TENSOR ALGEBRA

<u>**WHAT THE HECK'S A TENSOR….?**</u>

Relativity has made its foundation on the might of Tensor calculas and differential geometry.

So, that's what we are going to look into now. When you hear the word tensor the first questions that pops up into your head is, "what the heck is a tensor?", you would have heard many definitions like it's something which transforms like a tensor(meme) or it's a multi-dimensional array, etc. But, the truth is a bit of all (at least for now). For now, we can say that tensor is an "object" which obeys certain laws of transformations under a coordinate transformation. If, you are a curious person here's a heads up to you if you want to go a bit deeper into this whole tensor thing, study books or attend lectures on multi-linear algebra.

I’ve also provided a bit deeper insight into tensors in my blog(I’ll put up the link soon).

<u>Coordinate Transformations</u>**:**

$$x^i = (x^1, \ldots\ldots, x^n) \;\; ; \quad x'^i = (x'^1, \ldots\ldots\ldots, x'^n)$$

I’ve mentioned a coordinate change from normal basis to the primed basis, but both these coordinates live in the same space, connected by ‘n’ equations

$L : x'^i = B^i(x^1, \ldots, x^n)$; where B^i is just a continuous, differentiable function of the un-primed coordinates.

We define the Jacobian by,

$$J = \det \begin{pmatrix} \frac{\partial\varphi\text{^}1}{\partial x\text{^}1} & \cdots & \frac{\partial\varphi\text{^}1}{\partial x^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial\varphi\text{^}N}{\partial x\text{^}1} & \cdots & \frac{\partial\varphi\text{^}N}{\partial x\text{^}N} \end{pmatrix}$$

Contravariant Tensors:

These are the type of tensors which transform like a vector component under a coordinate transformation. Denoted by $A^{pq\dots n}$ and they transform like:

$$A'^{ij} = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}A'^{pq}$$

A little exercise for the reader to prove the following equation (it's quite trivial):

$$A^{rs} = \frac{\partial x^r}{\partial x'^i}\frac{\partial x^s}{\partial x'^j}A'^{ij}$$

$A^{ij} \rightarrow (2{,}0)$. We can generalize to (N,0) type contravariant tensor.

$$A^{ij} = \begin{pmatrix} A^{11} & \cdots & A^{1N} \\ \vdots & \ddots & \vdots \\ A^{N1} & \cdots & A^{NN} \end{pmatrix}$$

COVARIENT TENSORS:

They are tensors transform like basis vectors during a coordinate transformation. Denoted by A_{ij}..

And they transform like:

$$A'_{ij} = \frac{\partial x^p}{\partial {x'}^i}\frac{\partial x^q}{\partial {x'}^j}A_{pq}$$

Similarly, we can show that:

$$A_{rs} = \frac{\partial {x'}^i}{\partial x^r}\frac{\partial {x'}^j}{\partial x^s}A'_{ij}$$

$A_{ij} \to (0,2)$. We can generalize this too to get (0, N) type covariant tensor.

$$A_{ij} = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix}$$

MIXED TENSORS:

These are the tensors which have contra and covariant components → (p,q) type tensor.

$$A'^{i}_{j} = \frac{\partial x'^{i}}{\partial x^{p}} \frac{\partial x'^{j}}{\partial x^{s}} A^{r}_{s}$$

$A^{r}_{s} = \frac{\partial x^{r}}{\partial x'^{i}} \frac{\partial x'^{j}}{\partial x^{s}} A^{i}_{j}$ (we follow the same process here that we did before)

$$A^{i}_{j} = \begin{pmatrix} A^{1}_{1} & \cdots & A^{1}_{N} \\ \vdots & \ddots & \vdots \\ A^{N}_{1} & \cdots & A^{N}_{N} \end{pmatrix}$$

Okay, let's take a breath here and sink it all in. Now, you must've formed picture that tensors are just arrays, but one couldn't be more wrong if they think tensors, are just arrays. As an analogy, asking if tensors are just arrays is just like asking is a cute girl, girl? So, you probably have some idea about tensors by now let's learn how to do some operations on them.

<u>"ADDITION" OF TENSORS:</u>

$A'^{ij}_{k} + B'^{ij}_{k} = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\left(A^{pq}_{r} + B^{pq}_{r}\right)$ | just do the coordinate change first and then, add.

Now, we define $\quad C'^{ij}_{k} = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}C^{pq}_{r}$

$$A^{ij}_{k} + B^{ij}_{k} = C^{ij}_{k}$$

This is our desired result.

<u>CONTRACTION:</u>

It is way of reducing the components of a tensor by takin out the similar ones.

It is better explained by an example:

$$A'^{ij}_{kim} = \frac{\partial x'^i}{\partial x^p}\frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\frac{\partial x^S}{\partial x'^i}\frac{\partial x^t}{\partial x'^m}A^{pq}_{rpt}$$

$$= \frac{\partial x'^j}{\partial x^q}\frac{\partial x^r}{\partial x'^k}\frac{\partial x^t}{\partial x'^m}A^{pq}_{rpt}$$

Now, we define a tensor B of type (1,2):

$$B'^{j}_{km} = \frac{\partial x'^j}{\partial x^p}\frac{\partial x^q}{\partial x'^k}\frac{\partial x^r}{\partial x'^m}B^{p}_{qr}$$

Now we denote A^{pq}_{rpt} by B^{p}_{qr} and A'^{ij}_{kim} by B'^{j}_{km}. We'll get the same result a tensor of type (1,2).

The tensor $A'^{ij}_{kim}$is called the contracted is the contracted tensor of a given tensor, this process is known as *contraction*.

Little later we'll use this to contract tensors with a tensor called the metric tensor.

TYPES OF TENSORS:

1) Symmetric Tensor: $A_{ij} = A_{ji}$

2) Skew-Symmetric Tensor: $A_{\mathrm{ij}} = -A_{ji}$

3) Reciprocal Tensor: $B^{\mathrm{ij}} = \frac{cofactor\ of\ A_{ij}\ in\ |A_{ij}|}{|A_{ij}|}$

$$A_{ij}B^{jk} = \begin{cases} 1; k = j \\ 0; k \neq j \end{cases}$$

Can also represented by a Kronecker-delta.

METRIC TENSOR

Ladies and gentlemen, we are now, entering into the proper GR stuff. Now, we'll be dealing with Riemannian geometry. A Riemannian manifold is a smooth(C^∞) manifold with a positive definite inner product $g_p$on the tangent space T_pM at each point p. A bit on metric spaces ;

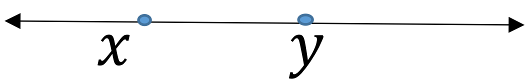
Metric Space (M, τ_M, g) : Where M is a set and $\tau_M$is a topology on M and g is a symmetric bilinear form (called as the metric):

$g(x, y) \to R$; where R is the real no. space.

$g(x, y)$ can be interpreted as distance between x and y

Positive definite meaning $g(x, y) \geq 0$ $and\ is$ 0 $iff\ x = y$.

Example: (R, $\tau_{std.}$, g) it's the real no. line equipped with standard topology then, metric is defined on it as



$$g(x, y) := |y - x|$$

In multi-calc you guys learnt that a line element ds can be represented by:

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2$$

And then you "divided" it by dt (for some parameter t), to use it in line integrals and stuff. In, a similar fashion when in Riemannian geometry we write line element like:

$ds^2 = g_{ij}dx^i dx^j$ | i, j = 1,2,,……..,n : n being the dimension of the manifold.

Where g_{ij} is our long time waited…you guessed it *metric tensor*.

Intuitively you can think metric tensor as a factor which acts when coordinate system is changed, so that our ds act as an invariant under the transformation.

Mathematically : G(e) =$(J\varphi)^T$ $(J\varphi)$

Where $(J\varphi)$ is our good ol' Jacobian matrix (it seems as if our intuition was not that far off)

$$J\varphi = \begin{pmatrix} \frac{\partial \varphi^1}{\partial x^1} & \cdots & \frac{\partial \varphi^1}{\partial x^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi^N}{\partial x^1} & \cdots & \frac{\partial \varphi^N}{\partial x^N} \end{pmatrix}$$ | Det(g) = g | just a notational step

$$g^{ij} = \frac{cofactor\ of\ g_{ij} in\ g}{g}$$

Length of a curve:

Again, keep multi-variable calc. methods we define it by:

$$L = \int_{t_1}^{t_2} dt \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}$$

You can clearly see resemblance with multi-calc here, if you can't what are you doing here, go revise multi-calc through your multi-calc notes.

CONTRACTION:

Déjà vu huh, well now we do it with our new friend metric tensor:

$$A_i = g_{ij}A^j$$

$$B^j = g_{ij}B_i$$

These are also called associated-tensors. But we won't use that terminology here.

ORTHAGONAITY CONDITION:

$$g_{ij}A^iB^j = 0$$

And guess what this expression means:

$A = \sqrt{g_{ij}A^iA^j}$ | yup, you guessed it, it's the magnitude of a given vector

Unit vectors can be shown as:

$$g_{ij}A^iA^j = 1$$

__ANGLE BETWEEN VECTORS:__

A^i and B^j are two non-null contravariant tensor and the angle between them Φ can be calculated by:

$$\Phi = \cos^{-1}\frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}}$$

Now, what we have done up until now, is that we have moved from our classical Euclidian(cartessian) geometry to our new Riemannian geometry by changing things accordingly. We are dealing with arbitrary finite no. of dimensions right now, when we start GR we'll confine ourselves to 4 dimensions. This, is a more general picture that we have painted here.

CHRISTOFFEL SYMBOLS:

The shit's getting real now. Before, you even ask Christoffel symbols are NOT tensors.

They are a function from a given jet bundle to R. But you don't have to worry about this definition, at least not at this stage. Just, think of them as the derivates of our metric tensor. They describe a metric connection and it is a special kind of the affine connection to surfaces or other manifolds endowed with a metric, allowing distances to be measured on that surface. Affine connection is a geometric object on a smooth manifold which connects nearby tangent spaces.

If you didn't understand a word up there, think of Christoffel symbols as a function which helps you make better calculation on surface of manifolds involving curvatures, distances, etc.

CHRIS. SYMBOL OF 1ST KIND:

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

This, object isn't really used that much.

CHRIS. SYMBOL OF 2nd KIND:

If we contract out 1st kind Chris. Symbol by metric tensor we obtain Chris. Symbol of 2nd kind

$$\Gamma^i_{jk} = g^{km}[ij,m]$$

Mathematically,

$$\Gamma^{i}_{jk} = \frac{g^{km}}{2}\left(\frac{\partial g_{ik}}{\partial x^{j}} + \frac{\partial g_{jk}}{\partial x^{i}} - \frac{\partial g_{ij}}{\partial x^{k}}\right)$$

Now, that's something we use a lot in our computations. It literally shows up everywhere in our calculations like Ricci tensor, covariant derivative, etc.

COVARIENT DERIVATIVE:

Partial differentiation of a tensor rank ≥ 1, does not necessarily produces a tensor. That's why mathematicians developed the notion of covariant differentiation, which when applied to a tensor produces a tensor.

$$A_{t,s} = \frac{\partial A^{i}}{\partial x^{j}} - \Gamma^{r}_{st} A_{r}$$

So, it seems we take the ordinary derivative and makes some corrections to it with 2nd kind Chris. Symbol. So, that it returns us a tensor (specifically a (0,2) tensor).

Covariant derivative of a contravariant tensor of type let's say (0,1) returns a (1,1) type tensor.

DIVEGENCE:

- Contravariant Tensor :

$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + \Gamma^i_{\lambda j} A^\lambda$$

Now, we contract i and j indices (we do this because we want to obtain a scalar at the end of the end):

$$div(A^i) = \nabla_i A^i = A^i_{,i}$$

$$\nabla_i A^i = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{\lambda i} A^\lambda$$

For calculation purposes I recommend the formula:

$$\nabla_i A^i = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} A^i)$$

- *Covariant Tensors:*

$$\nabla^i A_i = g^{jk} A_{j,k} = \left(g^{jk} A_j\right)_{,k} \ \mid\ g^{jk}_{,k} = 0$$

$$A^k_{,k} = div\left(A^k\right)$$

=> $\boxed{\nabla^i A_i = \nabla_i A^i}$

This, is an important result.

LAPLACIAN :

Div(grad(φ)) = Laplacian(φ)

Grad(φ) = $\varphi_{,k} \rightarrow$ (0,1) tensor

$$\nabla^2 \varphi = \mathrm{div}(\varphi_{,k})$$

This following equation is left as an exercise(how original, Ik) but, it's pretty straight forward just use the formula and simplify:

$$\nabla^2 \varphi = \frac{1}{\sqrt{g}} \partial_m \left(\sqrt{g}\, g^{km} \frac{\partial \varphi}{\partial x^j} \right)$$

[*hint*: okay, it's not that trivial so use the 2nd formula I mentioned for divergence]

<u>*CURL :*</u>

Curl of a covariant tensor is a skew-symmetric tensor.

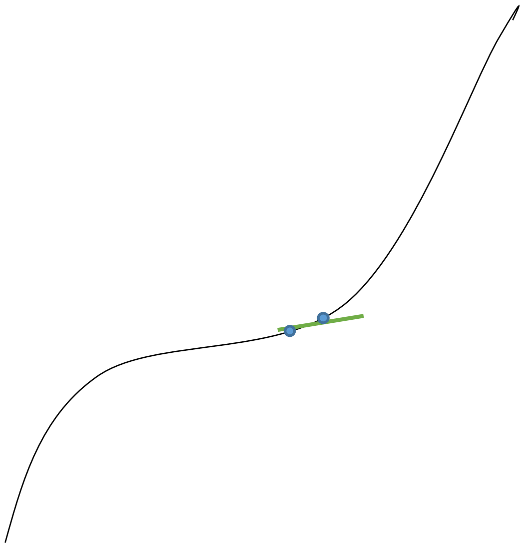
$$\nabla \times A_i = A_{i,j} - A_{j,i}$$

$$\nabla \times A_i = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

This one's an easy proof, just use the formulas terms cancel and you're left with this nice little expression.

PARALLEL TRANSPORT:

The little green line element, let’s call it dx^m it’s the amount by which we displace our contravariant vector V^nwhich lives on a curve. Let’s look at the covariant Derivative of the qty $V^n\, dx^m$:



$$V_{n,m}dx^m = \frac{\partial V^n}{\partial x^m}dx^m + \Gamma^n_{mr}V^r dx^m$$
$$= dV^n + \Gamma^n_{mr}V^r dx^m$$

If the above mentioned qty = 0(over a round trip), then the vector is called to be parallel to itself. Now, you must be thinking that if the vector is parallel to itself doesn’t that

Mean that the surface is flat, well it's technically wrong but, if you thought that, it's good sign. But, to prove a surface is flat we need more information about it than given to us just by the parallel transport, but we'll definitely get back to this topic soon enough.

REIMANN-CHRISTOFFEL CURVATURE TENSOR :

Since, the notion of just parallel transport isn't enough to say that a surface is flat or curved. These two mathematicians defined something relatively new at that time and quite elegant.

Extrinsic and Intrinsic curvature : Before we start playing around with our new tensor "toy". I must explain what I mean by curvature. From vector calc you guys will have some idea of curvature, it's something like the rate of change of tangent vector as we move it along the surface, I'll make it but clearer when we define our new tensor. So, there are two kinds of curvature :

1. <u>Extrinsic Curvature :</u> Take a sheet of paper right now and fold it into a cylinder, now you think it has curvature it does But, and it's a big but(that's what she said) you can still just "flatten" out to a "flat" surface. These surfaces have a special name they are called developable surface which can be flattened out to a "flat" surface. (for the mathematically inclined they are isometric to cartesian system and they have gaussian curvature = 0, I'll introduce in this section don't worry). Your, big take a from all this that surface which can be "flattened" out is said to have zero intrinsic curvature, meaning if we look at it closely after "flattening" it, it's not curved any more.
2. <u>Intrinsic Curvature:</u> Well, like the name suggests they are just the opposite of extrinsic curvature, meaning surfaces which can't be "flattened" out to plane. What we deal in GR is intrinsic curvature.

<u>*REIMANN – CHRIS. TESOR:*</u>

We’ve defined covariant derivative of covariant tensor (0,1) by:

$$D_{i,j} = \frac{\partial D_i}{\partial x^j} - \Gamma_{ij}^{\beta} D_\beta$$

Now, we take the covariant derivative of the covariant derivative:

$$D_{(i,j),k} = \frac{\partial y}{\partial x} - \Gamma_{ik}^{\beta} D_{\beta,j} - \Gamma_{jk}^{\beta} D_{i,\beta}$$

Now, we substitute the value of $D_{i,j}$in the previous equation to obtain :

$$D_{(i,j),k} = \frac{\partial B_i}{\partial x^j}\frac{\partial B_i}{\partial x^k} - \frac{\partial B_\beta}{\partial x^j}\Gamma_{ik}^{\beta} - \frac{\partial B_\beta}{\partial x^k}\Gamma_{ij}^{\beta} - \Gamma_{jk}^{\beta}\frac{\partial B_i}{\partial x^\beta} - D_\beta \frac{\partial}{\partial x^k}\Gamma_{ij}^{\beta} + D_\beta \Gamma_{\gamma j}^{\beta}\Gamma_{ik}^{\gamma}$$
$$+ D_\beta \Gamma_{i\gamma}^{\beta}\Gamma_{jk}^{\gamma}$$

Now, we swap the variables j and k :

$$D_{(i,k),j} = \frac{\partial B_i}{\partial x^k}\frac{\partial B_i}{\partial x^j} - \frac{\partial B_\beta}{\partial x^k}\Gamma^\beta_{ij} - \frac{\partial B_\beta}{\partial x^j}\Gamma^\beta_{ik} - \Gamma^\beta_{kj}\frac{\partial B_i}{\partial x^\beta} - D_\beta \frac{\partial}{\partial x^j}\Gamma^\beta_{ik}$$
$$+ D_\beta \Gamma^\beta_{\gamma k}\Gamma^\gamma_{ij} + D_\beta \Gamma^\beta_{i\gamma}\Gamma^\gamma_{kj}$$

Now, we subtract these two expressions and get:

$$= -\,\partial_k \Gamma^\beta_{ij} + \partial_j \Gamma^\beta_{ik} + \Gamma^\beta_{\gamma j}\Gamma^\gamma_{ik} - \Gamma^\beta_{\gamma k}\Gamma^\gamma_{ij} := R^\beta_{ijk}$$

$$D_{(i,j),k} - D_{(i,k),j} = D_\beta R^\beta_{ijk}$$

Where R^β_{ijk} is our new tensor aka the reimann-chris Tensor.

So, intuitively you can think we are taking the second covariant derivative w.r.t. keeping one component constant and interchanging the two and then, taking their difference. So, basically, you’re trying to get more and more information about the intrinsic geometry of the surface. I highly recommend the beginners here to go the third covariant derivative and doing the same process, just for practice and who doesn't like to play around with tensors.

RICCI TENSOR :

When we contract the two components 1 covariant and 1 contravariant, we obtain the ricci tensor. You, ask why we do this, well because we can. But, that's not all, we do this because it is used on GR. Why ricci tensor and why not the one with the big name, you ask, the reasons will become apparent when we start to derive Einstein's field equations.

$$R_{ij} = R^{\beta}_{ij\beta}$$

$$R_{ij} = -\,\partial_\beta \Gamma^{\beta}_{ij} + \partial_j \Gamma^{\beta}_{i\beta} + \Gamma^{\beta}_{\gamma j}\Gamma^{\gamma}_{i\beta} - \Gamma^{\beta}_{\gamma\beta}\Gamma^{\gamma}_{ij}$$

PROPERTIES OF R_{ij}:

- $R_{ij} = R_{ji}$

○ $R_j^i = g^{i\beta} R_{\beta j}$

○ $R := g^{j\beta} R_{\beta j} = R_j^j$ | it is called the Ricci scalar.

GAUSSIAN CURVATURE(k):

$$R_{1212} = g_{1\beta} R^{\beta}_{212}$$

$$k = \frac{R_{1212}}{g}$$

Now, I encourage you guys to go back and read the extrinsic and intrinsic curvature part.

GEODESICS :

If you're coming from a variational calculus background. You write a functional integral and you maximize it, like you write an action in lagrangian mechanics. You find the differential equation of the path which gives us the shortest path (well, because nature is lazy).In GR freely falling observers are inertial observers, the path of their trajectory is referred to as geodesics.

$$S = \int_{t_1}^{t_2} dt \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}$$

$$L = \sqrt{g_{ij} dx^i dx^j}$$

Now, we use our good ol' euler-lagrange equation :

$$\frac{\partial L}{\partial \dot{x}^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) = 0$$

With a couple of steps of calculations, we obtain:

$$\boxed{\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0}$$

We’ll use this equation to calculate the trajectory of inertial observers, mostly throughout the book.

EINSTEIN TENSOR:

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$$

We’ll encounter this tensor, a bit later in the book for now just keep it in the back of your mind.

SYMMETRIES IN GR:

We’ll use a tool called killing vectors, they help us to study symmetries in manifolds. They are a pretty deep topic in differential geometry, but we’ll deal with their elementary idea here.

An infinitesimal translation can be called a symmetry, if it leaves the line element unchanged:

$$\delta(ds^2) = \delta\left(g_{ab}dx^a dx^b\right) = 0$$

⇨ $\delta g_{ab}dx^a dx^b + g_{ab}\left[\delta(\partial x^a)dx^b + dx^a\delta\left(dx^b\right)\right] = 0$

Let, ξ be a tangent vectors some curve $X^a(\lambda)$ meaning $\overrightarrow{\xi^a} = \frac{dX^a}{d\lambda}$. Thence, infinitesimal translation along $\overrightarrow{\xi^a}$ is an infinitesimal translation along the curve from P to P'. $P = (X^a) \quad P' = (X^a + \delta X^a)$

Let, P be parametrized by $P = (X^1, X^2)$ $P' = (X^1 + \delta X^1, X^2 + \delta X^2)$

⇨ $\delta X^1 = \frac{dX^1}{d\lambda}\delta\lambda = \xi^1\delta\lambda,\ X^{a\prime} = X^a + \xi^a\delta\lambda$

now, we expand our metric tensor and we'll consider only till first order because we are physicists(don't quote me on that).

⇨ $g_{ab}(P') \approx g_{ab}(P') + \frac{\partial g_{ab}}{\partial\lambda}\delta\lambda + \cdots$

$$= g_{ab}(P') + \frac{\partial g_{ab}}{\partial x^\alpha}\frac{dX^\alpha}{d\lambda}\delta\lambda + \cdots$$

Since, δ and d commute:

$$\delta(dX^a) = d(\delta X^a) = d\left(\vec{\xi}^a \delta\lambda\right) = d\vec{\xi}^a\ \delta\lambda$$

$\frac{\partial \vec{\xi}^a}{\partial X^\alpha} dX^\alpha \delta\lambda = \xi^a_{,\alpha} dX^\alpha \delta\lambda$ | where we have just used compactification notation, for convenience.

$\Rightarrow \quad g_{ab,\alpha}\xi^\alpha \delta\lambda dX^a dX^b + g_{ab}\left[\xi^a_{,\alpha} dX^a dX^b + \xi^b_{,\gamma} dX^\gamma dX^a\right] = 0$

$\Rightarrow \quad \left[g_{ab,\alpha}\xi^\alpha + g_{\gamma b}\xi^\gamma_{,a} + g_{a\gamma}\xi^\gamma_{,b}\right] dX^a dX^b \delta\lambda = 0$

Now, we've seen Einstein's field equations before and they are invariant under ξ iff:

$$g_{ab,\alpha}\xi^\alpha + g_{\gamma b}\xi^\gamma_{,a} + g_{a\gamma}\xi^\gamma_{,b} = 0$$

They can be written covariantly like:

$$\xi_{a;b} + \xi_{b;a} = 0$$

This is called the *killing equation.* We can use it to calculate killing vectors to any given metric, if the metric even admits one.

Lie Derivatives :

FOR MATH PROS: we define the lie brackets by:

$[u, v] = \lim_{t \to 0} \frac{\Phi_{t*} v_{\Phi_{t(p)}} - v_p}{t} = [u, v]$ | where $\Phi_{t*}$is the push forward of the flow Φ_t. u,v are vector fields and Φ is a diffeomorphism.

the *lie derivative* of vector field u w.r.t vector field v is defined by:

$$L_u v = [u, v]$$

sly, we can define derivative for functions on a vector field $u: C^\infty(M) \to C^\infty(M), f \to u(f)$

$$u(f) = \lim_{t \to 0} (\Phi_t^* f - f)/\, t$$

Now, let's define for the majority here: It can be thought of as the variation of a tensor/form under an infinitesimal translation along the direction of $\vec{\xi}$, is called the lie derivative.

$$L_\xi U_{ab} = T_{ab,\alpha}\xi^\alpha + T_{\gamma b}\xi^\gamma_{,a} + T_{a\gamma}\xi^\gamma_{,b}$$

For our metric:

$$L_\xi g_{ab} = g_{ab,\alpha}\xi^\alpha + g_{\gamma b}\xi^\gamma_{,a} + g_{a\gamma}\xi^\gamma_{,b}$$

$$= \xi_{a,b} + \xi_{b,a}$$

If $\vec{\xi}$ is a killing vector of our metric then, $L_{\vec{\xi}}\, g_{ab} = 0$.

If a solution admits a time-like killing vector, it is possible to choose the time component of basis vector such that it is aligned with $\vec{\xi}$. Now, our time coordinate lines coincides with worldline to which $\vec{\xi}$ is tangent.

$$\vec{\xi}^a = (\xi^0, 0,0,0)$$

If $\xi^0 = const$. And unity:

$$\vec{\xi}^a = (1,0,0,0)$$

$$if, \frac{\partial g_{ab}}{\partial x^0} = 0$$

This means that our metric accepts a time-like killing vector which can be used to make the metric time-independent.

Sly, for the existence of a space-like killing vector $\frac{\partial g_{ab}}{\partial x^i} = 0$.

The map which leaves the metric unchanged is referred as an *isometry*.

Ex: killing vector of flat space-time:

In cartesian coordinates: $\vec{\xi}_{a,b} + \vec{\xi}_{b,a} = 0$

All the Christoffel symbols vanish in cartesian coordinate system.

$$\xi_{a,\beta\gamma} = 0$$

This differential equation has a general solution which looks like:

$$\xi_a = C_a + \epsilon_{a\gamma} X^{\gamma} \rightarrow A)$$

$$by\ subbing\ this\ into\ our\ differential\ quation : \epsilon_{a\gamma} X_{\beta}^{\gamma} + \epsilon_{\beta\gamma} X_{\alpha}^{\gamma}$$

$$= \epsilon_{a\gamma}\delta_{\beta}^{\gamma} + \epsilon_{\beta\gamma}\delta_{\alpha}^{\gamma} = \epsilon_{a\beta} + \epsilon_{\beta\alpha} = 0$$

This equation is satisfied if:

$$\epsilon_{a\beta} = -\epsilon_{\beta\alpha}$$

The general killing vector field from A), can be written as a linear combination of ten killing vectors ξ_a^A, A = 1,2,…,10.

$\xi_a^A = C_a^A + \epsilon_{a\gamma}^A X^{\gamma}|$ again A goes from 1 to 10.

Here we'll choose them as :

$$C_a^1 = (1,0,0,0), C_a^2 = (0,1,0,0), C_a^3 = (0,0,1,0), C_a^4 = (0,0,0,1)$$

Other C's from 5 through 10 have the value of 0.

$$\epsilon^5_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon^6_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\epsilon^7_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \epsilon^8_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\epsilon^9_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \epsilon^{10}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The other ϵ's 1 through 5 equal zero.

CONSERVED QTY. IN GEODESICS:

$$\frac{dV^a}{d\tau} + \Gamma^a_{\beta\gamma} V^\beta V^\gamma = 0$$

$$\xi_a\left[\frac{dV^a}{d\tau}+\Gamma^a_{\beta\gamma}V^\beta V^\gamma\right]=\frac{d(\xi_a V^a)}{d\tau}-V^a\frac{d\xi_a}{d\tau}+\Gamma^a_{\beta\gamma}V^\beta V^\gamma\xi_a$$

$$=\frac{d(\xi_a V^a)}{d\tau}-V^\beta V^\gamma\left[\frac{\partial\xi_\beta}{\partial x^y}-\Gamma^a_{\beta y}\xi_a\right]=0$$

$$=>\ \frac{d(\xi_a V^a)}{d\tau}-V^\beta V^\gamma\left(\xi_{\beta,\gamma}\right)=0$$

Now, we know that $V^\beta V^\gamma$ is anti-symmetric while $\xi_{\beta,\gamma}$ is symmetric:

=> $V^\beta V^\gamma\left(\xi_{\beta,\gamma}\right)=0$

This means: $\xi_a V^a = const.$

Which is basically telling us that there exist a conserved qty. for each killing vector. We can also write this as:

$$g_{a\mu}\xi^\mu V^a = const.$$

If the metric is asymptotically flat, $g_{\mu\nu}$ will become $\eta_{\mu\nu}$ at infinity

$$\eta_{00}V^0 = const. => V^0 = const. which\ means\ \frac{E}{c} = const.$$

If there are space-like killing vectors:

$$g_{a1}\xi^1 V^a = const. => g_{a1}V^1 = const.$$

$$p^a = mcV^a => \frac{p^1}{mc} = const.$$

SCHWARZSCHILD GEOMETRY:

Now, we actually start to use the tools we gathered by doing tensor analysis in GR to get some practical results. To do anything we must know the metric of that space or surface here we derive a spherically symmetric time -independent metric, which is also know as the Schwarzschild metric and know a bit about it’s singularity, which corresponds to a black hole. NOTE: Schwarzschild is metric is the generic metric to a spherically symmetric body, black hole’s a special, which we’ll discus in brief a bit later.

We first start by writing a general expression of a 4-D metric:

$$ds^2 = A(t,r)dt^2 - B(t,r)dt\, v.dv - C(t,r)(v.dv)^2 - D(t,r)dv^2$$

Where I’ve put v instead of our x,y and z 3-D cartesian coordinates.

Now, we do a coordinate change to spherical coordinates for obvious reasons:

$x^1 = r\sin\theta\cos\Phi \quad x^2 = r\sin\theta\sin\Phi$

$x^3 = r\cos\theta$

I’m going to derive the metric a bit differently here, but I’ll hint the way which is used in other GR. I use this method because it’s a bit shorter and easy to do.

I’ll use the formula $g = (J\varphi)(J\varphi)^T$

$$J\varphi = \begin{pmatrix} \sin\Phi\cos\theta & r\cos\Phi\cos\Phi & -r\sin\Phi\sin\theta \\ \sin\Phi\sin\theta & r\cos\Phi\sin\theta & r\sin\Phi\cos\theta \\ \cos\Phi & -r\cos\Phi & 0 \end{pmatrix}$$

$$(J\varphi)^T = \begin{pmatrix} \sin\Phi\cos\theta & \sin\Phi\sin\theta & \cos\Phi \\ r\cos\Phi\cos\Phi & r\cos\Phi\sin\theta & -r\cos\Phi \\ -r\sin\Phi\sin\theta & r\sin\Phi\cos\theta & 0 \end{pmatrix}$$

Now, $g = (J\varphi)(J\varphi)^T$

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

We get,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\Phi^2$$

Now, you can perform the same calculation in 4D with extra dimension of time(t) and you'll obtain the following equation:

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\Phi^2)$$

Where A and B are the constants which we need to find in order to convert this generic form of metric into our Schwarzschild metric.

Now, we add a bit of physics to our process, by considering the empty space solution for Einstein's field equation:

$$R_{\mu\nu} = 0$$

You can refer to the Ricci tensor formula we derived earlier.

Now,

$$g_{00} = A(r)$$

$$g_{11} = -B(r)$$

$$g_{22} = -r^2$$

$$g_{33} = -r^2 \sin^2 \theta$$

$$g^{00} = \frac{1}{A(r)}$$

$$g^{00} = \frac{1}{A(r)}$$

$$g^{11} = \frac{-1}{B(r)}$$

$$g^{22} = \frac{-1}{r^2}$$

$$g^{33} = \frac{-1}{r^2 \sin^2 \theta}$$

Now, we compute Christoffel symbols of 2nd kind:

$$\Gamma^0_{00} = 0$$

$$\Gamma^0_{0i} = \frac{1}{2} g^{0\nu}(\partial_i g_{\nu 0} + \partial_0 g_{\nu i} - \partial_\nu g_{0i}) = \frac{1}{2} g^{00} \partial_i g_{00}$$

$$\Gamma^0_{0j} = 0$$

$$\Gamma^i_{ii} = \frac{1}{2} g^{ii} \partial_i g_{ii}$$

This, is a generic solution for a metric with similar non-vanishing components.

$$\Gamma^1_{00} = \frac{1}{2B(r)}\frac{dA(r)}{dr} \; \Gamma^0_{01} = \frac{1}{2A(r)}\frac{dA(r)}{dr}$$

$$\Gamma^0_{01} = \frac{1}{2A(r)}\frac{dA(r)}{dr} \; \Gamma^1_{11} = \frac{1}{2B(r)}\frac{dB(r)}{dr}$$

$$\Gamma^1_{22} = -\frac{r}{B(r)} \quad \Gamma^1_{33} = \frac{-r\sin^2\theta}{B(r)}$$

$$\Gamma^2_{21} = \frac{1}{r}, \;\; \Gamma^2_{33} = -\sin\theta\cos\theta \;, \; \Gamma^3_{31} = \frac{1}{r}, \; \Gamma^3_{32} = \cot\theta$$

These are all the non-vanishing Christoffel symbols of our metric.

Now, let's calculate our Ricci tensor (which is just the derivatives and products of our Christoffel symbols) in flat space:

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} = \; 0 \; \rightarrow 1)$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} = \ 0 \ \rightarrow 2)$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = \ 0 \ \rightarrow 3)$$

$$R_{33} = R_{22} \sin^2\theta = 0 \quad \rightarrow 4)$$

We multiply equation 1) by B/A and add it to equation 2) to get:

$$A'B + B'A = 0$$

- ⇨ $AB = Const.$
- ⇨ $AB = \ \lambda$
- ⇨ $B = \frac{A}{\lambda}$

  Substitute this into equation 3):

  $$\frac{d(rA)}{dr} = \lambda$$

  $$A(r) = \lambda\left(1 + \frac{k}{r}\right),\ B(r) = \left(1 + \frac{k}{r}\right)^{-1}$$

Now, we use the weak field approximation. Our A(r) term here under weak field approximation, will be approximately equal to our Newtonian potential i.e.

$\frac{A(r)}{c^2} \rightarrow 1 + \frac{2\Phi}{c^2}$

Where, $\Phi = -\frac{GM}{r}$,

$k = -\frac{2GM}{c^2}$,

…Finally, we can write our long time waited Schwarzschild metric:

$$\boldsymbol{ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\Phi^2)}$$

This is one of the most beautiful equations in physics, because it defines not only spherically symmetric bodies in GR but, also predicts the existance of a new object called the black hole, which we’ll explore in a bit. Notice that our metric has singularity at $r = 2GM/c^2$ which we'll explore deeply in the black holes chapter.

GEODESICS IN SCHAWRZSCHILD METRIC:

There are two methods I recommend to calculate geodesics of any metric:

1)killing vector method

2)The Lagrangian method

! These are informal names. My personal favourite is the first one because it uses killing vectors and they paint a better picture in one's mind(at least in mine). That's why I'll use the first method and tell you how to proceed with the second one, so that you can try both and decide what works for you.

1)killing vector method:

First we calculate the Christoffel symbols, because they are use in the geodesic formula, this might be a good time to revisit it:

$$\Gamma^1_{00} = \frac{GM}{r^3}(r-2GM),\ \Gamma^1_{11} = -\frac{GM}{r(r-2GM)}$$

$$\Gamma^0_{01} = \frac{GM}{r(r-2GM)},\ \Gamma^1_{33} = -r(r-2GM)\sin^2\theta$$

$$\Gamma^2_{12} = \frac{1}{r},\ \ \Gamma^1_{22} = -r(r-2GM)$$

$$\Gamma^3_{13} = \ \frac{1}{r},\ \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{23} = \cot\theta$$

Now substituting them into our geodesic equation we get a set of 4 ODEs, one for each coordinate.

$$\frac{d^2t}{d\beta^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\beta}\frac{dt}{d\beta} = 0$$

$$\frac{d^2\theta}{d\beta^2} + \ \frac{2}{r}\frac{d\theta}{d\beta}\frac{dr}{d\beta} - \sin^2\theta\, cos\,\theta\left(\frac{d\Phi}{d\beta}\right)^2 = 0$$

$$\frac{d^2 r}{d\beta^2} + \frac{GM}{r^3}(r - 2GM)\left(\frac{dt}{d\beta}\right)^2 - \frac{GM}{r(r - 2GM)}\left(\frac{dr}{d\beta}\right)^2$$
$$-(r - 2GM)\left[\left(\frac{d\theta}{d\beta}\right)^2 + \sin^2\theta\,\left(\frac{d\Phi}{d\beta}\right)^2\right] = 0$$

$$\frac{d^2\Phi}{d\beta^2} + \frac{2}{r}\frac{d\Phi}{d\beta}\frac{dr}{d\beta} + 2\cot\theta\,\frac{d\theta}{d\beta}\frac{d\Phi}{d\beta} = 0$$

Now, we apply the knowledge that we attained in the killing vectors chapter(might be a good time to read it, if you didn't):

$$K_\mu \frac{dx^\mu}{d\beta} = const.$$

$\varepsilon = -g_{\mu\nu}\frac{dx^\mu}{d\beta}\frac{dx^\nu}{d\beta}$ (our geodesic const.)

GRAVITATIONAL POTENTIAL:

Conservation of direction of angular momentum of the angular momentum, means that the particle will move in a particular plane. We can choose a plane, and then rotate it according to our needs and wants. In that plane:

$\theta = \frac{\pi}{2}$ (also called the equitorial plane)

$$\frac{d\theta}{d\beta} = 0$$

Two other killing vectors correspond to energy and the magnitude of the angular momentum.

Time-like killing vector | $K^{\mu} = (1,0,0,0)^T$

$$k^{\mu} = k^{\nu} g_{\mu\nu} = \left(-\left(1 - \frac{2GM}{r}\right), 0,0,0\right)$$

$K_{\mu} \frac{dx^{\mu}}{d\beta} = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\beta} = E$ (const. of motion)

Sly. $L_{\mu} = \partial_{\Phi} (0,0,0,1)^T$

$$L_\mu = (0,0,0, r^2 \sin^2\theta) \;\mid\; r^2 \frac{d\Phi}{d\beta} = L$$

Now, we substitute this into the equation for ε.

$$-\varepsilon = -\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\beta}\right)^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{dr}{d\beta}\right)^2 + r^2\left(\frac{d\Phi}{d\beta}\right)^2$$

Multiplying by $\left(1 - \frac{2GM}{r}\right)$, putting the values for E and L:-

$$-E^2 + \left(\frac{dr}{d\beta}\right)^2 + \left(1 - \frac{2GM}{r}\right)\left(\frac{L^2}{r^2} + \varepsilon\right) = 0$$

We can rearrange this into :

$$\frac{1}{2}\left(\frac{dr}{d\beta}\right)^2 + V(r) = \frac{1}{2}E^2$$

Where V(r) is our potential, given by:

$$V(r) = \frac{1}{2}\varepsilon - \frac{\varepsilon GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

<u>MASSIVE PARTICLE TRAJECTORIES:</u>

For massive particles, our $\varepsilon = 1$.

If we assume the particle is executing circular orbits, if(minimum potential) $\frac{dv}{dr} = 0 \Leftrightarrow r = const.$

Subbing all this in our potential equation we get, for

$$GMr^2 - L^2 r + 3GML^2\gamma = 0$$

As,$r \ \rightarrow \infty : -GML^2/r^3 \rightarrow 0$

$\gamma = 1$ in General Relativity.

$$r_{circ.} = \frac{L^2 \ \pm \sqrt{L^4 - 12GM^2L^2}}{2GM}$$

$$r_{circ} = \frac{L^2}{GM}, 3GM$$

We will use that value of r_{circ} that minimizes the potential, in order to maximize stability of our circular orbit. One can check that $3GM$ is not our stable solution that we are looking for. The discriminant is zero(which gives us the 1st solution) is when:

$$L = \sqrt{12GM}$$

$$r_{circ} = r_{isco} = 6GM$$

This last equation that we wrote give us, our smallest possible stable circular orbit around any body which can be described by our Schwarzschild radius.

r_{isco} means inner most stable circular orbit, it's important for measurement purposes usually of rotating black holes, that we'll discuss in brief later in the book. Now, it's safe to say that:

Stable if: $r_{circ} > 6GM$

Unstable if: $r_{circ} \in (3GM, 6GM)$

MASSLESS PARTICLE TRAJECTORY:

For massless particles like photon, $\varepsilon = 0$

Subbing this into our potential equation with our minimum potential condition $\frac{dv}{dr} = 0$ and we obtain the potential:

$$V(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

And our minimum radius is:

$$r_{circ.} = 3GM$$

A photon with a given energy E comes in from infinity($r = \infty$) and it’s path is tilted if it doesn’t cross the singularity of the metric at $r = 2GM/c^2$, it goes off to infinity but id it does it falls into potential and there’s no coming back.

TESTS OF GENERAL RELATIVITY:

"It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with the experiment, it's wrong." -Richard Feynman

So, we must test general relativity in order to prove that it's correct.

Einstein suggested three tests for GR:

1) Gravitational Red shift of spectral lines.
2) Deflection of light by sun
3) Precession of the point in the orbit of a planet closest to the Sun aka Perihelia.

Another test has been added but this wasn't given by Einstein

4) Gravitational time delay of radar signal.

GRAVITATIONAL REDSHIFT :

We’ll consider two non-inertial observers here, for dramatic effects let’s name them Romeo and Juliet. In, later stages of book it will become more clear why we chose these two specific names. Both are stuck at spatial coordinates $(r_R, \theta_R, \varphi_R)$ and $(r_J\ \ , \theta_J, \varphi_J)$ respectively.

Now, we know that:

$$\frac{d\tau}{dt} = \sqrt{\left(1 - \frac{2GM}{r}\right)}$$

Suppose Romeo decides to send “Hey, I cheated on my math exam lol” to Juliet through his “cool” Photon technology which emits a light pulse which travels to Juliet, such that Romeo measures the time between two successive crests of the light wave to be $\Delta\tau_R$.Each crest follow the same path to Juliet , except they are separated by:

$$\Delta t = \frac{1}{\sqrt{\left(1 - \frac{2GM}{r_R}\right)}} (\Delta\tau_R)$$

This separation doesn't change along the photon trajectory, but Juliet here, measures a time between successive crests as:

$$\Delta\tau_J = \sqrt{\left(1 - \frac{2GM}{r_J}\right)} \Delta t$$

$$\Rightarrow \quad \Delta\tau_J = \sqrt{\frac{(1-2GM/r_J)}{(1-2GM/r_R)}} \Delta\tau_R$$

Since, $\Delta\tau_i$ measures the proper time interval between two crests of an EM wave. Their, frequencies will be :

$$\frac{\nu_J}{\nu_R} = \frac{\Delta\tau_R}{\Delta\tau_J}$$

$$= \sqrt{\frac{(1 - 2GM/r_R)}{\left(1 - 2GM/r_J\right)}}$$

This, is our ever wanted formula. Now, Juliet didn't receive the correct message, the light got red-shifted and she received "Hey, I cheated on you lol" now, she's pissed. It appears as though Romeo didn't quite realised the 4D spacetime curvature of the situation(pun intended), and maybe next Romeo just might study a bit hard.

DEFLECTION OF LIGHT :

Consider a light ray approaching from infinity. Using the equations we derived earlier(with $\varepsilon = 0$) we find that,

$$\frac{1}{L^2}\left(\frac{dr}{d\beta}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right) = \frac{E^2}{L^2}$$

We define the impact parameter as:

$$b := \frac{L}{E}$$

$$\frac{d\Phi}{dr} = \pm \frac{1}{r^2}\left\{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right\}^{-1/2}$$

Now, simply by doing integration we find :

$$\Delta\Phi = 2\int_{r_1}^{\infty} \frac{dr}{r^2}\left\{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right\}^{-1/2}$$

Where r = r_1 is the turning point, which is the radius where ${}^{1}\!/_{b^2} = \left(1 - {}^{2GM}\!/_{r}\right)/r^2$

For the deflection of light by the Sun, the impact parameter can’t be smaller than the stellar radius, $b \geq R_\odot \approx 7 * 10^{10} cm$, thus $\frac{2GM_\odot}{c^2 b} \leq 10^{-6}$.

This equation is solved by 1) change of parameter $w = \frac{b}{r}$ and 2) expanding $\frac{GM}{r}$ in Taylor series and keeping the first order.

$$\Delta\Phi \approx \pi + \frac{4GM}{b}$$

The deflection angle is :

$$\delta\Phi_{de} = \Delta\Phi - \pi \approx \frac{4GM}{c^2 b}$$

Which is ~1.75 (arc-seconds). That's really small amount but, Sun isn't that big to make a noticeable deflection of light, but Black holes are..

TIME DELAY RADAR SIGNAL:

This time we solve for dt/dr, the result is:

$$\frac{dt}{dr} = \pm\frac{1}{b}\left(1 - \frac{2GM}{r}\right)^{-1}\left\{\left\{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)\right\}\right\}^{-1/2}$$

If one sends sends an EM signal in a gravitational field, which can be reflected back, it is possible to calculate the GR correction to the time of the returned signal. The GR time delay is:

$$(\Delta t)_{GR} \approx \frac{4GM}{c^3}\left[\log\left(\frac{4r_{other}r_{earth}}{r_1}\right)+1\right]$$

Where, $r_1 = b$ is the radius of closest approach to the center.

This experiment was proposed by Irwin Shapiro, also called Shapiro time delay.

s

PERIHILIA:

We'll examine the orbit of Mercury around Sun, in specific here.

We start by writing the lagrnagian for the given motion:

$$L = \frac{mv^2}{r} = \frac{m}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

$$L = \frac{m}{2}\left[\left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \left(1 - \frac{2GM}{r}\right)^{-1} \dot{r}^2 - (r\dot{\theta}^2 + r^2 \sin\ ^2\theta\dot{\Phi}^2\right]$$

$$\frac{\partial L}{\partial x^a} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^a}\right) = 0$$

a = 0:

$$\frac{d}{d\tau}\left(1 - \frac{2GM}{r}\right)\dot{t} = 0$$

a = 2:

$$\frac{d}{d\tau}\left(r^2\dot{\Phi}^2\right) - r^2 \sin\theta \cos\theta\ \dot{\Phi}^2 = 0$$

a = 3:

$$\frac{d}{d\tau}\left(r^2 \sin^2 \theta \dot{\Phi}\right) = 0$$

a =1 case is not necessary at least for our purposes.

We'll consider in equatorial from now i.e. $\theta = \pi/2$

And $\dot{\theta} = 0$

$$\frac{d}{d\tau}\left(r^2 \dot{\theta}\right) = 0$$

$r^2 \dot{\theta}$ is constant.

$\tau = 0, r^2\dot{\theta} = 0\ \forall\tau$. If $r \neq 0, \dot{\theta} = 0 \forall\tau$. Hence, $\theta = \pi/2$ which means planar motion is possible. In XY plane,

Integrating a = 3, equation:

$$r^2 \dot{\Phi} = h$$

h is a constant. Thus, we can say that h represents angular momentum.

Sly. Integrating a = 0, equation:

$$\left(1-\frac{2GM}{r}\right)\dot{t}=k$$

Again, k is here a constant.

Now, subbing this equation and $\theta=\pi/2$ into the equation for the lagrangian:

$\frac{k^2}{1-\frac{2GM}{r}}-\frac{\dot{r}^2}{1-\frac{2GM}{r}}-r^2\dot{\Phi}^2=1$ → N)

$$u=\frac{1}{r}$$

$$\dot{r}=\frac{dr}{d\tau}=\frac{d}{d\tau}\left(\frac{1}{u}\right)$$

$$=-\frac{1}{u^2}\left(\frac{du}{d\Phi}\right)\left(\frac{d\Phi}{d\tau}\right)=-\frac{1}{u^2}\frac{du}{d\Phi}hu^2$$

$$\dot{r} = -h\left(\frac{du}{d\Phi}\right)$$

$r^2\dot{\Phi} = h, u = \frac{1}{r}$ in the new equation for our lagrangian(N)) to obtain :

$$\frac{k^2}{1-2Mu} - \frac{h^2\, du/d\Phi}{1-2Mu} - h^2u^2 = 1$$

Multiplying by $1 - 2Mu^2/h^2$

$$\frac{k^2}{h^2} - \left(\frac{du}{d\Phi}\right)^2 - u^2(1-2Mu) = \frac{1-2Mu}{h^2}$$

Rearranging this we get:

$$\left(\frac{du}{d\Phi}\right)^2 + u^2 = \frac{k^2-1}{h^2} + \frac{2Mu}{h^2} + 2Mu^3$$

Dividing this by 2, and differentiating we get:

$$\frac{d^2u}{d\Phi^2} + u = \frac{m}{h^2} + 3Mu^2$$

This is the differential for Mercury according to general relativity.

Now, we’ll solve it like physicists which means we’ll use perturbative method.

$$\varepsilon = \frac{3M^2}{h^2}$$

$$u'' + u = \frac{m}{h^2} + \varepsilon\left(\frac{h^2u^2}{m}\right)$$

Now, we assume that it has a solution of type:

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2)$$

Now, we’ll differentiate this solution twice and sub it into the differential equation obtained by perturbation, to get :

$$u_0'' + u_0 - \frac{M}{h^2} + \varepsilon\left(u_1'' + u_1 - \frac{h^2u_0^2}{M}\right) + O(\varepsilon^2) = 0$$

For first approximation we equate the coffeciants of ε,ε^2,… to zero. Then, $u_0 = \frac{M}{h^2}(1 + \mathrm{e}\cos\Phi)$ is the zeroth order solution. Now, for coffecients of ε :-

$$u_1'' + u_1 = \frac{h^2u_0^2}{M} = \frac{M}{h^2}(1 + e\cos\Phi)^2$$

$u_1'' + u_1 = \frac{M}{h^2}\left(1 + \frac{1}{2}e^2\right) + \frac{2Me}{h}\cos\Phi + \frac{me^2}{2h^2}\cos 2\Phi$→ M')

We can try the solution, $u_1 = A + B\Phi\sin\Phi + C\cos 2\Phi$

$$u_1' = B\sin\Phi + B\Phi\cos\Phi - 2C\sin 2\Phi$$

$$u_1'' = 2B\cos 2\Phi - B\Phi\sin\Phi - 4C\cos 2\Phi$$

$u_1'' + u_1 = A + 2B\cos\Phi - 3Cs\cos 2\Phi$→ M)

Comparing M) and M’), :-

$$A = \frac{M}{h^2}\left(1 + \frac{1}{2}e^2\right)$$

$$B = \frac{Me}{h^2}$$

$$C = \frac{-Me}{h^2}$$

$$u_1 = \frac{M}{h^2}\left(1 + \frac{1}{2}e^2\right) + \frac{Me}{h^2}\Phi \sin \Phi - \frac{Me^2}{6h^2}\cos 2\Phi$$

$$u \approx u_0 + \varepsilon u_1$$

$$u \approx u_0 + \frac{\varepsilon M}{h^2}(1 + e\Phi \sin \Phi + e^2\left(\frac{1}{2} - \frac{1}{6}\cos 2\Phi\right))$$

We notice $e\Phi \sin \Phi$ term increases after each revolution, it becomes dominant.

Therefore,

$$u \approx \frac{M}{h^2}(1 + e\cos\Phi + \ \varepsilon e\sin\Phi)$$

$$u \approx \frac{M}{h^2}(1 + e\cos(\Phi(1 - e)))$$

From this last equation we see that orbit of Mercury is not exactly an ellipse.

Period $= \frac{2\pi}{1-\varepsilon} \approx 2\pi(1 + \varepsilon)$

Precession is given by:

Precession $\approx 2\pi\varepsilon = \frac{6\pi M_r^2 r}{h_r^2 r}$

M_r^1 is the mass of the Sun.

h_r is mercury's angular momentum in relativistic units.

<u>TO BLACK HOLES ASAP:</u>

A black hole has three properties:

1)Mass 2)Charge3)Spin

Here, in this section we'll only discuss black holes with mass, which can be described by our sweet sweet Schwarzschild metric.

First we write our Schwarzschild metric so, that we can explore it's singularities in depth here.

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\Phi^2)$$

Now, if we look at when $r = 2GM$, we notice our long wanted singularity. Now, we try to get rid of it by trying to change coordinates in clever ways.

You might have another singularity at $r = 0$, but this isn't a coordinate singularity, which we can shake off by just a coordinate change. This singularity is often referred to as "The

Singularity" of a given black hole.

Imagine a light ray propagating in the radial directions θ and Φ which are constant and for a light ray $ds^2 = 0$.

⇨ $ds^2 = 0 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$

⇨ $\frac{dr}{dt} = \pm\left(1 - \frac{2GM}{r}\right)$

Now we something peculiar happens as $r \to \frac{2GM}{c^2}, \frac{dr}{dt} \to \pm\infty$ which means in these coordinates we never a light ray go in.

Imagine a light ray going in a black hole, in these coordinates you'd see it "slowing down" until it approaches the coordinate singularity a.k.a our Schwarzschild radius(remember this happens only in this coordinate system).

TORTOISE COORDINATES:

We substitute $t = \pm r^* \pm const$.

Where $r^*$is defined as :

$$r^* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

⇨ $ds^2 = \left(1 - \frac{2GM}{r}\right)\left(-dt^2 + dr^{*2}\right) + r^2 d\Omega^2$| where we have substituted dΩ as our 3 spatial coordinates, just for lesser writing purposes.

Let’s take a break, here’s a pun to relax you: A man asked a physicist who’s the love of his life and if she/he has ever disappointed, physicist replied with a grin on his face saying, “ we physicists only fall for the beauty of 4D spacetime curvature(gravity) “ and his smile grew bigger as he replied “ No, she has never disappointed me but I haven’t fully understood why she behaves so, weirdly around small stuff”. Okay, back to physics now,

NULL GEODESICS:

$$\bar{u} = t + r^*$$

$$\bar{v} = t - r^*$$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) d\bar{u}^2 + (d\bar{u}dr + d\bar{v}dr) + r^2 d\Omega^2$$

These are called the Eddington-Finkelstein coordinates.

$$\frac{d\bar{u}}{dr} = \begin{cases} 0 : infalling \\ 2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} : outgoing \end{cases}$$

$$dt = \frac{1}{2}(d\bar{u} + d\bar{v})$$

Inserting this into our original null geodesic metric, we get:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) d\bar{u}\, d\bar{v} \ \rightarrow a)$$

For $r < 2GM/c^2, \frac{d\bar{u}}{dr} < 0$ meaning all the future direction paths are in the direction of decreasing r.

KRUSKAL COORDINATES:

$$r^* = r + r_s \ln\left(\frac{r}{r_s} - 1\right) = \frac{1}{2}(\bar{u} - \bar{v})$$

We rewrite the parenthesis expression in a):

$$\left(1 - \frac{r_s}{r}\right) = \frac{r_s}{r}\left(\frac{r}{r_s} - 1\right) \rightarrow \frac{r}{r_s} + \ln\left(\frac{r}{r_s} - 1\right) = \frac{1}{2r_s}(\bar{u} - \bar{v})$$

We replace $\left(\frac{r}{r_s} - 1\right)$ term with exponentials:

$$ds^2 = \frac{r_s \exp\left(-\frac{r}{r_s}\right)}{r} . \exp(\bar{u} - \bar{v})\, d\bar{u} d\bar{v}$$

Now our metric ($g_{12} component$) is singular at $r = r_s$, as $\bar{u} \to \infty$ and $\bar{v} \to \infty$. Now to deal with these new singularities we define new coordinates:

$$V = -\exp\left(-\frac{\bar{v}}{2r_s}\right)$$

$$U = \exp\left(\frac{\bar{u}}{2r_s}\right)$$

Which gives us yet another new metric:

$$ds^2 = -\frac{4r_s^3}{r}\exp\left(-\frac{r}{r_s}\right) dUdV$$

Let,

$T = \frac{1}{2}(U + V) \quad X = \frac{1}{2}(U - V)$ after this substitution we obtain the metric of the form:

$$\left(\frac{r}{r_s} - 1\right) \exp\left(\frac{r}{r_s}\right) = X^2 - T^2$$

$$\frac{t}{r_s} = 2 \tanh^{-1}\left(\frac{T}{X}\right)$$

REGIONS OF EXTENDED METRIC:

The extended Schwarzschild metric can be divided into 5 regions:

1) Region 1 is the original spacetime where we live, experiments happen, people do stupid stuff, etc.
2) Region 2 is $T = X$, anything which passes region 2 will fall into the singularity at r = 0.
3) Region 3 is time reversed version of region 2. Region 3 can be thought of just as the opposite of region 2. That's why it's called a *white hole*. There is no direct experiment to prove it's existence.

4) Region 4 has identical properties to our universe, i.e. it represents an asymptotically flat region which exists inside of the radius $r = r_s$
5) The singularity at r = 0, cannot be removed.
6) When you cross the region 1 and enter region 2, space and time axis flip their direction, making r=0 singularity your near future. That's why you can't escape after crossing the event horizon.
7) Suppose that 1 person from our 2 hypothetical people Romeo and Juliet decides to jump in a black hole, and want to enter region 3 through region 1, they'd have to exceed the speed of light.

Now, we'll try and embed the given space in a 3d flat space. Metric in cylindrical coordinates is:

$$d\gamma^2 = dr^2 + dz^2 + rd\Phi^2 = dr^2\left(1 + \left(\frac{dz}{dr}\right)^2\right) + r^2 d\Phi^2$$

Now, we compare this to our Schwarzschild metric and we get:

$$d\gamma^2 = dr^2\left(1-\frac{r_s}{r}\right)^{-1} + r^2 d\Phi^2$$

This means, $\left(\frac{dz}{dr}\right)^2 = \left(1-\frac{r_s}{r}\right)^{-1} - 1$

After integrating the non-Euclidian 2d-hyperboloid is embedded in the 3d Euclidian space by:

$z = \pm 2r_s\sqrt{\left(\frac{r}{r_s}-1\right)}$ for $r > r_s$

<u>PENSROSE DIAGRAMS:</u>

They are way to represent the entirety of our infinite spacetime on a flat 2d "graph".

First we write our Minkowski metric in natural units:

$$ds^2 = (dt + dr)(dt - dr) - r^2 d\Omega$$

The transformation that we our looking for must satisfy these 2 conditions:

1) they should preserve the light cone

2) map the infinite space to a finite 2d flat plane.

$dt \pm dr = 0$ they describe the propogation of a light like “thing”.

$Y^+ = f(t + r) \mid Y^- = f(t - r)$

These conditions are satisfied by our tanh function.

$Y^+ = \tanh(t + r) \mid Y^- = \tanh(t - r)$

What we did is, we mapped the entirety of spacetime on a triangle bounded by

$Y^+ = 1 \mid (t + r) \to \infty$

$Y^- = -1 \mid (t - r) \to -\infty$

SOME EXTRA STUFF:

(in Kruskal coordinates) if Juliet decides to jump In the black hole, Romeo will see Juliet getting closer and closer to the horizon but never actually passing it, and Juliet will see Romeo accelerating away from her. Which might just break her heart, but who cares she's the one who decide to jump in a black hole with studying the physics of it. That's why always study about black holes first then jump in one(given that you can) otherwise you might just get spaghettified.

Spaghettification is a process, due to which a person or a 'thing' falling in a black hole will get stretched out as your head and feet experience, large amount of gravity.

Wormholes are a hypothetical construct which lets you pass between region 1 and 4.