TO BLACK HOLES ASAP:

A black hole has three properties:

1)Mass 2)Charge3)Spin

Here, in this section we'll only discuss black holes with mass, which can be described by our sweet sweet Schwarzschild metric.

First we write our Schwarzschild metric so, that we can explore it's singularities in depth here.

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\Phi^2) \tag{1}$$

Now, if we look at when $r = 2GM$, we notice our long wanted singularity. Now, we try to get rid of it by trying to change coordinates in clever ways.

You might have another singularity at $r = 0$, but this isn't a coordinate singularity, which we can shake off by just a coordinate change. This singularity is often referred to as "The Singularity" of a given black hole.

Imagine a light ray propagating in the radial directions θ and Φ which are constant and for a light ray

$\mathrm{ds}^2 = 0$.

- $ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$
- $\frac{dr}{dt} = \pm \left(1 - \frac{2GM}{r}\right)$

Now we something peculiar happens as $r \to \frac{2GM}{c^2}$, $\frac{}{d}r dt \to \pm\infty$which means in these coordinates we never a light ray go in.
Imagine a light ray going in a black hole, in these coordinates you'd see it "slowing down" until it approaches the coordinate singularity a.k.a our Schwarzschild radius(remember this happens only in this coordinate system).
TORTOISE COORDINATES:
We substitute $t = \pm r^* \pm const.$
Where $r^*$is defined as :

$$r^* = r + 2GMln\left(\frac{r}{2GM} - 1\right) \qquad (2)$$

- $ds^2 = \left(1 - \frac{2GM}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2|$
  where we have substituted dΩ as our 3 spatial coordinates, just for lesser writing purposes.

Let's take a break, here's a pun to relax you:
A man asked a physicist who's the love of his

life and if she/he has ever disappointed, physicist replied with a grin on his face saying, “ we physicists only fall for the beauty of 4D spacetime curvature(gravity) “ and his smile grew bigger as he replied “ No, she has never disappointed me but I haven't fully understood why she behaves so, weirdly around small stuff”. Okay, back to physics now,

NULL GEODESICS:

$$\overline{u} = t + r^* \tag{3}$$

$$\overline{v} = t - r^* \tag{4}$$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) d\overline{u}^2 + (d\overline{u}dr + d\overline{v}dr) + r^2 d\Omega^2 \tag{5}$$

These are called the Eddington−Finkenstein coordinates.

$$\frac{d\overline{u}}{dr} = \begin{cases} 0 : infalling \\ 2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} : outgoing \end{cases} \tag{6}$$

$$dt = \frac{1}{2}(d\overline{u} + d\overline{v}) \tag{7}$$

Inserting this into our original null geodesic metric, we get:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) d\overline{u} d\overline{v} \rightarrow a) \tag{8}$$

For $r < 2GM/c^2$, $\frac{}{d}\overline{u}dr < 0$meaning all the future direction paths are in the direction of decreasing r.

KRUSKAL COORDINATES:

$$r^* = r + r_s \ln\left(\frac{r}{r_s} - 1\right) = \frac{1}{2}(\overline{u} - \overline{v}) \tag{9}$$

We rewrite the paranthesis expression in a):

$$\left(1 - \frac{r_s}{r}\right) = \frac{r_s}{r}\left(\frac{r}{r_s} - 1\right) \rightarrow \frac{r}{r_s} + \ln\left(\frac{r}{r_s} - 1\right) = \frac{1}{2r_s} \tag{10}$$

We replace $\left(\frac{}{r}r_s - 1\right)$term with exponentials:

$$ds^2 = \frac{r_s \exp\left(-\frac{r}{r_s}\right)}{r} . \exp\left(\overline{u} - \overline{v}\right) d\overline{u} d\overline{v} \tag{11}$$

Now our metric ($g_{12}component$)is singular at $r = r_s$, as $u \to \infty$and $v \to \infty$. Now to deal with these new singularities we define new coordinates:

$$V = -\exp\left(-\frac{\overline{v}}{2r_s}\right) \tag{12}$$

$$U = \exp\left(\frac{\overline{u}}{2r_s}\right) \tag{13}$$

Which gives us yet another new metric:

$$ds^2 = -\frac{4r_s^3}{r}\exp\left(-\frac{r}{r_s}\right)dUdV \tag{14}$$

Let,
$T = \frac{1}{2}(U + V) \quad X = \frac{1}{2}(U - V)$after this substitution we obtain the metric of the form:

$$\left(\frac{r}{r_s} - 1\right)\exp\left(\frac{r}{r_s}\right) = X^2 - T^2 \tag{15}$$

$$\frac{t}{r_s} = 2\tanh^{-1}\left(\frac{T}{X}\right) \tag{16}$$

REGIONS OF EXTENDED METRIC:

The extended Schwarzschild metric can be divided into 5 regions:

1. Region 1 is the original spacetime where we live, experiments happen, people do stupid stuff,etc.
2. Region 2 is $T = X$, anything which passes region 2 will fall into the singularity at r = 0.
3. Region 3 is time reversed version of region 2. Region 3 can be thought of just as the opposite of region 2. That's why it's called a *white hole*. There is no direct experiment to prove it's existence.
4. Region 4 has identical properties to our universe, i.e. it represents an asymptotically flat region which exists inside of the radius

   $r = r_s$
5. The singularity at r = 0, cannot be removed.
6. When you cross the region 1 and enter region 2, space and time axis flip their direction, making r=0 singularity your near

future. That's why you can't escape after crossing the event horizon.

7. Suppose that 1 person from our 2 hypothetical people Romeo and Juliet decides to jump in a black hole, and want to enter region 3 through region 1, they'd have to exceed the speed of light.

Now, we'll try and embed the given space in a 3d flat space. Metric in cylindrical coordinates is:

$$d\gamma^2 = dr^2 + dz^2 + rd\Phi^2 = dr^2 \left(1 + \left(\frac{dz}{dr}\right)^2\right) + r^2 d\Phi^2 \tag{17}$$

Now, we compare this to our Schwarzschild metric and we get:

$$d\gamma^2 = dr^2 \left(1 - \frac{r_s}{r}\right)^{-1} + r^2 d\Phi^2 \tag{18}$$

This means, $\left(_{\overline{d}} z dr\right)^2 = \left(1 - \frac{r_s}{r}\right)^{-1} - 1$
After integrating the non−eucledian 2d−hyperboloid is embedded in the 3d eucledian space by:

$z = \pm 2r_s\sqrt{\left(\frac{r}{r_s} - 1\right)}$ for $r > r_s$

PENSROSE DIAGRAMS:

They are way to represent the entirety of our infinite spacetime on a flat 2d “graph”.

First we write our minkowski metric in natural units:

$$ds^2 = (dt + dr)(dt - dr) - r^2 d\Omega \qquad (19)$$

The transformation that we our looking for must satisfy these 2 conditions:

1) they should preserve the light cone

2) map the infinite space to a finite 2d flat plane.

$dt \pm dr = 0$they describe the propogation of a light like “thing”.

$Y^+ = f(t + r)|\ Y^- = f(t - r)$

These conditions are satisfied by our tanh function.

$Y^+ = \tanh(t + r)|\ Y^- = \tanh(t - r)$

What we did is, we mapped the entirety of spacetime on a triangle bounded by

$Y^+ = 1|\ (t + r) \to \infty$

$$Y^{-} = -1|(t-r) \to -\infty$$

SOME EXTRA STUFF:

(in Kruskal coordinates) if Juliet decides to jump In the black hole, Romeo will see Juliet getting closer and closer to the horizon but never actually passing it, and Juliet will see Romeo accelerating away from her. Which might just break her heart, but who cares she's the one who decide to jump in a black hole with studying the physics of it. That's why always study about black holes first then jump in one(given that you can) otherwise you might just get spaghettified.

Spaghettification is a process, due to which a person or a 'thing' falling in a black hole will get stretched out as your head and feet experience, large amount of gravity.

Warmholes are a hypothetical construct which lets you pass between region 1 and 4.