GRAVITATIONAL POTENTIAL:

Conservation of direction of angular momentum of the angular momentum, means that the particle will move in a particular plane. We can choose a plane, and then rotate it according to our needs and wants. In that plane:

$\theta = \frac{\pi}{2}$(also called the equitorial plane)

$$\frac{d\theta}{d\beta} = 0 \tag{1}$$

Two other killing vectors correspond to energy and the magnitude of the angular momentum.
Time−like killing vector | $K^{\mu} = (1, 0, 0, 0)^T$
$k^{\mu} = k^{\nu} g_{\mu\nu} = \left(-\left(1 - \frac{2GM}{r}\right), 0, 0, 0\right)$
$K_{\mu}\frac{dx^{\mu}}{d\beta} = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\beta} = E$(const. of motion)
Sly. $L_{\mu} = \partial_{\Phi}\left(0, 0, 0, 1\right)^T$
$L_{\mu} = (0, 0, 0, r^2 \sin^2\theta)$ | $r^2\frac{d\Phi}{d\beta} = L$
Now, we substitute this into the equation for ε.

$$-\varepsilon = -\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\beta}\right)^2+\left(1-\frac{2GM}{r}\right)\left(\frac{dr}{d\beta}\right) \tag{2}$$

Multiplying by $\left(1-\frac{2GM}{r}\right)$, putting the values for E and L:−

$$-E^2+\left(\frac{dr}{d\beta}\right)^2+\left(1-\frac{2GM}{r}\right)\left(\frac{L^2}{r^2}+\varepsilon\right)=0 \tag{3}$$

We can rearrange this into :

$$\frac{1}{2}\left(\frac{dr}{d\beta}\right)^2+V\left(r\right)=\frac{1}{2}E^2 \tag{4}$$

Where V(r) is our potential, given by:

$$V\left(r\right)=\frac{1}{2}\varepsilon-\frac{\varepsilon GM}{r}+\frac{L^2}{2r^2}-\frac{GML^2}{r^3} \tag{5}$$

<u>*MASSIVE PARTICLE TRAJECTORIES:*</u>

For massive particles, our $\varepsilon = 1$.

If we assume the particle is executing circular orbits, if(minimum potential) $\frac{}{d}vdr = 0r = const.$

Subbing all this in our potential equation we get, for

$$GMr^2 - L^2r + 3GML^2\gamma = 0 \tag{6}$$

As,$r \to \infty$: $-GML^2/r^3 \to 0$
$\gamma = 1$in General Relativity.

$$r_{circ.} = \frac{L^2 \pm \sqrt{L^4 - 12GM^2L^2}}{2GM} \tag{7}$$

$$r_{circ} = \frac{L^2}{GM}, 3GM \tag{8}$$

We will use that value of r_{circ}that minimizes the potential, in order to maximize stability of our circular orbit. One can check that 3GMis not our stable solution that we are looking for. The discriminant is zero(which gives us the 1st solution) is when:

$$L = \sqrt{12GM} \tag{9}$$

$$r_{circ} = r_{isco} = 6GM \tag{10}$$

This last equation that we wrote give us, our smallest possible stable circular orbit around any body which can be described by our Schwarzschild radius.

r_{isco}means inner most stable circular orbit, it's important for measurement purposes usually of rotating black holes, that we'll discuss in brief later in the book. Now, it's safe to say that:

Stable if: $r_{circ} > 6GM$

Unstable if: $r_{circ} \in (3GM, 6GM)$

MASSLESS PARTICLE TRAJECTORY:

For massless particles like photon, $\varepsilon = 0$

Subbing this into our potential equation with our minimum potential condition $\frac{}{d}vdr = 0$and we obtain the potential:

$$V(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \tag{11}$$

And our minimum radius is:

$$r_{circ.} = 3GM \tag{12}$$

A photon with a given energy E comes in from infinity($r = \infty$) and it's path is tilted if it doesn't cross the singularity of the metric at $r = 2GM/c^2$, it goes off to infinity but id it does it falls into potential and there's no coming back.