A Road to the Grothendieck Spectral Sequence:Derived Functors

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1 Introduction

These notes are meant to be a nice introduction to Derived Functors. These notes are compiled from my blog post series 'A Road to the Grothendieck Spectral Sequence' and hence will really be geared to collecting results which lead to the result. However, I have decided to make the notes more comprehensive regardless.

An object I in a category C is said to be an injective object if for every morphism $f : X \to I$ and every monomorphism $i : X \to Y$, there exists a morphism $h : Y \to I$ extending the map f such that the diagram commutes.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & Y \arrow[dl, dotted, "h"] \\ I & \end{tikzcd}$$

In the abelian category setting, the importance lies in the fact that I is an inejctive object if and only if the Hom functor is $Hom_C(--, I)$ is exact. If an injective object is at the beginning of a short exact sequence in C, the sequence splits.

A category C has enough injectives if for every every object $X \in C$, there exists a monomorphism $X \hookrightarrow I$ for some injective object I.

An injective reslution of an object $X \in C$, an abelian category is a resolution $0 \to X \to I_0 \to I_1 \to \cdots$ where I_k are injective objects. In particular, this is a quasi isomorphism of chain complexes in C given by $X \to I_\bullet$ where X is the complex $0 \to X \to 0 \cdots$.

2 Derived Functors

Let's say A is an abelian category. Consider a short exact sequence in A:

$0 \to L \to M \to N \to 0$

An exact functor is a functor between abelian categories which preserves such sequences. Taking the direct sum, for example, preserves preserves a short exact sequence. Accordingly, we say that functors are left and right exact if they preserve the left and right parts of the short exact sequence. It is a well known that in the case of the category of modules over a ring R,the covariant Hom functor is left exact. $0 \to L \to M \to N \to 0$, then

$0 \to 0 \to Hom(A, L) \to Hom(A, M) \to Hom(A, N)$ where $F_A(X) = Hom(A, X) : A \to Ab$ is the Hom functor.

The tensor product functor $G(X) = X \otimes_R M$ where M is an $R - module$ is known to be right exact. Many of these facts and their proofs can be found in many standard texts on commutative algebra or homological algebra. Some of the arguments in these proofs tend to be quite arduous to work through. An easier way to prove it is to notice that the functors are adjoint and show the equivalent statement that Hom preserves limits which is not too difficult. Taking the dual, yields the statement for tensoring and infact, it yields the completely general version which states that left adjoint functors preserve finite colimits using the Yoneda Lemma. See my other post on adjoint functors if you wish to learn more.

The point of derived functors(which I'll shortly introduce) is to take these 'incomplete' exact sequences where we've 'lost data' to try and construct a long exact sequence. Remember that chain complex

$\cdots \to Cn+1 \xrightarrow{\partial} C_n \xrightarrow{\partial}$

equipped with 'boundary maps(which I've not labelled) allows us to compute homology $H_n = \frac{ker\partial}{Im\partial}$ which measures how far the sequence is from being exact. ALL the data is in the chain complex itself and the entire process of computing homology/cohomology is just a formalization which turns out to be quite handy. In the same manner, one should treat derived functors as a comfortable formalization using the data we possess. For an object $X \in A$, we know just one thing:that there is an injective resolution:

$0 \to X \to I_0 \to I_1 \to \cdots.$

Now, we take our left exact functor F(contravariant Hom for example) and apply it to the injective resolution to get $F(X \to I_\bullet)$

$0 \to F(X) \to F(I_0) \to F(I_1) \to \cdots$

Now, just 'take homology/cohomology' and call it the right derived functor

$R^iF(X) = H^i(F(X \to I_\bullet)).$

But wait, did you notice something?On the left hand side, I don't refer to the injective resolution. That is the essence of the construction, it is independent of the injective resolution of X upto canonical isomorphism. A proof of this can be found in any standard textbook on algebra in the homological algebra section(Aluffi, Dummit and Foote;I think Hatcher also proves it for the dual case in the section on cohomology). Let's take a closer look at this sequence. Since F is left exact, we get the following exact sequence:

$0 \to F(A) \to F(I_0) \to F(I_1)$

W get $R^0(F(X)) = F(X)$. If we F is exact, then the all $R^i(F(X)) = 0$ would be trivial for $i > 0$! I guess you could think of this as a way to encode approximation just like in homology/cohomology.

The converse isn't necessarily true. An object $X \in A$ is said to be $\mathbf{F-acyclic}$ for a left exact functor if $R^i(F(X)) = 0$ for $i > 0$.

The final step of this formalization is ensuring that we have the long exact sequence

Lemma 1. *If $0 \to L \to M \to N \to 0$ is a short exact sequence in an abelian category A with enough injectives and F is a left exact functor, there is an LES*

$0 \to L \to M \to N \to$

$\to R^1(F(L)) \to R^1(F(M)) \to R^1(F(N)) \to \cdots$

If we're given a short exact sequence $0 \to I \to A \to B \to 0$ in an abelian category where I is an injective object. From the map $I \to I$ and the monomorphism $I \hookrightarrow A$, we can extend to a map $A \to I$ such that the composition is the identity. Using the Splitting Lemma, one obtains a non-canonical splitting $A \simeq I \oplus B$. Applying F,

$0 \to F(A) \to F(I) \oplus F(B) \to F(B).$

The identity map $B \to B$ factors through the projection map $\pi : A \to B$, the same holds true after applying F, in particular, the last map is surjective!

$0 \to F(A) \to F(I) \oplus F(B) \to F(B) \to 0.$

Proof. Step 1:A morphism between objects with injective resolution induces a chain map between the resolutions

Let $\phi : A \to B$ be a morphism between two objects with injective resolutions $A_\bullet, B_\bullet$. In the figure below, the map ϕ_0 is constructed from the fact that $d_0 : A \to I_0$ is a monomorphism and there is a map $A \to B \to I'_0$ from A to an injective object.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\phi"] & I_0 \arrow[r] \arrow[d, dotted, "\phi_0"] & I_1 \arrow[r] \arrow[d, dotted, "\phi_1"] & I_2 \arrow[r] \arrow[d, "\phi_2"] & \cdots \\
0 \arrow[r] & B \arrow[r] & I'_0 \arrow[r] & I'_1 \arrow[r] & I'_2 \arrow[r] & \cdots
\end{tikzcd}$$

Now, there is a monomorphism $I_0/ker(d_1) = I_0/Im(d_0) = Coker(A \to I_0) \hookrightarrow I_1$. Next, note that by the commutativity of the square already defined, ϕ_0 takes $Im(d_0) = Ker(d_1)$ to $Ker(d'_1) = Im(d'_0)$ by the fact that $d'1d'_0 = 0$ by exactness of the lower sequence. This means that the map ϕ_0 induces a morphism $h_0 : Coker(A \to I_0) \to Coker(B \to I'_0)$ and by exactness, we can compose this with $B/Ker(d'_1)$ to get a map $Coker(A \to I_0) \to I'_1$. Since I'_1, we get the required map ϕ_1. Inductively continue this process to get the entire chain map. Note that all the maps defined from the injective object property are not unique.

Step 2:Proving that any two such extensions are chain-homotopic

Let f_n, g_n be two chain maps from $A \to I_\bullet$ to $B \to I'_\bullet$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "d_0"] \arrow[d, "\phi"] & I_0 \arrow[r, "d_1"] \arrow[dl, "\sum_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & I_1 \arrow[r, "d_2"] \arrow[dl, "\sum_1"'] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & I_2 \arrow[r, "d'_3"] \arrow[dl, "\sum_2"'] \arrow[d, shift right, "f_2"'] \arrow[d, shift left, "g_2"] & \cdots \\
0 \arrow[r] & B \arrow[r, "d'_0"'] & I'_0 \arrow[r, "d'_1"'] & I'_1 \arrow[r, "d'_2"'] & I'_2 \arrow[r, "d'_3"] & \cdots
\end{tikzcd}$$

To construct the chain homotopy $\sum_n : I_n \to I_{n-1}$ requires $d'_n \sum_n + \sum_{n+1} d_{n+1}$ =(1). We define the chain homotopy inductively too. Let's start with sum_1. By the commutativity of the first square f_0, g_0 take the same values on $Im(d_0)$, we get that $f_0 - g_0$ factors through $Coker(d_0)$. Using the exactness of the top row, we get a map $I_0 \to Coker(d_0) \hookrightarrow I_1$. Since I'_0 is an injective object, we get the required map $\sum_1 : I_1 \to I'_0$ and hence $f_0 - g_0 = \sum_1 d_1$.

Assume that we have defined the maps so $\sum_r$ so that (1) holds for all $r \leq n$. We must construct $\sum_{n+1}$. Notice that from (1), we have that $(f_n - g_n - d'_n \sum_n)d_n = d'_n f_{n-1} - d'_n g_{n-1} - d'_n \sum_n d_n = d'_n(f_{n-1} - g_{n-1} - \sum_n d_n)$

$= d'_{n+1} d'_n \sum_{n-1} = 0$ by exactness and commutativity. Use the same argument above with the new map $f_n - g_n - d'_n \sum_n$ to construct $\sum_{n+1}$.

Step 3:Exact sequence induces exact sequence in induced chain map from injective resolution(Horseshoe Lemma)

Finally, after all that annoying diagram chasing, we've gotten the most basic results. The gist of the story is that a morphism between two objects induces a unique chain map between their injective resolutions up to chain homotopy.

Let $0 \to A \to B \to C \to 0$ be an exact sequence with resolutions $A \to I_\bullet$ and $C \to K_\bullet$. We haven't specified the resolution $B \to J_\bullet$ as we're going to construct it. Define $J_n = I_n \oplus K_n$. Let's deal with the first row.

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & Coker(d_1) \arrow[r] \arrow[u] & Coker(e_1) \arrow[r] \arrow[u] & Coker(f_1) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & I_0 \arrow[r, "i"] \arrow[u] & J_0 \arrow[r, "\pi"] \arrow[u, dotted] & K_0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, "d_1"] & B \arrow[r] \arrow[u, dotted, "e_1"] \arrow[ul] \arrow[ur] & C \arrow[r] \arrow[u, "f_1"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Let $i : I_n \to J_n, \pi : J_n \to K_n$ be the usual inclusion and projection maps. We define the map e^1 by defining the map onto the components(note that this is an abelian category). The map from $B \to C \to K_0$ is just that. $A \hookrightarrow$ is a monomorphism. Since I_0 is an injective object, we can extend to a map $B \to I_0$. It is easy to see that it commutes.

Now, I leave the inductive step to the reader. Use the Snake Lemma to get an exact sequence of cokernels and repeat the procedure.

Step 4:Obtaining the long exact sequence

Now, that we have setup everything needed for the lemma, we see that a short exact sequence $0 \to A \to B \to C \to 0$ yields a short exact sequence in $Ch_\bullet(C)$ of the injective resolutions.

Chopping off A, B, C doesn't affect exactness upon replacement with 0. A short exact sequence of the form $0 \to I_n \to J_n \to K_n$ splits as we've already shown. We've also shown that on the application of the left exact functor F, it gets upgraded to a short exact sequence

$0 \to F(I_n) \to F(J_n) \to F(K_n) \to 0$. Now, apply the snake lemma to this short exact sequence of chain complexes ,take homology and voila,

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] & F(B) \arrow[r] & F(C) \arrow[dll] \\
& R^1F(A) \arrow[r] & R^1F(B) \arrow[r] & R^1F(C) \arrow[dll] \\
& \cdots & &
\end{tikzcd}$$

∎

Note that $R^0(F) \simeq F$. I don't think I've proven this but it is simple enough.

As we've already mentioned in the previous post, an object $A \in C$ is called F-acyclic if $R^i(F(A)) = 0$ for all $i > 0$. It's great that that all our objects have injective resolutions but it doesn't help that it is not really possible to calculate these resolutions. Towards this goal, let's define an acyclic resolution:

An object A is said to have an $F-acyclic$ resolution if there is an exact sequence

$0 \to A \to C_0 \to C_1 \to \cdots$ where C_i for $i \geq 0$ are all $F-acyclic$.

Theorem 1. *For any $F-acyclic$ resolution $A \to C_\bullet$, the following holds true:*

$R^i(F(A)) \simeq H^i(F(A \to C_\bullet))$

This isomorphism also satisfies the following natural property:

If we have the following commutative chain map for two acyclic resolutions $A \to I_\bullet$ and $B \to I'_\bullet$,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\phi"] & I_0 \arrow[r] \arrow[d, "\phi_0"] & I_1 \arrow[r] \arrow[d, "\phi_1"] & \cdots \\
0 \arrow[r] & B \arrow[r] & I'_0 \arrow[r] & I'_1 \arrow[r] & \cdots
\end{tikzcd}$$

Then, the following naturality square commutes

$$\begin{tikzcd}
R^nF(A) \arrow[r, "\simeq"] \arrow[d, "R^n(T\phi)"] & H^n(FI) \arrow[d, "H^n(T\phi_\bullet)"] \\
R^nF(B) \arrow[r, "\simeq"] & H^n(FI')
\end{tikzcd}$$

Proof. $0 \to A \to C_0 \to C_1 \to \cdots$

Let K_0 be the cokernel of the map $A \to C_0$. Similarly, let K_i be the cokernel of the map $C_{i-1} \to C_i$.We can break up the exact sequence into the following short exact sequences:

$0 \to A \to C_0 \to K_0 \to 0$ and

$0 \to K_{i-1} \to C_i \to K_i \to 0$ for $i \geq 1$.

Applying F to both types of exact sequences and using $R^iF(C_j) = 0$ for $i > 0$, we get

$R^1(F(A))$ is the cokernel of the map $F(C_0) \to F(K_0)$,

$R^i(F(A)) = R^{i-1}(F(K_0))$ for $i > 1$,

$R^1(F(K^{i-1}))$ is the cokernel of the map $F(C_i) \to F(K_i)$ for $i \geq 1$ and $R^i(F(K_{j-1})) \simeq R^{i-1}(F(K_j))$ for $i > 1$ and $j \geq 1$.

Take $j = 1$, for $i > 1$ use the first equality to get $R^i(F(A)) \simeq R^{i-1}(F(K_0))$. Use the second type of equality to get $R^i(F(A)) \simeq R^{i-2}(F(K_1))$. Use the equality again, repeatedly, to get $R^i(F(A)) \simeq R^1(F(K_{i-2}))$. So, the problem reduces to calculating $R^1(F(K_i))$ for $i \geq 0$ and additionally finding $R^1(F(A))$.

Step 1:Finding $R^1(F(K_i))$ for $i \geq 0$ and $R^1(F(A))$

Consider the part of the sequence $\cdots \to F(C_{i+1}) \to F(C_{i+2}) \to \cdots$. There exists a unique map $F(K_{i+1}) \to F(C_{i+2})$ that factors through $F(C_{i+1}) \to F(K_{i+1})$. From the diagram below,

$$\begin{tikzcd}
 & F(C_{i+1}) \arrow[r, "\simeq"] \arrow[d] & F(C_{i+1}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F(K_{i+1}) \arrow[r] & F(C_{i+2}) \arrow[r] & F(C_{i+3}) &
\end{tikzcd}$$

Using the Snake Lemma, we get the exact sequence,

$0 \to R^1(F(K_i)) \to coker(F(C_{i+1}) \to F(C_{i+2})) \to F(C_{i+3})$

Naturally, $R^1(F(K_i)) \simeq H^{i+2}(F(C_\bullet))$ and this gives from the previous equality, $R^i(F(A)) \simeq H^i(F(C_\bullet))$ for $i > 1$.

It remains to deal with the case of $R^1(F(A))$. We've seen that $R^1(F(A))$ is the cokernel of the map $F(C_0) \to F(K_0)$. Replace i by -1 in the above diagram and do the same procedure to get the result.

Step 2:Naturality

Left as an exercise to the reader.

□

3 Projective objects, dimension shifting

Before I continue on, I must make sure to inform the reader of the projective object. An object $P \in C$, an abelian category is said to be a projective object if for every epimorphism $B \to A$ and a map $\phi : P \to A$, ϕ can be extended to B so that the diagram commutes:

$$\begin{tikzcd}
B \arrow[r, two heads] & A \\
P \arrow[u, dotted, "\bar{\phi}"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

As you may have noticed, this is the dual of the injective object and in fact, everything that I have proven and will continue to prove has a dual statement in terms of projective resolutions, projective objects and so on. I must also note something else extremely important. In Lemma 1, the hypothesis can be weakened to A having an acyclic resolution instead of a injective resolution.

Theorem 2. *Suppose $A \in C$, an abelian category and we have the following exact sequence:*

$0 \to A \to I_0 \to I_1 \to \cdots \to I_m \to M \to 0$

If F is a left exact funtor, then there are canonical isomorphisms $R^n(F(M)) \simeq R^{n+m+1}(F(A))$ for $n \geq 1$and we have the right exact sequence:

$F(I^m) \to F(M) \to R^{m+1}(T(A)) \to 0.$

Proof. Take the kernel K_m of the morphism $I_m \to M$ and K_i of $I_{m-1} \to K_m$ and so on to split the sequence into many short exact sequences. Now, apply F to these sequences and use Lemma 1 to get the corresponding long exact sequence for each short exact sequences, notice that the terms involving I_k are destroyed and combine all equalities to get the result.

□