TESTS OF GENERAL RELATIVITY:

"It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with the experiment, it's wrong." –Richard Feynman

So, we must test general relativity in order to prove that it's correct.

Einstein suggested three tests for GR:

1. Gravitational Red shift of spectral lines.
2. Deflection of light by sun
3. Precession of the point in the orbit of a planet closest to the Sun aka Perihilia.

   Another test has been added but this wasn't given by Einstein
4. Gravitational time delay of radar signal.

GRAVITATIONAL REDSHIFT :

We'll consider two non–inertial observers here, for dramatic effects let's name them Romeo and Juliet. In, later stages of book it will become more clear why we chose these two specific names. Both are stuck at spatial coordinates

$(r_R, \theta_R, \varphi_R)$ and $(r_J, \theta_J, \varphi_J)$ respectively.

Now, we know that:

$$\frac{d\tau}{dt} = \sqrt{\left(1 - \frac{2GM}{r}\right)} \tag{1}$$

Suppose Romeo decides to send “Hey, I cheated on my math exam lol” to Juliet through his “cool” Photon technology which emits a light pulse which travels to Juliet, such that Romeo measures the time between two successive crests of the light wave to be
$\Delta\tau_R$.Each crest follow the same path to Juliet , except they are separated by:

$$\Delta t = \frac{1}{\sqrt{\left(1 - \frac{2GM}{r_R}\right)}}(\Delta\tau_R) \tag{2}$$

This separation doesn’t change along the photon trajectory, but Juliet here, measures a time between successive crests as:

$$\Delta\tau_J = \sqrt{\left(1 - \frac{2GM}{r_J}\right)}\Delta t(3)$$

$$\bullet\ \Delta\tau_J = \sqrt{\frac{\left(1-\frac{2GM}{r_J}\right)}{\left(1-\frac{2GM}{r_R}\right)}}\Delta\tau_R$$

Since, $\Delta\tau_i$measures the proper time interval between two crests of an EM wave. Their, frequencies will be :

$$\frac{\nu_J}{\nu_R} = \frac{\Delta\tau_R}{\Delta\tau_J} \tag{4}$$

$$= \sqrt{\frac{\left(1-\frac{2GM}{r_R}\right)}{\left(1-\frac{2GM}{r_J}\right)}} \tag{5}$$

This, is our ever wanted formula. Now, Juliet didn't receive the correct message, the light got red−shifted and she received "Hey, I cheated on you lol" now, she's pissed. It appears as though Romeo didn't quite realised the 4D spacetime curvature of the situation(pun intended), and maybe next Romeo just might study a bit hard.

DEFLECTION OF LIGHT :

Consider a light ray approaching from infinity.

Using the equations we derived earlier(with $\varepsilon = 0$) we find that,

$$\frac{1}{L^2}\left(\frac{dr}{d\beta}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) = \frac{E^2}{L^2} \tag{6}$$

We define the impact parameter as:

$$b \coloneqq \frac{L}{E} \tag{7}$$

$$\frac{d\Phi}{dr} = \pm\frac{1}{r^2}\left\{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right\}^{-1/2} \tag{8}$$

Now, simply by doing integration we find :

$$\Delta\Phi = 2\int_{r_1}^{\infty}\frac{dr}{r^2}\left\{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right\}^{-1/2} \tag{9}$$

Where r = $r_1$is the turning point, which is the radius where ${}_{1}b^2 = \left(1 - \frac{2GM}{r}\right)/r^2$

For the deflection of light by the Sun, the impact parameter can't be smaller than the stellar radius, b$\geq$ $R_{\odot} \approx 7 * 10^{10} cm$, thus ${}_{\overline{2}}GM_{\odot}c^{2}b \leq 10^{-6}$.

This equation is solved by 1) change of parameter $w = \frac{b}{r}$and 2) expanding ${}_{\overline{G}}Mr$in Taylor series and keeping the first order.

$$\Delta\Phi \approx \pi + \frac{4GM}{b} \tag{10}$$

The deflection angle is :

$$\delta\Phi_{de} = \Delta\Phi - \pi \approx \frac{4GM}{c^2 b} \tag{11}$$

Which is ~ 1.75(arc$-$seconds). That's really small amount but, Sun isn't that big to make a noticeable deflection of light, but Black holes are..

<u>TIME DELAY RADAR SIGNAL:</u>

This time we solve for dt/dr, the result is:

$$\frac{}{d}tdr = \pm\frac{1}{b}\left(1 - \frac{2GM}{r}\right)^{-1} \left\{\left\{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right.\right.\right. \tag{12}$$

If one sends sends an EM signal in a gravitational field, which can be reflected back, it is possible to calculate the GR correction to the time of the returned signal. The GR time delay is:

$$(\Delta t)_{GR} \approx \frac{4GM}{c^3}\left[\log\left(\frac{4r_{other}r_{earth}}{r_1}\right)+1\right] \tag{13}$$

Where, $r_1 = b$is the radius of closest approach to the center.
This experiment was proposed by Irwin Shapiro, also called Shapiro time delay.
s

<u>*PERIHILIA:*</u>
We'll examine the orbit of Mercury around Sun, in specific here.
We start by writing the lagrnagian for the given motion:

$$L = \frac{mv^2}{r} = \frac{m}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{14}$$

$$L = \frac{m}{2}\left[\left(1 - \frac{2GM}{r}\right)\dot{t^2} - \left(1 - \frac{2GM}{r}\right)^{-1}\dot{r^2} - (r\theta \tag{15}$$

$$\frac{\partial L}{\partial x^a} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x^a}}\right) = 0 \tag{16}$$

a = 0:

$$\frac{d}{d\tau}\left(1 - \frac{2GM}{r}\right)\dot{t} = 0 \tag{17}$$

a = 2:

$$\frac{d}{d\tau}\left(r^2\dot{\Phi^2}\right) - r^2 \sin\theta\cos\theta\dot{\Phi^2} = 0 \tag{18}$$

a = 3:

$$\frac{d}{d\tau}\left(r^2 \sin^2\theta\dot{\Phi}\right) = 0 \tag{19}$$

a =1 case is not necessary atleast for our purposes.

We'll consider in equatorial from now i.e. $\theta = \frac{\pi}{2}$

And $\dot{\theta} = 0$

$$\frac{d}{d\tau}\left(r^2\dot{\theta}\right) = 0 \tag{20}$$

$r^2\dot{\theta}$is constant.
$\tau = 0, r^2\dot{\theta} = 0 \forall \tau$. If $r \neq 0, \dot{\theta} = 0 \forall \tau$. Hence, $\theta = \frac{\pi}{2}$which means planar motion is possible.
In XY plane,
Integrating a = 3, equation:

$$r^2\dot{\Phi} = h \tag{21}$$

h is a constant. Thus, we can say that h represents angular momentum.
Sly. Integrating a = 0, equation:

$$\left(1 - \frac{2GM}{r}\right)\dot{t} = k \tag{22}$$

Again, k is here a constant.
Now, subbing this equation and $\theta = \frac{\pi}{2}$ into the equation for the lagrangian:
$\frac{k^2}{1-\frac{2GM}{r}} - \frac{\dot{r}^2}{1-\frac{2GM}{r}} - r^2\dot{\Phi}^2 = 1 \ \rightarrow$ N)

$$u = \frac{1}{r} \tag{23}$$

$$\dot{r} = \frac{dr}{d\tau} = \frac{d}{d\tau}(\frac{1}{u}) \quad (24)$$

$$= -\frac{1}{u^2}\left(\frac{du}{d\Phi}\right)\left(\frac{d\Phi}{d\tau}\right) = -\frac{1}{u^2}\frac{du}{d\Phi}hu^2 \quad (25)$$

$$\dot{r} = -h\left(\frac{du}{d\Phi}\right) \quad (26)$$

$r^2\dot{\Phi} = h$, $u = \frac{1}{r}$in the new equation for our lagrangian(N)) to obtain :

$$\frac{}{k}^2 1 - 2Mu - \frac{h^2\frac{du}{d\Phi}}{1-2Mu} - h^2u^2 = 1 \quad (27)$$

Multiplying by $1 - 2Mu^2/h^2$

$$\frac{k^2}{h^2} - \left(\frac{du}{d\Phi}\right)^2 - u^2(1-2Mu) = \frac{1-2Mu}{h^2} \quad (28)$$

Rearranging this we get:

$$\left(\frac{du}{d\Phi}\right)^2 + u^2 = \frac{k^2-1}{h^2} + \frac{2Mu}{h^2} + 2Mu^3 \quad (29)$$

Dividing this by 2, and differentiating we get:

$$\frac{d^2u}{d\Phi^2} + u = \frac{m}{h^2} + 3Mu^2 \tag{30}$$

This is the differential for Mercury according to general relativity.
Now, we'll solve it like physicists which means we'll use perturbative method.

$$\varepsilon = \frac{3M^2}{h^2} \tag{31}$$

$$u'' + u = \frac{m}{h^2} + \varepsilon \left(\frac{h^2 u^2}{m} \right) \tag{32}$$

Now, we assume that it has a solution of type:

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2) \tag{33}$$

Now, we'll differentiate this solution twice and sub it into the differential equation obtained by perturbation, to get :

$$u_0'' + u_0 - \frac{M}{h^2} + \varepsilon \left(u_1'' + u_1 - \frac{h^2 u_0^2}{M} \right) + O\left(\varepsilon^2\right) = 0 \tag{34}$$

For first approximation we equate the coffeciants of ε,ε^2,... to zero. Then, $u_0 = \frac{M}{h^2}(1+ ecos\Phi)$is the zeroth order solution. Now, for coffecients of ε :−

$$u_1^{''} + u_1 = \frac{h^2 u_0^2}{M} = \frac{M}{h^2}\left(1 + e\cos\Phi\right)^2 \quad (35)$$

$u_1^{''}+u_1 = \frac{M}{h^2}\left(1+\frac{1}{2}e^2\right)+\frac{2Me}{h}\cos\Phi+\frac{me^2}{2h^2}\cos 2\Phi\rightarrow$ M')

We can try the solution, $u_1 = A + B\Phi\sin\Phi + C\cos 2\Phi$

$$u_1^{'} = B\sin\Phi + B\Phi\cos\Phi - 2C\sin 2\Phi \quad (36)$$

$$u_1^{''} = 2B\cos 2\Phi - B\Phi\sin\Phi - 4C\cos 2\Phi \quad (37)$$

$u_1^{''} + u_1 = A + 2B\cos\Phi - 3Cs\cos 2\Phi\rightarrow$ M)

Comparing M) and M'), :−

$$A = \frac{M}{h^2}\left(1 + \frac{1}{2}e^2\right) B = \frac{Me}{h^2} \quad (38)$$

$$C = \frac{-Me}{h^2} \tag{39}$$

$$u_1 = \frac{M}{h^2}\left(1 + \frac{1}{2}e^2\right) + \frac{Me}{h^2}\Phi\sin\Phi - \frac{Me^2}{6h^2}\cos 2\Phi \tag{40}$$

$$u \approx u_0 + \varepsilon u_1 \tag{41}$$

$$u \approx u_0 + \frac{\varepsilon M}{h^2}(1 + e\Phi\sin\Phi + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\Phi\right)) \tag{42}$$

We notice $e\Phi\sin\Phi$term increases after each revolution, it becomes dominant. Therefore,

$$u \approx \frac{M}{h^2}(1 + e\cos\Phi + \varepsilon e\sin\Phi) \tag{43}$$

$$u \approx \frac{M}{h^2}(1 + e\cos(\Phi\,(1 - e))) \tag{44}$$

From this last equation we see that orbit of Mercury is not exactly an ellipse.
Period $= \frac{2\pi}{1-\varepsilon} \approx 2\pi(1+\varepsilon)$
Precession is given by:

Precession $\approx 2\pi\varepsilon = \frac{6\pi M_r^2 r}{h_r^2 r}$

$M_r^1$is the mass of the Sun.

$h_r$is mercury's angular momentum in relativistic units.