Lecture 5

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February 6th 2020

1 Introduction

Functions are the basic elements as 'points in a space'. Examples: Cantor sets, fractal sets.

Quite often, one is interested in spaces of functions like the class of continuous functions, L^2 functions. Spaces such as Hilbert and Banach spaces can be realized as a space of functions.

For example, let K be a compact set and let $C(K)$ be the set of continuous functions with support in K(i.e not zero on K). We can consider $C_C(\mathbb{R}^n)$ be the space of continuous functions with compact support. We can turn this into a normed vector space with the **sup/uniform/Chebyshev** norm
$\|f\| = \sup_x |f(x)|$

Question: Why is $C(K)$ is complete but $C_C(\mathbb{R}^n)$ not complete.

Proof. Let f be the limit of a Cauchy sequence of function $\{f_n\}$ which are ocntinuous and have compact support K. f is continuous. The result follows trivially. $C(K)$ is trivial under the uniform norm.

For the second question, the trick roughly is that you take a function which goes to 0 at ∞ and construct a sequence of functions with increasing compact support which converge to it. Consider the sequence of piece-wise functions defined by:
$f_n =$

$$\begin{cases} \frac{1-(x/n)^2}{1+x^2} & x \in [-n, +n] \\ 0 & x \notin [-n, +n] \end{cases}$$

This sequence $\{f_n\}$ converges to $\frac{1}{1+x^2}$ which has support $\mathbb{R}$ which is not compact. Dylan Alexander had a similar example. Karthik had another example of a slightly different flavor but I forgot it. $\square$

2 Functions and their Role in Integration

Every Reimann integrable function is continuous except for sets of measure zero. The aim is to partition the range such that pullback of open intervals $f^{-1}(a, b)$ is a measurable set , that is, an element of Ω.

The measurable sets should form a $\sigma-$ algebra of sets in Ω. Set theory comes into play in understanding these $\sigma-$ algebras. This property of pullbacks gives the defintion of a measurable function.

2.1 Remark

Let S be a set function which maps sets into $\mathbb{R}^+$. We wish for S to be countably sub-additive with respect to union, that is,
$S(\bigcup E_i) \leq S(E_1) + S(E_2) + \cdots$

Theorem 1 (Borel-Cantelli Lemma)**.** *Let $\{E_l\}$ be a countable collection of sets with the property that $\sum\limits_k S(E_l) < \infty$. The number of points in infinitely many E_l, say A, has size* 0

Proof. Write $A = \bigcap\limits_{k=1}^{\infty} (\bigcup\limits_{l=k}^{\infty} E_l)$

$\Rightarrow A \subset \bigcup\limits_{l=k}^{\infty} E_l = A_k$ $\{A_k\}$ is a decreasing sequence. $\Rightarrow S(A) \leq S(A_k) = S(\bigcup\limits_{l=k} E_l) \leq \sum\limits_{l=k}^{\infty} S(E_l) \to 0$ as $k \to \infty$

$\Rightarrow S(A) = 0$. Choosing S to be $m^\bullet$, the Lebesgue outer measure implies the result. $\square$

Rudin has an alternate proof using integration for the Lebesgue measure or the case that S is a measure. See Chapter , section for a proof.

Suppose $\{g_k\}$ is a sequence of functions. We aim to give up not only uniform convergence but also convergence. An interesting

question is to characterize the set of points where $g_k(x)$ converges.
At every point, define $f(x) = \sup_k g_k(x)$ and $h(x) = \inf_k g_k(x)$, If $\{g_k\}$ is measurable then $f(x)$ is measurable $\Rightarrow h(x)$ is measurable. Check the first chapter of Stein, Shakarachi if you want a proof of this. Even if $\{f_k(x)\}$ doesn't converge at any point, we wish to have functions based on f that can always be defined to converge.

$F(x) = \lim\sup(g_k(x)) = \inf_l \sup_{l \leq k} g_k(x)]$ and $H(x) = \lim\inf(g_k(x)) = \sup_l \inf_{l \leq k} x_k$

The set $X = \{x | G(x) = H(x)\}$ where g_k converges.

This will be useful to prove an interesting integration formula known as Fatou's lemma which is equivalent to the dominated convergence theorem.