0, since it is fixed once.
Solutions to 1) are(can be) given by:
$B_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$ for some vector potential C_μ
$\Gamma =$
$k\int_\Sigma d^\xi(\partial_\mu C_\nu)\partial_i x^\mu \partial_j x^\nu \hat{\epsilon}^{ij} = k\int_\Sigma d^2\xi \frac{\partial}{\partial\xi}[C_\nu \partial_j x^\nu \epsilon^{ij}] = k\oint_{\partial\Sigma} d\xi_i C_\nu \partial_j x^\nu \epsilon^{ij} =$
k $[\int_P dt C_\mu \frac{dx^\mu}{dt} - \int_{P_0} dt C_\mu \frac{dx^\mu}{dt}] \rightarrow 2)$
Second term in 2) is a common phase factor for all since it is same for all paths and thus can be dropped from the path integral.

Thus, $W(P, x, x') = e^{iS'_P(x,x')}$
$S'_P = S' + \int dt C_\mu \frac{dx^\mu}{dt}$
Thus, discussion about σ becomes irrelevant and the only relevant term is $B_{\mu\nu}$ now.
Today, we're interested in the case when $B_{\mu\nu}$ can't be written as a curl of some vector i.e. $B_{\mu\nu} \neq \partial_\mu C_\nu - \partial_\nu C_\mu$
$B_{\mu\nu}$ is the component version of a two-form
$B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$, $dB = 0$ but $B \neq dC$
Thus it is an element of $H^2(C, \mathbb{R})$

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Example: Monopole Field
$B_{\mu\nu} = \frac{1}{8\pi} \frac{\epsilon_{\mu\nu\alpha} \hat{x^\alpha}}{r^2} dx^\mu \wedge dx^\nu$
$dB = 0$ on $\mathbb{R}^3/\{0\}$. Integrating B over two-sphere surrounding the origin:
$\oint_{S^2} B = 1$, valid for $\mathbb{R}^3/\{0\}$, it has non contractible two-spheres.

Case of R when two points are removed

This leads us to think about two different types of non-contractible surfaces. S_1 surrounding the first point removed, S_2 surrounding the second point removed.
OR
we can define $S_1 + S_2$ surrounding both the points, but this surface can be decomposed upto smooth deformations in $(S_1 + S_2)$. We're saying that two-surfaces can be decomposed into copies of S_1 and S_2, hence S_1 and S_2 are generators of non-contractible surfaces.
NOTE!:In the first case surfaces can't be deformed into each other.

Homology comes in

If one is watching carefully, we just defined $H_2(C)$

$H_2(\mathbb{R}^3 - \{0\}) = \mathbb{Z}$ and hence there is one generator(unity). for $H_2(\mathbb{R}^3 - \{2pts\}) = \mathbb{Z} + \mathbb{Z}$.
In general, $B_{\mu\nu}$ can be defined for each generator proportional to the area element of generating surface with normalization condition as:
$\oint_S B_{\mu\nu} dS^{\mu\nu} = 1$
Given two two-surfaces Σ and Σ', where one is a small deformation of the other.Since, they've a common boundary we've 3-volume in between defined as $\partial V = \Sigma - \Sigma'$ and by stoke's theorem:

$$\Gamma_\Sigma - \Gamma'_\Sigma = K \int_{\partial V} d^\xi \frac{1}{2} B_{\mu\nu} \partial_i x^\mu \partial_j x^\nu \epsilon ij = k \int_{\partial V} B = K \int_V dB = 0 \tag{2}$$

Choice of surface is irrelevant. This only works on;y when there exists a V such that $\partial V = \Sigma - \Sigma'$. $H_2(C) \neq 0$ i.e. $\partial V \neq \Sigma - \Sigma'$

Overview(wrap-up)

$C = \mathbb{R}^3 - \{0\}$ and chose Σ,Σ' such that $\Sigma - \Sigma'$ is a two-sphere surrounding the origin, there's no volume such that $\partial V = \Sigma - \Sigma'$(origin is removed)

$\Gamma_\Sigma - \Gamma'_\Sigma = k \oint_{\mathbb{S}^2} B = k$

$e^{i\Gamma_\Sigma} = e^{i\Gamma'_\Sigma} e^{ik}$

If we impose $e^{ik} = 1$ and $k = 2\pi n$, $n \in \mathbb{Z}$. Gives us topological quantization rule.

If the configuration space C has non-contractible two-surfaces and it's possible to generalize writing path integrals:

- $$\psi(Q,t) = \int [dQ] e^{is(Q,t,Q',t') + i\Gamma_\Sigma} \psi(Q',t') \tag{3}$$
- $\Gamma_\Sigma = 2\pi n \int_\Sigma B$ where $\int_S B = 1$ and S is the generator of $H_2(C)$
- $dB = 0$ and $B \neq dC$

Γ_Σ is called the wess-zumino term.

Examples

- Analmous gauge theories
- Motion of charged particle in a monopole background.

Basic Overview of Spontaneous Symmetry Breaking

We start with an action which which is $U(1)$ symmetric i.e. invariant under $\phi \to \phi' = e^{i\theta}\phi$:
$S = \int d^4x[\partial_\mu \phi^* \partial^\mu \phi - a\phi^*\phi - b(\phi^*\phi)^2]$
The hamiltonian of the system corrosponds to:
$H = \int d^3x[\partial_0 \phi^* \partial^0 \phi + \nabla\phi^*\nabla\phi + a\phi^*\phi + b(\phi^*\phi)^2]$
To get the ground state we must minimize the hamiltonian. The field configuration $(\phi, \partial_0\phi)$ is called the ground state configuration. If $b < 0$, H is minimized by taking $\partial_0\phi = 0$, $\phi \to \infty$ in that case $H \to -\infty$. Which is a BS ground state(unphysical) thus $b > 0$(must).
Two sub-cases:

- $a > 0$ every term in H is positive and H is minimized by $\partial_0\phi = 0$ and $\phi = 0$
- $a < 0$.We write the hamiltonian as: $H = \int d^3x[\partial_0\phi^*\partial_0\phi + \nabla\phi^*\nabla\phi + b(\phi^*\phi - v^2)^2] - \int d^3x \frac{|a|^2}{4b}$

$v = \frac{|a|}{2b}$ the configuration which minimizes H is given by
$\partial_0 \phi = 0, \nabla \phi = 0$
$\phi^* \phi = v^2$ or $\phi = ve^{i\alpha}$
In the first case when $a < 0 ... \langle 0| \phi |0\rangle = 0$
In the second case when $a < 0 ... \langle 0| \phi |0\rangle = \sqrt{\frac{|a|}{2b}} e^{i\alpha} + \mathbb{O}(\hbar)$
In the second case our $U(1)$ symmetry is considered spontaneously broken.
example: $O(N)$ symmetry is broken into $O(N-1)$ symmetry.
Idea of Spontaneous symmetry breaking:

- G is the symmetry group of the lagrangian, G is the symmetry group of the hamiltonian.
- $G |0\rangle \neq |O\rangle$, $H |0\rangle = |0\rangle$ shows the G is spontaneously broken down $H \subseteq G$.

Topological Vortices

We start with the action for the abelian-Higgs model:
$S = \int d^4x[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - V(\phi^*\phi)]$
$V(\phi^*\phi) = \lambda(\phi^*\phi - v^2/2)^2$
$H = \int d^3x[\frac{1}{2}(E^2 + B^2) + \dot{\phi}^*\dot{\phi} + (D_i\phi)^*(D_i\phi) + V(\phi^*\phi)$
The minimization of H is done by:
$\dot{\phi} = 0,\ D_i\phi = 0$
$V(\phi^*\phi) = 0,\ E = B = 0$
$\phi = \frac{V}{\sqrt{2}}$ is the solution to these conditions.
The integrand has to vanish at spatial infinity(convergence). Which requires field to go to our solution at spatial infinity. Since V(potential) has to be zero at spatial infinity, the field shpuld be of the form $\phi = \frac{v}{\sqrt{2}}e^{i\theta}$.
We have a mapping $e^{i\theta} : S^2 \rightarrow U(1)$.
Now, we know that $\Pi_2(U(1)) = 0$ field has only one connected component minimization leads to vacuum again.

We get non-trivial topology if we consider vortices with finite energy density.
$= \int d^2x[\frac{1}{2}B^2 + (D_i\phi)^*(D_i\phi) + V(\phi^*\phi)]$(simplest case of a straight line vortex, along the x^3 axis)
for this to be finite we need $\phi \rightarrow \frac{v}{\sqrt{2}}e^{i\theta}$ as $r^2 = x_1^2 + x_2^2 \rightarrow \infty$ giving,
$e^{i\theta} : S^1 \rightarrow U(1)$ and $\Pi_1(U(1)) = \mathbb{Z}$.Winding no. for these maps is given by:
$Q = \frac{-i}{2\pi} \int d^2x\epsilon^{\mu\nu} \frac{\partial_\mu\phi^*\partial_\nu\phi}{\phi^*\phi}$
$Q = \frac{-i}{2\pi} \oint_{|x|\rightarrow\infty} \frac{\phi^* d\phi}{\phi^*\phi}$
the field of configuration of finite energy per unit length has an infinite no. of path connected components(Q). Simplest case of $Q = 1$ and we need $\phi \rightarrow \frac{x_1+ix_2}{r}$ as $r \rightarrow \infty$
Generally If a gauge symmetry group G is spontaneously broken to H then the relevant homotopy groups for vortices are $\Pi_1(G/H)$

For finiteness we would also need $(D_i\phi)^*(D_i\phi)$ need to vanish as $r^2 \to \infty$. This can be done by making an assumption about A_i such that ${}_i\phi$ is cancelled by $-ieA_i\phi$.

$Q = \frac{e}{2\pi}\oint_{|x|\to\infty} A$

thus, $\int d^2xF = \frac{2\pi}{e}Q$

A configuration with winding no. Q maybe interpreted as Q-flux tubes, each carrying magnetic flux of $\frac{2\pi}{e}$

The quantization of magnetic flux carried by the vortex is a result of topological origin of vortex due to $\pi_1(U(1)) \neq 0$

For $Q = 1$, our assumptions(ansatz) are:

$\phi = \frac{v}{\sqrt{2}}h(r)e^{i\theta}$ and $eA_i = \frac{-\epsilon_{ij}x^j}{r^2}f(r)$

Equation of motion for static field:

$\frac{d}{d\xi}(\frac{1}{\xi}\frac{d}{d\xi}) - \frac{h^2(f-1)}{\xi^2} = 0$

$\frac{1}{\xi}\frac{d}{d\xi}(\xi\frac{df}{d\xi}) - h\frac{(f-1)^2}{\xi^2} - \frac{M_H^2}{2M_v^2}h(h-1) = 0$

$\xi = evr$ particle mass $M_v = ev$

higgs scalar mass $M_H = \sqrt{2\lambda}v$

$f \to 1 + \alpha exp(-\xi)$

$h \to 1 - \beta exp(-M_H r)$

$\frac{M_H}{M_v} < 1$...Type 1 superconductors.
$\frac{M_H}{M_v} > 1$....Type 2 superconductors.
In Type 1 superconductors higgs field falls off more slowly than the gauge field. Flux is contained in the small region where higgs field is close to 0.
In Type 2 gauge field falls off more slowly, giving a flux tube in the superconductor.
While the exact solution to our equations of motion are unknown. They are the variational equations for energy equations:
$E = 2\pi v^2 \int \xi d\xi [\frac{1}{2}(\frac{f'^2}{\xi^2} + h'^2 + \frac{h^2(f-1)^2}{\xi^2}) + \frac{\lambda(h^2-1)^2}{4e^2}]$
Consider vortex around x^3. One can consider of this vortex as "bending of vortex".
In general one can get vortex any curve C. We know that vortices have to end in monopoles since the flux has to terminate in magnetic charges.
It is also possible for vortices to form closed loops, since there is no difficulty with flux conservation in that case.

We will consider very thin vortices which can be obtained when $M_H >> M_v$. Let C be the vortex as a curve in 3-d space with coordinates $z^i(\tau)$. The magnetic flux for that vortex is given by:
$F = 4g \int_C d\tau \frac{dz^k}{d\tau} \delta^{(3)}(x - z(\tau)) 1/2 epsilon_{ijk} dx^i \wedge dx^j \equiv F_v$
$g = 1/2e$ is the strength of the vortex. F is 0 except at our vortex(C).
Thus α is a closed form on $\mathbb{R}^3/\{C\}$
We wish to create an operator which creates a vortex. In $A_0 = 0$ the action for $U(1)$ gauge theory is:
$S = \int d^4x [\frac{1}{2}(\partial_0 A_i \partial_0 A_i) - \frac{1}{2} B^2]$
The equal-time commutation relation are:
$[A_i(x), E_j(y)] = i\delta^{(3)}(x - y)$
In hamiltonian dynamics we've to impose gauss law. An alternative is to require that physical states are invariant under gauge transformations.i.e. $(\nabla . E - J_0) |\psi\rangle = 0$ for all physical states $|\psi\rangle$.
The operator which an create a vortex line in our flux formula is give by:
$T(C) = exp(-i \int d^3x \alpha_i E_i)$

This yields,:

$T(C)^{-1}A_i(x)T(C) = A_i(x) + \alpha_i(x)$(canonical commutation relation)

Magnetic flux across a surface(Σ) can be measured by $\oint_C A$, where $\partial\Sigma = C$

Using our commutation we can write:

$T^{-1}(C)[\oint_C A]T(C) = \oint_C(A+\alpha) = \oint_C A + \oint_\Sigma F_v$

magnetic flux is shifted by the flux of a vortex by the operator $T(C)$ confirming the interpretation(vortex creation operator)

If the vortex line does not intersect the surface Σ then $\int_\Sigma F_v = 0$

If it intersects we get a contribution for each intersection:

$\int_\Sigma F_v = 4\pi g L(C, C')$, where $L(C, C')$ is called the gauss-link no., it is the no of times the curve C' link C

$L(C, C')$ can be a topological invariant if C is infinitely long or if it is closed loop.

$L(C, C') = \frac{-1}{4\pi}\oint_C \oint_C' \epsilon_{ijk}\frac{(x-y)^i}{|x-y|^3}dy^j dx^k$

now, we can write:

$[\oint_C A]T(C) = T(C)[\oint_C A] + T(C)4\pi g L(C', C)$

In abelian case Wilson Opertaor becomes:
$W(C) = exp[ie \oint_C A]$, now we consider another identity:
$exp[-ie \oint_C A]E_i(x)[ie \oint_C A] = E_i(x) + e \oint_C \delta^{(3)}(x - z(\tau))\frac{dz_i}{s\tau}d\tau$
Wilson operator creates a thin electric tube along C.
$T(C)$ creates a thin magnetic flux tube and $\oint_C A$ as magnetic flux through C.
The electric flux will be measured by $T(C)$.The commutation relations can be given as:
$W(C)T(C') = exp(i4\pi eg)T(C')W(C) = T(C')W(C)$
The additional phase factor is 1 because of the quantization of flux by the vortex, $eg = n/2$ for some integer n. We found another way of stating dirac quantization rule.The operators $W(C)$ and $T(C')$ provides a new way of specifying the phase of gauge theory, which produces superconducting phases.If we bring in a dual potential A' for the electric field such that $E_i = \epsilon_{ijk}\partial_j A'_k$.

Thus, $T(C') = exp(-i2\pi g \oint A')$
There exists some restrictions on such dual potential. But, these equation lead us to think $T(C')$ as the dual of $W(C)$.

Non-Abelian Case

Consider the construction of $T(C')$ operator in a $U(1)$ subgroup of $SU(N)$ gauge theory. There exists some central elements in $SU(2)$ which commute with other elements of $SU(N)$, these central elements form a cyclic group $\mathbb{Z}_N$ represented by $(N \times N)$ matrices $\{1, \omega, \omega^2, ..., \omega^{N-1}\}$ with $\omega = exp(2\pi i/N)$ as a matrix all these are proportional to identity and hence commuting with other elements of $SU(N)$.
example: Let t_a with $a = 1, 2, , ...(N^2 - 1)$ be the basis for the lie algebra of $SU(N)$ and normalization condition $Tr(t_a t_b) = \frac{1}{2}\delta_{ab}$ and with consider the following:
$t_{N^2-1} = \sqrt{\frac{N}{2(N-1)}} diag(\frac{1}{N}, \frac{1}{N}, ..., -1 + \frac{1}{N})$
$= \sqrt{\frac{N}{2(N-1)}} Y$, as for the group element:
$exp(2\pi i \sqrt{\frac{2(N-1)}{N}} t_{N^2-1} = exp(2\pi Y) = exp(2\pi i/N) = \omega$
Fields in adjoint representation transform as $\phi \to \phi' = g\phi g^{-1} = \phi^a t_a$.

These are indeed invariant under $\mathbb{Z}_N$ and also all representations obtained from the reduction of tensor products of the adjoint representation are also $\mathbb{Z}_N$ invariant. For all such fields(gauge fields too) the group of concern is $SU(N)/\mathbb{Z}_N$ and thus $\mathbb{Z}_N$ invariant. Which means integer eigenvalues of Y on the other hand matter fields aren't $\mathbb{Z}_N$ invariant as they transform as $\phi' = g\phi$ and thus for them Y has fractional eigenvalues.
$SU(N)$ is simply connected but the new group of interest $SU(N)/S\mathbb{Z}_N$ isn't.
$\Pi_1(SU(N)/\mathbb{Z}_N) = \mathbb{Z}_N$ thus the spontaneous symmetry of this theory will lead to $\mathbb{Z}_N$ vortices.
Let's discuss the breaking of $U(1)$ by higgs field which are in some representation of $SU(N)/\mathbb{Z}_N$. The covarient derivative would be of the form:
$D_\mu\phi = \partial_\mu\phi - ieYA_\mu\phi$
. There can be/are many components of higgs field in adjoint representation which aren't $\mathbb{Z}_N$ invariant and hence breaking the $U(1)$ symmetry and if it is broken we get vortices. And the since the eigenvalues of Y are integers , the magnetic flux of the vortices will be quantized in the units of $2\pi/e$.

We can create $T(C')$ and $W(C)$ as before but we can't use $\oint_C A$ as a gauge invariant operator. for wilson loops the flux of a vortex can be measured using $W(C)$?

well, for wilson loops in their fundamental representation, the values of Y are $1/N$ or $-1+(1/N)$. Thus, in this case:

$W_F(C)T(C') = exp(\frac{2\pi i L(C,C')}{N})T(C')W_F(C))T(C')W_F(C)$

At the level of $U(1)$ field there are also vrotices with fluxes which are N times the basic unit of $\frac{2\pi}{e}$ or multiple thereof.An operator $T(C')$ for them will commute with $W_F(C)$ and thus with $W(C)$'s in any representation(can't be detected). They have no meaning as we embed the chosen $U(1)$ subgroup in $SU(N)$, the only meaningful flux values are modulo N since, $\Pi_1[SU(N)/\mathbb{Z}_N] = \mathbb{Z}_N$,

't hooft-Polyakov Magnetic Monopoles

In a spontaneously broken gauge theory monopoles can appear as stable classical solutions of finite energy.These Solutions are called as 't hooft polyakov monopoles.
We start with a nonabelian gauge theory with symmetry G spontaneously broken to subgroup $H.H$ leaves the higgs-expectation value invariant, we understand that asymptotically a map from $S^2 \to G/H$. S^2 corrosponds to large radius sphere in $\mathbb{R}^3$. The homtopy classes of such maps $\Pi_2(G/H)$ are formulated by the exact sequence:
$... \to \Pi_2(G) \to \Pi_2(G/H) \to \Pi_1(H) \to \Pi_1(G) \to ...$
$... \to 0 \to \Pi_2(G/H) \to \Pi_1(H) \to \Pi_1(G) \to ...$
shows that if G is simply connected then $\Pi_2(G/H) = \Pi_1(H)$
For a simple example we start with $SU(2)$ gauge theory breaking down to $U(1)$ subgroup. The higgs can be taken as a triplet under $SU(2)$ and the action is given by:
$S = \int d^4x[-\frac{1}{4}F^a_{\mu\nu}F^a\mu\nu + \frac{1}{2}(D_\mu\phi)^a(D_\mu\phi)^a - V(\phi^a\phi^a)]$ where

$(D_\mu\phi)^a = \partial_\mu\phi^a + e\epsilon^{abc}A_\mu^b\phi^c$
$V(\phi^a\phi^a) = \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2$
Since, we know that higgs field is a triplet, the symmetry group is $SO(3)$ and is breaking down to $SO(2)$ and the configuration for static energy is given by:
$E = \int d^3x[\frac{1}{2}B_i^aB_i^a + \frac{1}{2}(D_i\phi)^a(D_i\phi)^a + V(\phi^a\phi^a)]$
The isotropy subgroup is $SO(2)$ corresponding to group rotations around $a = 3$ direction.The classical vacuum is at the minimum of V, we can take this as $\phi^a = \delta^{a3}v$
We have the higgs breaking to the gauge group $SO(2)$ $U(1)$. The unbroken $SO(2)$ direction will be taken as the electromagnetic $U(1)$ gauge theory of this model.
The gauge field components $W_i^\pm = (A_i^1 \mp iA_i^2)/\sqrt{2}$ and get mass $M_v = ev$
For finite energy configuration we need,
$B_i^a \to 0$, $(D_i\phi)^a \to 0$
$\phi^a\phi^a \to v^2$, as $r \equiv |x| \to \infty$
From the last condition we've that as $r \to \infty$, we've the mapping $\phi^a : S^2 \to S^2$.

$\Pi_2[SO(3)/SO(2)] = \Pi_2[S^2] = \mathbb{Z}$
$Q = \frac{1}{8\pi} \oint_{r\to\infty} \epsilon^{abc} \hat{\phi}^a d\hat{\phi}^b \wedge d\hat{\phi}^c$
$\hat{\phi}^a = \frac{\phi^a}{|\phi|}$
For the vacuum configuration $Q = 0$ with our asymptotic condition of $(D\phi)$ vanishing, we write:
$\oint \epsilon^{abc} \hat{\phi}^a D\hat{\phi}^b \wedge D\hat{\phi}^c = 0$ after bit of simplification:
$\oint F^a \hat{\phi}^a = \frac{-4\pi}{e} Q$ where
F^a is our field strength two-form. The unbroken direction is asymptotically provided with $Q \neq 0$ by $\hat{\phi}^a$ thus $F^a \hat{\phi}^a$ is the field strength of the unbroken group.
From our equation we find that total magnetic flux is indeed proportional to Q thus we are for sure discussing about monopoles, elementary monopoles have charge $g = \frac{1}{e}$ and $Q = -1$.
The asymptotic behavior of ϕ^a becomes:
$\phi^a \to -v\frac{x^a}{r}$
finiteness of energy requires that $(D_i\phi)$ vanish, with the behavior of ϕ^a stated above this leads to:
$A_i^a \to \epsilon_{aik} \frac{x^k}{er^2}$

suitable ansatz for the fields for all r is given by:
$\phi^a = -v\frac{x^a}{r}\frac{H(\xi)}{\xi}$
$A_i^a = \epsilon\frac{x^k}{er^2}[1 - K(\xi)]$
$\xi = evr$. If ϕ does not vanish anywhere, this leads to a contradiction, mapping $Q \neq O$ to $Q = 0$. This means that any configuration of non-zero Q must have zero for the higgs field ϕ^a at some point.
Also K must go to 1 as $r \to 0$ just avoid any singularities.
The static energy for our choice is given as:
$E =$
$\frac{4\pi v}{e}\int_0^\infty \frac{d\xi}{\xi^2}[\xi^2(\partial_\xi K)^2 + 1/2(\frac{\xi dH}{d\xi} - H)^2 + 1/2(K^2-1)^2 + K^2H^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2]$
The Static Equation of Motion are:

$$\xi^2\frac{d^2K}{d\xi^2} = KH^2 + K(K^2-1) \tag{4}$$

$$\xi^2\frac{d^2H}{d\xi^2} = 2K^2H + \frac{\lambda}{e^2}H(H^2-\xi^2) \tag{5}$$

Exact analytical solutions to these equations are still unknown, but one can find numerical solutions to these equations with our given conditions for our ansatz.

The energy of the solution is generally of the form.
$E = \frac{4\pi v}{e} f(\lambda/e^2)$ with $f(0) = 1$.The asymptotic behavior of the form:
$H - \xi\ exp(-M_H r)$, $K\ exp(-M_v r)$
$M_v = ev$ and $M_H = \sqrt{2\lambda} v$
There is a particular case in which we can solve these equations analytically it is known as the Bogomol'nyi-Prasad-Sommerfield(BPS) limit, energy functional obeys an inequality:
$E \geq \int d^3x[\frac{1}{2}B_i^a B_i^a + \frac{1}{2}(D_i\phi)^a(D_i\phi)^a] \geq \int d^3x B_i^a (D_i\phi)^a$
$\geq \frac{4\pi v}{e}$
It follows from some basic facts that $(B - D\phi)^2 \geq 0$ in the equation we've used the bianchi identity $(D_i B_i)^a = 0$ and the equation giving relating F^a and Q. In the BPS limit $\lambda \to 0$ still retaining the boundary condition $\phi^a\phi^a \to v^2$ as $r \to \infty$. The contribution of the potential term(V) is then 0. Possible to reach the inequality by taking:
$B_i^a = (D_i\phi)^a$

Solution to these equations are given by:
$H(\xi) = \xi \coth \xi - 1$
$K(\xi) = \frac{\xi}{\sinh \xi}$
We see that magnetic and electric charge obey the quantization condition $eg = Q$ but the dirac quantization condition was $eg = n/$ this due to the fact that this theory's starting group is $SO(3)$ instead of $SU(2)$, the theory can admit charges $e/2$ corresponding to the spinor representation of $SO(3)$ which is the fundamental representation of $SU(2)$ thus again we've our condition $eg =$ integer.
A bit more "mathy" explanation:
$0 \to \Pi_2[SO(3)/SO(2)] \to \Pi_1[SO(2)] \to \Pi_1[SO(3)] \to 0$
$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$
This shows that elements of $\Pi_2[SO(3)/SO(2)]$ should map onto even elements of $\Pi_1[SO(2)]$

THANK YOU FOR ATTENDING AND STUFF
HAPPY CHRISTMAS IF YOU'RE A CHRISTIAN IF NOT,
BE HAPPY YOU'VE A REASON TO DRINK
SAME GOES FOR THE NEW YEAR