May 3, 2022

Topological Aspects Of Gauge Theories

Pranat Sharma[1]

[1] *SR DAV public school*
Noida, Uttar Pradesh, India

Abstract

In this article we review the topological features of Gauge Theories and how it affects the physics of it. We will primarily discuss configuration spaces with mom-contractible loops and two-surfaces.Further it's discussed that how we (should) modify our action to fit these configuration spaces with non-trivial topology.Lastly we will review two research papers to understand the further developments and how Gauge Theories play a key role in them.

Contents

1 Introduction

Gauge Theories have always played a central role in describing physics at classical as well as quantum level. Even though we will mainly focus on gauge theories used in QFTs. We start our discussion with talking about basic formalism of abelian and non-abelian gauge theories using differential forms and differential geometry. We will proceed our discussion by talking about configuration spaces and their topology. It will be answered how their non-trivial topology affects the action and in general the "physics of it". We cite a paper by Hidenori Fukaya on explaining why QG is hard by referencing gauge theories on parallalizable frame bundles and how it's much different from working with theories like QCD.[1] We will also analyze of how the gauge theories and their topological aspects play an important role in understanding existence of particles like fluctons and this will be discussed briefly in section 7 along with the citation of an article by R. Cartas-Fuentevilla and J.M. Solano-Altamirano on flucton [2].Fluctons are topological excitations which can be thought of as generalization of instantons which takes the restriction of the base manifold to be self-dual,etc. We'll examine the physics of this "particle"during later sections of the article

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We'll start our discussion with brief overview of abelian and non-abelian gauge theories and provide a formalism for them using differential geometric setup.
The discussion is continued by talking about topics like configuration spaces with non-trivial topology such has non-trivial fundamental group, non-trivial second cohomology group. For non-trivial fundamental group we understand that our configuration has non contracitble loops in it. In order to deal with this problem we'll divide the solution into steps and finally see the emerging the well known "$\theta\nu$" term if you aren't already familiar with it from QCD, you will be.While, the non-trivial second cohomology group means there are non-contractible two surfaces in our configuration space and understand the "reshaping" of our action into weight factors and their dependence on path and doing further analysis to get an extra added term in our formula of action so it's modified according to our non trivial cohomology configuration space.

We will start with few mathematical preliminaries majorly discussing about differential k-forms, exterior derivatives and fibre bundles. We'll also have a brief and quick overview of parallalizable manifolds as the knowledge will be required in later sections of the article. For a detailed insight into these preliminaries, I recommend going over Dr. Fredrich Schuller's notes on differential geometry. https://people.utwente.nl/f.p.schuller?

tab=research

2 Mathematical Preliminaries

2.1 Differential Forms

Differential One-form Given T_pM and T_p^*M are tangent and co-tangent space of base manifold M. We denote the basis of the dual space(T_p^*M) as dx^i. A differential one-form is giving an element of this dual vector space at each point of the manifold i.e.

$$w = w_i dx^i$$

The components of a differential one-form(w_i) are covarient vectors.
Differential k-forms Taking the k-fold tensor product of T_pM and we get $(T_pM)^k$, element of this space defines a rank k tensor at the point p. Collection of all such tensors over the manifold defines a tensor field over it. A differential $k-$ form is obtained by taking anti-symmetrized k fold tensor product of T^*M [1].In local coordinates, k form w will be given by:

$$w = \frac{1}{k!} w_{i_1 i_2 \dots i_k} dx^{i_1} \wedge dx^{i_2} \dots dx^{i_k} \tag{2.1}$$

[2]

$w_{i_1 i_2 .. i_k}$ are antisymmetric in the $k-$indices.

2.2 Exterior Derivative

The exterior derivative d of a k form takes it to $k+1$ form, it can be given as follows in terms of local coordinates:

$$dw = \frac{1}{(k+1)!} w_{i_1 i_2 .. i_{k+1}} dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_{k+1}} \tag{2.2}$$

$$w_{i_1 i_2 \dots i_{k+1}} = \tfrac{\partial}{\partial x^{i_1}} w_{i_2 i_3 \dots i_{k+1}} - \tfrac{\partial}{\partial x^{i_2}} w_{i_1} i_3 \dots i_{k+1} + \dots + (-1)^k \tfrac{\partial}{\partial x^{i_{k+1}}} w_{i_1} i_2 \dots i_k$$

[1]cotangent bundle i.e disjoint union of co-tangent space at each point $p \in M$. We'll discuss them in detail in a later section.

[2]where $\wedge$ is the wedge product given by $(a \wedge b)(x_1, .., x_{n+m}) = \frac{1}{n!m!} \Sigma_{\pi \in S_{n+m}} sgn(\pi)(a \otimes b)(x_{\pi(1),\dots,x_{\pi(n+m)}})$

Example: Let's take one-form $\alpha = f dx + g dy$
and now we calculate the exterior derivative of this one-form:

$$d\alpha = \partial_x f dx \wedge dx + \partial_y f dy \wedge dx + \partial_x g dy \wedge dx + \partial_y g dy \wedge dy$$
$$d\alpha = \partial_y f dy \wedge dx + \partial_x g dy \wedge dx$$

Exterior derivative can be thought of as generalization of curl to a n dimensional manifold.

2.3 Fibre Bundles

A fibre bundle is a manifold E consisting of base manifold M(in physics spacetime manifold) and the fibre space F(in physics space of fields) and it is represented by the direct product $\mathbb{R}^4 \times F$ and the total space of fibre bundle E is $M \times F$.
We've a structure group G, which generates the coordinate transformation in the fibre space F. The coordinate transformation in fibre is what's referred to as *gauge transformation* in physics.

If take the structure group G itself as fibre i.e $F = G$, now we can call our fibre bundle as principal bundle(P).
We can define what's known as a connection one-form on fibre bundle and which is given by:

$$w = g^{-1} dg + g^{-1} A g$$

Where $A = A^a_\mu(x) t_a dx^\mu$, $A^a_\mu(x)$ is our vector potential and t_a are generators of G
w doesn't change under coordinate transformation $g \to hg$(fibre's direction) and thus we require the gauge field to transform as:

$$A \to h dh^{-1} + h A h^{-1}$$

We also define a curvature two-form Ω as follows:

$$\Omega = dw + w \wedge w = g^{-1}(dA + A \wedge A) g = g^{-1} F g \tag{2.3}$$

Where F is what's known as *field strength* For Example: Frame Bundles
If M be a smooth manifold. We consider the following space:

$$L_pM = \{(e_1, ..., e_{dimM})|e_1, ..., e_{dimM} \text{ is basis of } T_pM\ \}$$

We can then define the frame bundle on M as

$$LM \equiv \bigcup_{p\in M} L_pM \tag{2.4}$$

$dim(LM) = dim(M) + (dim(M))^2$
$LM \xrightarrow{\pi} M$ is a principal $GL(dim(M), \mathbb{R})-$ bundle.
For more details on this topic I suggest checking out Prof. Fredrich Schuller's notes on differential geometry.
https://people.utwente.nl/f.p.schuller?tab=research

2.4 Parallalizable Manifolds

If we start with a differentiable manifold M and define smooth vector fields($\{V_1, V_2, ..., V_n\}$) on it such that at each point $p \in M$, the tangent vectors
$\{V_1(p), .., V_n(p)\}$
provides a basis at tangent space of M at p.
For Example: $GL(4, \mathbb{R})$ has parallizable frame bundle and thus there exists a on-to-one map e between $v \in V$ and $t \in T_pM$ such that:

$$v^a = e^a_\mu t^\mu$$

the four component one-form on M is given by $e = (e^1_\mu dx^\mu, e^2_\mu dx^\mu, e^3_\mu dx^\mu, e^4_\mu dx^\mu)$ but this isn't invariant under coordinate transformation. So, we construct a coordinate transformation invariant one form on frame bundle $F(M)$ known as the *solder-form* as
:

$$\theta = g^{-1}e$$

The frame bundle is described by both solder one-form and our original gauge field.We can also define a two-from through solder one-forms which is known as the torsion two-from

$$\Theta = d\theta + w \wedge \theta \tag{2.5}$$

One doesn't encounter torsion and solder one-forms with non parallalizable frame bundles.

3 Abelian Gauge Theory

Let's start by discussing Maxwell's theory in a more "differential geometric" point of view.As we know, that our theory's gauge group is $U(1)$, so we can express the gauge potential as follows:

$$A = A_\mu dx^\mu$$

and we calculate the field strength tensor($F_{\mu\nu}$) as the exterior derivative of the one-form(gauge field) plus some commutator term but since right now we're dealing with abelian gauge theories, it will vanish and we'll be left with:
$F = dA$ and exterior derivative is calculated as follows:

$$F_{\mu\nu} = (\tfrac{\partial}{\partial x_\mu} A_\nu - \tfrac{\partial}{\partial x_\nu} A_\mu) dx^\mu \wedge dx^\nu = (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu$$

and we know that $d^2 = 0$ thus $dF = 0$
Thus, we get:

$$dF = \tfrac{1}{6}(\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} - \partial_\nu F_{\mu\alpha}) dx^\alpha \wedge dx^\mu \wedge dx^\nu = 0$$

this is well-known bianchi identity.
Now, we'll compute the hodge duals:

$$F = E_i dx^0 \wedge dx^i + \tfrac{1}{2}\epsilon_{ijk} B_k dx^i \wedge dx^j$$
$$^*F = B_i dx^0 \wedge dx^i + \tfrac{1}{2}\epsilon_{ijk} E_k dx^i \wedge dx^j$$

current(J) is one-form so it's hodge dual would give us three-form($*J$) which is given by :
$^*J = J_0 d^3x - \frac{1}{2}\epsilon_{ijk} J_k dx^0 \wedge dx^i \wedge dx^j$
now, let's compute $d * F$, since $*F$ is also two-form, it's exterior derivative is given as:
first we write $*F$ like:
$^*F = E_1 dx^2 \wedge dx^3 - E_2 dx^1 \wedge dx^2 + E_3 dx^1 \wedge dx^2 + B_1 dx^0 \wedge dx^1 + B_2 dx^0 \wedge dx^2 + B_3 dx^0 \wedge dx^3.$
now one does some laborious work and arrives at a result looking something like this

$$d^*F = (\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3) dx \wedge dy \wedge dz + (\partial_0 E_1 + \partial_3 B_2 - \partial_2 B_3) dt \wedge dy \wedge dz + (-\partial_0 E_2 + \partial_3 B_1 - \partial_1 B_3) dt \wedge dx \wedge dz + (\partial_3 E_3 + \partial_1 B_2 - \partial_3 B_1) dt \wedge dx \wedge dy$$

and which is hodge dual$J$ thus:

$$d^*F =^* J \tag{3.1}$$

Now, we can write the action of maxwell theory as:

$$S = -\frac{1}{2}\int_{\mathbf{M}} F \wedge^* F + \int_{\mathbf{M}} A \wedge^* J \tag{3.2}$$

$$F \wedge^* F = F^{\mu\nu} F_{\mu\nu} = B^2 - E^2 \text{ and}$$
$$A \wedge^* J = A_\mu J^\mu d^4x$$

So, now one can mentally establish the simplification of differential form formalism to the regular component form formalism they're familiar with.
The existence of a global A such that $F = dA$ is altogether a different discussion about whether the second cohomology of the configuration space for gauge theory is trivial or not. But, more on what exactly does that mean later.

4 Non-Abelian Gauge Theories

Here things work a little bit different since elements of our gauge group don't commute. So, we write the gauge potential in the form of:

$$A = (-it_a) A^a_\mu dx^\mu$$

now, field strength will be given by:
$F = dA + [A, A]$

$$F_{\mu\nu} = (-it_a)(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu + t_b t_c A^b_\mu A^c_\nu dx^\mu \wedge dx^\nu - t_c t_b A^c_\nu A^b_\nu dx^\nu \wedge dx^\mu$$

[3] $$F_{\mu\nu} = (-it_a)(\partial_\mu A_\nu - \partial_\nu A_\mu + f^{abc} A^b_\mu A^c_\nu) dx^\mu \wedge dx^\nu$$

This our well-known field strength tensor for non-abelian gauge theories.
Accordingly, we notice something while writing F is that it's initial expression can be written as:

[3] $[t_a, t_b] = i f_{abc} t_c$

$F = dA + A^2$, now we would compute dF the reason of calculation will become clear as we proceed.
$dF = dAA - AdA = (F - A^2)A - A(F - A^2) = FA - AF$
Further, we shift our focus to write an action for our theory. The action is given by what's known as yang-mills action which is:

$$S_{YM} = -\tfrac{1}{2e^2}\int_M F^a \wedge^* F^a$$
$$S_{YM} = -\frac{1}{e^2}\int_M Tr(F \wedge^* F)^4 \tag{4.1}$$

If the first homotopy group of our configuration space isn't trivial i.e. $\pi_1(C) \neq 0$. Then we compute of what's known as "instanton number". We'll see what exactly we mean by "configuration space" in the next section but for now, instanton number is written as:

$$\nu[A] = -\frac{1}{8\pi^2}\int_M (F \wedge F) = \int_M \Omega \tag{4.2}$$
$$d\Omega = -\tfrac{1}{8\pi^2}Tr(dFF + FdF)$$
$$d\Omega = -\frac{1}{8\pi^2}Tr((FA - AF)F + (FA - AF)dF) = 0 \tag{4.3}$$

Thus, this four-form is indeed closed. Now, we would want to examine its exactness i.e. we write $dk = \Omega$ for some three-form k. we can write k in the form:

$$k = aTr(AdA + bA^3)$$
$$dk = aTr(dAdA + bdAA^2 - bAdAA + bA^2dA) = aTr(dAdA + 3bdAA^2)$$
$$dk = aTr((F - A^2)(F - A^2) + 3b(F - A^2)A^2) = aTr(FF + (3b - 2)FA^2) \neq 0 \tag{4.4}$$

We chose $3b = 2$ and $a = -\frac{1}{8\pi^2}$
Then our three-form is given as:
$k = -\frac{1}{8\pi^2}(AdA + \frac{2}{3}A^3)$
This is called the chern-simons three-form. Now, we check the gauge-invariance of this three-form
$k[A^g] = -\frac{1}{8\pi}((A + \nu)F - \frac{1}{3}(A + \nu)^3)$
$= -\frac{1}{8\pi}(A\nu + \nu F - \frac{1}{3}(A^3 + \nu^3 + 3A\nu(A + \nu)))$
$k[A^g] = k[A] - \frac{1}{8\pi^2}Tr(\nu dA - \nu^2 A - \frac{1}{3}\nu^3)$

[4] $tr(t_a t_b) = \frac{1}{2}\delta^{ab}$

Thus our chern-simons form isn't gauge invariant. Here, $\nu = dgg^{-1}$ and $d\nu = \nu^2$
$= k[A] + d(\frac{1}{8\pi^2}Tr(\nu A)) + \frac{1}{24\pi^2}Tr(dgg^{-1})^3$
There are few points to consider here:

- The last term upon integration can give non-trivial contributions upon integration over 3-space of non-trivial topology.
- k will have singularities which we'll deal with by defining separate patch coordinates.
- k in different patch coordinates is related by gauge transformations on the overlap region.

5 Configuration Space

Let A the set of all gauge potentials i.e. set of all lie algebra value one-forms on $\mathbb{R}^4$ and G_* as the set of all gauge transformations defined as:
$G_* = \{g(x)|g(x) \to 1 as \sqrt{x_\mu x^\mu} \to \infty\}$
The gauge invariant set of transformations of configuration is $C = A/G_*$
There exists a natural splitting $A = C \times G_*$ for neighbhorhood of C, A is a fibre bundle over C with G_* as the fibre.
NOTE: Bundle need to be trivial so the splitting might not be globally true.The non-triviality of bundle is related to what's known as Gribov Ambiguity which we won't be covering in this paper.
Properties of A:

- A is an affine space which means any potential can be written as $A_i^{(0)} + h_i$, where h_i is a lie-algebra valued vector field.
- Any two points in A can be connected by a straight line:

$$A_i(x, \tau) = A_i^{(1)}(1 - \tau) + \tau A_i^{(2)}$$

- where $\tau \in [0,1]$ provides a straight line interpolation between the two potentials. Topologically, A is rather trivial with $\pi_1(A) = 1$ and $\pi_0(A) = 0$.

G_* is made up of maps $g(x) : \mathbb{R}^3 \to G$, G being our gauge group and the condition that $g(x) \to 1$ as $|x| \to \infty$. Structure of A/G_* and let's look at a line given in A:

$$A_i(x,\tau) = A_i(x)(1-\tau) + A_i^g\tau \text{ such that}$$
$$A_i(x,0) = A_i(x) \text{ and } A_i(x,1) = A_i^{g_1}(x)$$

Where $A_i^{g_1}(x)$ is the gauge transformed version of $A_i(x)$ by $g_1(x) \in G$. Since, $A^{g_1}i(x)$ is just the gauge transformed version of $A_i(x)$ thus they represent the same configuration i.e. a loop in C accordingly, we'd question it's contractibility. In, QCD $\pi_1(C) = \mathbb{Z}$ and we'll see how to deal with such a configuration space.

6 Path Integral for C with non-trivial topology

[3]

6.1 $\pi_1(C) \neq 0$

This means intuitively that our configuration space has non-contractible loops in it.Now, in order to deal with such configuration spaces, we divide the process into 5 parts as stated below:
$\pi_1(\mathbb{R}^3/\{0\}) = \mathbb{Z}$ with $C = \mathbb{R}^3/\{0\}$

- First we start with writing the winding no. of our path $c$as:
$\nu(c) = -\frac{1}{2\pi}\oint_c \frac{\epsilon_{ij}x^j}{x^2}dx^i = \oint_c \alpha$

- Now, we start by adding a topological term in our action by:

$$\psi(x,t) = \int [dx] e^{iS(x,t,x',t')} e^{i\theta \int_{x'}^{x} \alpha} \psi(x',t')$$

This newly added term has no affect on equations of motion.

- So, let's say we've a path P and we write it as "sum" of our non-contractible path with winding no. ν as C_ν with some arbitrary path p as
$P = p + C_\nu$ and hence we can simplify the following integral as:

$$\int_{p+C_\nu} \alpha = \nu + \int_p \alpha$$

Now, we'll modify our action using this new discovered fact and we'll expand our path as above mentioned sum and then see how it affects our topological term and action in general.

$$\psi(x,t) = [\Sigma_{P\sim p} e^{iS_p + i\theta \int_P \alpha} + \Sigma_{P\sim p+c_1} e^{iS_p + i\theta \int_P \alpha + i\theta} + ...]\psi(x',t')$$
$$\psi(x,t) = \Sigma_\nu \Sigma_{P\sim p+c_\nu} e^{iS_p + i\theta \int_P \alpha + i\nu\theta}\psi(x',t')$$

Here, in the second equation we're summing over all the paths P homotopic to $p + c_\nu$ and we obtain an overall phase factor of $e^{i\theta \int_P \alpha}$ which doesn't matter while calculating the matrix elements.Thus we can ignore it for the above stated purposes.

- $\psi(x,t) = \Sigma_\nu \Sigma_{P\sim p+c_\nu} e^{iS_p + i\nu\theta}\psi(x',t')$

It is unchanged under the transformation $\theta \to \theta + 2\pi$ where, $\theta \in [0, 2\pi]$
$\theta\nu$-term is a topological term, we can write our action as:

$$S = \int dt\mathcal{L} - \frac{\hbar\theta}{2\pi} \int dt \epsilon_{ij} \frac{x^i x^j}{x^2}$$

we can vary the action and see that this new topological term has no affect on equations of motion.

- Now, we generalize what we've learnt to a general C not just $\mathbb{R}^3/\{0\}$
So, to generalize our method we add a $k(c)$ term(in place of $e^{i\theta \int_x^{x'} \alpha}$) such that:

  - $k(c + \delta C) = k(c)$ i.e $k(c)$ should be a topological invariant for our paths.

- $k(-c) = k^*(c)$, $k(c)$ changes sign under the change of orientation.

- $k(c_1c_2) = k(c_1) + k(c_2)$ is just the basic composition law.

In general, we can write:

$$\psi(x,t) = \int [dx] e^{iS(x,t,x',t')} k(c) \psi(x',t') \tag{6.1}$$

$k(c)$ will contain new parameter which are to be treated as new coupling constants in our theory.

6.2 $\mathcal{H}^2(C) \neq 0$

The existence of non trivial second cohomology group means that there exists non-contractible two surfaces in our configuration and we must write our action in such a way to deal with this problem. As before we divide this process into steps so that it's easier to comprehend.

- First we try model a general structure of any action and notice how it can be generalized by the use of what we'll call as weight factor i.e.

$$\psi(x,t) = \int [dx] W(P,x,x') \psi(x',t')$$

  Where P is the path joining the two points x and x'.
  $W(P,x,x') = e^{iS_P(x,x')}$ for every path between x and x', e^{iS_P} gives a complex no. of unit modulus i.e. (set of paths) $\rightarrow U(1)$ and our path integral gives us sum over all paths with e^{iS_P} as weight factor for each path in summation.

- Now, we want our weight factors to have something to do with surface not paths. In order to do this we would do a trick of first selecting a standard path P_0 and P as our chosen path. We write $P - P_0$(closed loop) as the boundary of two-surface i.e. $\partial\Sigma = P - P_0$. If our loops isn't a boundary of a surface then we'd have to modify our construction. $x^\mu(t)$ gives the coordinate of the path and we want to write them in a more generalized manner such that $x^\mu(\sigma,t) = x(\xi)$ where $\xi^i = (\sigma,t)$ parametrize the surface Σ. Now, in our setup and we can start doing some cool math.

- We write the area element of Σ as

$$B_{\mu\nu} = \tfrac{1}{2}\partial_i x^\mu \partial_j x^\nu \epsilon^{ij}$$
$$i, j = 1, 2 \ \xi_2 = \sigma, \xi_1 = t$$

Thus, we can now write the weight factor as:

$$\Gamma(\Sigma) = k \int_\Sigma d^2\xi \tfrac{1}{2} B_{\mu\nu}\partial_i x^\mu \partial_j x^\nu \epsilon^{ij}$$

This integral must be invariant under small deformations in x^μ i.e.

$$\delta\Gamma(\Sigma) = \tfrac{k}{2} \int_\Sigma d^2\xi [\partial_\alpha B_{\mu\nu}\delta x^\alpha \partial_i x^\mu \partial_j x^\nu \epsilon^{ij} + B_{\mu\nu}\partial_i(\delta x^\mu)\partial_j x^\nu \epsilon^{ij} + B_{\mu\nu}\partial_i x^\mu \partial_j(\delta x^\nu)\epsilon^{ij}]$$

$$\delta\Gamma(\Sigma) = \frac{k}{2} \int_\Sigma d^2\xi [\partial_\alpha B_{\mu\nu} + \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\mu\alpha}]\partial_i x^\mu \partial_j x^\nu \epsilon^{ij}\delta x^\alpha + k \oint_{\partial\Sigma} d\xi_i B_{\mu\nu}\partial_j x^\nu \epsilon^{ij}\delta x^\mu \tag{6.2}$$

Here, we've used integration by parts and stokes law to arrive at this conclusion. Thus we require that

$$\partial_\alpha B_{\mu\nu} + \partial_\mu B_{\nu\alpha} + \partial_\nu B_{\mu\alpha} = 0 \tag{6.3}$$

Second term depends on $\partial\Sigma$ and since δx^μ is 0 for P_0 because P_0 is fixed standard path. Thus our chosen surface must satisfy (5.3).

- Now, we express $B_{\mu\nu}$ in terms of a vector potential by:

$$B_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$$
$$\Gamma = k \int_\Sigma d^2\xi (\partial_\mu C_\nu)\partial_i x^\mu \partial_j x^\nu \epsilon^{ij}$$
$$\Gamma = k \int_\Sigma d^2\xi \tfrac{\partial}{\partial \xi^i}[C_\nu \partial_j x^\nu \epsilon^{ij}] = k \oint_{\partial_\Sigma} d\xi_i C_\nu \partial_j x^\nu \epsilon^{ij}$$

We finally arrive at the conclusion that

$$\Gamma = k[\int_P dtC_\mu \frac{dx^\mu}{dt} - \int_{P_0} dtC_\mu \frac{dx^\mu}{dt}] \tag{6.4}$$

The second term in (6.4) is a phase factor for all paths and since can be dropped from the path integral.

- The weight factor can now be given by:

$$W(P, x, x') = e^{i\tilde{S}_P(x,x')}$$
$$\tilde{S}_P = \tilde{S} + \int C_\mu \dot{X}^\mu dt$$

Discussion for Σ becomes irrelevant and the only relevant term left is $B_{\mu\nu}$
The case of our interest is when the two-form B isn't exact but it is close i.e. $B_{\mu\nu} \in \mathcal{H}^2(C)$

$$B = \tfrac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$$
$$B = dC \text{ but } B \neq dC$$

- Example: Monopole field

$$B_{\mu\nu} = \frac{1}{8\pi} \frac{\epsilon_{\mu\nu\alpha\hat{x}^\alpha}}{r^2} dx^\mu \wedge dx^\nu \tag{6.5}$$

$dB = 0$ on $\mathbb{R}^3/\{0\}$. Integrating B over a two-sphere surrounding the origin,

$$\oint_{S^2} B = 1 \text{ and}$$
$$B \neq dC.$$

- We can generalize the path integral for a configuration space C with non-contractible two-surfaces as:

$$\psi(x, t) = \int [dx] e^{iS(x,t,x',t') + i\Gamma_\Sigma} \psi(x', t') \tag{6.6}$$

7 Further Developments and Applications

7.1 Quantum Gravity and Gauge Theories

The difficulty in quantizing gravity is a well known fact amd is well-explained by the negative mass dimension of the Newton constant.The case of gravity is special among other gauge theories.

GR is a gauge theory with general covariance and local lorenz symmetry while QCD is a gauge theory with $SU(3)$ invariance.

In, QCD we get static or stable solution of EOM against with respect to time evolution. However, in GR many solution show a time-dependence for ex, inflation unless we "fine tune" the cosmological constant. In GR, the frame bundle is parallalizable which leads to vierbein forms and torsion but we encounter no such thing in QCD. To examine the special case of gravity in gauge theories, we'll cite a paper by Hidenori Fukaya on why QG is so hard and he does this by giving it's comparison with QCD [1]. Here, we'll only discuss gauge theories for QG.

We start by writing the gauge group for gravity which should be $G = GL(4, \mathbb{R})$. The Principal Bundle for our gauge group could be the frame bundle $F(M)$ where M is our base manifold. But, what makes gravity so special is that it has a parallalizable frame bundle which opens the door for stuff for solder one-form, torsion tensor and affine connections and we'll discuss all that in bit of detail now.

For any $a \in T_v(F(M))$ then

$$< w, a >= 0 < \theta, z >= 0 \tag{7.1}$$

Thus, we mean that a can be by both(dual basis) w and θ. where,

$$w = g^{-1}dg + g^{-1}Ag \text{ (our good ol' connection one form)}$$
$$\theta = g^{-1}e \text{ is our solder one-form which we get from vierbian "stuff".}$$

Where e is a map between $v \in V$ and $t \in T_xM$ such that:

$$v^a = e^a_\mu t^\mu$$

In view of above, it is concluded that gravity is a special case which requires not only gauge fields but also vierbians. We would now introduce torsion two-form:

$$\Theta = d\theta + w \wedge \theta$$

Let's count the degrees of freedom in our theory:
So, our gauge connection($\in GL(4,\mathbb{R})$) in four different directions has $4^2(4) = 64$ d.o.f
Vierbian has $4^2 = 16$ d.o.f.
In total our theory has $64 + 16 = 80$ d.o.f but particles like gravitons need only 2 physical modes.So, from now on we would basically focus on reducing the d.o.f of our current theory. We use a method knows as reduction of principal bundle to try to reduce d.o.f of our theory.we know that:

$$GL(4,\mathbb{R}) = O(4) \times C \tag{7.2}$$

Where C is a component smoothly contractible to a point. To ignore C and use reduced structure group $O(4)$ for principal bundle is called reduction of principal bundle.
We need to find such a way such that reducing the principal bundle is equivalent to applying the metricity condition(from general relativity) so that we're left with 40 d.o.f(we'll see as to why a bit later).
Let's have a deep and close look at the equation of motion for the vierbian:

$$(D_\mu e^a_\mu)^a = (\partial_\nu \delta^a_b + [A_\nu]^a_b)e^b_\mu = 0\text{[5]} \tag{7.3}$$

A_ν is the original $GL(4,\mathbb{R})$ gauge field and D_ν is its covariant derivative.
Now, we'll decompose A_ν into symmetric and anti-symmetric part.

$$A_\nu = A^S + A^A_\nu$$

A^S_ν has $10(4) = 40$ d.o.f and $A^A_\nu = 6(4) = 24$ d.o.f. Thus we can write:

$$(\partial_\nu \delta^a_b + [A_\nu]^a_b)e^b_\mu = -(A^S_\nu + \Delta A^A_\nu)^a_b e^b_\mu \tag{7.4}$$

ΔA^A_ν represents the $SO(4)$ gauge ambiguity(24 d.o.f) i.e. $64 - 24 = 40$ non-trivial constraints freezes A^S_ν as function of e^a_μ and A^A_ν existing as the connection of the remaining $O(4)$ group and we've frame bundle reduction $GL(4,\mathbb{R}) \to O(4)$.

[5]Consider this condition upto $O(4)$ gauge transformations to allow the gauge ambiguity w.r.t A^A_ν

We can define the affine connection as follows:

$$\Gamma^{\rho}_{\mu\nu} = -(A^S_\nu + \Delta A^A_\nu)^a_b e^b_\mu [e^{-1}]\rho_a$$
$$(\overline{D}_\nu e_\mu)^a \equiv (\partial_\nu \delta^a_b + [A^A_\nu]^a_b) e^b_\mu = \Gamma^\rho_{\mu\nu} e^a_\rho \tag{7.5}$$

Where $\overline{D}_\nu$ is the covariant derivative w.r.t $O(4)$ gauge field and from this postulate the metricity condition can also be obtained and it can be shown that:

$$\partial_\rho(g_{\mu\nu}) = \Gamma^\lambda_{\mu\rho} g_{\lambda\nu} + \Gamma^\lambda_{\nu\rho} g_{\mu\lambda}$$

Now, we left with 40 d.o.f and affine connection can be written as:

$$\Gamma^\lambda_{\mu\nu} = [e^{-1}]^\lambda_a (\overline{D}_\nu \rho_\mu)^a = (A^A_\nu)^a_b \eta_{ca} e^b_\mu e^c_\sigma g^{\sigma\lambda} + (\partial_\nu e^a_\mu) \eta_{ca} e^c_\sigma g^{\sigma\lambda} \tag{7.6}$$

Equivalence states that we can make the affine connection locally 0 by coordinate transformations i.e. $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ or equivalently the torsion $\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 0$.
From (6.3) we can write:

$$(\overline{D}_\nu e^\mu)^a - (\overline{D}_\mu e_\nu)^a = 0$$

From Equivalence principle $\Theta = 0$ and it has 24 d.o.f which happens to be the same as $O(4)$ gauge field. Thus we can eliminate A_μ only vierbian with 16 d.o.f are enough to describe. Furthermore, lorenz gauge reduces 6 d.o.f and we get the "regular" GR form where we only need metric and affine connection becomes christoffel symbol. Gauge fixing of general covariance, we lose 4 d.o.f and 4 from gauss's law constraints i.e. $10-4-4=2$ d.o.f which we initially wanted.Now we jump ahead to see the construction of actions here:
$\sigma^b_a = (e^{-1})^\mu_a (D_\nu e_\mu)^b dx^\nu$ and we can construct:

$$S_\theta = \int_M Tr[\sigma \wedge \sigma \wedge F]$$

Where I've used the subscript θ to show that it's analogous to θ term which we encounter in QCD. The leading term would be the cosmological term and next to leading is Einstein-Hilbert action and the matter fields and all three are given as follows:

$$S_\Lambda = \Lambda M_p^2 \int_M e^a \wedge e^b \wedge e^c \wedge e^d \epsilon_{abcd}$$
$$S_{EH} = M_p^2 \int_M e^a \wedge e^b \wedge (\overline{D}A^A)^c_d \epsilon_{abc}$$
$$S_m = \int_M d^4x \overline{\psi} g^{\mu\nu} \gamma_a e^a_\mu (\partial_\nu + [A^A_\nu]^b_c \eta^{cd} \gamma_b \gamma_d) \psi(x)$$

M_p is the plank scale and the total action would be:

$$S = S_\Lambda + S_{EH} + S_m \tag{7.7}$$

So, we saw how and why gravity is so difficult to deal with since it has *Parallalizable frame bundle which introduces vierbian forms and things like torsion tensor and affine connections which one usually won't encounter in gauge theories for QCD.* For more details, I suggest checking out the paper yourself [1].

7.2 Fluctons

If I were to say in one line, I'd say fluctons are just generalization of instantons but let's disect this statement and to see as to why we need them and where they come in. Here we're citing paper by R.Cartas-Fuentevilla and J.M. Solano-Altamirano on Fluctons [2].

The self-duality associated with instantons require very restrictive conditions on our base manifold i.e. self-duality, manifolds with no two dim anti-self dual cohomology, non-compact gauge groups for lorentzian manifolds, etc. Under certain condition there does not exist instanton on the $S^2 \times S^2$ manifold.

It has been found that chern-simons wavefunction in quantum yang mills theory represents the *topological excitations* defined on the whole space of gauge connections of the theory and in specific self-dual subspace corresponding to instantons. So, we directly start from the topological YM-action, a symplectic structure is constructed on complete space of gauge connections leading to a quantum hamiltonian that admits chern-simons wavefunction as an eigenstate with zero energy. Thus now we neither require self-duality in the gauge connections nor the YM field equations to establish the chern-simons state. Thus, let's begin with topological YM action:

$$S_{TYM}(A) = \beta \int_M Tr(F \wedge F)$$

and this action depends on the structure on M.
Here, our "EOM" given by:

$$dF + [A, F] = 0 \tag{7.8}$$

where:

$$F = dA + A \wedge A$$

If we define an operator D as $D = d + [A,]$ then EOM are given by $Ker(DD)$(i.e. parametrized by the complete space of gauge connections). If G is our symmetry gauge group then we construct the moduli space A/G.

If A is a smooth connection satisfying our "EOM" then gauge transformed($g^{-1}Ag + g^{-1}dg$) *isn't* necessarily smooth if g isn't sufficiently smooth. Hence, we only chose smooth connections and that can be done by using gauge fixing condition:

$$d^*\delta A + [A,^* \delta A] = 0$$

and we'll obtain a *Flucton Moduli Space(F.M.S)*
EOM and gauge fixing conditions are elliptic [2] and thus our moduli space is finite dimesnional.
Now, we'd look at the bonuses of developing a theory for "topological excitations" in this manner.
BONUSES:

- We didn't need to invoke neither YM equations nor self-duality to get an elliptic system.

- Instanton can be obtained as a particular case because the self-duality condition $F = \pm^* F$ will reduce S_{TYM} to conventional YM action, bianchi identity. Where we've YM equations:

  $$d^*F + [A,^* F] = 0$$

  Then our F.M.S reduces to Instanton Moduli space(I.M.S).
  Lesson Learnt F.M.S $\subset A$(infinite dimensional) more generally

$$I.M.S \subset F.M.S \subset A$$

- *Unbroken Topological Invariance* For flucton model we don't need neither self-dual connection nor self-dual deformations to maintain unbroken toplogical inavriance of action. Since:

  $$\int_M Tr(F \wedge F) \to \int_M Tr(F + \delta F) \wedge (F + \delta F)$$

  The action is unbroken because deformed action is topological in character as the original one.
  Only condition on tangent space of F.M.S is our gauge fixing condition.

Now, we look at the *Consequences of Starting from* S_{TYM}

- Fluctons in arbitrary background and self-dual deformations.

- Fluctons in self-dual background and arbitrary deformations.

- Self-dual background and Self-dual deformations.
  which is just our instanton case.

Acknowledgements

I'd like to thank everyone who supported and encouraged me to complete this article and a special thanks to Dr. Christian Ferko for valuable guidance and for helping with typesetting.

A Homotopy and Cohomology

A.1 Homtopy Theory

Two paths f and g in a given topological space X are called *homotopic*, iff there exists a continuous function $H(s,t) : [0,1] \times [0,1] \to X$ such that

$$H(0,t) = f(t)$$
$$H(1,t) = g(t)$$

Homotopically equivalent classes of closed paths based on a specific point under the group operation(A.1).This is called the *fundamental group* of the space X based at the point x and is denoted by $\pi_1(X,x)$

$$(f.g)(t) = \begin{cases} f(2t)t \in [0,1/2], \\ g(2t-1)t \in [1/2,1] \end{cases} \tag{A.1}$$

Properties of homotopy group:

- For a path-connected space, $\pi_1(X,x)$ is independent of the basepoint x i.e.

$$\pi_1(X,x) = \pi_1(X,x')$$

- For simply connected space X, $\pi_1(X) = 0$ i.e. it's trivial.

- From the above point we can infer that $\pi_1(X) \neq 0$ would mean the existense of non-contractible loops in our space X.

A.2 Cohomology of Lie Groups

Cohomology is mathematical which can be used to determine the existance of non-contractible surfaces in our space(here specifically lie group). For example $\mathcal{H}^2(C) \neq 0$ means that our configuration has non-contractible two-surfaces.In above reference, we begin their discussion in bit more detail

Suppose $g \in G$ where G is our lie-group and the representation of that element g can be given by $g = exp(i\theta^a t^a)$. Now, we can construct lie-algebra valued one form as follows:

$$\omega = g^{-1}dg = -it^a E_i^a d\theta^i = -it^a E^a$$

where, $E^a = E_i^a(\theta)d\theta^i$ are the frame fields.

we can build $Tr(\omega^k)$ and these will vanish if k is even because of cyclicity of trace.

$$Tr(\omega^2) = Tr(t^a t^b)E^a \wedge E^b = -Tr(t^a t^b)E^b \wedge E^a = -Tr(\omega^2)$$

For odd values we define $\Omega^{(k)} = Tr(\omega^k)$ these are non-zero generally and:

$$d\omega = d(g^{-1}dg) = -g^{-1}dgg^{-1}dg = -\omega^2$$
$$d\Omega^{(k)} = -Tr(\omega^2\omega^{k-1} - \omega\omega^2\omega^{k-2}...) = 0$$

Thus $\Omega^{(k)}$ are closed but not exact and hence will be an element of the cohomology group. Example: $U(1)$ here $g = exp(i\theta)$ and $\Omega^{(1)} = id\theta$, integrating the form over a circle with constraints $\theta \in [0, 2\pi]$

$$-i \oint \Omega^{(1)} = 2\pi,$$

Thus $\Omega^{(1)}$ is closed but not exact if it were exact it could be written in the form like $\Omega^{(1)} = d\alpha$ for some one-form α which is periodic function of θ. This shows that $\Omega^{(1)}$ is an element of $\mathcal{H}^1(U(1), \mathbb{R})$ [4].

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