Lecture 4

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1 Homework problem discussion

Find a complete metric space where a closed and bounded set is not compact. One common example that many gave is the discrete metric on $\mathbb{N}$. Beckner was surprised that nobody gave the example of a unit ball in an infinite-dimensional Hilbert space. This is how it proceeds. We use the **Riesz Lemma** which states that if X is a normed space and S is a closed proper subspace then for every $\epsilon \in (0,1)$, there exists some $x \in X$ with $\|x\| = 1$ such that $\|x - s\| \geq \epsilon$ for all $s \in S$.

Let X be an infinite dimensional normed vector space. First choose some $x_1 \in X$ with norm 1, $\|x\| = 1$. Consider the subspace S_1 spanned by x_1. Using the Riesz Lemma, there exists some x_2 such that $\|x_2 - s_1\|$ for all $s_1 \in S_1$. Proceed inductively to get a sequence $\{x_1, \cdots\}$ where each x_i lies in the unit closed ball but clearly $\|x_i - x_j\| \geq \epsilon$ for $i \neq j$. So, this sequence doesn't have a convergent subsequence in the clsoed unit ball.

2 Abstract Measure on ω and Lebesgue Outer Measure in $\mathbb{R}^n$

Consider a $\sigma-$algebra of sets S which are then called measurable sets.

Consider a measure $\mu : S \to [0, \infty]$ defined on S with there being atleast one set of finite measure and also satisfying the $\sigma-$additive property(see Stein or Rudin for definition). This leads to the monotone convergence property.

A real valued function on the space ω given by $f : \omega \to \mathbb{R}$ is said to be measurable if $f^{-1}[(a, b)] \in S$, that is, if the pullback of intervals is measurable.

In $\mathbb{R}^n$, one must define what the Lebesgue measurable sets are-essentially using Lebesgue outer measure and "approximation" by open sets. Certainly, we want the family of Lebesgue meausrable sets to include the Borel $\sigma-$algebra (eg the $\sigma-$algebra generated by all open and closed sets).

Here we want to define when a set E is Lebesgue measurable. If it is, we define $m(E) = m^{\bullet}(E)$ where $m^{\bullet}(E) = inf_{E \subset \bigcup B_k} \sum vol(B_k)$ where $\bigcup B_k$ is an open rectangle cover of E.

Structural point:For theorems that use the meausre of sets, ask the question if the result still holds true when you replace the measure $m(E)$ by $m^{\bullet}(E)$. $m^{\bullet}(E)$ is defined for all subsets of $\mathbb{R}^n$.

Theorem 1. *$m^{\bullet}(E)$ is countable sub-additive, that is, $m^{\bullet}(\bigcup E_k) \leq \sum m^{\bullet}(E_k)$*

Proof. We use the so-called $\frac{1}{2^n}$ trick. For every E_k, assume that $m^{\bullet}(E) < \infty$ else the result is trivial. Choose a rectangular cover $\bigcup_{i=1}^{\infty} T_{k,i}$ for each E_k such that
$vol(T_{k,i}) \leq m^{\bullet}(E_k) + \frac{\epsilon}{2^k}$ □

Not all subsets of $\mathbb{R}^n$ are Lebesgue-measurable. This is a consequence of set theory properties of the rational numbers and the Axiom of Choice.

3 Construction Of Vitalli

Start with the unit interval $[0, 1]$ and partition it into equivalence classes defined by the relation $x\ y$ iff. $x - y \in \mathbb{Q}$.Obviously, one class contains all the rational numbers.

Using the Axiom of Choice, we can form a set V which has one element/representative from each equivalence class.

Theorem 2. *V is not measurable*

Proof. Check Stein,Shakarchi Chapter 1 □