SYMMETRIES IN GR:

We'll use a tool called killing vectors, they help us to study symmetries in manifolds. They are a pretty deep topic in differential geometry, but we'll deal with their elementary idea here.

An infinitesimal translation can be called a symmetry, if it leaves the line element unchanged:

$$\delta\left(ds^2\right) = \delta\left(g_{ab}dx^a dx^b\right) = 0 \tag{1}$$

- $\delta g_{ab}dx^a dx^b + g_{ab}\left[\delta\left(\partial x^a\right)dx^b + dx^a\delta\left(dx^b\right)\right] = 0$

Let, ξ be a tangent vectors some curve $X^a(\lambda)$meaning $\rightarrow \xi^a = \frac{dX^a}{d\lambda}$. Thence, infinitesimal translation along $\rightarrow \xi^a$is an infinitesimal translation along the curve from P to $P^{'}$. $P = \left(X^a\right) P^{'} = \left(X^a + \delta X^a\right)$

Let, P be parametrized by $P = \left(X^1, X^2\right) P^{'} = \left(X^1 + \delta X^1, X^2 + \delta X^2\right)$

- $\delta X^1 = \frac{dX^1}{d\lambda}\delta\lambda = \xi^1\delta\lambda$, $X^{a^{'}} = X^a + \xi^a\delta\lambda$

now, we expand our metric tensor and we'll consider only till first order because we are physicists(don't quote me on that).

- $g_{ab}\left(P^{'}\right) \approx g_{ab}\left(P^{'}\right) + \frac{\partial g_{ab}}{\partial\lambda}\delta\lambda + \ldots$

$$= g_{ab}\left(P'\right) + \frac{\partial g_{ab}}{\partial x^{\alpha}} \frac{dX^{\alpha}}{d\lambda} \delta\lambda + \ldots \tag{2}$$

Since, δ and d commute:

$$\delta\left(dX^{a}\right) = d\left(\delta X^{a}\right) = d\left(\overrightarrow{\xi}^{a} \delta\lambda\right) = d\overrightarrow{\xi}^{a} \delta\lambda \tag{3}$$

$\frac{\partial \overrightarrow{\xi}^{a}}{\partial X^{\alpha}} dX^{\alpha} \delta\lambda = \xi^{a}_{,\alpha} dX^{\alpha} \delta\lambda|$ where we have just used compactification notation, for convenience.

- $g_{ab,\alpha} \xi^{\alpha} \delta\lambda dX^{a} dX^{b} + g_{ab} \left[\xi^{a}_{,\alpha} dX^{a} dX^{b} + \xi^{b}_{,\gamma} dX^{\gamma} d\right.$
  0
- $\left[g_{ab,\alpha} \xi^{\alpha} + g_{\gamma b} \xi^{\gamma}_{,a} + g_{a\gamma} \xi^{\gamma}_{,b}\right] dX^{a} dX^{b} \delta\lambda =$
  0

Now, we've seen Einstein's field equations before and they are invariant under ξ iff:

$$g_{ab,\alpha} \xi^{\alpha} + g_{\gamma b} \xi^{\gamma}_{,a} + g_{a\gamma} \xi^{\gamma}_{,b} = 0 \tag{4}$$

They can be written covariantly like:

$$\xi_{a;b} + \xi_{b;a} = 0 \tag{5}$$

This is called the *killing equation.* We can use it to calculate killing vectors to any given metric, if the metric even admits one.

Lie Derivatives :

FOR MATH PROS: we define the lie brackets by:

$[u, v] = \lim_{t \to 0} \frac{\Phi_{t*} v_{\Phi_{t(p)}} - v_p}{t} = [u, v]|$ where $\Phi_{t*}$is the push forward of the flow Φ_t. u,v are vector fields and Φ is a diffeomorphism.

the *lie derivative* of vector field u w.r.t vector field v is defined by:

$$L_u v = [u, v] \tag{6}$$

sly, we can define derivative for functions on a vector field $u : C^{\infty}(M) \to C^{\infty}(M), f \to u(f)$

$$u(f) = \lim_{t \to 0} (\Phi_t^* f - f)/t \tag{7}$$

Now, let's define for the majority here: It can be thought of as the variation of a tensor/form under an infinitesimal translation along the direction of

ξ, is called the lie derivative.

$$L_\xi U_{ab} = T_{ab,\alpha}\xi^\alpha + T_{\gamma b}\xi^\gamma_{,a} + T_{a\gamma}\xi^\gamma_{,b} \tag{8}$$

For our metric:

$$L_\xi g_{ab} = g_{ab,\alpha}\xi^\alpha + g_{\gamma b}\xi^\gamma_{,a} + g_{a\gamma}\xi^\gamma_{,b} \tag{9}$$

$$= \xi_{a,b} + \xi_{b,a} \tag{10}$$

If $\rightarrow\xi$is a killing vector of our metric then, $L_{\overrightarrow{\xi}} g_{ab} = 0$.

If a solution admits a time$-$like killing vector, it is possible to choose the time component of basis vector such that it is aligned with ξ. Now, our time coordinate lines coincides with worldline to which $\rightarrow\xi$is tangent.

$$\overrightarrow{\xi}^a = (\xi^0, 0, 0, 0) \tag{11}$$

If $\xi^0 = const.$And unity:

$$\overrightarrow{\xi}^a = (1, 0, 0, 0) \tag{12}$$

$$if, \frac{\partial g_{ab}}{\partial x^0} = 0 \tag{13}$$

This means that our metric accepts a time−like killing vector which can be used to make the metric time−independent.
Sly, for the existance of a space−like killing vector $\frac{}{\partial}g_{ab}\partial x^i = 0$.
The map which leaves the metric unchanged is referred as an *isometry.*
Ex: killing vector of flat space−time:
In cartesian coordinates: $\rightarrow\xi_{a,b} + \overrightarrow{\xi}_{b,a} = 0$
All the Christoffel symbols vanish in cartesian coordinate system.

$$\xi_{a,\beta\gamma} = 0 \tag{14}$$

This differential equation has a general solution which looks like:

$$\xi_a = C_a + \epsilon_{a\gamma}X^{\gamma} \rightarrow A) \tag{15}$$

$$bysubbingthisintoourdifferentialquation : \epsilon_{a\gamma}X^{\gamma}_{\beta} \tag{16}$$

This equation is satisfied if:

$$\epsilon_{a\beta} = -\epsilon_{\beta\alpha} \tag{17}$$

The general killing vector field from A), can be written as a linear combination of ten killing vectors
ξ_a^A, A = 1,2,. . .,10.
$\xi_a^A = C_a^A + \epsilon_{a\gamma}^A X^\gamma|$ again A goes from 1 to 10.
Here we'll choose them as :

$$C_a^1 = (1,0,0,0)\,, C_a^2 = (0,1,0,0)\,, C_a^3 = (0,0,1,0)\,, C \tag{18}$$

Other C's from 5 through 10 have the value of 0.

$$\epsilon_{a\beta}^5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \epsilon_{a\beta}^6 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

$$
\epsilon^{7}_{a\beta} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \epsilon^{8}_{a\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{20}
$$

$$
\epsilon^{9}_{a\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \epsilon^{10}_{a\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{21}
$$

The other ϵ's 1 through 5 equal zero.

CONSERVED QTY. IN GEODESICS:

$$
\frac{dV^a}{d\tau} +^{a}_{\beta\gamma} V^{\beta} V^{\gamma} = 0 \tag{22}
$$

$$
\xi_a \left[\frac{dV^a}{d\tau} +^{a}_{\beta\gamma} V^{\beta} V^{\gamma} \right] = \frac{d(\xi_a V^a)}{d\tau} - V^a \frac{d\xi_a}{d\tau} +^{a}_{\beta\gamma} V^{\beta} V^{\gamma} \tag{23}
$$

$$= \frac{d(\xi_a V^a)}{d\tau} - V^\beta V^\gamma \left[\frac{\partial \xi_\beta}{\partial x^y} -^{a}_{\beta y} \xi_a\right] = 0 \quad (24)$$

$$=> \frac{d(\xi_a V^a)}{d\tau} - V^\beta V^\gamma \left(\xi_{\beta,\gamma}\right) = 0 \quad (25)$$

Now, we know that $V^\beta V^\gamma$is anti−symmetric while $\xi_{\beta,\gamma}$is symmetric:
$=> V^\beta V^\gamma \left(\xi_{\beta,\gamma}\right) = 0$
This means: $\xi_a V^a = const.$
Which is basically telling us that there exist a conserved qty. for each killing vector. We can also write this as:

$$g_{a\mu}\xi^\mu V^a = const. \quad (26)$$

If the metric is asymptotically flat, $g_{\mu\nu}$will become $\eta_{\mu\nu}$at infinity

$$\eta_{00}V^0 = const. => V^0 = const.whichmeans\frac{E}{c} = \quad (27)$$

If there are space−like killing vectors:

$$g_{a1}\xi^1 V^a = const. => g_{a1}V^1 = const. \tag{28}$$

$$p^a = mcV^a => \frac{p^1}{mc} = const. \tag{29}$$