SCHWARZSCHILD GEOMETRY:

Now, we actually start to use the tools we gathered by doing tensor analysis in GR to get some practical results. To do anything we must know the metric of that space or surface here we derive a spherically symmetric time $-$independent metric, which is also know as the Schwarzschild metric and know a bit about it's singularity, which corresponds to a black hole. NOTE: Schwarzschild is metric is the generic metric to a spherically symmetric body, black hole's a special, which we'll discus in brief a bit later.

We first start by writing a general expression of a $4-$D metric:

$$ds^2 = A(t,r)\,dt^2 - B(t,r)\,dtv.dv - C(t,r)\,(v.dv)^2 - D(t,r)\,dv^2 \tag{1}$$

Where I've put v instead of our x,y and z $3-$D cartesian coordinates.

Now, we do a coordinate change to spherical coordinates for obvious reasons:

$$x^1 = r\sin\theta\cos\Phi \qquad x^2 = r\sin\theta\sin\Phi$$

$$x^3 = r\cos\theta \tag{2}$$

I'm going to derive the metric a bit differently here, but I'll hint the way which is used in

other GR. I use this method because it's a bit shorter and easy to do.
I'll use the formula $g = (J\varphi)\,(J\varphi)^T$

$$J\varphi = \begin{pmatrix} \sin\Phi\cos\theta & r\cos\Phi\cos\Phi & -r\sin\Phi\sin\theta \\ \sin\Phi\sin\theta & r\cos\Phi\sin\theta & r\sin\Phi\cos\theta \\ \cos\Phi & -r\cos\Phi & 0 \end{pmatrix} \tag{3}$$

$$(J\varphi)^T = \begin{pmatrix} \sin\Phi\cos\theta & \sin\Phi\sin\theta & \cos\Phi \\ r\cos\Phi\cos\Phi & r\cos\Phi\sin\theta & -r\cos\Phi \\ -r\sin\Phi\sin\theta & r\sin\Phi\cos\theta & 0 \end{pmatrix} \tag{4}$$

Now, $g = (J\varphi)\,(J\varphi)^T$

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{5}$$

We get,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\Phi^2 \tag{6}$$

Now, you can perform the same calculation in 4D with extra dimension of time(t) and you'll obtain the following equation:

$$ds^2 = A\,(t,r)\,dt^2 - B\,(t,r)\,dr^2 - r^2(d\theta^2 + \sin^2\theta d\Phi^2) \tag{7}$$

Where A and B are the constants which we need to find in order to convert this generic form of metric into our Schwarzschild metric. Now, we add a bit of physics to our process, by considering the empty space solution for Einstein's field equation:

$$R_{\mu\nu} = 0 \tag{8}$$

You can refer to the Ricci tensor formula we derived earlier.
Now,

$$g_{00} = A(r)$$

$$g_{11} = -B(r) \tag{9}$$

$$g_{22} = -r^2 \tag{10}$$

$$g_{33} = -r^2 \sin^2 \theta \tag{11}$$

$$g^{00} = \frac{1}{A(r)} \tag{12}$$

$$g^{00} = \frac{1}{A(r)} \tag{13}$$

$$g^{11} = \frac{-1}{B(r)} \tag{14}$$

$$g^{22} = \frac{-1}{r^2} \tag{15}$$

$$g^{33} = \frac{-1}{r^2 \sin^2 \theta} \tag{16}$$

Now, we compute Christoffel symbols of 2$^{\text{nd}}$ kind:

$$\Gamma^0_{00} = 0 \tag{17}$$

$$\Gamma^0_{0i} = \frac{1}{2} g^{0\nu} \left(\partial_i g_{\nu 0} + \partial_0 g_{\nu i} - \partial_\nu g_{0i} \right) = \frac{1}{2} g^{00} \partial_i g_{00} \tag{18}$$

$$\Gamma^0_{0j} = 0 \tag{19}$$

$$\Gamma^i_{ii} = \frac{1}{2} g^{ii} \partial_i g_{ii} \tag{20}$$

This, is a generic solution for a metric with similar non−vanishing components.

$$\Gamma^1_{00} = \frac{1}{2B(r)} \frac{dA(r)}{dr} \Gamma^0_{01} = \frac{1}{2A(r)} \frac{dA(r)}{dr} \tag{21}$$

$$\Gamma^0_{01} = \frac{1}{2A(r)} \frac{dA(r)}{dr} \quad \Gamma^1_{11} = \frac{1}{2B(r)} \frac{dB(r)}{dr}$$

$$\Gamma^1_{22} = -\frac{r}{B(r)} (22) \quad \Gamma^1_{33} = \frac{-r \sin^2 \theta}{B(r)}$$

$$\Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta \cos\theta,\ \Gamma^3_{31} = \frac{1}{r},$$

$$\Gamma^3_{32} = \cot\theta$$

These are all the non−vanishing Christoffel symbols of our metric.

Now, let's calculate our Ricci tensor (which is just the derivatives and products of our Christoffel symbols) in flat space:

$$R_{00} = -\frac{A^{''}}{2B} + \frac{A^{'}}{4B}\left(\frac{A^{'}}{A} + \frac{B^{'}}{B}\right) - \frac{A^{'}}{rB} = 0 \ \rightarrow 1)$$

$$R_{11} = \frac{A^{''}}{2A} - \frac{A^{'}}{4A}\left(\frac{A^{'}}{A} + \frac{B^{'}}{B}\right) - \frac{B^{'}}{rB} = 0 \ \rightarrow 2)$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A^{'}}{A} - \frac{B^{'}}{B}\right) = 0 \ \rightarrow 3)$$

$$R_{33} = R_{22}\sin^2\theta = 0 \quad \rightarrow 4)$$

We multiply equation 1) by B/A and add it to equation 2) to get:

$$A^{'}B + B^{'}A = 0 \tag{23}$$

- $AB = Const.$
- $AB = \lambda$
- $B = \frac{A}{\lambda}$

  Substitute this into equation 3):

  $\frac{d(rA)}{dr} = \lambda$

  $A\left(r\right) = \lambda\left(1 + \frac{k}{r}\right),\ B\left(r\right) = \left(1 + \frac{k}{r}\right)^{-1}$

Now, we use the weak field approximation. Our A(r) term here under weak field approximation, will be approximately equal to our Newtonian potential i.e.

$\frac{A(r)}{c^2} \rightarrow 1 + \frac{2\Phi}{c^2}$
Where, $\Phi = -\frac{GM}{r}$,
$k = -\frac{2GM}{c^2}$,
...Finally, we can write our long time waited Schwarzschild metric:

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 \tag{24}$$

This is one of the most beautiful equations in physics, because it defines not only spherically symmetric bodies in GR but, also predicts the existance of a new object called the black hole, which we'll explore in a bit. Notice that our metric has singularity at
r = 2GM/c^{2}which we'll explore deeply in the black holes chapter.

GEODESICS IN SCHAWRZSCHILD METRIC:

There are two methods I recommend to calculate geodesics of any metric:
1)killing vector method
2)The Lagrangian method
! These are informal names. My personal

favourite is the first one because it uses killing vectors and they paint a better picture in one's mind(at least in mine). That's why I'll use the first method and tell you how to proceed with the second one, so that you can try both and decide what works for you.

1)killing vector method:

First we calculate the Christoffel symbols, because they are use in the geodesic formula, this might be a good time to revisit it:

$\Gamma^1_{00} = \frac{GM}{r^3}(r - 2GM)$, $\Gamma^1_{11} = -\frac{GM}{r(r-2GM)}$

$\Gamma^0_{01} = \frac{GM}{r(r-2GM)}$, $\Gamma^1_{33} = -r\,(r - 2GM)\sin^2\theta$

$\Gamma^2_{12} = \frac{1}{r}$, $\Gamma^1_{22} = -r(r - 2GM)$

$\Gamma^3_{13} = \frac{1}{r}$, $\Gamma^2_{33} = -\sin\theta\cos\theta$

$\Gamma^3_{23} = \cot\theta$

Now substituting them into our geodesic equation we get a set of 4 ODEs, one for each coordinate.

$$\frac{d^2t}{d\beta^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\beta}\frac{dt}{d\beta} = 0 \tag{25}$$

$$\frac{d^2\theta}{d\beta^2}+\frac{2}{r}\frac{d\theta}{d\beta}\frac{dr}{d\beta}-\sin^2\theta\cos\theta\left(\frac{d\Phi}{d\beta}\right)^2=0 \tag{26}$$

$$\frac{d^2r}{d\beta^2}+\frac{GM}{r^3}\left(r-2GM\right)\left(\frac{dt}{d\beta}\right)^2-\frac{GM}{r\left(r-2GM\right)}\left(\frac{dr}{d\beta}\right. \tag{27}$$

$$\frac{d^2\Phi}{d\beta^2}+\frac{2}{r}\frac{d\Phi}{d\beta}\frac{dr}{d\beta}+2\cot\theta\frac{d\theta}{d\beta}\frac{d\Phi}{d\beta}=0 \tag{28}$$

Now, we apply the knowledge that we attained in the killing vectors chapter(might be a good time to read it, if you didn't):

$$K_\mu\frac{dx^\mu}{d\beta}=const. \tag{29}$$

$\varepsilon=-g_{\mu\nu}\frac{dx^\mu}{d\beta}\frac{dx^\nu}{d\beta}$(our geodesic const.)