TENSOR ALGEBRA

WHAT THE HECK'S A TENSOR....?

Relativity has made it's foundation on the might of Tensor calculas and differential geometry. So, that's what we are going to look into now. When you hear the word tensor the first questions that pops up into your head is, "what the heck is a tensor?", you would have heard many definitions like it's something which transforms like a tensor(meme) or it's a multi−dimensional array, etc. But, the truth is a bit of all(at least for now). For now, we can say that tensor is an "object" which obeys certain laws of transformations under a coordinate transformation. If, you are a curious person here's a heads up to you if you want to go a bit deeper into this whole tensor thing, study books or attend lectures on multi−linear algebra. I've also provided a bit deeper insite into tensors in my blog(I'll put up the link soon).

Coordinate Transformations :

$$x^i = \left(x^1, \ldots.., x^n\right) \;\; ; \quad x'^i = \left(x'^1, \ldots\ldots.., x'^n\right)$$

I've mentioned a coordinate change from

normal basis to the primed basis, but both these coordinates live in the same space, connected by 'n' equations

$L : x'^{i} = B^{i}\left(x^{1}, \ldots, x^{n}\right)$; where $B^{i}$is just a continuous, differentiable function of the un−primed coordinates.

We define the Jacobian by,

$$\mathrm{J} = \det \begin{pmatrix} \partial\varphi\hat{}1\partial x\hat{}1 & \cdots & \frac{\partial\varphi\hat{}1}{\partial x^{N}} \\ \\ \vdots & \ddots & \vdots \\ \frac{\partial\varphi\hat{}N}{\partial x\hat{}1} & \cdots & \frac{\partial\varphi\hat{}N}{\partial x\hat{}N} \end{pmatrix}$$

<u>Contravariant Tensors:</u>

These are the type of tensors which transform like a vector component under a coordinate transformation. Denoted by $\mathrm{A}^{pq\ldots n}$and they transform like :

$$A'^{ij} = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}A'^{pq} \tag{1}$$

A little exercise for the reader to prove the following equation(it's quite trivial):

$$A^{rs} = \frac{\partial x^r}{\partial x'^i}\frac{\partial x^s}{\partial x'^j}A'^{ij} \tag{2}$$

$A^{ij} \to (2,0)$. **We can generalize to (N,0) type contravariant tensor.**

$$A^{ij} = \begin{pmatrix} A^{11} & \cdots & A^{1N} \\ \vdots & \ddots & \vdots \\ A^{N1} & \cdots & A^{NN} \end{pmatrix} \tag{3}$$

COVARIENT TENSORS:

They are tensors transform like basis vectors during a coordinate transformation. Denoted by A_{ij}..

And they transform like:

$$A'_{ij} = \frac{\partial x^p}{\partial x'^i}\frac{\partial x^q}{\partial x'^j}A_{pq} \tag{4}$$

Similarly we can show that:

$$A_{rs} = \frac{\partial x'^{i}}{\partial x^{r}}\frac{\partial x'^{j}}{\partial x^{s}}A'_{ij}$$

$A_{ij} \rightarrow (0,2)$ **.We can generalize this too to get (0,N) type covariant tensor.**

$$A_{ij} = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix} \tag{5}$$

MIXED TENSORS:

These are the tensors which have contra and covariant components $\rightarrow$ (p,q) type tensor.

$$A'^{i}_{j} = \frac{\partial x'^{i}}{\partial x^{p}}\frac{\partial x'^{j}}{\partial x^{s}}A^{r}_{s} \tag{6}$$

$A^{r}_{s} = \frac{\partial x^{r}}{\partial x'^{i}}\frac{\partial x'^{j}}{\partial x^{s}}A^{i}_{j}$ **(we follow the same process here that we did before)**

$$A^i_j = \begin{pmatrix} A^1_1 & \cdots & A^1_N \\ \vdots & \ddots & \vdots \\ A^N_1 & \cdots & A^N_N \end{pmatrix} \tag{7}$$

Okay, let's take a breath here and sink it all in. Now, you must've formed picture that tensors are just arrays, but one couldn't be more wrong if they think tensors are just arrays. As an analogy, asking if tensors are just arrays is just like asking is a cute girl, girl? So, you probably have some idea about tensors by now let's learn how to do some operations on them.

<u>"ADDITION" OF TENSORS:</u>

$A'^{ij}_k + B'^{ij}_k = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\left(A^{pq}_r + B^{pq}_r\right) \quad |$ just do the coordinate change first and then, add.

Now, we define $\quad C'^{ij}_k = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}C^{pq}_r$

$$A_k^{ij} + B_k^{ij} = C_k^{ij} \tag{8}$$

This is our desired result.

<u>CONTRACTION:</u>

It is way of reducing the components of a tensor by takin out the similar ones. It is better explained by an example:

$$A'^{ij}_{kim} = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\frac{\partial x^S}{\partial x'^i}\frac{\partial x^t}{\partial x'^m}A^{pq}_{rpt} \tag{9}$$

$$= \frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\frac{\partial x^t}{\partial x'^m}A^{pq}_{rpt}$$

Now, we define a tensor B of type (1,2):

$$B'^{j}_{km} = \frac{\partial x'^j}{\partial x^p}\frac{\partial x^q}{\partial x'^k}\frac{\partial x^r}{\partial x'^m}B^p_{qr} \tag{10}$$

Now we denote $A^{pq}_{rpt}$by $B^p_{qr}$and $A'^{ij}_{kim}$by B'^{j}_{km}. We'll get the same result a tensor of type (1,2).

The tensor A'^{ij}_{kim} is called the contracted is the contracted tensor of a given tensor, this process is known as ***contraction.***

Little later we'll use this to contract tensors with a tensor called the metric tensor.

"TYPES" OF TENSORS:

1. Symmetric Tensor: $A_{ij} = A_{ji}$

2. Skew−Symmetric Tensor :

$$A_{ij}B^{jk} = \begin{cases} 1; k = j \\ 0; k \neq j \end{cases} \tag{11}$$

Can also represented by a Kronecker−delta.

3. METRIC TENSOR

Ladies and gentlemen we are now, entering into the proper GR stuff. Now, we'll be dealing with Reimannian geometry. A reimannian manifold is a smooth(

C$^\infty$) manifold with a positive definite inner product $g_p$on the tangent space $T_pM$at each point p.

A bit on metric spaces ;
<u>Metric Space</u>

<u>*CHRIS. SYMBOL OF 1ST KIND:*</u>

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{20}$$

This, object isn't really used that much.

<u>*CHRIS. SYMBOL OF 2nd KIND:*</u>

If we contract out 1$^{\mathrm{st}}$ kind Chris. Symbol by metric tensor we obtain Chris. Symbol of 2$^{\mathrm{nd}}$ kind

$$\Gamma^i_{jk} = g^{km}[ij,m] \tag{21}$$

Mathematically,

$$\Gamma^i_{jk} = \frac{g^{km}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{22}$$

Now, that's something we use a lot in our computations. It literally shows up everywhere in our calculations like ricci tensor, covariant derivative, etc.

<u>*COVARIENT DERIVATIVE:*</u>

Partial differentiation of a tensor rank $\geq$ 1, does not necessarily produces a tensor. That's why mathematicians developed the notion of covariant differentiation , which when applied to a tensor produces a tensor.

$$A_{t,s} = \frac{\partial A^i}{\partial x^j} - \Gamma^r_{st} A_r \tag{23}$$

So, it seems we take the ordinary derivative and makes some corrections to it with 2$^{\text{nd}}$ kind Chris. Symbol. So, that it returns us a tensor (specifically a (0,2) tensor).

Covarient derivative of a contravariant tensor of type let's say (0,1) returns a (1,1) type tensor.

DIVEGENCE:

- Contravarient Tensor :

$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{\lambda j} A^\lambda \tag{24}$$

Now, we contract i and j indices (we do this because we want to obtain a scalar at the end of the end):

$$div\left(A^i\right) = \nabla_i A^i = A^i_{,i} \tag{25}$$

$$\nabla_i A^i = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{\lambda i} A^\lambda \tag{26}$$

For calculation purposes I recommend the formula:

$$\nabla_i A^i = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}A^i) \tag{27}$$

- *Covarient Tensors:*

  $\nabla^i A_i = g^{jk}A_{j,k} = \left(g^{jk}A_j\right)_{,k} \mid g^{jk}_{,k} = 0$

  $A^k_{,k} = div\left(A^k\right)$

  $=> \quad \square \nabla^i A_i = \nabla_i A^i$

  This, is an important result.

LAPLACIAN :

Div(grad(φ)) = laplacian(φ)

Grad(φ) = $\varphi_{,k} \rightarrow$ (0,1) tensor

$$\nabla^2\varphi = div(\varphi_{,k}) \tag{28}$$

This following equation is left as an exercise(how original, Ik) but, it's pretty straight forward just use the formula and simplify:

$\nabla^2\varphi = \frac{1}{\sqrt{g}}\partial_m(\sqrt{g}g^{km}\frac{\partial\varphi}{\partial x^j})$

[*hint*: okay, it's not that trivial so use the 2$^{\text{nd}}$ formula I mentioned for divergence]

CURL :

Curl of a covariant tensor is a skew−symmetric tensor.

$$\nabla \times A_i = A_{i,j} - A_{j,i} \tag{29}$$

$$\nabla \times A_i = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \tag{30}$$

This one's a easy proof, just use the formulas terms cancel and you're left with this nice little expression.

PARALLEL TRANSPORT:

The little green line element, let's call it dx^mit's the amount by which we displace our contravariant vector V^nwhich lives on a curve. Let's look at the covariant Derivative of the qty $V^n dx^m$:

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$$V_{n,m}dx^m = \frac{\partial V^n}{\partial x^m}dx^m + \Gamma^n_{mr}V^r dx^m$$

$$= dV^n + \Gamma^n_{mr}V^r dx^m \tag{31}$$

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If the above mentioned qty = 0(over a round trip), then the vector is called to be parallel to itself. Now, you must be thinking that if the vector is parallel to itself dosen't that
Mean that the surface is flat, well it's technically wrong but, if you thought that, it's good sign. But, to prove a surface is flat we need more information about it than given to us just by the parallel transport, but we'll definitely get back to this topic soon enough.

REIMANN–CHRISTOFFEL CURVATURE TENSOR :

Since, the notion of just parallel transport isn't enough to say that a surface is flat or curved. These two mathematicians defined something relatively new at that time and quite elegant.

Extrinsic and Intrinsic curvature : Before we start playing around with our new tensor "toy". I must explain what I mean by curvature. From vector calc you guys will have some idea of curvature, it's something like the rate of change of tangent vector as we move it along the surface, I'll make it but clearer when we define our new tensor. So, there are two kinds of curvature :

1. Extrinsic Curvature : Take a sheet of paper right now and fold it into a cylinder, now you think it has curvature it does But, and it's a big but(that's what she said) you can still just "flatten" out to a "flat" surface. These surfaces have a special name they are called developable surface which can be flattened out to a "flat" surface. (for the mathematically inclined they are isometric

to cartesian system and they have gaussian curvature = 0, I'll introduce in this section don't worry). Your, big take a from all this that surface which can be "flattened" out is said to have zero intrinsic curvature, meaning if we look at it closely after "flattening" it, it's not curved any more.

2. Intrinsic Curvature: Well, like the name suggests they are just the opposite of extrinsic curvature, meaning surfaces which can't be "flattened" out to plane. What we deal in GR is intrinsic curvature.

REIMANN – CHRIS. TESOR:

We've defined covariant derivative of covariant tensor (0,1) by:

$$D_{i,j} = \frac{\partial D_i}{\partial x^j} - \Gamma^{\beta}_{ij} D_{\beta} \tag{32}$$

Now, we take the covariant derivative of the covariant derivative:

$$D_{(i,j),k} = \frac{\partial y}{\partial x} - \Gamma^{\beta}_{ik} D_{\beta,j} - \Gamma^{\beta}_{jk} D_{i,\beta} \tag{33}$$

Now, we substitute the value of $D_{i,j}$in the previous equation to obtain :

$$D_{(i,j),k} = \frac{\partial B_i}{\partial x^j}\frac{\partial B_i}{\partial x^k} - \frac{\partial B_\beta}{\partial x^j}\Gamma^\beta_{ik} - \frac{\partial B_\beta}{\partial x^k}\Gamma^\beta_{ij} - \Gamma^\beta_{jk}\frac{\partial B_i}{\partial x^\beta} - D \tag{34}$$

Now, we swap the variables j and k :

$$D_{(i,k),j} = \frac{\partial B_i}{\partial x^k}\frac{\partial B_i}{\partial x^j} - \frac{\partial B_\beta}{\partial x^k}\Gamma^\beta_{ij} - \frac{\partial B_\beta}{\partial x^j}\Gamma^\beta_{ik} - \Gamma^\beta_{kj}\frac{\partial B_i}{\partial x^\beta} - D \tag{35}$$

Now, we subtract these two expressions and get:

$$= -\partial_k\Gamma^\beta_{ij} + \partial_j\Gamma^\beta_{ik} + \Gamma^\beta_{\gamma j}\Gamma^\gamma_{ik} - \Gamma^\beta_{\gamma k}\Gamma^\gamma_{ij} := R^\beta_{ijk} \tag{36}$$

$$D_{(i,j),k} - D_{(i,k),j} = D_\beta R^\beta_{ijk} \tag{37}$$

Where $R^\beta_{ijk}$is our new tensor aka the reimann−chris Tensor.

So, intuitevly you can think we are taking the second covariant derivative w.r.t. keeping one component constant and interchanging the two and then, taking their difference. So, basically you're trying to get more and more information about the intrinsic geometry of the surface. I highly recommend the beginners here to go the third covariant derivative and doing the same process, just for practice and who doesn't like to play around with tensors.

RICCI TENSOR :

When we contract the two components 1 covariant and 1 contravariant, we obtain the ricci tensor. You, ask why we do this, well because we can. But, that's not all, we do this because it is used on GR. Why ricci tensor and why not the one with the big name, you ask, the reasons will become apparent when we start to derive Einstein's field equations.

$$R_{ij} = R^{\beta}_{ij\beta} \tag{38}$$

$$R_{ij} = -\partial_{\beta}\Gamma^{\beta}_{ij} + \partial_{j}\Gamma^{\beta}_{i\beta} + \Gamma^{\beta}_{\gamma j}\Gamma^{\gamma}_{i\beta} - \Gamma^{\beta}_{\gamma\beta}\Gamma^{\gamma}_{ij} \tag{39}$$

<u>PROPERTIES OF $*0R_{ij}$:$R_{ij} = R_{ji}$ $R^{i}_{j} = g^{i\beta}R_{\beta j}$ R</u>