

TASI LECTURES ON TOPOLOGICAL FIELD THEORIES AND DIFFERENTIAL COHOMOLOGY

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Abstract

These are lecture notes expanding upon a set of lectures given by G.M. at the TASI 2023 School. Part I is an introduction to topological field theory, including extended topological field theory. Part II is an introduction to generalized Abelian gauge theories and their relation to differential cohomology.

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1 General Plan

This is a write-up based on lectures given by one of us at the Theoretical Advanced Study Institute (TASI) 2023 summer school [TAS23]. The lectures are meant to be very elementary introductions to two distinct (but closely related) subjects, both of which play an important role in modern discussions of generalized notions of symmetry and topological effects in quantum field theory, supergravity, and string/M-theory. Accordingly, the notes are divided into two parts. The first part addresses topological field theory. This was the topic of the first two TASI lectures. The videos are at [Moo23b, Moo23c]. The second part concerns generalized Abelian gauge theories and differential cohomology. This was the topic of the last two TASI lectures. The videos are at [Moo23d, Moo23e].

We comment here on some other pedagogical sources for this material. For the part on topological field theory, some older reviews include [Tho91, BBRT91]. At some points, we will closely follow the review [MS06] and the closely related chapter in [ABC⁺09]. Three other useful reviews are [Bar05, DKR11, CR17, CDZR23]. A recent set of lectures, the Simons Lectures On Categorical Symmetry [C⁺24], has much useful material.

For part two on differential cohomology, a key reference is the original paper by J. Cheeger and J. Simons [CS85], which still makes for good reading. An important foundational paper by M. Hopkins and I. Singer is [HS02]. Pedagogical material and reviews can be found in [Fre00, FMS06a, FMS06b, Moo11, Sza12, CFLS19, Deb23]. There are two books on the subject: [ADH21] and [BB14]. Some other references with summaries or alternative

viewpoints are [FN22, App. A], [Fre24, Sec. 4], [Bun12, Sch13, BNVar] and [HTY20, Sec. 2].

We supplement the text with several exercises for the reader. These exercises can be found in gray bounded boxes labeled by the rather suggestive weightlifting icon (). Numerous important facts appear in red bounded boxes labeled by the book icon () – a green signal for some reflection.

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2 Preliminary: Classical And Quantum Mechanics

Although most of these lecture notes deal with highly abstract mathematical ideas, it is worthwhile to take a moment and reflect on the basic physical principles that gave rise to them. In this section we give a lightning overview of classical and quantum mechanics.

Physics is the study of nature based on the idea that what we can observe is governed by mathematical rules. We aim to discover and describe these rules by measurement and experiment. The goal of a theoretical physicist is to explain (“postdict”) the results of a measurement, or, better, to predict the results of a measurement.

In physics, what we measure is called an *observable*. Examples include ticks of a clock (measuring time intervals), temperature, particle positions, particle momenta, energy, force, etc. Let \mathcal{O} be some set describing a mathematical model for observables. The description of

a physical system is called its *state*. Let \mathcal{S} be some set describing a mathematical model for the states of a physical system. Then, from the mathematical perspective, a measurement is described by a pairing of \mathcal{S} and \mathcal{O} .

In classical mechanics the result of a measurement is, at least naively, a real number. Let us temporarily adopt this notion, although we will revise it. If the result of a measurement is a real number then it is described by a real-valued function:

$$\mathcal{S} \times \mathcal{O} \rightarrow \mathbb{R} . \quad (2.1)$$

Let us describe this pairing for a general system described by classical mechanics.

A classical mechanical system is based on a symplectic manifold (X, ω) . Here X is a manifold. We will assume it is finite-dimensional for simplicity. ω is a symplectic form on X . This means ω is a smooth real-valued 2-form - we will write $\omega \in \Omega^2(X)$ - which is closed $d\omega = 0$ and *nondegenerate*. The term nondegenerate means that in local coordinates we can write

$$\omega = \frac{1}{2!} \omega_{\mu\nu}(x) dx^\mu dx^\nu , \quad (2.2)$$

and at each point $x \in X$ the anti-symmetric matrix $\omega_{\mu\nu}(x)$ is invertible. Note that this implies, in finite dimensions, that the dimension of X is even: $\dim_{\mathbb{R}} X = 2\ell$ for some positive integer ℓ . Note that ω^ℓ is a nowhere zero form of maximal degree. Thus X is also orientable. The *Liouville measure* is

$$\mu_{\text{Liouville}} := \frac{\omega^\ell}{\ell!} . \quad (2.3)$$

In finite dimensions we can equivalently say that ω^ℓ is nowhere-vanishing. This implies that the determinant of $\omega_{\mu\nu}(x)$ is nonzero, and hence it is invertible. A crucial theorem of symplectic geometry is that a symplectic form has no local invariants. That is, in a neighborhood of any point we can skew-diagonalize $\omega_{\mu\nu}(x)$. Put differently, there exists a local coordinate system (p_i, q^i) , $i = 1, \dots, \ell$ with

$$\omega = \sum_{i=1}^{\ell} dp_i \wedge dq^i . \quad (2.4)$$

Such coordinates are known as *Darboux coordinates*. They reader should ask herself/himself if such a coordinate system is unique. It is also good to compare the case with Riemannian geometry based on a symmetric rank two matrix $ds^2 = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ where $g_{\mu\nu}(x)$ is invertible. Here there are local invariants, such as the curvature, and in any neighborhood with nonzero curvature the best one can do to simplify the metric is use Gaussian normal coordinates around a point p so that in a neighborhood of p we have

$$g_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{1}{3} R_{\mu\lambda\nu\rho}(p) x^\lambda x^\rho + \mathcal{O}(x^3) , \quad (2.5)$$

where $R_{\mu\nu\rho}(p)$ is the Riemannian curvature at p . An important special case of a classical system is the cotangent bundle to a vector space, $T^*\mathbb{R}^\ell$, where there is a global system of Darboux coordinates. q^i are coordinates on \mathbb{R}^ℓ and p_i are the dual coordinates on the fiber of the cotangent bundle. For example, a particle moving in \mathbb{R}^ℓ would be described, in classical physics, by such a symplectic manifold. One generally refers to $q \in \mathbb{R}^\ell$ as the “position” and p in the fiber, (isomorphic to the dual space $(\mathbb{R}^\ell)^\vee$) as the “momentum.”

In the context of classical physics the set of observables \mathcal{O} is the set of real-valued functions on X . A complete description of a physical system is a point on X so $\mathcal{S} = \mathcal{X}$. The measurement pairing (2.1) is simply the evaluation map:

$$\wp(x, f) := \text{ev}(x, f) := f(x) \in \mathbb{R}. \quad (2.6)$$

This is not quite the end of the story. It is useful to consider a more general notion of a physical state from an algebraic point of view. The algebraic viewpoint is based on the notion of a C^* -algebra:

Definition 2.7 (C^* -algebra). A C^* -algebra is a complex associative Cauchy complete normed algebra with an anti-linear involution, denoted $*$ which satisfies $(ab)^* = b^*a^*$ together with the C^* identity

$$\|a^*a\| = \|a\|^2. \quad (2.8)$$

We can introduce the C^* -algebra $\mathcal{A}(X)$ of continuous complex valued functions on X (we assume X is compact here).¹ The $*$ -involution of the C^* algebra is complex-conjugation of the function. The multiplication is pointwise multiplication of functions. The norm of a function f is the supremum over $x \in X$ of $|f(x)|$. The C^* identity is readily checked.

In the theory of C^* -algebras there is a notion of *states* (motivated by the physical examples). They are positive linear functionals on the algebra, unital, if \mathcal{A} is unital. For the case $\mathcal{A}(X)$ the positive linear functionals are the positive probability measures on X . In this language the physical states corresponding to points $x \in X$ are Dirac measures supported at x .

It is interesting to note that the observables also have a natural interpretation in terms of the algebra $\mathcal{A}(X)$. Note that the observables are the real points in $\mathcal{A}(X)$, that is, the fixed points under the $*$ -involution, so we can write:

$$\mathcal{O} = (\mathcal{A}(X))_{\mathbb{R}}. \quad (2.9)$$

Moreover, we can encode the existence of the symplectic form algebraically by noting that

¹When X is noncompact one should take $\mathcal{A}(X)$ to be the continuous functions that vanish at ∞ . This means that for any $\epsilon > 0$ there is a compact set $K \subset X$ so that $|f(x)| < \epsilon$ for all $x \in X - K$. Note that the algebra is nonunital.

it induces a *Poisson bracket* on the algebra of functions:

$$\{f, g\}(x) := \omega^{\mu\nu}(x) \partial_\mu f(x) \partial_\nu g(x) . \quad (2.10)$$

If we adopt positive normalized probability measures on X as our set \mathcal{S} of classical states then the pairing (2.1) must be modified so that the codomain is the space \mathcal{P} of probability measures on the real line:

$$\mathcal{S} \times \mathcal{O} \rightarrow \mathcal{P} . \quad (2.11)$$

On a Borel measurable set $E \subset \mathbb{R}$ the probability that a measurement in a state $\mu \in \mathcal{S}$ of an observable $f \in \mathcal{A}(X)_\mathbb{R}$ will take values in E is

$$\wp(\mu, f)(E) = \mu(f^{-1}(E)) . \quad (2.12)$$

and the expected value of the measurement is

$$\langle f \rangle = \int_X f \mu . \quad (2.13)$$

For later comparison with quantum mechanics it is useful to rewrite (2.12). Given f and E we can define the characteristic function

$$P_f(E)(x) = \begin{cases} 1 & x \in f^{-1}(E) \\ 0 & \text{otherwise} \end{cases} \quad (2.14)$$

Note that, while it is not a continuous function, it satisfies $(P_f(E))^2 = P_f(E)$ and so behaves as a projection operator on measurable functions. Now (2.12) can be written:

$$\wp(\mu, f)(E) = \int_X P_f(E) \mu . \quad (2.15)$$

This should be compared to the quantum mechanical Born rule (2.19) below.

The description (2.12) and (2.15) of classical measurement is actually closer to reality than the real-number-valued pairing (2.6). In a real experiment – even in the description of classical physics – we can never exactly specify a physical state. The result of an experiment is only a statistical distribution on the set of possible answers. Indeed experimentalists typically describe their results with error bars corresponding to statistical and systematic errors. Finally, time evolution of a classical physical system is described by choosing a particular observable – the Hamiltonian $H \in \mathcal{A}(X)_\mathbb{R}$. Poisson bracket with H induces a one-parameter family of automorphisms of $\mathcal{A}(X)$ preserving the real points. For example, for the standard example of the harmonic oscillator $X = \mathbb{R}^2$ with $\omega = dp \wedge dq$ and $H = \frac{1}{2}(p^2 + q^2)$.

If we write our state relative to the Liouville measure

$$\mu = \rho \frac{\omega^\ell}{\ell!} , \quad (2.16)$$

then

$$\frac{d\rho}{dt} = -\{H, \rho\} . \quad (2.17)$$

Finally, thanks to Gelfand duality one can replace the symplectic manifold X by $\mathcal{A}(X)$ with its Poisson algebra. Thus everything can be derived by starting from the Poisson algebra of functions.

Let us now proceed to the quantum mechanical description of states and observables. In classical mechanics $\mathcal{A}(X)$ is a commutative C^* algebra. In quantum mechanics we consider more generally a noncommutative C^* algebra \mathcal{A} .² In quantum systems with a semiclassical limit the noncommutative algebra should be a noncommutative deformation of $\mathcal{A}(X)$. The set of states \mathcal{S} will be the positive unital linear functional on \mathcal{A} where, if ℓ is a linear functional, it is positive if $\ell(a^*a) \geq 0$ for all $a \in \mathcal{A}$. All C^* algebras can be represented (nonuniquely) as a subalgebra of the algebra of bounded operators on a complex Hilbert space \mathcal{H} . We will take \mathcal{H} to be separable. Normal positive linear functionals ℓ are represented by a *density matrix* usually denoted ρ . The density matrix will be a positive traceclass operator in \mathcal{A} and $\ell(a) = \text{Tr}_{\mathcal{H}}(a\rho)$. Since our state is a unital functional $\text{Tr}_{\mathcal{H}}\rho = 1$.

Following the definition in classical mechanics the set \mathcal{O} of observables is the subspace of real points in \mathcal{A} , that is, the fixed points under the $*$ -involution. If \mathcal{A} is represented as a C^* subalgebra of $B(\mathcal{H})$ then \mathcal{O} is the set of bounded self-adjoint operators on \mathcal{H} which lie in \mathcal{A} .³

In quantum mechanics we again have a pairing of the form (2.11) known as the Born rule. To write the Born rule, we need to recall the spectral theorem for self-adjoint operators [Hal63, RS72, Rud90]. The spectral theorem asserts that for a self-adjoint operator T , there is an associated projection-valued measure P_T . If T has a discrete spectrum of eigenvalues $\{\lambda_i\}$, then for a Borel-measurable set $E \subset \mathbb{R}$, we have

$$P_T(E) = \sum_{\lambda_i \in E} P(\lambda_i) , \quad (2.18)$$

where $P(\lambda_i)$ is the projector onto the eigenspace for λ_i . The Born Rule can then be stated

²From some points of view it is better to replace a C^* algebra by a W^* algebra. The latter is an abstract version of a von-Neumann algebra.

³In the classical case $\mathcal{A}(X)_{\mathbb{R}}$ forms a subalgebra of $\mathcal{A}(X)$. In the noncommutative case \mathcal{O} is not a subalgebra of \mathcal{A} since $(ab)^* = b^*a^* = ba$ need not be the same as ab when a and b do not commute. One can define a *Jordan algebra* structure on \mathcal{O} where $a \circ b := \frac{1}{2}(ab + ba)$. This is a commutative, but not associative algebra structure. Jordan algebras have found many interesting applications in mathematics, but thus far have had played a limited role in physics.

as:

$$\wp_{\rho,T}(E) = \text{Tr}_{\mathcal{H}}(P_T(E)\rho) . \quad (2.19)$$

In this probability distribution the expected value of T in the state ρ is

$$\langle T \rangle_{\rho} = \text{Tr}_{\mathcal{H}}(T\rho) . \quad (2.20)$$

Returning to classical mechanics for a moment, the set of states \mathcal{S} – viewed as the set of probability measures on X is a convex set: If $t \in (0, 1)$ and μ_1 and μ_2 are positive unital measures then so is $t\mu_1 + (1-t)\mu_2$. It is therefore of interest to look at the extremal points of \mathcal{S} . These are precisely the Dirac measures, i.e., the physical states of minimal uncertainty. In the quantum mechanical case where \mathcal{A} is noncommutative the extremal points of the convex set of physical states are given by rank-one projection operators. These are known as *pure states*. Here is where we encounter a profound difference between classical and quantum mechanics. In classical mechanics, in a pure state $\mu(p)$ (given by a Dirac measure supported at $p \in X$) the variance of any physical observable $f \in \mathcal{A}(X)_{\mathbb{R}}$ is zero. The variance is defined by:

$$\sigma_{f,\rho}^2 := \langle (f - \langle f \rangle)^2 \rangle_{\rho} = \langle f^2 \rangle_{\rho} - \langle f \rangle_{\rho}^2 , \quad (2.21)$$

and if ρ corresponds to a Dirac measure this is clearly zero.

However, in the noncommutative case in a pure state the variance $\sigma_{T,\rho}^2$ is in general nonzero.

Exercise 2.1: Generalized Uncertainty Relation

In fact, for any angle θ and any state ρ and any pair of observables T_1, T_2 it is easy to show that

$$\sigma_{T_1,\rho}^2 \sigma_{T_2,\rho}^2 \geq \frac{1}{4} \left(\langle (e^{i\theta} T_1 T_2 + e^{-i\theta} T_2 T_1) \rangle_{\rho} - 2 \cos \theta \langle T_1 \rangle_{\rho} \langle T_2 \rangle_{\rho} \right)^2 . \quad (2.22)$$

Prove (2.22).

Hint: Consider a^*a where $a = T_1 + e^{i\theta}\lambda T_2$ where λ is any real number.

If we specialize (2.22) to $\theta = \pi/2$ we get the Heisenberg uncertainty relation as usually stated:

$$\sigma_{T_1,\rho}^2 \sigma_{T_2,\rho}^2 \geq \frac{1}{4} (\langle i[T_1, T_2] \rangle_{\rho})^2 . \quad (2.23)$$

So, if $T \in \mathcal{A}$ does not commute with any other operator there will be a fundamental limit on the variance $\sigma_{T,\rho}^2$, even if ρ is a pure state. Thus, in classical mechanics the statistical uncertainty of measurement can be eliminated in principle, but that is not so in quantum mechanics.

We now examine the set of pure states in closer detail. Pure states are rank-one projection

operators. Such operators are in 1–1 correspondence with one-dimensional subspaces (often called “lines”) in Hilbert space. Thus, the set of pure states can be identified with

$$P\mathcal{H} = (\mathcal{H} - \{0\})/\mathbb{C}^\times . \quad (2.24)$$

In many textbooks on quantum mechanics pure states are associated with “normalizable vectors in Hilbert space.” This identification should be used with caution, and can lead to confusion. First of all, by definition, any vector in Hilbert space is normalizable. Now, a non-zero vector $\psi \in \mathcal{H}$ determines a line via:

$$L_\psi := \{z\psi \mid z \in \mathbb{C}\} \subset \mathcal{H} . \quad (2.25)$$

This line depends neither on the normalization nor the phase of ψ , i.e., $L_\psi = L_{\xi\psi}$ for any nonzero complex number $\xi \in \mathbb{C}^\times$. This is another way of understanding that the space of pure states is the projective Hilbert space (2.24). Note that the projective Hilbert space is not a vector space. Given any line $L \in \mathcal{H}$, one can define the corresponding rank-one projection operator onto that line as the unique projection operator. One can write an explicit formula for this projection operator by choosing any nonzero vector $\psi \in L$ and writing in Dirac’s bra-ket notation:

$$P_L := \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} . \quad (2.26)$$

Note that a rescaling $\psi \mapsto \xi\psi$ for $\xi \in \mathbb{C}^\times$ produces the same projector.

In quantum mechanics textbooks, physical states are often represented by nonzero vectors $\psi \in \mathcal{H}$ normalized so that $\langle\psi, \psi\rangle = 1$, which eliminates some of the scaling ambiguity $\psi \mapsto \xi\psi$, but retains the ambiguity of scaling by phases. The equivalence class of normalized vectors under rescaling by a phase is sometimes referred to – rather confusingly – as a “ray in Hilbert space.” We would discourage this terminology. For more about the relation between the projective structure of quantum mechanics and its generalization to field theory, see [Fre23b].

Exercise 2.2: Quantum mechanics of a particle on the real line

As an exercise consider the quantum mechanics of a particle on the real line. Let T be the position operator, consider a pure state associated with a normalized wavefunction $\psi(q)$ and let $E = [a, b]$. Show that the Born rule reproduces the standard answer for the probability of measuring q to be in E :

$$p_{\rho, T}(E) = \int_a^b |\psi(q)|^2 dq . \quad (2.27)$$

In quantum mechanics, time evolution is modeled by the action of a coherent family of unitary operators $U(t_1, t_2)$ where $U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3)$. In physical systems with a time-translation symmetry, $U(t_1, t_2) = U(t_1 - t_2)$ for a one-parameter family of operators with $U(t) = e^{-itH}$, where H is a self-adjoint operator, the Hamiltonian. The action of time translation on any state $\rho \in \mathcal{S}$ is:

$$\rho \mapsto U(t)\rho U(t)^{-1} . \quad (2.28)$$

A one-parameter group of unitary operators is generated by a self-adjoint operator H via $U(t) = \exp[-itH]$ (when H is unbounded some care is needed here) so that

$$\frac{d\rho}{dt} = -i[H, \rho] . \quad (2.29)$$

a formula to be compared with the classical evolution (2.17) above.

For pure states ρ the time evolution is induced by a linear evolution on a representative family of vectors ψ in Hilbert space. The evolution

$$\psi \mapsto \psi(t) := U(t)\psi , \quad (2.30)$$

induces the quantum evolution (2.29) on the family of projectors $\rho(t) = P(t)$ where $P(t)$ is the projector onto the line through $\psi(t)$. If $U(t)$ is generated by the Hamiltonian H we obtain the Schrödinger equation

$$i\hbar \frac{d\psi}{dt} = H\psi , \quad (2.31)$$

where we have set $\hbar = 1$ in our previous discussion. Of course, the same time-dependent density matrix is obtained if we rescale $\psi(t) \rightarrow e^{i\alpha(t)}\psi(t)$. Then the Schrödinger equation becomes

$$i\hbar \frac{d\psi}{dt} = (H + \hbar\dot{\alpha})\psi , \quad (2.32)$$

If we insist that the Hamiltonian is time-independent then $\dot{\alpha}$ must be constant and the freedom corresponds to shifting H by the unit operator times a real number. This corresponds to a shift of all the energy levels by a uniform constant. Such a shift is not physically meaningful unless one includes gravity as a dynamical degree of freedom.

Because the action of time translation is linear in (2.30), quantum states can be superposed. The principle that physical states can be linearly combined is a fundamental aspect of quantum information theory and leads to the distinctive quantum phenomenon of entanglement. On the other hand, quantum theory is projective, and projective space is nonlinear, so one might wonder how one describes linear superposition from the projective viewpoint.⁴ Given two pure states corresponding to linearly independent lines $\ell_1, \ell_2 \subset \mathcal{H}$, we obtain a two-dimensional subspace $S \subset \mathcal{H}$ spanned by the lines. A choice of linear

⁴We thank D. Freed for raising this interesting point and for discussions about it.

superposition is then simply a choice of a pure state corresponding to a line ℓ in S . If we choose nonzero vectors $\psi_i \in \ell_i$ (for $i = 1, 2$), then the traditional textbook description of linear superposition says that such a superposition “is” a vector $z_1\psi_1 + z_2\psi_2$, where $z_i \in \mathbb{C}$. The same vector can also be represented by $\tilde{z}_1\psi_1 + \tilde{z}_2\psi_2$ provided that $z_i = \lambda\tilde{z}_i$ for some $\lambda \in \mathbb{C}^\times$. Thus we recover the set of linear combinations as points in \mathbb{CP}^1 . Put another way, two linearly independent pure states define points $\ell_1, \ell_2 \in \mathbb{P}(\mathcal{H})$. Standard projective geometry tells us that two such points determine a line $\mathbb{CP}^1 \subset \mathbb{P}(\mathcal{H})$. This is the line of quantum superpositions. It is also interesting to describe the linear combination of states in a choice-free way in terms of rank-one projection operators. Suppose then that we are given two different rank-one projection operators P_i , $i = 1, 2$. (P_i are the projectors onto the lines ℓ_i .) One then notes that the equations

$$P_1Q = Q, \quad \text{and} \quad QP_2 = Q, \quad (2.33)$$

define a one-dimensional subspace of $\text{End}(\mathcal{H})$. The choice of linear combination is – essentially – a choice of operator Q in this one-dimensional subspace. In the case where P_1, P_2 are orthogonal projection operators, so $P_1P_2 = P_2P_1 = 0$, we can form a projector of the form

$$P = \alpha P_1 + Q + Q^\dagger + \delta P_2, \quad (2.34)$$

where $0 \leq \alpha, \delta \leq 1$, provided $\text{Tr}(QQ^\dagger) \leq \frac{1}{4}$. Relative to an orthonormal (ON) basis for ℓ_1, ℓ_2 , the resulting \mathbb{CP}^1 family of projection operators can be written as the Bott projector:⁵

$$P(z) = \frac{1}{1+|z|^2} \begin{pmatrix} 1 & z^* \\ z & |z|^2 \end{pmatrix}, \quad (2.35)$$

and the choice of Q corresponds to the choice of $z \in \mathbb{C} \cup \{\infty\}$.⁶

An important aspect of quantum mechanics concerns how to “combine” two physical systems described by operator algebras associated with two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 . The “combined” system has Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, and not $\mathcal{H}_1 \oplus \mathcal{H}_2$. This simple fact will lead to many interesting aspects of topological field theory. More generally, the Hilbert spaces

⁵So named because this family of projection operators plays a central role in discussions of Bott periodicity in algebraic K -theory.

⁶We record here the formula in the case where $P_1P_2 \neq 0$. Again, we choose an operator Q in the line determined by equation (2.33), and then take $P = \alpha_{\pm}P_1 + Q + Q^\dagger + \alpha_{\mp}P_2$, where,

$$\alpha_{\pm} = \frac{1}{2} \left((1 - 2 \operatorname{Re}(t)) \pm \sqrt{(1 - 2 \operatorname{Re}(t))^2 - 4x} \right),$$

and $t = \text{Tr}(Q)$ and $x = \text{Tr}(QQ^\dagger)$ and $x \leq \frac{1}{4}(1 - 2 \operatorname{Re}(t))^2$. The reader might be puzzled by the use of both roots α_{\pm} as the coefficient of P_1 . Together, they parametrize the entire \mathbb{CP}^1 of families corresponding to a “linear superposition of states”. To recover the case of orthogonal projectors, set $t = 0$ and $x = \frac{1}{1+|z|^2}$ or $x = \frac{|z|^2}{1+|z|^2}$.

of quantum mechanics should be super-Hilbert spaces, that is, they should be \mathbb{Z}_2 -graded. Then one uses the graded tensor product. The \mathbb{Z}_2 -gradings are always present when the physical system involves fermions, and conversely, a nontrivial \mathbb{Z}_2 -grading is a hallmark of the presence of fermions. In these lectures, for simplicity, we will generally restrict to the ungraded case.

Finally, we note that if the spectrum of the Hamiltonian H is unbounded above, but bounded below, then the time evolution operator e^{-itH} can be viewed as the boundary value of a holomorphic family of operators $e^{-i\tau H}$ defined on the half-plane with $\text{Im}(\tau) \leq 0$. What is called “Euclidean time evolution” is the evolution by the semigroup of operators with τ on the negative imaginary axis. If we define $\tau = -i\tau_E$ with $\tau_E \in \mathbb{R}_+$ then the “Euclidean signature time evolution operator is the family of operators $e^{-\tau_E H}$. One can think of the one-dimensional manifold of time as endowed with a complex family of metrics. The formal substitution $t \rightarrow -i\tau_E$ taking the Lorentzian to Euclidean signature metric $-dt^2 \rightarrow d\tau_E^2$ is known as *Wick rotation*. Although topological field theory is metric-independent it is meant to model Wick-rotated QFT in Euclidean signature metrics.

PART I: TOPOLOGICAL FIELD THEORY

3 Basic Picture In TFT: Heuristic Motivation

What is physics? Physics means many things to many people, but one common and central goal one will often encounter is the desire to describe or predict the future, given a knowledge of the present. In cosmology, on the other hand, one might wish to predict the past, given a knowledge of the present. In any case we are trying to describe some kind of time evolution. In quantum systems, such predictions are based on the computation of amplitudes. In topological field theory, one abstracts certain primitive aspects of such amplitudes in a system of axioms and then describes the resulting mathematical structure defined by just those axioms.

An important aspect of topological field theory, and indeed much of the literature of the past few decades, is an emphasis on defining theories in general spacetime and general spatial manifolds. In other words we consider spacetimes which are not necessarily Euclidean or Minkowski spacetimes. This is in contrast with most textbooks on quantum field theory (QFT), which generally assume spacetime to be Minkowski space. Thus, in a theory defined on n -dimensional spacetimes, we wish to consider the spacetime to be some general n -dimensional manifold M_n . A spatial manifold should be a manifold N_{n-1} of dimension $(n - 1)$. If we are engaged in prediction, then we might have a past manifold N_{n-1}^0 and a future manifold N_{n-1}^1 , and given a knowledge of a quantum state in the past, we wish to say something about the quantum state in the future (or vice versa). We should even allow the possibility that the past and future spatial manifolds might have different topologies. The spacetime evolution must be some n -dimensional manifold that “interpolates” between the past and the future, somewhat as in [Figure 1](#).

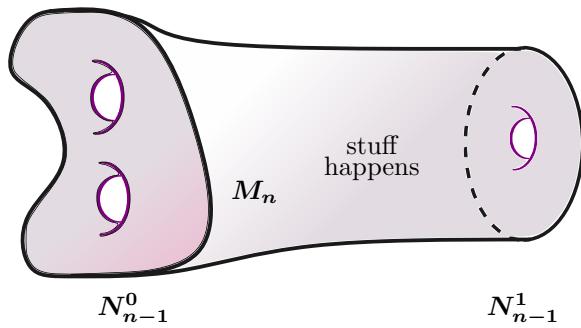


Figure 1: Picture of “time” evolution. Note that M_n need not be a cylinder, i.e., diffeomorphic to $N_{n-1} \times [0, 1]$, and there might not be a globally well-defined vector field of time.

In quantum theory, there is a Hilbert space associated with the initial and final spatial manifolds:⁷

⁷The phrase “Hilbert space of states” is an abuse of terminology: a state in quantum theory is a density matrix: it is a positive operator of trace one. Given a nonzero vector $|\psi\rangle$ in \mathcal{H} , then $P = \frac{|\psi\rangle\langle\psi|}{\|\psi\|^2}$ is a projection operator, which is a more appropriate descriptor of the state.

$$\begin{aligned} N_{n-1}^0 &\rightsquigarrow \mathcal{H}_0 : \text{ Hilbert space from which we describe initial states.} \\ N_{n-1}^1 &\rightsquigarrow \mathcal{H}_1 : \text{ Hilbert space from which we describe final states.} \end{aligned}$$

In other words, part of the description of a “theory” is to give a “function” on $(n - 1)$ -dimensional manifolds valued in Hilbert spaces.⁸ We could call this “function” F_{states} . Given an $(n - 1)$ -dimensional manifold N_{n-1} , there should be a corresponding Hilbert space $F_{\text{states}}(N_{n-1})$, the “Hilbert space of states on the spatial manifold N_{n-1} .” So we have:

$$\begin{aligned} \mathcal{H}_0 &= F_{\text{states}}(N_{n-1}^0) , \\ \mathcal{H}_1 &= F_{\text{states}}(N_{n-1}^1) . \end{aligned} \tag{3.1}$$

In quantum mechanics, the time evolution of quantum states is given by a linear map from initial to final Hilbert space. Therefore, to the interpolating history M_n we will want to associate a linear map

$$F_{\text{ampl}}(M_n) : F_{\text{states}}(N_{n-1}^0) \rightarrow F_{\text{states}}(N_{n-1}^1) . \tag{3.2}$$

The linear map depends on the interpolating spacetime and the stuff that happened in between. In QFT, F_{ampl} is often computed from a path integral, and indeed, the axioms for F_{ampl} below are intended to capture properties one expects such path integrals to satisfy. Most path integrals of interest are extremely difficult to compute. It can therefore be helpful to extract or axiomatize the key properties such path integrals are expected to possess. In topological quantum field theory (TQFT) we aim to choose axioms that allow us to ignore many of the complexities of nontrivial interacting QFT’s while retaining the key aspects and consequences of *locality*. One way we can eliminate much of the complexity of general QFT is to take as axiomatic that the values of F_{states} and F_{ampl} only depend (up to a suitable notion of isomorphism) on the diffeomorphism class of the manifolds N_{n-1}^0 , N_{n-1}^1 , and M_n , respectively.

The assumption that F_{states} and F_{ampl} only depend on the diffeomorphism class of N_{n-1}^0 , N_{n-1}^1 , and M_n , is a drastic assumption. It will not be true of “most” quantum field theories. It is certainly not true of typical quantum field theories studied in textbooks on QFT. It is certainly not true in the standard model of particle physics. Nevertheless, the axiomatic definition for TFT we will give below turns out to define an amazingly rich mathematical structure, one that continues to be explored and refined in current research. Moreover, it naturally suggests a conceptual framework for the description of general quantum field theories. This hypothetical formulation, sometimes known as *geometrical field theory* or *functorial field theory* is discussed briefly in [Section 28](#). Giving a fully satisfactory definition of quantum field theory along such lines is an extremely ambitious goal for the future. One

⁸Our final definition of a topological field theory will be more elastic and allow F_{states} to take values which are not vector spaces. See [Section 22](#) below.

notable case in which the formulation of a class of nontrivial QFT's has been achieved along these lines is G. Segal's formulation of 2d conformal field theory (CFT) [Seg02]. Indeed, Segal's axioms for 2d conformal field theory predated Atiyah's formal definition of topological quantum field theory [Ati89], and indeed Atiyah's definition was motivated, in part, by Segal's approach to 2d CFT.

As we have mentioned, topological field theory is now a mathematical structure that is investigated on its own merits. It also provides a key conceptual framework for exploring many subjects in mathematics, especially in low-dimensional topology and in enumerative algebraic geometry. Somewhat surprisingly, in addition to playing an important role in mathematics, topological field theory has also played an important role in some mathematical discussions of condensed matter physics (such as anyons, topological materials, and phases of matter). It has even played a notable role in some discussions of quantum gravity.

The statement that F_{states} and F_{ampl} only depend on diffeomorphism classes implies, among other things, that they do not depend on a choice of metric on these manifolds. Since there is no choice of metric, there is no choice of a signature of the metric, so the terms “spatial” and “time evolution” above are being used rather loosely. We should think of the axioms of TFT as capturing the behavior of Euclidean, i.e., Wick-rotated field theory. Since there is no metric dependence, we are throwing away the dependence on length scales and energy scales. Therefore we expect any Hamiltonian in this theory to vanish. Moreover there will not be particle excitations or waves. This has the virtue of clarifying the essential physical ideas related to locality, as well as making things computable. Remarkably, we have not completely thrown out the baby with the bathwater - the resulting structure is quite nontrivial.

We now come to our first axiomatization of locality. In quantum mechanics, the statespace of a “union” of two separate systems is the tensor product of the constituent statespaces. For example, the statespace of one qubit is \mathbb{C}^2 , and that of N qubits is $(\mathbb{C}^2)^{\otimes N}$. In our context, that means we should say there is an isomorphism of statespaces:⁹

$$F_{\text{states}}\left(N_{n-1} \coprod N'_{n-1}\right) \cong F_{\text{states}}(N_{n-1}) \otimes F_{\text{states}}(N'_{n-1}). \quad (\text{LOC 1})$$

Remark 3.3.

1. It follows from (LOC 1) that $F_{\text{states}}(\emptyset_{n-1}) \cong \mathbb{C}$.¹⁰
2. The isomorphism class of the vector space $F_{\text{states}}(N_{n-1})$ depends only on the diffeomorphism class of N_{n-1} . It is in this sense that $F_{\text{states}}(N_{n-1})$ is a (smooth) “topological

⁹In fermionic theories, the spaces will be \mathbb{Z}_2 -graded, i.e., “super-vector spaces” (in the mathematical sense), and the tensor product will be a \mathbb{Z}_2 -graded product.

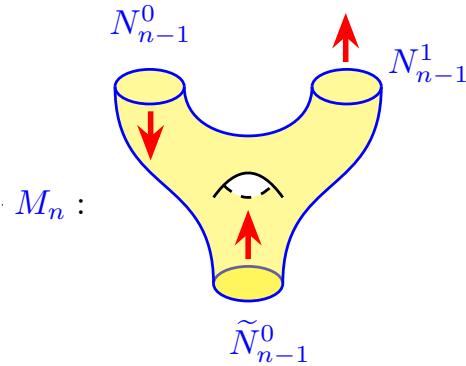
¹⁰Take $N'_{n-1} = \emptyset_{n-1}$. Then (LOC 1) implies that $F_{\text{states}}(N_{n-1}) = F_{\text{states}}(N_{n-1} \coprod \emptyset_{n-1}) = F_{\text{states}}(N_{n-1}) \otimes F_{\text{states}}(\emptyset_{n-1})$. We will prove below that in a TQFT $F_{\text{states}}(N_{n-1})$ is finite dimensional. It then follows that $F_{\text{states}}(\emptyset_{n-1}) \cong \mathbb{C}$, as we are working over the category of complex vector spaces. Here \cong stands for “is isomorphic to,” and implies an isomorphism of \mathbb{C} -vector spaces.

invariant.” It is interesting to compare with traditional topological invariant vector spaces associated to manifolds. For example, for homology we have

$$H_k(M \coprod M') \cong H_k(M) \oplus H_k(M') , \quad (3.4)$$

and similarly for cohomology. Note that for the fundamental group, one is at a loss to define $\pi_1(M \coprod M')$ since the definition of π_1 requires the choice of a basepoint. Classical invariants of algebraic topology depend very differently on underlying manifolds than what quantum mechanics would dictate. For this reason, the spaces $F_{\text{states}}(N_{n-1})$ are called “Quantum Invariants.”

Now consider an n -manifold M_n with boundary, see [Figure 2](#). We want to think of this as a spacetime connecting past and future states. In general the boundary of M_n will have several connected components. We need to choose which spatial slices are components of the “past” spatial manifold and which are components of the “future” spatial manifold. We indicate this with **red arrows** pointing in (past) or out (future) of the manifold.



[Figure 2](#): An n -manifold M_n with boundary. The different connected components of the boundary carry an arrow indicating whether they are “ingoing” (i.e., “initial” or “past”) spatial components or “outgoing” (i.e., “final” or “future”) spatial components. Note well that these arrows, sometimes referred to as a *co-orientation*, do not imply that M_n or its boundaries are oriented, or even orientable. In this example the initial spatial manifold is the disjoint union $N_{n-1}^0 \coprod \tilde{N}_{n-1}^0$.

So, for the example of [Figure 2](#), our quantum amplitudes F_{ampl} will give a linear map:

$$F_{\text{ampl}}(M_n) : F_{\text{states}}(N_{n-1}^0 \coprod \tilde{N}_{n-1}^0) \longrightarrow F_{\text{states}}(N_{n-1}^1) . \quad (3.5)$$

The next aspect of locality that we wish to axiomatize is the idea that a sum over a complete set of states is tantamount to an insertion of the identity. This leads to the idea of “gluing.” The gluing axiom is motivated by the Feynman path integral. For those who have not studied path integrals in detail we give an extremely brief overview in section [3.1](#) below.

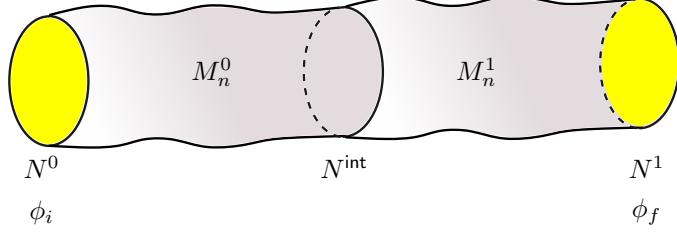


Figure 3: Gluing.

Suppose we have a spacetime M_n^0 interpolating from spatial slice N_{n-1}^0 to N_{n-1}^{int} and another one M_n^1 interpolating from spatial slice N_{n-1}^{int} to N_{n-1}^1 as in 3. We can glue the two n -manifolds along the common spatial boundary N_{n-1}^{int} to produce a spacetime interpolating from N_{n-1}^0 to N_{n-1}^1 and we demand that:

$$F_{\text{ampl}}(M_n^1) \circ F_{\text{ampl}}(M_n^0) = F_{\text{ampl}}(M_n^1 \circ_{N^{int}} M_n^0), \quad (\text{LOC 2})$$

where $\circ_{N^{int}}$ denotes gluing along N^{int} , and on the LHS, \circ denotes composition of linear maps.

The amplitudes expressed in terms of path integrals that motivate this axiom might not yet have mathematical definitions. Nevertheless, we certainly axiomatize certain properties we would definitely want these path integrals to satisfy. The above gluing axiom is an example of such a property.

We will next put these general ideas on a nice and precise mathematical foundation. Our first step in this direction is to introduce the idea of “bordism.”

3.1 Path Integral Motivation For The Gluing Axiom: A Quick Review Of Sigma Models And Free Field Quantization.

Let us begin¹¹ with a basic example in quantum mechanics, the harmonic oscillator. In the Hamiltonian formulation phase space is \mathbb{R}^2 with $\omega = dp \wedge dq$ where (p, q) are Darboux coordinates on \mathbb{R}^2 and

$$H = \frac{p^2}{2m} + \frac{\omega^2 q^2}{2}, \quad (3.6)$$

where q is the position of a particle of mass m on a spring and ω^2 is a measure of the stiffness of the spring. The equation of motion is governed by an action principle:

$$S[q(t)] = \int \frac{m}{2}(\dot{q}^2 - \omega^2 q^2) dt. \quad (3.7)$$

¹¹This material makes use of several concepts from Part 2 and in general is meant as a reminder for readers who have had a course on quantum field theory. Some readers might wish to return to this remark later.

3.1 Path Integral Motivation For The Gluing Axiom: A Quick Review Of Sigma Models Section 3

Upon quantization p and q are replaced by $\hat{p}, \hat{q} \in \mathcal{A}$ with $\hat{p}^* = \hat{p}$, $\hat{q}^* = \hat{q}$ and

$$[\hat{p}, \hat{q}] = -i\hbar 1 , \quad (3.8)$$

where 1 is the unit in \mathcal{A} . Now one completes - in some way - the algebra of functions of \hat{p} and \hat{q} subject only to this relation. The spectrum of \hat{p} and \hat{q} is unbounded and when realized on a Hilbert space they become unbounded operators. It is better to work with bounded operators so we consider the one parameter families of operators:¹²

$$\begin{aligned} T(\alpha) &= \exp[i\alpha\hat{p}] , \\ M(\beta) &= \exp[i\beta\hat{q}] . \end{aligned} \quad (3.9)$$

These are unitary operators for $\alpha, \beta \in \mathbb{R}$. These operators satisfy the relations

$$\begin{aligned} T(\alpha_1)T(\alpha_2) &= T(\alpha_1 + \alpha_2) , \\ M(\beta_1)M(\beta_2) &= M(\beta_1 + \beta_2) , \\ T(\alpha)M(\beta) &= \exp[i\hbar\alpha\beta]M(\beta)T(\alpha) . \end{aligned} \quad (3.10)$$

They generate a group of operators known as the Heisenberg group. As a group it is simply an extension

$$1 \rightarrow U(1) \rightarrow \text{Heis}(\mathbb{R} \oplus \mathbb{R}) \rightarrow \mathbb{R} \oplus \mathbb{R} \rightarrow 0 . \quad (3.11)$$

The group $\text{Heis}(\mathbb{R} \oplus \mathbb{R})$ has a unique (up to isomorphism) unitary representation in which phases ζ in the central $U(1)$ acts as scalar multiplication by ζ . One such representation (the Stone-von Neumann representation) is $\mathcal{H} = L^2(\mathbb{R})$ with

$$\begin{aligned} (\hat{q} \cdot \psi)(q) &= q\psi(q) , \\ (\hat{p} \cdot \psi)(q) &= -i\hbar \frac{d}{dq}\psi(q) . \end{aligned} \quad (3.12)$$

Another representation is motivated by the standard analysis of the spectrum of the harmonic oscillator Hamiltonian using creation and annihilation operators. Setting $m = \omega = \hbar = 1$ for simplicity these are

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}) \\ \hat{a}^* &= \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}) \\ [\hat{a}, \hat{a}^*] &= 1 \end{aligned} \quad (3.13)$$

¹²Readers who think that unbounded operators pose no technical challenges are invited to diagonalize $\hat{q}^n\hat{p} + \hat{p}\hat{q}^2$, for $n > 1$. That that it is formally self-adjoint.

3.1 Path Integral Motivation For The Gluing Axiom: A Quick Review Of Sigma Models Section 3

In these terms the Harmonic oscillator Hamiltonian is $H = \hat{a}^* \hat{a} + \frac{1}{2}$. The groundstate line is determined by

$$\hat{a}\psi_0 = 0 , \quad (3.14)$$

which implies $\psi_0(q) \sim e^{-\frac{1}{2}q^2}$, and the eigenlines of the Hamiltonian of energy $n + 1/2$ are lines through $(\hat{a}^*)^n \psi_0$. Thus the Hilbert space is a completion of a space of vectors of the form of a polynomial in q times a decaying quadratic exponential. This will be an important model for the field-theoretic situation discussed below.

The presentation in terms of creation and annihilation operators motivates the Bargmann-Fock representation of the Heisenberg group. Now \mathcal{H} will be a completion of the space of complex polynomials in a variable $z \in \mathbb{C}$. Then $\rho(\hat{a}^*)$ acts on a polynomial as multiplication by z so $\rho(\hat{a}^*)$ is the operator d/dz . The inner product can be taken to be

$$\langle f_1, f_2 \rangle := \int_{\mathbb{C}} (f_1(z))^* f_2(z) e^{-|z|^2} \frac{dz \wedge d\bar{z}}{2\pi i} , \quad (3.15)$$

where the normalization of the integral is a choice of convention.

As we have mentioned, the representation of the Heisenberg group is unique up to isomorphism. The reader is invited to construct the isomorphism between the Stone-von Neumann and Bargmann-Fock representations.

We can of course consider more general Hamiltonians on the phase space $(\mathbb{R}^2, dpdq)$. In general the spectrum cannot be written down so easily, the quantum mechanics will still be described in terms of the representation of the Heisenberg group above. The time evolution of a state, from time t_1 to t_2 with a time-independent Hamiltonian is determined from

$$\psi_2 = e^{-i(t_2-t_1)H} \psi_1 , \quad (3.16)$$

and in the Stone-von Neumann representation it will be represented by an integral kernel:

$$\psi_2(q_2) = \int K(q_2, q_1; M_1(t_2, t_1)) \psi_1(q_1) dq_1 . \quad (3.17)$$

Here $M_1(t_2, t_1)$ is the 1-dimensional manifold given by the time interval $[t_1, t_2]$ in the real line representing time. Feynman showed that if H is derived from an actional functional of paths then $K(q_2, q_1; M_1(t_2, t_1))$ can be given an integral representation as an integral over an infinite-dimensional space of all paths:

$$\mathcal{P}(q_2, q_1; M_1(t_2, t_1)) , \quad (3.18)$$

defined to be the set of all continuous paths $q : M_1(t_2, t_1) \rightarrow \mathbb{R}$ such that $q(t_1) = q_1$ and

$q(t_2) = q_2$. Then we have

$$K(q_2, q_1; M_1(t_2, t_1)) = \int_{\mathcal{P}(q_2, q_1; M_1(t_2, t_1))} e^{\frac{i}{\hbar} S[q(t)]} [dq(t)] , \quad (3.19)$$

where $[dq(t)]$ is a suitable measure $\mathcal{P}(q_2, q_1; M_1(t_2, t_1))$. For finite-dimensional quantum systems the measure can be defined in a mathematically rigorous way and is known as a *Wiener measure*. Note the kernel function depends on the specific Hamiltonian (or action) but we have suppressed this in the notation.

Now we can discuss gluing. Consider an intermediate time $t_1 < t_{\text{int}} < t_2$. In this case the gluing of the two 1-manifolds $M_1(t_{\text{int}}, t_1)$ and $M_1(t_2, t_{\text{int}})$ is simply given by taking the union of the intervals. The gluing formula can be written as:

$$K(q_2, q_1; M_1(t_2, t_1)) = \int dq_{\text{int}} K(q_2, q_{\text{int}}; M_1(t_2, t_{\text{int}})) K(q_{\text{int}}, q_1; M_1(t_{\text{int}}, t_1)) . \quad (3.20)$$

This follows from the representation of time evolution in terms of a one-parameter family of evolution operators e^{-itH} together with insertion of a resolution of the identity operator of the form $1 = \int dq_{\text{int}} |q_{\text{int}}\rangle \langle q_{\text{int}}|$. It also follows from the path integral representation of the kernel because any path $q(t)$ must take some value q_{int} at time t_{int} so we can write the set of all paths as a union over $q_{\text{int}} \in \mathbb{R}$ of the product

$$\mathcal{P}(q_2, q_{\text{int}}; M_1(t_2, t_{\text{int}})) \times \mathcal{P}(q_{\text{int}}, q_1; M_1(t_{\text{int}}, t_1)) . \quad (3.21)$$

The idea is that we can first do the path integral over the two factors, and then integrate over q_{int} to get an integral over all paths. All the considerations above generalize to the Wick-rotated theory with Euclidean time evolution. The Euclidean signature action for a particle with potential $V(q)$ is

$$S_E[q(t)] = \int \left(\frac{1}{2} m \dot{q}^2 + V(q) \right) dt , \quad (3.22)$$

and the Euclidean evolution kernel is

$$K(q_2, q_1; M_1(t_2, t_1)) = \int_{\mathcal{P}(q_2, q_1; M_1(t_2, t_1))} e^{-\frac{1}{\hbar} S_E[q(t)]} [dq(t)] . \quad (3.23)$$

which is formally convergent if S_E is bounded below and unbounded above at “infinity” in field space.

 **Exercise 3.1: Gaussian Integrals**

- a.) The path integrals involving Gaussian expressions in terms of q and its derivatives can be done by formal generalization of the formula for Gaussian integration on Euclidean \mathbb{R}^N

$$\int_{\mathbb{R}^N} \frac{d^N q}{(\sqrt{2\pi})^N} \exp \left[-\frac{1}{2} q^{\text{tr}} Q q + q^{\text{tr}} \cdot j \right] = \left(\frac{1}{\det Q} \right)^{1/2} \exp \left[+ \frac{1}{2} j^{\text{tr}} Q^{-1} j \right]. \quad (3.24)$$

where Q is a symmetric nondegenerate quadratic form with positive real part. The exponential is just the value of the “action” evaluated at the stationary point. In infinite dimensions one needs to give some mathematical meaning to the determinant. This can be done, for example, using the Ray-Singer ζ -regularized determinant described in [Section 48.2](#). All of perturbative quantum field theory is derived from relentless application of the formula for a Gaussian integral.

- b.) As an exercise one should consider again the harmonic oscillator and show, by evaluating the Gaussian integral formally that

$$K(q_2, q_1; M_1(t_2, t_1)) = \left(\frac{m\omega}{2\pi i \hbar \sin \omega(t_2 - t_1)} \right)^{1/2} \exp \left[\frac{i m \omega}{2\hbar \sin \omega(t_2 - t_1)} ((q_1^2 + q_2^2) \cos \omega(t_2 - t_1) - 2q_1 q_2) \right] \quad (3.25)$$

The time-dependence of the prefactor is derived from the ζ -function evaluation of the operator $m \left(-\left(\frac{d}{dt} \right)^2 + \omega^2 \right)$ acting on real-valued functions on $[t_1, t_2]$ vanishing at the boundaries. An overall constant must be determined by demanding that the kernel give a delta function $\delta(q_1, q_2)$ as the time interval shrinks to zero.

- c.) Verify the gluing formula explicitly using the above kernel function by explicitly doing the Gaussian integral over q_{int} .

We would now like to generalize the above discussion to field theory. To do this we introduce a class of field theories known as *nonlinear σ -models*. We choose a Riemannian target space \mathcal{X} so that the fields of the theory, when defined on an n -dimensional spacetime M_n , are $\mathcal{X}^{M_n} = \text{Map}(M_n \rightarrow \mathcal{X})$. That space of maps is to be thought of as the generalization of the space of continuous maps from \mathbb{R} (thought of as parametrizing time) to \mathbb{R} (thought of as parametrizing position of a particle). To write the classical action we endow M_n with a metric (of Euclidean or Lorentzian signature) so that

$$d\phi(p) : T_p M_n \rightarrow T_{\phi(p)} \mathcal{X}, \quad (3.26)$$

3.1 Path Integral Motivation For The Gluing Axiom: A Quick Review Of Sigma Models Section 3

is a linear map between spaces with metric. Therefore we can define a norm-square $\|d\phi(p)\|^2$. The classical sigma model action is proportional to $\int_{M_n} \|d\phi\|^2 \text{vol}(g)$. Written out in local coordinates so that the metric on M_n is $g_{\mu\nu}(x)dx^\mu \otimes dx^\nu$ and the metric on \mathcal{X} is $G_{ab}(\phi)d\phi^a \otimes d\phi^b$, we have

$$\int_{M_n} \|d\phi\|^2 \text{vol}(g) = \int_{M_n} G_{ab}(\phi(x))g^{\mu\nu}(x)\partial_\mu\phi^a(x)\partial_\nu\phi^b(x)\sqrt{|\det g_{\mu\nu}(x)|}d^n x . \quad (3.27)$$

In general, this action describes a nontrivial interacting theory. The phase space is formally $T^*\mathcal{X}^{M_n}$. The classical mechanical model is well-defined with an infinite dimensional phase space. However, in general one does not expect such theories to have a well-defined quantization. There are three notable exceptions:

1. If $n = 1$ and \mathcal{X} is finite dimensional we are describing a particle moving on a Riemannian manifold. We are back in the world of quantum mechanics of finite dimensional systems and everything can be done rigorously.
2. If $n = 2$ there is renormalization group evolution of the metric $G_{ab}d\phi^a \otimes d\phi^b$ [Fri80, Fri85]. The modern view of QFT is that a theory is defined by perturbation around conformal fixed points. The case of 2d conformal field theories has been given a rigorous formulation by G. Segal.
3. For any value of n if we take \mathcal{X} to be a finite-dimensional vector space with a translation invariant metric then we can completely describe the theory. This is the case of the free scalar field.

We now describe the case of the free scalar field in some more detail. For simplicity we take $\mathcal{X} = \mathbb{R}$ with its usual Euclidean metric. In this case the nonlinear sigma model action can be generalized in a useful way to:¹³

$$S[\phi] = \frac{1}{2} \int_{M_n} (d\phi \star d\phi + m^2 \phi \star \phi) , \quad (3.28)$$

where the second term makes use of the fact that $\phi(x)$ is a real number and we take m to be a positive real number.¹⁴ Formally, the Feynman path integral would express the kernel as

$$K(\phi_f, \phi_i) = \int_{\mathcal{X}^{M_n}(\phi_f, \phi_i)} d\mu e^{-S[\phi]} , \quad (3.29)$$

where $\mathcal{X}^{M_n}(\phi_f, \phi_i)$ is the infinite-dimensional vector space of “all” maps from $M_n \rightarrow \mathcal{X}$

¹³For simplicity we will assume M_n oriented so we can define Hodge $*$ and integration of n -forms. It is absolutely not necessary to assume M_n is orientable to define the quantum theory of the free massive scalar field.

¹⁴The symbol \star is the Hodge star, which is discussed at length in [Section 29](#).

3.1 Path Integral Motivation For The Gluing Axiom: A Quick Review Of Sigma Models Section 3

such that ϕ restricts to ϕ_i on N_{n-1}^0 and ϕ_f on N_{n-1}^1 . The measure $d\mu$ can be constructed from the rigorous free field quantization we are about to describe.

One approach – that taken in most textbooks – is to quantize the field on a cylinder $M_n = N_{n-1} \times \mathbb{R}$ and decompose the field $\phi(x, t)$, with $x \in N_{n-1}$ in terms of the orthonormal basis of $\Omega^0(N_{n-1})$ (the vector space of infinitely differentiable functions on N_{n-1}) provided by the eigenmodes of the operator $(d^\dagger d + m^2)$ acting on functions on N_{n-1} :

$$(d^\dagger d + m^2) f_k(x) = \omega_k^2 f_k(x) , \quad k = 1, 2, 3, \dots \quad (3.30)$$

Since the $\{f_k\}_{k=1}^\infty$ form an ON basis the general field on spacetime can be expanded as

$$\phi(x, t) = \sum_k q_k(t) f_k(x) . \quad (3.31)$$

Substitution in the action gives an infinite system of decoupled quantum degrees of freedom. The conjugate momenta to the degrees of freedom $q_n(t)$ are obtained from the expansion of $(\star d\phi)$ for a solution of the equation of motion in spacetime, then restricted to N_{n-1} . The symplectic form on phase space splits into a sum of forms for single particle motion with coordinates and momenta (q_k, p_k) leading to quantization:

$$[\hat{p}_j, \hat{q}_k] = -i\hbar\delta_{j,k} . \quad (3.32)$$

The Hamiltonian derived from the action is simply

$$H = \frac{1}{2} \sum_k (p_k^2 + \omega_k^2 q_k^2) . \quad (3.33)$$

Thus the system decomposes into an infinite collection of harmonic oscillators. The Hilbert space is, formally, an infinite tensor product of the Stone-von Neumann or the Bargmann-Fock Hilbert spaces for these systems. A basis of states can be labeled by sequences of nonnegative integers ℓ_k which eventually become zero corresponding to

$$\prod_{j=1}^{\infty} H_k(\sqrt{\omega_k} q_k) e^{-\frac{1}{2}\omega_k q_k^2} , \quad (3.34)$$

where ω_k is the positive square root of ω_k^2 and H_k are Hermite polynomials. In the Bargmann-Fock basis it would correspond to symmetric polynomials $\prod_k z_k^{\ell_k}$, making case with the Fock space, the symmetric polynomials on $\bigoplus_k L^2(\mathbb{R})_{q_k}$.

A second approach to deriving $\mathcal{H}(N_{n-1})$ begins with the phase space of the theory on N_{n-1} :

$$T^* \Omega^0(N_{n-1}) \cong \Omega^0(N_{n-1}) \times \Omega^{n-1}(N_{n-1}) . \quad (3.35)$$

3.1 Path Integral Motivation For The Gluing Axiom: A Quick Review Of Sigma Models Section 3

Here $\Omega^{n-1}(N)$ is the space of $(n-1)$ -forms and it is dual to $\Omega^0(N_{n-1})$, the dual pairing being:

$$\langle \xi, \eta \rangle = \int_{N_{n-1}} \xi \eta . \quad (3.36)$$

The phase space is thus of the form $V \oplus V^\vee$ with $V = \Omega^0(N_{n-1})$ and $V^\vee = \Omega^{d-1}(N_{n-1})$ with standard symplectic form:

$$\omega(\xi_1 \oplus \eta_1, \xi_2 \oplus \eta_2) = \langle \xi_1, \eta_2 \rangle - \langle \xi_2, \eta_1 \rangle . \quad (3.37)$$

In general, to a symplectic vector space of the form $V \oplus V^\vee$ we can associate a Heisenberg group extension:

$$1 \rightarrow \mathsf{U}(1) \rightarrow \mathsf{Heis}(V \oplus V^\vee) \rightarrow V \oplus V^\vee \rightarrow 0 . \quad (3.38)$$

We can think of the Heisenberg group as generated by expressions $T(\xi)$, $\xi \in V$ and $M(\eta)$, $\eta \in V^\vee$ with the multiplication rules:

$$\begin{aligned} T(\xi_1)T(\xi_2) &= T(\xi_1 + \xi_2) , \\ M(\eta_1)M(\eta_2) &= M(\eta_1 + \eta_2) , \\ T(\xi)M(\eta) &= \exp[i\pi\langle \xi, \eta \rangle]M(\eta)T(\xi) . \end{aligned} \quad (3.39)$$

In the case that V is finite-dimensional there is a unique irreducible representation of the Heisenberg group where $\mathsf{U}(1)$ acts as scalar multiplication. This representation is known as the Stone-von Neumann representation.¹⁵ The representation space is $L^2(V)$ in some translation-invariant metric on V . Formally, one would like to invoke a theorem like this to define the Hilbert space of the free scalar field on N_{n-1} as $L^2(\Omega^0(N_{n-1}))$. Since $\Omega^0(N_{n-1})$ is infinite-dimensional more care is needed. One needs to include a positive quadratic form on $\Omega^0(N_{n-1})$ and the Hilbert space is a completion of functions which are differential polynomials in ϕ times a decaying exponential defined by the quadratic form q . This indeed is in accord with the wavefunctions (3.34) which makes use of a specific form q . As opposed to the finite-dimensional case different choices of q can lead to non-isomorphic representations.

Returning to our general discussion above we see that for the free massive scalar field $F_{\text{states}}(N_{n-1})$ should be something like “ $L^2(\mathcal{X}^{N_{n-1}})$.” So a quantum “statevector” would be a functional Ψ assigning a complex number $\Psi[\phi]$ to every $\phi \in \mathcal{X}^{N_{n-1}}$ subject to some kind of normalizability constraint.

We can now generalize our discussion of (Euclidean) time evolution for spacetimes of cylindrical form $M_n = N_{n-1} \times [t_1, t_2]$ by $\exp[-tH]$. If $\Psi_f = e^{-(t_2-t_1)H}\Psi_i$ then evolution once

¹⁵The Stone-von Neumann representation is discussed in more detail in Section 42.

again can be written in terms of a kernel:

$$\Psi_f[\phi_f] = \int_{\phi_i \in \mathcal{X}^{N^0}} K(\phi_f, \phi_i; M_n) \Psi_i[\phi_i] \mathcal{D}\phi_i , \quad (3.40)$$

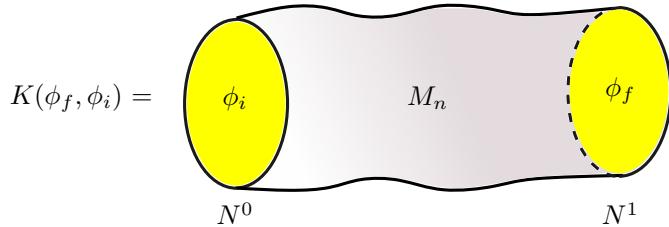
where now the integral is a functional integral over spatial field configurations $\phi_i \in \mathcal{X}^{N_{n-1}}$. Once again, the Feynman path integral expresses K as a functional integral over field configurations on M_n with boundary conditions ϕ_i, ϕ_f as illustrated in [Figure 4](#).

Once we have a path integral representation of the kernel it is straightforward – at least formally – to generalize K to general manifolds M_n with boundary

$$\partial M_n = N_{n-1}^0 \cup N_{n-1}^1 , \quad (3.41)$$

where we impose boundary condition $\phi = \phi_i$ on N_{n-1}^0 and $\phi = \phi_f$ on N_{n-1}^1 . We still have [Figure 4](#), but now M_n can be a more general manifold than a cylinder and the initial and final spatial slices can be different manifolds. The kernel defines a linear map $\Psi_i \rightarrow \Psi_f$ which we can write as a linear operator

$$\Psi_f = F_{\text{ampl}}(M_n)(\Psi_i) . \quad (3.42)$$



[Figure 4](#): The propagator.

If we cut M_n along some intermediate $(n-1)$ -manifold as in [Figure 3](#) the same kind of reasoning as in the quantum mechanics of a single particle, based on the path integral representation of the kernel, suggests that we ought to have

$$K(\phi_f, \phi_i; M_n) = \int K(\phi_f, \phi_{\text{int}}; M_n^1) K(\phi_{\text{int}}, \phi_i; M_n^0) \mathcal{D}\phi_{\text{int}} . \quad (3.43)$$

This is the motivation for the gluing axiom ([LOC 2](#)).

Exercise 3.2: Checking The Gluing Formula

- a.) Treating the path integral (3.29) as a formal Gaussian integral show that the

amplitude can be evaluated as:

$$K(\phi_f, \phi_i) = (\text{Det}(d^\dagger d + m^2))^{-1/2} \exp \left[-\frac{1}{2} \langle \phi_\partial, A_{M_n} \phi_\partial \rangle \right]. \quad (3.44)$$

where ϕ_∂ is the function ϕ_f on N_{n-1}^1 and ϕ_i on N_{n-1}^0 and the determinant is computed for the operator acting on functions vanishing on the boundary. (The operator $d^\dagger d + m^2$ preserves the space of such functions. Choose a collar neighborhood with direct sum metric. The second derivative of a sine function is a sine function.)

The operator A_{M_n} is defined as follows: For ϕ_∂ defined on the boundary of M_n solve the (Euclidean) equation of motion $(d^\dagger d + m^2)\phi = 0$ with Dirichlet boundary condition $\phi|_{\partial M_n} = \phi_\partial$. Let $\iota : \partial M \rightarrow M$ be the inclusion. For a solution of the equation of motion with Dirichlet boundary condition ϕ_∂ consider $\iota^*(\star d\phi) \in \Omega^{n-1}(\partial M_n)$. Intuitively this is the normal derivative of the solution. The map $\phi_\partial \rightarrow \iota^*(\star d\phi)$ defines the linear map $A_{M_n} : \Omega^0(\partial M_n) \rightarrow \Omega^{n-1}(\partial M_n)$.

- b.) Describe what needs to be proven to check the gluing formula (3.43): Write the gluing formula and do the Gaussian path integral over ϕ_{int} . Express the result in terms of well-defined mathematical quantities.

Remark 3.45. A_{M_n} only depends on the *germ* of the Riemannian manifold ∂M_n in M_n , and in fact only depends on a finite number of normal derivatives of the metric.

Remark 3.46. We mentioned that making sense of $L^2(\Omega^0(N_{n-1}))$ depends on a choice of positive quadratic form. A natural choice of quadratic form is $\langle \phi, A_{M_n} \phi \rangle$ where $\partial M_n = N_{n-1}$. Note that the quadratic form depends on M_n .

Remark 3.47. A useful way to think of the phase space is as the space of solutions to the equations of motion $(d^\dagger d + m^2)\phi = 0$ on a cylinder with $\widetilde{\partial M_n} = N_{n-1} \times \{0, 1\}$. For a solution f of the equation of motion we can identify the “position” at “time zero” as the value of f on N_{n-1} and the conjugate “momentum” on N_{n-1} as $(\star df)|_{N_{n-1}}$. One can specify the momentum and position on an “initial manifold” instead of Dirichlet boundary conditions in the past and future.

4 Bordism

For much more about bordism theory (with a view to applications in TFT), see [Fre12a, Fre19]. We also recommend the classic textbook [MS74].

In our heuristic discussion, we did not distinguish between compact and noncompact manifolds. Noncompact manifolds are very natural in physics, but do introduce complications of choosing boundary conditions at infinity. For this reason, the rigorous mathematical theory is generally restricted to compact manifolds. We will abbreviate “compact manifold without boundary” to “closed manifold.” We will assume our manifolds as smooth, i.e., \mathcal{C}^∞ manifolds. A few brief comments about the nontrivial generalization to more general topological spaces can be found in section 22.

Definition 4.1 (Bordism). Let N_{n-1}^0 and N_{n-1}^1 be closed $(n-1)$ -manifolds. A **bordism from N_{n-1}^0 to N_{n-1}^1** is a quadruple $(M_n, p, \theta_{\text{in}}, \theta_{\text{out}})$ consisting of the following collection of data:

- (a) A compact n -manifold with boundary M_n .
- (b) A disjoint decomposition of boundary components into two sets, called the “in” and “out” components:

$$\partial M_n = (\partial M_n)^{\text{in}} \coprod (\partial M_n)^{\text{out}} . \quad (4.2)$$

We can express this in the data as a map $p : \partial M_n \rightarrow \{\text{in}, \text{out}\}$.

- (c) There exist disjoint collar neighborhoods:¹⁶

$$\theta_{\text{in}} : N_{n-1}^0 \times [0, 1) \xrightarrow{\text{embedding}} M_n , \quad (4.3)$$

$$N_{n-1}^0 \times \{0\} \xrightarrow{\text{diffeomorphism}} (\partial M_n)^{\text{in}} , \quad (4.4)$$

$$\theta_{\text{out}} : N_{n-1}^1 \times (-1, 0] \xrightarrow{\text{embedding}} M_n , \quad (4.5)$$

$$N_{n-1}^1 \times \{0\} \xrightarrow{\text{diffeomorphism}} (\partial M_n)^{\text{out}} . \quad (4.6)$$

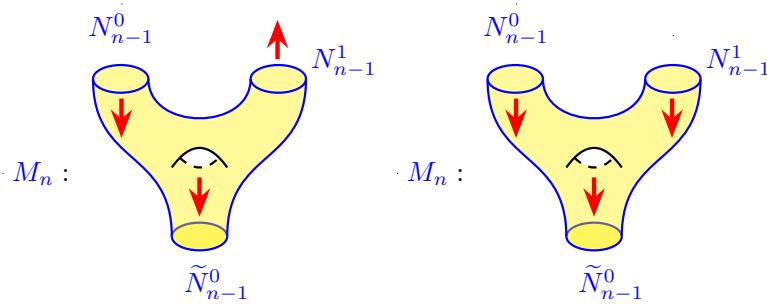


Figure 5: Same manifold, different bordism!!

¹⁶A collar is an open neighborhood which is the Cartesian product of a boundary component with a half-open interval. The collar neighborhood theorem (Brown [Bro62]) states that the boundary of any manifold admits a collar.

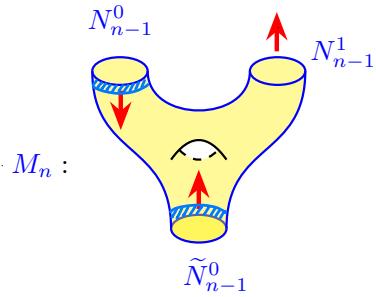


Figure 6: In this example, N_{n-1}^0 is a disjoint union of two (ingoing) circles.

Remark 4.7. The “in” and “out” designations – introduced in [Section 3](#) via arrows – are important, as illustrated in [Figure 5](#).

Remark 4.8. Intuitively one should think of the in and out collar neighborhoods as small neighborhoods of the boundary as illustrated in [Figure 6](#): θ_{in} takes these circles times $[0, 1)$, and embeds them into M_n as shown by the shaded parts in [Figure 6](#), namely, as small collar neighborhoods around the boundary. θ_{out} is similar.

Remark 4.9. The manifolds discussed here need not be connected. In the definition of a bordism nothing is said about $(\partial M_n)^{\text{in}}$ or $(\partial M_n)^{\text{out}}$ being connected. Moreover, we must stress that at this point we have said nothing about orientation! Our manifolds are unoriented. They might, or might not, be orientable.

Remark 4.10. Some authors use the term “cobordism” for what we are calling “bordism.” The former term is more traditional but the “co” prefix carries technical implications that are misleading, so we prefer the term “bordism.”

A lot of information encoded in our definition of a bordism is not important in the constructions we are going to make. For example we do not care about most of the details of the collar neighborhoods provided by the embeddings θ_{in} and θ_{out} . We would prefer to focus on an equivalence class of bordisms where we mod out the unimportant information. Therefore we define:

Definition 4.11 (Diffeomorphism of bordisms). A diffeomorphism of bordisms,

$$\begin{array}{ccc}
 & (M_n, p, \theta_{\text{in}}, \theta_{\text{out}}) & \\
 N_{n-1}^0 & \xrightarrow{\psi} & N_{n-1}^1 \\
 & (M'_n, p', \theta'_{\text{in}}, \theta'_{\text{out}}) &
 \end{array} \tag{4.12}$$

is a diffeomorphism $\psi : M_n \rightarrow M'_n$, so that $p' \circ \psi = p$ and we have commutative diagrams:

$$\begin{array}{ccc}
 & M_n & \\
 \theta_{\text{in}} \nearrow & \downarrow \psi & \theta_{\text{out}} \nearrow \\
 N_{n-1}^0 \times [0, 1] & & N_{n-1}^1 \times (-1, 0] \\
 \theta'_{\text{in}} \searrow & & \theta'_{\text{out}} \searrow \\
 & M'_n & M'_n
 \end{array} \tag{4.13}$$

One of the reasons it is useful to incorporate the data of θ_{in} , θ_{out} in the definition of a bordism is that it allows us to glue two bordisms:

$$\begin{aligned}
 (M_n^{12}, p^{12}, \theta_{\text{in}}^{12}, \theta_{\text{out}}^{12}) & : N_{n-1}^1 \rightarrow N_{n-1}^2, \\
 (M_n^{23}, p^{23}, \theta_{\text{in}}^{23}, \theta_{\text{out}}^{23}) & : N_{n-1}^2 \rightarrow N_{n-1}^3,
 \end{aligned} \tag{4.14}$$

into a single bordism $(M_n^{13}, \dots) : N_{n-1}^1 \rightarrow N_{n-1}^3$. We use the collar neighborhoods near N_{n-1}^2 provided by θ_{out}^{12} for M_n^{12} and θ_{in}^{23} for M_n^{23} . We can define a topological space:

$$(N_{n-1}^2 \times (-1, 0] \amalg N_{n-1}^2 \times [0, 1]) / \sim, \tag{4.15}$$

where we identify $(x, 0)$ for the left factor with $(x, 0)$ for the right factor. The resulting manifold $N_{n-1}^2 \times (-1, 1)$ has a smooth structure. We then use this model to glue M_n^{12} to M_n^{23} using the inverse of the collar embeddings. We could generalize the identification to $(x, 0)_{\text{left}} \sim (f(x), 0)_{\text{right}}$ for some diffeomorphism f of N_{n-1}^2 but nothing new is gained because f could be absorbed into θ_{out}^{12} or θ_{in}^{23} .

Remark 4.16. We say that two manifolds N_{n-1}^0 and N_{n-1}^1 are *bordant* if there exists a bordism from N_{n-1}^0 to N_{n-1}^1 . Bordism defines an equivalence relation on manifolds. Let us verify the three axioms for an equivalence relation:

1. The cylinder $N_{n-1} \times [0, 1]$ defines a bordism of N_{n-1} to itself.
2. If $(M_n, p, \theta_{\text{in}}, \theta_{\text{out}})$ is a bordism $N_{n-1}^1 \sim N_{n-1}^2$ then we can define the *opposite bordism*

from $N_{n-1}^2 \sim N_{n-1}^1$ by composing p with the map that exchanges *in* and *out* and exchanging θ_{in} with θ_{out} .

3. The gluing procedure described above shows that the relation is transitive.

Remark 4.17. If a manifold N_{n-1} is bordant to \emptyset_{n-1} it is said to be *null bordant*. Put differently, N_{n-1} is the boundary of an n -manifold. Note that the sphere is null bordant, because we can fill it in with a disk. Note that any oriented surface is null bordant, because it is the boundary of a handlebody. These two examples bring up an important point: An important topological invariant of manifolds (and more general topological spaces) is the Euler characteristic. We conclude that the Euler characteristic is not a bordism invariant. On the other hand, it can be shown that some invariants, such as Stiefel-Whitney and Pontryagin numbers are bordism invariants. Using that fact it can be shown that \mathbb{RP}^n are null bordant if and only if $n+1$ is a power of 2.¹⁷ Another bordism invariant of great importance is the index of an elliptic operator.

Remark 4.18. Any closed n -manifold M_n defines a bordism $\emptyset_{n-1} \rightarrow \emptyset_{n-1}$.

Example: Consider one-dimensional bordisms between two zero-dimensional manifolds N^0 and $N^1 =$ both of which are disjoint union of two points, as shown below:

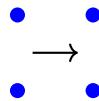


Figure 7: Two disjoint unions of two points.

Let us find all the bordisms from N^0 to N^1 (up to diffeomorphism). There are three obvious bordisms, see Figure 8:

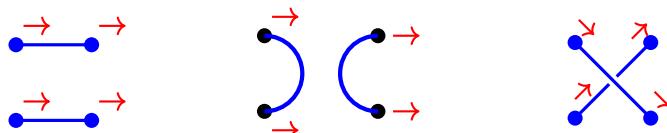


Figure 8: Three obvious bordisms from a disjoint union of two points to a disjoint union of two points.

But since a circle is a bordism from \emptyset to \emptyset , we can take a disjoint union with any number of circles, e.g., see Figure 9:

¹⁷Being null bordant is equivalent to the vanishing of all Stiefel-Whitney numbers. Let $a \in H^1(\mathbb{RP}^n; \mathbb{Z}_2)$ be the generator of the tautological real line bundle $\gamma^1 \rightarrow \mathbb{RP}^n$, i.e., $w_1(\gamma^1) = a$. The total Stiefel-Whitney class of (the tangent bundle of) \mathbb{RP}^n is $w(T\mathbb{RP}^n) = \sum_{i=1}^n w_i = (1+a)^{n+1}$, so the i^{th} Stiefel-Whitney class is $w_i = \binom{n+1}{i} a^i$. The Stiefel-Whitney number for a partition $i_1 + \dots + i_r = n$ is $\langle w_{i_1} \cdots w_{i_r}, [\mathbb{RP}^n] \rangle = \prod_{j=1}^r \binom{n+1}{i_j} \bmod 2$, since $\langle a^n, [\mathbb{RP}^n] \rangle = 1$. All the Stiefel-Whitney numbers will vanish if and only if the binomial coefficients $\binom{n+1}{i}$ for $0 < i < n+1$ are even. This is true if and only if $n+1$ is a power of 2.

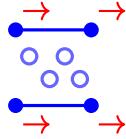


Figure 9: Infinitely many bordisms upon including disjoint union with arbitrarily many circles.

and hence there are infinitely many inequivalent bordisms. So the inequivalent bordisms are obtained from the three obvious types of Figure 8 with n disjoint circles, where n can be any nonnegative integer.

4.1 Bordism Groups and Bordism Rings

We showed above that bordism is an equivalence relation on n -manifolds. The set of bordism classes in n dimensions – denoted Ω_n – forms an Abelian monoid, with disjoint union defining the monoidal product:

$$[N] + [N'] := [N \amalg N'] . \quad (4.19)$$

Note there is something to check here, namely that the formula is well-defined and the sum really makes sense on bordism classes. The identity element of this monoid is the empty n -dimensional manifold \emptyset_n . Now consider the picture of Figure 10.

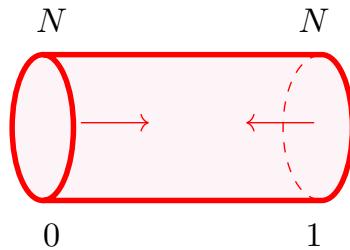


Figure 10: The bordism from $N \amalg N$ to \emptyset .

This shows that

$$[N] \amalg [N] \cong [\emptyset_n] . \quad (4.20)$$

Therefore, every element has an inverse and hence Ω_n is an Abelian group (which is completely 2-torsion). It is straightforward to see that $\Omega_0 \cong \mathbb{Z}/2\mathbb{Z}$. The generator is the bordism class of a single point. Note that $\Omega_1 = \{0\}$ is the trivial group because every circle bounds a disk. But $[\mathbb{RP}^2] \in \Omega_2$ is nontrivial: if $\exists M_3$ such that $\partial M_3 = \mathbb{RP}^2$, then gluing yields $M_3 \cup_{\mathbb{RP}^2} M_3$ closed 3-fold with Euler characteristic $2\chi(M_3) - 1$. But the Euler characteristic of a three-dimensional manifold (orientable or not) must vanish by Poincaré duality. But an odd integer can't vanish, so no M_3 exists. In fact, by the classification theorem for surfaces

[Zee62, Gal08, GX12] one can prove that $\Omega_2 \cong \mathbb{Z}/2\mathbb{Z}$. A key thing to prove is that $\mathbb{RP}^2 \# \mathbb{RP}^2$ is the boundary of a 3-manifold. This is true because $\mathbb{RP}^2 \# \mathbb{RP}^2$ can be presented as a circle bundle over a circle, and there is the boundary of a disk bundle.

We can turn $\Omega_* := \bigoplus_n \Omega_n$ into a \mathbb{Z} -graded ring, where the grading is given by the dimension and the ring product is

$$[M_k] \cdot [\widetilde{M}_k] := [M_k \times \widetilde{M}_k]. \quad (4.21)$$

(Again one needs to check that this equation makes sense – that is one must check we get the same result if we use different representatives of the bordism classes.) For unoriented bordism, a celebrated theorem of René Thom states that the ring has the structure of a polynomial ring over \mathbb{Z}_2 with infinitely many generators:

$$\Omega_* = \mathbb{Z}_2[x_2, x_4, x_5, x_6, \dots]. \quad (4.22)$$

There is a single generator in every dimension n not of the form $2^k - 1$.¹⁸ The even-dimensional generators are bordism classes of real projective spaces. The odd dimensional generators, have representatives known as *Dold manifolds*. These are \mathbb{Z}_2 quotients of $\mathbb{S}^a \times \mathbb{CP}^b$ where the \mathbb{Z}_2 action takes $(x, z) \rightarrow (-x, \bar{z})$.

Bordism groups play an especially important role in the application of TFT to the mathematical theory of topological phases of matter.

4.2 Oriented Bordism

It turns out that we can equip manifolds with topological and geometrical structures and still define a notion of bordism of manifolds equipped with that structure. One simple example is orientation. We define oriented bordism carefully below. One can also define bordism for Spin, Pin, etc. manifolds. The crucial element is a *tangential structure*. Bordism still defines an equivalence relation and hence there are bordism rings.

To define an oriented bordism we modify the definition of bordism slightly. Now, N_0, N_1, M are all oriented. The embeddings θ_0 and θ_1 are required to be orientation preserving and we identify bordisms (M, \dots) and (M', \dots) by oriented diffeomorphisms.

The condition that θ_0 and θ_1 are orientation preserving must be treated with care. Note that if we are given a sum of oriented real vector spaces there is no natural orientation on the direct sum. However, if we are given an exact sequence

$$0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0. \quad (4.23)$$

¹⁸Integers of the form $2^k - 1$ are called Mersenne numbers; if they are prime, they are called Mersenne primes. A necessary (but not sufficient) condition for $2^k - 1$ to be prime is that k be prime. A counterexample to the converse is $k = 11$ for which $2^{11} - 1 = 23 \cdot 89$. If k is composite, $2^k - 1$ is a binomial number, i.e., it can be factorized.

Then there is a canonical isomorphism $\text{DET } V_2 \cong \text{DET } V_1 \otimes \text{DET } V_3$ so if two of the three spaces are oriented, we can determine an orientation on the third. In particular, an orientation on a submanifold and the ambient manifold determines an orientation on the normal bundle.

In more concrete terms, when defining θ_0, θ_1 we orient $[0, +1)$ and $(-1, 0]$ with the standard orientation on \mathbb{R} , $+dx$ and then we take the product orientation on $[0, +1) \times N$ and $(-1, 0] \times N$. So, if y^1, \dots, y^{n-1} are local coordinates on N^0 and θ_{in} looks like the identity at $x = 0$ then $dx \wedge dy^1 \cdots dy^{n-1}$ should be an oriented volume form on M . Similarly $dx \wedge dy^1 \cdots dy^{n-1}$ should be an oriented volume form on M for an outgoing boundary N^1 . Note that if we wish to change an outgoing boundary N^1 to an ingoing boundary by a change of the map p then we must use the map $x \rightarrow -x$ from $(-1, 0] \rightarrow [0, 1)$. This is orientation reversing and hence we must reverse the orientation of N^1 to make it ingoing.

Definition 4.24 (Dual Oriented Bordism). To every oriented bordism $M : N^0 \rightarrow N^1$ there is a *dual oriented bordism* $M^\vee : (N^1)^\vee \rightarrow (N^0)^\vee$. Let us write it out carefully, since it can cause confusion. N^\vee denotes N with the opposite orientation. M^\vee is the manifold with the same orientation. However, we exchange ingoing and outgoing boundaries. Moreover,

$$\theta_0^\vee(t, y_1) = \theta_1(-t, y_1) \quad \forall t \in [0, +1) \quad \& \quad y_1 \in N^1 . \quad (4.25)$$

$$\theta_1^\vee(t, y_0) = \theta_0(-t, y_0) \quad \forall t \in (-1, 0] \quad \& \quad y_0 \in N^0 . \quad (4.26)$$

Note that the relation between θ_0^\vee and θ_1 involves an orientation-reversing transformation $t \rightarrow -t$ and hence we require orientation reversal on Y since X^\vee has the same orientation as X . Forgetting about orientations we also obtain a notion of dual bordism for the unoriented case.

Once again we can define oriented bordism groups Ω_n^{SO} , for $n \geq 0$, the oriented bordism ring $\Omega_\bullet^{\text{SO}}$.

Example 1. Let us consider the oriented bordism group Ω_0^{SO} . There are two kinds of points pt_+ and pt_- . It is easy to see that $\Omega_0^{\text{SO}} \cong \mathbb{Z}$. The isomorphism takes the difference of the number of $+$ and $-$ points.

Example 2. It should be fairly obvious that in dimensions 1 and 2 we again have zero bordism groups. It is also true that $\Omega_3^{\text{SO}} = 0$. In dimension 4 we have $\Omega_4^{\text{SO}} \cong \mathbb{Z}$. The isomorphism is provided by the signature and a generator can be taken to be \mathbb{CP}^2 .

A summary of the main facts regarding the oriented bordism ring $\Omega_\bullet^{\text{SO}}$ is the following. (See Milnor and Stasheff [MS74]. Several further references are provided in Freed's notes [Fre12a], near Theorem 2.24.)

Theorem 4.27.

1. All torsion elements in $\Omega_{\bullet}^{\text{SO}}$ have order two.
2. $\Omega_{\bullet}^{\text{SO}}/\text{torsion}$ is a ring with one generator in degrees $4k$, $k \geq 1$.
3. There is an isomorphism

$$\Omega_{\bullet}^{\text{SO}} \otimes \mathbb{Q} \cong \mathbb{Q}[y_4, y_8, \dots], \quad (4.28)$$

under which y_{4k} corresponds to the oriented bordism class of \mathbb{CP}^{2k} .

4. There are characteristic classes of the tangent bundle of Y , the Stiefel-Whitney classes $w_i(Y) \in H^i(Y; \mathbb{Z}_2)$ and the Pontryagin classes $p_i(Y) \in H^{4i}(Y; \mathbb{Z})$ (the latter depending on the orientation of Y) such that Y_1 and Y_2 are bordant iff all the Stiefel-Whitney and Pontryagin numbers are the same. We defined the Stiefel-Whitney classes above and the Pontryagin numbers are similarly the collection

$$\langle p_{i_1}(Y) \cup \dots \cup p_{i_k}(Y), [Y] \rangle \in \mathbb{Z}. \quad (4.29)$$

A key example is that of complex projective space \mathbb{CP}^n , which is the space of pure states in a quantum Hilbert space of dimension $(n+1)$. The integral cohomology ring is $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[a]/(a^{n+1})$ where a has degree 2. With \mathbb{Z}_2 coefficients we have $H^*(\mathbb{CP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[a]/(a^{n+1})$ (where we use the same symbol for the mod-two reduction of a). The total Stiefel-Whitney class is

$$w(\mathbb{CP}^n) = \sum_i w_i(\mathbb{CP}^n) = (1+a)^{n+1}, \quad (4.30)$$

and the Stiefel-Whitney numbers (valued in \mathbb{Z}_2) computed from this measure the unoriented bordism class of \mathbb{CP}^n , while the total Pontryagin class is

$$p(\mathbb{CP}^n) = \sum_i p_i(\mathbb{CP}^n) = (1+a^2)^{n+1}, \quad (4.31)$$

and the Pontryagin numbers (valued in \mathbb{Z}) computed from this measure the oriented bordism class of \mathbb{CP}^n .

5 Using Bordism To Define A Topological Field Theory

Now we can give a more precise version of **(LOC 1)** and **(LOC 2)** above. A TFT is an association to any bordism $(\{M_n, p, \theta_{\text{in}}, \theta_{\text{out}}\})$ of a corresponding linear map $F_{\text{ampl}}(\{M_n, p, \theta_{\text{in}}, \theta_{\text{out}}\})$ from $F_{\text{states}}(N_{n-1}^0)$ to $F_{\text{states}}(N_{n-1}^1)$. Henceforth, we will usually drop the subscript “states” and “amplitude” on F because it should be clear which one is meant by context. So in this

language our rules for TFT are:

$$F : \left\{ \text{closed } (n-1)\text{-folds} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Vector spaces} \\ (\text{over } \mathbb{C}, \text{ in these lectures}) \end{array} \right\}, \quad (5.1)$$

such that:

$$F(N \coprod N') \cong F(N) \otimes F(N'), \quad (\text{LOC 1}')$$

$$F : \left(\begin{array}{c} \text{Bordism} \\ N^0 \rightarrow N^1 \end{array} \right) \longmapsto F(M_n, p, \theta_{\text{in}}, \theta_{\text{out}}) \in \text{Hom}(F(N^0), F(N^1)). \quad (5.2)$$

For vector spaces V, W we denote the vector space of linear transformations from V to W as $\text{Hom}(V, W)$. We can therefore demand that F maps a composition of bordisms to a composition of linear transformations:

$$F(M'_n \circ M_n) = F(M'_n) \circ F(M_n). \quad (\text{LOC 2}')$$

(We have suppressed θ_{in} and θ_{out} in the gluing of bordisms encoded in $(\text{LOC 2}')$, but in fact, the details of the gluing will rely on these diffeomorphisms.) Now we make the crucial assumption:

Box 5.1: Crucial Assumption

The value of F on a bordism only depends on the diffeomorphism equivalence class of the bordism.

Corollaries:

- 1.) Consider the cylinder $M_n = N \times [0, 1]$ where N is $(n-1)$ -dimensional, $\theta_{\text{in}}|_{N \times \{0\}} = \text{Id}_N$ and $\theta_{\text{out}}|_{N \times \{1\}} = \text{Id}_N$. There are different bordisms that we can associate with this data, depending one whether we consider the boundaries $N_{n-1} \times \{0\}$ and $N_{n-1} \times \{1\}$ to be ingoing or outgoing. Let us first consider $N_{n-1} \times \{0\}$ to be ingoing and $N_{n-1} \times \{1\}$ to be outgoing. If we glue two such cylinders we get another one “twice as long.” But a cylinder which is twice as long as the original cylinder is diffeomorphic to the original

cylinder and hence:

$$F(M_n) \circ F(M_n) = F\left(\begin{array}{c} 0 & & 2 \\ \text{---} & \text{---} & \text{---} \\ \textcolor{red}{\rightarrow} & & \textcolor{red}{\rightarrow} \\ \hline N_{n-1} & 1 & N_{n-1} \end{array}\right) \quad (5.3)$$

$$\xrightarrow{\text{topological !!}} F\left(\begin{array}{c} 0 & & 1 \\ \text{---} & \text{---} & \text{---} \\ \textcolor{red}{\rightarrow} & & \textcolor{red}{\rightarrow} \\ \hline N_{n-1} & & N_{n-1} \end{array}\right). \quad (5.4)$$

(5.5)

It follows that $P := F(M_n)$ satisfies $P = P^2$ and is hence a projector. In particular, we can use the projector to decompose any statespace:

$$F(N_{n-1}) = PF(N_{n-1}) \oplus (1 - P)F(N_{n-1}). \quad (5.6)$$

Now, given any amplitude associated with a bordism we can always glue a small cylinder to any ingoing or outgoing boundary. Hence any amplitude is invariant under the compression by P . Therefore all amplitudes vanish on the image of $(1 - P)$ since $P(1 - P) = (1 - P)P = 0$. Therefore, one can assume without loss of generality that $F\left(\begin{array}{c} \text{---} \\ \textcolor{red}{\rightarrow} \end{array}\right)$ is the identity map. Now, if we compare to the usual Euclidean time evolution by $\exp[-\tau H]$ in QFT, where τ is the Euclidean time, we see that in a topological field theory the Hamiltonian is zero.

- 2.) *Dualizability.* Now consider another bordism associated with the cylinder $M_n = N_{n-1} \times [0, 1]$ but now we consider both boundaries to be ingoing. So we have the picture:

$$F\left(\begin{array}{c} \textcolor{blue}{\text{---}} \\ \textcolor{red}{\rightarrow} \end{array}\right) : V \otimes V \longrightarrow \mathbb{C}. \quad (5.7)$$

where $V = F(N_{n-1})$. We could make a diffeomorphism where the two ingoing circles are exchanged. F should be invariant and therefore $F(M_n)$ defines a symmetric bilinear form $V \otimes V \xrightarrow{b} \mathbb{C}$.

One could also consider both boundaries of the cylinder to be outgoing. This yields

the following picture:

$$F\left(\begin{array}{c} \text{red circle} \\ \text{blue circle} \end{array}\right) : \mathbb{C} \xrightarrow{\tilde{b}} V \otimes V , \quad (5.8)$$

yielding another symmetric bilinear map \tilde{b} , this time from \mathbb{C} to $V \otimes V$.

Now consider the “S-diagram,” see [Figure 11](#). This gives a map (recall $V = F(N)$):

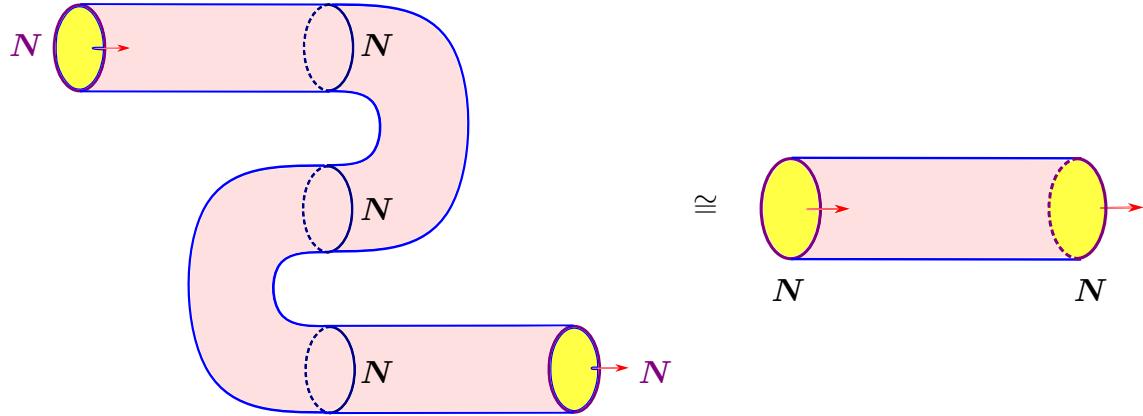


Figure 11: The S -diagram.

$$V \xrightarrow{\text{Id} \otimes \tilde{b}} V \otimes (V \otimes V) \cong (V \otimes V) \otimes V \xrightarrow{b \otimes \text{Id}} V . \quad (5.9)$$

Because of the equivalence in [Figure 11](#), the composition must be the identity. This implies that b is nondegenerate and $V = F(N)$ is finite-dimensional. This is part of something called *dualizability*.¹⁹

Proof. Choose a basis $\{v_i\}$ for V and define $b(v_i, v_j) = b_{ij}$. Then $\tilde{b}(1) = \tilde{b}^{ij}v_i \otimes v_j$. We learn that $\tilde{b}^{ij}\tilde{b}_{jk} = \delta_k^i \implies b_{ij}$ is invertible. If V is infinite-dimensional, the sums defining b and \tilde{b} are infinite sums and might not correspond to well-defined vectors. Indeed they can't both correspond to well-defined vectors. For example, if V is a Hilbert space, we could choose an ON basis $\{v_i\}$. Then b_{ij} must decay rapidly as $i, j \rightarrow \infty$, and consequently \tilde{b} will not converge. \square

As an added bonus note that if v_i is any basis and v^i is the dual basis with respect to b then

$$\tilde{b} : 1 \mapsto \sum_i v_i \otimes v^i \quad (5.10)$$

¹⁹See, however, remarks in [Section 28](#), where we relax dualizability. In the proof below we use the canonical isomorphism of vector spaces $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$. In the language of monoidal categories introduced below we are using the fact that the associator in the category of vector spaces is trivial.

- 3.) For any one-dimensional complex vector space L , $\text{Hom}(L, L) \cong \mathbb{C}$ canonically: every linear transformation $L \rightarrow L$ is of the form $v \mapsto T(v) = z_0 v$ where $v \in L$ and $z_0 \in \mathbb{C}$. So we can naturally associate to T a complex number, and vice versa. As noted above, a closed n -manifold M_n can be viewed as a bordism from the empty $(n-1)$ -manifold to the empty $(n-1)$ -manifold, i.e., $M_n : \emptyset_{n-1} \rightarrow \emptyset_{n-1}$, and therefore, by (5.2),

$$F(M_n) \in \text{Hom}(F(\emptyset_{n-1}), F(\emptyset_{n-1})) \xrightarrow[\text{(LOC 1)}]{\text{Locality}} \text{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C} , \quad (5.11)$$

and the corresponding complex number $F(M_n) \in \mathbb{C}$ is called *the partition function of the theory on M_n* .

We should explain the relation to the usual “partition function” of quantum statistical mechanics, defined as $Z = \text{Tr}_{\mathcal{H}} e^{-\beta H}$ where β is the inverse absolute temperature. It can be shown (see, e.g., [FHS10]) that if \mathcal{H} is the Hilbert space on a spatial slice N then Z can be expressed as a Euclidean signature path integral over $N \times S^1$ with a product metric and S^1 has radius β . The modern notion of a partition function of a QFT is the generalization where $N \times S^1$ is replaced by a general n -manifold.

- 4.) It follows that

$$F(N \times S^1) = \dim_{\mathbb{C}}(F(N)) , \quad (5.12)$$

$$F\left(\text{torus with 3 handles}\right) = \tilde{b}^{ij} b_{ji} = \sum_i \delta_i^i = \dim_{\mathbb{C}}(F(N)) . \quad (5.13)$$

- 5.) Consider a bordism from nothing to something, see Figure 12:

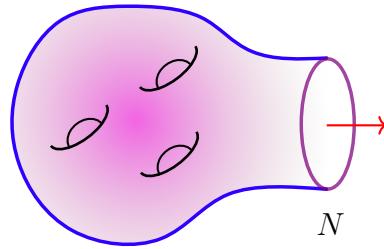


Figure 12: A bordism from nothing to something.

$$M_n : \emptyset_{n-1} \rightarrow N_{n-1} . \quad (5.14)$$

Then $F(M_n) \in \text{Hom}(\mathbb{C}, F(N_{n-1}))$. Now, there is a canonical vector in \mathbb{C} , namely the complex number 1. There to M_n we associate a canonical vector in $F(N_{n-1})$ namely

$$\Psi_{\mathbf{HH}}(M_n) := F(M_n)(1) . \quad (5.15)$$

This can be visualized as

$$F\left(\text{Diagram}\right) \in \text{Hom}(F(\mathbb{C}), F(N)) \cong \text{Hom}(\mathbb{C}, F(N)) \cong F(N) . \quad (5.16)$$

We call $\Psi_{\text{HH}} \in F(N)$ the “state created by the bordism.” It formalizes the idea of the Hartle-Hawking state [HH83] from quantum gravity. (In the latter context, one might, or might not, wish to sum over topologically distinct bordisms with the same N , with a suitable weighting of the topologies.)

By reversing the bordism so that M_n defines a bordism from N to \emptyset_{n-1} we similarly define a *Hartle-Hawking covector* $\Psi_{\text{HH}}^\vee(M_n) \in F(N)^\vee$.

- 6.) *A Gluing Formula For Partition Functions.* Consider a closed n -manifold M_n . Suppose we cut it into two pieces by gluing along some $(n-1)$ -dimensional manifold N so that we can view M_n as a gluing of $M_n^1 : \emptyset \rightarrow N$ and of $M_n^2 : N \rightarrow \emptyset$. Then the partition function can be viewed as the pairing of a HH vector and covector:

$$F(M_n) = \langle \Psi_{\text{HH}}^\vee(M_n^2), \Psi_{\text{HH}}(M_n^1) \rangle . \quad (5.17)$$

- 7.) *Representations Of Diffeomorphism Groups.* Now we claim that $F(N_{n-1})$ is a representation of the diffeomorphism group of N_{n-1} . If $f \in \text{Diff}(N_{n-1})$ then we can consider the bordism $B_f : N_{n-1} \rightarrow N_{n-1}$ where we take $M_n = N_{n-1} \times [0, 1]$ and with $N_{n-1} \times \{0\}$ the in-boundary and $N_{n-1} \times \{1\}$ the out boundary. We take θ_{in} to be the identity map on $N_{n-1} \times \{0\}$ and θ_{out} to be defined by the diffeomorphism f . Then $F(B_f) \in \text{End}(F(N_{n-1}))$ and moreover $B_{f_2} \circ B_{f_1}$ is diffeomorphic to $f_2 \circ f_1$, so

$$F(B_{f_2}) \circ F(B_{f_1}) = F(B_{f_2 \circ f_1}) , \quad (5.18)$$

since, by assumption, F only depends on diffeomorphism equivalence class of the bordism.

- 8.) *Representations Of Mapping Class Groups.* In fact, more is true. We can give $\text{Diff}(N_{n-1})$ a topology so that it is a topological group. One natural choice is the *compact-open topology* where the open sets are generated by sets of diffeomorphisms $\mathcal{O}(K, U)$ where $K \subset N$ is compact and $U \subset N$ is open. The set $\mathcal{O}(K, U)$ is the set of diffeomorphisms such that $f(K) \subset U$.

Now, for any topological group G , the connected component of the identity, G_0 , is a normal subgroup, a.k.a. an invariant subgroup. It is an easy, but good, exercise to

prove that. Therefore, one has an exact sequence

$$1 \rightarrow G_0 \rightarrow G \rightarrow \pi_0(G) \rightarrow 1 , \quad (5.19)$$

so $\pi_0(G) \simeq G/G_0$ is a group. (Note that for a general topological space \mathfrak{X} , $\pi_0(\mathfrak{X})$ does not admit any natural group structure.) Applying this construction to the topological group $G = \text{Diff}(N)$ we obtain the *mapping class group* of N , defined to be $\pi_0(\text{Diff}(N))$.

Now, if there is a continuous path in $\text{Diff}(N)$ of the identity to a diffeomorphism then B_f is diffeomorphic to the trivial cylinder. Therefore $F(B_f) = \text{Id}_{F(N)}$ when f is in the identity component of $\text{Diff}(N)$. Therefore the representation factors through to a representation of the group of components $\pi_0(\text{Diff}(N))$, that is the mapping class group. See Remark 5.23(3) below for some discussion of mapping class groups.

- 9.) The value of $V = F(\mathbb{S}^{n-1})$ is not just a finite-dimensional vector space but it is also an algebra. To prove this consider the n -dimensional disk and cut out two smaller n -dimensional disks. Consider the boundaries of the small disks to be ingoing and that of the large disk to be outgoing. We have defined a bilinear multiplication

$$m : V \otimes V \rightarrow V , \quad (5.20)$$

In fact, much more is true. These algebras have the structure of *E_n -algebras*. To define the term we introduce the *operad of little disks*. Let $\mathcal{D}^n(k)$ be the moduli space of k small disjoint n -dimensional labeled disks inside the unit n -disk. Note that by scaling the unit disk to be sufficiently small we can "plug in" collections of little disks inside little disks to get a map

$$\mathcal{D}^n(k) \times \mathcal{D}^n(\ell_1) \times \cdots \mathcal{D}^n(\ell_k) \rightarrow \mathcal{D}^n(\ell_1 + \cdots + \ell_k) . \quad (5.21)$$

A key fact about $\mathcal{D}^n(k)$ is that it has the homology of the configuration space of k points in the disk. A fact, which is intuitively clear is that $\mathcal{D}^n(2)$ has the homotopy type of S^{n-1} . (Move one disk to the center and then all that matters is the direction to the center of the other disk.) Now an *E_n -algebra* or *algebra over the operad of little disks* is a space X with a system of multiplication operations:

$$\mathcal{D}^n(k) \times X^k \rightarrow X , \quad (5.22)$$

which is invariant under the action of the symmetric group S_k that permutes the little disks and the factors of X and is compatible with the operation (5.21). Evidently topological field theory provides us with such families of multiplications with $X = F(\mathbb{S}^{n-1})$. Combining the simplest operadic operations with diffeomorphisms we learn

that $F(\mathbb{S}^{n-1})$ is a commutative and associative algebra for $n > 1$. (We will discuss $n = 1$ below.)

Remark 5.23.

1. The finite-dimensionality result of item 5 above leads to the inconvenient fact that many “topological field theories” found in the physics literature are not strictly topological field theories as we have defined them. In particular, the original example of Witten [Wit88b], his field-theoretical formulation of the Donaldson invariants does not fit the above axiom system. Another notable example is Chern-Simons theory for noncompact groups. See [Section 27.3](#) on “non-examples” below for further discussion.
2. In many discussions of QFT, the overall normalization of the path integral gets no respect. This is not the case in TFT, where the overall normalization has a definite meaning, and for some manifolds, it is even quantized. This is clear from the result

$$F_{\text{ampl}}(N \times \mathbb{S}^1) = \dim_{\mathbb{C}}(F_{\text{states}}(N)) . \quad (5.24)$$

Note that the RHS is an integer. One of the many applications is to produce topological invariants. The partition function $F(M)$ is a topological invariant and often an enumerative invariant, counting geometrical objects of some kind. For examples: (1) In supersymmetric quantum mechanics (SQM), the partition function on the circle (with periodic boundary conditions for fermions) is the Euler character of the target space. (2) In Donaldson-Witten theory, the partition function (without insertions of observables) is a signed sum over instanton solutions. (3) In topological string theory, the partition function counts the number of holomorphic maps into a target space.

3. *Mapping class groups of compact orientable surfaces.* The mapping class groups of compact orientable surfaces are very well-studied. One example, which can be exhibited very explicitly is the mapping class group of the torus. This group has played a very notable role in many physical applications. We can exhibit the torus as $N = \mathbb{T}^2 = (\mathbb{R} \oplus \mathbb{R})/\mathbb{Z} \oplus \mathbb{Z}$, i.e., as the quotient of the plane by translations along the x and y directions. If (σ^1, σ^2) denote the coordinates on \mathbb{T}^2 , identified as $\sigma^i \sim \sigma^i + n$ for $n \in \mathbb{Z}$ and $i = 1, 2$. The linear transformation on $\mathbb{R} \oplus \mathbb{R}$,

$$\begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) , \quad (5.25)$$

defines a diffeomorphism, which is compatible with the $\mathbb{Z} \oplus \mathbb{Z}$ group action, and descends to a diffeomorphism on the torus. The projection to the quotient group $\pi_0(\text{Diff}(\mathbb{T}^2))$ is nontrivial because it acts nontrivially on $H_1(\mathbb{T}^2; \mathbb{Z})$. The reader should

be warned that, although this example is pervasive, it is somewhat atypical and unrepresentative since we have presented a $\text{GL}(2, \mathbb{Z})$ subgroup of $\text{Diff}(\mathbb{T}^2)$ rather than as a quotient group. In general, $\pi_0(G)$ is not a subgroup of G and the sequence (5.19) above does not split.

There is a very famous class of 3d topological field theories, the Chern-Simons theories, described in some more detail in [Section 27.2](#). It turns out that they only give projective representations of the mapping class groups. This is not a contradiction since the 3d Chern-Simons theories have a “framing anomaly” – meaning they are not strictly speaking topological field theories.

4. *Hartle-Hawking state for a free massive scalar field.* Note that the result of Exercise 3.1 applied to the case of a Riemannian bordism from \emptyset to a Riemannian manifold N_{n-1} defines the Hartle-Hawking state. Of course, this QFT is not a topological field theory so the state depends on details of the Riemannian metric.

Exercise 5.1: Characters Of The Mapping Class Group Representation

For a spatial manifold N show that the characters of the representation ρ of the mapping class group are obtained by evaluating F on the mapping cylinder $M_f := (N \times [0, 1])/(x, 0) \sim (f(x), 1)$:

$$\text{Tr}_{F(N)}\rho(f) := F(M_f) . \quad (5.26)$$

5.1 Oriented Topological Field Theory

The definition of an oriented topological field theory parallels that we have given thus far of a topological field theory, except that one uses oriented spatial manifolds and oriented bordisms to define the amplitudes.

Running the S -duality argument in the oriented case one learns that there is a canonical isomorphism

$$F(N^\vee) \cong F(N)^\vee . \quad (5.27)$$

That is, reversing the orientation of the spatial manifold gives the dual state space. The analog of equation (5.7) is the dual pairing of $F(N)$ and $F(N)^\vee$. This extends to amplitudes: If $M : N^0 \rightarrow N^1$ we defined above a dual bordism $M^\vee : (N^1)^\vee \rightarrow (N^0)^\vee$. Recall that if $T : V \rightarrow W$ is a linear transformation of vector spaces there is a canonically defined dual transformation $T^\vee : W^\vee \rightarrow V^\vee$, defined by $\langle T(\ell), v \rangle = \langle \ell, T(v) \rangle$. We have indeed

$$F(M^\vee) = F(M)^\vee . \quad (5.28)$$

When we cut a closed n -manifold in two along N then the partition function is a pairing

$$F(M_n) = \langle F(M_1^\vee), F(M_2) \rangle . \quad (5.29)$$

In physics it is often important to include the data of a unitary structure. Hilbert spaces of statevectors should have a (complete) positive inner product. Evolution in Lorentzian signature is via unitary evolution. The Wick rotation of e^{itH} is a contraction semigroup $e^{-\tau H}$. Some of this structure can be incorporated in a TFT: A nondegenerate sesquilinear form on a complex vector space defines an isomorphism $V^\vee \cong \overline{V}$. So now (5.7) describes a Hilbert space inner product and should satisfy some positivity properties. Now in terms of amplitudes we see that if $F(M) : F(N^0) \rightarrow F(N^1)$ then

$$\overline{F(M^\vee)} : F(N^1) \rightarrow F(N^0) \quad (5.30)$$

and we can require that $\overline{F(M^\vee)} = F(M)^\dagger$. In particular, the partition function of a closed manifold M should change under change of orientation by complex conjugation:

$$Z(M^\vee) = Z(M)^* . \quad (5.31)$$

6 Example 1: Topological Field Theory In One Dimension

What data do we need to define a one-dimensional topological field theory? We first address the case when our bordisms are unoriented. Then we comment on how the picture changes when they are oriented.

Unoriented One-Dimensional TFT

One-dimensional bordisms will connect zero-dimensional manifolds on their boundary. There exists a unique connected 0-dimensional unoriented manifold, namely, the humble point. All other zero-dimensional manifolds are finite disjoint unions of points. (They are finite because we assume our manifolds to be compact.) By the rule (**LOC 1**), we know all statespaces associated to zero-dimensional manifolds once we know the value of F_{states} on an unoriented point. That value must be some finite-dimensional vector space V , i.e., $F_{\text{states}}(\text{pt}) = V$. We could work over any field, but for definiteness, we will take it to be a complex vector space. Thus, the statespaces are:

$$F\left(\underbrace{\text{pt} \coprod \text{pt} \cdots \coprod \text{pt}}_{n \text{ times}}\right) \cong V^{\otimes n} . \quad (6.1)$$

Note that the diffeomorphism group is the symmetric group S_n and indeed, $V^{\otimes n}$ is a repre-

sentation of S_n acting as a permutation of the factors.

The vector space V has a nondegenerate bilinear form because of the bordism:

$$F\left(\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}\right) : V \otimes V \xrightarrow{b} \mathbb{C}. \quad (6.2)$$

Running the bordism the other way produces a state $\tilde{b}(1) \in V \otimes V$. Moreover, by the S -diagram of Figure 11, b must be nondegenerate. Indeed, \tilde{b} can be computed from b . Recall that if we choose a basis, then $\tilde{b}^{ij} b_{jk} = \delta_k^i$.

That's all the data we need to specify! We now have the basic data to compute any amplitude we like, such as $V^{\otimes 4} \rightarrow V^{\otimes 4}$, see Figure 13.

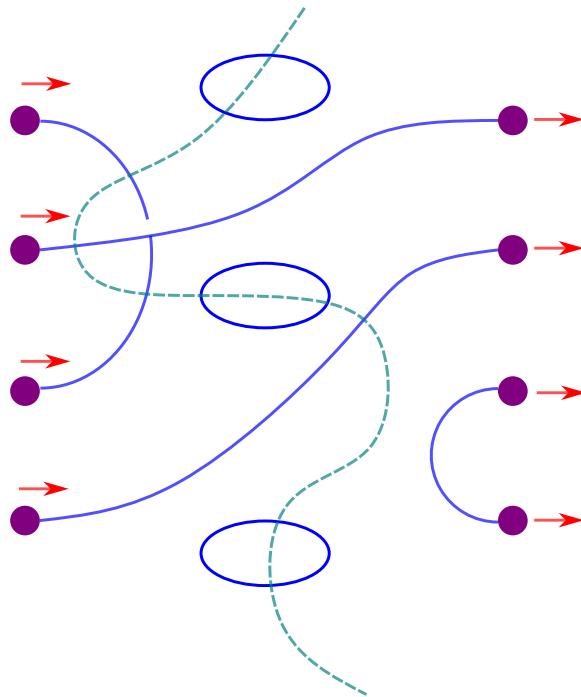


Figure 13: A bordism B leading to a linear map $F_{\text{ampl}}(B) : V^{\otimes 4} \rightarrow V^{\otimes 4}$. The dotted line shows one way of cutting up the bordism so that $F_{\text{ampl}}(B)$ can be computed in terms of the defining data (V, b) . It is a nontrivial fact, proven using Morse-theoretic ideas described below, that no matter how we cut along intermediate channels, we will produce the same linear operator $F_{\text{ampl}}(B)$.

Oriented One-Dimensional TFT

It is interesting to contrast the above description with the topological field theory we obtain using oriented bordisms. In one-dimensional oriented bordism theory, the one-dimensional manifolds carry an orientation. There are therefore two kinds of points: A positively oriented point pt_+ is a source and a negatively oriented point pt_- is a sink. The

basic bordisms we can draw are:

$$\begin{array}{c} \text{Generating objects: } \text{pt}_+ \text{ and } \text{pt}_- \\ \text{Generating 1-morphisms: } \\ \text{pt}_+ \text{--- pt}_- \quad \text{pt}_- \text{--- pt}_+ \end{array} \quad (6.3)$$

The generating relations are:

$$\begin{array}{ccc} \text{pt}_+ & \bullet & \text{pt}_+ \\ & \swarrow \curvearrowright & \downarrow \curvearrowright \\ & & \text{pt}_+ \end{array} = \text{pt}_+ \bullet \text{pt}_+ ,$$

and

$$\begin{array}{ccc} & \bullet & \text{pt}_- \\ & \uparrow \curvearrowright & \\ \text{pt}_- & \bullet & \text{pt}_- \end{array} = \text{pt}_- \bullet \text{pt}_- .$$
(6.4)

We can now define $F_{\text{states}}(\text{pt}_+) = V$. In contrast to the unoriented case, this is the only data we need. It follows, again from the S -diagram argument of Figure 11, that $F_{\text{states}}(\text{pt}_-) = V^\vee$ is the linear dual. All amplitudes can be computed just in terms of those data.

Exercise 6.1: Bordisms to linear maps and amplitudes

- (a)) Show that the linear map associated with the bordism B of Figure 13 is given by

$$F(w_1 \otimes w_2 \otimes w_3 \otimes w_4) = (\dim V)^3 b(w_1, w_3) w_2 \otimes w_4 \otimes \tilde{b}(1) , \quad (6.5)$$

where $w_i \in V$, $i = 4$ is a choice of four vectors. One then extends to arbitrary elements of $V^{\otimes 4}$ by linearity.

- (b)) Now endow the bordism of Figure 13 with some orientations, and compute the resulting amplitude.

7 Example 2: Oriented TFT In Dimension Two

7.1 Defining Data And Frobenius Algebras

For $n = 2$, to avoid complications with the classification of unoriented surfaces, we work with oriented bordisms. There exists a unique connected oriented 1-manifold without boundary, namely, an oriented \mathbb{S}^1 . Suppose $F(\mathbb{S}^1) = V$. There are two algebraic structures:

1. Canonical bordism $\mathbb{S}^1 \rightarrow \emptyset$:

$$\text{the disk } = \begin{array}{c} \text{a blue circle with a red arrow inside} \\ (7.1) \end{array}$$

2. Multiplication: Pair of pants:

$$\begin{array}{c} \text{a blue shape resembling a pair of pants with three holes} \\ m : V \otimes V \rightarrow V . \quad (7.2) \end{array}$$

A useful way to view the pair of pants is shown in Figure 14. We have disks within disks. An advantage of this viewpoint is that it readily generalizes to higher dimensions.

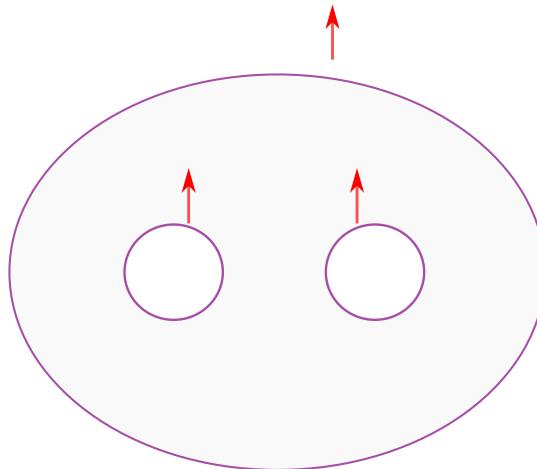


Figure 14: Disks within disks.

Figure 14 has some nice immediate corollaries:

Corollary 1: $m : V \otimes V \rightarrow V$ is a commutative and associative multiplication. The dual bordism to (7.1) is the unit element under the algebra multiplication on V . The reader should prove associativity by cutting a bordism from three circles to one in different ways.

The commutativity follows from a simple diffeomorphism permuting two incoming disks. The reader should prove that for the dual of (7.1) we have $F(1) = 1_V$ using a simple picture.

Corollary 2: In the $n = 2$ case, the bilinear form $b(v_1, v_2) = \theta(v_1 \cdot v_2)$ is nondegenerate. Proving this is a good exercise.²⁰

Definition 7.3 (Frobenius algebra). An associative algebra with a linear functional $\theta : V \rightarrow \mathbb{C}$ such that the bilinear form $b(v_1, v_2) = \theta(v_1 \cdot v_2)$ is nondegenerate, is called a **Frobenius algebra**.

For an elementary exposition of Frobenius algebras, see for example, [Koc03].

Now, any orientable bordism from $\amalg_1^{k_{in}} S^1$ to $\amalg_1^{k_{out}} S^1$ can be decomposed into the composition of bordisms chosen from the five elementary bordisms shown in Figure 15.

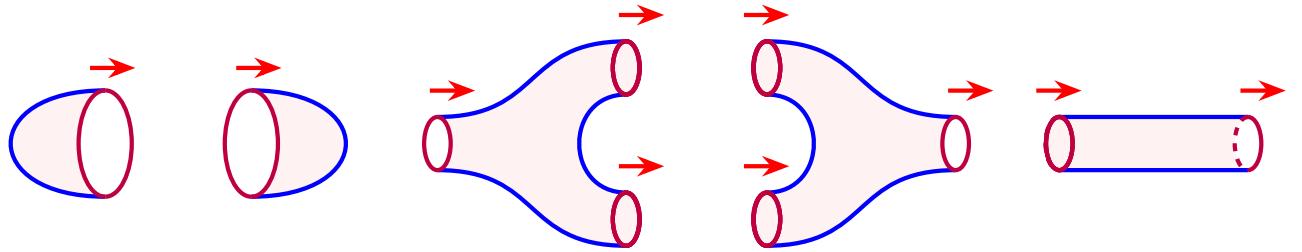


Figure 15: Elementary pieces.

The data (V, m, θ) determines the linear transformations associated with these elementary bordisms and therefore we can compute any amplitude using just (V, m, θ) . The only one we have not seen is the dual to the pair of pants from one circle to two circles. The dual of m takes $V^\vee \rightarrow V^\vee \otimes V^\vee$. But using the nondegenerate pairing provided by the trace we get a linear transformation $V \rightarrow V \otimes V$.

7.2 The Sewing Theorem: Morse Theory Proof

A given bordism B can be cut up into elementary bordisms in many distinct ways. No matter how we cut it up, we should get the same answer for $F(B)$. This would seem to place constraints on the linear transformations we associate to the elementary bordisms.

Theorem 7.4 (Sewing Theorem). *If (V, m, θ) is a commutative Frobenius algebra, then all amplitudes are well-defined. That is, for any bordism B the value of $F(B)$ does not depend on the decomposition of B into a gluing of elementary bordisms. Put differently: To give an $n = 2$ dimensional TFT is to give a commutative finite-dimensional Frobenius algebra.*

²⁰Answer to the exercise: θ corresponds to capping off the maps at the waist. Then the surface is homeomorphic to a cylinder. Apply the S -diagram argument.

For a long time, the Sewing Theorem was a folk theorem attributed variously to D. Friedan, R. Dijkgraaf, G. Segal, and perhaps others. It appears, for example, in R. Dijkgraaf's Ph.D. thesis [Dij89]. A careful proof can be found in the appendix of the expository paper [MS06]. We now briefly review the proof from [MS06]. The basic idea is to make use of Morse-Cerf theory.

Recall that a *Morse function* on a manifold M_n is a smooth real-valued function on M_n that has isolated nondegenerate critical points. A *critical point* $p \in M_n$ is a point where $df(p) = 0$. The associated *critical value* is $f(p)$. A critical point is called *nondegenerate* if the *Hessian at p* , defined to be

$$\left. \frac{\partial^2 f}{\partial x^i \partial x^j} \right|_p , \quad (7.5)$$

in local coordinates x^i near p , is a nondegenerate symmetric matrix. The nondegeneracy condition does not depend on what coordinate system we use near p . Because p is a critical point, we don't need to use covariant derivatives to define the second derivative.

A Morse function gives a decomposition of M_n into level sets $f^{-1}(t) = N_t \subset M_n$, for $t \in \mathbb{R}$, to be thought of as "time slices," or spatial slices. Now $f^{-1}(t)$ will be a nice smooth manifold unless t is a critical value. A Morse function on a bordism $M_n : N_{n-1}^0 \rightarrow N_{n-1}^1$ is *excellent* if it is constant on N_{n-1}^0 , N_{n-1}^1 , and the critical points can be ordered so that the critical values are

$$c_0 = f(N_{n-1}^0) < c_1 < \cdots < c_N < c_f = f(N_{n-1}^1) . \quad (7.6)$$

The spatial slices $f^{-1}(t)$ are all diffeomorphic for all t in an interval $c_i < t < c_{i+1}$. However, the slices for values of t in two successive intervals, for example $c_{i-1} < t_1 < c_i$ and $c_i < t_2 < c_{i+1}$ will differ by a (simple) change of topology. For example, in Figure 16 and Figure 17, the topology change as t increases through a critical value are readily seen.

Example: $n = 2$. See Figure 17.

Now we can change time-slicings by considering a path of smooth functions f_s which are excellent Morse functions for generic s . A crucial theorem of Cerf [Cer70] states that in a natural topology²¹ on the set of infinitely differentiable real functions on M_n , i.e., on $C^\infty(M_n \rightarrow \mathbb{R})$, the set of excellent Morse functions is open and dense, but disconnected. Moreover, we can define a function $f : M_n \rightarrow \mathbb{R}$ to be "good" if it is Morse everywhere except for one or two critical points, and, (in the case that M is two-dimensional) one has:

- One critical point with a function locally of the form $\pm y^2 + x^3$,
- Two critical points with the same value.

Then, a remarkable result [Cer68, Cer70] states that the set of excellent and good functions is a *connected set*. The good but not excellent functions form a real codimension-one subset.

²¹The topology is known as the *Whitney topology*. See [Fre12a, Ch. 20].

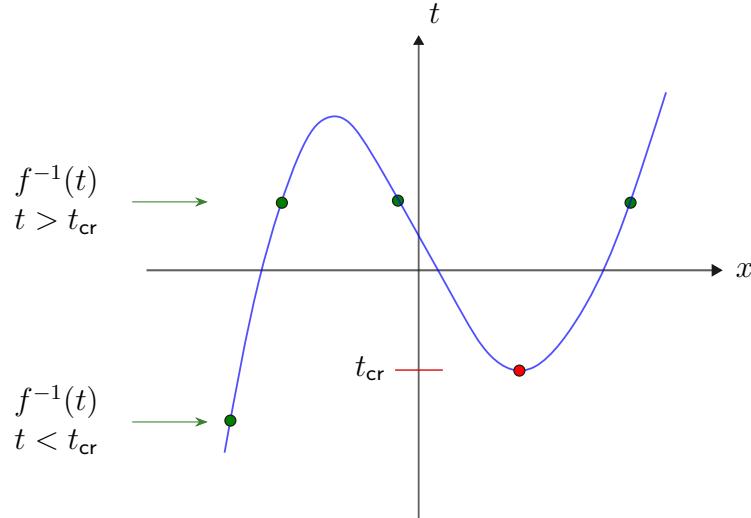


Figure 16: A one-dimensional bordism between two points labeled by a Morse function whose value is t .

It follows from the results quoted above that a generic path of \mathcal{C}^∞ functions, f_s with path parameter s , in the space of excellent and good functions, will cross a finite set of critical values s_1, \dots, s_k where the functions are good but not excellent.

Example: Consider the case $n = 1$, and consider the path of functions $f_s(x) = \frac{x^3}{3} - sx$, for $s \in (-1, 1)$. For $s \neq 0$, the function is excellent, but for $s = 0$, it is only good. Note that as s changes from positive to negative the nature of the function changes:

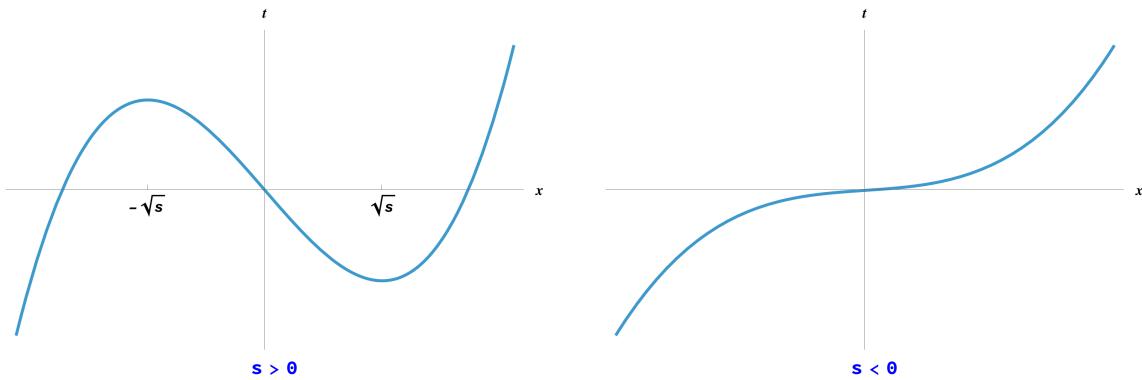


Figure 18: The function $f_s(x)$ for $s > 0$ (left), and $s < 0$ (right).

It now follows from Cerf-Morse theory that in the 2d field theory, any two changes of time slicings are related by a sequence of elementary changes in the decomposition of the bordism into the the elementary pieces of Figure 15. (Each of these elementary pieces corresponds to an excellent Morse function with a single critical point.) Invariance under those elementary changes is guaranteed by the algebraic axioms of a commutative Frobenius algebra. This is how the sewing theorem is proven. See [MS06] for more complete details.

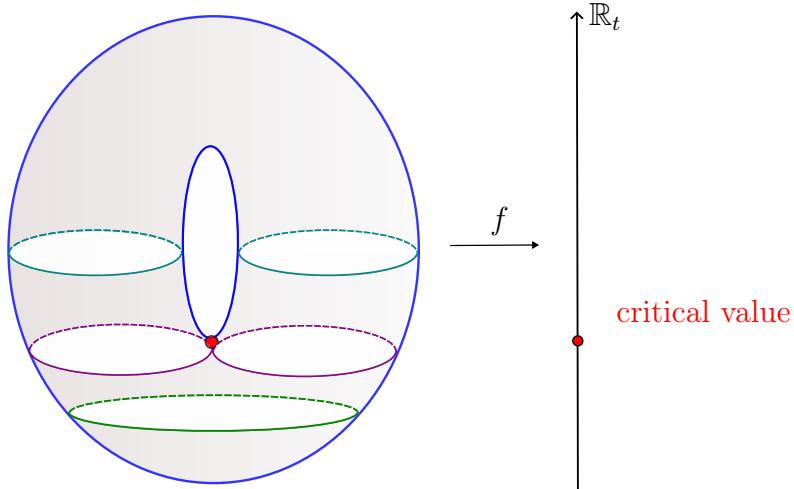


Figure 17: A two-dimensional bordism from \emptyset to \emptyset labeled by a Morse function whose value is t .

7.3 Computing Amplitudes

One of the pleasant properties in 2d TFT is that one can immediately write down all the “amplitudes” in the theory.

The key is to introduce the “characteristic element” H . This is defined by *** FIGURE: CHARACTERISTIC ELEMENT *** which corresponds to the map

$$1 \rightarrow \sum (-1)^{\phi^\mu} \phi_\mu \otimes \phi^\mu \rightarrow \sum \phi^\mu \phi_\mu \quad (7.7)$$

and thus it is given by

$$H = \sum_{\mu} \phi^\mu \cdot \phi_\mu \quad (7.8)$$

where ϕ_μ is any basis for \mathcal{C} and ϕ^μ is a dual basis wrt the trace $\theta(\phi_\mu \phi^\nu) = \delta_\mu^\nu$. (with these choices the formulae hold in the \mathbb{Z}_2 -graded case).

Exercise 7.1:

- a.) Show that H is independent of the choice of basis.
- b.) Compute $\theta(H)$
- c.) Show that in the \mathbb{Z}_2 -graded case $\theta(H) = STr(1) = Sdim(\mathcal{C})$

In the left (=right) regular representation, H is a “handle-adding operator” because of *** FIGURE OF HANDLE-ADDING OPERATOR *** .

Note that we have

$$\theta(\phi H) = \text{Tr}_{\mathcal{C}} \left(L(\phi) \right) \quad (7.9)$$

where the trace is in the regular representation.

One can now immediately write down all amplitudes in the theory. In particular, for a genus g surface with have

$$Z(M_g) = \theta(H^g) = \text{Tr}_c(L(H))^{g-1} \quad (7.10)$$

Exercise 7.2: The general amplitude

Write similar formula for “matrix elements” with arbitrary ingoing and outgoing states, that is, write the amplitude for $\phi_1 \otimes \cdots \otimes \phi_n$ to evolve into $\phi'_1 \otimes \cdots \otimes \phi'_m$. ^a

^aAnswer: The image of $\phi_1 \otimes \cdots \otimes \phi_n$ for n ingoing circles on a genus g surface is $\sum_{\nu_1, \dots, \nu_m} \theta(\phi_1 \cdots \phi_n H^g \phi'^{\nu_1} \cdots \phi'^{\nu_m}) \phi'_{\nu_1} \otimes \cdots \otimes \phi'_{\nu_m}$ if there are m outgoing circles.

7.3.1 Summing over topologies

In quantum gravity one must face the issue of whether in the path integral over metrics (“geometries”) one should also sum over topologies, and how to weight them.

In string theory, we have a theory of quantum gravity in two dimensions coupled to a sigma model embedding the string into spacetime. In this case one is definitely obliged to sum over topologies. This sum is the analog of the Feynman diagram expansion of field theory in the spacetime.

With this motivation, let us ask what the sum over topologies might look like in our baby theory. We want to understand

$$Z_{\text{string}} = Z(M_0) + Z(M_1) + \cdots \quad (7.11)$$

Evidently, from our general formula, we can write the sum formally as

$$Z_{\text{string}} = \theta \left(\frac{1}{1 - H} \right). \quad (7.12)$$

Suppose we have a Frobenius algebra (A, θ) . Let us now define a new Frobenius algebra by $\tilde{\theta} := \lambda^{-2}\theta$. Then it is easy to check that

$$\tilde{H} = \lambda^2 H \quad (7.13)$$

and hence the new genus g amplitude is

$$\tilde{Z}(M_g) = \lambda^{-2+2g} Z(M_g) = \lambda^{-\chi(M_g)} Z(M_g) \quad (7.14)$$

Thus, we can interpret the scale of θ as the string coupling constant. If we define a theory

by $\widehat{\theta}(1) = 1$ then we have

$$Z_{\text{string}} = g^{-2} \widehat{\theta} \left(\frac{1}{1 - g^2 \widehat{H}} \right) \quad (7.15)$$

where $\theta(1) = g^{-2}$ is the string coupling constant.

It is unusual that perturbation theory is convergent – usually perturbation series are only asymptotic series. Since the series is convergent we can analytically continue to all values of g^2 , thus defining the nonperturbative sum.

7.4 Semisimple Frobenius Algebras

One can be even more explicit about the amplitudes in a 2d TFT in the case where the algebra (\mathcal{C}, θ) is *semisimple*.

The most useful characterization of semisimplicity is the following. The structure constants ²²

$$\phi_\mu \phi_\nu = N_{\mu\nu}^\lambda \phi_\lambda , \quad (7.16)$$

defines a set of matrices via the left-regular representation, $L(\phi_\mu)$, with matrix elements $N_{\mu\nu}^\lambda$. Since \mathcal{C} is commutative these are commuting matrices. Then:

Definition 7.17 (Semisimple Frobenius Algebra). \mathcal{C} is semi-simple iff the matrices $L(\phi_\mu)$ are simultaneously diagonalizable.

Thus in the semisimple case we can find a matrix S such that:

$$N_{\mu\nu}^\lambda = \sum_x S_\nu^x \Lambda_x^{(\mu)} (S^{-1})_x^\lambda \quad (7.18)$$

where $\Lambda_x^{(\mu)}$ are the different eigenvalues of $L(\phi_\mu)$.

Now choose a basis such with the index μ running over values $\mu = 0, \dots, n$, and take $\phi_0 = 1_{\mathcal{C}}$, the multiplicative identity. Putting $\mu = 0$ in equation (7.18) leads to a trivial identity, but putting $\nu = 0$ and using $N_{\mu\nu}^\lambda = N_{\nu\mu}^\lambda$ so that $N_{\mu 0}^\lambda = \delta_\mu^\lambda$ we see that $S_0^x \Lambda_x^{(\mu)} = S_\mu^x$ since this is the matrix inverse of S^{-1} . So:

$$S_0^x \Lambda_x^{(\mu)} = S_\mu^x \quad \text{no sum on } x . \quad (7.19)$$

Plugging back into (7.18) we get:

$$N_{\mu\nu}^\lambda = \sum_x \frac{S_\nu^x S_\mu^x (S^{-1})_x^\lambda}{S_0^x} . \quad (7.20)$$

²²The structure constants $N_{\mu\nu}^\lambda$ need not be integral. But in many interesting examples there is a basis for the algebra in which they are in fact integral.

Note that $\theta(\phi_\mu \phi_\nu \phi_\lambda) = N_{\mu\nu}^\lambda Q_{\lambda\lambda} := N_{\mu\nu\lambda}$ is totally symmetric on μ, ν, λ . Suppose we further restrict attention to a basis $\{\phi_\mu\}$ so that $\theta(\phi_\mu) = \delta_{\mu,0}$. Then taking the trace of (7.16) we learn that $Q_{\mu\nu} = N_{\mu\nu}^0$ and then (7.20) gives

$$Q_{\mu\nu} = \sum_x S_\nu^x S_\mu^x \frac{(S^{-1})_x^0}{S_0^x} \quad (7.21)$$

so that

$$N_{\mu\nu\lambda} = \sum_x S_\mu^x S_\nu^x S_\lambda^x \frac{(S^{-1})_x^0}{(S_0^x)^2} \quad (7.22)$$

If we form the linear combinations

$$\epsilon_x = \sum_\mu S_0^x (S^{-1})_x^\mu \phi_\mu \quad (7.23)$$

then the ϵ_x serve as a set of basic idempotents, that is,

$$\begin{aligned} \mathcal{C} &= \bigoplus_x \mathbb{C}\epsilon_x \\ \epsilon_x \epsilon_y &= \delta_{x,y} \epsilon_y \end{aligned} \quad (7.24)$$

Moreover, if we choose the natural normalization $\theta(\phi_\mu) = \delta_{\mu,0}$ then

$$\theta_x := \theta_{\mathcal{C}}(\epsilon_x) = S_0^x (S^{-1})_x^0 \quad (7.25)$$

Here θ_i are some nonzero complex numbers. The unordered set $\{\theta_x\}$ is the only invariant of a finite dimensional commutative semisimple Frobenius algebra.

Note that in this case the characteristic element is simply

$$H = \sum_x \frac{1}{\theta_x} \epsilon_x \quad (7.26)$$

and hence the vacuum amplitude on a genus $g \geq 0$ surface is

$$Z_g = \sum_x \theta_x^{1-g} \quad (7.27)$$

Remarks:

1. If we consider the sum over all genera then the sum only converges when $|\theta_x| > 1$ (with conditional convergence on the unit circle), in which case it is:

$$Z_{\text{string}} = \sum_x \frac{\theta_x}{1 - \theta_x^{-1}} \quad (7.28)$$

2. Note that nothing has fixed the overall normalization of the matrix S at this point. In some cases S will be unitary so that $(S^{-1})_x^0 = (S_0^x)^*$. Moreover, if the matrix elements S_0^x can be taken to be real then we have a nice simplification of (7.22):

$$N_{\mu\nu\lambda} = \sum_x \frac{S_\mu^x S_\nu^x S_\lambda^x}{S_0^x} \quad (7.29)$$

This is how the Verlinde formula is usually stated.

 **Exercise 7.3:**

Show that the eigenvalues $\Lambda_x^{(\rho)}$ satisfy the algebra

$$\Lambda_x^{(\mu)} \Lambda_x^{(\nu)} = \sum_{\lambda} N_{\mu\nu}^{\lambda} \Lambda_x^{(\lambda)} \quad (7.30)$$

 **Exercise 7.4:**

A natural question in field theory is whether the vacuum amplitudes of a theory completely determine all the amplitudes in the theory.

Investigate this for the case of a semisimple 2d TFT.

7.4.1 Example: Commutative Frobenius Algebras Arising From Finite Group Theory

Let G be a finite group. The space of complex-valued functions $C[G]$ is a C^* algebra (see below) with the obvious product given by pointwise multiplication

$$f_1 \cdot f_2(g) := f_1(g) f_2(g) . \quad (7.31)$$

Let \mathcal{C} be the subspace of class functions, that is, functions such that

$$f(hgh^{-1}) = f(g) \quad \forall g, h \in G . \quad (7.32)$$

This is the space of functions on the the (finite) set of conjugacy classes of G .

There are two natural bases of functions for \mathcal{C} . One makes it clear that \mathcal{C} is a Frobenius algebra in a natural way, and the other makes it clear that this Frobenius algebra is semisimple.

The first natural basis for the space of class functions is given by the characters of the distinct irreps χ_μ , μ labels the distinct irreps of G .

Under the pointwise product

$$\chi_\mu \chi_\nu = \sum_{\lambda} N_{\mu\nu}^{\lambda} \chi_{\lambda} , \quad (7.33)$$

where $N_{\mu\nu}^\lambda$ are the *fusion coefficients*, (they are also known as ‘‘Littlewood-Richardson coefficients’’). They are determined by the Clebsch-Gordon series

$$T^\mu \otimes T^\nu = \bigoplus_\lambda N_{\mu\nu}^\lambda T^\lambda , \quad (7.34)$$

and are nonnegative integers. The natural trace is

$$\theta(\chi_\mu) = \delta_{\mu,0} , \quad (7.35)$$

where $\chi_0 = 1$ corresponds to the identity representation. Since for every rep μ there is a rep μ^* with $\chi_\mu \chi_{\mu^*} = \chi_0 + \dots$, we conclude $\langle f, g \rangle = \theta(fg)$ is nondegenerate, and hence that \mathcal{C} is indeed a Frobenius algebra.

Another natural basis of class functions are the delta functions on conjugacy classes:

$$\delta_C(g) = \begin{cases} 1 , & \text{if } g \in C , \\ 0 , & \text{if } g \notin C . \end{cases} \quad (7.36)$$

where C is a conjugacy class. Note that in this basis the pointwise product is diagonal. Thus it is clear that \mathcal{C} is semi-simple.

We can of course expand one basis in terms of another:

$$\chi_\mu = \sum_i \chi_\mu(C_i) \delta_{C_i} . \quad (7.37)$$

Now recall a standard result from group representation theory: the orthogonality relations on the characters of the irreducible representations:

$$\frac{1}{|G|} \sum_g \chi_\mu(g) \chi_\nu(g^{-1}) = \delta_{\mu,\nu} . \quad (7.38)$$

Since G is finite we can, WLOG, assume the representation T^μ is unitary. Therefore the matrix

$$S_{i\mu} = \sqrt{\frac{m_i}{|G|}} \chi_\mu(C_i) , \quad (7.39)$$

where m_i is the order of the class $|C_i|$, is a unitary matrix.

Now we have:

$$\chi_\mu = \sum_i \sqrt{\frac{|G|}{m_i}} S_{i\mu} \delta_{C_i} , \quad (7.40)$$

and therefore since multiplication is diagonal in the basis δ_{C_i} , $S_{i\mu}$ is the matrix which diagonalizes the fusion rules in the character basis.

Now, using (7.25) we compute

$$\theta_x = |S_{0x}|^2 = \frac{(\dim V_x)^2}{|G|} , \quad (7.41)$$

and hence the partition function on a compact Riemann surface of genus g is

$$Z_g = |G|^{1-g} \sum_x \frac{1}{(\dim V_x)^{2g-2}} , \quad (7.42)$$

where the sum runs over irreducible representations of G . The first factor is relatively uninteresting (it can be absorbed in the scale of the string coupling) but the second is interesting.

What geometrical object is the sum in (7.42) counting?

We will answer this question in a few lectures.

Exercise 7.5:

- a.) Show that the center of the group algebra $C[G]$ with the convolution product is \mathcal{C} , the space of class functions.
- b.) Show that the matrix $S_{i\mu}$ is a kind of Fourier transform between these two product structures on \mathcal{C} . Note that the basis of characters of irreps χ_μ diagonalize the convolution product:

$$\chi_\mu * \chi_\nu = \frac{\delta_{\mu\nu}}{n_\nu} \chi_\nu . \quad (7.43)$$

- c.) Show that the invariants θ_x of this Frobenius algebra are given by

$$\theta(\epsilon_\mu) = \frac{(\dim V_\mu)^2}{|G|} . \quad (7.44)$$

Need to incorporate subsection below into material above

7.5 Semisimplicity

Choose an ordered basis $\{v_i\}$ for an algebra V . Consider the operator L_i defined by left-multiplication by v_i . It has matrix elements:

$$L_i(v_j) = v_i v_j = N_{ij}{}^k v_k . \quad (7.45)$$

If the algebra V is commutative, then $[L_i, L_j] = 0$. If the L_i are all diagonalizable, we say the algebra V is *semisimple*. In this case, there is a basis of idempotents $\{\varepsilon_i\}$:

$$\varepsilon_i \varepsilon_j = \delta_{ij} \varepsilon_i . \quad (7.46)$$

Therefore, for a semisimple commutative Frobenius algebra, the only invariants of the Frobenius algebra are the number of idempotents (i.e., the dimension), and the values of the trace applied to the idempotents are $\theta(\varepsilon_i) = \theta_i$.

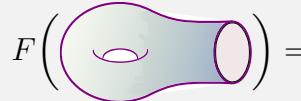
Remark 7.47.

1. If we view this model as a baby model of string theory with a zero-dimensional target “spacetime,” then the target spacetime is a finite union of points, the Frobenius algebra V is the space of functions on spacetime, and θ_i can be interpreted as defining the value of the dilaton/string coupling on the i^{th} component “universe.”
2. We can also use the $n = 1, 2$ theories as topological models of quantum gravity. In this context, they are useful playgrounds for exploring the role of topology change and “baby universes” in Euclidean Quantum Gravity (EQG). An important paper on this is by D. Marolf and H. Maxfield [MM20], with clarifications, generalizations and further extensions in the paper by A. Banerjee and G. W. Moore [BM22], which in turn inspired a general framework for EQG laid out by D. Friedan [Fri23].

 **Exercise 7.6: Frobenius algebra**

Suppose (V, θ) is a semisimple Frobenius algebra and let F be the associated oriented two-dimensional topological field theory.

- (a) Show that the state produced by a handle is



$$F\left(\text{handle}\right) = \sum_i \theta_i^{-1} \varepsilon_i . \quad (7.48)$$

- (b) Suppose Σ_g is a connected genus g surface without boundary. Show that

$$F(\Sigma_g) = \sum_i \theta_i^{1-g} . \quad (7.49)$$

- (c) Deduce that the sum over connected vacuum to vacuum amplitudes, i.e., the sum over connected bordisms $\emptyset \rightarrow \emptyset$ is

$$F_{\text{connected}} = \sum_i \frac{\theta_i}{1 - \theta_i^{-1}} = \sum_i \lambda_i . \quad (7.50)$$

Show that with a suitable weighting of topologies, the full vacuum to vacuum amplitude, that is, the sum over all bordisms $\emptyset \rightarrow \emptyset$, is

$$F_{\text{vac} \rightarrow \text{vac}} = \prod_i e^{\lambda_i}. \quad (7.51)$$

Exercise 7.7: Unit element for algebra multiplication

Show that $F(\text{circle})$ defines the unit element for the algebra multiplication in V by illustrating a suitable change of Morse function.

Exercise 7.8: Associativity of algebra multiplication

Illustrate the change of Morse function that implies the multiplication on V is associative.

Exercise 7.9: Non-semisimple Frobenius algebra

Consider a compact orientable manifold X with all odd Betti numbers $b_i(X) = 0$. Consider the commutative algebra given by the cohomology group $H^\bullet(X; \bullet \mathbb{C})$.

- (a) Show that the trace defined by integral with the fundamental class defines a Frobenius algebra structure, but that it is not semisimple.
- (b) Show that for $X = \mathbb{CP}^1$ we get $\mathbb{C}[x]/(x^2)$ with $\theta(x) = 1$. Compute all the amplitudes in the topological field theory associated to this Frobenius algebra.
- (c) Which bordisms $\emptyset \rightarrow \mathbb{S}^1$ create “states” which are the zero-vector? These give examples where there is not a quantum mechanical interpretation of the “states” created by some topology.

Remark 7.52. The last exercise above brings up an important point. If we follow the traditional mathematical definition of an n -dimensional topological field theory, the vector in $F(N_{n-1})$ created by a bordism with $N_{n-1} = \partial M_n$ might well be the zero vector: $F(M_n) = 0$. As we stressed in [Section 2](#), in traditional quantum mechanics a (pure) state is represented by a rank-one projection operator P . If $\psi \in \text{im } P$ and ψ is nonzero, we can make the projector onto the line through ψ : $P = \frac{|\psi\rangle\langle\psi|}{\|\psi\|^2}$. But this cannot be done if $\psi = 0$. So the term “statespace” for $F(N_{n-1})$ is inaccurate. There are further examples in three-dimensional Chern-Simons theory on M_3 , where $F(M_3) = 0$ for interesting bordisms $\emptyset \rightarrow \mathbb{S}^2$ [FG91]. (The partition function vanishes for certain Lens spaces and one can cut out a 3-dimensional ball from such Lens spaces to produce an example.) There is very interesting literature applying Chern-Simons theory to quantum information theory [SSW16, BFLP16]. We expect that

further research in this direction will likely need to address this phenomenon.

8 Open-Closed Oriented Two-Dimensional TFT And The Emergence Of Categories

If we think of TFT for oriented two-dimensional manifolds as a baby model of topological string theory with zero-dimensional target “spacetime,” it is natural to ask about the extension to open strings. It turns out that this generalization is extremely interesting and introduces several conceptually new ingredients.

At first, it might seem like it is no big deal. For example, the spatial manifolds we consider will simply consist of disjoint unions of both circles and intervals. But the intervals call for boundary conditions. Let us simply label the boundary conditions by symbols a, b, c, \dots in some set \mathcal{B}_0 of boundary conditions.

Now our field theory must assign “open string statespaces” to oriented intervals labeled by a pair of boundary conditions:

$$F\left(\begin{array}{c} b \\ | \\ a \end{array}\right) := \mathcal{O}_{ab} = \left(\begin{array}{l} \text{a vector space of open string states} \\ \text{with boundary conditions } a \text{ and } b \end{array} \right). \quad (8.1)$$

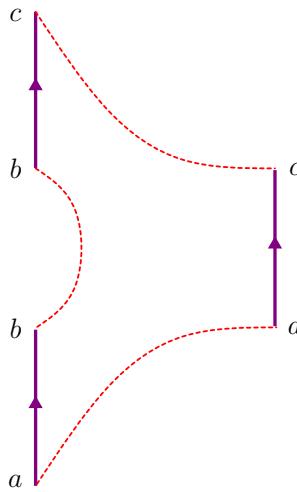


Figure 19: Associativity.

It follows from consideration of the bordism shown in Figure 19, that

1. \mathcal{O}_{aa} is an associative (but not necessarily commutative) *algebra*.

2. \mathcal{O}_{ab} is a *bimodule* for $\mathcal{O}_{aa} \times \mathcal{O}_{bb}$.
3. There is an *associative* multiplication:

$$\mathcal{O}_{ab} \times \mathcal{O}_{bc} \rightarrow \mathcal{O}_{ac} . \quad (8.2)$$

The proof of associativity is given in [Figure 20](#).

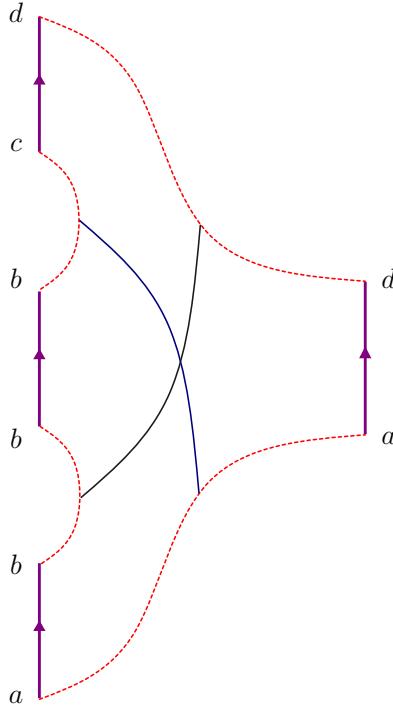


Figure 20: Proof of associativity.

Moreover for any boundary condition a , the half-disk with boundary condition a creates a state $1_{aa} \in \mathcal{O}_{aa}$, which is a unit in the algebra \mathcal{O}_{aa} , see [Figure 21](#).

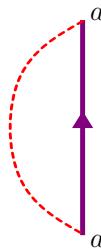


Figure 21: Bordism creating $1_{aa} \in \mathcal{O}_{aa}$.

The structure we have encountered is precisely that of a *category*. The formal definition of a category is given in [Section 9](#). Comparing with the definition given there, we see that for 2d open-closed TFT, there is a category \mathcal{C} of boundary conditions whose set of objects

C_0 is the set of boundary conditions, labeled a, b, c, \dots , and whose hom spaces $\text{Hom}(a, b)$ are the vector spaces of open string states \mathcal{O}_{ab} . The composition in the category is the multiplication provided by the pair of chaps in Figure 22.

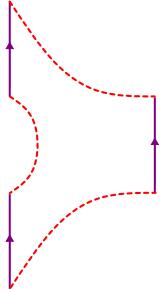


Figure 22: Multiplication.

A natural question arises as to the analog of a Frobenius algebra structure and the sewing theorem (cf. [Theorem 7.4](#)) for the open-closed case. This was provided in [[MS06](#), Sec. 2.2]. The proof of the sewing theorem (in Appendix A.2 of that paper) again makes use of Morse theory, but now for manifolds with boundary. A general formulation of Morse theory for manifolds with boundary has been discussed in [[KM07](#), Sec 2.4], and in [[BNR12](#)].

9 Some Background On Categories

Now that we have seen how naturally categories emerge from TFTs with boundary, it is useful to take a step back and review some basic background mathematical material about categories. Some pedagogical references are [[ML78](#), [GM03](#), [ABC⁺09](#), [Lur09b](#), [Fre12a](#), [Rie17](#), [Fre19](#)].²³

We begin with the formal definition:

Definition 9.1 (Category). A category \mathcal{C} is a collection of data (C_0, C_1, p_0, p_1, m) where²⁴

(a) C_0, C_1 are sets.

C_0 : “the set of objects” (also denoted as $C_0 = \text{Obj}(\mathcal{C})$),

C_1 : “the set of morphisms.”

(b) $C_1 \xrightarrow[p_0]{p_1} C_0$ the codomain and domain maps.

Denote $\{f \in C_1 \mid p_1(f) = y \text{ and } p_0(f) = x\} =: C(x, y) := \text{Hom}_C(x, y)$.

(c) Define $C_2 := C_{1p_1} \times_{p_0} C_1 = \{(f, g) \in C_1 \times C_1 \mid p_0(f) = p_1(g)\}$, the set of composable pairs of morphisms $m : C_2 \rightarrow C_1$. Denote $m(f, g) := f \circ g$ satisfying conditions:

²³Ironically, in the 2023 lectures by G.M. at the TASI school, the definition of a category was first introduced in a lecture on square dancing [[MR23](#)].

²⁴If we restrict C_0 to be a set, it is called a *small category*. That will always be the case in these notes. We will also sometimes write $\text{Obj}(\mathcal{C})$ for C_0 .

(α) $\forall x \in C_0, \exists$ morphism $1_x \in C(x, x)$ such that

$$\begin{aligned} \forall f \in \mathbf{Hom}(y, x), \quad & 1_x \circ f = f, \\ \forall g \in \mathbf{Hom}(x, y), \quad & g \circ 1_x = g. \end{aligned} \tag{9.2}$$

(β) Consider the set of 3 composable morphisms:

$$C_3 = \{(f, g, h) \in C_1 \times C_1 \times C_1 \mid p_0(f) = p_1(g) \text{ and } p_0(g) = p_1(h)\}. \tag{9.3}$$

The diagram

$$\begin{array}{ccc} & C_2 & \\ m \times \mathbf{Id} \nearrow & & \searrow m \\ C_3 & & C_1 \\ \mathbf{Id} \times m \searrow & & \nearrow m \\ & C_2 & \end{array}$$

commutes, i.e.,

$$(f \circ g) \circ h = f \circ (g \circ h). \tag{9.4}$$

Remark 9.5. In [Section 16](#) and [Section 21.2](#), we will encounter the notion of the **opposite category** to a category \mathcal{C} , denoted \mathcal{C}^{op} . The object and morphism sets are the same:

$$C_0^{\text{op}} = C_0, \quad C_1^{\text{op}} = C_1. \tag{9.6}$$

But the source and target maps are reversed:

$$p_0^{\text{op}} = p_1, \quad p_1^{\text{op}} = p_0, \tag{9.7}$$

which has the effect that all arrows are reversed. If (f, g) is a composable pair of morphisms in \mathcal{C} , then (g, f) is a composable pair in \mathcal{C}^{op} and

$$m^{\text{op}}(g, f) := m(f, g). \tag{9.8}$$

In general, it is often useful to think of a category as a directed graph, with

| | | |
|-----------|---|------------------------------|
| Objects | : | Vertices of the graph. |
| Morphisms | : | Oriented edges of the graph. |

For us, a very important category is the **bordism category** $\text{Bord}_{\langle n-1, n \rangle}$ with

| | | |
|------------------|---|---------------------------------------|
| Objects | : | Smooth, closed, $(n - 1)$ -manifolds. |
| Morphisms | : | Bordisms (up to diffeomorphism). |
| Compositions m | : | Gluing of bordisms. |

 **Exercise 9.1: Identity morphism in $\text{Bord}_{\langle n-1, n \rangle}$**

What is the identity morphism in $\text{Bord}_{\langle n-1, n \rangle}$?

Another important category for us is **VECT**, the category of finite-dimensional complex vector spaces:

| | | | | |
|-------------|---|------------------|---|---|
| VECT | : | Objects | : | finite-dimensional \mathbb{C} -vector spaces. |
| | | Morphisms | : | \mathbb{C} -linear transformations between vector spaces. |
| | | Compositions m | : | composition of linear maps. |

Remark 9.9. *Categories and Algebras.* While a directed graph is one way to visualize a category, another helpful way to think about categories is in terms of algebras. Recall that an associative *algebra* A over a field κ is a κ -vector space A equipped with a bilinear pairing $A \times A \rightarrow A$, denoted $(a, b) \mapsto ab$, which is associative: $(ab)c = a(bc)$ for all $a, b, c \in A$, and (if A is *unital*) consists of a unit element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$. An algebra can be viewed as a category with one object: $*$, such that $\text{End}(*) \cong A$, and the composition of morphisms corresponds to multiplication in the algebra. This leads to a natural generalization: A *linear category* (also known as an *algebroid*) is a category whose morphisms are vector spaces (or more generally, modules) and where the composition is bilinear. More precisely, given a field κ , for two objects X, Y in a κ -linear category \mathcal{C} , the set of morphisms $\text{Hom}_{\mathcal{C}}(X, Y)$ is a vector space over κ , and the composition of morphisms is a κ -bilinear map: $\circ : \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$. Every object $X \in \text{Obj}(\mathcal{C})$ has an identity morphism $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$. Two familiar examples are:

1. The category of vector spaces over κ is a κ -linear category – in our notation above, this is VECT_{κ} (above, we took $\kappa = \mathbb{C}$). This is essentially just saying that for vector spaces V_1, V_2 , the set of linear transformations $\text{Hom}(V_1, V_2)$ is itself a vector space such that composition of linear transformations is bilinear.
2. The category of modules over a κ -algebra A is a κ -linear category.

Conversely, any linear category with a finite set of objects may be viewed as an algebra. The endomorphism algebra of each object provides a subalgebra, but the full algebra need not be a direct sum of these subalgebras.

With one more idea from category theory, we can nicely formalize one key aspect of TFT:

Definition 9.10 (Functor). Let \mathcal{C}, \mathcal{D} be two categories. A **functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of maps:

$$\begin{aligned} F_0 : C_0 &\rightarrow D_0 , \\ F_1 : C_1 &\rightarrow D_1 , \end{aligned} \tag{9.11}$$

such that $\forall x, y \in C_0$,

$$F_1 : \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{D}}(F_0(x), F_0(y)) , \tag{9.12}$$

and $\forall f, g \in C_1$,

$$\text{either } F_1(f \circ g) = F_1(f) \circ F_1(g) \quad (\text{covariant functor}) , \tag{9.13}$$

$$\text{or } F_1(f \circ g) = F_1(g) \circ F_1(f) \quad (\text{contravariant functor}) . \tag{9.14}$$

With the concept of functor, we can summarize a lot of what we have said so far about topological field theory by simply saying that an n -dimensional TFT is a functor

$$F : \text{Bord}_{\langle n-1, n \rangle} \rightarrow \text{VECT} . \tag{9.15}$$

The equation

$$F(f \circ g) = F(f) \circ F(g) , \tag{9.16}$$

captures (**LOC 2**). But we have not yet captured everything we said above in the math definition. What about (**LOC 1**)? That is,

$$F(N \coprod N') \cong F(N) \otimes F(N') . \tag{9.17}$$

To incorporate (**LOC 1**) into our math definition of a field theory, we need four more definitions from category theory:

Definition 9.18 (Natural Transformation). Given categories \mathcal{C}, \mathcal{D} and two functors,

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \swarrow \Downarrow \tau \searrow & \mathcal{D} \\ & G & \end{array}$$

a *natural transformation* (a.k.a. “morphism of functors”) denoted

$$\tau : F \implies G , \tag{9.19}$$

is a collection of maps τ_x , indexed by $x \in C_0 = \text{Obj}(\mathcal{C})$, such that, for all $x, y \in C_0$ and all $f \in \text{Hom}_C(x, y)$, there is a commutative diagram:

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \tau_x \downarrow & & \downarrow \tau_y \\ G(x) & \xrightarrow{G(f)} & G(y) \end{array}$$

Example: The k^{th} integral cohomology is a contravariant functor:

$$H_{\mathbb{Z}}^k : \text{TOP} \rightarrow \text{ABGROUP} , \quad (9.20)$$

where **TOP** is the category of topological spaces (as objects) and continuous maps between topological spaces (as morphisms), and **ABGROUP** is the category of Abelian groups (as objects) and group homomorphisms (as morphisms). On morphisms, for $f : X \rightarrow Y$ a continuous map between topological spaces X and Y ,

$$H_{\mathbb{Z}}^k(X \xrightarrow{f} Y) = f^* : H^k(Y; \mathbb{Z}) \rightarrow H^k(X; \mathbb{Z}) . \quad (9.21)$$

Then the cup product is a **natural** transformation between²⁵ $H_{\mathbb{Z}}^{k_1} \otimes H_{\mathbb{Z}}^{k_2}$ and $H_{\mathbb{Z}}^{k_1+k_2}$:

$$H^{k_1}(X; \mathbb{Z}) \otimes H^{k_2}(X; \mathbb{Z}) \rightarrow H^{k_1+k_2}(X; \mathbb{Z}) , \quad (9.22)$$

where $\tau_X =$ cup product. Similarly, Steenrod squares are natural transformations.

HH Exercise 9.2: Natural transformations and VECT

For $V \in \text{Obj}(\text{VECT})$ define a functor $F_V : \text{VECT} \rightarrow \text{VECT}$ by

$$F_V(W) := \text{Hom}(V, W) \oplus V , \quad (9.23)$$

$$F_V(W_1 \xrightarrow{T} W_2) : \text{Hom}(V, W_1) \oplus V \xrightarrow{\circ T \oplus \text{Id}} \text{Hom}(V, W_2) \oplus V . \quad (9.24)$$

²⁵The tensor product $A \otimes B$ of two Abelian groups $A, B \in \text{Obj}(\text{ABGROUP})$ is defined as the Abelian group equipped with a bilinear map $\otimes : A \times B \rightarrow A \otimes B$ satisfying the following universal property: for every Abelian group $C \in \text{Obj}(\text{ABGROUP})$ and every bilinear map $f : A \times B \rightarrow C$, there exists a unique group homomorphism $\tilde{f} : A \otimes B \rightarrow C$ such that $f(a, b) = \tilde{f}(a \otimes b)$, $\forall a \in A, b \in B$. This property characterizes the tensor product up to unique isomorphism. Alternatively, consider the free Abelian group $\mathbb{Z}[A \times B]$ generated by pairs $(a, b) \in A \times B$, and the subgroup $\mathcal{R} \subset \mathbb{Z}[A \times B]$ generated by relations that encode bilinearity: (1) $(a + a', b) \sim (a, b) + (a', b)$, (2) $(a, b + b') \sim (a, b) + (a, b')$, and (3) $(ka, b) \sim k(a, b) \sim (a, kb)$ for all $k \in \mathbb{Z}$, $a, a' \in A$, $b, b' \in B$. Then, the tensor product is the quotient group $A \otimes B := \mathbb{Z}[A \times B]/\mathcal{R}$, a typical element of which is the class $a \otimes b$ of $(a, b) \in A \times B$. More formally, the tensor product of Abelian groups is well-defined because Abelian groups are \mathbb{Z} -modules (see footnote 96).

Show that the evaluation map,

$$\begin{aligned} \tau_W : F_V(W) &\longrightarrow W \\ A \oplus v &\longmapsto A(v) , \end{aligned} \tag{9.25}$$

is a natural transformation of F_V to the identity functor $\text{Id} : \text{VECT} \rightarrow \text{VECT}$.

Definition 9.26 (Isomorphism of functors). An isomorphism of functors $\tau : F_1 \rightarrow F_2$ is a natural transformation τ , such that there is a natural transformation $\tau' : F_2 \rightarrow F_1$ with commutative diagrams:

$$\begin{array}{ccc} & F_2(X) & \\ \nearrow \tau_X & & \searrow \tau'_X \\ F_1(X) & \xrightarrow{\text{Id}_{F_1(X)}} & F_1(X) \end{array} \quad \text{and} \quad \begin{array}{ccc} & F_1(X) & \\ \nearrow \tau_X & & \searrow \tau'_X \\ F_2(X) & \xrightarrow{\text{Id}_{F_2(X)}} & F_2(X) \end{array} \tag{9.27}$$

Definition 9.28 (Equivalence of categories). An equivalence of categories C and D is a pair of functors

$$F : C \rightarrow D , \quad \text{and} \quad G : D \rightarrow C , \tag{9.29}$$

with isomorphisms of $F \circ G$ and $G \circ F$ to the identity functors $\text{Id} : C \rightarrow C$ and $\text{Id} : D \rightarrow D$ respectively.

Many, many, important results in mathematics are statements of equivalence of categories. See, e.g., [Moo23a, p. 556-7], for a list of examples.

Definition 9.30 (Tensor category (a.k.a. monoidal category)). A tensor category is a category with a functor

$$\otimes : C \times C \rightarrow C , \tag{9.31}$$

and an isomorphism of the functors:

$$\begin{array}{ccc} & C \times C & \\ \nearrow \otimes_{12} \times \text{Id} & & \searrow \otimes \\ C \times C \times C & \Downarrow \mathcal{A} & C \\ \searrow \text{Id} \times \otimes_{23} & & \nearrow \otimes \\ & C \times C & \end{array} \tag{9.32}$$

\mathcal{A} is known as the **associator**. It is a morphism

$$\mathcal{A}_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z) , \tag{9.33}$$

and it must satisfy the **pentagon identity**,

$$\begin{array}{ccccc}
 & ((x_1 \otimes x_2) \otimes x_3) \otimes x_4 & \longrightarrow & (x_1 \otimes x_2) \otimes (x_3 \otimes x_4) \\
 & \swarrow & & \searrow & \\
 (x_1 \otimes (x_2 \otimes x_3)) \otimes x_4 & & & & x_1 \otimes (x_2 \otimes (x_3 \otimes x_4)) \\
 & \searrow & & \swarrow & \\
 & x_1 \otimes ((x_2 \otimes x_3) \otimes x_4) & & &
 \end{array} \tag{9.34}$$

Finally, there is an identity object $\mathbb{1}_C \in \text{Obj}(\mathcal{C})$, and natural transformations:

$$\iota_L : \mathbb{1}_C \otimes (\bullet) \longrightarrow \text{Id} , \tag{9.35}$$

$$\iota_R : (\bullet) \otimes \mathbb{1}_C \longrightarrow \text{Id} , \tag{9.36}$$

satisfying some natural compatibility conditions. See [EGNO16] for a complete treatment of the subject.

A *monoidal functor* (a.k.a. *tensor functor*) between monoidal (tensor) categories $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor that preserves structure in the sense that there are isomorphisms

$$F(X \otimes Y) \xrightarrow{\theta_{X,Y}} F(X) \otimes F(Y) , \tag{9.37}$$

$$F(\mathbb{1}_C) \xrightarrow{\eta_C} \mathbb{1}_{\mathcal{D}} , \tag{9.38}$$

satisfying some natural conditions. Again, the reader should consult [EGNO16] for details. The words “monoidal” and “tensor” are synonymous.

Remark 9.39. A **braiding** is an isomorphism of $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ with $\otimes \circ \sigma : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ where $\sigma : (X, Y) \rightarrow (Y, X)$ is the exchange functor. This amounts to the data of isomorphisms:

$$\Omega_{X,Y} : X \otimes Y \longrightarrow Y \otimes X . \tag{9.40}$$

If $\Omega_{Y,X} \circ \Omega_{X,Y} = \text{Id}_{X \otimes Y}$, the category is a **symmetric monoidal category**.

Now, **VECT** is a tensor category, using the tensor product of vector spaces. The associator is trivial, so this is a symmetric tensor category. Less trivially, the category of super-vector spaces is also a symmetric tensor category. Moreover **Bord**_(n-1,n) is a symmetric tensor category where the monoidal structure is defined using the disjoint union.²⁶

²⁶Categories in which there is a braiding which is not necessarily symmetric, known as *braided tensor categories*, play an important role in physical mathematics, particularly in the mathematics of 2+1 dimensional anyons and three-dimensional topological field theories. We will not go into this subject in any detail in these notes, confining ourselves to just the briefest remarks in Section 11.

Hi Exercise 9.3: Monoidal unit

What is the monoidal unit $\mathbb{1}_C$ in VECT and in $\text{Bord}_{\langle n-1, n \rangle}$?

We can now summarize concisely in a simple definition the notion of topological field theory we have been using as follows:

Definition 9.41 ([Topological Field Theory \(TFT\)](#)). An n -dimensional TFT is a symmetric tensor functor

$$F : \text{Bord}_{\langle n-1, n \rangle} \longrightarrow \text{VECT}. \quad (9.42)$$

Remark 9.43.

1. We will eventually generalize this definition to a much more elaborate statement in equation (23.3) below.
2. An important example of topological field theory (in any dimension) is provided by finite group gauge theory. After reviewing some background material on principal G -bundles in [Section 12](#), we begin describing these in [Section 14](#).
3. In a topological field theory F of any dimension $n > 1$, the value $F(\mathbb{S}^{n-1})$ on a sphere is always a commutative algebra. The basic multiplication is obtained simply by considering the bordism

$$\mathbb{S}^{n-1} \coprod \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}, \quad (9.44)$$

given by cutting out two n -disks within a larger n -disk. This observation goes back, at least, to [\[Abr96\]](#). These are always commutative algebras and in closely related contexts, they have more structure and are known as E_n algebras. For a review of E_n algebras, see [\[Lur14\]](#).

4. There is a very useful analogy of this definition of a topological field theory with a group representation. Recall that a group representation of a group G with carrier space V – a vector space – is a homomorphism $\phi : G \rightarrow \text{End}(V)$. The homomorphism property means that ϕ is compatible with group multiplication and composition of linear transformations. For topological field theory, we replace the group G by a bordism category, the carrier space V by a category VECT , and the homomorphism by a functor. Once again the functor is – by definition – compatible with the composition in the bordism category and in the target category.

10 Sums And Products Of Theories

Now that we have a definition of a topological field theory we can describe a notion of sum and products of theories.

Products Of Theories. A familiar notion from Lagrangian field theory is that if one has two sets of fields Φ_1 and Φ_2 with actions $S[\Phi_1]$ and $S[\Phi_2]$, one can define a new theory with fields Φ_1, Φ_2 and exponentiated action:

$$e^{-S[\Phi_1]-S[\Phi_2]} . \quad (10.1)$$

The amplitudes will, rather obviously, be products of the amplitudes from the constituent theories. (Of course, this will not be true if we deform (10.1) by adding interaction terms between Φ_1 and Φ_2 .) This procedure defines a “multiplication” of Lagrangian theories. It is also sometimes referred to as *stacking* theories. The procedure naturally generalizes to a notion of a product of field theories. In the case of topological field theories, F_1 and F_2 , we can define $F_1 \otimes F_2$ by taking the tensor products of the statespaces and amplitudes in the obvious way.

An important consequence of the notion of multiplication is the notion of an *invertible field theory*. We define the trivial field theory F to have statespace $F(N) = \mathbb{C}$ on all spatial manifolds and $F(M) : \mathbb{C} \rightarrow \mathbb{C}$ is the identity map on all bordisms M . (In particular $F(N)$ is a trivial representation of the diffeomorphism group of N .) We say F is *invertible* if there is another field theory G such that $F \otimes G$ is the trivial field theory. Note that G inverts F . An immediate consequence is that for an invertible field theory, $F(N)$ is a one-dimensional vector space on any spatial manifold N , and $F(M)$ is always invertible, hence can be expressed as multiplication of \mathbb{C} by a nonzero complex number, for any bordism M .

A natural source of invertible theories is given by classical actions of Lagrangian field theories. These are not in general topological field theories, so our bordisms will be endowed with field configurations we denote as Φ . We can take the Hilbert space associated to spatial manifolds to be \mathbb{C} . A bordism $M_n : N_{n-1}^0 \rightarrow N_{n-1}^1$ will be endowed with fields Φ , and the amplitude involves multiplication by the exponentiated action $e^{-S[\Phi]}$. Local actions will satisfy the crucial locality condition (**LOC 2**). Nonlocal actions will not. Invertible topological field theories are therefore local topological terms in field theory actions. It was exactly in this context that the notion of invertible field theory was introduced in [FM04]. A rather thorough classification of invertible theories has been described by Freed and Hopkins [FH16].

Although classical actions furnish natural examples of invertible field theories, not all invertible theories are of this type. A simple example is the exponentiated eta invariant of the Dirac operator, which satisfies the locality properties of an invertible theory [DF94]. For further discussion, see [Wit99a, Wit15b, WY19].

Sums of theories. A more subtle notion is the sum of field theories F_1 and F_2 . If N_{n-1} is a connected spatial manifold, we define:

$$(F_1 + F_2)(N_{n-1}) := F_1(N_{n-1}) \oplus F_2(N_{n-1}) . \quad (10.2)$$

Note that the formula for $F_1 + F_2$ on disconnected manifolds will be more complicated. Moreover, if $M_n : N_{n-1}^0 \rightarrow N_{n-1}^1$ is a connected bordism between connected spatial manifolds, then we likewise have

$$(F_1 + F_2)(M_n) := \begin{pmatrix} F_1(M_n) & 0 \\ 0 & F_2(M_n) \end{pmatrix}, \quad (10.3)$$

in the natural block diagonalization provided by (10.2).

Here are some examples of sums of theories:

- Given a nonzero complex number $\theta \in \mathbb{C}^\times$, we can define an invertible 2d orientable topological field theory by the rule that

$$\Sigma \longmapsto F_\theta(\Sigma) = \theta^{\frac{1}{2}\chi(\Sigma)}, \quad (10.4)$$

on closed oriented Riemann surfaces Σ . We can call these *Euler character theories*. It follows from the discussion of [Section 7.5](#) that the general semisimple 2d oriented TFT is a direct sum of Euler character theories.

- A closely related (but nontopological) example is that of 2d Yang-Mills theory described in [Section 27.3](#), where the sum is over the unitary dual of the gauge group.²⁷
- A simple example of a sum of theories is provided by a scalar field sigma model, where the target space is a disjoint union of connected manifolds.

Sums of theories naturally arise when describing long-distance limits, or IR limits of theories. Thus they appear naturally in discussions of spontaneous symmetry breaking. Another example is in long-distance limits of renormalization group (RG) flow as in [[HKMM01](#), Fig. 1]. The concept has also been useful in elucidating the long-distance behavior of 2d adjoint QCD [[KORS20](#)]. Finally, we mention that the notion of sums of theories is closely related to the subject of “decomposition” of theories, especially in the examples of theories admitting a non-faithful action of a (generalized) symmetry group. For a sampling of the literature on this, see [[PS05](#), [HHP⁺06](#), [CDH⁺07](#), [GS18](#), [Sha19](#), [TU19](#), [ES20](#), [KKSS22](#), [Sha22](#), [Sha23e](#), [Sha23g](#), [SS24](#)].

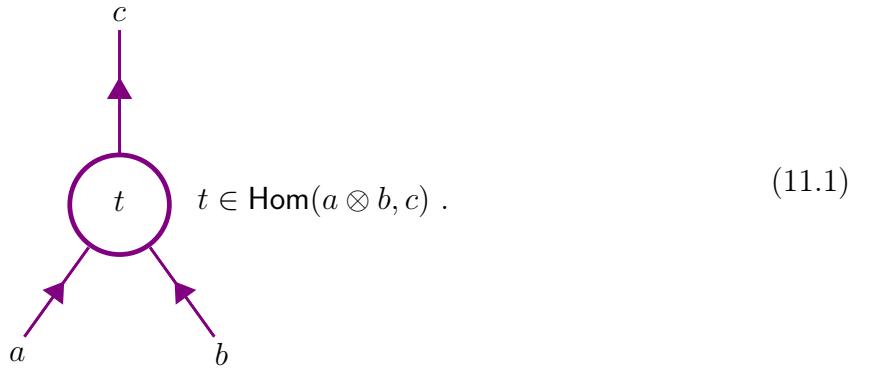
11 Some Brief Remarks On Anyons

The notion of braiding introduced in equation (9.40) turns out to be particularly well suited to the description of *anyons* – particles with exotic statistics that can exist in 2+1 dimensional systems. One common way of describing such particles uses 3d Chern-Simons theory

²⁷The *unitary dual* of a group is the set of unitary irreducible representations of the group.

as a description of the long-distance, low-energy physics of the anyonic excitations. General ideas of low energy effective theory predict that the low-energy physics of a gapped system that breaks parity symmetry should be described by some Chern-Simons theory (which has only one derivative in the action). Chern-Simons theory turns out to be particularly well-suited to fractional quantum Hall systems.²⁸ In this description, anyons can be “fused” and the mathematical embodiment of that is to model anyons as objects in a braided tensor category. In general, for anyons, $\Omega_{y,x} \circ \Omega_{x,y}$ is not the identity.

The “fusion of the anyons” can be pictured as:



The associator is:

$$= \sum_{t_3, t_4} F_{t_1 t_2}^{t_3 t_4} \quad (11.2)$$

Exercise 11.1: Pentagon diagram

Write out the pentagon diagram using the above notation.

The physics of anyons is believed to be described by a special kind of braided tensor category, known as a *modular tensor category*. The connection of modular tensor categories to anyons in gapped materials is that at long distances, the dominant term in an effective action describing the system with anyons should be a Chern-Simons term, but 3d Chern-Simons

²⁸For some reviews, see [BF10, Ton16a, Ton16b, Sim23, CMR⁺25].

theory is “holographically dual” to 2d rational conformal field theory, and the monodromy behavior of 2d rational conformal field theories is governed by a modular tensor category [MS89a, MS89b]. This was the argument originally used in [MR91].

We comment briefly on a (rough) definition of a modular tensor category (MTC). An MTC is a semisimple braided tensor category equipped with an extra piece of data known as a “ribbon structure” (related to the spin of the anyons) and a compatible notion of duals (related to the CPT conjugate of the anyon) allowing one to define a trace on the endomorphism of any object in the category. The ribbon structure and the “ S -matrix,”

$$S_{ij} := \text{Tr}(\Omega_{X_i, X_j} \circ \Omega_{X_j, X_i}), \quad (11.3)$$

is invertible.²⁹ For more discussion of modular tensor categories, see [Wal91, BK00, Row05, Kit05, Sti08, Row16, RW17].

Modular tensor categories played a crucial role in combinatorial constructions of the invariants of 3-manifolds predicted by Chern-Simons-Witten theory [RT90, RT91] and indeed the three-dimensional topological field theory associated with a modular tensor category is often referred to as *Reshetikhin-Turaev* theory, or *Reshetikhin-Turaev-Witten* theory.

There are numerous reviews and books on this topic. A very useful one is [KZ22]. See also the recent book [Sim23]. For some historical remarks about the role of (modular) tensor categories in anyon physics, see [Moo24a, Moo24b].

12 Some Background On Principal Bundles

For a group G , a *G -torsor* or a *principal homogeneous space* is a set T with a free and transitive G -action on T . The points of G can be put into one-to-one correspondence with those of T by picking a point $t_0 \in T$ and identifying the point $g \cdot t_0$ with $g \in G$, but to make this correspondence, one needs to choose a point $t_0 \in T$, so the one-to-one correspondence is not natural.

Example 1: A good example is the set of points $T = \theta + \mathbb{Z} \subset \mathbb{R}$, where θ is a non-integer real number. The group $G = \mathbb{Z}$ acts by translations on the set T , but there is no natural basepoint in T .

Example 2: Another good example is affine space \mathbb{A}^n . This is a space with a transitive action of \mathbb{R}^n without fixed points. Thus, if $p \in \mathbb{A}^n$ and $v \in \mathbb{R}^n$, then there is a notion of translating p by v , so $p \mapsto p + v$, and given any two points $p_1, p_2 \in \mathbb{A}^n$, there is a unique

²⁹In the original definition, S_{ij} and a matrix T derived from the ribbon structure were constrained to satisfy the relations of the modular group $\text{SL}(2, \mathbb{Z})$. Later, it was realized [Tur92, Tur16] that one only need require the invertibility of the “ S -matrix,” and then the relations of the modular group would follow. This same observation was made by I. Frenkel in collaboration with G.M., and announced by G.M. at a conference in Princeton in November 1989, but was not published.

vector $v \in \mathbb{R}^n$ such that $p_2 = p_1 + v$, but there is no natural origin. In the case of \mathbb{A}^2 , think of the map of the world flattened to a plane. There is no natural choice of capital city which one could call “the origin.”

For G a topological group and X a topological space, a principal G -bundle over X is a continuous map of topological spaces $\pi : P \rightarrow X$ such that

1. P admits a continuous and free right G -action such that $\pi(p \cdot g) = \pi(g)$ and the fibers $\pi^{-1}(x)$ are G -torsors.
2. $\pi : P \rightarrow X$ is locally trivial: for all $x \in X$, there exists $\mathcal{U}_x \subset X$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(\mathcal{U}_x) & \xrightarrow{\phi_{\mathcal{U}_x}} & \mathcal{U}_x \times G \\ & \searrow \pi & \swarrow \\ & \mathcal{U}_x & \end{array} \quad (12.1)$$

commutes, and $\phi_{\mathcal{U}_x}$ is G -equivariant.

A key example for us is the following: choose $g_0 \in G$ and let \mathbb{Z} act on $\mathbb{R} \times G$ by $n : (x, g) \mapsto (x + n, g_0^n g)$. Taking equivalence classes of \mathbb{Z} -orbits on $\mathbb{R} \times G$ defines a space we can call P_{g_0} . Similarly, taking equivalence classes of the \mathbb{Z} -action on \mathbb{R} gives the quotient space \mathbb{S}^1 , and we have a continuous map:

$$\begin{aligned} P_{g_0} := (\mathbb{R} \times G) / \mathbb{Z} &\xrightarrow{\pi} \mathbb{R} / \mathbb{Z} = \mathbb{S}^1 \\ [(x, g)] &\longmapsto [x]. \end{aligned} \quad (12.2)$$

An intuitive way of thinking of P_{g_0} is that we are considering $[0, 1] \times G$ and gluing the “boundary” $\{0\} \times G$ to $\{1\} \times G$ by left multiplication by g_0 , as shown in [Figure 23](#).

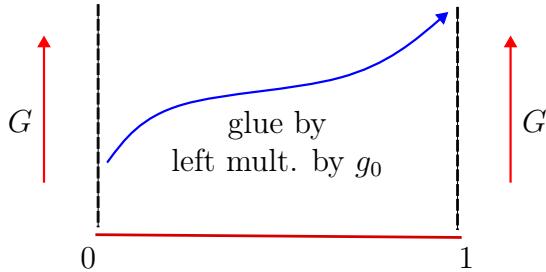


Figure 23: Constructing a bundle over \mathbb{S}^1 .

Denote this principal G -bundle over \mathbb{S}^1 by $\pi : P_{g_0} \rightarrow \mathbb{S}^1$, or just P_{g_0} for short.

Definition 12.3 (Bundle Map). A bundle map, or morphism of principal G -bundles over

X is a fiber-preserving G -equivariant map:

$$\begin{array}{ccc} P_1 & \xrightarrow{\psi} & P_2 \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & X & \end{array} \quad (12.4)$$

Exercise 12.1: Principal G -bundles over \mathbb{S}^1

Show that the bundle map

$$\begin{array}{ccc} \mathbb{R} \times G & \xrightarrow{\psi_h} & \mathbb{R} \times G \\ & \searrow & \swarrow \\ & \mathbb{R} & \end{array} \quad (12.5)$$

given by $\psi_h : (x, g) \mapsto (x, hg)$ induces an isomorphism of bundles over the circle $\psi_h : P_{g_0} \cong P_{hg_0h^{-1}}$. Therefore, isomorphism classes of principal G -bundles over \mathbb{S}^1 are labeled by *conjugacy classes* of elements of G . The automorphism group of P_g is $Z(g) = \{h \in G \mid hgh^{-1} = g\}$. Moreover, show that the automorphism group of P_{g_0} (that is the group of bundle isomorphisms of P_{g_0} with itself) is just:

$$\text{Aut}(P_{g_0}) \cong Z(g_0), \quad (12.6)$$

where $Z(g_0)$ is the centralizer of g_0 within G .

Remark 12.7.

1. Note that a G -equivariant map $\psi : X \rightarrow Y$ between two G torsors must be invertible: Equivariance implies that $\psi(xg) = \psi(x)g$, so if we choose a basepoint $x_0 \in X$ so that every element of X is expressible as x_0g for a unique group element g , then we can use $\psi(x_0)$ as a basepoint for Y . Then $\psi^{-1}(\psi(x_0)g) = x_0g$ defines the inverse map. When upgraded to families, we learn that every bundle map of principal G -bundles has an inverse bundle map, so it defines an isomorphism of bundles.
2. In general, let G be a finite group. Isomorphism classes of principal G -bundles over a topological space X are in one-to-one correspondence with elements of

$$\text{Hom}(\pi_1(X, x_0), G)/G, \quad (12.8)$$

where the quotient identifies $\phi \sim \phi'$ if there exists $g \in G$ such that $\phi(\gamma) = g\phi'(\gamma)g^{-1}$ for all $\gamma \in \pi_1(X, x_0)$. Note that setting $X = \mathbb{S}^1$, we recover the claim that isomorphism classes are in one-to-one correspondence with conjugacy classes of G .

13 Groupoids

A special class of categories known as *groupoids* is extremely useful when working with gauge theories, and will be used quite frequently in the material below. We begin with the formal definition:

Definition 13.1 (Groupoid). A groupoid is a category $\mathcal{C} = (C_0, C_1, p_0, p_1, m)$ in which every morphism (i.e., every element of C_1) is invertible (an isomorphism).

Here are some key examples:

1. A group G can be viewed as an example of a category. The category has one object, call it $*$. The morphisms $* \rightarrow *$ are in one-to-one correspondence with the group elements $g \in G$, that is, $C_1 = G$. The composition of morphisms $m : C_1 \times C_1 \rightarrow C_1$ is group multiplication. The identity and associativity axioms are part of the definition of a category. The fact that group elements have inverses implies that this category is in fact a groupoid.
2. The previous example can be generalized as follows: Consider a set X with a (left) G -action. We can define the category denoted as $X//G$ whose objects are elements $x \in X$, so $C_0(X//G) = X$, and whose morphisms are pairs $(x, g) \in X \times G$. The morphism (x, g) is considered as a morphism $(x, g) : x \mapsto g \cdot x$. It is a good exercise to draw a picture of this category for a finite group acting on a finite set and to write the general group law for composition of morphisms. The previous example is the case of a totally ineffective action of a group G on a single point: $\text{pt}/\!/G$. This category turns out to be important in the theory of principal G -bundles. The origin of the name “groupoid” comes from this generalization of the notion of group. Categories presented in the form $X//G$ are sometimes called *action groupoids* or *transformation groupoids*.
3. Given G and a topological space X , we can form the category $\mathbf{Bun}_G(X)$ of principal G -bundles over X . The objects are principal G -bundles and the morphisms are bundle maps of principal G -bundles. It follows from Remark 1 above that this category is a groupoid. It is sometimes called the *stack of principal G -bundles over X* .³⁰
4. Let X be a topological space. We can define a groupoid known as the *fundamental groupoid* of X . The objects are the points $x \in X$, and the morphisms are continuous paths $\gamma : [0, 1] \rightarrow X$ considered up to homotopy with fixed endpoints. The source and target maps are $p_0(\gamma) = \gamma(0)$ and $p_1(\gamma) = \gamma(1)$. It is necessary to consider the paths

³⁰A stack is an equivalence class of groupoids. Equivalently, a groupoid is a presentation of a stack. For example, a smooth manifold is a (smooth) stack, and an open covering by coordinate charts furnishes a presentation as a groupoid. See [FHT07, App. A] and [Hop99] for details.

up to homotopy (with fixed endpoints) in order for the strict associativity condition to hold.

Exercise 13.1: Groupoids and automorphism groups

1. Show that in a groupoid \mathcal{G} , the homsets $\text{Hom}(x, x)$ are groups for every object $x \in C_0(\mathcal{G})$. This is called the *automorphism group of the object x* .
2. Show that if $\mathcal{G} = X//G$, the category determined by the group action of G on a set X , then the automorphism group of an object in $X//G$ can be identified with the stabilizer group of a point of X .

14 Finite Group Gauge Theory

Let G be a finite group. In general, the functor defining finite group gauge theory for gauge group G is defined by “quantizing” in some way the groupoid of principal G -bundles over X , denoted $\text{Bun}_G(X)$, and introduced in [Section 13](#). Here X is a smooth manifold, of a dimension that depends on the physical quantity under consideration. The fact that $\text{Bun}_G(X)$ is a groupoid is very important because some principal G -bundles have nontrivial automorphisms, and we should “quotient” by that group of automorphisms in a gauge theory.

We can illustrate the general outlook of the previous paragraph by first writing the formula for the partition function. This will be a sum over the set of isomorphism classes of principal G -bundles. When G is a finite group and X is a compact space, the set of isomorphism classes in $\text{Bun}_G(X)$ has finite cardinality. We denote the set of isomorphism classes by $[\text{Bun}_G(M_n)]$ (sometimes we will also denote it by $\pi_0(\text{Bun}_G(M_n))$ below). For each isomorphism class, choose a representative principal G -bundle P , and let $\text{Aut}(P)$ denote the automorphism group of P . Isomorphic bundles P will have isomorphic groups $\text{Aut}(P)$, so the order $|\text{Aut}(P)|$ only depends on the isomorphism class of P . With these notations in place, the partition function of the theory on a compact n -dimensional manifold without boundary, M_n is:

$$F(M_n) := \sum_{[\text{Bun}_G(M_n)]} \frac{1}{|\text{Aut}(P)|}. \quad (14.1)$$

Similarly, we can also write down the statespaces on closed $(n-1)$ -dimensional manifolds

N_{n-1} as a linear space obtained by “quantizing” $\text{Bun}_G(N_{n-1})$. The explicit formula is:³¹

$$F(N_{n-1}) := \text{Fun}([\text{Bun}_G(N_{n-1})] \rightarrow \mathbb{C}) . \quad (14.2)$$

The discussion of how to compute amplitudes in this theory – which is quite necessary to define the functor and check the gluing properties – is best deferred to after we introduce BG below.

Remark 14.3. A useful observation is that for a finite group G , there is a unique connection on P and the isomorphism class of the bundle is completely determined by the holonomy. (See [Section 36](#) below for discussions of connections and holonomy.) So, for X a connected space, we can choose any basepoint $x_0 \in X$, and there is a 1–1 correspondence:

$$[\text{Bun}_G(X)] \cong \text{Hom}(\pi_1(X, x_0), G)/G , \quad (14.4)$$

where the quotient by G is conjugation action on the homomorphism. That is, if $\varphi : \pi_1(X, x_0) \rightarrow G$ is a group homomorphism and $g \in G$, then $(g \cdot \varphi)([\gamma]) := g\varphi([\gamma])g^{-1}$, for all homotopy classes $[\gamma]$ of closed loops γ based at x_0 .

Example 1: Let us consider G -gauge theory for $n = 1$. According to the above, the partition function on the circle is a sum over isomorphism classes of G -bundles over \mathbb{S}^1 . We have just seen that the isomorphism classes are in 1–1 correspondence with conjugacy classes. The Boltzmann weight $B(P_{g_0})$ of a bundle P_{g_0} should therefore be a class function of g_0 , and can be written as

$$B(P_g) = \frac{\chi_\rho(g)}{|Z(g)|} , \quad (14.5)$$

where χ_ρ is the character for some element ρ in the representation ring of G , and $Z(g)$ is the centralizer of g .

The orthogonality relations for characters imply that the sum in the partition function projects to the identity isotypical component of ρ . Therefore, without loss of generality, we can take ρ to be some multiple of the trivial representation. Taking that multiple to be one,

³¹We will use the notation $\text{Fun}(X \rightarrow Y)$ for the vector space of continuous functions from $X \rightarrow Y$, when Y is a topological vector space. Usually, we are taking $Y = \mathbb{C}$, and sometimes, we will simply write $\text{Fun}(X)$ for the vector space of continuous complex-valued functions on X . We use the notation $\text{Map}(X \rightarrow Y)$ for the topological space of continuous maps between topological spaces X and Y with compact-open topology. The compact-open topology on the space of continuous maps $\text{Map}(X \rightarrow Y)$ between two topological spaces X, Y can be defined as the topology derived from a basis of open sets $\mathcal{O}(K, U)$ labeled by a compact set $K \subset X$ and an open set $U \subset Y$. The formula for these open sets is $\mathcal{O}(K, U) = \{f : X \rightarrow Y \mid f(K) \subset U\}$. It is the topology such that a sequence of functions converges uniformly on compact sets. It is also the topology so that $\text{Map}(X \times Y \rightarrow Z)$ is homeomorphic to $\text{Map}(X \rightarrow \text{Map}(Y \rightarrow Z))$ for all topological spaces X, Y, Z .

for simplicity we have $\chi_\rho(g) = 1$. This recovers the definition in (14.1). Note that we can rewrite the sum as:

$$F_{\text{ampl}}^{1d}(\mathbb{S}^1) = \sum_{\text{C.C.}} \frac{1}{|Z(g)|} \quad (14.6)$$

$$= \sum_{P_g} \frac{1}{|G|} = 1 . \quad (14.7)$$

In the second line, we are summing over all bundles P_{g_0} and dividing by the “volume of the gauge group.” Since the result is simply 1, we expect that $F_{\text{statespace}}^{1d}(\text{pt}) = \mathbb{C}$. This too, is in accord with (14.2), since there is a unique isomorphism class of principal G -bundles over a point. Therefore, the statespace is the vector space of functions on a point, and is hence a copy of \mathbb{C} .

Example 2: For now, we just note that for the two-dimensional case since $\mathsf{Bun}_G(\mathbb{S}^1)$ is in 1–1 correspondence with the conjugacy classes on G , we can identify $F_{\text{statespace}}^{2d}(\mathbb{S}^1)$ with the space of class functions on G .

Using the formulation of Section 24 below, we will see that the composition of Figure 24,

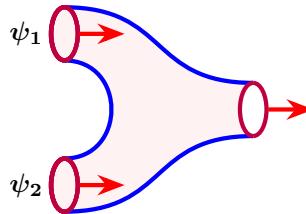


Figure 24: Composition $m(\psi_1, \psi_2)$.

works out to be (by equation (24.36) below) the convolution product

$$m(\psi_1, \psi_2)(g) = \frac{1}{|G|} \sum_{g_1 g_2 = g} \psi_1(g_1) \psi_2(g_2) . \quad (14.8)$$

A natural basis of the vector space of class functions is given by the characters χ_μ in the irreps $\mu \in \text{Irrep}(G)$. The orthogonality relations for characters of irreps of G , denoted χ^μ (where μ labels the distinct irreps), is

$$\frac{1}{|G|} \sum_{h \in G} \chi^\mu(h) \chi^\nu(h^{-1}g) = \delta_{\mu\nu} \frac{\chi^\nu(g)}{\chi^\nu(1)} . \quad (14.9)$$

It follows that a basis of idempotents of the Frobenius algebra is $\varepsilon_\mu = \chi_\mu(1)\chi_\mu$.

The Frobenius trace must be proportional to the evaluation of the character function at

$g = 1$:

$$\theta(\psi) = \lambda \cdot \psi(1) . \quad (14.10)$$

Any $\lambda \in \mathbb{C}^\times$ defines a Frobenius structure. (The definition (24.36) below gives $\lambda = 1/|G|$.) Applying equation (7.49) from Exercise 7.6, we obtain the partition function of the finite group gauge theory on a genus g Riemann surface:

$$F(\Sigma_g) = \lambda^{1-g} \sum_{\substack{\text{irreps } \mu \\ \text{of } G}} (\dim V_\mu)^{2-2g} , \quad (14.11)$$

it can be shown that this is

$$(\lambda|G|)^{1-g} \frac{\#\text{Hom}(\pi_1(\Sigma_g, x_0), G)}{|G|} , \quad (14.12)$$

so it is indeed a sum of the Boltzmann weights $1/|G|$ over isomorphism classes of G -bundles, up to an overall factor $(\lambda|G|)^{1-g}$. We can attribute this factor to multiplication by an invertible TFT, namely, the Euler theory introduced in (10.4) above. Again, with the definition (24.36), this overall factor is simply 1.

15 Classifying Spaces

We will now present another perspective on the finite group gauge theory discussed above.

For any topological group G , there is a topological space (only defined up to homotopy equivalence) denoted BG with the property that there is a natural one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{homotopy classes} \\ \text{of maps} \\ f : M \rightarrow BG \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{isomorphism classes} \\ \text{of principal } G\text{-bundles} \\ P \rightarrow M \end{array} \right\} . \quad (15.1)$$

One way to construct BG is to find a G -space (i.e., a space with a G action on it) which is a “CW complex,”³² such that it

³²At several points in these notes when making technical comments, we refer to the notion of a CW complex [Whi49, Sec. 5]. Informally, a CW complex is a generalization of both a simplicial complex and of a manifold. The definition states – roughly – that it is a space that can be obtained from a finite set of points by successively attaching open balls, or cells, (of various dimensions) by continuous maps. The “C” stands for “closure finite,” and the “W” stands for “weak topology.”

- (a) is weakly contractible,³³ and,
- (b) admits a free G -action.

For any topological group G , such a space exists, although it is not unique. It is, however, unique up to homotopy equivalence, and any such space is often denoted EG . Given EG , since the G action (say, on the right) is free, we can define $BG := EG/G$, so that there is a principal G -bundle, known as the *universal bundle* $\pi : EG \rightarrow BG$, where π is the quotient map.³⁴ The correspondence above in one direction is then simply the pullback map $f \mapsto f^*(EG)$, which defines a principal G -bundle over the domain of f . Homotopic maps f define isomorphic bundles over M .

Example 1: Take $G = \mathbb{Z}$. Then we can take $EG = \mathbb{R}$ with \mathbb{Z} acting by translations. This clearly satisfies properties (a) and (b), so BG is any topological space homotopy equivalent to \mathbb{S}^1 . Similarly, if $G = \mathbb{Z}^n$, then we can model BG with an n -dimensional torus. This example is misleadingly simple, as we see by contemplating the next case.

Example 2: Consider $G = \mathbb{Z}_2$. The simplest model for EG is the unit sphere in an infinite-dimensional real Hilbert space, and $B\mathbb{Z}_2 \cong \mathbb{RP}^\infty$. A heuristic argument for that is the following: There is a natural free action of \mathbb{Z}_2 on the k -dimensional sphere. Of course, \mathbb{S}^k is not contractible. However note that the homotopy groups $\pi_j(\mathbb{S}^k, *) = 0$ for $j < k$. (We say that \mathbb{S}^k is $(k-1)$ -connected.) “So,” the “limit” $\lim_{k \rightarrow \infty} \mathbb{S}^k$ should, in fact, have vanishing homotopy groups and be connected. In fact, this turns out to be rigorously true if we interpret the limit $\lim_{k \rightarrow \infty} \mathbb{S}^k$ as the unit sphere in an infinite-dimensional countable Hilbert space.

Example 2 above shows that, given a topological group G , it is not completely trivial to construct a space EG or BG , so it is good to have a systematic construction. Such a construction exists and is due to Milnor [Mil56a, Mil56b]. The construction begins by identifying a group G as a category with one object, all of whose morphisms are invertible:

$$C_0 = \{\bullet\} , \quad C_1 = \text{Hom}(\bullet, \bullet) = G . \quad (15.2)$$

³³A space X is *weakly contractible* if all its homotopy groups are trivial, i.e., $\pi_i(X) = \{1\}$ for all i . This is equivalent to the statement that the canonical map $X \rightarrow \text{pt}$ is a weak homotopy equivalence. The space X is *contractible* if the map $X \rightarrow \text{pt}$ is a homotopy equivalence. This technical distinction is important in situations where the Whitehead theorem (introduced in footnote 50) may not apply. By the Whitehead theorem, a weakly contractible CW complex is contractible. Two examples of spaces that are weakly contractible but not contractible are the so-called long line, and the double comb space. Both these spaces do not have the homotopy type of a CW complex. For discussions of such examples, see for example, [SS78].

³⁴Note that, if a space X satisfies properties (a) and (b), then so does any finite product of X with itself. This is in part the reason that classifying spaces are only defined up to homotopy equivalence.

The composition of arrows in C_1 is defined by group multiplication:

$$g_1 \cdot g_2 = g_1 g_2 . \quad (15.3)$$

Now, assume G is a finite group. We construct a CW complex:

$$0\text{-skeleton} = * , \quad (15.4)$$

$$1\text{-skeleton} = G . \quad (15.5)$$

Next, the aim is to attach disks of higher dimension so that all the higher homotopy groups vanish, and $\pi_1(BG, *) \cong G$, but $\pi_j(BG) = 0$ for $j > 1$.

We view C_2 as defining triangles which we wish to fill in:

$$\left(\begin{array}{l} \text{arrow } \xrightarrow{g} \text{ with reversed orientation} \\ \text{is } \xleftarrow{g^{-1}} \end{array} \right) \quad (15.6)$$

It is important that the three orange dots above are meant to represent the same object in our category. They have been written as distinct points on the page for visual purposes. If we do not fill in these triangles then π_1 would be more complicated than the group G .

Now we have built an oriented simplicial complex with a single 0-vertex, 1-simplices corresponding to group elements, and 2-simplices corresponding to arbitrary pairs of group elements. This complex will have a nontrivial second homotopy group π_2 , and will not be homotopic to BG . We can start with this complex and add 3-simplices to produce a space with $\pi_2 = 0$. To do that, we take triples of group elements and consider the 3-simplex they define, see [Figure 25](#). Filling that in will define a new complex with $\pi_2 = 0$, but will have nontrivial π_3 . So we keep going. Some readers will wish to have a more systematic description of the construction we have just described. We tackle that in [Section 16](#).

Remark 15.7. Given the existence of BG , we can view finite G gauge theory as a homotopy-theoretic version of a nonlinear σ model with target space BG . The definitions (14.1) and

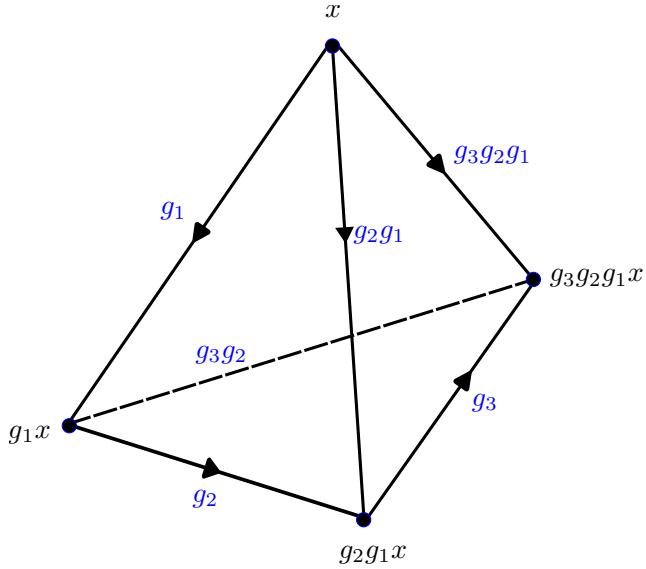


Figure 25: An oriented 3-simplex determined by g_1, g_2, g_3 . Edges are labeled in blue color.

(14.2) above can now be written as:

$$F(M_n) = \sum_{\pi_0(\text{Map}(M_n \rightarrow BG))} \frac{1}{|\text{Aut } \phi|}, \quad (15.8)$$

$$F(N_{n-1}) = \text{Fun}(\pi_0(\text{Map}(N_{n-1} \rightarrow BG))). \quad (15.9)$$

where in the sum in (15.8), ϕ is any representative ‘‘sigma model map $\phi : M_n \rightarrow BG$ ’’ taken from each component.

This is part of a larger story of the theory of homotopy sigma models that we will sketch in [Section 24](#) below.

16 Simplicial Sets And The Construction of BG

In this section, we give a more systematic description of the construction of BG outlined in the previous section. As a bonus, we will learn about *simplicial sets*. Simplicial sets play an important role in algebraic topology and are needed to incorporate the higher-form fields and differential cohomology in the general definition of background fields. Our treatment will be very brief. For more details, see the Wikipedia article [[Wik24](#)], the nLab article [[nLa25b](#)], and [[GM03](#), [Friar](#), [Rie11](#), [Por21](#)]. For a very thorough presentation, see [[Lur25](#)].

We begin by reviewing the standard n -simplex, identified as a subset of \mathbb{R}^{n+1} :

$$\Delta_n = \{(t_0, \dots, t_n) \mid t_i \geq 0 \text{ and } \sum_{i=0}^n t_i = 1\} \subset \mathbb{R}^{n+1}. \quad (16.1)$$

There is an ordered set of $n + 1$ vertices v_i given by putting $t_i = 1$ and all other coordinates to zero. For each $j = 0, \dots, n$, there is the j^{th} face map:

$$\delta_{n,j} : \Delta_{n-1} \rightarrow \Delta_n . \quad (16.2)$$

Note that in Δ_n , the set of points with $t_j = 0$ is a copy of an $(n - 1)$ -simplex. It is the simplex “opposite” to the j^{th} vertex, and we call it the j^{th} face. The map $\delta_{n,j}$ maps Δ_{n-1} to the j^{th} face. In equations, we have:

$$\delta_{n,j} : (t_0, \dots, t_{n-1}) \mapsto \begin{cases} (0, t_0, \dots, t_{n-1}) , & j = 0 , \\ (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}) , & 0 < j < n , \\ (t_0, \dots, t_{n-1}, 0) , & j = n . \end{cases} \quad (16.3)$$

On the other hand, for $0 \leq j \leq n$ we also have the degeneracy maps:

$$\sigma_{n,j} : \Delta_{n+1} \rightarrow \Delta_n , \quad (16.4)$$

given by “composing” the j^{th} and $(j + 1)^{\text{th}}$ coordinates:

$$\sigma_{n,j} : (t_0, \dots, t_{n+1}) \mapsto \begin{cases} (t_0, \dots, t_j + t_{j+1}, t_{j+2}, \dots, t_{n+1}) , & 0 \leq j < n , \\ (t_0, \dots, t_{n-1}, t_n + t_{n+1}) , & j = n . \end{cases} \quad (16.5)$$

It is now straightforward, if tedious, to check the following relations on the maps $\delta_{n,j}$ and $\sigma_{n,j}$:

$$\begin{aligned} \delta_{n+1,j} \circ \delta_{n,i} &= \delta_{n+1,i} \circ \delta_{n,j-1} , & i < j , \\ \sigma_{n,j} \circ \sigma_{n+1,i} &= \sigma_{n,i} \circ \sigma_{n+1,j+1} , & i \leq j , \\ \sigma_{n,j} \circ \delta_{n+1,i} &= \begin{cases} \delta_{n,i} \circ \sigma_{n-1,j-1} , & i < j , \\ \text{Id} , & i = j, j + 1 , \\ \delta_{n,i-1} \circ \sigma_{n-1,j} , & i > j + 1 . \end{cases} \end{aligned} \quad (16.6)$$

Next, in order to motivate the general definition of a simplicial set, let us recall how one begins the definition of singular homology of a topological space M . The set of singular n -simplices $\mathcal{S}(M)_n$ is the set of all continuous maps $\phi : \Delta_n \rightarrow M$. In singular homology theory, one defines $C_n(M)$ to be the free Abelian group generated by $\mathcal{S}(M)_n$, and the space of singular cochains is $C^n(M) := \text{Hom}(C_n(M), \mathbb{Z})$. These will play some role in [Part II](#).

Note that precomposition of ϕ with $\delta_{n,j}$ defines maps:

$$d_{n,j} : \mathcal{S}(M)_n \rightarrow \mathcal{S}(M)_{n-1} , \quad 0 \leq j \leq n . \quad (16.7)$$

That is, for $\phi \in \mathcal{S}(M)_n$, we have $d_{n,i}(\phi) := \phi \circ \delta_{n,i}$.³⁵ Similarly, precomposition with $\sigma_{n,j}$ defines maps,

$$s_{n,i} : \mathcal{S}(M)_n \rightarrow \mathcal{S}(M)_{n+1} , \quad 0 \leq j \leq n . \quad (16.8)$$

The relations (16.6) above on $\delta_{n,j}$ and $\sigma_{n,j}$ imply the relations,

$$\begin{aligned} d_i d_j &= d_{j-1} d_i , \quad i < j , \\ s_i s_j &= s_{j+1} s_i , \quad i \leq j , \\ d_i s_j &= \begin{cases} s_{j-1} d_i , & i < j , \\ \text{Id} , & i = j, j+1 , \\ s_j d_{i-1} , & i > j+1 , \end{cases} \end{aligned} \quad (16.9)$$

where we have intentionally streamlined the notation and dropped the subscript n .

The mathematical structure provided by the sets of simplices in a topological space turns out to be a useful construction in other contexts and so we have the definition:

Definition 16.10 (Simplicial set). A *simplicial set* \mathcal{X} consists of the following data:

1. A collection of sets \mathcal{X}_n , labeled by nonnegative integers n . The sets \mathcal{X}_n are known as the “set of n -simplices in the simplicial set.”
2. A collection of maps, known as face maps:

$$d_{n,j} : \mathcal{X}_n \rightarrow \mathcal{X}_{n-1} , \quad 0 \leq j \leq n . \quad (16.11)$$

3. A collection of maps, known as degeneracy maps:

$$s_{n,j} : \mathcal{X}_n \rightarrow \mathcal{X}_{n+1} , \quad 0 \leq j \leq n , \quad (16.12)$$

such that the maps satisfy the relations of equation (16.9).

Remark 16.13.

1. A simplicial set is not a set with special properties. It is the collection of data above satisfying the degeneracy relations. Nevertheless, an ordinary set can be considered

³⁵Note that if we extend $d_{n,i}$ to $C_n(M)$ by linearity, then the boundary map $\partial : C_n(M) \rightarrow C_{n-1}(M)$ is just $\partial = \sum_j (-1)^j d_{n,j}$. Then, $\partial^2 = 0$, and the homology of the resulting complex is the n^{th} singular homology of M .

as a special case of a simplicial set. Suppose S is a set. We can give it the *discrete topology* where every subset is open. Then $\mathcal{X}(S)_n$ is naturally identified with S itself, since any continuous map from an n -simplex to S must map to a single element of S . Note that all the degeneracy and face maps are simply the identity map on S .

2. There is a much more elegant categorical definition of a simplicial set. To motivate it, let us introduce the ordered set $[n]$ of integers from 0 to n :

$$[n] := \{0, 1, \dots, n\} . \quad (16.14)$$

Suppose we have an ordered set $S \subset [n]$ of k integers between 0 and n , where $k \leq n$. Then there is a sub-simplex $\Delta_S \subset \Delta_{[n]}$, where all coordinates with $j \notin S$ are equal to zero. Thus, the j^{th} vertex is $\Delta_{\{j\}}$, and $\Delta_{[n]} = \Delta_n$. The image of $\delta_{n,j}(\Delta_{[n-1]})$ in $\Delta_{[n]}$ is the sub-simplex $\Delta_{S_{n,j}}$, where $S_{n,j}$ is the image of the “coface map” $\delta^{n,j} : [n-1] \rightarrow [n]$ that preserves order and “misses” j . More is true: If $T \subset [n-1]$, then

$$\delta_{n,j}(\Delta_T) = \Delta_{\delta^{n,j}(T)} \subset \Delta_{[n]} . \quad (16.15)$$

Similarly, we can define the “co-degeneracy map”:

$$\sigma^{n,j} : [n+1] \rightarrow [n] , \quad (16.16)$$

that preserves order, and maps j and $j+1$ in $[n+1]$ to j in $[n]$, for $0 \leq j \leq n$. Analogous to (16.15), we have

$$\sigma_{n,j}(\Delta_T) = \Delta_{\sigma^{n,j}(T)} \subset \Delta_{[n]} . \quad (16.17)$$

These maps satisfy the relations (16.6). Now it turns out that any order-preserving map $[n] \rightarrow [m]$ can be written as a composition of the coface and codegeneracy maps $\delta^{\ell,j}$ and $\sigma^{\ell,j}$ for various ℓ 's and j 's. Therefore, we can define a category Δ whose objects are the sets $[n]$ and whose morphisms are order-preserving maps. We can now define a simplicial set to be a functor $\mathcal{X} : \Delta^{\text{op}} \rightarrow \text{SET}$. (Recall the definition of the opposite category in equation (9.6) et. seq.) Thus, $\mathcal{X}([n]) = \mathcal{X}_n$, and

$$\mathcal{X}(\delta^{n,j}) = d_{n,j} : \mathcal{X}_n \rightarrow \mathcal{X}_{n-1} , \quad (16.18)$$

and so on. It is now possible to define a category of simplicial sets sSET , where the objects are simplicial sets and the morphisms are natural transformations $\tau : \mathcal{X} \Rightarrow \mathcal{X}'$. Concretely, a morphism is a set of maps $\tau_n : \mathcal{X}_n \rightarrow \mathcal{X}'_n$ commuting with the face and degeneracy maps. The category sSET will be useful in [Section 21](#).

3. Given a topological space M , our motivating example of the spaces of n -simplices

$\mathcal{S}(M)_n$ form a simplicial set $\mathcal{S}(M)$ known as the *singular set of M* .

Example 1: Let G be a group. We form the simplicial set $\mathcal{X}(G)$ where $\mathcal{X}(G)_n := G^n$, is the Cartesian product of n copies of G , and the set is often written as:

$$\bullet \leftarrow G \xleftarrow{\begin{matrix} d^0 \\ d^1 \\ d^2 \end{matrix}} G \times G \xleftarrow{\quad} G \times G \times G \xleftarrow{\quad} \dots \quad (16.19)$$

The face maps are:

$$\begin{aligned} d_0(g_1, \dots, g_n) &= (g_2, \dots, g_n) , \\ d_1(g_1, \dots, g_n) &= (g_1g_2, g_3, \dots, g_n) , \\ d_j(g_1, \dots, g_n) &= (g_1, \dots, g_{j-1}, g_jg_{j+1}, g_{j+2}, \dots, g_n) , \quad 1 < j < n-1 , \\ d_{n-1}(g_1, \dots, g_n) &= (g_1, g_2, \dots, g_{n-2}, g_{n-1}g_n) , \\ d_n(g_1, \dots, g_n) &= (g_1, \dots, g_{n-1}) , \end{aligned} \quad (16.20)$$

and the degeneracy maps are:

$$s_j(g_1, \dots, g_n) = (g_1, \dots, g_j, 1_G, g_{j+1}, \dots, g_n) . \quad (16.21)$$

Example 2: Given a category \mathcal{C} , we can form the *nerve of the category*, $N(\mathcal{C})$ as the simplicial set whose set of n -simplices is the set of strings of n composable morphisms. More precisely, $N(\mathcal{C})_0$ is the set of objects of \mathcal{C} , $N(\mathcal{C})_1$ is the set of morphisms of \mathcal{C} , and for $n > 1$, we let $N(\mathcal{C})_n$ to be the set of collections of n morphisms which can be concatenated to make a morphism. The degeneracy maps $s_j : N(\mathcal{C})_n \rightarrow N(\mathcal{C})_{n-1}$ insert the identity morphism in the j^{th} slot, while the face maps $d_j : N(\mathcal{C})_n \rightarrow N(\mathcal{C})_{n-1}$ are defined by:

$$d_j(f_1, \dots, f_n) = \begin{cases} (f_2, \dots, f_n) , & j = 0 , \\ (f_1, \dots, f_j \circ f_{j+1}, \dots, f_n) , & 0 < j < n , \\ (f_1, \dots, f_{n-1}) , & j = n . \end{cases} \quad (16.22)$$

Notice that if we regard a group G as a category with one object, that is, as the groupoid $\text{pt}/\!/G$, then $N(\text{pt}/\!/G)$ is just the simplicial set $\mathcal{X}(G)$ constructed in the previous example.

Now, to any simplicial set \mathcal{X} , we can associate the *geometric realization*. This is defined as a quotient of the disjoint union of $\mathcal{X}_n \times \Delta_n$ over all $n \geq 0$ modulo relations for the face and degeneracy maps:

$$|\mathcal{X}| = \left(\coprod_{n=0}^{\infty} \mathcal{X}_n \times \Delta_n \right) / \sim , \quad (16.23)$$

where the relations are given by:

$$(x, \delta_{n+1,j}(t)) \sim (d_{n+1,j}(x), t) , \quad (16.24)$$

for all $x \in \mathcal{X}_{n+1}$, $t \in \Delta_n$ and $0 \leq j \leq n+1$, together with

$$(x, \sigma_{n+1,j}(t)) \sim (s_{n+1,j}(x), t) , \quad (16.25)$$

for all $x \in \mathcal{X}_{n+1}$, $t \in \Delta_{n+1}$ and $0 \leq j \leq n+1$. We make this a topological space by taking the discrete topology on \mathcal{X}_n , the standard topology on Δ_n , and then using the quotient topology.

The construction of the geometric realization raises an obvious question: Given a topological space Y , consider the simplicial set given by the singular set $\mathcal{S}(Y)$. The singular set has a geometric realization $|\mathcal{S}(Y)|$ producing another topological space. How are these topological spaces related? Note that there is a canonical map:

$$|\mathcal{S}(Y)| \rightarrow Y , \quad (16.26)$$

given by evaluation: $(\phi, t) \mapsto \phi(t)$, where ϕ is a map $\phi : \Delta_n \rightarrow Y$ and $t \in \Delta_n$. (The map respects the equivalence relation in (16.23).) The references above show that the map $|\mathcal{S}(Y)| \rightarrow Y$ induces an isomorphism of homotopy groups, and when Y is a CW complex (see footnote 32), the map is a homotopy equivalence. *Thus, in homotopy theory, we may pass freely between simplicial sets to topological spaces.*

Finally, given a group G , we form the simplicial set $\mathcal{X}(G)$ and take its geometric realization to form a concrete model for BG :

$$BG = |\mathcal{X}(G)| . \quad (16.27)$$

Note that if G is a topological group, $\mathcal{X}(G)$ is not the same simplicial set as $\mathcal{S}(G)$.

17 Group Cohomology

The cohomology of BG defines the group cohomology. Concretely,

$$C^n(G; A) = \{\phi : G^n \rightarrow A\} , \quad (17.1)$$

where A is Abelian. The differential

$$\delta : C^n(G; A) \longrightarrow C^{n+1}(G; A) , \quad (17.2)$$

is given by

$$\begin{aligned} (\delta\phi)(g_1, \dots, g_{n+1}) &= \phi(g_2, \dots, g_{n+1}) - \phi(g_1g_2, g_3, \dots, g_{n+1}) + \phi(g_1, g_2g_3, \dots, g_{n+1}) \\ &\quad \pm \dots + (-1)^{n+1}\phi(g_1, \dots, g_n) . \end{aligned} \quad (17.3)$$

Exercise 17.1: Differential in group cohomology

- (a) Check $\delta^2 = 0$.
- (b) Write out $\delta\phi = 0$ for the first few cases.

Definition 17.4 (Group Cohomology). $H^n(G; A) := \ker \delta / \text{im } \delta$.

18 Digression: Projective Representations In Quantum Mechanics

One of the most important group cohomologies in physics is $H^2(G; \text{U}(1))$, which classifies isomorphism classes of central extensions. It plays an important role in the discussion of symmetries in quantum systems.

Recall the Born rule summarized in [Section 2](#), in particular equation [\(2.19\)](#). These are the physical probability distributions. They must be preserved by a “symmetry.” An *automorphism* of a quantum system is a bijective map of states and observables preserving the Born rule. One can show that it is determined by a bijective correspondence on *pure states* preserving the overlap function on pure states:

$$\mathcal{O}(P_1, P_2) := \text{Tr}(P_1 P_2) . \quad (18.1)$$

It is called the overlap function because if we represent the pure states by vectors ψ_1 and ψ_2 , so that

$$P_i = \frac{|\psi_i\rangle\langle\psi_i|}{\|\psi_i\|^2} , \quad (18.2)$$

then

$$\mathcal{O}(P_1, P_2) = \frac{|\langle\psi_1|\psi_2\rangle|^2}{\|\psi_1\|^2\|\psi_2\|^2} . \quad (18.3)$$

As discussed in [Section 2](#), the pure states form a *projective* Hilbert space and the overlap function is related to the Fubini-Study metric on projective space:

$$\theta(P_1, P_2) = \cos^2 \frac{d_{\text{FS}}(P_1, P_2)}{2} . \quad (18.4)$$

So the automorphism group of a quantum system is the group of isometries of complex projective space. Wigner's Theorem relates this to (anti-)linear operators on \mathcal{H} .

Let $\text{Aut}(\mathcal{H})$ be the group of unitary and antiunitary operators on \mathcal{H} . We define

$$\begin{aligned}\pi : \text{Aut}(\mathcal{H}) &\longrightarrow \text{Aut(QM)} \\ g &\longmapsto \pi(g) : P \mapsto gPg^{-1}.\end{aligned}\tag{18.5}$$

Wigner's theorem asserts that π is surjective and the kernel is the group $\text{U}(1)$ acting as scalars on \mathcal{H} .

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{U}(1) & \longrightarrow & \text{Aut}(\mathcal{H}) & \xrightarrow{\pi} & \text{Aut(QM)} \longrightarrow 1 \\ & & & & \uparrow \varphi & & \\ & & & & G & & \end{array}\tag{18.6}$$

Dynamical evolution is described in terms of a 1-parameter group of automorphisms α_t on states and observables. For example, given a Hamiltonian, $\alpha_t(\rho) = e^{-itH}\rho e^{itH}$. Only a subgroup of $\text{Aut}(\text{QM})$ will commute with the flows on the states and observables. Thus, we have a homomorphism $\varphi : G \rightarrow \text{Aut}(\text{QM})$. By Wigner's theorem, $\forall g \in G$, we can choose a $U(g) \in \text{Aut}(\mathcal{H})$, such that

$$\pi(U(g)) = \varphi(g).\tag{18.7}$$

Because π has a kernel, the choice is not unique. Two choices $U(g)$ and $\tilde{U}(g)$ are related by a phase: $\tilde{U}(g) = b(g)U(g)$, where $b(g) \in \text{U}(1)$. Now, since $\varphi(g_1)\varphi(g_2) = \varphi(g_1g_2)$, and since π is a group homomorphism, we deduce:

$$\pi(U(g_1)U(g_2)) = \pi(U(g_1g_2)).\tag{18.8}$$

Equation (18.8) does not imply that $U(g_1)U(g_2)$ is the same as $U(g_1g_2)$, again because π has a kernel. All we can conclude is that:

$$U(g_1)U(g_2) = c(g_1, g_2)U(g_1g_2),\tag{18.9}$$

for some function $c : G \times G \rightarrow \text{U}(1)$. Note that if we make a different choice $U(g) \rightarrow \tilde{U}(g)$, then the function c changes accordingly.

Now assume (for simplicity) that the $U(g)$ are \mathbb{C} -linear for $g \in G$. Then, $c \in Z^2(G; \text{U}(1)) := \ker(\delta) \cap C^2(G; \text{U}(1))$.

Exercise 18.1: Cocycle in group cohomology

Show that $c \in Z^2(G; \text{U}(1))$.

The pullback group, which is, as a set:

$$\tilde{G} = \mathrm{U}(1) \times G , \quad (18.10)$$

has the group law:

$$(z_1, g_1) \cdot (z_2, g_2) := (z_1 z_2 c(g_1, g_2), g_1 g_2) . \quad (18.11)$$

The group \tilde{G} is the group which is linearly represented on \mathcal{H} :

$$T((z, g)) = z U(g) . \quad (18.12)$$

A good example is a spin-1/2 qubit where the $\mathrm{SO}(3)$ isometry of \mathbb{CP}^1 is represented on the Hilbert space \mathbb{C}^2 by the central extension

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{SU}(2) \longrightarrow \mathrm{SO}(3) \longrightarrow 1 . \quad (18.13)$$

Exercise 18.2: \mathbb{C} -antilinear symmetries

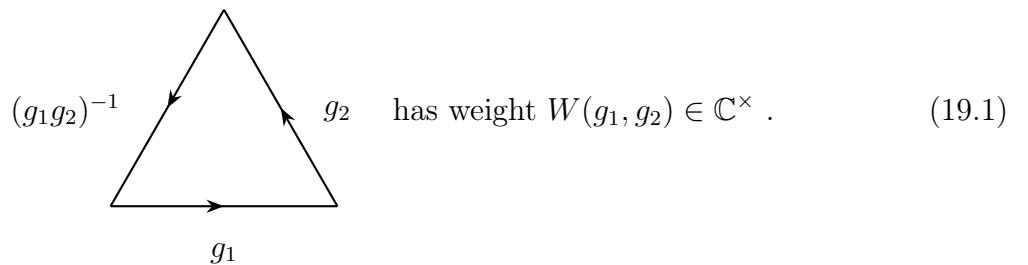
Generalize the above discussion to include symmetries which are \mathbb{C} -antilinear.

19 Dijkgraaf-Witten Theory

Dijkgraaf and Witten [DW90] gave a “lattice gauge theory model” of topological finite G gauge theory, and an important generalization thereof based on group cohomology.

Let us just describe it for 2 dimensions, and we just explain how to compute the partition function. The construction generalizes to “fully extended” (see below) n -dimensional theories.

Let M_2 be a compact oriented surface. Choose a triangulation Δ on M_2 . Require that the gauge field be flat, so that the plaquette Boltzmann weight is only determined by two group elements, as in (19.1).



Some choices must be made for this to be unambiguous.

$$F(M_2) = \sum_{\text{Map(Edges} \rightarrow G)} \prod_{\Delta} W(g_1, g_2) . \quad (19.2)$$

Now demand invariance under triangulation:

$$\begin{array}{ccc} \text{Diagram showing two configurations of a square with edges labeled } g_1, g_2, g_3. & & (19.3) \\ \text{Left: } g_1g_2g_3 \quad \text{Right: } g_1g_2g_3 \\ \text{Left: } g_1 \quad \text{Right: } g_1 \end{array}$$

which implies that $W(g_1, g_2)$ is a group cocycle. Then one checks, using the cocycle identity, that we have invariance under refinement of the triangulation:

$$\begin{array}{ccc} \text{Diagram showing two configurations of a triangle with edges labeled } g_1, g_2. & & (19.4) \\ \text{Left: } g_1g_2 \quad \text{Right: } g_1g_2 \\ \text{Left: } g_1 \quad \text{Right: } g_1 \end{array}$$

It is a mathematical theorem [Pac91] that all triangulations can be obtained by these two moves. So $F(M_2)$ is independent of triangulation. Moreover, $F(M_2)$ depends only on the group cohomology class determined by $W(g_1, g_2)$.

In n -dimensions, we use a simplicial decomposition and the Boltzmann weights are an n -cocycle over G valued in \mathbb{C}^\times . In particular, in 3 dimensions, the theory is determined by an element of $H^3(G; \mathrm{U}(1))$. Using techniques discussed in [Section 39.1](#) below, one can prove that for a finite group one has

$$H^3(G; \mathrm{U}(1)) \cong H^4(G; \mathbb{Z}) . \quad (19.5)$$

Now, 3d Chern-Simons-Witten theory can be formulated for any compact gauge group G . Quite generally, the Chern-Simons action is determined by a “level” which is an element of $k \in H^4(BG; \mathbb{Z})$. It follows that $n = 3$ Dijkgraaf-Witten theory is simply Chern-Simons-Witten theory for a finite gauge group. This was, in fact, the original motivation of Dijkgraaf and Witten [DW90].

20 Higher Categories

Now we want to start describing *extended* TFT. The idea is to take locality to its logical limit. Some physics students will be resistant to the need to extend to higher categories, so we give five motivations for doing so.

The first way proceeds by considering the gluing formula relating partition functions to pairings of vectors in statespaces. The basic idea is summarized by the picture:

$$F\left(\text{Manifold with boundary}\right) = \left\langle F\left(\text{Manifold with boundary}\right), F\left(\text{Manifold with boundary}\right) \right\rangle. \quad (20.1)$$

In this picture, the closed manifold on the left-hand side is n -dimensional in an n -dimensional quantum field theory. We could ask if the same picture also applies when the closed manifold on the left-hand-side is $(n-1)$ -dimensional. So, suppose the compact manifold N_{n-1} without boundary can likewise be glued together from pieces:

$$N_{n-1} = N_{n-1}^0 \bigcup_{\Sigma_{n-2}} N_{n-1}^1. \quad (20.2)$$

Here N_{n-1}^i are compact manifolds with boundary Σ_{n-2} , and we are gluing them along Σ_{n-2} to produce the compact manifold without boundary N_{n-1} . Now, we know that $F(N_{n-1}) = F_{\text{statespace}}(N_{n-1})$ is the space of states on the closed spatial manifold N_{n-1} . It is natural to expect that there should be mathematical objects $F(N_{n-1}^i)$ and $F(\Sigma_{n-2})$, which can be “glued” to produce the vector space $F(N_{n-1})$.

A second way of motivating higher categories comes from considering 2d open/closed string theory: We have already seen that intervals are morphisms – which we now call “1-morphisms” – in a bordism category, and the field theory F associates to these “1-morphisms,” open string state spaces $\mathcal{O}_{ab} = \text{Hom}_{\mathfrak{B}}(a, b)$, which are “1-morphisms” in a category \mathfrak{B} of boundary conditions. It is natural to ask if we can view bordisms such as that shown in Figure 26, as “2-morphisms” between the intervals (“1-morphisms”) in an extension of the bordism category, and whether those are represented by the field theory as some kind of “2-morphisms” in a target category.

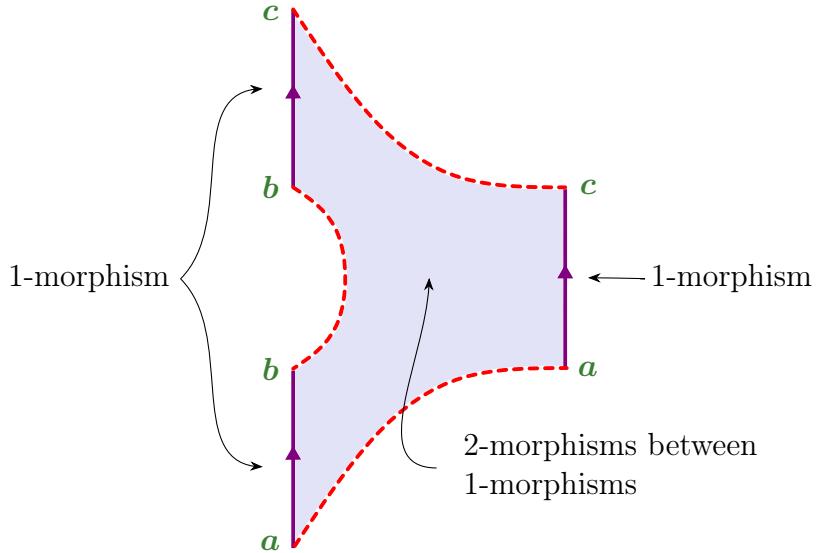


Figure 26: Higher categories from 2d open/closed string theory.

A third way of motivating these ideas comes from thinking of defects within defects, see [Figure 27](#). See the ICM 2010 lecture by Kapustin [[Kap10](#)] for a discussion of this. Briefly, in the n -dimensional spacetime of [Figure 27](#), there are theories in the regions to the left and right of the codimension 1 (i.e., $(n - 1)$ -dimensional) domain wall in the middle. We can consider the boundary conditions A and B as domain walls between a theory and the empty or trivial theory. The domain walls are labeled by P, Q, \dots and can be thought of as morphisms between a category of theories. But there can be codimension 2 (i.e., $(n - 2)$ -dimensional) domain walls within domain walls separating P and Q as shown in the picture. These “2-morphisms” will carry labels α, β, \dots . Then there can be codimension 3 domain walls between α and β . And so on.

For a concrete realization of the above idea in the context of supersymmetric 2d Landau-Ginzburg theory, see [[GMW15a](#), Sec. 3.3.3].

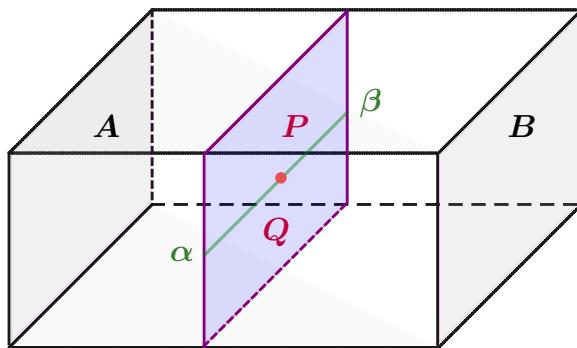


Figure 27: Defects within defects.

A fourth way comes from thinking about the proper categorical interpretation of the fundamental group: Let X be a topological space. We form a category whose objects are the points of X . The morphisms $x_1 \rightarrow x_2$ are paths in X , $\gamma : x_1 \rightarrow x_2$ considered up to homotopy with fixed endpoints: $\text{Hom}(x_1, x_2) = \mathcal{P}(x_1, x_2)/\text{homotopy}$. Then $\pi_1(X, x_0) = \text{Hom}(x_0, x_0) := \text{Aut}(x_0)$. This (important) category is called the *fundamental groupoid* $\pi_{\leq 1}(X)$. But we could make a more elaborate object if we decline to consider paths only up to homotopy. We could consider $\pi_{\leq 2}(X)$ where 1-morphisms between objects $(x_1 \rightarrow x_2) \in \mathcal{P}(x_1, x_2)$ and “2-morphisms” are homotopies of paths. And so on, up to $\pi_\infty(X)$.

A fifth way originates from Morse theory. Let us revisit topology change induced by a saddle, such as the one shown in [Figure 17](#) with a single critical point of Morse index 1. Focusing on the neighborhood of this critical point, we see that it resembles [Figure 28](#).

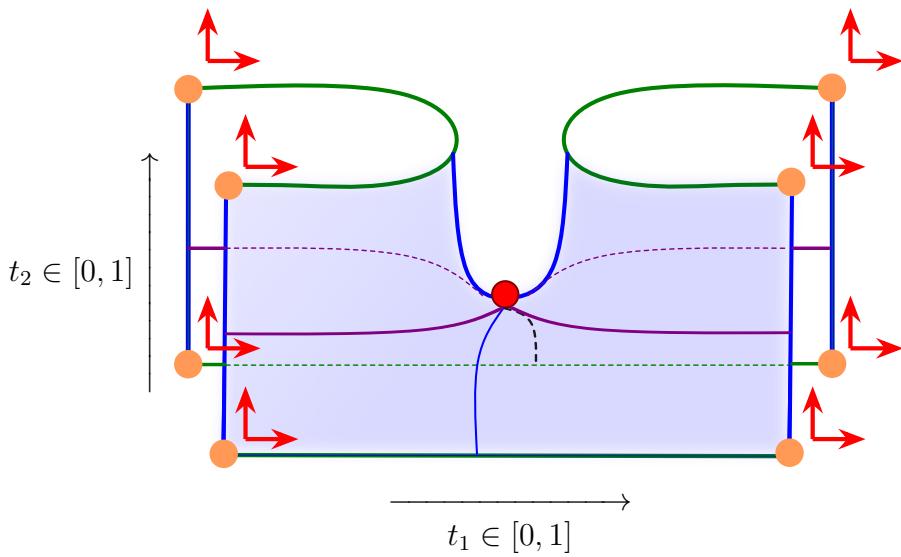


Figure 28: Topology induced by a saddle.

We can start to see the structure of a higher category here: The objects (now called “0-morphisms”) are the orange points. At time $t_2 = 0$, we have a bordism from two points to two points colored in green. It is the evolution along t_1 . Recall that the bordism is a 1-morphism in the category $\text{Bord}_{\langle 0,1 \rangle}$. At time $t_2 = 1$, we have another bordism between the same pair of objects we had at $t_2 = 0$, but the green bordism at $t_2 = 0$ is a different 1-morphism in $\text{Bord}_{\langle 0,1 \rangle}$ from that at $t_2 = 1$. The saddle itself is a 2-morphism between these two 1-morphisms. The saddle in this figure is an example of a 2-morphism in a 2-category one could denote by $\text{Bord}_{\leq 2}$. Note that this example shows that in extending the bordism category to a higher category, we must make use of manifolds with corners. For a foundational discussion of the theory of manifolds with corners, see [[Joy09](#), [Joy15](#)].

In general, if we think of categories in terms of directed graphs, when we add 2-morphisms, we introduce a new kind of arrow, see [Figure 29](#).

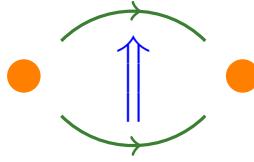


Figure 29: 2-morphisms.

These can be composed in several different ways, for example, two ways of composing 2-morphisms can be illustrated as shown in Figure 30.

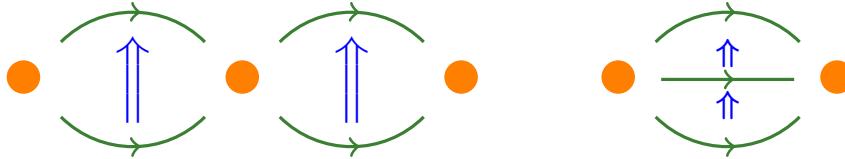


Figure 30: Two ways of composing 2-morphisms.

Given the above heuristic ideas, one can give technically precise definitions of higher categories. The simplest one uses an inductive procedure:

Definition 20.3 (Strict n -category). A **strict n -category** \mathcal{C} is a collection of objects $C_0(\mathcal{C})$ (the “0-morphisms”) and for each pair of objects $x, y \in C_0(\mathcal{C})$, an $(n-1)$ -category $\text{Hom}(x, y)$, such that, for all $x, y, z \in C_0(\mathcal{C})$, there exist (strict) $(n-1)$ -bifunctors:

$$c_{x,y,z} : \text{Hom}(x, y) \times \text{Hom}(y, z) \rightarrow \text{Hom}(x, z) , \quad (20.4)$$

which are (strictly) associative.³⁶

If we understand a “0-category” to be a set, then a 1-category as defined in Section 9 satisfies (9.4), and is hence a strict 1-category. We can then proceed to define n -categories inductively using the above definition once we agree that a *strict functor* between n -categories \mathcal{C} and \mathcal{D} is a map of objects $\phi_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$, such that for all $x, y \in \mathcal{C}_0$,

$$\phi_1 : \text{Hom}(x, y) \rightarrow \text{Hom}(\phi_0(x), \phi_0(y)) , \quad (20.5)$$

which is a (strict) $(n-1)$ -functor.

Remark 20.6.

1. The inductive definition thus provides a collection of sets $C_0(\mathcal{C}), \dots, C_n(\mathcal{C})$, where $C_k(\mathcal{C})$ is the set of k -morphisms in \mathcal{C} , for $0 \leq k \leq n$. There are source and target maps:

$$p_0, p_1 : C_k(\mathcal{C}) \rightarrow C_{k-1}(\mathcal{C}) , \quad (20.7)$$

³⁶A “bi-functor” means that $c_{x,y,z}$ is an $(n-1)$ -functor if we hold all the k -morphisms in $\text{Hom}(x, y)$ fixed and consider it as an $(n-1)$ -functor $\text{Hom}(y, z) \rightarrow \text{Hom}(x, z)$, or if we hold all the k -morphisms in $\text{Hom}(y, z)$ fixed and consider an $(n-1)$ -functor $\text{Hom}(x, y) \rightarrow \text{Hom}(x, z)$.

and many kinds of compositions. If we choose

$$\begin{aligned}
 x_0, y_0 &\in C_0(\mathcal{C}) , \\
 x_1, y_1 &\in C_0(\mathsf{Hom}(x_0, y_0)) , \\
 x_2, y_2 &\in C_0(\mathsf{Hom}(x_1, y_1)) , \\
 &\vdots \\
 x_{n-1}, y_{n-1} &\in C_0(\mathsf{Hom}(x_{n-2}, y_{n-2})) ,
 \end{aligned} \tag{20.8}$$

then $\mathsf{Hom}(x_k, y_k)$ is an $(n - k - 1)$ -category and the ℓ -morphisms in this category are such that

$$\begin{aligned}
 C_\ell(\mathsf{Hom}(x_k, y_k)) &\subset C_{\ell+1}(\mathsf{Hom}(x_{k-1}, y_{k-1})) \subset C_{\ell+2}(\mathsf{Hom}(x_{k-2}, y_{k-2})) \subset \cdots \\
 &\cdots \subset C_{\ell+k}(\mathsf{Hom}(x_0, y_0)) \subset C_{\ell+k+1}(\mathcal{C}) .
 \end{aligned} \tag{20.9}$$

2. The adjective “strict” used above refers to the associativity of the compositions. It means that all the many associativity relations between k -morphisms hold as equalities. This notion turns out to have limited utility. One should relax the strict associativity conditions to allow for suitable equivalences between different orders of compositions (much like one allows for an associator in a tensor category). When this is done, much new data and many new conditions are introduced. The subject of higher categories can get quite technical. For a sampling of the literature, see the works of Barwick et al. [BDG⁺16], Bergner [Ber06], Lurie’s kerodon.net [Lur25] and [Lur09b], Schommer-Pries [SPar], Rezk [Rez00], Riehl et al. [Rie14, Rie17, RV22] and references in the surveys by Freed [Fre12b, Fre19], and Teleman [Tel16].
3. It makes sense to take $n = \infty$ in the definition of an n -category. There is then a useful refinement of the notion of a higher category to a (p, q) category:

All k -morphisms with $k > p$ are the identity.

All k -morphisms with $k > q$ are invertible.

In this notation, a $(0, 0)$ category is a set. A $(1, 1)$ category is a category in the normal sense. One commonly encounters the term “ (∞, n) -category”. So it is an ∞ -category where all k -morphisms for $k > n$ are invertible. A good example is $\pi_\infty(X)$, which is an $(\infty, 0)$ -category. An important notion when discussing the cobordism hypothesis below is the idea of truncating to invertible morphisms. If \mathcal{C} is an (∞, n) -category and $k < n$ then $\mathcal{G}_k(\mathcal{C})$ is the (∞, k) category where we simply delete the noninvertible ℓ -morphisms with $\ell > k$.

Example 1: A good example of a 2-category is the category $\mathcal{C} = \mathbf{ALG}(\mathbf{VECT})$. This is the

prime example of what is sometimes called a *Morita 2-category*. The notation is meant to indicate that we are considering the category of “algebra objects” in the category VECT . In this case, an “algebra object” is just a finite-dimensional complex algebra, so the set $C_0(\text{ALG}(\text{VECT}))$ of “0-morphisms” is the set of finite-dimensional complex algebras. Given two such algebras A_1, A_2 , we have a 1-category $\text{Hom}(A_1, A_2)$. The objects of $\text{Hom}(A_1, A_2)$ are $A_1 - A_2$ bimodules.³⁷ The objects of $\text{Hom}(A_1, A_2)$ are the 1-morphisms of $\text{ALG}(\text{VECT})$. The composition of these 1-morphisms is the composition of bimodules:

$$\text{Hom}(A_1, A_2) \times \text{Hom}(A_2, A_3) \rightarrow \text{Hom}(A_1, A_3) , \quad (20.10)$$

defined by $\mathbf{M}_1 \circ \mathbf{M}_2 := \mathbf{M}_1 \otimes_{A_2} \mathbf{M}_2$. The 1-morphisms of the 1-category $\text{Hom}(A_1, A_2)$ are bimodule maps. Indeed, given two $A_1 - A_2$ bimodules, $\mathbf{M}_1, \mathbf{M}_2$

$$\text{Hom}_{\text{Hom}(A_1, A_2)}(\mathbf{M}_1, \mathbf{M}_2) , \quad (20.11)$$

is simply the linear space of $A_1 - A_2$ bimodule maps $T : \mathbf{M}_1 \rightarrow \mathbf{M}_2$. This means that T is a linear map which commutes with the left and right A_1, A_2 actions, respectively. These bimodule maps are 2-morphisms in the 2-category $\text{ALG}(\text{VECT})$. It is a good exercise to work out the two different compositions of 2-morphisms shown in Figure 30.

Example 2: A second useful example of a 2-category is CAT . The set of 0-morphisms is the set of (small) \mathbb{C} -linear categories. Given two such categories C_1 and C_2 , the homset between C_1 and C_2 is the set of functors from C_1 to C_2 . So functors are 1-morphisms. Given two functors $F_{1,2} : C_1 \rightarrow C_2$, we have a set $\text{Hom}(F_1, F_2)$. It is the set of natural transformations $\tau : F_1 \Rightarrow F_2$. So the 2-morphisms in CAT are natural transformations.

Example 3: *The extended bordism category.* By generalizing the discussion of the saddle shown in Figure 28 above, we can define an n -category $\text{Bord}_{\leq n}$ taking the k -morphisms to be bordisms (which are manifolds with corners) with k -times. This can even be generalized to an (∞, n) -category by taking $(n + 1)$ -morphisms to be diffeomorphisms of bordisms preserving all the initial and final k -bordisms. Note that these are invertible, whereas the k -morphisms with $k \leq n$ are in general not invertible. The $(n + 2)$ -morphisms are isotopies of diffeomorphisms, etc.

Monoidal Structure: The notion of a monoidal (tensor) category can be extended to n -categories. For the 2-category $\text{ALG}(\text{VECT})$, the monoidal structure is the usual \otimes product of algebras, bimodules, and linear maps. For $\text{Bord}_{\leq n}$, the monoidal structure is disjoint

³⁷An $A_1 - A_2$ bimodule is a vector space \mathbf{M} which is a left A_1 -module and a right A_2 -module where the actions of the two algebras commute. In other words it is a left $A_1 \times A_2^{\text{op}}$ -module.

union. In general, if done honestly, the definition of the monoidal structure beyond the first two levels requires hard work.

∞ -Groupoids: An ∞ -groupoid is a higher category in which all the k -morphisms are invertible. Thus it is an $(\infty, 0)$ -category in the above notation. A good example is the extension of the fundamental groupoid $\pi_{\leq 1}(X)$ of a topological space X to the ∞ -category $\pi_{\infty}(X)$ mentioned above. In some sense, this is the only kind of example. The reason is that from an ∞ -groupoid, we may produce a simplicial set. We take the nerve of the category $\pi_{\leq 1}(X)$ but also add simplices associated with the higher morphisms. For example, the 2-morphism of Figure 29 defines a 2-simplex in a natural way. We have already seen that in homotopy theory we can freely pass between simplicial sets and topological spaces, so we have a triangle of concepts – see Figure 31 – equivalent in the eyes of a homotopy theorist.

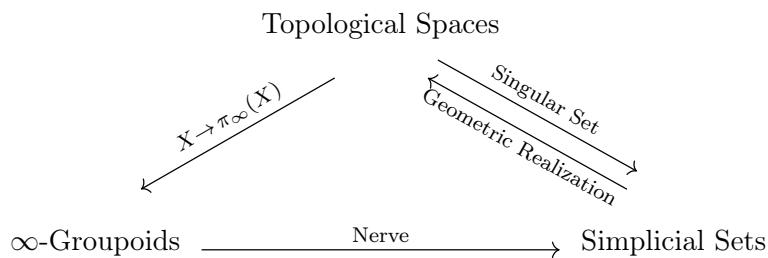


Figure 31: Triangle of concepts.

Thus for example, given a simplicial set \mathcal{X} , we can go around the above triangle:

$$\mathcal{X} \rightarrow |\mathcal{X}| \rightarrow \pi_{\infty}(|\mathcal{X}|) \rightarrow N(\pi_{\infty}(|\mathcal{X}|)) , \quad (20.12)$$

and \mathcal{X} should be equivalent to $N(\pi_{\infty}(|\mathcal{X}|))$ in the category of simplicial sets. For (much) more information, see [Lur25].

21 Background Fields

In physics, we often study quantities like partition functions as functions of other physical parameters. In quantum theory, we might also study operator algebras as functions of parameters. These parameters often obey locality properties, and when they do they are known as *background fields*, or more simply as *fields*.

21.1 Some Motivating Examples

In order to explain the mathematical definition of what a field is it is useful to begin with some examples.

Example 1: Scalar field theories. Consider a compact Riemannian manifold (X, g_X) , known as the *target space*. A scalar field theory is a theory whose dynamical fields are maps $\xi : M_n \rightarrow X$, where the domain M_n is a Riemannian manifold with metric g_{M_n} , to be thought of as the spacetime of the field theory.

The action is a function of the dynamical field ξ and also of the background fields (g_{M_n}, g_X) . In order to write it, we note that the differential $d\xi : T_p M_n \rightarrow T_{\xi(p)} X$ is an element of $\text{Hom}(T_p M_n, T_{\xi(p)} X)$, and the metrics on the “worldvolume” M_n and the “target” X define a metric on $\text{Hom}(T_p M_n, T_{\xi(p)} X)$. The action is then

$$S[\phi; g_{\mu\nu}, g_X] = \int_{M_n} \|d\xi\|^2 \text{vol}(g_{M_n}) . \quad (21.1)$$

We have chosen length scales both on the target and the domain manifold and set them to 1 so that all quantities here are dimensionless. The action we have written is sometimes called the *nonlinear sigma model action* for historical reasons.³⁸

A simple case is where the target space X is a circle of radius R , so $g_X = R^2(d\phi)^2$ on $X = \mathbb{S}^1$, where $\phi \sim \phi + 2\pi$ is a periodic target space coordinate. It is customary to choose a local logarithm $\xi = e^{i\phi}$, and then the action, written out with indices, is:

$$S[\phi; g_{\mu\nu}, R] = R^2 \int_{M_n} \sqrt{g(x)} g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) d^n x . \quad (21.2)$$

At least formally, the partition function is a function of the background fields g_{M_n} and g_X :

$$Z[g_{M_n}, g_X] = \int d\phi e^{-S[\phi; g_{\mu\nu}, g_X]} . \quad (21.3)$$

In general, there will be difficulties in giving a sensible definition to this partition function – even by the standards of physicists. But there are two notable exceptions:

First, if $n = 2$ so M_n is a two-dimensional manifold, then using ideas of the renormalization group it can be nicely defined [Fri80, Fri85].

Second, if the target space is flat (so X is a torus), then we are working with free field theory, and rigorous definitions are available. Returning to the simple case where $X = \mathbb{S}^1$, we can express the function explicitly as

$$Z[g_{\mu\nu}, R] = \frac{R}{\sqrt{\det' \Delta}} \sum_{\omega \in \mathcal{H}_{\mathbb{Z}}^1} e^{-R^2 \|\omega\|^2} , \quad (21.4)$$

where the sum is over the lattice $\mathcal{H}_{\mathbb{Z}}^1$ or harmonic 1-forms on (M_n, g_{M_n}) with integral peri-

³⁸It goes back to the work of M. Gell-Mann and M. Levy [GML60] and F. Gürsey [G60] on early field theory models of pion dynamics.

ods, and $\det' \Delta$ is the determinant of the scalar Laplacian acting on the space of functions orthogonal to the constant functions. Note that in the $R \rightarrow \infty$ limit, we expect to obtain the real-valued scalar field. The theta function goes to 1 exponentially fast in R , but there is an overall factor of R . In this noncompact limit of the target space, the finite quantity of interest is the partition function per unit volume of the target space.

In QFT textbooks, another set of background fields is also typically introduced. Let \mathcal{O} be the algebra of local operators, choose a linear basis $\mathcal{O}_n(x)$ of local operators, and let $J \in \text{Map}(M_n \rightarrow \mathcal{O}^\vee)$. Then we can extend the action by

$$S[\phi; g_{\mu\nu}, g_X; J] = \int_{M_n} \|d\xi\|^2 \text{vol}(g_{M_n}) + \int_{M_n} \sum_n J^n(x) \mathcal{O}_n(x) \text{vol}(g_{M_n}) , \quad (21.5)$$

where $J^n(x)$ are the components relative to a dual basis. This is known as *adding source terms*. These too, are background fields, although in the quantum theory, it might be necessary to study the partition function only as a formal series in J^n .

Example 2: *The Quantum Rotor.* We consider a one-dimensional scalar field theory, where the target space is a circle with round metric, but there is an additional parameter B , which is a real number. The action is

$$S = \int_{M_1} \frac{1}{2} I \dot{\phi}^2 dt + \frac{B}{2\pi} \dot{\phi} dt . \quad (21.6)$$

Here, I plays the role of the radius squared and the second term does not affect the classical theory but plays a very important role in the quantum theory. There are many physical interpretations of this action, including that of a charged massive particle constrained to move on a circle that links a solenoid carrying flux B , and the dimensional reduction of Yang-Mills theory with a θ -term. The reader should note that the two terms in the action behave differently under time-reversal $t \mapsto -t$. Thus, the action depends not only on I and B but on another quantity, the orientation \mathfrak{o} of the worldvolume M_1 . The quantum partition function is thus a (very interesting) function of (I, B, \mathfrak{o}) . As we will argue below, \mathfrak{o} should also be considered as a background field. As in the previous case, it can also be coupled to other background fields. For expository discussions of this very interesting and crucial example, we refer the reader to [GKKS17, App. D], [Moo19, Sec. 2.1], and [Moo23a, Sec. 15.4].

Example 3: *Spin Theories.* Theories that involve fermions typically require a choice of spin or spin^c structure on the spacetime. A simple example is supersymmetric quantum mechanics, where the dynamical fields are maps of a $(1|1)$ supermanifold³⁹ to a Riemannian target space (X, g_X) . The partition function on the circle depends on the choice of spin

³⁹For mathematical expositions of supermanifolds and supergeometry, see, for example, [Lei80, DeW92, Fre99, DF99b, DF99c, Man97, HST11, Wit12]. For physical expositions, see [GGRS01, WB92, Ter05].

structure. If it is the nonbounding (a.k.a. Ramond, a.k.a. periodic) spin structure, then the partition function is the *Witten index*. It is a topological invariant of the target space X . In fact, it is just the Euler character of X [Wit82a, Wit82b, Wit82c]. By contrast, the partition function with the other spin structure, the bounding (a.k.a. Neveu-Schwarz, a.k.a. anti-periodic) spin structure, then the partition function is a complicated function of (X, g_X) . Thus, supersymmetric quantum mechanics is an example of a *spin theory*, namely, a theory that depends on a choice of spin structure on the spacetime of definition. Once again, spin structures will be considered to be examples of background fields.

Another, important, example of a spin theory is the *Arf theory*. This is a two-dimensional TFT defined on spin manifolds. The partition function for a closed, oriented, spin manifold (M_2, \mathfrak{s}) is the *Arf invariant*, which is valued in $\{\pm 1\}$. The Arf invariant can be defined in many ways. One way [Ati71], identifies it as the mod-two index of the Dirac operator. (In terms of holomorphic geometry, the spin structure defines a square root of the canonical bundle K , and the invariant can be defined as $\dim H_{\text{hol}}^0(M_2; K^{1/2})$, since the Riemannian metric on M_2 defines a complex structure.) Another way identifies it as the Arf invariant of the quadratic function $q : H_1(M_2; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$, where q on a circle is 0 or 1, according to whether the circle inherits bounding or nonbounding spin structure from \mathfrak{s} , respectively. The Hilbert space on the circle with bounding spin structure is the super-Hilbert space $\mathbb{C}^{1|0}$, and on the circle with non-bounding spin structure is the super-Hilbert space $\mathbb{C}^{0|1}$. For much more information about this theory, see [Gun12, DG18, KTT19].⁴⁰

21.2 Freed-Hopkins-Teleman Definition Of A Field

In all the above examples we have encountered various fields. These include traditional fields like the scalar field $\xi : M_n \rightarrow X$ and the Riemannian metric g_{M_n} in [Example 1](#), but, as has been argued by D. Freed, M. Hopkins and C. Teleman, in [[FT12](#), [FH13](#)] that, for example, the orientation σ of [Example 2](#), and the spin structure \mathfrak{s} of [Example 3](#) should equally be considered to be “fields.” Moreover, the parameters like I and B in [Example 2](#) could also be rendered spacetime dependent – an old idea known as the *Janus construction* – and these two should be considered to be “fields.” Usually, they are constrained to be constant as functions of spacetime. Quite generally, given any symmetry, one should specify background fields associated with that symmetry. For example, if a theory has a global symmetry G , then it is natural to make the theory equivariant by defining the theory coupled to a principal G -bundle with connection over spacetime.

Some general locality properties we wish our fields to satisfy are:

1. They should pull back under smooth maps. In order to incorporate nondegenerate

⁴⁰The Arf theory was first used in quantum field theory in the theory of the 2d boson/fermion correspondence [[ÁGMN⁺86](#), [ÁGBM⁺87](#)]. It was found that this term needed to be included in the theory of the 2d scalar field to make it a spin theory with a partition function equal to that of a fermion with a fixed spin structure.

metrics and orientations in the definition, we restrict this condition to the condition that they pull back (or push forward) under local diffeomorphisms.

2. They should satisfy a sheaf property: if they are defined on open sets \mathcal{U} and \mathcal{V} , and agree on $\mathcal{U} \cap \mathcal{V}$, then there is a unique extension to $\mathcal{U} \cup \mathcal{V}$.

In general, the TFT functor gives the answer to the path integral: the dynamical fields (if there are any) have been integrated out and are not present in the definition. However, the background fields are present and we study how the partition functions, statespaces, amplitudes, etc. depend on these background fields. It is useful to begin with a classical field theory (which, as we have noted in [Section 10](#) above, defines an invertible field theory) with fields \mathcal{F} then declare another set of fields \mathcal{F}^{bck} to be the background fields. We must have some kind of map

$$\pi : \mathcal{F} \rightarrow \mathcal{F}^{\text{bck}} . \quad (21.7)$$

The quantization procedure (e.g., by path integration) then integrates over the fiber of this map to produce a functor that depends on \mathcal{F}^{bck} .

We can give a precise mathematical definition of “fields” following Appendix A of [\[FT12\]](#) and section 3 of [\[FH13\]](#) by formalizing them as a “sheaf on Man_n .” This is the same thing as a functor:

$$\mathcal{F} : \text{Man}_n^{\text{op}} \longrightarrow \text{sSET} , \quad (21.8)$$

where Man_n is the category whose objects are smooth n -dimensional manifolds and morphisms are local diffeomorphisms between n -manifolds. The superscript op means the opposite category (see equation (9.6) et. seq.) and sSET is the category of simplicial sets and maps of simplicial sets as described in [Section 16](#). (We ignore paradoxes of set theory here.)

In [Example 1](#) above, the fields ξ, g_{M_n} and g_X are all set-valued. For example, the field ξ is defined by a sheaf $\mathcal{F}_{\text{sigma model field}}$ whose value on an n -manifold is⁴¹

$$\mathcal{F}_{\text{sigma model field}}(M_n) := \text{Map}(M_n \rightarrow X) . \quad (21.9)$$

Here $\text{Map}(M_n \rightarrow X)$ is a set, but as we have seen, a set can be turned into a simplicial set in a natural way, so these examples fit the FHT definition (21.8).

The use of the codomain of simplicial sets (rather than sets) in the definition (21.8) becomes important for fields with gauge symmetries if one wants to preserve the notation of locality. A simple example will illustrate the main point.⁴² Let us consider principal \mathbb{Z}_2 -bundles in one-dimension (equivalently, double-covers). View \mathbb{S}^1 as the union of two open sets D_{\pm} corresponding to the “northern and southern hemispheres.” Then D_{\pm} are intervals

⁴¹A more complete discussion would specify precisely what kind of maps. We typically use C^∞ maps, but this is not sufficient for quantum theory.

⁴²We thank D. Freed for pointing out this illuminating example.

and $D_+ \cap D_-$ is a disjoint union of two intervals around the equator. Suppose we tried to restrict the field so that we only consider isomorphism classes of principal bundles. Then we would have exactly one point associated with each 1-manifold D_+ , D_- , and $D_+ \cap D_-$. This is not enough data to capture the two inequivalent double covers of the circle. If on the other hand, the target space of the field is the groupoid of principal \mathbb{Z}_2 -bundles, then associated to D_\pm is a category with one object but two invertible morphisms between the objects: The identity map of the double cover and the deck transformation. Now on D_\pm , we can compare these invertible morphisms with a 2-morphism and recover the distinct double covers of the circle.

Moving on to gauge theories, for a topological group G , we can define a field \mathcal{F} that assigns to an n -manifold M_n the groupoid $P_G(M_n)$ of principal G -bundles over M_n . In order to fit the FHT definition of a field, we should take \mathcal{F} to be the nerve of that category so we take:

$$\mathcal{F}(M_n) = N(P_G(M_n)) . \quad (21.10)$$

This will suffice as the definition of a field in G -gauge theory for a finite group G , since principal G -bundles come with a unique connection. In general, for G -gauge theory, we wish to include the connection as one of the fields. Therefore, we consider $P_G^\nabla(M_n)$, the groupoid of principal G -bundles over M_n equipped with connection. So, in general the field of a gauge theory in the FHT definition is:

$$\mathcal{F}(M_n) = N(P_G^\nabla(M_n)) . \quad (21.11)$$

As we will see in [Part II](#) when studying higher-form fields whose isomorphism classes on M_n are the differential cohomology groups $\check{H}^\ell(M_n)$, these groups are the group of isomorphism classes of a (higher) groupoid. (See [Section 40.2](#), especially [Section 40.2.3](#), for further discussion.) Once again, the field should be viewed in terms of a sheaf valued in simplicial sets, for the same reason as for gauge theories – locality. However, in this case, it is even more important to do so because of the phenomena of “ghosts for ghosts” in theories of higher-form fields. One can take the nerve of that (higher) groupoid and that is the value of $\mathcal{F}(M_n)$.

21.3 Decorating Bordisms With Fields

The definition of background fields allows us to put \mathcal{F} -structures on our bordisms to define an enhanced category $\text{Bord}_{\langle n-1, n \rangle}^{\mathcal{F}}$. In practice, this means that all the manifolds encountered in the definition of bordisms are equipped with orientation, (s)pin structure, Riemannian metric, etc. in a compatible way. To define the structures on the boundary components of the bordism, we must evaluate \mathcal{F} on germs of the boundary components. Thus, for example, if \mathcal{F} is a spin structure, we only work with spin structures in n -dimensions. Then, we define

a TFT with such background fields as:

$$\mathcal{F} : \text{Bord}_{\langle n-1, n \rangle}^{\mathcal{F}} \longrightarrow \text{VECT} . \quad (21.12)$$

This works well for discrete structures like orientations, and principal G -bundles with finite G . This viewpoint was quite useful in a recent paper clarifying some issues about global symmetries in Class \mathcal{S} theories and topological twisting of $d = 4$, $\mathcal{N} = 2$ theories [MSS24].

Remark 21.13.

1. In example 3 of [Section 20](#), we encountered the (∞, n) -category $\text{Bord}_{\leq n}$. If we endow the bordisms with the structure of a field \mathcal{F} , we define (∞, n) categories $\text{Bord}_{\leq n}^{\mathcal{F}}$. These will be important in the definition of fully local TFT in [Section 23](#) below.
2. The background fields of a topological field theory should be – in some sense – invariant under homotopies. Examples include orientations and spin structures. One technical way to express this is to use the concept of a *tangential structure*. See [Fre19], Definition 1.40 and section 24 of [Fre23a] for expositions. In particular in [Fre23a], a notion of locally constant \mathcal{F} is developed and proven to be equivalent to tangential structure. On the other hand, if the background fields include, for example, scalar fields, gauge fields, Riemannian metrics, conformal structures,... the theory will no longer be topological. We will return to this in [Section 28](#) below.

22 Generalizing Domain And Codomain

Let us return to our preliminary definition (9.42) of a topological field theory. In this point of view, a field theory is a functor. The codomain of the functor is VECT . That is natural because the traditional viewpoint in physics is that partition functions should be complex numbers and spaces of states should be constructed from compact operators on Hilbert spaces. However, the mathematical definition of a field theory admits natural generalizations where one changes either the domain or the codomain category.

One way in which the domain category can be generalized is by including fields as in [Section 21](#). Another way arises in the work of Freed and Hopkins [FH16] on invertible field theories. These theories factor through a quotient of the bordism category where all morphisms become invertible. That allows them to be viewed as morphisms of “spectra” (see [Section 50](#) below), which in turn opens up all the tools of stable homotopy theory.

Instead of taking a quotient of the domain bordism category, one can try to extend the domain to include non-smooth manifolds. The use of C^∞ manifolds in the definition of the bordism category is intended to guarantee that the gluing axioms work smoothly.

Of course, with extended topological field theory, one works with manifolds with corners, but one could ask if these manifolds could admit singularities of some kind or even be CW complexes. One early contribution in this direction is the lecture series by Frank Quinn in [Qui95]. Another recent contribution to this question is [KKN23]. Moreover, section 4.3 of Lurie’s famous paper [Lur09c] extends the cobordism hypothesis to include manifolds with certain allowed singularities. Finally, we note that there is a small but interesting literature where one uses something very like a “one-dimensional topological field theory,” but with graphs rather than one-manifolds to describe categories related to the representation theory of exceptional groups. For examples, see [GSZ21] and references therein. Finally, we remark that it might be fruitful to interpret the “ XZ -calculus” of quantum information theory [CD08, DC09, vdW20, GST24] in terms of a 1d topological field theory with a domain consisting of colored graphs instead of manifolds.

Another kind of generalization of the notion of field theory is obtained by changing the codomain category, which until now has been VECT . Of course, one could consider vector spaces over different “fields” (in the sense of abstract algebra). Moreover, one should generalize from the category of vector spaces to super-vector spaces, especially when working with theories with fermions. A more drastic example of a change of codomain category arises in the study of Higgs branches of Class \mathcal{S} supersymmetric 4d QFTs [MT11]. The codomain category has as objects complex algebraic groups. The Hom space between two such groups G_1 and G_2 is the set of holomorphic symplectic manifolds with a holomorphic \mathbb{C}^\times action scaling the symplectic form and a holomorphic Hamiltonian $G_1 \times G_2$ action. This is very far from the linear categories we have thus far met.

23 Fully Local TFT: The Cobordism Hypothesis

We have thus far stressed that the most primitive physical notions of locality are captured in the definition of an n -dimensional TFT by taking it to be a functor $F : \text{Bord}_{\langle n-1, n \rangle} \rightarrow \text{VECT}$. As we have seen, we can generalize the bordism category by including background fields to $\text{Bord}_{\langle n-1, n \rangle}^{\mathcal{F}}$. We can also generalize the codomain to a be a general symmetric monoidal n -category \mathcal{C} , so we can consider a more general viewpoint of a field theory as a monoidal functor,

$$F : \text{Bord}_{\langle n-1, n \rangle}^{\mathcal{F}} \rightarrow \mathcal{C}. \quad (23.1)$$

But we have also seen that we can generalize $\text{Bord}_{\langle n-1, n \rangle}^{\mathcal{F}}$ to an (∞, n) -category $\text{Bord}_{\leq n}^{\mathcal{F}}$ and hence we are led to a final generalization of a fully extended field theory as a monoidal (∞, n) -functor with a codomain \mathcal{C} which is a symmetric monoidal (∞, n) -category:

Definition 23.2 (Fully Local Field Theory). Let \mathcal{C} be a symmetric monoidal (∞, n) -category and \mathcal{F} be a sheaf on Man^{op} . Then a fully local field theory with background

fields \mathcal{F} is a monoidal functor between (∞, n) -categories:

$$F : \mathbf{Bord}_{\leq n}^{\mathcal{F}} \rightarrow \mathcal{C} . \quad (23.3)$$

Remark 23.4.

1. The terms “extended field theory” and “fully extended field theory” are also used for a “fully local field theory.”
2. The theory will be topological if \mathcal{F} is “locally constant” in the sense described in [Fre23a, Sec. 24.4]. The definition is only fully rigorous in the topological case.

Monoidal n -categories have a distinguished 0-morphism $1_{\mathcal{C}}$, the unit under \otimes , and one defines the looped category:

$$\Omega\mathcal{C} := \mathsf{Hom}(1_{\mathcal{C}}, 1_{\mathcal{C}}) , \quad (23.5)$$

which is a monoidal $(n - 1)$ -category, and therefore has a monoidal unit $1_{\Omega\mathcal{C}}$, so we can define,

$$\Omega^2\mathcal{C} := \mathsf{Hom}(1_{\Omega\mathcal{C}}, 1_{\Omega\mathcal{C}}) , \quad (23.6)$$

which is a monoidal $(n - 2)$ -category. We can iterate the looping process to produce an $(n - k)$ -category $\Omega^k\mathcal{C}$. At each stage, we get a monoidal category with a unit $1_{\Omega^{k-1}\mathcal{C}}$ for that category.

Let \emptyset_{k-1} be the empty $(k - 1)$ -manifold. Then $F(\emptyset_{k-1}) = 1_{\Omega^{k-1}\mathcal{C}}$. If M_k is a compact k -manifold without boundary, then it defines a k -morphism $\emptyset_{k-1} \rightarrow \emptyset_{k-1}$. Therefore, $F(M_k)$ must be a k -morphism in \mathcal{C} . Indeed, consulting (20.8) and (20.9), we see that it should be a 0-morphism, i.e., an object, in the $(n - k)$ -category:

$$\mathsf{Hom}(1_{\Omega^{k-1}\mathcal{C}}, 1_{\Omega^{k-1}\mathcal{C}}) = \Omega^k\mathcal{C} . \quad (23.7)$$

The right hand side is an $(n - k)$ -category. That is,

$$F(M_k) \in C_0(\Omega^k\mathcal{C}) = \mathsf{Obj}(\Omega^k\mathcal{C}) . \quad (23.8)$$

For a sanity check, let us suppose that $\Omega^n\mathcal{C} = \mathbb{C}$. (Remember that a monoidal “0-category” is just a set with a multiplication, hence a monoid.) Then for a compact n -manifold without boundary, $F(M_n)$ – the partition function – is indeed a complex number. Moreover, if we take $\Omega^{n-1}\mathcal{C} = \mathsf{VECT}$, then for a compact $(n - 1)$ -manifold without boundary N_{n-1} , we learn that $F(N_{n-1})$ is an object in VECT , that is, a vector space, in harmony with the previous discussions.

Remark 23.9. As we have just seen, it is very natural to assume that \mathcal{C} satisfies:

$$\Omega^n\mathcal{C} = \mathbb{C} , \quad \text{and} \quad \Omega^{n-1}\mathcal{C} = \mathsf{VECT} . \quad (23.10)$$

However, when proceeding to the next categorical level, there can be many different choices of \mathcal{C} that satisfy (23.10). For example at the next level, two natural choices are:

1. $\Omega^{n-2}\mathcal{C} = \text{ALG}(\text{VECT})$.
2. $\Omega^{n-2}\mathcal{C} = \text{CAT}$.

For example, when discussing finite gauge theory in $n = 2$ dimensions, in the first case, $F(\text{pt})$ will be an object in the Morita 2-category of algebras, bimodules, and bimodule maps. In other words, it will be an algebra. In the second case, $F(\text{pt})$ will be an object in the 2-category of categories. In other words, it will be a category. In the case of two-dimensional finite group gauge theory based on a finite group G , one can show that (see Section 24 below for more details):

$$\begin{aligned} F(\text{pt}) &= \mathbb{C}[G] , \\ F(\text{pt}) &= \text{Rep}(G) , \end{aligned} \tag{23.11}$$

where $\mathbb{C}[G]$ is the group algebra, and $\text{Rep}(G)$ is the category of representations of G .

The main idea behind (23.3) is that one has taken locality to its logical conclusion by chopping up spacetime into smaller spaces of increasingly high codimension. The most primitive level of locality says that we can “insert a complete set of states along a spatial slice,” if we cut an n -dimensional spacetime into “past” and “future” parts by cutting along an $(n - 1)$ -dimensional spatial slice. Now, we can consider the space of states on an $(n - 1)$ -dimensional spatial slice and divide that into two pieces by cutting along an $(n - 2)$ -dimensional manifold, as in the discussion around (20.2) above. And so on. This is the reason for the terminology *fully local* used above.

One of the main results in the study of extended TFTs is the proof of the “cobordism hypothesis.” The “cobordism hypothesis” is an idea going back to J. Baez and J. Dolan [BD95]. It states, very roughly, that a fully extended TFT is “completely determined by its value on a point”. Recall that $F(\text{pt}) \in \text{Obj}(\mathcal{C})$ is a 0-morphism in an n -category. A good example is the case of one dimensional oriented TFT where all amplitudes are determined by a single piece of data: $F(\text{pt}_+) = V$. A precise version of the cobordism hypothesis was stated and proved by J. Lurie [Lur09c] generalizing some important unpublished work of M. Hopkins and J. Lurie. Here is a very brief sketch of that result. We follow here the expositions [Fre12b, Tel16, Fre19].

An important part of the cobordism hypothesis is to study all the fully extended theories at once, so we should study the *category of theories* with fixed dimension n , background fields \mathcal{F} , and codomain \mathcal{C} . Thus, we consider the category of theories, which we denote as

$$\text{Hom}^\otimes(\text{Bord}_{\leq n}^{\mathcal{F}}, \mathcal{C}) . \tag{23.12}$$

It turns out that this category is itself an (∞, n) -category. The 0-morphisms are the different theories. A 1-morphism between two theories F_0 and F_1 , written $\eta : F_0 \Rightarrow F_1$, assigns to

every k -dimensional bordism W_k , a $(k+1)$ -morphism in \mathcal{C} in the hom-space:

$$\eta(W_k) \in \text{Hom}_{k+1}(F_0(W_k), F_1(W_k)) , \quad (23.13)$$

where Hom_{k+1} are the $(k+1)$ -morphisms in \mathcal{C} . In the simplest case of $n=1$, a field theory is a functor between 1-categories (thought of as $(\infty, 1)$ -categories) and a 1-morphism is just a natural transformation between those functors. The k -morphisms in (23.12) for $k > 1$ are defined similarly. See the above references for details.

The next step is a remarkable claim: All the morphisms in the category of theories (23.12) are invertible! Thus (23.12) is actually an $(\infty, 0)$ -category, i.e., an ∞ -groupoid. As we have seen in Figure 31 above, an ∞ -groupoid \mathcal{G} determines a space, via the geometric realization of its nerve: $\mathcal{G} \rightarrow |N(\mathcal{G})|$, and this space in turn determines the ∞ -groupoid, up to a suitable notion of equivalence. In our case the space,

$$|N(\text{Hom}^\otimes(\text{Bord}_{\leq n}^{\mathcal{F}}, \mathcal{C}))| , \quad (23.14)$$

is the moduli space of theories of dimension n with background fields \mathcal{F} and codomain \mathcal{C} .

In order to get some sense of why (23.12) is a groupoid, we need to introduce the notion of dualizability. Suppose first that D is a symmetric monoidal category and $x \in \text{Obj}(D)$. Then x is said to be *dualizable*, if there exists another object x^\vee in D , and morphisms

$$e : x^\vee \otimes x \rightarrow 1_D , \quad (23.15)$$

$$c : 1_D \rightarrow x \otimes x^\vee , \quad (23.16)$$

such that the composition of morphisms,

$$x \rightarrow 1_D \otimes x \xrightarrow{c \otimes \text{Id}_x} (x \otimes x^\vee) \otimes x \cong x \otimes (x^\vee \otimes x) \xrightarrow{\text{Id}_x \otimes e} x \otimes 1 \rightarrow x , \quad (23.17)$$

just produces $\text{Id}_x : x \rightarrow x$. Similarly, we require the composition of morphisms:

$$x^\vee \rightarrow x^\vee \otimes 1_D \xrightarrow{\text{Id}_{x^\vee} \otimes c} x^\vee \otimes (x \otimes x^\vee) \cong (x^\vee \otimes x) \otimes x^\vee \xrightarrow{e \otimes \text{Id}_{x^\vee}} 1_D \otimes x^\vee \rightarrow x^\vee , \quad (23.18)$$

just produces $\text{Id}_{x^\vee} : x^\vee \rightarrow x^\vee$.⁴³ The morphisms c and e are called the *coevaluation* and *evaluation*, respectively. The data (x^\vee, e, c) are called *duality data for x* . These data are unique up to canonical isomorphism. We have $(x \otimes y)^\vee \cong x^\vee \otimes y^\vee$. Moreover, if $f : x \rightarrow y$ is a morphism between dualizable objects there is a dual morphism (unique up to canonical isomorphism) $f^\vee : y^\vee \rightarrow x^\vee$.

Example 1: As a good example, in **VECT**, only the finite-dimensional spaces have duality

⁴³Careful readers will recognize that there is a distinction between left- and right- dualizable. We will ignore that subtlety here.

data and the dual to a finite-dimensional vector space is the standard dual space $V^\vee = \text{Hom}(V, \mathbb{C})$. The evaluation morphism is the canonical linear transformation $e : V^\vee \otimes V \rightarrow \mathbb{C}$. To express the coevaluation, we choose a basis $\{v_i\}$ for V , which determines a dual basis $\{v_i^\vee\}$ for V^\vee , and then, the linear transformation $c : \mathbb{C} \rightarrow V \otimes V^\vee$ is determined by its value on $1 \in \mathbb{C}$:

$$c(1) = \sum_i v_i \otimes v_i^\vee. \quad (23.19)$$

Note that this will only give a well-defined element of $V \otimes V^\vee$ if V is finite-dimensional.

Example 2: For another example of dualizability, consider the category of oriented bordisms $\text{Bord}_{\langle 0,1 \rangle}^{\text{or}}$. A positively oriented point, $x = \text{pt}_+$ is dualizable. Its dual object x^\vee is the oppositely oriented point $x^\vee = \text{pt}_-$. The duality data e, c are provided by the standard bordisms given by the pictures (6.3) above.

At this point, we can indicate why $\text{Hom}^\otimes(\text{Bord}_{\langle 0,1 \rangle}^{\text{or}}, \mathcal{C})$ is a groupoid. Here \mathcal{C} is an ordinary tensor 1-category. If F is a theory, i.e., an object in $\text{Hom}(\text{Bord}_{\leq 1}^{\text{or}}, \mathcal{C})$, then $F(\text{pt}_+)$ is a dualizable object in \mathcal{C} . Moreover, if $\eta : F_0 \Rightarrow F_1$ is a natural transformation of the tensor functors, then there is a morphism $\eta(\text{pt}_+) : F_0(\text{pt}_+) \rightarrow F_1(\text{pt}_+)$. It is shown in [Fre19, Lemma A.44] that $\eta(\text{pt}_-)^\vee$ is inverse to $\eta(\text{pt}_+)$. From this, it follows that $\eta : F_0 \Rightarrow F_1$ is invertible.

Dualizability has a generalization to symmetric monoidal (∞, n) -categories known as *full dualizability*. The basic idea is that we demand that the dualizability data itself is dualizable in the appropriate category. More technically, recall that for an (∞, n) -category \mathcal{C} , we let $\mathcal{G}_k(\mathcal{C})$ be the (∞, k) -category where we delete the noninvertible ℓ -morphisms with $\ell > k$. Then, if \mathcal{C} is a symmetric monoidal (∞, n) -category an object x (“0-morphisms”) of \mathcal{C} is said to be *fully dualizable* if:

1. x is dualizable in $\mathcal{G}_1(\mathcal{C})$.
2. The evaluation and coevaluation morphisms e, c of x are dualizable in $\mathcal{G}_2(\mathcal{C})$.
3. And so forth up to, but not including, the n -morphisms.

A key observation is that the image of a TFT only contains fully-dualizable objects. This follows from the same style of argument we used in the S -diagram of Figure 11 above, to show that statespaces in a TQFT are finite-dimensional. Indeed, “fully dualizable” is just a generalization of this finite-dimensional property. Thus, the field theory functor factorizes through a functor to the subcategory \mathcal{C}^{fd} of fully dualizable objects and morphisms.

The next step is to recognize that there is a very special kind of background field \mathcal{F} known as an *n-framing*. In general, if $\pi : V \rightarrow X$ is a rank r vector bundle, then a *framing* is an isomorphism of V with $X \times \mathbb{R}^r$. It is the same as a trivialization of $\pi : V \rightarrow X$. If $k < n$, an *n-framing* of a smooth k -manifold W_k is an framing of $TW_k \oplus \mathbb{R}^{n-k}$, that is, we

have a trivialization, that is an isomorphism:

$$TW \oplus (W \times \mathbb{R}^{n-k}) \rightarrow W \times \mathbb{R}^n . \quad (23.20)$$

Thus, an n -framing of a point is a vector space \mathbb{R}^n with the point at the origin. One should think of this vector space as being diffeomorphic to a neighborhood of a point in an n -manifold. The neighborhood can be arbitrarily small so it really represents a germ of a point within an n -dimensional manifold. In general, all the k -dimensional bordisms W_k in the extended bordism category $\text{Bord}_{\leq n}^{\text{fr}}$ of framed manifolds should be thought of as having such germs of n -dimensional neighborhoods. In $\text{Bord}_{\leq n}^{\text{fr}}$, the framings are only considered up to homotopy. Consider again the case where W_0 is a point. The isomorphism of $T\mathbb{R}^n \oplus T(\text{pt}) \rightarrow T\mathbb{R}^n$ has a determinant of either sign, and the sign is the only homotopy invariant of the framing. Thus, there are two points pt_\pm in $\text{Bord}_{\leq n}^{\text{fr}}$.

We can now state part 1 of the cobordism hypothesis: There is an equivalence τ of ∞ -groupoids between the groupoid of n -dimensional field theories defined on bordisms with n -framing and codomain \mathcal{C} , and the ∞ -groupoid $\mathcal{G}_0(\mathcal{C}^{\text{fd}})$:⁴⁴

$$\tau : \text{Hom}^\otimes(\text{Bord}_{\leq n}^{\text{fr}}, \mathcal{C}) \rightarrow \mathcal{G}_0(\mathcal{C}^{\text{fd}}) . \quad (23.21)$$

τ is determined on 0-morphisms simply by:

$$\tau_0 : F \mapsto F(\text{pt}_+) . \quad (23.22)$$

This suffices to determine how τ_k acts on the k -morphisms with $k > 0$. (We saw an example above for $n = 1$.) As we have stressed, ∞ -groupoids and spaces are “the same” in the eyes of a homotopy theorist. At the level of spaces, the map (23.22) (and its higher k analogs) determines a continuous map of spaces:

$$|N(\text{Hom}^\otimes(\text{Bord}_{\leq n}^{\text{fr}}, \mathcal{C}))| \rightarrow |\mathcal{G}_0(\mathcal{C}^{\text{fd}})| , \quad (23.23)$$

and the cobordism hypothesis asserts this to be a homotopy equivalence. As noted in (23.14), the LHS is the moduli space of theories.

Let us test (23.23) in the case of 1-dimensional theories. All 1-manifolds admit 1-framings. If we choose $\mathcal{C} = \text{VECT}$, then \mathcal{C}^{fd} will pick out the subcategory of finite-dimensional vector spaces. Next, the groupoid $\mathcal{G}_0(\mathcal{C}^{\text{fd}})$ is obtained by dropping the non-invertible morphisms. Thus $\mathcal{G}_0(\mathcal{C}^{\text{fd}})$ is the category whose objects are finite-dimensional vector spaces and morphism spaces $\text{Hom}_{\mathcal{G}_0(\mathcal{C}^{\text{fd}})}(V_1, V_2)$ are zero, if V_1 and V_2 have different dimensions, while $\text{Hom}_{\mathcal{G}_0(\mathcal{C}^{\text{fd}})}(V_1, V_2)$ is the $\text{GL}(r, \mathbb{C})$ -torsor of linear isomorphisms $V_1 \rightarrow V_2$ if $\dim V_1 = \dim V_2 = r$. The category $\mathcal{G}_0(\mathcal{C}^{\text{fd}})$ is thus equivalent to the category whose objects

⁴⁴ $\mathcal{G}_0(\mathcal{C}^{\text{fd}})$ is sometimes called the *core* of \mathcal{C} .

are nonnegative integers r , with morphism spaces

$$\text{Hom}(r_1, r_2) = \begin{cases} 0 & , \quad r_1 \neq r_2 \\ \text{GL}(r, \mathbb{C}) & , \quad r_1 = r_2 = r . \end{cases} \quad (23.24)$$

The space associated to this groupoid is just:

$$\coprod_{r \geq 0} \text{BGL}(r, \mathbb{C}) . \quad (23.25)$$

Recall that this is supposed to be the moduli space of theories. Being a moduli space means that any family of theories parametrized by a space S is equivalent to a continuous map $S \rightarrow \coprod_{r \geq 0} \text{BGL}(r, \mathbb{C})$. Such a map is the same thing as a vector bundle over S . Recall that a single theory is determined by $F(\text{pt}_+) \in \text{Obj}(\text{VECT})$, so a family of theories is just a vector bundle over S . So indeed, (23.25) is the right answer for the moduli space of one-dimensional theories!

Remark 23.26. An analogy ([Fre19, Thm. 4.40]) is quite helpful. The oriented bordism group in 0 dimensions is $\Omega_0^{\text{SO}} \cong \mathbb{Z}$, generated by pt_+ (or pt_-). Consider the 0-dimensional oriented bordism invariants valued in an Abelian monoid \mathbf{M} , written additively. Such bordism invariants should be viewed as elements F in the set of monoid homomorphisms $\text{Hom}(\Omega_0^{\text{SO}}, \mathbf{M})$. Now $F(\text{pt}_+) + F(\text{pt}_-) = F(\emptyset) = 0$, so $F(\text{pt}_+)$ is an invertible element of \mathbf{M} . It follows that F factors through a monoid homomorphism to the set $\mathcal{G}(\mathbf{M}) \subset \mathbf{M}$ of invertible elements of \mathbf{M} . This is the analog of the statement that a field theory factors through \mathcal{C}^{fd} . Now note that $\text{Hom}(\Omega_0^{\text{SO}}, \mathbf{M})$ is itself a commutative monoid: $(F + G)(Y) := F(Y) + G(Y)$, but since F is invertible it is an Abelian group. This is the analog of the claim that $\text{Hom}^\otimes(\text{Bord}_{\leq n}^{\text{fr}}, \mathcal{C})$ is an ∞ -groupoid. Likewise the invertible elements $\mathcal{G}(\mathbf{M}) \subset \mathbf{M}$ form an Abelian group. The analog of the cobordism hypothesis is that $F \rightarrow F(\text{pt}_+)$ defines an isomorphism of Abelian groups, and indeed this isomorphism is equivalent to the statement that Ω_0^{SO} is the free Abelian group generated by an oriented point. One statement of part 1 of the cobordism hypothesis that one will encounter is that $\text{Bord}_{\leq n}^{\text{fr}}$ is the “free symmetric monoidal n -category generated by a fully dualizable object.”

Of course, we are interested in manifolds which are not framed and might not even admit a framing. The cobordism hypothesis can be extended to theories defined on such manifolds as follows. Suppose we are given a homomorphism $\varphi : G \rightarrow \text{O}(n)$ and let $\text{Bord}_{\leq n}^\varphi$ be the extended bordism category where the bordisms have a reduction, along φ , of the $\text{O}(n)$ structure on their tangent bundle to a G -structure. In particular, if $G = \text{O}(n)$ and φ is the identity homomorphism, then $\text{Bord}_{\leq n}^\varphi$ is the extended bordism category of unoriented n -manifolds. If $\varphi : \text{SO}(n) \rightarrow \text{O}(n)$ is the standard embedding, then $\text{Bord}_{\leq n}^\varphi$ is the oriented bordism category. If $\varphi : \text{Spin}(n) \rightarrow \text{O}(n)$ is the standard double cover of the $\text{SO}(n)$ subgroup

of $O(n)$, then $Bord_{\leq n}^\varphi$ is the extended spin bordism category. In this notation, the framed bordism category corresponds to the case where $G = \{1\}$ is the trivial group.

An n -dimensional theory F defined on manifolds with tangential structure determined by $\varphi : G \rightarrow O(n)$ can be presented as a factorization of a theory F^{fr} defined on framed manifolds as in the following diagram:

$$\begin{array}{ccc} & \text{Bord}_{\leq n}^\varphi & \\ \nearrow & & \searrow \\ \text{Bord}_{\leq n}^{\text{fr}} & \xrightarrow{F^{\text{fr}}} & \mathcal{C} \end{array} \quad (23.27)$$

To state part 2 of the cobordism hypothesis, note that the group $O(n)$ acts on n -framings: The action by $g \in O(n)$ is post-composition of the trivialization (23.20) with $g : W_k \times \mathbb{R}^n \rightarrow W_k \times \mathbb{R}^n$ for $g \in O(n)$. Part 2 of the cobordism hypothesis says that factorizations (23.27) correspond to “ $O(n)$ fixed point structures on $F^{\text{fr}}(\text{pt}_+)$.” In the case of $n = 1$ with $\mathcal{C} = \text{VECT}$, the $O(1) \cong \mathbb{Z}_2$ action takes $V \rightarrow V^\vee$. An “ $O(1)$ fixed point structure on V ” is equivalent to a nondegenerate bilinear form on V . In this way, we recover the description of one-dimensional unoriented theories of Section 6. In general, there is a lot to unpack in making sense of “ $O(n)$ fixed point structures on $F^{\text{fr}}(\text{pt}_+)$.” One will find both data and conditions. For a detailed discussion for “low” values of n , see [SPar].

Remark 23.28.

1. A great deal of the content of the cobordism hypothesis, as formulated above, is buried in the elusive notions of “fully dualizable object” and “ $O(n)$ fixed point structure.” It might be a fruitful exercise to explain the meaning of these terms more fully in “physical” terminology.
2. Equation (23.14) realizes, at least in the setting of topological field theory, an old dream of many physicists, namely, that there is a well-defined “space of n -dimensional quantum field theories”. This notion is essential to the Wilsonian viewpoint on QFT and renormalization group flow [WK74]. In the case of $n = 2$, it exhibits extremely interesting properties as described by Friedan, Zamolodchikov et. al. It is therefore interesting to note that in the case of $\mathcal{N} = (0, 1)$ supersymmetric field theories, S. Stolz and P. Teichner [ST04, ST11] (following work of Segal [Seg88b, Seg07]) have proposed another definition of a “space of theories”. They proved the 1d version of their conjecture for SQM with time-reversal symmetry, identifying the space of theories with an Ω spectrum which turns out to be KO. (See Section 50.2 below for a brief discussion

of the meaning of “spectrum” in topology.)⁴⁵ In the case of two dimensional $\mathcal{N} = (0, 1)$ SQFTs, the Ω spectrum is conjecturally the spectrum of Topological Modular Forms (TMF) [Hop95, Hop02, Goe09] (see also Section 50.4.1). The theories studied by Stolz and Teichner are not topological. Nevertheless, It would be quite interesting to relate these two notions of a “space of theories.”

24 Homotopy Sigma Models (Of Finite Type)

Homotopy Sigma Models⁴⁶ are a very useful class of topological field theories that generalize finite group gauge theory and Dijkgraaf-Witten theory. These theories prove to be excellent laboratories for exploring general ideas in topological field theory. They also have applications to a wide variety of other theories by providing “higher” symmetry structures of those theories, as we will briefly indicate in Section 26. Closely related to this, these spaces also serve to define background fields for theories with “higher group symmetry.” Thus, a thorough study of these theories is well worth the effort. At least in the case of four-dimensional TQFT, they essentially exhaust the possible fully extended TQFTs according to [DR18, JF20a, DHJF⁺24].

The study of these TFTs goes back to [Kon88, Qui95] and the study of “finite path integral” quantization of these theories goes back to [Fre92b]. For the case where the codomain \mathcal{C} is a “Morita n -category” (constructed from algebra objects), a fairly complete sketch of the fully local theory can be found in Freed-Hopkins-Lurie-Teleman [FHLT09]. A full description of the quantization procedure is work to appear by C. Scheimbauer and T. Walde [SWnd]. Here, we are following Appendix A of [FMT22]. Some useful remarks can also be found in [Tur10] and [C⁺24].

Before briefly describing these theories, we review some relevant mathematical background in Section 24.1 and Section 24.2 below.

24.1 π -finite Spaces

When we discussed finite group gauge theory we introduced the space BG . It has a basepoint $*$, and the property that

$$\pi_q(BG, *) \cong \begin{cases} \{1\} , & q > 1 , \\ G , & q = 1 . \end{cases} \quad (24.1)$$

⁴⁵In particular, the SQM is equipped with an antilinear operator T that squares to the identity: $T^2 = 1$ on the Hilbert space. This corresponds to a Pin_- structure [Fre19]. In view of the Freed-Hopkins classification of invertible phases [FH16], the gravitational anomaly in such theories is characterized by an element in the Anderson dual of the three-dimensional Pin_- bordism homology group, i.e., $n \in (I_{\mathbb{Z}} \Omega^{\text{Pin}-})^3 \cong \mathbb{Z}/8\mathbb{Z}$, reflecting the mod-8 periodicity of the real KO spectrum.

⁴⁶These theories have other names such as “finite homotopy theories” and “finite total homotopy TQFTs.” See, for example, [FHLT09, MP23].

There is a generalization available when $G = A$ is an Abelian group.

For every Abelian group A and integer $n > 1$, we can define an “Eilenberg-MacLane space” $K(A, n)$ as a path-connected space, such that

$$\pi_q(K(A, n)) \cong \begin{cases} \{1\} , & q \neq n \geq 0 , \\ A , & q = n . \end{cases} \quad (24.2)$$

One can show that such spaces exist and are unique up to homotopy equivalence [Spa81]. In fact, they can be constructed explicitly. For $A = \mathbb{Z}$, one attaches successively higher-dimensional cells to \mathbb{S}^n to kill the higher homotopy groups.

Example 1: $K(\mathbb{Z}, 1) = \mathbb{S}^1$. Again, this is deceptively simple and atypical, as we see in the next example.

Example 2: Note that we cannot take $K(\mathbb{Z}, 2) \stackrel{?}{\cong} \mathbb{S}^2$. After all, $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$! This and the higher homotopy groups,

$$\pi_{j \geq 4}(\mathbb{S}^2) = \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \dots \quad (24.3)$$

show that \mathbb{S}^2 will not do. But it is a good start. To construct $K(\mathbb{Z}, 2)$, one can inductively attach higher and higher cells to kill the higher homotopy groups. Another model for $K(\mathbb{Z}, 2)$ is the following: Consider the set of pure states in an n -dimensional Hilbert space:

$$\mathbb{CP}^n = \mathbb{S}^{2n+1}/\mathbb{U}(1) . \quad (24.4)$$

The long exact sequence of homotopy groups implies:

$$\pi_2(\mathbb{CP}^n) \cong \mathbb{Z} , \quad (24.5)$$

$$\pi_j(\mathbb{CP}^n) = 0 \quad j = 3, \dots, 2n + 1 . \quad (24.6)$$

Once again, as in our heuristic description of $B\mathbb{Z}_2$, one would like to take a limit as $n \rightarrow \infty$. Indeed, it turns out that we can identify $K(\mathbb{Z}, 2)$ with the space of pure states in ∞ -dimensional complex Hilbert space.

In general, $K(A, n)$ for a general Abelian group A and integer n will have some kind of infinite-dimensional model. While the $K(A, n)$ are not groups, they are “almost” groups. They are known as “homotopy groups” or “groups up to homotopy” or simply “ H -groups.” In much of the literature of the past few years, they have been used in attempts to generalize the notion of symmetry.

To define an H -group⁴⁷ let us first recall the definition of a group. We can define a group

⁴⁷We follow here the very nice exposition of [Spa81, p. 33].

formally as a set G with a map $\mu : G \times G \rightarrow G$ such that we have a commutative diagram:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{\mu \times \text{Id}} & G \times G \\ \text{Id} \times \mu \downarrow & & \downarrow \mu \\ G \times G & \xrightarrow{\mu} & G \end{array} \quad (24.7)$$

This defines an associative multiplication map μ , we have $\mu \circ (\text{Id} \times \mu) = \mu \circ (\mu \times \text{Id})$. Next, G has a basepoint $e \in G$, so that we can define a map $c : G \rightarrow G$ that takes every element to this basepoint. This map has the property that the composite maps given by

$$G \xrightarrow{(\text{Id}, c)} G \times G \xrightarrow{\mu} G , \quad (24.8)$$

and

$$G \xrightarrow{(c, \text{Id})} G \times G \xrightarrow{\mu} G , \quad (24.9)$$

are equal to the identity map $\text{Id} : G \rightarrow G$. Finally, to encode inverses, we assume there is a map $\varphi : G \rightarrow G$ such that the composite maps

$$G \xrightarrow{(\text{Id}, \varphi)} G \times G \xrightarrow{\mu} G , \quad (24.10)$$

and

$$G \xrightarrow{(\varphi, \text{Id})} G \times G \xrightarrow{\mu} G , \quad (24.11)$$

are equal to the map $c : G \rightarrow G$.

Now we generalize. An *H-group* (we also use the synonymous term *H-space*) is a based topological space P such that there exist continuous maps $c : P \rightarrow P$ (defined by taking all points to the basepoint), $\mu : P \times P \rightarrow P$, and $\varphi : P \rightarrow P$, so that the diagrams above defining a group are all true up to homotopy. That is, we replace “are equal to” by “are homotopic to” in the above definition of a group.

A crucial property of an *H*-group is that if X is a pointed space, then the set of basepoint preserving homotopy classes of maps $[X, P]$ is a group. Conversely, if P is a space with basepoint such that the sets $[X, P]$ admit a group structure for all based spaces X , then P is an *H-space*. A key example of an *H*-space is the based loop space ΩX of a pointed space X . One can generalize the notion of a principal fiber bundle with group G to a fibration by *H*-spaces. The Eilenberg-Maclane spaces are *H*-spaces, so fibrations by Eilenberg-Maclane spaces are generalizations of principal bundles. One can show that⁴⁸

$$\Omega K(A, n) \cong K(A, n - 1) . \quad (24.12)$$

⁴⁸Proof: Ω is adjoint to suspension. If X and Y are pointed spaces then $[\Sigma X, Y] \cong [X, \Omega Y]$, where Σ is the suspension. Apply this to $X = \mathbb{S}^k$, so $\Sigma X \cong \mathbb{S}^{k+1}$ and note that $\pi_q(X)/\pi_1(X) \cong [\mathbb{S}^q, X]$.

Now, for a pointed topological space Z , we define PZ to be the space of continuous paths $[0, 1] \rightarrow Z$, and then we have the path space fibration $\pi : PZ \rightarrow Z$, given by evaluation of the path at $t = 1$. Note that PZ is contractible and that the fibers are homotopy equivalent to ΩZ . Apply this to $Z = K(A, n + 1)$. Then the fibers of $\pi : PK(A, n + 1) \rightarrow K(A, n + 1)$ are Eilenberg-MacLane spaces $K(A, n)$ and hence, the path fibration is the analog of the universal bundle for the H -space $K(A, n)$. Therefore, the classifying space is:

$$BK(A, n) \cong K(A, n + 1). \quad (24.13)$$

It follows from (24.13) that fibrations over X with fiber $K(A, n)$ are classified by homotopy classes of maps

$$X \longrightarrow K(A, n + 1). \quad (24.14)$$

Since $K(A, 1) = BA$ is the classifying space of principal A -bundles, equation (24.13) shows that we can write $K(A, n) = B^n A$. This notation emphasizes the role of the Eilenberg-MacLane spaces as classifying spaces and we will sometimes use it.

Definition 24.15 (π -finite space). A π -finite space \mathcal{X} is a topological space with a finite set of connected components, each of which has a finite set of nonzero homotopy groups $\pi_j(\mathcal{X})$, each of which is a finite group.

When \mathcal{X} is connected, it is homotopy equivalent to a “Postnikov tower,” which is an iterated fibration of Eilenberg-MacLane spaces (see Bott-Tu, p. 250-251 [BT82]):

$$\begin{array}{ccccc} & & \vdots & & \\ & K(\pi_3, q_3) & \longrightarrow & \mathcal{X}^{(3)} & \\ & \downarrow & & & \\ K(\pi_2, q_2) & \longrightarrow & \mathcal{X}^{(2)} & \longrightarrow & BK(\pi_3, q_3) \cong K(\pi_3, q_3 + 1) \\ & \downarrow & & & \\ \mathcal{X}^{(1)} = K(\pi_1, q_1) & \longrightarrow & & & BK(\pi_2, q_2) \cong K(\pi_2, q_2 + 1) \end{array} \quad (24.16)$$

\vdots

The horizontal arrows are classifying maps of the $K(\pi_k, q_k)$ bundle over $\mathcal{X}^{(k-1)}$. So from (24.14), we see that the fibration $\mathcal{X}^{(k)} \rightarrow \mathcal{X}^{(k-1)}$ is classified by the homotopy classes of maps $\mathcal{X}^{(k-1)} \rightarrow K(\pi_{k+1}, q_{k+1} + 1)$. A (defining) property of Eilenberg-MacLane spaces implies that

the set of homotopy classes is just the cohomology group $H^{q_{k+1}+1}(\mathcal{X}^{(k-1)}; \pi_{k+1})$.⁴⁹ The classes are known as *Postnikov invariants*. For a π -finite space, the tower will terminate at some finite stage. The full tower \mathcal{X} has j^{th} homotopy group π_j . If all the Postnikov invariants vanish then \mathcal{X} is just $\prod_j K(\pi_j, q_j)$. Otherwise, it is a twisted product of Eilenberg-MacLane spaces, and in general is not homotopy equivalent to a direct product of Eilenberg-MacLane spaces.⁵⁰

As the simplest nontrivial example, a 2-stage Postnikov decomposition of the form

$$\begin{array}{ccc} K(A, 2) & \longrightarrow & \mathcal{X} \\ & & \downarrow \\ & & K(G, 1) \longrightarrow K(A, 3), \end{array} \quad (24.17)$$

where A is an Abelian group and G is a (possibly) non-Abelian group. The data that classifies \mathcal{X} is therefore G, A and an element of $H^3(G; A)$.⁵¹ These spaces – sometimes referred to as “two-groups” – are closely related to (non)-extensions of G by A and are important in the subject of “crossed modules” [Bro82]. A 2-stage Postnikov tower plays the role of a classifying space for crossed modules. In theories with “two-group symmetry,” the background fields will be homotopy classes of maps to (24.17).

More generally, just as H -spaces, and the $K(A, n)$ in particular, generalize the notion of groups, so too π -finite spaces generalize groups. Indeed they are sometimes referred to as “higher groups.” These spaces will be “target spaces” for the finite homotopy sigma models. Those theories are in turn quite useful in the “quiche picture” of finite symmetry in QFT described very briefly in Section 26 and in [FMT22]. In this sense these “higher groups” serve to generalize standard symmetry groups in quantum field theory.

24.2 A Correspondence Course

In general, a *correspondence* between two sets R_1 and R_2 is a space S and a pair of maps:

$$\begin{array}{ccc} & S & \\ f_1 \swarrow & & \searrow f_2 \\ R_1 & & R_2 \end{array} \quad (24.18)$$

⁴⁹In general the homotopy classes of maps $S \rightarrow K(A, n)$ form the Abelian group $H^n(S, A)$. This is part of the theory of spectra and generalized cohomology theories mentioned briefly in Section 50.

⁵⁰Two spaces can have the same homotopy groups and fail to be homotopy equivalent. The standard example is $\mathbb{S}^2 \times \mathbb{RP}^3$ and $\mathbb{RP}^2 \times \mathbb{S}^3$. Both are free \mathbb{Z}_2 quotients of $\mathbb{S}^2 \times \mathbb{S}^3$, so the homotopy groups can be computed by the standard long exact sequence and are the same. Other simple and interesting examples of this phenomenon are discussed in [Thu11]. A foundational theorem in algebraic topology, the Whitehead theorem, states that if a map $f : X \rightarrow Y$ between CW complexes induces an isomorphism of homotopy groups then X is homotopy equivalent to Y .

⁵¹In general, given a homomorphism $G \rightarrow \text{Aut}(A)$ one can define group cohomology with twisted A -coefficients. Thus, the Postnikov invariant also includes the data of this twisting.

It generalizes the notion of a function from R_1 to R_2 . In the case of a function F from R_1 to R_2 , S would be the graph of F and f_1 , and f_2 would be the projections to the domain and codomain respectively. To check things like the gluing axioms for the amplitudes we shall write below, we would like to be able to compose correspondences. That is, we want to go from two concatenated correspondences

$$\begin{array}{ccccc} & S_{12} & & S_{23} & \\ f_1 \swarrow & & \searrow f_{12} & & \swarrow f_{23} \\ R_1 & & R_2 & & R_3 \\ \end{array} \quad (24.19)$$

to construct a new correspondence based on a set S_{13} and maps g_1, g_2 , such that

$$\begin{array}{ccccc} & S_{13} & & S_{23} & \\ g_1 \swarrow & & \searrow g_2 & & \\ S_{12} & \xleftarrow{f_1} & R_2 & \xrightarrow{f_{23}} & S_{23} \\ f_1 \swarrow & \searrow f_{12} & & \swarrow f_{23} & \searrow f_3 \\ R_1 & & R_2 & & R_3 \\ \end{array} \quad (24.20)$$

Therefore, given

$$\begin{array}{ccc} S_{12} & & S_{23} \\ & \searrow f_{12} & \swarrow f_{23} \\ & R_2 & \end{array} \quad (24.21)$$

one wants to generate a correspondence between S_{12} and S_{23}

$$\begin{array}{ccc} & S_{13} & \\ g_1 \swarrow & & \searrow g_2 \\ S_{12} & & S_{23} \end{array} \quad (24.22)$$

so that we can form the diagram (24.20).

In the world of topological spaces and continuous maps, a natural way to complete concatenated correspondences nicely to a new correspondence is via the *homotopy fiber product*. Given continuous maps $f_1 : S_1 \rightarrow R$ and $f_2 : S_2 \rightarrow R$ between topological spaces, one defines the homotopy fiber product $S_1 \times_h S_2$ to be the set of triples

$$(s_1, s_2, \gamma) \in S_1 \times S_2 \times \text{Map}([0, 1] \rightarrow R), \quad (24.23)$$

where γ is a path in R (i.e., a homotopy) from $f_1(s_1)$ to $f_2(s_2)$.⁵² The subscript h stands for “homotopy” in the homotopy fiber product. The homotopy fiber product fits into the

⁵²The space of continuous maps $\text{Map}([0, 1] \rightarrow R)$ is given the compact-open topology. See footnote 31.

following diagram:

$$\begin{array}{ccc}
 & S_1 \times_h S_2 & \\
 p_1 \swarrow & & \searrow p_2 \\
 S_1 & & S_2 \\
 & f_1 \searrow & \swarrow f_2 \\
 & R &
 \end{array} \tag{24.24}$$

with p_1, p_2 the obvious projection maps. This can then be used to concatenate correspondences in a useful way.

The homotopy fiber product should be contrasted with the ordinary fiber product, where one demands equality $f_1(s_1) = f_2(s_2)$, that is,

$$S_1 \times_{f_1 f_2} S_2 = \{(s_1, s_2) \mid f_1(s_1) = f_2(s_2)\}. \tag{24.25}$$

The extra data of the homotopy γ in the homotopy fiber product gives a much bigger – but much more useful – space that behaves better than the ordinary fiber product in some standard constructions in algebraic topology. Since there can be different paths in R from $f_1(s_1)$ to $f_2(s_2)$, the homotopy fiber product can have some interesting topology. For example, if $S_1 = S_2 = \{r_0\}$, where r_0 is an element of R , the homotopy fiber product is just the loop space of R based at r_0 .

The homotopy fiber product is closely related to a standard manipulation in algebraic topology that associates a fibration to any continuous map $f : S \rightarrow R$. See, e.g., [BT82, p. 249]. One defines the space,

$$E_f := \{(s, \gamma)\} \subset S \times \text{Map}([0, 1] \rightarrow R), \tag{24.26}$$

such that $\gamma(0) = f(s)$. It is not difficult to show that if $S \subset R$, then E_f is homotopic to S . Then, the fibration associated to the map $f : S \rightarrow R$ is the fibration: $\pi : E_f \rightarrow R$ defined by $\pi(s, \gamma) = \gamma(1)$. The fiber of this map above a fixed point $r \in R$ – known as the *homotopy fiber of f* – is the set of pairs (s, γ) so that γ is a path from $f(s)$ to r . It therefore only changes by homotopy equivalence as r is varied within a path-connected component of R . Therefore the homotopy fiber is only defined up to homotopy equivalence. Since $\pi : E_f \rightarrow R$ is a fibration, there is a long exact sequence of homotopy groups. This is an example of how the homotopy fiber behaves well in homotopy theory. Finally, note that if we take $S_1 = S$ and $S_2 = \{r\} \in R$ in the definition of the homotopy fiber product, we reproduce the homotopy fiber above r of a map $f : S \rightarrow R$.

Example: Consider $S = [0, \pi]$, $R = \mathbb{U}(1)$ and the map $f : [0, \pi] \rightarrow \mathbb{U}(1)$ given by $f(x) = e^{ix}$. The image of the map is the upper hemisphere of the unit circle in the complex plane. The ordinary fiber of f above a point in the unit circle is a single point $s \in S$, determined by

the principal value of the logarithm, for the fiber above a point r in the upper hemisphere, but is the empty set above points r in the lower hemisphere. This discontinuous behavior will wreak havoc on constructions in algebraic topology relying on continuity. By contrast, the homotopy fiber above any point $r \in U(1)$ is the set of pairs (s, γ) where $s \in [0, \pi]$ and γ is a continuous path in $U(1)$ from e^{is} to r . This set of pairs is always homotopic to the space of paths between two points on the circle so the homotopy fiber above r varies continuously with r . Notice that in this example, the homotopy fiber has infinitely many connected components labeled by winding number and is thus topologically interesting.

Exercise 24.1: Correspondences and principal G -bundles

For topological spaces X and Y , let Y^X be the space of maps from X to Y equipped with the compact-open topology. Let T be the two-dimensional torus, and slice that torus into two pieces T_L and T_R both diffeomorphic to cylinders. The left and right parts are separated by S which is a disjoint union of circles. Consider the concatenation of correspondences:

$$\begin{array}{ccccc} & & \mathcal{X}^T & & \\ & \swarrow g_1 & & \searrow g_2 & \\ \mathcal{X}^{\emptyset} & & \mathcal{X}^{T_L} & & \mathcal{X}^{T_R} \\ f_1 \swarrow & & \searrow f_{12} & & \swarrow f_{23} \\ & & \mathcal{X}^S & & \\ & & \searrow f_3 & & \\ & & \mathcal{X}^{\emptyset} & & \end{array} \quad (24.27)$$

- (a) Work through the natural maps in this correspondence and verify that the diagram commutes.
- (b) Let $\mathcal{X} = BG$. Explain how the correspondence (24.27) describes the construction of a principal G -bundle over the torus from gluing two principal G -bundles over two cylinders.

24.3 Statespaces And Amplitudes For The Finite Homotopy Sigma Models

The data needed to define a finite homotopy sigma model consists of:

1. A nonnegative integer n , the spacetime dimension.
2. A π -finite space \mathcal{X} . (See Section 24.1.)
3. A cocycle $\lambda \in Z^n(\mathcal{X}; \mathbb{C}^\times)$.⁵³

⁵³The isomorphism class of the theory only depends on the cohomology class of λ .

4. A symmetric monoidal n -category \mathcal{C} . We will assume that $\Omega^n \mathcal{C} = \mathbb{C}$ and $\Omega^{n-1} \mathcal{C} = \text{VECT}$.

We denote the resulting fully local theory by $\sigma_{\mathcal{X}, \lambda, \mathcal{C}}^{(n)}$. An important point about these theories is that their “quantization” always involves some version of a finite path integral over some “dynamical fields.” The functorial description of field theories we have described thus far involves a description of “the answer” to the path integral, and its expected properties, without concerning ourselves with how path integrals are to be done. But in the homotopy sigma models, we do have a notion of what the “dynamical fields” are. The homotopy models should be thought of as “sigma models” where the fields are only distinguished up to homotopy – hence the name. Thus there will be a description by “dynamical fields.” (In [FMT22] the dynamical fields are called “semiclassical data.”) A useful notation for what follows is the following: For any manifold M_n , let \mathcal{X}^{M_n} be the space of all continuous maps $M_n \rightarrow \mathcal{X}$. The “finite path integral” – at every categorical level – will be considered to be some kind of “quantization” of mapping spaces of the form \mathcal{X}^{M_n} . The exact meaning of this “quantization” is somewhat involved and can be found in [FHLT09] and the upcoming work of Scheimbauer and Walde [SWnd]. (For a telegraphic description, see equation (A.8) of [FMT22].)

Let us begin by describing the spaces of states on closed spatial manifolds N_{n-1} . In order to motivate the definition consider the standard quantum field theory of an n -dimensional sigma model with target space X . Typically, we take X to be a finite-dimensional Riemannian manifold. If N_{n-1} is a closed spatial manifold, the sigma model fields at fixed time are elements of the mapping space $X^{N_{n-1}}$. If N_{n-1} and X are endowed with Riemannian metrics, the mapping space inherits a Riemannian structure: A tangent vector α to a sigma model field $\phi \in X^{N_{n-1}}$ is a section of $\phi^*(TX)$, and $\|\alpha\|^2 = \int_{N_{n-1}} |\alpha|^2 \text{ vol}$. The Hilbert space of the theory on N_{n-1} is morally something like $L^2(X^{N_{n-1}})$. States would be derived from normalizable wavefunctionals $\Psi[\phi(x)]$ for $\phi \in \mathcal{X}^{N_{n-1}}$. In the TFT setting, with target space \mathcal{X} , the fields are only considered up to homotopy and so our Hilbert space should consist of wavefunctions defined on the set of homotopy classes of maps $N_{n-1} \rightarrow \mathcal{X}$. If \mathcal{X} is a π -finite space and N_{n-1} is closed, the set of homotopy classes of maps will be a finite set and the Hilbert space will be finite dimensional.⁵⁴ The set of homotopy classes of “sigma model field configurations” is just $\pi_0(\mathcal{X}^{N_{n-1}})$, and so we have the definition of the statespace:

$$\sigma_{\mathcal{X}, \mathcal{C}}^{(n)}(N_{n-1}) := \text{Fun}(\pi_0(\mathcal{X}^{N_{n-1}})) . \quad (24.28)$$

where the notation on the right-hand side indicates the vector space of complex-valued functions.

Remark 24.29. In equation (24.28), we have taken λ to be a trivial cocycle. When it

⁵⁴Note that in a homotopy sigma model, \mathcal{X} need not be finite dimensional and it is typically not finite dimensional.

is nontrivial, the data of λ defines a line bundle $L_\lambda \rightarrow \mathcal{X}^{N_{n-1}}$, and instead of a space of functions, one considers a space of sections up to homotopy. For simplicity, we will generally take λ to be trivial, and just denote the theory by $\sigma_{\mathcal{X},c}^{(n)}$.

Examples:

- (1) Let G be a finite group (not necessarily Abelian) and take $\mathcal{X} = K(G, 1) = BG$. The finite homotopy sigma model $\sigma_{\mathcal{X},c}^{(n)}$ is just the finite group gauge theory studied in [Section 14](#). In this case, we have:

$$\pi_0(\mathcal{X}^{N_{n-1}}) = \left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{principal } G\text{-bundles} \\ \text{over spatial manifold } N_{n-1} \end{array} \right\}. \quad (24.30)$$

and equation (24.28) reduces to (14.2). If we consider a nontrivial cocycle λ , then the finite gauge theory is generalized to Dijkgraaf-Witten theory.

- (2) Let A be a finite Abelian group, and take $\mathcal{X} = K(A, p)$. Then, as noted in footnote 49, $\pi_0(\mathcal{X}^{N_{n-1}}) \cong H^p(N_{n-1}; A)$. In this case, one can think of \mathcal{X} as classifying a “higher” notion of principal bundle sometimes called a “ p -principal bundle.” See [BM96, Bun12]. So the “space of states of $\sigma_{\mathcal{X},c}^{(n)}$ ” on the spatial manifold N_{n-1} is the vector space of functions from the finite Abelian group $H^p(N_{n-1}; A)$ to the complex numbers.

We leave it as a challenge to the reader to work out similarly explicit formulations of the space of states when the target space is a higher group.

Now, in order to define amplitudes associated with a bordism (see [Figure 32](#)), we form

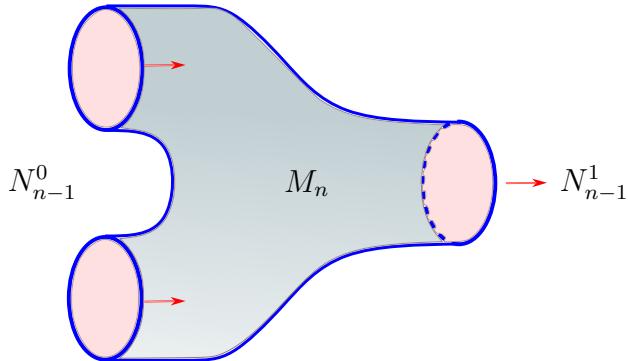


Figure 32: A bordism.

a correspondence:

$$\begin{array}{ccc} & \mathcal{X}^{M_n} & \\ p_0 \swarrow & & \searrow p_1 \\ \mathcal{X}^{N_{n-1}^0} & & \mathcal{X}^{N_{n-1}^1} \end{array} \quad (24.31)$$

where p_0 and p_1 are given by restricting the field $\phi \in \mathcal{X}^{M_n}$ to the in- and out- boundaries. The idea is that the linear map,

$$\sigma_{\mathcal{X}, \mathcal{C}}^{(n)}(M_n) : \sigma_{\mathcal{X}, \mathcal{C}}^{(n)}(N_{n-1}^0) \rightarrow \sigma_{\mathcal{X}, \mathcal{C}}^{(n)}(N_{n-1}^1) , \quad (24.32)$$

is given by the pullback and pushforward:

$$\sigma_{\mathcal{X}, \mathcal{C}}^{(n)}(M_n) := (p_{1,*}) \circ p_0^* , \quad (24.33)$$

The map p_0^* is straightforward pullback:

$$p_0^* : \text{Fun}(\pi_0(\mathcal{X}^{N_{n-1}^0})) \rightarrow \text{Fun}(\pi_0(\mathcal{X}^{M_n})) , \quad (24.34)$$

by the standard equation $p_0^*(F)(\phi) = F(p_0(\phi))$. Here $\phi : M_n \rightarrow \mathcal{X}$ is a “sigma model map” on the bulk space M_n , and F is a locally constant functional on the sigma model maps on the boundary. Note that evaluation of the pullback function $\pi_0^*(F)$ only involves the values of F on sigma model maps on the boundary that extend to the bulk.

The map,

$$p_{1,*} : \text{Fun}(\pi_0(\mathcal{X}^{M_n})) \rightarrow \text{Fun}(\pi_0(\mathcal{X}^{N_{n-1}^1})) , \quad (24.35)$$

is more involved. It uses the “homotopy cardinality.” If Ψ is a locally constant function on $\pi_0(\mathcal{X}^{M_n})$, then the pushforward function is defined by its values:

$$p_{1,*}(\Psi)(\phi_\partial) := \sum_{[\phi] \in \pi_0(p_1^{-1}(\phi_\partial))} \left(\prod_{i=1}^{\infty} |\pi_i(p_1^{-1}(\phi_\partial), \phi)|^{(-1)^i} \right) \Psi(\phi) . \quad (24.36)$$

Here $\phi_\partial : N_{n-1}^1 \rightarrow \mathcal{X}$ is a sigma model field on the outgoing boundary. We are defining a locally constant function $p_{1,*}(\Psi)$ on these sigma model fields. To evaluate it, we consider the fiber $p_1^{-1}(\phi_\partial)$, that is, the sigma model fields $\phi : M_n \rightarrow \mathcal{X}$ that extend ϕ_∂ to a field on the bulk. For each connected component of the set $p_1^{-1}(\phi_\partial)$, we choose a representative ϕ in that component and evaluate the locally constant function Ψ on that ϕ , and weight the result by the homotopy cardinality – that is, by the alternating product of orders of homotopy groups for that component.

Using properties of the homotopy fiber product described in [Section 24.2](#), one can check the crucial gluing properties. (See comments between (A.12) and (A.13) of [\[FMT22\]](#) and references therein. See also [\[MP23\]](#).) It is essential that one uses the homotopy cardinality for this to work.

Remark 24.37. Taking $N^0 = N^1 = \emptyset$ gives the partition function on a compact n -manifold

without boundary as a corollary:

$$\sigma_{\mathcal{X}, \mathcal{C}}^{(n)}(M_n) = \sum_{[\phi] \in \pi_0(\mathcal{X}^{M_n})} \left(\prod_{i=1}^{\infty} |\pi_i(\mathcal{X}^{M_n}, \phi)|^{(-1)^i} \right). \quad (24.38)$$

If \mathcal{X} is π -finite, the factors in the infinite product eventually all become equal to 1. The alternating product of these factors – the homotopy cardinality – is a manifestation of the “ghosts for ghosts” phenomenon discussed later in the case of generalized Abelian gauge theories in [Section 48](#).

Examples:

1. For finite G gauge theory (with G not necessarily Abelian) with $\mathcal{X} = K(G, 1)$, the partition function (14.1) is a special case of (24.38). The sum over $\pi_0(\mathcal{X}^{M_n})$ is a sum over the principal G -bundles over M_n . The weighting factor accounting for automorphisms is only nonzero for π_1 and $|\pi_1(K(G, 1)^{M_n})| = |G|$ when M_n is connected. These are just the “global gauge transformations,” so we are correctly dividing by the “order of the gauge group.”
2. For the “higher” forms of bundles with $\mathcal{X} = K(A, p)$, $p > 1$, the partition function is also sum over isomorphism classes of “higher” bundles weighted by automorphisms. The bundles are classified by,

$$\pi_0(K(A, p)^{M_n}) \cong H^p(M_n; A). \quad (24.39)$$

Moreover, because the Eilenberg-MacLane spaces form a spectrum, we have

$$\begin{aligned} \pi_\ell(K(A, p)^{M_n}) &= [S^\ell M_n, K(A, p)] \\ &= [M_n, \Omega^\ell K(A, p)] = [M_n, K(A, p - \ell)] \\ &= H^{p-\ell}(M_n; A). \end{aligned} \quad (24.40)$$

So,

$$\begin{aligned} \sigma_{\mathcal{X}, \mathcal{C}}^{(n)}(M_n) &= \sum_{H^p(M_n; A)} \frac{1}{|H^{p-1}(M_n; A)|} \cdot \frac{|H^{p-2}(M_n; A)|}{|H^{p-3}(M_n; A)|} \dots \\ &= \prod_{j=0}^p |H^{p-j}(M_n; A)|^{(-1)^j}. \end{aligned} \quad (24.41)$$

Hi Exercise 24.2: Convolution product from Finite Homotopy σ Models

Derive equation (14.8) as a special case of (24.33) and (24.36).

Let us go to the next categorical level, that is, we consider the value of the field theory on closed manifolds of dimension M_{n-2} . In this case, we need to know more about the codomain category \mathcal{C} . Among the many choices for $\Omega^{n-2}\mathcal{C}$, two natural choices are often adopted in the literature:

1. $\Omega^{n-2}\mathcal{C} = \text{ALG(VECT)}$.
2. $\Omega^{n-2}\mathcal{C} = \text{CAT}$.

The answers are expressed in terms of the finite groupoid $\pi_{\leq 1}(\mathcal{X}^{M_{n-2}})$ whose objects are the points of $\mathcal{X}^{M_{n-2}}$ (i.e., maps $M_{n-2} \rightarrow \mathcal{X}$) and whose morphisms are paths between points.

When $\Omega^{n-2}\mathcal{C} = \text{CAT}$ is the 2-category of linear categories then $\sigma_{\mathcal{X}, \mathcal{C}}^{(n)}(M_{n-2})$ is the category of “locally constant” vector bundles over $\mathcal{X}^{M_{n-2}}$. Recall that we only consider the maps in $\mathcal{X}^{M_{n-2}}$ and paths of maps up to homotopy. This implies the local constancy. Thus, we get the category of locally constant vector bundles over the groupoid $\pi_{\leq 1}(\mathcal{X}^{M_{n-2}})$. More concretely, locally constant bundles are vector bundles over $\pi_0(\mathcal{X}^{M_{n-2}})$ with an action of $\pi_1(\mathcal{X}^{M_{n-2}}, \phi_0)$, where, at the cost of some naturalness, we have chosen a basepoint ϕ_0 in each component.

When $\Omega^{n-2}\mathcal{C} = \text{ALG(VECT)}$, we consider the groupoid path algebra on $\pi_{\leq 1}(\mathcal{X}^{M_{n-2}})$. Paths generate the algebra and the multiplication is nonzero only when paths compose, in which case the product is the composed path. So we write:

$$\sigma_{\mathcal{X}, \mathcal{C}}^{(n)}(M_{n-2}) = \text{Alg}(\pi_{\leq 1}(\mathcal{X}^{M_{n-2}})) , \quad (24.42)$$

(where Alg denotes a specific algebra in the category ALG). This algebra can be expressed (noncanonically) as a direct sum of group algebras:

$$\text{Alg}(\pi_{\leq 1}(\mathcal{X}^{M_{n-2}})) = \bigoplus_{[\phi_0] \in \pi_0(\mathcal{X}^{M_{n-2}})} \mathbb{C}[\pi_1(\mathcal{X}^{M_{n-2}}, \phi_0)] . \quad (24.43)$$

Again, this presentation is slightly noncanonical because we chose a basepoint ϕ_0 in each connected component of $\mathcal{X}^{M_{n-2}}$.

It is worth unpacking the above answers a bit more in the case where $\mathcal{X} = BG$. Note that $BG^{M_{n-2}} = \text{Bun}_G(M_{n-2})$ is the moduli stack of principal G -bundles over M_{n-2} .

1. If we take $\Omega^{n-2}\mathcal{C} = \text{CAT}$, then the category $\sigma_{BG, \mathcal{C}}^{(n)}(M_{n-2})$ is the category of vector bundles over $\pi_0(\text{Bun}_G(M_{n-2}))$ together with an action of $\text{Aut}(P_x)$ on the fibers over P_x , where x runs over the finite set $\pi_0(\text{Bun}_G(M_{n-2}))$. Specializing further to $\sigma_{BG}^{(2)}$, we

have $\mathbf{Bun}_G(\text{pt}) = \text{pt}/\!/G$, and $\pi_1 \cong G$, so the vector bundle is just a G -representation, and we get $\mathbf{Rep}(G)$, thus recovering (23.11) above. On the other hand, for $\sigma_{BG}^{(3)}$, if $M_{n-2} = \mathbb{S}^1$, then

$$\sigma_{BG}^{(3)}(\mathbb{S}^1) = \mathbf{Vect}(\pi_{\leq 1}(BG^{\mathbb{S}^1})) = \mathbf{Vect}(G/\!/G) := \mathbf{Vect}_G(G) , \quad (24.44)$$

where the G action on G is by conjugation. Generating objects in this category consist of a group element g and a representation of the centralizer of g . If we include a cocycle λ , then vector bundles are replaced by twisted vector bundles.

2. The algebra is therefore,

$$\sigma_{BG,C}^{(n)}(M_{n-2}) = \bigoplus_{[P] \in \pi_0(\mathbf{Bun}_G(M_{n-2}))} \mathbb{C}[\pi_1(\mathbf{Bun}_G(M_{n-2}), P)] , \quad (24.45)$$

specializing this further to $n = 2$ dimensions we have $\mathbf{Bun}_G(\text{pt}) = \text{pt}/\!/G$ and

$$\pi_1(\mathbf{Bun}_G(M_{n-2}), P) \cong G , \quad (24.46)$$

so we recover $\sigma_{BG}^{(2)}(\text{pt}) = \mathbb{C}[G]$ quoted in (23.11) above. If we include a cocycle λ then the group algebra products are twisted.

Let us see how the statespace of M_{n-1} “factorizes” when divided into two pieces by cutting along M_{n-2} . In this discussion we take $\Omega^{n-2}\mathcal{C} = \mathbf{CAT}$. It is useful first to rederive (24.28) from the correspondence:

$$\begin{array}{ccc} & \mathcal{X}^{M_{n-1}} & \\ p_0 \swarrow & & \searrow p_1 \\ \mathcal{X}^{\emptyset_{n-2}} & & \mathcal{X}^{\emptyset_{n-2}} \end{array} \quad (24.47)$$

The evaluation of the field theory on the bordism $M_{n-1} : \emptyset_{n-2} \rightarrow \emptyset_{n-2}$ is an object in $\mathbf{Hom}(1_{\Omega^{n-2}\mathcal{C}}, 1_{\Omega^{n-2}\mathcal{C}}) = \mathbf{VECT}$, that is, it is a vector space. The formula for the vector space is $p_{1,*} \circ p_0^*$ applied to the generating vector bundle over $\mathcal{X}^{\emptyset_{n-2}}$. That is a vector bundle over a point, so we wish to evaluate $p_{1,*} \circ p_0^*(\mathbb{C})$. The pull-back $p_0^*(\mathbb{C})$ gives the rank-one vector bundle which is the trivial rank-one bundle over each component of $\mathcal{X}^{M_{n-1}}$. The pushforward gives a vector space. It is the sum of \mathbb{C} over the components $\pi_0(\mathcal{X}^{M_{n-1}})$. A function on $\pi_0(\mathcal{X}^{M_{n-1}})$ gives a complex number for each connected component of $\mathcal{X}^{M_{n-1}}$ and hence the pushforward bundle is naturally identified with the statespace (24.28).

Now suppose that we can write M_{n-1} by gluing together two bordisms $M_{n-1}^0 : \emptyset_{n-2} \rightarrow M_{n-2}$ and $M_{n-1}^1 : M_{n-2} \rightarrow \emptyset_{n-2}$. The vector space we get is obtained by composing the

push-pull morphisms in:

$$\begin{array}{ccccc}
 & \mathcal{X}^{M_{n-1}^1} & & \mathcal{X}^{M_{n-1}^2} & \\
 p \swarrow & & \searrow r_L & & \swarrow r_R & \searrow p \\
 \mathcal{X}^{\emptyset_{n-2}} & & \mathcal{X}^{M_{n-2}} & & \mathcal{X}^{\emptyset_{n-2}}
 \end{array} \tag{24.48}$$

That is, the vector space is:

$$(p_* \circ r_R^*) \circ (r_{L,*} \circ p^*)(\mathbb{C}) . \tag{24.49}$$

This makes sense: $p^*(\mathbb{C})$ is the trivial bundle over $\mathcal{X}^{M_{n-1}^1}$ and $r_{L,*}$ will push it forward to those maps in $\mathcal{X}^{M_{n-2}^1}$ which can be restrictions from maps on M_{n-1}^1 . Then r_R pulls back those maps to those which are restrictions of maps M_{n-1}^2 . Altogether we get a trivial bundle over the components of maps from M_{n-1} .

If instead $\Omega^{n-2}\mathcal{C} = \text{ALG}(\text{VECT})$, then $F(M_{n-2})$ is an algebra, $F(M_{n-1}^1)$ is a left-module, and $F(M_{n-1}^2)$ is a right-module for the algebra and the vector space factorizes as:

$$F(M_{n-1}) = F(M_{n-1}^1) \otimes_{F(M_{n-2})} F(M_{n-1}^2) . \tag{24.50}$$

The modules over the path algebra of the groupoid $\pi_{\leq 1}(\mathcal{X}^{M_{n-2}})$ are bundles over that groupoid, so the discussion is similar to the previous case.

Alternatively, one can choose not to think, and just rely on the formal theory of correspondences.

We will not continue to higher codimension systematically. Such a systematic procedure is given in the literature for proceeding to higher categorical levels. Here we simply mention one very important special case: Consider finite group gauge theory in 3 dimensions. In this case one must choose a 3-category \mathcal{C} for the codomain. A natural choice is the 3-category of tensor categories. Tensor categories are algebra objects in the 2-category of categories. The canonical procedure produces

$$\sigma_{BG,\mathcal{C}}^{(3)}(\text{pt}) = \text{Vect}[G] , \tag{24.51}$$

where $\text{Vect}[G]$, the category of vector bundles over G , is turned into a tensor category as follows: For G finite, a vector bundle over G amounts to a choice of a finite-dimensional vector space V_g for each $g \in G$. Then the tensor product of two such vector spaces is defined by the convolution product:

$$(V_1 \otimes V_2)_g \cong \bigoplus_{\substack{(g_1, g_2) \\ g = g_1 g_2}} V_{1,g_1} \otimes V_{2,g_2} , \tag{24.52}$$

where the direct sum on the right hand side is over pairs (g_1, g_2) such that $g = g_1 g_2$. If we include a twisting by $\lambda \in Z^3(BG; \mathbb{C}^\times)$, then it is shown in [FHLT09, Sec. 4.1] that λ is equivalent to a choice of complex line bundle over $L \rightarrow G \times G$, and (24.52) is replaced by a

twisted convolution product:

$$(V_1 \otimes V_2)_g \cong \bigoplus_{\substack{(g_1, g_2) \\ g = g_1 g_2}} L_{g_1, g_2} \otimes V_{1, g_1} \otimes V_{2, g_2}. \quad (24.53)$$

Remark 24.54. Electromagnetic Duality. There is a finite group analog of the electromagnetic duality discussed in more detail in Part II of these lectures. The theory $\sigma_{\mathcal{X}, \mathcal{C}}^{(n)}$ for $\mathcal{X} = B^p A$ is equivalent to the theory with $\mathcal{X} = B^{n-p-1} A^\vee$, where A^\vee is the Pontryagin dual group $A^\vee = \text{Hom}(A, \text{U}(1))$, at least, up to a product with an invertible Euler theory. The need for the Euler theory can already be seen by considering the partition function (24.41), since it shows that for M_n closed⁵⁵

$$\frac{\sigma_{K(A, p), \mathcal{C}}^{(n)}(M_n)}{\sigma_{K(A^\vee, n-p-1), \mathcal{C}}^{(n)}(M_n)} = (|A|^{\chi(M_n)})^{(-1)^p}. \quad (24.58)$$

The paper [Liu21] shows that, in fact, for the fully extended TFTs, we have:

$$\sigma_{K(A, p), \mathcal{C}}^{(n)} \cong \mathcal{E}_\theta \otimes \sigma_{K(A^\vee, n-p-1), \mathcal{C}}^{(n)}, \quad (24.59)$$

where \mathcal{E}_θ is the Euler theory introduced in (10.4) with $\theta = |A|^{(-1)^p}$. This electric-magnetic duality is closely related to that of the BF theories studied in Section 52 below. Moreover, the duality has an elegant extension to arbitrary finite homotopy theories. See [Liu21] for the details.

25 Defects

25.1 General Remarks On Defects

An important further generalization of the notion of field theory we have thus far explained is the inclusion of *defects*. Intuitively, these are modifications of a theory obtained by inserting

⁵⁵To prove (24.58), note first that the perfect pairing,

$$H^{n-j}(M_n; A^\vee) \times H^j(M_n; A) \rightarrow \text{U}(1), \quad (24.55)$$

implies that $|H^{n-j}(M_n; A^\vee)| = |H^j(M_n; A)|$. Therefore, we can write:

$$\sigma_{K(A^\vee, n-p-1), \mathcal{C}}^{(n)}(M_n) = |H^{p+1}(M_n; A)| |H^{p+2}(M_n; A)|^{-1} \dots \quad (24.56)$$

Now, we claim that:

$$\prod_{j=0}^n |H^j(M_n; A)|^{(-1)^j} = |A|^{\chi(M_n)}. \quad (24.57)$$

This is easily proved by using a finite chain complex to model the cohomology of M_n (e.g., by use of a finite CW decomposition) and invoking the Euler-Poincaré principle to see that $|C^j(M_n) \otimes A| = |A|^{\text{rank } C^j(M_n)}$.

(possibly nonlocal) observables or by enforcing (dynamical) fields to have certain singularities. If \mathcal{D} is the defect, then the modified theory $F_{\mathcal{D}}$ will have new partition functions, statespaces, etc., on closed manifolds.

A simple example of a defect in standard Lagrangian field theory is obtained by inserting a local field at a point in spacetime. Another famous class of examples in gauge theory are the *Wilson defects*. These are defined by choosing a representation R of the gauge group and a closed curve γ in spacetime. Denote the defect by $\mathcal{D} = W_{R,\gamma}$. In terms of the path integral of Yang-Mills theory, the modification by \mathcal{D} changes the measure in the Yang-Mills path integral by multiplying by the character in the representation R of the holonomy around γ . The path integral of Yang-Mills theory associated to a principal bundle $P \rightarrow M_n$ can be written as:

$$F^{\text{YM}}(P \rightarrow M_n; g_{\mu\nu}) = \int_{\mathcal{A}(P)/\mathcal{G}(P)} e^{-\int_{M_n} \text{Tr} F \wedge \star F}, \quad (25.1)$$

where $\mathcal{A}(P)/\mathcal{G}(P)$ is the (stack) of isomorphism classes of connections⁵⁶ on P , F is the fieldstrength of a connection, Tr is a positive definite Ad-invariant bilinear form on the Lie algebra (whose normalization defines the Yang-Mills coupling constant), and \star is the Hodge star. The modification of the path integral for the defect $\mathcal{D} = W_{R,\gamma}$ is simply:

$$F_{\mathcal{D}}^{\text{YM}}(P \rightarrow M_n; g_{\mu\nu}) = \int_{\mathcal{A}(P)/\mathcal{G}(P)} e^{-\int_{M_n} \text{Tr} F \wedge \star F} \text{Tr}_R \text{Hol}(\nabla, \gamma). \quad (25.2)$$

In general, a defect has a support on a subset \mathcal{Z} in a spacetime. For local fields, the defect would be supported on the spacetime points where the local fields are included in the path integral. For a Wilson defect, the support would be along γ , etc. The support \mathcal{Z} need not be smooth – it could be a stratified space, such as that illustrated in Figure 33.

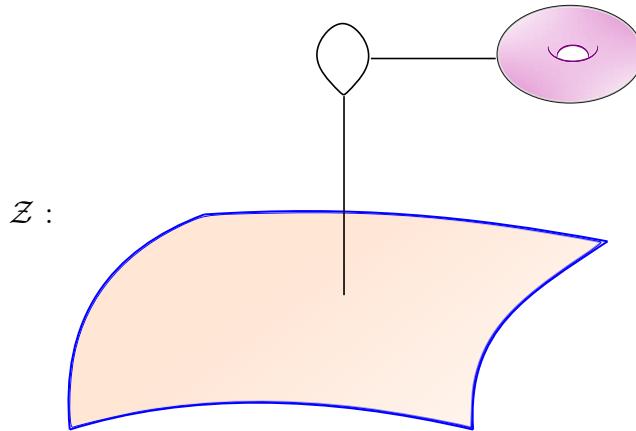


Figure 33: A stratified space.

⁵⁶Here and below, we will use the notation $\mathcal{A}(P)$ for the affine space of gauge fields (connections) on a principal G -bundle $P \rightarrow M_n$. See Section 36 below for a discussion of the meaning of connections. The group of automorphisms of P covering the identity on M_n , i.e., $\text{Aut}(P)$, is known in physics as the group of gauge transformations, denoted $\mathcal{G}(P)$. The quotient (stack) $\mathcal{A}(P)/\mathcal{G}(P)$ is the “space” of gauge-inequivalent Yang-Mills field configurations.

When working with defects whose support is a stratified space, one first describes defects whose support \mathcal{Z} is a manifold, then for the stratified space, one pieces them together working up in codimension. The various strata of the defects carry tangential structures, and the issue of which tangential structures are allowed is complex. For example, one could have oriented defects in unoriented theories, or the other way around. A general discussion can be found in [FMT22, Sec. 2.5]. For some recent progress, see [Mü125].

In the case of topological field theories, we study *topological defects*. The new amplitudes with \mathcal{D} only depend on the isotopy class of the support \mathcal{Z} of the defect. In fully extended field theory, there are two ways of labeling defects using *local labels* and *global labels*. These are terms introduced in [FMT22].

To describe the global label of a defect \mathcal{D} with connected support \mathcal{Z} , we consider a neighborhood $\mathcal{U}_{\mathcal{Z}}$ of \mathcal{Z} inside spacetime which, moreover, does not intersect any other defects. The boundary of $\mathcal{U}_{\mathcal{Z}}$ is a closed $(n - 1)$ -dimensional manifold, and the bordism $\emptyset \rightarrow \partial\mathcal{U}_{\mathcal{Z}}$ creates a vector $\psi_{\mathcal{D}} \in F(\mathcal{U}_{\mathcal{Z}})$. In a topological theory, $\psi_{\mathcal{D}}$ will not depend on the precise choice of $\mathcal{U}_{\mathcal{Z}}$. This is a very natural way to describe defects: Surround them by a small neighborhood, and describe them by the kind of state created on the boundary of that neighborhood.⁵⁷ The vector $\psi_{\mathcal{D}}$ is the *global label* of the defect \mathcal{D} . One can use the global label to define the partition function $F_{\mathcal{D}}(M_n)$ on a closed n -dimensional manifold in the presence of the defect \mathcal{D} as follows: The complement $W_n = M_n - \mathcal{U}_{\mathcal{Z}}$ can be used to construct a bordism $W_n : \partial\mathcal{U}_{\mathcal{Z}} \rightarrow \emptyset$. Therefore, $F(W_n)$ is a linear functional on the vector space $F(\partial\mathcal{U}_{\mathcal{Z}})$. Then we say,

$$F_{\mathcal{D}}(M_n) := \langle F(W_n), \psi_{\mathcal{D}} \rangle . \quad (25.3)$$

For the finite homotopy sigma models $\sigma_{\chi,c}^{(n)}$, one can evaluate $F_{\mathcal{D}}$ fairly explicitly using (24.33) with $N_{n-1}^0 = \partial\mathcal{U}_{\mathcal{Z}}$ and $N_{n-1}^1 = \emptyset$. One can extend (25.3) to define $F_{\mathcal{D}}$ on manifolds with corners of lower dimension.

⁵⁷In a nontopological theory, one would wish to choose some kind of inverse limit on a system of neighborhoods shrinking to the support \mathcal{Z} , along the lines of equation (28.9) below.

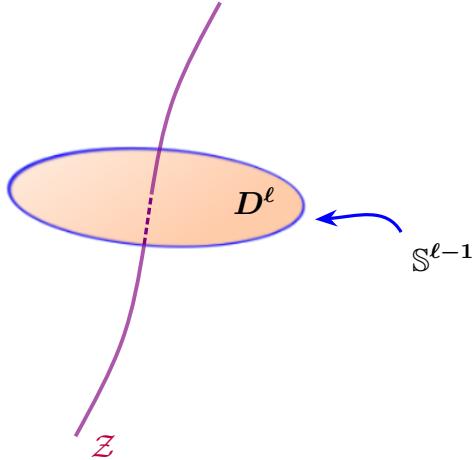


Figure 34: A linking sphere.

In the spirit of the cobordism hypothesis, one would wish to have a more local description of defects. Let us now assume that \mathcal{Z} is smooth and of codimension ℓ , as in Figure 34. Then there is a disk bundle ν of $\mathcal{Z} \subset M_n$ with fibers diffeomorphic to the closed disk D^ℓ . The boundary of a fiber above a point $p \in \mathcal{Z}$ is the linking sphere at p . It is diffeomorphic to $S^{\ell-1}$ as in Figure 34. The definition of the local label at p is motivated by (the TFT version of) Kaluza-Klein reduction. The field theory $\mathbf{KK}(F)$ defined by $\mathbf{KK}(F)(M_n) = F(M_n \times S^{\ell-1})$ is an $n - (\ell - 1)$ -dimensional field theory. Taking the radial direction to be in the interval $r \in [0, \epsilon]$, the effect of the defect is to define a boundary theory $\mathcal{B}(\mathcal{D})$ in the $n - (\ell - 1)$ -dimensional theory $\mathbf{KK}(F)$. Therefore,

$$\mathbf{KK}(F)([0, \epsilon] \times \mathcal{Z}) , \quad (25.4)$$

in the presence of the boundary $\mathcal{B}(\mathcal{D})$ at $r = 0$, is an object in:

$$\mathbf{Hom}(\mathbf{KK}(F)(\emptyset), \mathbf{KK}(F)(\mathcal{Z})) . \quad (25.5)$$

Recalling the definition of $\mathbf{KK}(F)$, we are motivated to define the *local label* of \mathcal{D} to be the morphism,

$$\psi_{\mathcal{D}, p} \in \mathbf{Hom}(1_{\Omega^{\ell-1}\mathcal{C}}, F(\partial\nu_p)) , \quad (25.6)$$

determined by (25.5). This is the definition found in [FMT22].

As in the cobordism hypothesis, one would like to deduce the global label, and indeed the entire functor $F_{\mathcal{D}}$ from the local labels $\psi_{\mathcal{D}, p}$. This is, in general, a highly nontrivial task, especially when one takes into account background fields. This has yet to be done in all its glory. But a very detailed sketch of how it should be done can be found in section 2.5 of [FMT22]. The story is summarized in figure 8 of that paper.

When defects are “in parallel,” they satisfy an analog of the operator product expansion. Thus, there is a whole calculus of defects, and in some cases, these can result in very

interesting algebraic structures. Some examples include:

1. Multiplication of Wilson/'t Hooft line defects in GL-twisted $d = 4$, $\mathcal{N} = 4$ super-Yang-Mills (reproducing results in Geometric Langlands (GL) theory) [KW06, Wit09, Wit15a].
2. Multiplication of defects in Class \mathcal{S} theories [GMN10, NY20, GMNY23].
3. Multiplication of interfaces between $d = 2$ QFTs with $\mathcal{N} = (2, 2)$ supersymmetry [GMW15a, GMW15b, KM20, KM24].
4. In very interesting work, M. Dedushenko and N. Nekrasov have described how algebras derived from multiplication of supersymmetric domain walls include interesting quantum algebras such as Yangians, and generalizations thereof. See [DN21, DN23].

In the next section, we specialize to the finite homotopy theories and briefly describe defects in this context.

25.2 Defects And Domain Walls In The Theories $\sigma_{\mathcal{X}, \mathcal{C}}^{(n)}$

As we have mentioned, the finite homotopy theories introduce notions of dynamical fields. As a consequence, the reference [FMT22] defines three new kinds of labels for defects in these theories:

1. Classical labels.
2. Semiclassical local labels.
3. Semiclassical global labels.

The classical label of a defect with smooth support \mathcal{Z} of codimension ℓ is an element of $\pi_0(\mathcal{X}^{\mathbb{S}^{\ell-1}})$. Although often used to describe defects in the literature, these labels often leave out many important aspects of \mathcal{D} . We will see an example below.

We leave the description of the local and global semiclassical labels to Appendix A of [FMT22]. One of the main virtues of the formalism is that it defines an unambiguous product on parallel defects. The central idea generalizes the use of the pair of pants to define a product in 2d TFT. Given defects \mathcal{D}_1 and \mathcal{D}_2 of codimension ℓ with “parallel” support, one considers disk bundles around each of these defects and embeds the ℓ -dimensional disk into a larger disk of dimension ℓ . From this, one constructs a correspondence that defines a product. Some interesting examples can be found in Section 4 and Appendix A of [FMT22].

An especially important class of defects are the *interfaces* between theories. As with all defects, these can be given local and global semiclassical descriptions. An interface between a trivial theory and a theory $\sigma_{\mathcal{X}, \mathcal{C}}^{(n)}$ is a *boundary theory*. These play an especially important role in the literature on TFT.

We conclude by examining the classical and global labels of some defects in the finite homotopy theories defined by $\mathcal{X} = K(A, p)$, where A must be an Abelian group if $p > 1$. Recall that $p = 1$ is the case of finite gauge theory, where the group need not be Abelian. We assume $p > 0$.

The classical labels are determined by

$$\pi_0(\mathcal{X}^{\mathbb{S}^{\ell-1}}) = H^p(\mathbb{S}^{\ell-1}; A) = \begin{cases} A, & \text{for } p = \ell - 1, \\ 0, & \text{otherwise.} \end{cases} \quad (25.7)$$

Note that for $p = 1$, we only find interesting classical labels for codimension $\ell = 2$. The label is a group element: This is the classic way of describing an 't Hooft defect, namely, by specifying the holonomy created by the defect on a small linking circle. The Wilson defects are not visible with classical labels.

Let us now move on to global labels of defects supported on a smooth closed codimension ℓ submanifold of spacetime $\mathcal{Z} \subset M_n$. We assume the normal bundle can be trivialized, so that we can identify the boundary of a small neighborhood $\mathcal{U}_{\mathcal{Z}}$ with

$$\mathcal{U}_{\mathcal{Z}} \cong \mathcal{Z} \times \mathbb{S}^{\ell-1}. \quad (25.8)$$

Referring back to (24.28), we see that the global labels are vectors $\psi_{\mathcal{D}}$ in the vector space of functions,

$$\psi_{\mathcal{D}} : H^p(\mathcal{Z} \times \mathbb{S}^{\ell-1}; A) \rightarrow \mathbb{C}. \quad (25.9)$$

The Künneth theorem states that this group sits in an extension,

$$\begin{aligned} 0 \rightarrow & H^p(\mathcal{Z}; A) \otimes_{\mathbb{Z}} A \oplus H^{p-(\ell-1)}(\mathcal{Z}; A) \otimes_{\mathbb{Z}} A \rightarrow \\ & \rightarrow H^p(\mathcal{Z} \times \mathbb{S}^{\ell-1}; A) \rightarrow \\ & \rightarrow \text{Tor}(H^{p+1}(\mathcal{Z}; A), A) \oplus \text{Tor}(H^{p+2-\ell}(\mathcal{Z}; A), A) \rightarrow 0. \end{aligned} \quad (25.10)$$

As we will discuss in Section 52 below, if we take $A = \mathbb{Z}/N\mathbb{Z}$, then the finite homotopy sigma model $\sigma_{B^p A}^{(n)}$ is closely related to the n -dimensional ‘BF theory’ with a p -form and an $(n-p-1)$ -form gauge field. The latter theory has Wilson and 't Hooft defects, as explained in Section 52. Suppose that \mathcal{Z} is an orientable cycle of dimension p , then it is natural to expect that functions concentrated on the component

$$H^p(\mathcal{Z}; \mathbb{Z}/N\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \cong \mathbb{Z}/N\mathbb{Z}, \quad (25.11)$$

correspond to states created by the Wilson defect measuring the holonomy of the p -form gauge field along \mathcal{Z} . Similarly, if \mathcal{Z} is a cycle of dimension $n-p-1$, then the codimension

is $\ell = p + 1$, and it is natural to expect that functions supported on the component

$$H^{p-(\ell-1)}(\mathcal{Z}; A) \otimes A \cong H^0(\mathcal{Z}; A) \otimes A \cong A , \quad (25.12)$$

correspond to states created by 't Hooft defects measuring the holonomy of an $(n-p-1)$ -form gauge field along \mathcal{Z} . This can, in principle, be checked by evaluating correlation functions of such defects using equation (25.3) and comparing with the BF correlator (52.13) below. We will not attempt this rather interesting exercise here.

Equation (25.10) suggests the existence of defects other than the standard 't Hooft and Wilson defects. They do not appear to have been investigated.⁵⁸

26 Symmetry Action Of A TQFT On A QFT: The Quiche Picture

One of the central themes of 20th century physics has been that symmetry is a fundamental concept in formulating the laws of nature, and that symmetry as a tool is one of the most effective ways we have to understand physical systems. The past ten years have witnessed a strong movement in theoretical physics to expand notions of symmetry in ways which – one hopes – will prove to be equally fundamental and powerful. These expanded notions include “generalized symmetries,” “categorical symmetries,” and “noninvertible symmetries.” The TASI 2023 lectures of S.-H. Shao [Sha23a, Sha23b, Sha23c, Sha23d] and T. Dumitrescu [Dum23a, Dum23b, Dum23c, Dum23d] addressed these ideas. Some recent reviews include [CDIS22, Fre22a, Fre22b, SN23, Sha23f, BH23, BBFT⁺23, C⁺24, Sha25].

Generalized/Categorical/Noninvertible symmetries have thus far proven to be useful in constraining dynamics of some nontrivial physical systems, but have not achieved the kind of fundamental status analogous to that of symmetry in the traditional sense. The field is evolving vigorously and future developments – especially nontrivial and surprising applications – might alter this judgment.

The paper [FMT22] provides a unified view of the generalized, categorical, and noninvertible symmetries, at least for symmetries “of finite type.” In practice, “finite type” means that the symmetry is implemented using the finite homotopy sigma models studied in Section 24. One of the main ideas of [FMT22] goes back to D-brane physics and holography from the 1990’s in which a physical system is divided into “bulk” and “boundary” systems, and a gauge symmetry in the “bulk” is related to a global symmetry on the “boundary.” For a relatively recent discussion in the context of holography, see [HO18a, HO18b]. A second key idea in [FMT22] is that, as we have seen in examples above, for a fully local topological field theory, $F(\text{pt})$ plays the role of some kind of algebra (perhaps at a higher categorical

⁵⁸The Kontsevich-Segal inverse limit procedure discussed in equation (28.9) suggests that every global label should come from some defect.

level). An outcome of the theory of defects sketched in [Section 25](#) above is that a boundary theory then plays the role of a module.

In the history of group theory [Wus84], symmetries were first studied in the context of examples. It was only slowly, and with some difficulty, that the abstract notion of a group emerged. It is useful to separate groups from their actions on modules because universal relations can be proven using the structure of groups, which then applies to all modules. Combining this observation with the second idea mentioned above, one is led to try to identify an $(n+1)$ -dimensional TFT $\sigma^{(n+1)}$ as a generalization of a group and an n -dimensional QFT $F^{(n)}$ – not necessarily topological! – as an analog of a module. The preliminary picture (which will need to be modified) is shown in [Figure 35](#).



Figure 35: Left-half plane with half-plane decorated with TFT $\sigma^{(n+1)}$ and quantum field theory $F^{(n)}$ on the right boundary.

This picture is not entirely satisfactory: The system is really $(n+1)$ -dimensional, and we are attempting to describe symmetries of n -dimensional quantum field theories. To make the system n -dimensional, we should “cap off” the left half-plane in [Figure 35](#) with a boundary theory, and in order to avoid introducing new local degrees of freedom, we introduce a topological boundary theory on the left. Thus a more accurate picture is that shown in [Figure 36](#). It is sometimes referred to as “the sandwich.”

In [Figure 36](#), note that the boundary theory on the right $\tilde{F}^{(n)}$ is a boundary theory for $\sigma^{(n+1)}$. Because the “bulk” theory is topological, the interval can be shrunk to an arbitrarily small size, and so the system shown is an n -dimensional field theory. In the “action” on a field theory $F^{(n)}$, there should be an isomorphism θ_M to $F^{(n)}$. This isomorphism is more data in the definition of the action. The reader may well ask, “the action of what?”

The proposal of [FMT22] is that the action of a group is replaced in field theory by the action of a *quiche*:

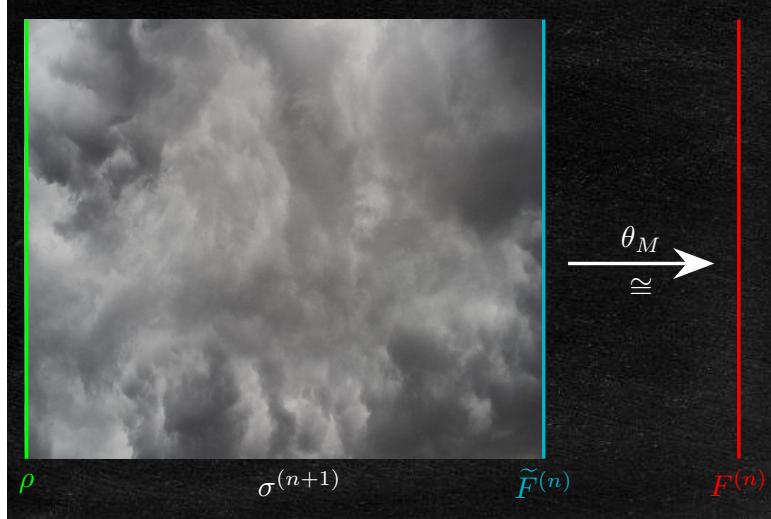


Figure 36: A cylindrical manifold diffeomorphic to $[0, 1] \times M$, where M is of dimension $\leq n$. An $(n + 1)$ -dimensional topological field theory $\sigma^{(n+1)}$ is supported on $[0, 1] \times M$. There is a topological boundary theory ρ on the left boundary and a not necessarily topological boundary theory $\tilde{F}^{(n)}$ on the right. Because $\sigma^{(n+1)}$ is topological, the entire configuration is isomorphic to the value of some n -dimensional field theory $F^{(n)}$ evaluated on M . This picture is known as the “sandwich” picture of a finite symmetry action on an n -dimensional field theory.

Definition 26.1 (Quiche). An n -dimensional *quiche* is a pair (ρ, σ) where σ is an $(n + 1)$ -dimensional topological field theory and ρ is a topological boundary theory.⁵⁹

A quiche may then be pictured as in Figure 37. The central proposal of [FMT22] is that in field theory, it is the quiche (ρ, σ) which should be regarded as the generalization of the notion of a group. The boundary theory \tilde{F} together with the isomorphism θ_M provide the notion of a module action. In this way, one separates the algebraic structure from its action on different modules. Thus the second crucial definition from [FMT22] is:

Definition 26.2 (Action of quiche on field theory). Let F be an n -dimensional field theory (not necessarily topological) and let (ρ, σ) be an n -dimensional quiche. We say that (ρ, σ) **acts on** F if there is a boundary theory \tilde{F} for σ and a coherent set of isomorphisms for all manifolds M of dimension $\leq n$,

$$\theta_M : \sigma_{\rho, \tilde{F}}([0, 1] \times M) \rightarrow F(M) . \quad (26.3)$$

Here $\sigma_{\rho, \tilde{F}}([0, 1] \times M)$ means that the theory σ is evaluated on $[0, 1] \times M$ with boundary theories ρ on $\{0\} \times M$ and \tilde{F} on $\{1\} \times M$.

This viewpoint becomes especially powerful when one starts to include topological defects in the quiche, and thus we replace the preliminary picture of Figure 35 with Figure 36.

⁵⁹We will write ρ on the left and the “physical” theory on the right. Many authors do the reverse.



Figure 37: A quiche. The TFT $\sigma^{(n+1)}$ is evaluated on $\mathbb{R}_+ \times M$ with a topological boundary theory ρ on the left.

Remark 26.4. The picture of Figure 36 is commonly called the “sandwich” picture of symmetry action, and the TFT σ is commonly called the “Symmetry TFT” or “SymTFT.” While the use is very common, it is sub-optimal. First, the term “SymTFT” refers to σ , and does not account for the important data ρ . Second, the terminology does not clearly distinguish between the abstract algebraic symmetry and the module it acts upon. The term “quiche” is meant to evoke an open sandwich, but one which resides in the quantum world.

Remark 26.5. The relation of the sandwich construction to symmetries of quantum field theories has many precedents. An incomplete list includes [Wit98d, FFRS04, BM04, MS06, GSW14, KWZ17, FT18, GK20, ABGE⁺21].

Two examples where the above picture has been worked out quite thoroughly are:

- A reworking of the standard notion [Wig13] of G -symmetry in quantum mechanics, for G a finite group. In this case, σ is the finite homotopy sigma model $\sigma_{BG,C}^{(2)}$, where $\Omega^2\mathcal{C}$ is the Morita 2-category of algebras, bimodules, and bimodule maps.
- A description of the different four-dimensional Yang-Mills theories with gauge Lie algebra \mathfrak{g} (and adjoint matter). In this case, we have $\sigma = \sigma_{B^2 A}^{(5)}$, where $A \subset Z(\tilde{G})$ and \tilde{G} is the compact, connected, and simply connected gauge group with Lie algebra \mathfrak{g} . The data ρ contains crucial information usually summarized via topological couplings in the gauge theory. For further details, see [GMN10, AST13, KS14, GSW14, FMT22, Fre22b].

An important open problem for the future is the generalization of the quiche construction from finite symmetries to continuous symmetries. Some notable attempts in this direction include [BS24, GGE24, DZDRG24, BDZM24, ADGE⁺25, BDZM25].

27 A Survey Of Some Famous Examples Of TFTs

27.1 Some Brief Historical Remarks

The subject of topological quantum field theory began with Edward Witten's reformulation of the Donaldson invariants of smooth four-dimensional manifolds in the framework of quantum field theory [Wit88b]. Witten, following some promptings from Michael Atiyah, showed that, with a suitable choice of background fields, certain correlation functions in $d = 4$, $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $SU(2)$ reproduce the Donaldson invariants. See [LM05, Moo12, Moo17a] for expository reviews, and [MSS24] for a description of topological twisting in the spirit of these lectures.

After Witten's insight [Wit89b] that the Jones polynomial of knots in \mathbb{S}^3 can also be formulated using Wilson defects in three-dimensional Chern-Simons theory, Atiyah [Ati89] gave a famous definition of topological field theory equivalent to our definition (9.42) of a topological field theory. Atiyah's formulation was strongly influenced by Graeme Segal's formulation of conformal field theory [Seg87, Seg88a, Seg88c, Seg02] as a functor on a bordism category on two-manifolds equipped with conformal structure. Segal's definition of conformal field theory may be regarded as the beginning of the functorial formulation of field theory.⁶⁰

If one attempts to fit the original example of Donaldson-Witten theory (topologically twisted super-Yang-Mills) into the Atiyah-Segal framework, one runs into trouble. The statespaces should be instanton Floer homology, but the Floer homology groups present technical difficulties and must come in several flavors [DFK02].⁶¹ Ironically, the Donaldson-Witten theory is not a topological field theory, and certainly not a fully extended topological field theory in the mathematical sense discussed above. Similar remarks apply to other famous topological field theories, such as the Vafa-Witten [VW94] and Rozansky-Witten [RW96] theories. The opposite appears to be the case for the 4d GL-twisted $\mathcal{N} = 4$ super-Yang-Mills theory, that is, for Kapustin-Witten theory [KW06]. Here, the theory can be defined in low dimensions, but not in dimensions 3 and 4. These theories are all examples

⁶⁰In a previous paper by the present authors, it was pointed out to us by Y. Tachikawa that there were important precedents for the functorial formulation of field theory going back some time. In particular, a perusal of the original papers of J. Schwinger [Sch48] and S. Tomonaga [Tom46] is most illuminating. Schwinger and Tomonaga formulated relativistic field theory in terms of families of unitary operators $U(\sigma, \sigma')$ associated to spatial slices σ, σ' and satisfying gluing axioms which we recognize as being essentially the rules of gluing that define a functor. An important advance relative to the older works is that Schwinger and Tomonaga were working in a very restricted (but important) example of spacetime, namely, Minkowski space. Thus, all the bordisms they considered were in fact invertible. It is very important that, in general, bordisms are not invertible. Another important difference is that the Atiyah-Segal axioms are aiming to capture the properties of Euclidean signature field theory. The entire question of Wick rotation is a nontrivial one. Atiyah and Segal were most directly influenced by the Feynman path integral approach to field theory.

⁶¹This is also true infrared description using Seiberg-Witten theory, although in that case, much more can be said [KM07, Lin16].

of *partially defined topological field theories*. They satisfy some of the axioms, but are not defined on all elements of the bordism category.

Another important class of topological field theories in the development of the subject is the topological sigma models introduced in [Wit88c, Wit91c]. The subject is vast and central to topological string theory and (homological) mirror symmetry, among many other things. It is beyond the scope of these lectures. For some expository material, see [GY97, HKK⁺03, ABC⁺09].

A common theme of the above examples is that they are *cohomological topological field theories*. They are based on \mathcal{Q} -closed sectors of nontrivial nontopological field theories, where \mathcal{Q} is an odd operator with $\mathcal{Q}^2 = 0$. For some expository discussion of cohomological topological field theories, see [vB90, Wit91b, BBRT91, Roc93, CMR94a, CMR94b, Moo95, Sza00, LM05, PZ16, Pes16, JJ24].

27.2 Chern-Simons Theories

One of the most famous and important examples of a topological field theory is pure 3d Chern-Simons theory. This is a huge topic and itself could be the focus of an entire book. It is a topic that continues to influence a great deal of current research. We only include a few remarks here for completeness.⁶²

Three-dimensional Chern-Simons theories were made famous by the renowned and profound paper of E. Witten [Wit89b] on the quantum field theory interpretation of the Jones polynomial, and related topological invariants of knots, links, and three-manifolds.

27.2.1 Abelian Chern-Simons Theory

Perhaps the simplest example of a Chern-Simons theory is that with gauge group $U(1)$, so we begin by describing this theory. The dynamical field is a gauge theory with a $U(1)$ gauge field, i.e., a connection on a principal $U(1)$ -bundle $P \rightarrow M_3$, with M_3 a 3-dimensional oriented manifold.⁶³ (So one of the background fields is an orientation of M_3 .) Locally, the gauge field is described by a real 1-form A with globally defined fieldstrength $F \in \Omega^2(M_3)$.⁶⁴ Locally, $F = dA$. The exponentiated action in the path integral is

$$\exp\left(i\frac{k}{4\pi} \int_{M_3} A \wedge dA\right), \quad (27.1)$$

⁶²For more on 3d Chern-Simons theory, see the TASI 2019 lectures [Moo19] and the many references therein.

⁶³See Section 12 and Section 36 for some mathematical background on principal G -bundles and connections (and Section 37 for the $G = U(1)$ case). We will take A (and hence, F) to be antihermitian, which is at variance with some physics conventions – this point is explained in much more detail in footnote 92.

⁶⁴Notation: For any manifold M_n , $\Omega^p(M_n)$ is the vector space of all globally defined smooth p -forms. We write $\Omega_{\mathbb{Z}}^p(M_n)$ for those with integral periods. If the forms are valued in a vector bundle $E \rightarrow M_n$, we write $\Omega^p(M_n; E)$.

where we have normalized A so that F has periods in $2\pi\mathbb{Z}$. The action does not look gauge invariant, but under “small” gauge transformations $A \mapsto A + d\epsilon$,

$$A \wedge dA \mapsto A \wedge dA + d(\epsilon dA) , \quad (27.2)$$

so if M_3 is compact and without boundary, $\int_{M_3} A \wedge dA$ is gauge-invariant. But under “large” gauge transformations $A \mapsto A + \omega$, with $\omega \in \Omega^1_{\mathbb{Z}}$, it is not well-defined.⁶⁵ One way to attempt to deal with that problem is to exhibit M_3 as a boundary of a four-manifold $\partial M_4 = M_3$. One can prove that it is possible to extend $P \rightarrow M_3$ and its connection to a principal $U(1)$ -bundle $P \rightarrow M_4$.⁶⁶ Once the bundle extends topologically, there is no problem extending the connection (as one proves using a partition of unity). Then the identity $F \wedge F = d(A \wedge dA)$ and Stokes’ Theorem motivates a provisional definition:

$$\int_{M_3} A \wedge dA \stackrel{?}{=} \int_{M_4} F \wedge F . \quad (27.3)$$

There is a fundamental problem with this definition: The extension from M_3 to M_4 is not unique. There can be different extensions as illustrated in Figure 38.

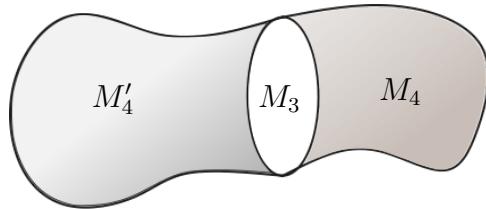


Figure 38: Non-uniqueness of the extension of M_3 to a four-manifold.

In general,

$$\int_{M_4} F \wedge F \neq \int_{M'_4} F \wedge F . \quad (27.4)$$

What saves the day is that on the closed manifold $M_4 \cup M'_4$,

$$\int_{M_4 \cup M'_4} F \wedge F \in (2\pi)^2 \mathbb{Z} , \quad (27.5)$$

⁶⁵We will say more about small and large gauge transformations in Section 37.

⁶⁶The obstruction to such an extension would be a nontrivial element of the bordism group $\Omega_3^{SO}(BU(1))$. (Bordism groups were introduced towards the end of Section 4.) However, as we discuss in footnote 71, $\Omega_3^{SO}(BU(1)) \cong H_3(BU(1); \mathbb{Z})$. Now $BU(1) \cong K(\mathbb{Z}, 2) \cong \mathbb{C}\mathbb{P}^\infty$, the homology groups of which are trivial in odd degree and isomorphic to \mathbb{Z} in even degree, since $\mathbb{C}\mathbb{P}^\infty$ has zero cells in every odd dimension and one cell in every even dimension. In particular, it follows that $\Omega_3^{SO}(BU(1)) \cong \{0\}$.

and hence

$$\left(\frac{1}{2\pi} \int_{M_4} F \wedge F \right) \bmod 2\pi\mathbb{Z}, \quad (27.6)$$

is well-defined. Therefore, if k is an even integer, the action appearing in equation (27.1) is not well-defined as a real number, but only as an element of $\mathbb{R}/2\pi\mathbb{Z}$. Even though the action is not a well-defined number, the exponentiated action,

$$\exp \left(i \frac{k}{4\pi} \int_{M_3} A \wedge dA \right), \quad (27.7)$$

is a well-defined phase. So, at least formally, the exponentiated action makes sense as a weighting factor in the path integral.

Note that the path integral ‘‘measure’’ $\exp \left(i \frac{k}{4\pi} \int_{M_3} A \wedge dA \right)$ is metric-independent. So we expect the path integral,

$$F(M_3) = \int_{\mathcal{A}/\mathcal{G}} [dA] \exp \left(i \frac{k}{4\pi} \int_{M_3} A \wedge dA \right), \quad (27.8)$$

to be a topological invariant, and to define a topological field theory.

Remark 27.9.

1. In the above discussion, we have been forced to take k to be an even integer. However, in some cases (e.g., applications to the quantum Hall effect), one wishes to make sense of such expressions for odd integers k . One can do this if one endows M_3 with a spin structure. The price one pays is that the value of the Chern-Simons action depends on the spin structure. When M_3 has a spin structure, we can find a bordant spin manifold M_4 , and again the principal $U(1)$ -bundle can be extended to M_4 . The key observation is that equation (27.5) can be refined to the statement that on the closed manifold $M_4 \cup M'_4$, we have

$$\int_{M_4 \cup M'_4} \frac{1}{2!} \left(\frac{F}{2\pi} \right)^2 \in \mathbb{Z}. \quad (27.10)$$

This follows from the Atiyah-Singer index theorem. The extra factor of 2 renders the path integral weight in (27.8) well-defined for k an odd integer, but the weight will depend on the spin structure on M_3 . If we change the spin structure by $\epsilon \in H^1(M_3; \mathbb{Z}/2\mathbb{Z})$, then the exponentiated action changes by a sign:

$$\exp \left[i\pi k \int_{M_3} \epsilon \cup \frac{F}{2\pi} \right]. \quad (27.11)$$

(For an explanation, see [BM05, Sec. 2.1.1].) Thus, the background fields in the theory

with k an odd integer include both an orientation and a spin structure.

- The path integral (27.8) is formally metric-independent, but in fact is not truly metric-independent. One way to see this is that one must use gauge fixing to deal with the gauge symmetry. The determinants of the resulting Faddeev-Popov ghosts turn out to have metric dependence. Another, rather elegant way to derive the metric dependence, is to view the Chern-Simons theory as the strong coupling (equivalently, long-distance) limit of Maxwell-Chern-Simons theory with Euclidean action

$$\int_{M_3} \left(\frac{1}{2e^2} F \wedge \star F + \frac{ik}{4\pi} A \wedge dA \right) . \quad (27.12)$$

One finds an overall dependence on the metric [Wit89b]

$$F(M_3) = e^{2\pi i \frac{c}{24} \omega_{CS}(g)} \times \underbrace{\widetilde{F}(M_3)}_{\substack{\text{metric} \\ \text{independent}}} , \quad (27.13)$$

with $c = \text{sign}(k)$, and $\omega_{CS}(g)$ is the “gravitational Chern-Simons term,” with

$$d\omega_{CS}(g) = \text{Tr } R \wedge R . \quad (27.14)$$

One would like to consider the gravitational Chern-Simons term to be a “local counterterm,” but now one has difficulty defining precisely the gravitational Chern-Simons term (again because the Chern-Simons term is not well-defined as a real number).

- The metric dependence of the previous remark is known as the “framing anomaly.” There are various approaches to dealing with it. The upshot is that the topological Chern-Simons theory depends on more background fields. These extra background fields can be taken to be a “2-framing” in the sense of Atiyah [Ati89, Ati90] or a p_1 -structure [BHMV95, App. A].⁶⁷ We briefly explain here some details of what these terms mean.⁶⁸ Fix a positive integer $n \in \mathbb{Z}_{\geq 0}$, and let M_n be an oriented n -manifold. A p_1 -structure on M_n is a lift of the classifying map $M_n \rightarrow B\mathbf{GL}^+(n, \mathbb{R})$ of the oriented tangent bundle through the map $X_n \rightarrow B\mathbf{GL}^+(n, \mathbb{R})$, where X_n is the homotopy fiber⁶⁹ of the map $B\mathbf{GL}^+(n) \xrightarrow{p_1} K(\mathbb{Z}, 4)$ that realizes the first Pontryagin class, such that the diagram

$$\begin{array}{ccc} & X_n & \\ \nearrow \tilde{\tau}_{M_n} & \downarrow \pi & \\ M_n & \xrightarrow{\tau_{M_n}} & B\mathbf{GL}^+(n, \mathbb{R}) \xrightarrow{p_1} K(\mathbb{Z}, 4) \end{array} \quad (27.15)$$

⁶⁷In [Seg02], this is referred to as a *rigged structure*. See also [BN09, Sec. 2.3].

⁶⁸V.S. thanks D. Freed for a discussion about these issues.

⁶⁹See Section 24.2 for a definition of the homotopy fiber.

commutes. A point of X_n is a pair $b : \text{pt} \rightarrow B\text{GL}^+(n, \mathbb{R})$ with $H : [0, 1] \rightarrow K(\mathbb{Z}, 4)$ such that $H(0) = p_1 \circ b$ and $H(1) = \text{pt}$, i.e., H is a homotopy from $p_1 \circ b$ to the basepoint in $K(\mathbb{Z}, 4)$. This lift exists if and only if the composition $p_1 \circ \tau_{M_n} : M_n \rightarrow K(\mathbb{Z}, 4)$ is nullhomotopic. The set of homotopy classes of lifts (27.15) is a torsor over $H^3(M_n; \mathbb{Z})$.

For a closed, oriented 3-manifold M_3 , a p_1 -structure always exists. In this case, for connected M_3 , $H^3(M_3; \mathbb{Z}) \cong \mathbb{Z}$, so the set of p_1 -structures (up to homotopy) forms a torsor over the integers. Projectively, 3d Chern-Simons theory is a theory of oriented manifolds, i.e., it does not depend on the p_1 -structure. More precisely, on a closed, connected 3-manifold M_3 , shifting the p_1 -structure by an integer $k \in \mathbb{Z}$ multiplies the Chern-Simons partition function by $\exp(2\pi i \frac{c}{24} k)$ where c is the central charge of the associated modular tensor category. For a discussion of why p_1 structures may be considered more natural in defining, for instance, a fractional Chern-Simons theory,⁷⁰ see [Fre23a, Sec. 36.4].

4. The basic example of 3d $U(1)$ Chern-Simons theory can be generalized to include multiple $U(1)$ gauge fields A^I , $I = 1, \dots, r$, with action

$$S = \frac{1}{4\pi} \int k_{IJ} A^I \wedge dA^J . \quad (27.16)$$

These theories prove to be very useful long-distance descriptions of the physics of various fractional quantum Hall states. The matrix k_{IJ} must be a symmetric integral matrix and it determines an integral lattice. The lattice must be even. (With “spin topological fields theories” – where we include a spin structure as one of the background fields – the lattice can also be an odd lattice.) The quantum amplitudes can be expressed in terms of invariants of this lattice [BM05, Sti08, FHLT09, KS10].

27.2.2 Non-Abelian Chern-Simons Theory

A much richer theory emerges when we generalize to 3d Chern-Simons theories with non-Abelian gauge group. Now take a connection on a non-Abelian principal G -bundle $P \rightarrow M_3$ for G a compact simple group. Locally (i.e., on a contractible patch $\mathcal{U}_\alpha \subset M_3$), the connection is $d + A$, with $A \in \Omega^1(\mathcal{U}_\alpha, \mathfrak{g})$, where $\mathfrak{g} = \text{Lie}(G)$. We can construct the Chern-Simons form:

$$d\text{Tr} \left(AdA + \frac{2}{3} A^3 \right) = \text{Tr}(F \wedge F) . \quad (27.17)$$

⁷⁰By “fractional” Chern-Simons theory, we mean a situation where the level is allowed to be a rational or real number. Recall that the standard exponentiated Chern-Simons action functional is well-defined for integer level.

In the case where (M_3, A) can be extended to a G -bundle with connection on a bounding four-manifold (M_4, A') , we can define the Chern-Simons action as:

$$\frac{k}{4\pi} \int_{M_3} \text{Tr} \left(AdA + \frac{2}{3} A^3 \right) := \frac{k}{4\pi} \int_{M_4} \text{Tr}(F \wedge F) . \quad (27.18)$$

As in the Abelian case, there can be different extensions leading to different values for the right-hand-side. For a suitable notion of trace (e.g., $\text{Tr} = \text{Tr}_N$ for $G = \text{SU}(N)$), k must be an integer, and then (for M_3 compact and without boundary) the exponentiated action,

$$\exp \left(i \frac{k}{4\pi} \int_{M_3} \text{Tr} \left(AdA + \frac{2}{3} A^3 \right) \right) = \exp \left(i \frac{k}{4\pi} \int_{M_4} \text{Tr}(F \wedge F) \right) , \quad (27.19)$$

is a well-defined function on the space of gauge equivalence classes of connections. The logarithm is not well-defined.

Exercise 27.1: Non-Abelian Chern-Simons 3-form

Compute the change of $\text{Tr}(AdA + \frac{2}{3} A^3)$ under a gauge transformation $d+A \mapsto g^{-1}(d+A)g$ to see why the 3-form $\text{Tr}(AdA + \frac{2}{3} A^3)$ is not globally well-defined on M_3 .

Remark 27.20.

1. The above definition is not entirely satisfactory because there can be situations in which the principal G -bundle and M_3 cannot be extended to a principal G -bundle over an oriented bounding four-manifold M_4 .⁷¹ A better definition, intrinsic to the 3-manifold M_3 is provided by differential cohomology. See [Section 51](#) below.
2. The action also makes sense when M_3 has a nonempty boundary, but in this case, the exponentiated action must be considered as a section of a line bundle over the space of isomorphism classes of gauge fields, rather than a well-defined function on that space. Again, the viewpoint of differential cohomology turns out to be very useful. See [Section 37](#) below for details.
3. One can define a nice Chern-Simons-Witten TFT for any compact group, and indeed, the theories for finite groups G are special cases of the finite homotopy theories studied above. (Note that the discussion above using local connections does not immediately apply.) In general, a 3d Chern-Simons theory is determined simply by a choice of a compact Lie group G , a “level” $k \in H^4(BG; \mathbb{Z})$, and an orientation.

⁷¹Technically, the bordism group $\Omega_3^{\text{SO}}(BG)$ can be nonzero for some compact groups. See, for example, [\[JF17\]](#) for a discussion of interesting properties of $H^3(BG; \mathbb{R}/\mathbb{Z})$ where G is the Monster group. The exact sequence (39.10) below and the fact that $H_2(BG; \mathbb{Z}) = 0$ for G a compact group shows that $H_3(BG; \mathbb{Z})$ is nonzero. Then the Atiyah-Hirzebruch Spectral Sequence shows that $\Omega_3^{\text{SO}}(BG) \cong H_3(BG; \mathbb{Z})$ for a compact group G .

4. It is generally expected that 3d Chern-Simons theory for a compact group can be formulated as a fully local extended TQFT. One associates to a closed one-manifold \mathbb{S}^1 , the relevant modular tensor category which can be viewed as a 2-category. The 3-category associated to a point is more subtle. Some comments for the case of an Abelian group appear in [FHLT09]. Remarks for the non-Abelian case appear in [Hen15] and [Tel22].
5. One can also extend 3d Chern-Simons theory to noncompact groups. Here the flavor is quite different and they are typically not TFTs for the simple reason that statespaces are infinite-dimensional. But they are very similar to topological field theories. For more about this very active research topic, see [Wit10, AK11, Dim16, AK18, BFM⁺22]. More references can be found in [BFM⁺22, Sec. 8.2]. One of the many reasons Chern-Simons theory for noncompact groups is of great interest is its relation to 3d quantum gravity. See, for example, [AT86, Wit88d, Wit89a, CC94]. For some of the most recent developments on this idea, we refer to [CEZ23, CEZ24].
6. Chern-Simons theories can also be “upgraded” to higher-dimensional theories by using extra background fields: for example, for a suitably normalized closed form ω , we can contemplate a higher-dimensional theory defined by an action such as,

$$\int \omega \text{Tr} \left(A dA + \frac{2}{3} A^3 \right) . \quad (27.21)$$

One needs to be careful here to get a well-defined propagator and we will not go into details. This generalization goes back at least as far as the famous paper of A. Schwarz [Sch79] (see p. 13). When ω is a Kähler form on a 4d Kähler manifold, one obtains a 5-dimensional theory which was studied in [LMNS95a, LMNS95b, LMNS96, LMS97] and further developed in N. Nekrasov’s Ph.D. thesis [Nek96]. More recently, theories of this nature have played a major role in works by K. Costello and collaborators, and have proved especially useful in providing new insights into integrable models. For examples, see [Cos13a, Cos13b, CWY17, CWY18]. Curiously, a similar expression makes an appearance in an effective action for 4d topological insulators with nontrivial first Chern class of the band structure bundle [Moo17b].

7. Another way to generalize Chern-Simons theory to higher dimensions is to replace the closed 2-form F of Maxwell theory by an ℓ -form $F \in \Omega^\ell(M_n)$. Now $dF = 0$ implies $F = dA$, locally, where A is a locally defined $(\ell - 1)$ -form. In n dimensions, we can introduce another locally defined higher form “gauge potential” B of degree $n - \ell$ and consider the renowned “BF-action”:

$$\exp \left(2\pi i N \int_{M_n} B \wedge F \right) . \quad (27.22)$$

If the periods of F and a corresponding field strength associated to B have quantized periods, then N must be quantized. This action can be used to define different theories depending on how one treats the zeromodes. One natural way to make sense of such BF theories makes use of differential cohomology, studied in Part II. See in particular, Section 41.3 and Section 52.

8. The BF theories also admit a non-Abelian generalization. These arise quite naturally in studying 2d non-Abelian Yang-Mills theory [Wit91a]. An interesting relation of non-Abelian BF theory to Koszul duality is indicated in [KM24, App. A]. There is a relation of 2d non-Abelian BF theory to 2d quantum gravity that is analogous to the relation of 3d non-Abelian Chern-Simons theory to 3d quantum gravity. It appears this was first noticed in a remarkably prescient paper of T. Fukuyama and K. Kamimura [FK85]. The connection was further explored in [CW89, CW90]. More recently, the subject has risen to importance in studies of 2d gravity and holography. See [SSS19, Sec. 3.3] and [SW19] for more extensive discussion.

27.3 Two-Dimensional Yang-Mills Theory

A very interesting example of a completely solved field theory which is not topological is two-dimensional Yang-Mills theory. The exact solution began with the work of A. Migdal [Mig75]. In modern terms, one can define the theory as follows. One chooses a compact Lie group G and defines two fields: A connection on a principal G -bundle over a two-manifold $P \rightarrow \Sigma$, together with a zero-form $\phi \in \Gamma[\mathbf{ad} P]$. The theory has a background field μ , which is an area form on the two-dimensional spacetime Σ . With a suitably normalized bilinear form on the Lie algebra, the action is,

$$S = \int \mathbf{tr}(\phi F) + \mu \mathbf{tr}(\phi^2) . \quad (27.23)$$

This will be defined on a bordism category where each connected component Σ_c of the two-dimensional morphisms Σ is equipped with a total area A_c for that component, where A_c is a positive real number. The areas are additive under gluing.

The Hilbert space of this theory on the circle turns out to be the space of L^2 class functions on G , that is

$$F(\mathbb{S}^1) = L^2(G)^G , \quad (27.24)$$

where the G action is by conjugation. One should think of this Hilbert space as the space of functions of the one gauge-invariant quantity we can assign to the circle, namely, the holonomy. Since the holonomy is only defined up to conjugation, we must use class functions of the holonomy. A natural basis for $F(\mathbb{S}^1)$ is the set of characters χ^μ in the irreducible representations of G , labeled by μ .

For a gauge group of positive dimension, there will be infinitely many irreducible representations, so $\mathcal{H}(\mathbb{S}^1)$ is not a topological field theory in the sense we have discussed.

Nevertheless, 2d Yang-Mills theory for a positive dimensional gauge group behaves very much like a topological field theory. For example, the amplitude associated to a cylinder with one ingoing and one outgoing circle, in the case where G is finite, is,

$$F([0, 1] \times \mathbb{S}^1) = \sum_{\mu} \chi^{\mu, \vee} \otimes \chi^{\mu}, \quad (27.25)$$

and for a gauge group with positive dimension, is,

$$F([0, 1] \times \mathbb{S}^1; A) = \sum_{\mu} \chi^{\mu, \vee} \otimes \chi^{\mu} e^{-Ac_2(\mu)}, \quad (27.26)$$

where $c_2(\mu)$ is the quadratic Casimir, and A is a dimensionless measure of the area. There is a multiplication defined by the pair of pants with total area A . (The picture is the same as in [Figure 24](#).)

In equation [\(14.8\)](#) above, we found that for G finite, the pants multiplication of characters χ_{μ} (where μ indexes the distinct irreducible representations of G) is:

$$m_P(\chi^{\mu}, \chi^{\nu}) = \delta_{\mu\nu} \frac{\chi^{\mu}}{\chi^{\mu}(1)}. \quad (27.27)$$

The generalization to the case where G has finite dimension is:

$$m_{P,A}(\chi^{\mu}, \chi^{\nu}) = \delta_{\mu\nu} \frac{\chi^{\mu}}{\chi^{\mu}(1)} e^{-Ac_2(\mu)}. \quad (27.28)$$

One can show that the partition function can be expressed as:

$$F(\Sigma_g, A) = \sum_{R:\text{irrep}} (\dim R)^{2-2g} e^{-AC_2(R)}. \quad (27.29)$$

Note that this expression does not have a good $A \rightarrow 0$ behavior for $g = 0$. So the limit $\mu \rightarrow 0$ only gives a partially defined topological field theory. See [\[Wit92, CMR94a, CMR94b\]](#) for more details (or [\[Moo95\]](#) for a short summary).

Some time ago, D. Gross and W. Taylor made a very interesting attempt to express two-dimensional Yang-Mills theory as a string theory [\[GT93\]](#). Details of the $1/N$ expansion of $SU(N)$ theory are given in [\[CMR94a, CMR94b\]](#). For a very careful description of the theories in the spirit of these notes, taking careful account of the infinite-dimensional topological vector space theory needed in a rigorous description of these theories, see [\[RS18, Wed24\]](#).

28 Geometric Approach To Field Theory

We now return to some of the motivating remarks from [Section 3](#) concerning geometric field theory. The question we address here is: Can the basic formalism of TFT be generalized to nontopological field theories? Put differently, we ask: Can a general QFT be viewed as a “representation” of some geometric bordism category?

The reason the above questions are interesting is that, while quantum field theory is the most successful scientific framework for describing the laws of nature ever conceived, nevertheless, a completely satisfactory description of what quantum field theory “is” remains a subject of much debate and little research. See, for example, [\[BFM⁺22, Sec. 2.1\]](#), and the Snowmass whitepaper [\[Ded22\]](#), and references therein for some overview remarks. Among the various approaches to a mathematical formulation of QFT described there, the one closest to Part I of these lectures seeks to define a QFT as some kind of functor out of some kind of bordism category. One might call such a description of QFT “geometric field theory,” in contrast to “topological field theory,” because one now includes fields such as metrics in the definition of the bordism category. (Technically, the sheaf on Man_n of background fields should not be flat.) It has also been called “functorial field theory” because, ultimately, a “field theory” is defined to be a certain kind of functor. Another name which has been suggested is “dynamical field theory,” because typical theories have local dynamics of locally defined degrees of freedom. One of the most significant papers exploring this approach to general quantum field theory is the paper of M. Kontsevich and G. Segal [\[KS21\]](#). Another important approach, also with a geometric (and sheaf-theoretic) flavor, is that of factorization algebras [\[CG16, CG21\]](#). Some overlapping ideas can also be found in [\[Los23, Mne25, GL25\]](#).

In order to appreciate some of the issues involved, it is worth starting with the one-dimensional case, namely, quantum mechanics, where we include a Riemannian metric in the fields defining the bordism category. Note that the only invariant of a Riemannian metric on a one-dimensional manifold is the length of each connected component.

At first sight, what one should assign to a point would appear to be a complex Hilbert space \mathcal{H} . Or should it be the dual Hilbert space \mathcal{H}^\vee ? These are isomorphic, by the \mathbb{C} -antilinear Riesz isomorphism,

$$R : \mathcal{H} \rightarrow \mathcal{H}^\vee , \quad (28.1)$$

where if $\psi \in \mathcal{H}$, then R_ψ is the linear functional,

$$R_\psi(\xi) := \langle \psi, \xi \rangle . \quad (28.2)$$

However, when the Hilbert space is graded by a Hamiltonian, the isomorphism will not be natural in families (such as manifolds of control parameters, i.e., background fields, deforming the theory). This is one (of the many) motivations to replace the spatial point

by a “germ” of a point in a 1-dimensional manifold. For the moment, we identify a germ with an infinitesimal open neighborhood of the point in some 1-manifold. Because the neighborhood is infinitesimal, it can be any 1-manifold. The 1-manifold is oriented (forward time evolution). The point needs to carry a co-orientation $\pm \frac{\partial}{\partial t}$. Then we can say,

$$F\left(\text{pt}, \frac{\partial}{\partial t}\right) = \mathcal{H} , \quad F\left(\text{pt}, -\frac{\partial}{\partial t}\right) = \mathcal{H}^\vee . \quad (28.3)$$

If the bordism is diffeomorphic to an interval of length L , then composition of bordisms implies that for $t_2 - t_1 = L > 0$, if the bordism $[t_1, t_2] : t_1 \rightarrow t_2$ has a metric of length L then it follows from gluing that,

$$F([t_1, t_2]) = e^{-LH} , \quad (28.4)$$

where H is an operator on \mathcal{H} . For Wick-rotated versions of unitary theories H will be self-adjoint (not necessarily bounded), and identified with the Hamiltonian.⁷²

(Here we have limited attention to systems with time-translation invariance, corresponding to a time-independent Hamiltonian.) In topological quantum mechanics, the Hamiltonian is zero. But for general quantum mechanics, it is nonzero, and hence there are dynamics.

An important remark is that, as we have repeatedly stressed, the S -diagram argument based on Figure 11 continues to hold when we include metrics. However, the cylinder must have positive length $L > 0$ and therefore, the compositions of the cup and cap morphisms will not produce an identity operator. Thus, the “dualizability” condition that the statespace is finite-dimensional will no longer hold. Of course, many quantum mechanical systems of interest have infinite-dimensional statespaces.

When \mathcal{H} is infinite-dimensional and H is an unbounded operator, there are further important subtleties because the domain of H cannot be the entire Hilbert space. Rather, there will be a dense subspace $\check{E} \subset \mathcal{H}$ on which expectation values of H and its powers will make sense. Technically, \check{E} will be a “nuclear Fréchet space.”⁷³ When combined with the idea that a field theory should be a map out of a bordism category where the objects are germs of points and the morphisms are germs of bordisms, what is naturally associated to a point is a “rigged Hilbert space,”

$$\check{E} \subset \mathcal{H} \subset \widehat{E} , \quad (28.5)$$

where \widehat{E} is the dual vector space to \check{E} . This is an important part of the treatment of geometric field theory given by Kontsevich and Segal [KS21]. The triple (28.5) is known as a *Gelfand triple* and a Hilbert space equipped with a Gelfand triple is known as a “rigged

⁷²For some interesting examples where H is not self-adjoint see [Los23, LL22, LL23, BLM24].

⁷³See [Cos11a, App. 2] and [Wed24, App. A] for quick summaries of the theory of nuclear topological vector spaces and nuclear maps.

Hilbert space.”⁷⁴

In general dimensions, Kontsevich and Segal argue that rather than incorporating general Riemannian metrics as background fields on the bordisms (as in Section 21), it is important to consider a class of complex metrics which have the nice property that the real part of the action for all nonzero real generalized Abelian gauge fields (see equation (31.5) below) is positive. For any manifold M , this defines an infinite-dimensional complex manifold $\text{Met}_{\mathbb{C}}(M)$, and Kontsevich and Segal propose replacing $\text{Riem}(M)$ in $\mathcal{F}(M) = \text{Riem}(M)$ by $\text{Met}_{\mathbb{C}}(M)$ when defining the space of background metrics. Moreover, they require that the field theory “functor” should depend holomorphically on the complex metric.

As an example, returning to the one-dimensional case $\text{Riem}([t_1, t_2]) \cong \text{Diff}^+([t_1, t_2])$ and the metric is $f^*((dt)^2)$. The Kontsevich-Segal domain $\text{Met}_{\mathbb{C}}([t_1, t_2])$ can be identified with the infinite-dimensional manifold of smooth embeddings $f : [t_1, t_2] \rightarrow \mathbb{C}$ such that $f(t_1) = 0$ and $\text{Re}(f'(t)) > 0$ for all $t \in [t_1, t_2]$. The “holomorphy” of the field theory functor is, in this case, related to the fact that the time-evolution operator of quantum mechanics, $U(t) = e^{-itH/\hbar}$, admits an analytic continuation to the lower-half of the complex plane as a function of t (for H which is bounded below). The complex metric on the interval is $f^*((dz)^2)$.

In general, a complex domain has a notion of boundary known as a *Shilov boundary* (it enforces a maximum modulus principle). A very remarkable property of the Kontsevich-Segal domain is that while the Shilov boundary of $\text{Met}_{\mathbb{C}}(M)$ contains real metrics, it only contains those real metrics of Lorentzian signature!

When one tries to define the bordism “category” $\text{Bord}_{\langle n-1, n \rangle}^{\mathcal{F}}$, with the sheaf \mathcal{F} on Man_n given by $\mathcal{F}(M) = \text{Met}_{\mathbb{C}}(M)$, one encounters a subtlety with composition of morphisms. To do this smoothly, one needs to know all the normal derivatives of the spatial metric along the $(n-1)$ -dimensional spatial slice N_{n-1} along which the gluing takes places.⁷⁵ The upshot is that the objects in the “category” $\text{Bord}_{\langle n-1, n \rangle}^{\mathcal{F}}$ should be *germs* of compact $(n-1)$ -dimensional (“spatial”) manifolds N_{n-1} . The germ $[N_{n-1}]$ of N_{n-1} is an equivalence class of open n -dimensional manifolds M_n , containing N_{n-1} in their interior, with the equivalence relation that $M_n \sim M'_n$ if there is an M''_n with a diffeomorphism (equal to the identity on N_{n-1}) to open submanifolds of M_n and M'_n . As we have remarked, the germ should be equipped with a co-orientation indicating the direction of forward “time” evolution. The morphisms in the “category” should likewise be defined in terms of germs. We have been using scare-quotes on the word “category” thus far because of the very crucial fact that objects do not have an identity morphism. This is closely related to the fact that the S-duality diagram of Figure 11 does not imply that the statespaces are fully dualizable. A

⁷⁴The notion of “rigged Hilbert space” was introduced by Gelfand [GYV16] in order to give precise mathematical meaning to the bra- and ket- eigenstates of unbounded operators in quantum mechanics. For an overview see, e.g., [Rob66b, Rob66a, dLM05]. The name “rigged Hilbert space” is an unfortunate translation from the Russian. The term has, regrettably, become standard. The name “equipped Hilbert space” would be more accurate. See [Tri10] for additional references.

⁷⁵Indeed, on the field theory side, it is a property already seen in the quantization of free fermion field theories that the quantization procedure depends on the first $[n/2]$ normal derivatives of the metric [Seg99].

careful formulation of the category with germs of manifolds can be found in section 2 of the paper of R. Wedeen [Wed24].⁷⁶

A crucial step in the geometric approach to field theory is the formulation of local operators and their correlation functions. The Kontsevich-Segal definition of local operators at a point $x \in M_n$ is derived from standard intuition in field theory and is particularly famous in the formulation of vertex operators in 2d conformal field theory.⁷⁷ It is worth taking a moment to recall this intuition before giving the formal definition. Consider an amplitude associated with a bordism $M_n : N_{n-1}^0 \rightarrow N_{n-1}^1$, but where a local operator \mathcal{D}_x has been inserted at a point x in the interior of M_n . We can denote this amplitude in the presence of the defect as,

$$F(M_n; \mathcal{D}_x) : F(N_{n-1}^0) \rightarrow F(N_{n-1}^1) . \quad (28.6)$$

Now, we can surround x by an n -dimensional ball $D \subset \text{Int}(M_n)$. The field theory associates an amplitude,

$$F(M_n - D) : F(N_{n-1}^0) \otimes F(\partial D) \rightarrow F(N_{n-1}^1) , \quad (28.7)$$

to the bordism $M_n - D : N_{n-1}^0 \amalg \partial D \rightarrow N_{n-1}^1$. On the other hand, the effect of the local operator is to create a state $\psi_D \in F(\partial D)$, and we can say that,

$$F(M_n; \mathcal{D}_x)(\xi) = F(M_n - D)(\xi \otimes \psi_D) , \quad (28.8)$$

for all $\xi \in F(N_{n-1}^0)$. Now, the state ψ_D does not intrinsically characterize the defect \mathcal{D}_x because it contains irrelevant information about the details of what goes on in the interior of D away from x . However, if D' is a smaller ball around x in the interior of D then the amplitude associated to the bordism $D - D' : \partial D' \rightarrow \partial D$ takes $\psi_{D'} \rightarrow \psi_D$. More formally, we have a directed system of (germs of) small balls $x \in D \subset M_n$ with $D < D'$ if $x \in \text{Int}(D') \subset D$. (Note the ordering, which some readers might find confusing.) The co-orientation on ∂D is directed outward. Then the (germ of the) bordism $D - \text{Int}(D')$ from $\partial D' \rightarrow \partial D$ is associated by the field theory to a family of (nuclear) linear maps $F_{D',D} : \mathcal{H}(\partial D') \rightarrow \mathcal{H}(\partial D)$ which are coherent in the sense that if $D < D' < D''$, then $F_{D'',D} = F_{D'',D'} \circ F_{D',D}$. Therefore, we can take the inverse limit:

$$\mathcal{O}_x = \varprojlim_D \mathcal{H}(\partial D) , \quad (28.9)$$

to obtain a vector space of observables. This is the Kontsevich-Segal definition of local observables in the field theory F .

⁷⁶A criticism of this approach to defining a bordism category with metrics can be found at the beginning of [BEP15].

⁷⁷A similar discussion in the context of a functorial formulation of field theory was described by A. Losev in [Los23] and further developed in [GL25]. See also [Mne25].

Let us spell out the meaning of the Kontsevich-Segal definition in the case of quantum mechanics. The ball surrounding a point t has $F(D) \cong \mathcal{H} \otimes \mathcal{H}^\vee$, which can be identified with the space $\mathsf{HS}(\mathcal{H})$ of Hilbert-Schmidt operators on \mathcal{H} . For $t_1 < t_2$, the map from $F([-t_1, t_1]) \rightarrow F([-t_2, t_2])$ considered as a map of Hilbert-Schmidt operators is the map,

$$\mathcal{O} \mapsto \mathcal{O} = e^{-\Delta t H} \mathcal{O} e^{-\Delta t H}, \quad (28.10)$$

where $\Delta t = t_2 - t_1$, and H is the Hamiltonian. An element of the direct limit (28.9) in this case is equivalent to a map $\Phi : [0, 1] \rightarrow \mathsf{HS}(\mathcal{H})$ such that,

$$\Phi(t_2) = e^{-\Delta t H} \Phi(t_1) e^{-\Delta t H}. \quad (28.11)$$

Thus, the evolution of $\Phi(t)$ to larger times is essentially evolution by the heat kernel e^{-tH} , with $t > 0$. In the (typical) case where the spectrum of H is bounded below and unbounded above the heat kernel operator e^{-tH} is a smoothing operator. To gain some intuition for what this means let H be the Laplacian acting on functions on a smooth Riemannian manifold. Under evolution by the heat kernel, the eigenfunctions of the Laplacian with eigenvalue λ scale to zero exponentially fast like $e^{-\lambda t}$. Distributions at time $t = 0$ will evolve to smooth functions at any positive time. By the same token, inverse heat flow will take a typical smooth function and make it increasingly oscillating. Applying this intuition to (28.11), we see that if we fix a typical Hilbert-Schmidt operator $\Phi(t)$ at some time t_2 then at earlier times t_1 it will become less-well behaved. The net effect is that the Kontsevich-Segal definition of the space of observables at time $t = 0$ is a nuclear Frechet space of unbounded operators on the Hilbert space. This is consistent with standard ideas in quantum mechanics. The paper [KS21] goes on to make contact with the Wightman axioms of QFT. While the definition is in accord with physical intuition and makes good mathematical sense, fundamental questions remain to be resolved. For example, in Segal's mathematical definition of two-dimensional conformal field theory [Seg87, Seg88a, Seg02], it is an important open question whether one can prove that the tangent space to moduli spaces of CFTs can be identified with exactly marginal operators.

Another approach to defining field theories based on some notion of a bordism category equipped with Riemannian metrics has been explored in [GP20]. Moreover, the approach has been extended to a definition of field theory, together with an analog of the cobordism hypothesis in [GP21]. The issue of the lack of dualizability in general quantum mechanics and quantum field theories is addressed by considering freely generated symmetric monoidal categories – in essence attaching duals to statespaces even when they do not exist in the domain of functional analysis. Whether this more formal approach will be fruitful in defining general interacting quantum field theories remains to be seen.

Remark 28.12.

1. Geometric field theory offers a natural framework for describing the renormalization group. If $(M, g_{\mu\nu})$ is a bordism equipped with metric, then we can scale the metric by $g_{\mu\nu} \rightarrow \lambda^2 g_{\mu\nu}$ where $\lambda \in \mathbb{R}$. Then we define a family of field theories:

$$\mathcal{F}_\lambda(M, g_{\mu\nu}) := \mathcal{F}(M, \lambda^2 g_{\mu\nu}) . \quad (28.13)$$

The long-distance “limit” is, formally, $\lim_{\lambda \rightarrow \infty} \mathcal{F}_\lambda$. Even in the simplest case of free field theories, it is clear that taking such a limit is not entirely trivial. There is a standard piece of folklore that if \mathcal{F} is a “gapped” theory, then the long-distance limit is a topological field theory. In fact, strictly speaking, there are counterexamples to this claim [FMT24]. So a more precise formulation of the folklore is needed.

2. An interesting challenge to the standard Wilsonian paradigm of the renormalization group and effective field theories is presented by the phenomenon of statistical models with fracton excitations [NH18, PCY20, GR22, You24]. It has been proposed that theories with fracton excitations can be defined by defining theories equipped with foliations [SSWC17]. Defining the bordism category for manifolds equipped with foliations is an interesting open problem.
3. The general framework of geometric field theory proved to be quite useful in clarifying how one defines a topological twisting of a general $d = 4$, $\mathcal{N} = 2$ field theory [MSS24].
4. It is worth noting that the standard model of fundamental interactions based on gauge group $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\Gamma$, where $\Gamma \subset \mathbb{Z}_6$, which describes the strong and electroweak interactions – so far as we know – cannot fit into the framework of geometric field theory because the theory is not UV complete.

We conclude that the geometrical approach to QFT is promising but much more remains to be done, even for free field theories. In Part II of these lectures, we turn to a description of an interesting but tractable class of field theories. Even though these theories are free field theories, they present a large number of interesting and instructive features. These are the Generalized Abelian Gauge Theories. They are all generalizations of Maxwell’s formulation of electricity and magnetism and these generalizations appear very naturally in supergravity and string theory, especially when one considers branes in these theories. Giving a complete formulation of these theories along the lines sketched above is a very interesting problem for future research. It is nontrivial, but should be tractable.

PART II: DIFFERENTIAL COHOMOLOGY

29 Preliminary Math Background: Hodge Theory

Before we turn to the study of higher form gauge fields and generalized Abelian gauge theory, it is useful to review the Hodge star and some basic statements of Hodge theory.

Let $(M_n, g_{\mu\nu})$ be an n -dimensional manifold with a metric $g_{\mu\nu}$ defining a section in $\Gamma(\text{Sym}_+^2 T^* M_n)$, where the subscript $+$ stands for positive definite. Assume M_n is oriented, and choose a nowhere zero volume form $\text{vol}(g)$. In local coordinates around a point $p \in M_n$,

$$\text{vol}(g) = \sqrt{|\det g_{\mu\nu}|} \underbrace{dx^1 \wedge \cdots \wedge dx^n}_{\substack{\text{determined by orientation} \\ \text{of } T_p^* M_n}} . \quad (29.1)$$

Now we define a linear operator called the “Hodge star,” as a map,⁷⁸

$$\star : \Omega^\ell(M_n) \longrightarrow \Omega^{n-\ell}(M_n) . \quad (29.2)$$

To do this, we first introduce the local inner product on $\Lambda^\ell T_p^*(M_n)$. For

$$\alpha = \frac{1}{\ell!} \alpha_{\mu_1 \dots \mu_\ell} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_\ell} \in \Lambda^\ell T_p^*(M_n) , \quad (29.3)$$

we define

$$(\alpha, \beta)_p := \frac{1}{\ell!} g^{\mu_1 \nu_1} \cdots g^{\mu_\ell \nu_\ell}(p) \alpha_{\mu_1 \dots \mu_\ell}(p) \beta_{\nu_1 \dots \nu_\ell}(p) . \quad (29.4)$$

Then the formula

$$\alpha \wedge \star \beta|_p := (\alpha, \beta)_p \text{vol}(g) , \quad (29.5)$$

defines $\star \beta$ since it holds for all α .

Exercise 29.1: Hodge Star *

Here are eight exercises on \star . Solve them, and the Hodge star will never strike fear into your heart:

⁷⁸Although we have assumed that M_n is oriented, the Hodge star can, in fact, be defined on an unoriented manifold by twisting with the orientation bundle – a real rank-one vector bundle built as follows. At any point $p \in M_n$, the tangent space $T_p M_n$ has two possible orientations: these choices are the connected components of the set $\Lambda^n T_p^* M_n \setminus \{0\}$. The set of these orientation choices over each point of M_n forms the orientation double cover \widetilde{M}_n of M_n , which is a 2-sheeted cover of M_n , or equivalently, a principal \mathbb{Z}_2 bundle (see [Section 12](#)) over M_n . The orientation bundle $\text{Or}(M_n)$ then is the real line bundle associated to this \mathbb{Z}_2 bundle by letting \mathbb{Z}_2 act on the fiber \mathbb{R} by multiplication by ± 1 . The Hodge star can then be defined as $\star : \Omega^\ell(M_n) \rightarrow \Omega^{n-\ell}(M_n; \text{Or}(M_n))$, where $\Omega^k(M_n; \text{Or}(M_n))$ (see also footnote [64](#)) is the space of k -forms on M_n twisted by the orientation bundle, or equivalently, sections of $\Lambda^k T^* M_n \otimes \text{Or}(M_n)$.

Exercise 1. Show that $\star^2 : \Omega^\ell \rightarrow \Omega^\ell$ acts as multiplication by the sign

$$\star^2|_{\Omega^\ell} = (-1)^{\ell(n-\ell)} \cdot \text{sign}(\det g_{\mu\nu}) , \quad (29.6)$$

and this works for any signature.^a

Exercise 2. Show that in local coordinates,

$$\star(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}) = \frac{|\det g_{\mu\nu}|^{1/2}}{(n-k)!} g^{\mu_1\mu'_1} \cdots g^{\mu_k\mu'_k} \varepsilon_{\mu'_1 \cdots \mu'_k \nu_1 \cdots \nu_{n-k}} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_{n-k}} , \quad (29.7)$$

with $\varepsilon_{12\cdots n} = +1$. Thus $\varepsilon_{\mu'_1 \cdots \mu'_k \nu_1 \cdots \nu_{n-k}}$ takes values in $\{0, \pm 1\}$.

Exercise 3. For a product metric $g_1 \oplus g_2$ on $M_1 \times M_2$ and product orientation $\text{vol}(g_1) \wedge \text{vol}(g_2)$, with $\omega_1 \in \Omega^{k_1}(M_1)$ and $\omega_2 \in \Omega^{k_2}(M_2)$, show that

$$\star_{g_1 \oplus g_2}(\omega_1 \wedge \omega_2) = (-1)^{k_2(n_1 - k_1)} (\star_{g_1} \omega_1) \wedge (\star_{g_2} \omega_2) , \quad (29.8)$$

where $n_1 = \dim M_1$.

Exercise 4. Show that under a conformal transformation,

$$g'_{\mu\nu} = \lambda^2 g_{\mu\nu} \implies \star_{g'_{\mu\nu}}|_{\Omega^k} = \lambda^{n-2k} \star_{g_{\mu\nu}} . \quad (29.9)$$

A corollary of this result is that the Hodge star is conformally invariant when acting on the middle cohomology of an even-dimensional manifold.

Exercise 5. On $\mathbb{M}^{1,1}$ with line element $g_{\mu\nu} dx^\mu dx^\nu = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1$ and volume form $\text{vol}(g) = dx^1 \wedge dx^0$, show that $\star dx^\pm = \pm dx^\pm$, where $x^\pm := x^0 \pm x^1$.

Exercise 6. On $\mathbb{M}^{1,3}$ with metric $g_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ and volume form $\text{vol}(g) = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, show that

$$\star(dx^0 \wedge dx^i) = -\frac{1}{2} \varepsilon^{ijk} dx^j \wedge dx^k , \quad (29.10)$$

$$\star(dx^i \wedge dx^j) = \varepsilon_{ijk} dx^0 \wedge dx^k . \quad (29.11)$$

Exercise 7. On Euclidean \mathbb{R}^D with volume form

$$dx^1 \wedge \cdots \wedge dx^D = r^{D-1} dr \wedge \Omega_{D-1}, \quad (29.12)$$

show that

$$\star d \left(\frac{1}{r^{D-2}} \right) = -(D-2)\Omega_{D-1}. \quad (29.13)$$

Introduce the unit volume form on \mathbb{S}^{D-1} ,

$$\omega_{D-1} = \frac{\Omega_{D-1}}{\nu_D}, \quad (29.14)$$

where

$$\nu_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \quad (29.15)$$

Then

$$d\omega_{D-1} = \delta^{(D)}(0) dx^1 \wedge \cdots \wedge dx^D. \quad (29.16)$$

In general, we will write a δ -function supported form Poincaré dual to $W \subset M$ as $\eta(W \hookrightarrow M)$. So we can write this equation as

$$d \left(\frac{1}{\nu_D} \star \frac{dr}{r^{D-1}} \right) = \eta(\{0\} \hookrightarrow \mathbb{R}^D). \quad (29.17)$$

Exercise 8. In $D = 4$, show that the variation of the Hodge \star (acting on a 2-form) with respect to an arbitrary infinitesimal variation $\delta g_{\mu\nu}$ of the metric is given by^b

$$(\delta_g \star)_{\mu\nu\rho\sigma} = |\det g_{\mu\nu}|^{1/2} \varepsilon_{\mu\nu\rho\sigma} (\delta_g g^{\alpha\rho'}) g_{\rho\rho'} - \frac{1}{4} |\det g_{\mu\nu}|^{1/2} \varepsilon_{\mu\nu\rho\sigma} g_{\rho'\sigma'} \delta_g g^{\rho'\sigma'} \quad (29.18)$$

$$= \left(-|\det g_{\mu\nu}|^{1/2} \varepsilon_{\mu\nu\alpha\sigma} g^{\alpha\alpha'} \delta_\rho^{\beta'} + \frac{1}{4} |\det g_{\mu\nu}|^{1/2} \varepsilon_{\mu\nu\rho\sigma} g^{\alpha'\beta'} \right) \delta g_{\alpha'\beta'}. \quad (29.19)$$

In the second equality, we used $\delta g^{\mu\nu} = -g^{\mu\mu'} g^{\nu\nu'} \delta g_{\mu'\nu'}$.

^a**Solution to Exercise 1:** Choose an ON basis for $T_p^* M_n$, namely, e^1, \dots, e^n , so that the orientation volume form is $e^1 \wedge \cdots \wedge e^n$ and $(e^\alpha, e^\beta) = \eta^\alpha \delta^{\alpha\beta}$, where $\eta^\alpha \in \{\pm 1\}$. For a multi-index $I = (\alpha_1 < \alpha_2 < \cdots < \alpha_\ell)$, let $I_c = (\beta_1 < \beta_2 < \cdots < \beta_{n-\ell})$. Define a sign $s(I, I_c)$ by

$$e^I \wedge e^{I_c} = s(I, I_c) e^1 \wedge \cdots \wedge e^n.$$

Note that $e^I \wedge e^{I_c} = (-1)^{\ell(n-\ell)} e^{I_c} \wedge e^I$. So,

$$s(I, I_c) s(I_c, I) = (-1)^{\ell(n-\ell)}.$$

Now

$$\begin{aligned}\star e^I &= \eta^{\alpha_1} \wedge \cdots \wedge \eta^{\alpha_\ell} s(I, I_c) e^{I_c} , \\ \star e^{I_c} &= \eta^{\beta_1} \wedge \cdots \wedge \eta^{\beta_\ell} s(I_c, I) e^I .\end{aligned}$$

So,

$$\star^2 e^I = \eta^I \wedge \cdots \wedge \eta^n (-1)^{\ell(n-\ell)} e^I = \text{sign}(\det g_{\mu\nu}) (-1)^{\ell(n-\ell)} e^I .$$

^bFor a solution, see [CMRS23, App. E].

Let M_n be an n -dimensional smooth manifold. The de Rham complex is defined to be:

$$\Omega^\bullet(M_n) = \bigoplus_k \Omega^k(M_n) , \quad (29.20)$$

with a degree one operator d that squares to zero. If M_n is equipped with a Euclidean signature Riemannian metric and an orientation, then we can define a nondegenerate and nonnegative inner product on $\Omega^\bullet(M_n)$ by,

$$(\alpha, \beta) = \int_{M_n} \alpha \wedge \star \beta . \quad (29.21)$$

Now, one can compute the adjoint d^\dagger of the exterior derivative with respect to this inner product. One finds that the adjoint of $d : \Omega^k \rightarrow \Omega^{k+1}$ is $d^\dagger : \Omega^{k+1} \rightarrow \Omega^k$, given by⁷⁹

$$d^\dagger = (-1)^{n(k+1)+1} \star d \star . \quad (29.22)$$

Note that since Hodge star squares to a constant, we have $(d^\dagger)^2 = 0$.

Now assume in addition that M_n is compact. Then there is an orthogonal decomposition, known as the Hodge decomposition of Ω^\bullet , where:⁸⁰

$$\Omega^k(M_n) = \mathcal{H}^k(M_n) \oplus \text{im}(d : \Omega^{k-1} \rightarrow \Omega^k) \oplus \text{im}(d^\dagger : \Omega^{k+1} \rightarrow \Omega^k) , \quad (29.23)$$

where $\mathcal{H}^k(M_n)$ is the real vector space of *harmonic forms*:

$$\mathcal{H}^k(M_n) := \{\alpha \mid d\alpha = 0 \text{ and } d^\dagger \alpha = 0\} . \quad (29.24)$$

Remark 29.25. The Hodge decomposition leads to a nice connection to K. Intriligator's TASI 2023 lectures [Int23a, Int23b, Int23c, Int23d]. Consider the quantum mechanics of a

⁷⁹If one performs the same computation, formally, in Lorentzian signature, one finds $d^\dagger = (-1)^{n(k+1)} \star d \star$. Also, note that the definition of d^\dagger in (29.22) holds even if M_n is unorientable, because \star appears twice in it and d^\dagger maps even forms to odd forms and vice versa. See also footnote 78.

⁸⁰See [War83, Ch. 6] a proof.

particle moving on a Riemannian manifold $(M_n, g_{\mu\nu})$. The wavefunctions are in $\Omega^0(M_n) \otimes \mathbb{C}$, and should be L^2 -normalizable with respect to the Riemannian measure. In the language of Part I of these lectures, we are considering an $n = 1$ field theory based on maps,

$$\phi : \mathbb{R}_t \rightarrow (M_n, g_{\mu\nu}) . \quad (29.26)$$

In the supersymmetric case, the target space is replaced by a supermanifold built from the cotangent bundle of M_n , and the Hilbert space of the supersymmetric quantum mechanics is $(\Omega^\bullet(M_n) \otimes \mathbb{C})_{L^2}$, where the subscript L^2 means normalizability with respect to the inner product introduced above. The supersymmetry operators are d and d^\dagger . The Hamiltonian is the Laplacian on forms: $H = dd^\dagger + d^\dagger d$. The vector space of supersymmetric ground states – also known as BPS states – is the (complexified) vector space of normalizable harmonic forms. The connection to Part I of these lectures (and to K. Intriligator's lectures [Int23a, Int23b, Int23c, Int23d]) is that in this one-dimensional field theory, $F(\text{pt}) = (\Omega^\bullet(M_n) \otimes \mathbb{C})_{L^2}$. In Part II of these lectures, we are studying dynamical fields on a spacetime M_n . Same mathematics. Very different physics.

Exercise 29.2: Spectrum Of The Hodge Laplacians

The material in this exercise will be useful when we discuss partition functions in Section 48 below. Suppose M_n is smooth, compact, and oriented. For $0 \leq k \leq n - 1$, define:

$$V_k := \text{im}(d^\dagger : \Omega^{k+1} \rightarrow \Omega^k) , \quad (29.27)$$

and for $1 \leq k \leq n$, define

$$\tilde{V}_k := \text{im}(d : \Omega^{k-1} \rightarrow \Omega^k) . \quad (29.28)$$

(1) Show that

$$\begin{aligned} d : V_s &\rightarrow \tilde{V}_{s+1} & 0 \leq s \leq n-1 , \\ d^\dagger : \tilde{V}_{s+1} &\rightarrow V_s & 0 \leq s \leq n-1 , \end{aligned} \quad (29.29)$$

are isomorphisms. Note that the proof of these assertions is the same as the proof that the Witten index in SQM with target space (M_n, g) is invariant under deformations of the metric.

It is not difficult to show that spectrum $\{\mu_i^{(k)}\}_{i=1,2,\dots}$ of $d^\dagger d$ on V_k is positive, discrete, and unbounded.

- (2) Show that $\{\mu_i^{(k)}\}_{i=1,2,\dots}$ is identical to the spectrum of dd^\dagger on \tilde{V}_{k+1} .
- (3) The Hodge Laplacian on Ω^k is $\Delta_k := dd^\dagger + d^\dagger d$. Show that the spectrum of Δ_k

is:

- (a) $\{\mu_i^{(0)}\} \cup \{0\}$ on Ω^0 .
- (b) The union of $\{\mu_i^{(k)}\} \cup \{\mu_i^{(k-1)}\}$ (and union zero if there are harmonic forms) on Ω^k for $1 \leq k \leq n-1$.
- (c) $\{\mu_i^{(n-1)}\}$ on Ω^n .

30 Classical Generalized Maxwell Theory

We begin with Maxwell theory on $\mathbb{M}^{1,3}$ ((3 + 1)-dimensional Minkowski spacetime). Start with the classical fieldstrength $F \in \Omega^2(\mathbb{M}^{1,3})$. This is a globally defined 2-form. The spacetime splitting leads to a splitting given by a field decomposition

$$F = \frac{1}{2}\varepsilon_{ijk}B_idx^j \wedge dx^k + E_idx^0 \wedge dx^i. \quad (30.1)$$

We define the electric and magnetic fields separately:

$$\begin{aligned} F_m &:= \frac{1}{2}\varepsilon_{ijk}B_idx^j \wedge dx^k, \\ F_e &:= E_idx^0 \wedge dx^i, \end{aligned} \quad (30.2)$$

so that $F = F_m + F_e$. The components E_i and B_i determine vectors and (pseudo-) vectors \mathbf{E} and \mathbf{B} respectively, valued in \mathbb{R}^3 , which are functions of space and time so that, so that we can write half of the vacuum Maxwell equations on $\mathbb{M}^{1,3}$ as:

$$dF = 0 \iff \begin{cases} \nabla \cdot \mathbf{B} = 0, \\ \frac{\partial \mathbf{B}}{\partial x^0} + \nabla \times \mathbf{E} = 0. \end{cases} \quad (30.3)$$

In order to write the other half of the Maxwell equations in terms of F , we must use the Hodge \star operator reviewed in [Section 29](#). Then, the other half of Maxwell's equations in vacuum are:

$$d\star F = 0 \iff \begin{cases} \nabla \cdot \mathbf{E} = 0, \\ \frac{\partial \mathbf{E}}{\partial x^0} - \nabla \times \mathbf{B} = 0. \end{cases} \quad (30.4)$$

There is a natural generalization of Maxwell theory, which we will call “**Classical Generalized Maxwell theory**”, which has the following equations of motion: Consider a manifold (of any dimension n) with nondegenerate metric (of any signature) $(M_n, g_{\mu\nu})$. The physically observable field of the theory is an ℓ -form, $F \in \Omega^\ell(M_n)$, and the on-shell field

satisfies:

$$dF = 0 , \quad (30.5)$$

$$d \star F = 0 . \quad (30.6)$$

Note that if $\ell = 0$, then $dF = 0$ implies that F is a constant, whereas if $\ell = n$, then $d \star F = 0$ implies that F is a constant. So, the most interesting theories arise for $0 < \ell < n$.

Electric-Magnetic Duality: Some authors refer to $dF = 0$ as the “Bianchi identity” and $d \star F = 0$ as the “equation of motion.” But the properties of Hodge \star imply that there is a symmetry of the equations. If we set $\tilde{F} = \star F \in \Omega^{n-\ell}(M_n)$, then \tilde{F} satisfies the same form of the equations with $\ell \leftrightarrow (n - \ell)$, where “equation of motion” and “Bianchi identity” have been exchanged.

Exercise 30.1: Dual Fieldstrength in Maxwell Theory

Returning to $\ell = 2$ on Minkowski space $\mathbb{M}^{1,3}$, work out \tilde{F} in terms of \mathbf{E} and \mathbf{B} .

Exercise 30.2: E and B fields in Generalized Maxwell Theory

When there is a global space-time splitting, define the analogs of electric and magnetic fields for classical generalized Maxwell theory, and write the transformation $F \rightarrow \tilde{F}$ in terms of these fields.

Solutions of the equations of motion. In Minkowski space $\mathbb{M}^{1,n-1}$, a natural ansatz is given by

$$F = f e^{ik \cdot x} , \quad (30.7)$$

with $f \in \Lambda^\ell(\mathbb{R}^n)^\vee$ a constant tensor. The equations are linear, so the superposition principle applies, and we can make real combinations of such solutions. Assume $k \neq 0$. Then plugging (30.7) into $dF = 0$ shows that $k \wedge f = 0$. Since we could take k to be the first vector in a basis, it follows that $f = k \wedge a$ for some rank $(\ell - 1)$ constant tensor a .⁸¹ Then, the equation $d \star F = 0$ implies $k \wedge \star(k \wedge a) = 0$. Wedging this with a shows that $(k, k)(a, a) = 0$. Assuming (as we may) that $(a, a) \neq 0$ it follows that $k^2 = 0$. The condition $k^2 = 0$ implies that in Lorentzian signature, waves propagate at the speed of light. The space of solutions is an infinite-dimensional linear space. What that space exactly is depends on the boundary conditions at infinity.

It follows from the Hodge decomposition of Section 29 that the space of solutions in Euclidean space is the space of harmonic ℓ -forms: $\mathcal{H}^\ell(M_n)$. Note that this space is isomorphic

⁸¹This provides a proof of the Stokes lemma.

to $\mathcal{H}^{n-\ell}(M_n)$ by Poincaré duality. This already shows that there is a close relation between Poincaré duality and electromagnetic duality.

Remark 30.8. *Self-dual theories.*

1. There is an important generalization of Classical Generalized Maxwell Theory called the theory of the self-dual (SD) or anti-self-dual (ASD) field. Suppose $n = 2\ell$. Then

$$\star : \Omega^\ell \rightarrow \Omega^\ell , \quad \star^2 = (-1)^\ell \text{sign}(\det g_{\mu\nu}) . \quad (30.9)$$

When $\star^2 = 1$, we can impose (anti-)self-duality equations on real fieldstrengths:

$$F = \star F \quad (\text{SD field}) \quad \text{or} \quad F = -\star F \quad (\text{ASD field}) . \quad (30.10)$$

It follows from (30.9) that $\star^2 = 1$ for

$$\begin{aligned} \text{Euclidean signature} &: n = 0 \bmod 4 , \\ \text{Lorentzian signature} &: n = 2 \bmod 4 . \end{aligned} \quad (30.11)$$

If we also impose $dF = 0$, then the other Maxwell equation comes for free. This is the classical theory of the (anti-)self-dual field.

2. As an example of a self-dual theory, in two-dimensional Minkowski space $\mathbb{M}^{1,1}$ with line element $ds^2 = -(dx^0)^2 + (dx^1)^2$ and volume element $dx^1 \wedge dx^0$, we can apply the above with $n = 2$ and $\ell = 1$:

$$F = \pm \star F \implies F = \underbrace{\phi(x^0, x^1)}_{\mathbb{R}\text{-valued function}} dx^\pm , \quad (30.12)$$

$$dF = 0 \implies \partial_\mp \phi = 0 . \quad (30.13)$$

So the theory of the $(1+1)\text{d}$ self-dual field is the theory of the chiral self-dual scalar field. Higher-dimensional SD (ASD) fields are generalizations of the theory of the chiral self-dual scalar field.

So far we have introduced generalized Abelian gauge theories via their equations of motion. It is natural to ask about an action principle. We address this next.

31 Action Principle For Generalized Maxwell Fields

Let us return to the non-self-dual field. To write an action principle, we must break manifest electromagnetic duality. We want to use Stokes' lemma, so if we prefer $F \in \Omega^\ell(M_n)$ to

$\tilde{F} \in \Omega^{n-\ell}(M_n)$, then,

$$dF = 0 , \quad (31.1)$$

implies that locally, we can write,

$$F = dA , \quad A \in \Omega^{\ell-1}(\mathcal{U}) , \quad (31.2)$$

in some contractible neighborhood (patch) $\mathcal{U} \subset M_n$. The obstruction to writing $F = dA$ globally is the degree ℓ de Rham cohomology:

$$H_{\text{dR}}^\ell(M) = \ker(d : \Omega^\ell \rightarrow \Omega^{\ell+1}) / \text{im}(d : \Omega^{\ell-1} \rightarrow \Omega^\ell) . \quad (31.3)$$

For a fixed cohomology class $h \in H_{\text{dR}}^\ell(M_n)$, let us choose a representative F_0 with $h = [F_0]$. Then every closed form F with $[F] = h$ can be written as

$$F = F_0 + dA , \quad (31.4)$$

where $A \in \Omega^{\ell-1}(M_n)$ is a globally well-defined $(\ell - 1)$ -form. We can now write an action principle for the field A . The action depends on a choice of de Rham cohomology class h :

$$S_h[A] = \int \pi \lambda F \wedge \star F . \quad (31.5)$$

where $\pi \lambda$ is a coupling constant, with the factor of π introduced for later convenience in [Section 48](#). Stationarity, $\delta S_h[A] = 0$, yields the other Maxwell equation:

$$d \star F = 0 . \quad (31.6)$$

Now a theorem due to Hodge states that for every de Rham cohomology class, there is a unique harmonic representative in that class and consequently, we have an isomorphism of real vector spaces:

$$\underbrace{H_{\text{dR}}^\ell(M_n)}_{\ell^{\text{th}} \text{ de Rham cohomology}} \cong \underbrace{\mathcal{H}^\ell(M_n)}_{\substack{\text{space of} \\ \text{harmonic forms} \\ \text{of degree } \ell}} . \quad (31.7)$$

One proves this by combining the Hodge decomposition with the existence of a Green's function (a.k.a. Green function [[Wri06](#)]) for the Laplacian $\{d, d^\dagger\}$. Again, see [[War83](#), Ch. 6] for a proof.

Now, if F_h is the harmonic representative of h then we have the Hodge decomposition

$$F = F_0 + dA = F_h + d\alpha , \quad (31.8)$$

and Hodge theory guarantees that for any F_0 , there exists an A that sets $\alpha = 0$. On the other hand, substitution into the action gives,

$$S_h[A] = \pi\lambda \int F_h \wedge \star F_h + \pi\lambda(d\alpha, d\alpha) . \quad (31.9)$$

Now $(d\alpha, d\alpha) \geq 0$ and is equal to zero only when $d\alpha = 0$ (almost everywhere). Therefore, there exists a unique minimum of the action (31.5) on a compact Euclidean M_n , and the minimum is the unique harmonic form in the specified cohomology class h .

One benefit of the action (31.5) is that it gives a coupling to gravity (i.e., a coupling to the background field given by a Riemannian metric). This coupling defines an energy-momentum tensor. In general, the energy-momentum tensor T_F is defined by the variation of the action with respect to the background metric: in local coordinates,

$$\delta S = \int_{M_n} \text{vol}(g) \frac{1}{2} \delta g^{\mu\nu} T_{F,\mu\nu} . \quad (31.10)$$

In the present case, one can show that,

$$T_F \in \Gamma(\text{Sym}^2 T^* M_n) , \quad (31.11)$$

is defined by saying that for all $v \in T_p M_n$,

$$(\pi\lambda)^{-1} T_F(v \otimes v)|_p = (\iota_v F, \iota_v F)_p - \frac{1}{2}(v, v)_p (F, F)_p . \quad (31.12)$$

Exercise 31.1: T_F in Generalized Maxwell Theory

Derive (31.12).

Remark 31.13.

1. In the case $\ell = 1$, we can write $F = d\phi$ locally, but the scalar field ϕ might not be globally defined. If we take $\phi \in \mathbb{R}/2\pi\mathbb{Z}$, then the factor λ has the meaning of the squared radius of the target space circle of ϕ . Recall that, in general, the form of the nonlinear σ -model action,

$$\int g_{ij}(X) h^{\mu\nu} \partial_\mu X^i \partial_\nu X^j \text{vol}(h) , \quad (31.14)$$

shows that the kinetic term defines a $(\text{length})^2$ on the target space.

2. More generally, without constraints on the periods of F , λ is meaningless and can be absorbed via a redefinition of F . But if periods of F are quantized, λ has physical meaning. For example for $\ell = 1$, it is proportional to the squared radius of the target

circle. For $\ell = 2$, it is proportional to the inverse square of the electric coupling, and so forth.

3. **Electric-Magnetic Duality:** Even though the action principle (31.5) has broken manifest electromagnetic duality, physical quantities such as the energy-momentum tensor are electromagnetically dual. In ordinary Maxwell theory, the familiar equations for energy and momentum density are $\mathbf{E}^2 + \mathbf{B}^2$ and $\mathbf{E} \times \mathbf{B}$ respectively, and are thus clearly electric-magnetic dual. Equation (31.16) below generalizes that statement to all generalized Maxwell fields.
4. The above remark raises the obvious question of whether there is an action which is manifestly electromagnetically dual. This is a subtle and difficult problem. If we write an action like

$$\int_{M_n} (F \wedge \star F + \tilde{F} \wedge \star \tilde{F}), \quad (31.15)$$

then we have doubled the number of degrees of freedom, and thereby changed the theory. If we try to introduce Lagrange multipliers to enforce the duality relation, we introduce again new degrees of freedom. The issue is closely related to writing a Lorentz invariant action for the self-dual field, something we comment on in Section 31.1 below.

Exercise 31.2: Electric-Magnetic duality invariance of T_F

Show that equation (31.12) is invariant under electric-magnetic duality:

$$T_F = T_{\tilde{F}}. \quad (31.16)$$

31.1 An Action Principle For The Self-Dual Field?

In the self-dual case, $F = \pm \star F$ and $n = 2 \bmod 4$, so we have $\ell = 1 \bmod 2$ odd, hence $\int F \wedge \star F = \int F \wedge F = 0$. Certainly there is no obvious local and Lorentz invariant action for a self-dual field. Many people infer that there is no action principle for the self-dual field. This is not true. In fact, there exist (many) actions for the self-dual field, but in each case, more data needs to be provided.

There is a very large literature on attempts to write an action for the self-dual field. For some recent papers addressing this issue, together with extensive references to the many previous works on the subject, see [Sen19, Lam23].

One of the many approaches to formulating a Lagrangian for the self-dual field uses the “holographic” duality between Chern-Simons theory on a manifold with boundary and the chiral edge modes on the boundary [BM06a]. There is a close analogy to a theta function: The function exists and is independent of duality frame, but to write an explicit formula

one must choose a suitable Lagrangian decomposition of a cohomology space. We sketch the idea of [BM06a] in the simplest case where there is no nontrivial topology, that is we consider the theory on $\mathbb{R}^{1,4s+1}$. Similar considerations work in Euclidean signature and on manifolds with nontrivial topology. The argument proceeds along the following lines:

1. One begins by observing that $V := \Omega^\ell(M_n)$ with $\ell = 2s+1$ has a symplectic structure:

$$\omega(\phi_1, \phi_2) := \int_{M_n} \phi_1 \wedge \phi_2 . \quad (31.17)$$

2. First, as in the ordinary non-self-dual theory, to formulate an action, we restrict attention to fields $R \in V_{\text{cl}} := \Omega_d^\ell(M)$. We will vary within this space. Note that it is a Lagrangian subspace of V . We call the fieldstrength R so that it will not be confused with the classical self-dual field we see in the semiclassical physics of this theory. We will do the path integral over closed fields R modulo gauge transformations.
3. Next, we *choose* another Lagrangian subspace $V_m \subset V$, assumed to be maximal Lagrangian and transversal to V_{cl} (i.e., $V_{\text{cl}} \cap V_m = \{0\}$) and moreover,

$$V = V_m \oplus \star V_m , \quad (31.18)$$

is a decomposition into maximal Lagrangian subspaces. It is important to demand that V_m and $\star V_m$ are transverse. (This condition can be slightly relaxed.) This is the extra piece of data one must introduce to obtain an action principle.

4. Now, given (31.18), there is a unique decomposition of any $R \in V_{\text{cl}}$ as:

$$R = R_m + R_e , \quad (31.19)$$

with $R_m \in V_m$ and $R_e \in V_e := \star V_m$.

5. The Lorentzian signature action for the ϵ -self-dual field is then:

$$S = \pi \int (R_e \star R_e + \epsilon R_e R_m) . \quad (31.20)$$

There are two nice features of this action:

6. First, the action is stationary iff the ϵ -self-dual field $\mathcal{F} := R_e - \epsilon \star R_e$ is closed:

$$d\mathcal{F} = 0 . \quad (31.21)$$

Thus, the set of stationary points of the action is the set of solutions of the self-dual equations of motion for \mathcal{F} .

7. The proof that the variation of the action gives (31.21) goes as follows:

$$\begin{aligned}\delta S &= \pi \int 2\delta R_e \star R_e + \epsilon \delta R_e R_m + \epsilon R_e \delta R_m \\ &= \pi \int 2\delta R_e \star R_e + 2\epsilon \delta R_e R_m \\ &= 2\pi \int \delta R (\star R_e + \epsilon R_m) \\ &= 2\pi \int d(\delta c) (\star R_e + \epsilon R_m) ,\end{aligned}\tag{31.22}$$

where in the second line, we used the fact that both R and δR are in V_{cl} , which is Lagrangian. In the third line, we notice that $\star R_e + \epsilon R_m \in V_m$, and hence, we can replace δR_e by δR . In the fourth line, we use the fact that variations of R in V_{cl} are exact. Now integration by parts gives $d(\star R_e + \epsilon R_m) = 0$. Finally, $dR = dR_e + dR_m = 0$, so $d(\star R_e + \epsilon R_m) = 0$ is equivalent to $d(\star R_e - \epsilon R_e) = 0$, which is equivalent to $d(R_e - \epsilon \star R_e) = 0$.

A nice feature of the above action is that if we consider the action as a functional of both the metric and the field R then, varying the metric holding R fixed, the action varies into

$$\delta S = \frac{\pi}{2} \int \text{vol}(g) \delta g^{\mu\nu} T(\mathcal{F})_{\mu\nu} ,\tag{31.23}$$

where $T(\mathcal{F})_{\mu\nu}$ is the standard energy-momentum tensor for the ϵ -self-dual field $\mathcal{F} := R_e + \epsilon \star R_e$.

A common choice of V_m is given by choosing some timelike direction ξ , and letting V_m be the forms annihilated by $\iota(\xi)$. It is commonly said that there is no Lorentz invariant action for the self-dual field, but this is not really true, as the following example shows. Let us consider the self-dual scalar on $\mathbb{R}^{1,1}$ so $\star dx^\pm = \pm dx^\pm$. We can choose V_m to be of the form,

$$V_m = \{R = f(x)dx^+ + (G \cdot f)(x)dx^- \} ,\tag{31.24}$$

where $f(x)$ is any suitably normalizable function and $(G \cdot f)(x) := \int G(x, y)f(y)d^2y$. Then V_m will be Lagrangian if $G(x, y) = G(y, x)$ is symmetric, and it will be Lorentz invariant if $G(\phi_\lambda(x), y) = \lambda^{-2}G(x, \phi_\lambda^{-1}(y))$, where $\phi_\lambda(x^+, x^-) = (\lambda x^+, \lambda^{-1}x^-)$. An example of such a kernel function would be $G(x, y) = (x^+ - y^+)^{-2}$. This leads to the action for a self-dual scalar field,

$$S = \pi \int \partial_+ \phi \partial_- \phi d^2x - \pi \int d^2x d^2y \partial_- \phi(x) G^{-1}(x, y) \partial_- \phi(y) .\tag{31.25}$$

While equation (31.25) is Lorentz invariant, it is nonlocal. It is possible that with a suitable definition of “local,” there is no local and Lorentz invariant Lagrangian for the self-

dual field. A first step to proving such a no-go theorem would be showing that there is no “local” and Lorentz invariant choice for V_m .

Remark 31.26.

1. Standard folk wisdom states that there is no description of a non-Abelian analog in terms of elementary fields and a local action. (There are local fields in the non-Abelian analog, i.e., the energy-momentum tensor.) There is, however, a hypothetical quantum theory of a non-Abelian self-dual field. Current folklore asserts that it is impossible to write fundamental fields that describe this theory and there is no generally accepted description in terms of an action principle. About twenty years ago, people said similar things about the non-Abelian M2-brane theory, but then along came the ABJM description [ABJM08]. One moral from [ABJM08] is that folklore can be overturned. It would be good to state and prove a rigorous no-go theorem about classical non-Abelian self-dual actions. The hypotheses of no-go theorems merely define the limits of our imagination.
2. In unpublished work, I. Nidaiev [Nid17, Nid19] began with the formulation of [BM06a] for the self-dual field and refined it into a formulation very close to that of the work of A. Sen [Sen19]. For further developments, see [Hul23, HL25a, HL25b].

32 Many Maxwell Fields

32.1 Kaluza-Klein Reduction Of The Generalized Maxwell Field

One way to motivate the consideration of theories with several Maxwell fields is to consider the reduction of the theory of a single Maxwell field $F \in \Omega^\ell(M_n \times K)$, where K is a compact manifold and we assume there is a direct sum metric on $M_n \times K$.

We will show that if we consider a family of background fields where the metric of K is scaled to zero with a real parameter t^2 , and we take the limit $t \rightarrow 0$, then the correlation functions of the generalized Maxwell theory coincides – up to exponential accuracy in t – with those of a generalized Maxwell theory in n dimensions with fields,

$$F \in \bigoplus_{p+q=\ell} \Omega^p(M_n) \otimes \mathcal{H}^q(K) , \quad (32.1)$$

where $\mathcal{H}^q(K)$ is the real vector space of harmonic q -forms on K .

In order to justify the claim above, we introduce an ON eigenbasis of the Laplacian on $\Omega^q(K)$, denoted $\phi_s^q \in \Omega^q(K)$, where s labels the eigenmodes. It is convenient to change the index s to reflect the Hodge decomposition on K , so we separate the eigenmodes into:

$$\phi_h^q , \quad h = 1, \dots, b_q(K) , \quad \phi_e^q , \quad e = 1, \dots, \infty , \quad \phi_c^q , \quad c = 1, \dots, \infty , \quad (32.2)$$

corresponding to the harmonic, exact, and co-exact eigenmodes, respectively. Then the general section $F \in \Omega^\ell(M_n \times K)$ can be written as:

$$\begin{aligned} F &= \sum_{p=0}^{\ell} \sum_s F_s^p \phi_s^{\ell-p} \\ &= \sum_{p=0}^{\ell} \sum_{h=1}^{b_q(K)} F_h^p \phi_h^q + \sum_{p=0}^{\ell-1} \sum_{e=1}^{\infty} F_e^p \phi_e^q + \sum_{p=1}^{\ell} \sum_{c=1}^{\infty} F_c^p \phi_c^q . \end{aligned} \quad (32.3)$$

Here F_s^p are pulled back from M_n , so we can regard them as $F_s^p \in \Omega^p(M_n)$. We stress that F_h^p, F_e^p, F_c^p are general sections and the index h, e, c indicates the nature of the form (harmonic, exact, or coexact) it multiplies in the Fourier decomposition (32.3).

Now, imposing that F is exact shows that $dF_h^p = 0$ and

$$F_c^p = -\frac{1}{\sqrt{\mu_e^{(\ell-p+1)}}} dF_e^{p-1} , \quad (32.4)$$

where we are using the isomorphisms (29.29) to establish a 1–1 correspondence between the indices c and e .

Substitution into the action (31.5) gives an action,

$$S = \pi\lambda \int_{M_n} \sum_{p=0}^{\ell} \int F_h^p \star F_h^p + \pi\lambda \sum_{p=0}^{\ell-1} \sum_{e=1}^{\infty} \int_{M_n} \left(\frac{1}{\mu_e^q} dF_e^p \star dF_e^p + F_e^p \star F_e^p \right) , \quad (32.5)$$

subject to the constraint that $dF_h^p = 0$. We stress again that the fields F_e^p are general sections of $\Omega^p(M_n)$. The path integral for F_e^p computes a determinant of a Laplacian of

$$d^\dagger d + \mu_e^q , \quad (32.6)$$

on $\Omega_{L^2}^p(M_n)$ and will be 1, up to exponentially small terms in the scale t of the metric on K . Similar arguments apply other observables. This justifies the Kaluza-Klein reduction to a Maxwell field valued in (32.1) and satisfying the equations of motion $dF = 0$ and $d \star F = 0$.

32.2 Coupling Constants

The previous section has taught us that it is natural to broaden the definition of a generalized Maxwell field to a field,

$$F \in \Omega^\bullet(M_n; V) , \quad (32.7)$$

where V is a graded vector space, and one can impose a condition on the total grading. One could also generalize further to the case where V is a flat bundle. For simplicity, in the

following discussion, we will simply take V to have degree zero and,

$$F \in \Omega^\ell(M_n; V) , \quad (32.8)$$

where the fixed vector space V has grading zero.

In order to write an energy momentum tensor, we should assume that V is endowed with a positive definite metric. This also allows us to write a Lagrangian. By choosing a basis e_I for V we can express the theory as a theory of several generalized Maxwell fields F_I , where $F = F^I e_I$. The Lagrangian is:

$$S = \pi \int_{M_n} \lambda_{IJ} F^I \star F^J . \quad (32.9)$$

Remark 32.10.

1. The λ_{IJ} is a positive definite real symmetric metric on V . If we make no assumptions about the periods of the F^I , it can be diagonalized and scaled to the unit matrix. On the other hand, if, as in string theory, one assumes some quantization conditions on the periods, then λ_{IJ} becomes meaningful. For example, in the Kaluza-Klein example above, it is more natural to replace the ON basis ϕ_h^q (which depends on a metric) to a metric-independent basis of harmonic forms with integral periods.
2. When there are several Maxwell fields an additional coupling constant can be added so that the action reads:

$$S = \pi \int_{M_n} (\lambda_{IJ} F^I \star F^J + \theta_{IJ} F^I F^J) , \quad (32.11)$$

where θ_{IJ} is an anti-symmetric form on V when ℓ is odd and a symmetric form when ℓ is even. These terms do not affect the equations of motion but have profound effects on the quantum system.

3. It is natural to generalize λ_{IJ} and θ_{IJ} to be functions on M_n . When the generalized Maxwell field is embedded in a larger theory, these fields often have their own kinetic terms. The θ_{IJ} are known as axion fields.

32.3 Manifestly Electromagnetically Dual Equations Of Motion

On application of the generalization of Maxwell theory to fields of the form (32.7) is that it gives a nice framework for writing equations of motion that are manifestly electric-magnetic duality-invariant. For simplicity, we consider the case of $n = 0 \bmod 4$ dimensions with Lorentzian signature with $\ell = n/2$. (This includes the original case of Maxwell theory.) We are following the discussion in sections 2.5.3 – 2.5.4 of [Moo10].

We will introduce a field,

$$\mathbb{F} \in \Omega^\ell(M_n) \otimes V , \quad (32.12)$$

where V is a real vector space. In order to be able to write an energy-momentum tensor, V will need to be endowed with a positive definite symmetric metric.

Our field will be closed, $d\mathbb{F} = 0$ and will satisfy a self-duality condition of the form,

$$\mathbb{F} = (\star \otimes J)\mathbb{F} , \quad (32.13)$$

where

$$J : V \rightarrow V , \quad (32.14)$$

is a linear transformation. Under the conditions specified above with $\ell = n/2$ and even integer and Lorentzian signature consistency requires $J^2 = -1$. In other words, the real vector space V admits a complex structure. It follows that V must have even dimension:

$$\dim_{\mathbb{R}} V = 2r , \quad (32.15)$$

for some $r \in \mathbb{Z}_{\geq 0}$. We claim that the equations:

$$\begin{aligned} d\mathbb{F} &= 0 , \\ \mathbb{F} &= (\star \otimes J)\mathbb{F} , \end{aligned} \quad (32.16)$$

describe a manifestly electric-magnetically dual formulation of a collection of s independent generalized Maxwell fields of degree ℓ .

For compatibility of the energy-momentum tensor and the self-duality condition, the metric should be compatible with the metric:

$$(J(v), J(v')) = (v, v') \quad \forall v, v' \in V , \quad (32.17)$$

where (\cdot, \cdot) denotes the inner product on V . It follows that V has a symplectic structure ω defined by⁸²

$$\omega(v, v') := (v, J(v')) . \quad (32.18)$$

To lighten the notation, we write $\langle v, v' \rangle := \omega(v, v')$. We can therefore introduce a symplectic basis (aka a Darboux basis) denoted $\{\alpha_I, \beta^I\}$ where $I = 1, \dots, r$. Our convention is that

$$\begin{aligned} \langle \alpha_I, \alpha_J \rangle &= 0 , \\ \langle \beta^I, \beta^J \rangle &= 0 , \\ \langle \alpha_I, \beta^J \rangle &= \delta_I^J . \end{aligned} \quad (32.19)$$

⁸²In general when a real vector space has a compatible metric, complex structure and symplectic structure, any two structures will imply the third.

The linear span of α_I is a maximal Lagrangian subspace $L_1 \subset V$, while that for β^I is another $L_2 \subset V$, and we have a Lagrangian decomposition:

$$V \cong L_1 \oplus L_2 . \quad (32.20)$$

Upon complexification, the \mathbb{C} -linear extension of J has eigenvalues $\pm i$. We let $V^{0,1}$ denote the $-i$ eigenspace and $V^{1,0}$ the $+i$ eigenspace, so:

$$V \otimes_{\mathbb{R}} \mathbb{C} \cong V^{0,1} \oplus V^{1,0} . \quad (32.21)$$

Given a Darboux basis, we can define a basis for $V^{0,1}$ of the form,

$$f_I := \alpha_I + \tau_{IJ}\beta^J , \quad I = 1, \dots, r . \quad (32.22)$$

while $V^{1,0}$ is then spanned by,

$$\bar{f}_I := \alpha_I + \bar{\tau}_{IJ}\beta^J , \quad I = 1, \dots, r . \quad (32.23)$$

Compatibility of J with the symplectic structure now implies $\langle f_I, f_J \rangle = 0$ and hence $\tau_{IJ} = \tau_{JI}$. It is useful to split τ_{IJ} into its real and imaginary parts:

$$\tau_{IJ} = X_{IJ} + i Y_{IJ} . \quad (32.24)$$

Positive definiteness of (\cdot, \cdot) implies Y_{IJ} is positive definite. It will be convenient to denote the matrix elements of the inverse by Y^{IJ} , so,

$$Y^{IJ}Y_{JK} = \delta_K^I . \quad (32.25)$$

Exercise 32.1: Darboux basis and symplectic structure

1. Find the change of basis between f_I, \bar{f}_I and α_I, β^I in terms of the matrices X, Y .
2. Compute the action of J in the basis α_I, β_I in terms of the matrices X, Y .

Now, given a Darboux basis we can expand:

$$\mathbb{F} = \alpha_I F^I - \beta^I G_I . \quad (32.26)$$

Then, the self-dual equation allows one to solve for G_I in terms of $F^I, \star F^I$, and the matrices X, Y :

$$G_J = -Y_{JK} \star F^K - X_{JK} F^K . \quad (32.27)$$

Then, the equations of motion $dF^I = 0$ and $dG_I = 0$ can be derived from a Lagrangian:

$$\int_{M_n} Y_{JK} F^J \star F^K + X_{JK} F^J F^K . \quad (32.28)$$

Remark 32.29.

1. Of course, one could change the sign of the duality condition in (32.16) to an anti-self-duality condition. The change in the Lagrangian is a change in the axion coupling.
2. The exchange of duality basis $\alpha_I, \beta^J \mapsto \beta^J, -\alpha_I$ corresponds to the electromagnetic duality we have thus far described. In fact, we see that in this theory, there is a natural duality group $\text{Sp}(V)$, which is the group of symplectic linear transformations of V and the equations (32.16) are manifestly $\text{Sp}(V)$ -invariant because they do not refer to a specific Lagrangian decomposition. If one makes a choice of Lagrangian decomposition of V , then it is possible to write an action principle for the equations. This is compatible with the considerations of Section 31.1.
3. If one imposes a quantization condition on the periods of F^I , then in general, only a discrete subgroup of $\text{Sp}(V)$ will be compatible with that quantization condition. This typically happens in string theory, where the theory of branes implies quantization conditions.
4. If we specialize to a single self-dual 3-form $H \in \Omega^3(M_4 \times \Sigma)$, where Σ is a closed Riemann surface, then Kaluza-Klein reduction gives a family of Maxwell fields F^I , $I = 1, \dots, b_1(\Sigma)$. The space of harmonic one forms $\mathcal{H}^1(\Sigma)$ carries a natural geometric symplectic structure: The intersection form. The matrix τ_{IJ} relative to a Darboux basis has the geometric interpretation of a period matrix for the Riemann surface Σ . If we require that the periods of F^I are integral, then the duality group becomes $\text{Sp}(2b_1, \mathbb{Z})$, and thus we have geometrized the duality transformations. This geometric interpretation of duality symmetry appears naturally in string theory, in F-theory, and has been used in precisely this context in [Wit95, Ver95].

33 Electric And Magnetic Currents

Recall that in vacuum, we had

$$dF = 0 , \quad \text{and} \quad d \star F = 0 . \quad (33.1)$$

Now, let us introduce magnetic and electric currents as external sources in these equations. The currents are:

$$\begin{aligned} J_m &\in \Omega^{\ell+1}(M_n) && \text{magnetic ,} \\ J_e &\in \Omega^{n-\ell+1}(M_n) && \text{electric .} \end{aligned} \quad (33.2)$$

The response of fields to background currents is given by Maxwell's equations with sources

$$d \star F = J_e , \quad (33.3)$$

$$dF = J_m . \quad (33.4)$$

Remark 33.5.

1. A nonzero magnetic current obstructs the existence of a gauge potential: $J_m \neq 0$ implies $F \neq dA$, even locally, at least not where the support of J_m is. So, if one does not excise the region of support of J_m , it is not possible to regard Maxwell theory as a theory of a connection on a $U(1)$ line bundle. One will have to start talking about *gerbes* and *gerbe connections*, best described in the context of differential cohomology. See [Section 40.1.1](#) below.
2. Current conservation automatically follows from these equations, as the currents are closed:

$$dJ_m = 0 , \quad dJ_e = 0 . \quad (33.6)$$

3. If F is smooth, then equations (33.4) say that J_m and J_e are cohomologically trivial, i.e., $[J_m] = 0$ and $[J_e] = 0$. This may seem confusing: don't the cohomology classes have something to do with the charge? The answer is that they do, but the charge is measured in a relative cohomology group, see [Box 33.1](#).

Box 33.1: Relative Cohomology

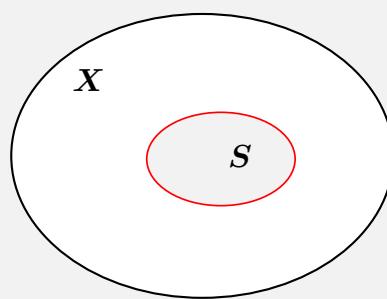


Figure 39: A topological space X and a subspace $S \subset X$.

(See also Bott-Tu [BT82].) Fix a positive integer $k \in \mathbb{Z}_{\geq 1}$. Given an inclusion $\iota : S \hookrightarrow X$ (see Figure 39), the *relative cochain complex* is

$$C_{\text{dR}}^k(X, S) = \Omega^k(X) \oplus \Omega^{k-1}(S) , \quad (33.7)$$

with differential

$$d(\omega, \theta) := (d\omega, \iota^*\omega - d\theta) . \quad (33.8)$$

Define the k^{th} *relative cohomology group of the pair* (X, S) :

$$H_{\text{dR}}^k(X, S) := \ker d / \operatorname{im} d . \quad (33.9)$$

(This is not to be confused with cohomology with a nontrivial coefficient group.) Now, the short exact sequence

$$\begin{aligned} 0 \rightarrow \Omega^{k-1}(S) &\rightarrow C^k(X, S) \rightarrow \Omega^k(X) \rightarrow 0 \\ \theta &\longmapsto (0, \theta) \\ (\omega, \theta) &\longmapsto \omega \end{aligned} \quad (33.10)$$

induces a long exact sequence in cohomology

$$\cdots \longrightarrow H^{k-1}(S) \longrightarrow H^k(X, S) \longrightarrow H^k(X) \xrightarrow{\text{restriction}} H^k(S) \longrightarrow \cdots \quad (33.11)$$

Exercise 33.1: Relative Cohomology

For the differential (33.8):

- (a) Check that $d^2 = 0$.
- (b) Show that “closed” means $d\omega = 0$ and $\omega|_S$ is trivialized.

Let us now apply the above ideas about relative cohomology to the question of how the charges are related to cohomology classes of currents. The following discussion comes from [MW99].⁸³ Note that $(J_e, \star F)$ is exact, but $[(J_e, 0)]$ could be a nontrivial class in $H_{\text{dR}}^{n-\ell+1}(M_n, M_n^{-J_e})$, where $M_n^{-J_e} := M_n - \text{Supp}(J_e)$. There is a long exact sequence in

⁸³In this section, we are working with de Rham cohomology, i.e., with coefficients in \mathbb{R} . However, these results hold more generally for integral cohomology (i.e., coefficients in \mathbb{Z}), and there is an analogous version in K -Theory, as discussed in [MW99].

cohomology,

$$\begin{aligned} \cdots &\longrightarrow H_{\text{dR}}^{n-\ell}(M_n) \xrightarrow{\iota^*} H_{\text{dR}}^{n-\ell}(M_n^{-J_e}) \xrightarrow{\delta} H_{\text{dR}}^{n-\ell+1}(M_n, M_n^{-J_e}) \\ &\xrightarrow{\psi} H_{\text{dR}}^{n-\ell+1}(M_n) \longrightarrow \cdots \end{aligned} \quad (33.12)$$

Therefore,

$$\ker \psi \cong \text{im } \delta \cong H_{\text{dR}}^{n-\ell}(M_n^{-J_e}) / \iota^* H_{\text{dR}}^{n-\ell}(M_n) . \quad (33.13)$$

The electric charge group is by definition the kernel of ψ , and by the long exact sequence,

$$Q_e \cong H_{\text{dR}}^{n-\ell}(M_n^{-J_e}) / \iota^* H_{\text{dR}}^{n-\ell}(M_n) . \quad (33.14)$$

Here $H_{\text{dR}}^{n-\ell}(M_n)$ is the classical flux group: these are the fluxes not sourced by charge. This is the mathematical expression of the idea that the charge is measured by the “flux at ∞ ”. If $M_n = \mathbb{R}_t \times N_{n-1}$ and the support of J_e is compact for all time, we can identify Q_e as the kernel of the map,

$$H_{\text{cpct}, \text{dR}}^{n-\ell+1}(N_{n-1}) \rightarrow H_{\text{dR}}^{n-\ell+1}(N_{n-1}) , \quad (33.15)$$

where the subscript ‘cpct’ denotes compactly supported cohomology.

Box 33.2: Relative cohomology and compactly supported cohomology

The relative cohomology group $H_{\text{dR}}^{n-\ell+1}(M_n, M_n^{-J_e})$ is isomorphic to the compactly supported cohomology group $H_{\text{cpct}, \text{dR}}^{n-\ell+1}(M_n)$.

34 Branes

As before, $n \in \mathbb{Z}_{\geq 1}$ denotes the spacetime dimension, and fix $p \in \mathbb{Z}_{\geq 0}$ such that $p \leq n - 1$. p -branes are extended objects generalizing point particles with worldvolumes:

$$\mathcal{W} = \mathcal{S}_p \times \mathbb{R}_t \subset N_{n-1} \times \mathbb{R}_t = M_n . \quad (34.1)$$

In a generalized Maxwell theory, electrical branes can be viewed as objects that produce an electric current:

$$J_e = q_e \underbrace{\eta(\mathcal{W}_e \hookrightarrow M_n)}_{\begin{array}{c} \delta\text{-function representative} \\ n - (p_e + 1)\text{-form} \\ \text{Poincar\'e dual to } \mathcal{W}_e \end{array}} , \quad (34.2)$$

Then $d \star F = J_e$ implies electrically charged branes have $p_e = \ell - 2$.

A crucial solution exists when $\mathcal{S}_{p_e} = H_e$ is a hyperplane in spatial \mathbb{R}^{n-1} , see Figure 40.

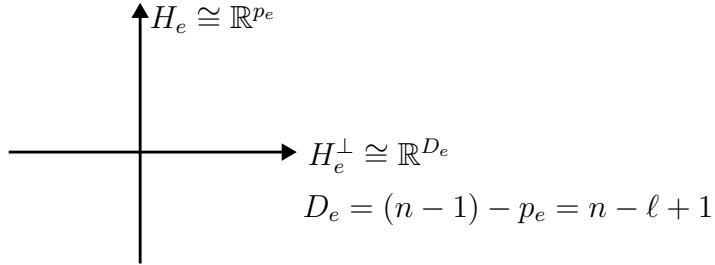


Figure 40: An flat electric p -brane as a hyperplane.

The field due to this configuration is given by,

$$F = \frac{q_e}{V_e} \frac{dt \wedge dr}{r^{D_e-1}} \wedge \text{vol}(H_e) . \quad (34.3)$$

From (33.12),

$$\begin{aligned} H^0(\mathcal{W}_e) &\xrightarrow{\iota^*} H^{n-\ell+1}(M_n, M_n - \mathcal{W}_e) \\ \mathbb{R} \ni q_e &\longmapsto [(J_e, 0)] . \end{aligned} \quad (34.4)$$

Similarly, magnetic branes would have a worldvolume:

$$\mathcal{W}_m = \mathcal{S}_m \times \mathbb{R}_t \subset \mathcal{N}_{n-1} \times \mathbb{R}_t = M_n , \quad (34.5)$$

and produce a magnetic current:

$$J_m = q_m \underbrace{\eta(\mathcal{W}_m \hookrightarrow M_n)}_{\substack{\delta\text{-function representative} \\ (\ell+1)\text{-form}}} , \quad (34.6)$$

Poincaré dual to \mathcal{W}_m

Then $dF = J_m$ implies $p_m = n - \ell - 2$.

In $\mathbb{M}^{1,n-1}$,

$$F = 2\pi q_m \omega_\perp , \quad (34.7)$$

where ω_\perp is the unit volume form for $\mathbb{S}^\ell \subset H_m^\perp \cong \mathbb{R}^{\ell+1}$.

35 Quantum Mechanics Of A Test Brane In An External Classical Generalized Maxwell Field

Let us recall a key insight from the theory of branes [Pol95, CHS91a, CHS91b, CHS91c, Joh02]. Branes are not just solitonic solutions in supergravity, but are dynamical objects which must be included in the description of the space of states in string theory. Unlike defects, branes can wiggle and move. These are among their internal degrees of freedom. So – similar to the AdS/CFT correspondence [Mal97, Wit98a, AGM⁺99] – we can change our point of view and consider the worldvolume theory of a test brane moving in a prescribed field configuration F .

Let's start with a charged particle moving in the field of an $\ell = 2$ Maxwell theory in a Lorentzian signature spacetime M_n . The gauge field is a nondynamical background field. In this context, one sometimes refers to the particle as a “test” particle, because the backreaction on the electromagnetic field on M_n is ignored. For an uncharged but massive particle with worldline $\mathcal{W}_1 \subset M_n$, the action is given by,

$$S = \int_{\mathcal{W}_1} T ds , \quad ds = \sqrt{-\left(\frac{dx}{d\tau}\right)^2} d\tau . \quad (35.1)$$

Here τ is the time coordinate along a worldline, and T is the mass of the test particle. Choosing boundary conditions so that the boundary variation vanishes, the equation of motion is the geodesic equation,⁸⁴

$$\frac{d}{d\tau} \left(T \frac{dx_\mu}{ds} \right) = 0 . \quad (35.2)$$

The action (35.1) has a natural generalization to arbitrary branes known as the *Dirac-Born-Infeld* (DBI) action.⁸⁵ It is simply the induced volume of the worldvolume times T , where T – known as the *tension* – has units of energy divided by spatial worldvolume. So in units where $\hbar = c = 1$, it has units of energy to the $(p+1)^{th}$ power, so the action is

⁸⁴Note that (35.2) is equivalent to

$$\frac{d}{d\tau} \left(\frac{T g_{\mu\nu}}{\sqrt{-\left(\frac{dx}{d\tau}\right)^2}} \frac{dx^\nu}{d\tau} \right) = 0 ,$$

which, after using the chain rule and standard expressions for the Christoffel symbols, reduces to the conventional form of the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 ,$$

for constant T .

⁸⁵The DBI action was originally introduced as a formulation of nonlinear electrodynamics intended to cure the divergence of electron self-energy. See [nLa25a] for more information.

dimensionless, as it must be.

Now let us return to the particle in $\ell = 2$ Maxwell theory, and let us suppose that the test particle also has electric charge. If the particle has charge q_e , the equations of motion must be modified to account for the Lorentz force:

$$\frac{d}{d\tau} \left(T \frac{dx_\mu}{ds} \right) = q_e F_{\mu\nu}^{\text{phys}} \frac{dx^\nu}{d\tau} . \quad (35.3)$$

When $F^{\text{phys}} = dA^{\text{phys}}$, where $A^{\text{phys}} \in \Omega^1(M_n)$ is a globally defined 1-form, then these equations can again be derived from an action where the DBI action is modified to:

$$S = \int_{\mathcal{W}_1} T ds + \int_{\mathcal{W}_1} q_e A^{\text{phys}} . \quad (35.4)$$

Let us now consider the quantum theory of a charged particle moving in a classical background electromagnetic field. The path integral over the possible trajectories of the particle will be weighted by:

$$\exp \left(\frac{i}{\hbar} \int_{\mathcal{W}_1} T ds \right) \cdot \underbrace{\chi(\mathcal{W}_1)}_{\substack{\text{some } U(1) \text{ factor} \\ \text{that depends on} \\ \text{the worldline}}} . \quad (35.5)$$

This $U(1)$ factor is a map $\chi : \mathcal{W}_1 \rightarrow U(1)$. If $F^{\text{phys}} = dA^{\text{phys}}$, where $A^{\text{phys}} \in \Omega^1(M_n)$ is a globally defined 1-form, then,

$$\chi(\mathcal{W}_1) = \exp \left(\frac{i}{\hbar} \int_{\mathcal{W}_1} q_e A^{\text{phys}} \right) , \quad (35.6)$$

produces the equation of motion (35.3). Now, we note two key properties which will be defining properties of the map,

$$\chi : \{\text{worldlines}\} \longrightarrow U(1) . \quad (35.7)$$

- (A) Suppose we have two particles. Their worldlines are \mathcal{W}_1 and \mathcal{W}'_1 so the combined worldline is $\mathcal{W}_1 \cup \mathcal{W}'_1$. Their actions should add if they are not interacting:

$$\chi(\mathcal{W}_1 \cup \mathcal{W}'_1) = \chi(\mathcal{W}_1)\chi(\mathcal{W}'_1) . \quad (35.8)$$

So if we restrict to 1-cycles, it makes sense to consider χ as a group homomorphism from the infinite-dimensional Abelian group of 1-cycles on M to $U(1)$:

$$\chi : Z_1(M_n) \longrightarrow U(1) , \quad (35.9)$$

i.e., $\chi \in \text{Hom}(Z_1(\mathcal{M}_n), \text{U}(1))$ is a homomorphism of Abelian groups.

(B) If we have a closed worldline and it is of the form $\mathcal{W}_1 = \partial\mathcal{B}_2$ for some disc \mathcal{B}_2 , then

$$\chi(\mathcal{W}_1) = \exp\left(\frac{i}{\hbar} \int_{\mathcal{B}_2} q_e F^{\text{phys}}\right). \quad (35.10)$$

This gives an extension of χ to cases where F^{phys} defines a nontrivial cohomology class and there is no globally defined 1-form A^{phys} with $F^{\text{phys}} = dA^{\text{phys}}$. At least it gives such an extension when \mathcal{W}_1 is the boundary of some 2-chain \mathcal{B}_2 . We will see momentarily that equation (35.10) implies that the periods of F^{phys} are quantized.

Equations (35.9) and (35.10) are in fact the two defining properties of a Cheeger-Simons character of degree two:

Definition 35.11 (Degree 2 Cheeger-Simons Character). A Cheeger-Simons character of degree 2 is a group homomorphism

$$\chi \in \text{Hom}(Z_1(M_n), \text{U}(1)), \quad (35.12)$$

with the special property that there exists $F \in \Omega^2(M_n)$ (specific to χ) such that whenever $\mathcal{W}_1 \in Z_1(M_n)$ is the boundary of a 2-chain, i.e., $\mathcal{W}_1 = \partial\mathcal{B}_2$, we have

$$\chi(\mathcal{W}_1) = \exp\left(i \int_{\mathcal{B}_2} F\right). \quad (35.13)$$

The set of such characters has a natural Abelian group structure:

$$(\chi_1 \cdot \chi_2)(\mathcal{W}) := \chi_1(\mathcal{W})\chi_2(\mathcal{W}), \quad (35.14)$$

and forms an infinite-dimensional Abelian group denoted $\check{H}^2(M_n)$, known as the Cheeger-Simons or Differential Cohomology group of degree $\ell = 2$.

Note that, by definition, a Cheeger-Simons character is defined on all 1-cycles, not just those which are boundaries. As we will soon see, there is more gauge-invariant information in the Cheeger-Simons character than is encoded in the fieldstrength F of the character. Therefore we can extend the action principle for a charged particle in an electromagnetic field if we say that the electromagnetic field is actually specified not by F but by a Cheeger-Simons character χ . Then we can use the action principle (35.5).

Remark 35.15.

1. We have absorbed the q_e/\hbar into F . So “ F^{phys} ” in our physical motivation is not exactly the same as “ F ” in the definition of the Cheeger-Simons group.

2. Properties (A) and (B) (i.e., equations (35.9) and (35.10)) above imply the quantization of the periods of F : Suppose spacetime has a closed 1-cycle that bounds a 2-chain \mathcal{B}_2 . Suppose it were to bound another 2-chain \mathcal{B}'_2 so that $\Sigma_2 = \mathcal{B}_2 \cup \overline{\mathcal{B}}'_2$ is a closed 2-cycle, see Figure 41.

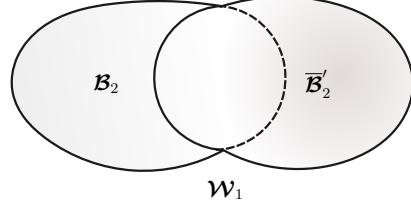


Figure 41: Worldline \mathcal{W}_1 bounded by two 2-chains \mathcal{B}_2 and $\overline{\mathcal{B}}'_2$.

Then,

$$\chi(\mathcal{W}_1) = \exp\left(i \int_{\mathcal{B}_2} F\right) = \exp\left(i \int_{\mathcal{B}'_2} F\right). \quad (35.16)$$

This implies that the fieldstrength satisfies:

$$\exp\left(i \int_{\Sigma_2} F\right) = 1, \quad (35.17)$$

otherwise, the two expressions would lead to different answers. Therefore,

$$F \in \Omega_{\mathbb{Z}'}^2(M_n) \hookrightarrow \Omega_d^2(M_n) := \ker(d), \quad (35.18)$$

where $\Omega_{\mathbb{Z}'}^2$ denotes the space of 2-forms with periods valued in $\mathbb{Z}' := 2\pi\mathbb{Z}$.

Remark 35.19.

- Any 2-form which has all periods of 2-cycles in $2\pi\mathbb{Z}$ must be closed. Here is a proof: Suppose $\omega \in \Omega_{\mathbb{Z}'}^2(M_n)$ is a 2-form with all periods in $\mathbb{Z}' := 2\pi\mathbb{Z}$. Let Σ_2 be a 2-cycle. Any other cycle homologous to it is of the form $\Sigma_2 \mapsto \Sigma_2 + \partial\mathcal{B}_3$. If we choose $\partial\mathcal{B}_3$ to be a small deformation then, since \mathbb{Z}' is discrete, the period cannot change. Therefore, using Stokes' theorem,

$$\int_{\mathcal{B}_3} d\omega = 0. \quad (35.20)$$

Now, we can choose \mathcal{B}_3 to be a degenerate chain whose image in M_n is 2-dimensional everywhere except for a small 3-ball around any point $x \in M_n$ (presuming M_n is connected). Since we can make the ball arbitrarily small, it must be that $d\omega|_x = 0$.

- The argument above is closely related to Dirac's quantization argument for the product

of electric and magnetic charge:

$$\frac{g^2 e^2}{\hbar c \hbar c} = \left(\frac{S}{2}\right)^2, \quad S \in \mathbb{Z}. \quad (35.21)$$

Reinstating \hbar and q_e , the worldline action on $\mathbb{R}^3 \setminus \{0\}$ has exponentiated action

$$e^{\frac{i}{\hbar} \int_{\gamma} q_e A} = e^{\frac{i}{\hbar} \int_{D_+} q_e F} = e^{-\frac{i}{\hbar} \int_{D_-} q_e F}, \quad (35.22)$$

but $D_+ \cup D_-$ is a closed 2-cycle enclosing the magnetic source $F = q_m \omega_2$, where ω_2 is the volume form (29.14). So,

$$e^{\frac{i}{\hbar} \int_{D_+ \cup D_-} q_e q_m \omega_2} = 1 \implies \frac{q_e q_m}{\hbar} \in 2\pi\mathbb{Z}. \quad (35.23)$$

The q_e and q_m are proportional to charges e, g that appear in Coulomb's law by factors depending on 2 and π .

36 Connections On Principal G -bundles

Recall from Section 12 that a principal G -bundle is a continuous family of G -torsors over M_n . It is a space P with a continuous map $\pi : P \rightarrow M_n$ invariant under a free right-action of G . For a point $p \in M_n$, the fiber over the point, $\pi^{-1}(p)$ is a G -torsor. Near any such point, there exists a neighborhood U such that $\pi^{-1}(U) \cong U \times G$, where \cong denotes a homeomorphism.

A *connection* is a rule for coherently lifting paths from paths in M to paths in P , see Figure 42.

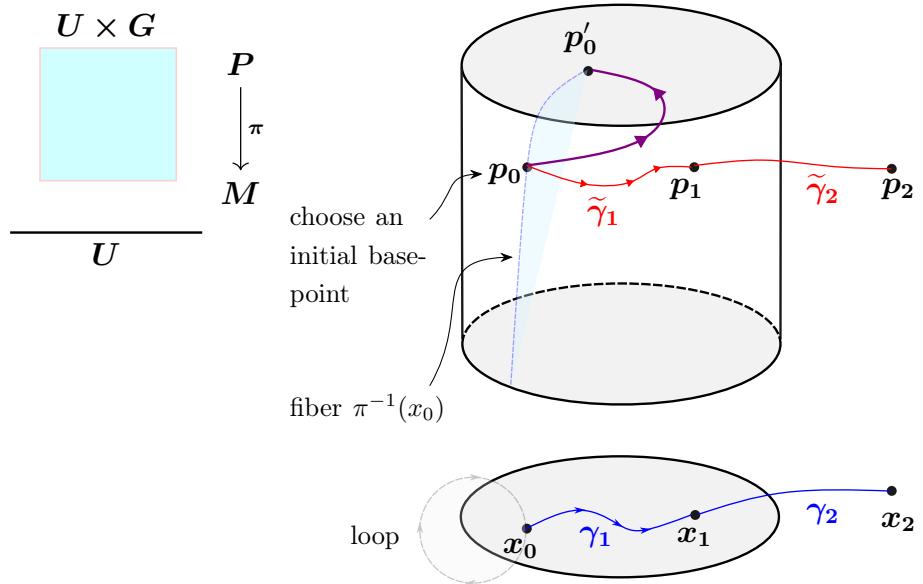


Figure 42: Coherent path lifting.

Consider a path $\gamma : [0, 1] \rightarrow M_n$, such that $\gamma(0) = x_0 \in M_n$. Pick an initial point $p_0 \in \pi^{-1}(x_0)$ in the fiber over x_0 . The lifted path $\tilde{\gamma}$ depends on the initial path γ and the choice of initial point p_0 . Since the path is lifted, we have

$$\pi(\tilde{\gamma}(t)) = \gamma(t) . \quad (36.1)$$

The connection needs to be coherent with respect to gluing: the concatenation of paths γ_1 and γ_2 is the composition $\gamma_1 \circ \gamma_2$. For reasons that will be explained soon, the connection is denoted ∇ .

Now, suppose we are given a connection ∇ , and we consider a loop from x_0 to itself where x_0 is a point in M_n . In doing so, we go from an initial point p_0 in the fiber $\pi^{-1}(x_0)$ above x_0 to another point p'_0 in the fiber $\pi^{-1}(x_0)$. Since these are two points in the same fiber of a G -bundle, they will be related by a G -action:

$$p'_0 = p_0 \cdot g . \quad (36.2)$$

If we change the initial point of the lifted path to $\tilde{p}_0 = p_0 \cdot h$ then the group element g is changed by conjugation with h . However, the conjugacy class of g is a well-defined, function of the connection ∇ and the path γ . Thus a connection defines a function:

$$\text{Hol}_{\nabla} : Z_1(M_n) \longrightarrow \text{ConjClass}(G) , \quad (36.3)$$

known as the *holonomy of the connection*. The value of a particular cycle $\text{Hol}_{\nabla}(\gamma)$ is called the *holonomy around* γ . (See the thick **violet** curve connecting p_0 and p'_0 in [Figure 42](#)). We also write $\text{Hol}(\nabla, \gamma)$ when ∇ is not fixed.

Note that if the group G is Abelian, the set of conjugacy classes is the same as the group and we can view the holonomy as valued in G .

Two connections are said to be *gauge equivalent* if one is obtained from the other by a bundle automorphism. The holonomy function is a gauge-invariant function of the connection. But it does not determine the connection up to gauge equivalence.⁸⁶ However, if we choose a basepoint $x_0 \in M_n$ and trivialize the fiber above P_0 , then the holonomy becomes a function from loops based at x_0 to G . This function does determine the equivalence class of the bundle and connection.⁸⁷

Theorem 36.4. *Let G be a Lie group. Let $P \rightarrow M_n$ be a principal G -bundle over a path-*

⁸⁶For a counterexample, consider flat G -connections on a torus. The isomorphism classes of such connections are pairs of group elements (g_1, g_2) up to equivalence by simultaneous conjugation $(g_1, g_2) \sim (hg_1h^{-1}, hg_2h^{-1})$. g_1, g_2 can be thought of as the holonomy around a, b cycles on the torus. One cannot lift a pair of conjugacy classes to such an equivalence class of (g_1, g_2) .

⁸⁷See [[Kob54](#), [Ana86](#), [Bar91](#), [CP93](#), [Lew93](#), [Men19](#)] for discussions. Some of these papers restrict to the case where the structure group G is compact. Some do not specify conditions on G for validity of the theorem. The references [[Bar91](#), [CP93](#)] only require that G be a Lie group.

connected space M_n with connection ∇ trivialized at a point $x_0 \in M_n$, and consider the holonomy map from loops based at x_0 :

$$\text{Hol}_{\nabla, x_0} : \Omega_{x_0}(M_n) \longrightarrow G . \quad (36.5)$$

If $\text{Hol}_{\nabla, x_0} = \text{Hol}_{\nabla', x_0}$ for two connections ∇ and ∇' , then $P \cong P'$, and ∇ and ∇' are gauge-equivalent.

The basic idea of the proof is the following: We will use the path lifting provided by the connections to define a bundle isomorphism $\varphi : P \rightarrow P'$. By assumption, P and P' have a trivialization over x_0 . So we take the map φ from $P_{x_0} \cong \{x_0\} \times G$ to $P'_{x_0} \cong \{x_0\} \times G$ to be the identity map. To define the map on the fibers above a point $x \in M_n$, we choose a path γ from x_0 to x . Choosing a point $(x_0, g_0) \in P_{x_0}$, the connection ∇ provides a unique lifted path $\tilde{\gamma}_{\nabla, g_0}(t) : [0, 1] \rightarrow P$ such that $\tilde{\gamma}_{\nabla, g_0}(0) = (x_0, g_0)$. Now, φ takes $p = \tilde{\gamma}_{\nabla, g_0}(1) \in P_x$ to $p' = \tilde{\gamma}_{\nabla', g_0}(1) \in P'_x$. This will be smooth and provide the desired bundle isomorphism so that the connections are isomorphic. The main thing to check is that φ is well-defined. This is so because $\text{Hol}_{\nabla, x_0} = \text{Hol}_{\nabla', x_0}$. Note that the key property that is used is continuity and ordinary differential equations. Thus the argument applies to the case that the structure group G is any finite-dimensional Lie group.

Remark 36.6. One can try to weaken the hypotheses of [Theorem 36.4](#), so that we only require that,

$$\chi \circ \text{Hol}_{\nabla, x_0} = \chi \circ \text{Hol}_{\nabla', x_0} , \quad (36.7)$$

for all characters χ on G . In this case, the modified theorem with weakened hypotheses is still true for compact groups G , but for noncompact groups, there are counterexamples [[Sen94](#)]. One could also consider generalizing the theorem to infinite-dimensional Lie groups, but this would require further considerations of functional analysis beyond the scope of this discussion. For some purposes, it is often useful to generalize the notion of holonomy of a connection (with irregular singularities) to include Stokes data. However in this case, the fact that extra conditions are needed to solve the Riemann-Hilbert problem suggests that a potential generalization of [Theorem 36.4](#) along such lines would be false.

There are other equivalent descriptions of connections which will be useful to us. We will be brief here. See textbooks on the differential geometry of fiber bundles for more details, e.g., [[KN96a](#), [KN96b](#), [Tau11](#)].

1. Horizontal Subspaces. The map $\pi : P \rightarrow M_n$ defines for each $p \in M_n$ a surjective linear map,

$$d\pi : T_p P \rightarrow T_x M_n , \quad (36.8)$$

where $x = \pi(p)$. The kernel of the map $V_p := \ker d\pi \subset T_p P$ is the *vertical tangent space*. We define a *horizontal tangent space* to be a choice of a complementary subspace

$H_p \subset T_p P$, so that,

$$T_p P \cong V_p \oplus H_p . \quad (36.9)$$

A connection, then, is a smooth choice of horizontal subspaces. The path lifting is defined by the condition that,

$$\tilde{\gamma}_* \left(\frac{d}{dt} \right) \subset H_{\tilde{\gamma}(t)} . \quad (36.10)$$

For a connection on a principal bundle, the horizontal spaces should be equivariant under the right G action on P .

2. **Global 1-form.** A connection can also be defined by a globally well-defined 1-form Θ on P , valued in the tangent space of the fiber, such that for vertical vector fields $v \in V_p$, we have $\Theta(v) = v$. For principal G -bundles, the tangent space of the fiber can be globally identified with the Lie algebra \mathfrak{g} of G , and we can view $\Theta \in \Omega^1(P; \mathfrak{g})$. Then for $X \in \mathfrak{g}$, the right G -action defines a vertical vector field $\xi(X)$ on P , and we require that $\Theta(\xi(X)) = X$. We also require that Θ be G -equivariant: $R_g^*(\Theta) = \text{Ad}(g)(\Theta)$.
3. **Covariant Derivative.** If E is a vector bundle associated to P via a representation of G , then the path-lifting condition (36.10) can be expressed in terms of a first order differential operator,

$$\nabla : \Gamma(E) \rightarrow \Omega^1(M_n; E) , \quad (36.11)$$

that satisfies the Leibniz rule:

$$\nabla(fs) = s \otimes df + f\nabla s , \quad (36.12)$$

where s is a section of E and f is a function on M_n . The local differential equation defining a lifting of a path in M_n to a path in E is:

$$\nabla_{\dot{\gamma}(t)} s(t) = 0 . \quad (36.13)$$

4. **Local Gauge Fields.** In local patches, the covariant derivative can be expressed relative to a coordinate basis dx^μ for the cotangent bundle as⁸⁸

$$\partial_\mu + iA_\mu , \quad (36.14)$$

where $A_\mu(x) \in \mathfrak{g}$ for the principal bundle and is in the endomorphisms of the representation space for an associated bundle. The A_μ are different in different patches and are related by gauge transformations on patch overlaps. If the bundle has been locally

⁸⁸We take our gauge fields to be “real,” as is standard in the physics literature. Many factors of i could be removed by taking them to be antihermitian, but we will not do this.

trivialized in patches

$$\begin{aligned}\varphi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) &\cong \mathcal{U}_\alpha \times G , \\ \varphi_\beta : \pi^{-1}(\mathcal{U}_\beta) &\cong \mathcal{U}_\beta \times G ,\end{aligned}\tag{36.15}$$

and the covariant derivatives are $d + iA^{(\alpha)}$ and $d + iA^{(\beta)}$ in the two patches, then on the patch overlap

$$d + iA^{(\beta)} = g_{\alpha\beta}^{-1} (d + iA^{(\alpha)}) g_{\alpha\beta} ,\tag{36.16}$$

where $g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \rightarrow G$ is the transition function. So the connections on the patch overlap are gauge equivalent.

In terms of the gauge fields, the holonomy can be expressed as a path-ordered exponential:⁸⁹

$$\text{Hol}_\nabla(\gamma) = P \exp \left(i \oint_\gamma A \right) .\tag{36.17}$$

Restricting now to $G = U(1)$, the holonomy function of a principal $U(1)$ -bundle with connection defines a character:

$$\text{Hol}_\nabla : Z_1(M_n) \longrightarrow U(1) ,\tag{36.18}$$

which can be expressed as:

$$\text{Hol}_\nabla(\gamma) = \exp \left(i \oint_\gamma A \right) ,\tag{36.19}$$

(where we have reverted to expressing the gauge field as a locally-defined real 1-form). Stokes' theorem implies that this character on $Z_1(M_n)$ satisfies the definition of a differential character. Thus, the holonomy function of a principal $U(1)$ -bundle with connection defines a degree two differential character. The converse is also true: see Box 36.1.

Box 36.1: Nontrivial Fact

Every Cheeger-Simons character is the holonomy function of some connection on some principal $U(1)$ -bundle on M_n , that is,

$$\chi(\mathcal{W}) = \text{Hol}(\nabla, \mathcal{W}) ,\tag{36.20}$$

for some connection ∇ on some principal $U(1)$ -bundle $P \xrightarrow{U(1)} M_n$. Informally, we can write $\nabla = d + iA$, and $\text{Hol}(\nabla, \mathcal{W}) = \exp(i \int_{\mathcal{W}} A)$, but A is not globally well-defined. Conversely, if we have a principal $U(1)$ -bundle with connection, then the holonomy can be viewed as a map $\chi : Z_1(M_n) \rightarrow U(1)$ as elaborated upon below.

⁸⁹For a review of path ordering, see [Moo19], App. H].

37 Degree Two Differential Characters And Connections On Principal U(1) Bundles

Using the interpretation of degree two characters in terms of the holonomy function of some connection on a line bundle, we can describe the full Abelian group $\check{H}^2(M_n)$. This group contains the gauge-invariant information of all possible principal U(1)-bundles with connection. We now unpack that information:

For each $x \in H^2(M_n; \mathbb{Z})$, there exists a principal U(1)-bundle $P_x \rightarrow M_n$ with $c_1(P_x) = x$.⁹⁰ Let $(\mathcal{A}/\mathcal{G})_x$ be the set of gauge equivalence classes of connections on P_x . (See footnote 56.) We have,

$$\check{H}^2(M_n) = \mathcal{A}/\mathcal{G} = \coprod_{x \in H^2(M_n; \mathbb{Z})} (\mathcal{A}/\mathcal{G})_x . \quad (37.1)$$

Specifying a point in \mathcal{A}/\mathcal{G} is precisely the same as specifying a holonomy function.

Now quite generally, for any principal G -bundle $P \xrightarrow{G} M_n$ for any Lie group G , the space of connections on P ,

$$\mathcal{A}(P) := \text{Conn}(P \xrightarrow{G} M_n) , \quad (37.2)$$

is an affine space modeled on $\Omega^1(M_n; \text{ad } P)$.⁹¹ This means that we can choose (any) basepoint connection ∇_0 on P , and then every other connection on P is of the form

$$\nabla = \nabla_0 + \alpha , \quad (37.3)$$

for some suitable $\alpha \in \Omega^1(M_n; \text{ad } P)$.

Applying the previous paragraph to the case $G = \text{U}(1)$, the adjoint bundle is trivial, i.e., $\text{ad } P = M_n \times \mathbb{R}$, and so, $\Omega^1(M_n; \text{ad } P) = \Omega^1(M_n)$, and $\nabla = \nabla_0 + i\alpha$, where α is a globally defined real 1-form on M_n .⁹² Gauge transformations are given by

$$\alpha \mapsto \alpha + \omega , \quad \omega \in \Omega_{\mathbb{Z}'}^1(M_n) . \quad (37.4)$$

Note that the periods have to be in $\mathbb{Z}' := 2\pi\mathbb{Z}$. The reason is that the shift $\nabla \mapsto \nabla + i\omega$ shifts the holonomy function by a factor $\exp[i \oint_{\gamma} \omega]$. But for a gauge transformation, the

⁹⁰The Chern class is defined for a complex vector bundle, which, after being endowed with a Hermitian metric on its fibers, may equally well be viewed as a U(1) bundle. What we have in mind here is the Chern class of the complex line bundle associated to the principal U(1)-bundle $P_x \rightarrow M_n$ by the fundamental representation. See Section 40.1 for a description of the first Chern class in terms of transition functions.

⁹¹Here $\Omega^k(M_n; E)$ is the space of smooth k -forms on M_n valued in a vector bundle $E \rightarrow M_n$.

⁹²There is an unfortunate clash between standard math and physics conventions at this point. Strictly speaking, the fibers of $\text{ad}(P)$ should be the Lie algebra of G , and for $G = \text{U}(1)$, the real Lie algebra should be identified with $i\mathbb{R}$ so that the exponentiated Lie algebra elements are phases.

holonomy function is gauge-invariant. Therefore, ω must have periods in \mathbb{Z}' . If ω has nontrivial periods, they are called “large gauge transformations,” whereas if $\omega = d\epsilon$ for a globally well-defined $\epsilon \in \Omega^0(M_n)$, they are called “small gauge transformations.”

The previous paragraph implies that, each component $(\mathcal{A}/\mathcal{G})_x$ of \mathcal{A}/\mathcal{G} associated with a fixed Chern class $x = c_1(P)$ is a torsor for $\Omega^1(M_n)/\Omega_{\mathbb{Z}'}^1(M_n)$, so

$$(\mathcal{A}/\mathcal{G})_x \cong \Omega^1(M_n)/\Omega_{\mathbb{Z}'}^1(M_n) . \quad (37.5)$$

The explicit isomorphism makes use of a basepoint connection, as in (37.3).

Now let us recall the Hodge decomposition (29.23): $\Omega^1 \cong \mathcal{H}^1 \oplus \text{im } d \oplus \text{im } d^\dagger$. Therefore,

$$\Omega_{\mathbb{Z}'}^1 \cong \mathcal{H}_{\mathbb{Z}'}^1 \oplus \text{im } d , \quad (37.6)$$

$$\Omega^1/\Omega_{\mathbb{Z}'}^1 \cong \mathcal{H}^1/\mathcal{H}_{\mathbb{Z}'}^1 \oplus \text{im } d^\dagger , \quad (37.7)$$

and hence,

$$(\mathcal{A}/\mathcal{G})_x \cong \mathcal{H}^1/\mathcal{H}_{\mathbb{Z}'}^1 \oplus \text{im } d^\dagger . \quad (37.8)$$

The conclusion from the above discussion is that, as an Abelian group, $\check{H}^2(M_n)$ can be written as the direct product:

$$\check{H}^2(M_n) = \mathbb{T} \times V \times \Gamma , \quad (37.9)$$

$$\mathbb{T} = \mathcal{H}^1/\mathcal{H}_{\mathbb{Z}'}^1 \cong \text{U}(1)^{b_1} = \text{a connected torus} , \quad (37.10)$$

$$V = \text{im}(d^\dagger : \Omega^2 \rightarrow \Omega^1) = \text{an } \infty\text{-dimensional vector space} , \quad (37.11)$$

$$\Gamma = H^2(M_n; \mathbb{Z}) = \text{a discrete Abelian group} . \quad (37.12)$$

The Abelian group Γ specifies the topological sector. It is finitely generated if M_n is compact. The connected torus \mathbb{T} is the space of Wilson lines connected to the identity (these are Wilson lines for topologically trivial flat gauge fields), as opposed to “discrete Wilson lines” which cannot be smoothly deformed to 1 in fieldspace. Finally, V is the infinite-dimensional vector space of the oscillator modes of the Maxwell field, or more precisely, the gauge-invariant information in the oscillator modes.

The decomposition (37.9) of $\check{H}^2(M_n)$ is noncanonical for two reasons: First, to identify the gauge equivalence class of connections on P_x with a quotient of $\Omega^1(M_n)$, we needed to choose a basepoint connection $\nabla_{0,x}$. Second, to identify the quotient of $\Omega^1(M_n)$ with a connected torus times a vector space, we need to introduce a gauge fixing condition. If we are willing to use the extra information of a Riemannian metric, then we can apply the Hodge decomposition, as we did above. Note that if $\alpha \in \text{im } d^\dagger$, then $d^\dagger \alpha = 0$, which is equivalent to $\partial^\mu \alpha_\mu = 0$. This is the well-known Landau gauge fixing. So the use of the Hodge decomposition to write (37.11) is equivalent to choosing the Landau gauge.

38 The Group Of Cheeger-Simons Characters

In Section 35, we used quantum Maxwell theory to motivate the definition of Cheeger-Simons differential characters of degree two. Now we wish to extend that idea to characters of all nonnegative degrees. From the physical point of view, we generalize from the Maxwell field with $\ell = 2$ to the generalized Abelian gauge field with any value of ℓ .

One physics application of this mathematics to be stressed below, is to the Hamiltonian quantization of the generalized Maxwell field. This involves a construction of the Hilbert space that is manifestly S -duality invariant as in Section 43 and Section 44. It also illustrates the noncommutativity of electric and magnetic fluxes, which in turn provides a Hamiltonian interpretation of generalized symmetries and their 't Hooft anomalies as described in Section 45.

Our physical motivation in the case $\ell = 2$ started by considering the Lorentz force of a test particle moving on a worldline \mathcal{W}_1 of a charged particle moving in a background electromagnetic field configuration. There is a natural generalization of the coupling $\exp(i \oint q_e A)$ to that for electric branes for a generalized Abelian gauge field.

As we have seen, for topologically trivial Maxwell fields, namely, fields where there is a globally well-defined one-form gauge potential $A \in \Omega^1(M_n)$ with $F = dA$, we have:

$$\chi(\mathcal{W}_1) = \chi_A(\mathcal{W}_1) := \exp\left(i \int_{\mathcal{W}_1} A\right). \quad (38.1)$$

Now, consider an electrically charged p -brane for a generalized Maxwell field that is topologically trivial. Topological triviality again means that we can write $F = dA$, where $A \in \Omega^{\ell-1}(M_n)$ is globally well-defined. Then the electric coupling is just:

$$\chi(\mathcal{W}_{p+1}) = \chi_A(\mathcal{W}_{p+1}) := \exp\left(i \int_{\mathcal{W}_{p+1}} A\right). \quad (38.2)$$

Note that we recover the value of $p = \ell - 2$ for an electrically charged p -brane because the above formulae only make sense for $p + 1 = \ell - 1$.

But what do we do if $0 \neq [F] \in H_{\text{dR}}^\ell(M_n)$, i.e., if F is cohomologically nontrivial? Now there is no globally defined A . If $\mathcal{W}_{\ell-1}$ is the boundary of an ℓ -chain (i.e., if the worldline is homologically trivial), then we can propose a definition:

$$\chi(\mathcal{W}_{\ell-1}) := \exp\left(i \int_{\mathcal{B}_\ell} F\right). \quad (38.3)$$

Now in the case when F is topologically trivial, (38.3) reduces to (38.2) by Stokes' Theorem. But in general, (38.3) is an extension of the definition of this term in the action to new field configurations where F is not exact. Once again, since a given cycle $\mathcal{W}_{\ell-1}$ will in

general be bounded by several different ℓ -chains which can be glued together to make a nontrivial ℓ -cycle, we find that this proposal only works if F has quantized periods, that is, if $F \in \Omega_{\mathbb{Z}'}^\ell(M_n) \subset \Omega_d^\ell(M_n)$. (Recall that $\mathbb{Z}' := 2\pi\mathbb{Z}$.)

But now, what does one do if F is cohomologically nontrivial and $[\mathcal{W}_{\ell-1}]$ is homologically nontrivial, i.e., if $\mathcal{W}_{\ell-1}$ cannot be written as the boundary of an ℓ -chain \mathcal{B}_ℓ ? Here, extra physical information is needed. One way to proceed is to assume that there is extra gauge-invariant information in the generalized Abelian gauge field background through which the p -brane moves. One declares the electromagnetic coupling to be a Cheeger-Simons character:

Definition 38.4 (Cheeger-Simons Character). A Cheeger-Simons character of degree ℓ is a group homomorphism,

$$\chi \in \text{Hom}(Z_{\ell-1}(M_n), \text{U}(1)) , \quad (38.5)$$

such that there exists $F \in \Omega^\ell(M_n)$, known as the fieldstrength of the character χ , such that if $\mathcal{W}_{\ell-1} = \partial\mathcal{B}_\ell$, then

$$\chi(\mathcal{W}_{\ell-1}) = \exp\left(i \int_{\mathcal{B}_\ell} F\right) . \quad (38.6)$$

An argument similar to the one around Figure 41 implies that $F \in \Omega_{\mathbb{Z}'}^\ell(M_n) \hookrightarrow \Omega_d^\ell(M_n)$.

Definition 38.7 (Differential Cohomology Group). Two Cheeger-Simons characters χ_1, χ_2 can be multiplied:

$$(\chi_1 \cdot \chi_2)(\mathcal{W}_{\ell-1}) := \chi_1(\mathcal{W}_{\ell-1})\chi_2(\mathcal{W}_{\ell-1}) \quad \forall \quad \mathcal{W}_{\ell-1} \in Z_{\ell-1}(M_n) . \quad (38.8)$$

The Abelian group of all Cheeger-Simons characters of degree ℓ is denoted by $\check{H}^\ell(M_n)$. It is also known as the differential cohomology group.

So we make a physical proposal that the proper way to describe the gauge-invariant information in generalized Maxwell theory is⁹³

$$\check{H}^\ell(M_n) = \left\{ \begin{array}{c} \text{gauge equivalence classes} \\ \text{of generalized Maxwell fields} \\ \text{with } \ell\text{-form fieldstrength on } M_n \end{array} \right\} . \quad (38.9)$$

Then, the action for the test brane in such a generalized Abelian gauge theory background whose gauge equivalence class is described by a differential character χ , is:

$$e^{-T \int_{\mathcal{W}_{\ell-1}} \text{vol}(\iota^*(g))} \chi(\mathcal{W}_{\ell-1}) , \quad (38.10)$$

⁹³This physical proposal turns out to be correct in spirit but technically wrong for Ramond-Ramond (RR) fields in type II string theory, where the appropriate setup is in fact, Differential K -Theory. It is, however, correct modulo a subtlety, for the C field of M-theory, where the relevant object is $\check{H}^4(M_{11})$. The subtlety has to do with a shift in the C field quantization, due to background topology.

where T is the tension of the brane and $\iota : \mathcal{W} \rightarrow M_n$ is the embedding of the worldvolume.

We now start elucidating the gauge-invariant information contained in a character χ that is independent of its associated fieldstrength F . We will explain this much more thoroughly in [Section 39](#) below.

If we just consider characters of the form χ_A for globally well-defined A , then we find

$$\chi_{A+\omega} = \chi_A , \quad (38.11)$$

for $\omega \in \Omega_{\mathbb{Z}'}^{\ell-1}(M_n)$. The shift $A \mapsto A + \omega$ can therefore be viewed as a gauge transformation. Indeed, when ω is exact, $\omega = d\Lambda$ for $\Lambda \in \Omega^{\ell-2}(M_n)$, these are standard gauge transformations in the theory of higher-form fields. The space of topologically trivial characters is the quotient space $\Omega^{\ell-1}/\Omega_{\mathbb{Z}'}^{\ell-1}$. Thanks to the Hodge decomposition, we can write this as a connected torus times $\text{im}(d^\dagger)$, just as in Maxwell theory. The torus

$$\mathcal{H}^{\ell-1}(M_n)/\mathcal{H}_{\mathbb{Z}'}^{\ell-1}(M_n) , \quad (38.12)$$

describes the topologically trivial flat Wilson surfaces. These have $F = 0$ but nonzero “holonomy,” generalizing the familiar situation in the Aharonov-Bohm effect for charged particles in an electromagnetic fields. Differential characters (i.e., gauge fields) described by such Wilson surfaces have nontrivial gauge-invariant information not encoded in the fieldstrength (which is zero).

There is also a more subtle kind of gauge-invariant information that cannot be deduced from the fieldstrength. As we will see in [Section 39.3](#) below, to a character χ , one can associate an integral cohomology class $c(\chi) \in H^\ell(M_n; \mathbb{Z})$ known as its *characteristic class*. The general definition of $\chi \mapsto c(\chi)$ is best left to the “chain complex” descriptions, and we are not defining it precisely just now. When this class is nonzero, it is impossible to write χ in the form χ_A for some globally well-defined form $A \in \Omega^{\ell-1}(M_n)$. We can consider the embedding of $H^\ell(M_n; \mathbb{Z})$ into $H_\mathbb{Z}^\ell(M_n; \mathbb{R})$, and the image of $c(\chi)$ under this embedding actually loses information. The image $c(\chi)_\mathbb{R}$ can be captured by the fieldstrength through the de Rham cohomology class $[F/2\pi]$, but the kernel cannot be captured by the fieldstrength. The gauge-invariant information in $c(\chi)$ not captured by $[F/2\pi]$ is the more subtle gauge-invariant information we mentioned above.

We now illustrate the two previous paragraphs with two examples of the group of differential characters:

Examples:

1. $\ell = 1$. A degree one Cheeger-Simons cohomology class is nothing other than a $U(1)$ -valued continuous function on the manifold. Note that $\mathcal{W}_{\ell-1}$ is just a cycle of the form

$\sum_i n_i p_i$, where p_i are points and given a $U(1)$ -valued function f on M_n , we set

$$\chi_f \left(\sum_i n_i p_i \right) := \prod_i f(p_i)^{n_i}, \quad (38.13)$$

which is well-defined because n_i are integers. To find the fieldstrength, note that

$$\chi_f(p_1 - p_2) = f(p_1)f(p_2)^{-1} = \exp \left[\int_{\gamma} d \log(f) \right] = \exp \left[i \int_{\gamma} (-if^{-1}df) \right], \quad (38.14)$$

where γ is a 1-chain such that $\partial\gamma = p_1 - p_2$. We conclude that

$$F(\chi_f) = -if^{-1}df \in \Omega_{\mathbb{Z}'}^1. \quad (38.15)$$

To define the characteristic class, we take the canonical integral class $c \in H^1(U(1); \mathbb{Z})$ that integrates to one. (So in de Rham theory, $c = d\theta/2\pi$, if we parametrize $U(1)$ by $e^{i\theta}$.) We set

$$c(\chi) := f^*(c). \quad (38.16)$$

Put differently, the evaluation of $c(\chi) \in H^1(M_n; \mathbb{Z})$ around any 1-cycle is just the winding number. Now, one can write $f = e^{i\phi}$, where the scalar field ϕ is not necessarily globally well-defined. Rather, it can have nonzero periods $\oint_{\gamma} d\phi \in \mathbb{Z}'$. This is almost universally the way physicists write the theory of a periodic scalar field. Note that such a representation of f makes it clear that the de Rham cohomology class of $c(\chi)$ corresponds to that of $[F(\chi)/2\pi]$. In this case, $H^1(M_n; \mathbb{Z})$ never has any non-trivial torsion subgroup, so there is no further information in a degree 1 character. In conclusion, we have

$$\check{H}^1(M_n) \cong \text{Map}(M_n \rightarrow \mathbb{R}/\mathbb{Z}) \cong \text{Map}(M_n \rightarrow U(1)). \quad (38.17)$$

2. **$\ell = 2$** . This is the case described in our motivating example of Maxwell theory. Just to recapitulate: To give a differential character of degree two is equivalent to giving an isomorphism class of a principal $U(1)$ -bundle with connection:

$$\chi \longleftrightarrow [(P, \nabla)]. \quad (38.18)$$

Modeling the differential character as the holonomy function of a connection on a principal $U(1)$ line bundle,

$$\chi(\mathcal{W}_1) = \text{Hol}_{\nabla}(\mathcal{W}_1) \quad \forall \quad \mathcal{W}_1 \in Z_1(M_n), \quad (38.19)$$

for some ∇ on $P \xrightarrow{U(1)} M_n$. We define the characteristic class of the character to be

the first Chern class of the $U(1)$ -bundle P (recall footnote 90):

$$c(\chi) := c_1(P) \in H^2(M_n; \mathbb{Z}) . \quad (38.20)$$

This brings up an important new point. The de Rham cohomology class associated with $c_1(P)$ can be captured by $[F/2\pi]$, but the latter expression does not capture all the gauge-invariant information in $c_1(P)$. The reason for this will be described in Section 39. The reason is that there can be a nontrivial torsion subgroup of $H^2(M_n; \mathbb{Z})$. A simple example where such things appear is in the Lens spaces $\mathbb{S}^{2k+1}/\mathbb{Z}_n$. See Section 39.1 for further details.

3. $\ell > 2$. This is best left to models of differential cohomology discussed in Section 40.

Remark 38.21.

1. In the definition of differential characters one can replace $U(1)$ by a higher rank Abelian group $U(1)^d$ in the above (and subsequent) definitions. There is no universally accepted extension to the case of non-Abelian groups.
2. *Is differential cohomology the cohomology of some complex?* The answer is “yes,” and this follows from the work of Hopkins and Singer [HS02]. This will be explained in Section 40.2.
3. *Remark on notation for group structure on Abelian groups.* Whenever one works with Abelian groups, one must make a choice of whether to use multiplicative or additive notation. For example, when working with the cyclic group $\mathbb{Z}/n\mathbb{Z}$, it is natural to write the group product additively and think of the group operation as addition modulo n . On the other hand, working with the isomorphic group μ_n of n^{th} roots of unity, it would be perverse to use an additive multiplication. Of course, the isomorphism between $\mathbb{Z}/n\mathbb{Z}$ and μ_n involves an exponential map. When working with a differential character, one simultaneously works with both additive and multiplicative notations. To begin, it is defined as the group of homomorphisms from $Z_{\ell-1}(M_n)$ to $U(1)$. The codomain is naturally written in multiplicative notation. The domain is more naturally written with additive notation. Recall that to define the Abelian group $Z_{\ell-1}(M_n)$, we begin with the group of chains, $C_{\ell-1}(M_n)$, defined as the free Abelian group generated by continuous maps from $(\ell - 1)$ -simplices to the manifold M_n , i.e., $\phi : \Delta^{\ell-1} \rightarrow M_n$. Then one defines a boundary operation ∂ via a sequence of group homomorphisms,

$$\cdots \longleftarrow C_{\ell-2} \xleftarrow{\partial} C_{\ell-1} \longleftarrow \cdots \quad (38.22)$$

such that $\partial^2 = 0$. Therefore, $Z_{\ell-1}(M_n) := \ker(\partial : C_{\ell-1} \rightarrow C_\ell)$ is a subgroup of the Abelian group $C_{\ell-1}$ and is naturally written in additive notation. However, because

the group structure on $U(1)$ is multiplicative, in equation (38.8) above, we wrote (and will continue to write) the group structure on characters multiplicatively. We will also introduce a notation for (not necessarily topologically trivial) characters: $\chi = \chi_{\check{A}}$ where \check{A} is a set of data described in Section 40.1. Expressed this way, the natural group structure on \check{A} is additive and again, the relation to χ involves an exponential map. An important pitfall to avoid is that there is a much more subtle (and very important) multiplicative structure on characters which should not be confused with the Abelian group structure on characters. We will put a nontrivial graded ring structure on $\oplus_{\ell} \check{H}^{\ell}(M_n)$ in equation (41.1) below. The Abelian group structure multiplies two characters χ_1, χ_2 of the same degree ℓ , to produce a character $\chi_1 \cdot \chi_2$ of degree ℓ . But the ring structure multiplies two characters χ_1, χ_2 of arbitrary degrees ℓ_1, ℓ_2 , to produce a new character $\chi_1 \odot \chi_2$ of degree $\ell_1 + \ell_2$.

Exercise 38.1: \mathbb{R} -valued characters

Show that if we repeat the definition of a Cheeger-Simons character but for $\text{Hom}(Z_{\ell-1}(M_n), \mathbb{R})$ then the fieldstrength must have zero periods.

39 The Dancing Hexagon

We now analyze the structure of $\check{H}^{\ell}(M_n)$ as an Abelian group through a number of interlocked exact sequences. Our goal in this section is to unpack and understand the following diagram:

(39.1)

A dance based on this diagram was choreographed by K. Borkin and A. Selissons [Sto23], so we will refer to it as the “dancing hexagon.” The hexagon was introduced by J. Simons and

D. Sullivan in [SS07], where it is shown that it forms the basis for an axiomatic definition of differential cohomology.⁹⁴

39.1 Several Mathematical Preliminaries

Before we get going, we need a few math preliminaries. (See texts on group theory and algebraic topology for proofs.)

39.1.1 Abelian Groups

1. Abelian groups have a canonically defined subgroup called the *torsion subgroup*

$$\text{Tors}(A) = \{a \in A \mid \exists n \in \mathbb{Z} \text{ such that } na = 0\} , \quad (39.2)$$

where we use additive notation. As an example, the torsion subgroup of $\mathbf{U}(1)$ is the subgroup of roots of unity. It is isomorphic to \mathbb{Q}/\mathbb{Z} .

2. If A is a finitely generated Abelian group, then $\text{Tors}(A)$ is a finite Abelian group and,

$$0 \longrightarrow \text{Tors}(A) \xrightarrow{i} A \xrightarrow{\pi} \overline{A} \longrightarrow 0 , \quad (39.3)$$

where $\overline{A} \cong \mathbb{Z}^b$, where b is the “rank of A ” and $A \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^b$. The group of symmetries of a crystal is finitely generated. That group is a subgroup of the affine Euclidean group, which is not finitely generated.

3. If A is a compact topological Abelian group⁹⁵ then, as we have noted in equation (5.19), we have a short exact sequence (SES):

$$\begin{array}{ccccccc} 1 & \longrightarrow & A_0 & \longrightarrow & A & \longrightarrow & \pi_0(A) \longrightarrow 1 , \\ & & \downarrow & & \downarrow & & \\ & & \text{connected Abelian} & & \text{finite} & & \\ & & \text{group} \cong \mathbf{U}(1)^b & & \text{Abelian} & & \\ & & & & \text{group} \cong \oplus \mathbb{Z}/n\mathbb{Z} & & \end{array} \quad (39.4)$$

where A_0 is the connected component of the identity. (We write 1 and not 0 at the ends of the SES because it is more natural in this context to think about Abelian groups multiplicatively.) This is a compact (normal) subgroup of A . We have seen that $\pi_0(A) \cong A/A_0$ must be a group, and because A is compact, it must be a finite group. So, as a topological space, A is a disjoint union of $|\pi_0(A)|$ copies of an r -dimensional torus. As a simple example consider the homomorphism $\mathbf{U}(1) \times \mathbf{U}(1) \rightarrow \mathbf{U}(1)$ given by $(\zeta_1, \zeta_2) \mapsto (\zeta_1 \zeta_2)^n$ where $n > 1$ is an integer. Let $A \subset \mathbf{U}(1) \times \mathbf{U}(1)$ be the kernel

⁹⁴For a video of Jim Simons presenting the hexagon, see [Sim11].

⁹⁵Technically, we require the Pontryagin dual to be finitely generated.

of this homomorphism. The group of components of A is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. The connected component of the identity is the subtorus with $\zeta_1\zeta_2 = 1$.

39.1.2 Some Homological Algebra

We begin by defining the important notion of the *splitting of an short exact sequence*. Suppose $0 \rightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \rightarrow 0$ is a short exact sequence (SES) of modules over a commutative ring R .⁹⁶ A *left-splitting* (or *splitting on the left*) of the SES is a homomorphism $r : M_2 \rightarrow M_1$ (called a *retraction*) such that $r \circ f = \text{id}_{M_1}$. A *right-splitting* (or *splitting on the right*) of the SES is a homomorphism $s : M_3 \rightarrow M_2$ (called a *section*) such that $g \circ s = \text{id}_{M_3}$. A standard theorem in homological algebra – known as the *splitting lemma* – states that the existence of a left-splitting is equivalent to the existence of a right-splitting and either is equivalent to the existence of an isomorphism $M_2 \rightarrow M_1 \oplus M_3$. Given a retraction r , we have $M_2 \cong \text{im } f \oplus \ker r$, whereas given a section, we have $M_2 \cong \text{im } s \oplus \ker g$. All subsequent statements in this section can be read assuming $R = \mathbb{Z}$, i.e., we are working in the category of Abelian groups.

Recall that a *chain complex* of Abelian groups is a series of Abelian groups labeled by the integers with a *differential*, that is, a series of homomorphisms ∂

$$\cdots \leftarrow C_k \xleftarrow{\partial} C_{k+1} \xleftarrow{\partial} \cdots , \quad (39.5)$$

such that $\partial^2 = 0$. One can define the *homology with integer coefficients* $H_\bullet(C_\bullet; \mathbb{Z})$ from the quotients of kernels modulo images of ∂ :

$$H_j(C_\bullet; \mathbb{Z}) := \ker(\partial : C_j \rightarrow C_{j-1}) / \text{im}(\partial : C_{j+1} \rightarrow C_j) . \quad (39.6)$$

Then the homology with coefficients in an Abelian group A , whose graded components are denoted $H_j(C_\bullet; A)$, is the homology of the chain complex

$$\cdots \leftarrow C_k \otimes A \xleftarrow{\partial \otimes 1} C_{k+1} \otimes A \xleftarrow{\partial \otimes 1} \cdots \quad (39.7)$$

⁹⁶In other words, we are working in the category $R - \text{MOD}$ of left modules over the commutative ring R . Objects in this category are left R -modules and morphisms are R -linear maps. Of course, since R is commutative, every right R -module can be regarded as a left R -module and vice versa. An R -module M is said to be *free* if it has a basis $\{e_k\}$, i.e., if every element $m \in M$ can be written uniquely as a finite linear combination: $m = \sum_k r_k e_k$, with $r_k \in R$ and all but finitely many $r_k = 0$. When $R = \mathbb{Z}$, the category is isomorphic to the category **ABGROUP** of Abelian groups and group homomorphisms (which are \mathbb{Z} -linear maps). An Abelian group is naturally a \mathbb{Z} -module. However, note that any finite Abelian group with nonzero order is not a free \mathbb{Z} -module. For example, for $n \in \mathbb{Z}_{>0}$, the quotient group $\mathbb{Z}/n\mathbb{Z}$ is a \mathbb{Z} -module but not a free \mathbb{Z} -module: any element $x \in \mathbb{Z}/n\mathbb{Z}$ satisfies $n \cdot x = 0$, while also $0 = 0 \cdot x$, contradicting the uniqueness of representation in terms of a basis.

We define the corresponding *cochain complex* of Abelian groups

$$\cdots \rightarrow C^k \xrightarrow{d} C^{k+1} \xrightarrow{d} \cdots \quad (39.8)$$

by taking $C^k = \text{Hom}(C_k, \mathbb{Z})$ and the cohomology with coefficients in A is defined by tensoring with A . The relation of $H_\bullet(C_\bullet; A)$ to $H_\bullet(C_\bullet)$ and of $H^\bullet(C^\bullet; A)$ to $H^\bullet(C^\bullet)$ is not completely trivial and are governed by the Universal Coefficient Theorems, which say:

$$0 \longrightarrow H_k(C_\bullet; \mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow H_k(C_\bullet; A) \longrightarrow \text{Tor}(H_{k-1}(C_\bullet; \mathbb{Z}), A) \longrightarrow 0 , \quad (39.9)$$

and

$$0 \longrightarrow \text{Ext}(H_{k-1}(C_\bullet; \mathbb{Z}), A) \longrightarrow H^k(C^\bullet; A) \longrightarrow \text{Hom}(H_k(C_\bullet; \mathbb{Z}), A) \longrightarrow 0 . \quad (39.10)$$

For two finitely generated Abelian groups A, B , one can define associated Abelian groups $\text{Ext}(A, B)$ and $\text{Tor}(A, B)$. See [BT82] or any textbook on homological algebra (e.g., [Wei94, GM03]) for the definitions. We summarize some of their useful properties in Box 39.1.

Box 39.1: Useful Properties Of Tor And Ext

Tor satisfies the following useful properties:

- $\text{Tor}(\text{Free}, G) = 0$ for any free group as one of the arguments.
- $\text{Tor}(\mathbb{Z}_n, G) = \ker(G \xrightarrow{\times n} G)$.
- $\text{Tor}(A \oplus B, G) = \text{Tor}(A, G) \oplus \text{Tor}(B, G)$.
- $\text{Tor}(A, B) = \text{Tor}(B, A)$.
- $\text{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$, where $d = \gcd(m, n)$.
- Tor detects the common torsion of its arguments.

Ext satisfies the following useful properties:

- $\text{Ext}(\text{Free}, G) = 0$.
- $\text{Ext}(\mathbb{Z}_n, G) = G/nG$.
- $\text{Ext}(A \oplus B, G) = \text{Ext}(A, G) \oplus \text{Ext}(B, G)$.
- $\text{Ext}(A, G) = 0$ if G is a divisible group.^a
- $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$, where $d = \gcd(m, n)$.
- Ext also detects torsion.

Note that, unlike Tor , the order of arguments in Ext is important.

^aA divisible group is an Abelian group $(G, +)$ with the property that for every $g \in G$ and $n \in \mathbb{Z}_{>0}$, there exists $x \in G$ such that $nx = g$. Examples of divisible groups are: the group \mathbb{R} of real numbers

under addition, the group \mathbb{Q} of rational numbers under addition, \mathbb{C}^\times – the multiplicative group of complex numbers, and $U(1) \cong \mathbb{R}/\mathbb{Z}$. Any quotient of a divisible group is also divisible, so in particular, \mathbb{Q}/\mathbb{Z} is a divisible group.

It is worth noting that the sequences (39.9) and (39.10) split, but not naturally. Consequently, there are noncanonical isomorphisms:

$$H_k(C_\bullet; A) \cong H_k(C_\bullet; \mathbb{Z}) \otimes_{\mathbb{Z}} A \oplus \text{Tor}(H_{k-1}(C_\bullet; \mathbb{Z}), A) , \quad (39.11)$$

and

$$H^k(C^\bullet; A) \cong \text{Hom}(H_k(C_\bullet; \mathbb{Z}), A) \oplus \text{Ext}(H_{k-1}(C_\bullet; \mathbb{Z}), A) . \quad (39.12)$$

Remark 39.13.

1. The splitting of (39.9) can be explained as follows. Let $Z_k := \ker(\partial : C_k \rightarrow C_{k-1})$ and $B_k := \text{im}(\partial : C_{k+1} \rightarrow C_k)$. We have a short exact sequence:

$$0 \longrightarrow Z_k \longrightarrow C_k \longrightarrow B_{k-1} \longrightarrow 0 . \quad (39.14)$$

Since C_k is a free Abelian group (see footnote 96), it has a basis $\{e_i\}_{i \in I}$ labeled by an index set I , so that $C_k \cong \bigoplus_{i \in I} \mathbb{Z} \cdot e_i$. Now, without loss of generality, we can choose a basis for Z_k , and extend this basis to a basis for C_k . (This is always possible for a free Abelian group.) This allows us to write $C_k \cong Z_k \oplus B'_{k-1}$ where B'_{k-1} is the subgroup generated by the remaining basis elements. It follows by construction that $B'_{k-1} \cong B_{k-1}$, and that every element $c \in C_k$ can be written uniquely as $c = z + b'$ with $z \in Z_k$ and $b' \in B'_{k-1}$. We now define $r : C_k \rightarrow Z_k$ by $z + b' \mapsto r(z + b') = z$. The map r is a group homomorphism, and furthermore, for any $z \in Z_k$, $r(z) = z$, therefore $r|_{Z_k} = \text{id}$. This function r serves as a left-splitting or retraction (recall the splitting lemma above). Note that the choice of $r : C_k \rightarrow Z_k$ is dictated by the choice of complement B'_{k-1} to Z_k in C_k , which in turn depends on the choice of basis. Thus, the map r is highly noncanonical.

Let $r : C_k \rightarrow Z_k$ be a splitting of $Z_k \hookrightarrow C_k$. Then we have a composition of maps:

$$Z_k(C_\bullet \otimes A) \subset C_k \otimes A \xrightarrow{r \otimes 1} Z_k \otimes A \rightarrow H_k(C_\bullet; \mathbb{Z}) \otimes A , \quad (39.15)$$

that takes $B_k(C_\bullet \otimes A)$ to zero, and induces a map,

$$\rho : H_k(C_\bullet \otimes A) \rightarrow H_k(C_\bullet; \mathbb{Z}) \otimes A , \quad (39.16)$$

which satisfies $\rho \circ \alpha = \text{id}$, where α is the map,

$$\alpha : H_k(C_\bullet; \mathbb{Z}) \otimes_{\mathbb{Z}} A \rightarrow H_k(C_\bullet \otimes A) \cong H_k(C_\bullet; A) , \quad (39.17)$$

in the UCT (39.9). The map ρ yields a splitting of the UCT (39.9). A proof of the splitting of (39.10) proceeds along similar lines.

2. Here is an example to illustrate the noncanonical nature of the isomorphism (39.11) (equivalently, the non-natural or non-functorial splitting of (39.9)), using the chain complex associated with a manifold (see Section 16). Consider the real projective plane $\mathbb{RP}^2 \cong \mathbb{S}^2/\mathbb{Z}_2$, where the quotient by the \mathbb{Z}_2 action identifies antipodal points. The integral homology groups of \mathbb{RP}^2 and \mathbb{S}^2 are:

$$H_k(\mathbb{RP}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} , & \text{for } k = 0 , \\ \mathbb{Z}_2 , & \text{for } k = 1 , \\ 0 , & \text{otherwise .} \end{cases} \quad H_k(\mathbb{S}^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} , & \text{for } k = 0, 2 , \\ 0 , & \text{otherwise .} \end{cases} \quad (39.18)$$

The UCT (39.9) for \mathbb{RP}^2 at degree $k = 2$ with $A = \mathbb{Z}_2$ reads

$$0 \rightarrow H_2(\mathbb{RP}^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \xrightarrow{\phi} H_2(\mathbb{RP}^2; \mathbb{Z}_2) \xrightarrow{\psi} \text{Tor}(H_1(\mathbb{RP}^2; \mathbb{Z}), \mathbb{Z}_2) \rightarrow 0 . \quad (39.19)$$

Upon substituting the homology groups, and using the fact that $\text{Tor}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ (see Box 39.1), this yields

$$0 \rightarrow 0 \rightarrow H_2(\mathbb{RP}^2; \mathbb{Z}_2) \xrightarrow{\psi} \mathbb{Z}_2 \rightarrow 0 \implies H_2(\mathbb{RP}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2 . \quad (39.20)$$

Carrying out the same exercise for \mathbb{S}^2 , we find

$$0 \rightarrow \mathbb{Z} \otimes \mathbb{Z}_2 \xrightarrow{\phi} H_2(\mathbb{S}^2; \mathbb{Z}_2) \xrightarrow{\psi} \text{Tor}(0, \mathbb{Z}_2) \rightarrow 0 , \quad (39.21)$$

which, upon substituting the homology groups, yields

$$0 \rightarrow \mathbb{Z}_2 \rightarrow H_2(\mathbb{S}^2; \mathbb{Z}_2) \rightarrow 0 \implies H_2(\mathbb{S}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2 . \quad (39.22)$$

Now there is a map – called the pinch map – denoted by

$$p : \mathbb{RP}^2 \rightarrow \mathbb{S}^2 , \quad (39.23)$$

which collapses the 1-skeleton $\mathbb{RP}^1 \cong \mathbb{S}^1 \subset \mathbb{RP}^2$ to a point.⁹⁷ Importantly, the pinch

⁹⁷This terminology was introduced in Section 15.

map is not nullhomotopic. But it induces a map on the homology,

$$p_* : H_2(\mathbb{RP}^2; \mathbb{Z}_2) \rightarrow H_2(\mathbb{S}^2; \mathbb{Z}_2) , \quad (39.24)$$

which turns out to be zero. Here is why: First, note that the \mathbb{Z}_2 in $H_2(\mathbb{RP}^2; \mathbb{Z}_2)$ originated from the torsion in $H_1(\mathbb{RP}^2; \mathbb{Z})$, whereas the \mathbb{Z}_2 in $H_2(\mathbb{S}^2; \mathbb{Z}_2)$ originated from tensoring \mathbb{Z} with \mathbb{Z}_2 . So, if the isomorphism (39.11) (for $A = \mathbb{Z}_2$) were to split canonically, the induced map (39.24) would have to send the torsion summand in the homology of \mathbb{RP}^2 to the tensor summand in the homology of \mathbb{S}^2 , but there is no natural map that will do so. The pinch map (39.23) induces a map of UCT short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \overbrace{H_2(\mathbb{RP}^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2}^{=0} & \longrightarrow & H_2(\mathbb{RP}^2; \mathbb{Z}_2) & \xrightarrow{\cong} & \overbrace{\text{Tor}(H_1(\mathbb{RP}^2; \mathbb{Z}), \mathbb{Z}_2)}^{\mathbb{Z}_2} \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow \cong & & \downarrow 0 \\ 0 & \longrightarrow & H_2(\mathbb{S}^2; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2 & \xrightarrow{\cong} & H_2(\mathbb{S}^2; \mathbb{Z}_2) & \longrightarrow & \underbrace{\text{Tor}(H_1(\mathbb{S}^2; \mathbb{Z}), \mathbb{Z}_2)}_0 \longrightarrow 0 \end{array} \quad (39.25)$$

from which it is evident that the splitting of the UCT short exact sequence is not functorial. Recall that \mathbb{RP}^2 is built out of $\mathbb{RP}^1 \cup e^2$, where e^2 denotes the 2-cell, and the pinch map (39.23) collapses \mathbb{RP}^1 mapping it to the base point of \mathbb{S}^2 and retaining only the 2-cell, which is mapped homeomorphically to \mathbb{S}^2 . However, there is no integral 2-cycle in \mathbb{RP}^2 in the first place, since $H_2(\mathbb{RP}^2; \mathbb{Z}) = 0$.

Exercise 39.1: Pinch map and UCT cohomology

Use the pinch map (39.23) to exhibit the noncanonical nature of the UCT cohomology isomorphism (39.12).

A cochain complex B^\bullet of Abelian groups is sometimes called a *differential graded Abelian group*. This means B has an integer grading:

$$B = \bigoplus_{k \in \mathbb{Z}} B^k , \quad (39.26)$$

and there is a differential operator that raises degree by 1 and squares to zero: $d^2 = 0$. In more physical terms, B has a fermion number operator F which has a spectrum that is integers and has a \mathcal{Q} symmetry, i.e., a \mathcal{Q} operator with charge 1 satisfying $\mathcal{Q}^2 = 0$ and $[\mathcal{Q}, F] = \mathcal{Q}$ (this means \mathcal{Q} has charge 1 under the fermion number operator). This is entirely equivalent to the math description of a cochain complex.

When we have a differential graded subgroup $A^\bullet \subset B^\bullet$, there is an important construction

known as the *Bockstein map* that will be used below. To say that $A^\bullet \subset B^\bullet$ is a differential subgroup means that $d(A) \subset A$, where $d(A)$ uses the restriction of d to A . From this, we deduce that the quotient B/A is also a differential Abelian group. B/A is the set of equivalence classes written as $(b+A) \in B/A$. We can define a differential on B/A by setting $d(b+A) := db + A$. To see that this definition makes sense, one needs to check that the formula does not depend on representative b of the equivalence class $b+A$, and this is where we use $d(A) \subset A$. It follows immediately that $d^2 = 0$ on B/A . We have a sequence of differential-graded Abelian groups:

$$0 \longrightarrow A \xrightarrow{i} B \longrightarrow B/A \longrightarrow 0 . \quad (39.27)$$

Now suppose that $d(b+A) = 0 + A$, that is, suppose that $b+A$ is in the kernel of d acting on B/A . In terms of a representative $b \in B$ of $b+A$, this means that $db \in A$. Let us denote $s = db$. It is an element of A and need not be zero. In general, for an element $b+A$ in the kernel of d on B/A , it might well happen that there is no representative b that makes $s = 0$. Moreover, we also have $ds = 0$ because $d(db) = d^2b = 0$. Therefore, s determines a cohomology class of the cohomology of A . Note that if $b' \in B$ is another representative of $b+A$, then $b' = b+t$ where $t \in A$. Therefore, $s' = s + dt$, so s and s' differ by an image of d acting on A . Therefore, the cohomology classes of $[s]$ and $[s']$ in the cohomology of A are the same, and therefore, $[s]$ only depends on $b+A \in B/A$, and not on the choice of representative. If $b+A$ has degree k in B/A , then $b \in B$ has degree k , so s has degree $(k+1)$. So, we have constructed a degree 1 map,

$$\begin{aligned} \beta : H^k(B/A) &\longrightarrow H^{k+1}(A) \\ [b+A] &\longmapsto [s] = [db] . \end{aligned} \quad (39.28)$$

Note well that s is in the image of d acting on B , but not necessarily in the image of d acting on A . So it might well define a nonzero cohomology class in $H^\bullet(A)$! The map β is called the *Bockstein map*.

39.1.3 Some Algebraic Topology

Recall that in [Section 16](#) we defined simplicial sets $\mathcal{X}(M)$ associated with any manifold⁹⁸ M . The groups of singular cochains $C^k(M)$ are the free Abelian groups generated by $\mathcal{X}(M)_j$ and the face maps [\(16.7\)](#) can be extended by linearity to define a differential ∂ . From this complex, we can define the homology and cohomology groups $H_k(M; \mathbb{Z})$ and $H^k(M; \mathbb{Z})$. It is shown in books on algebraic topology that for a compact manifold, these Abelian groups are finitely generated.

Here is a simple example of some integral cohomology groups and corresponding groups

⁹⁸One can generalize manifold to a much wider class of topological spaces.

with coefficients:

Box 39.2: Example: Lens Space

The Lens space L_m is defined as the quotient $L_m := \mathbb{S}^3/\mathbb{Z}_m$ where $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ and the quotient by the \mathbb{Z}_m action identifies $(z_1, z_2) \sim (\omega z_1, \omega z_2)$ where $\omega \in \mu_m$ is an element of the multiplicative cyclic group of order m .

We have the free action of a cyclic group on a simply connected space. So $\pi_1(L_m; x_0) \cong \mathbb{Z}_m$. By the Hurewicz theorem $\pi_1(L_m; x_0) \cong H_1(L_m; \mathbb{Z})$. Note that earlier (just above equation (38.17)) we asserted that there is no torsion in H^1 , not H_1 ! By Poincaré duality $H^2(L_m; \mathbb{Z}) \cong H_1(L_m; \mathbb{Z}) \cong \mathbb{Z}_m$. (One could also use the Universal Coefficient Theorem with $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}) \cong \mathbb{Z}_m$.) This is an example where the second cohomology is pure torsion.

The group of isomorphism classes of $U(1)$ -bundles on any manifold is classified by $H^2(M; \mathbb{Z})$. Thus it is the finite group \mathbb{Z}_m in this case. So the characteristic class of degree two differential characters on a Lens space is entirely torsion.

There are many ways of producing cochain complexes that compute the cohomology of a manifold. One is singular cohomology as indicated above, but others are often more convenient. For example, one can use a cellular decomposition to produce much smaller cochain complexes with the same cohomology. For example, in the case of a Lens space, one can use the symmetry of the situation to produce an equivariant cell decomposition of \mathbb{S}^3 well adapted to the quotient. For example, specializing to $m = 2$, the \mathbb{Z}_2 -equivariant cell decomposition of \mathbb{S}^n is given by using the upper and lower hemispheres:

$$e_j^\pm = \{(x_1, \dots, x_j, x_{j+1}, 0, \dots, 0) \in \mathbb{S}^n \mid \text{sign}(x_{j+1}) = \pm\} . \quad (39.29)$$

The orientation on e_j^\pm from $\pm dx^1 \wedge \cdots \wedge dx^j$.

$$\left. \begin{aligned} \partial e_j^+ &= e_{j-1}^+ + e_{j-1}^- \\ \partial e_j^- &= -(e_{j-1}^+ + e_{j-1}^-) \end{aligned} \right\} i \leq j \leq n , \quad (39.30)$$

$$\partial e_0^\pm = 0 . \quad (39.31)$$

We introduce dual cochains, satisfying

$$c_j^\alpha(e_k^\beta) = \delta^{\alpha\beta}\delta_{jk} . \quad (39.32)$$

This yields the cochain model of $\mathbb{RP}^n = \mathbb{S}^n/\mathbb{Z}_2$:

$$\mathbb{Z}(c_0^+ - c_0^-) \xrightarrow{0} \mathbb{Z}(c_1^+ + c_1^-) \xrightarrow{2} \mathbb{Z}(c_2^+ - c_2^-) \xrightarrow{0} \mathbb{Z}(c_3^+ + c_3^-) \xrightarrow{2} \dots \quad (39.33)$$

Therefore,

$$H^j(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & j = 0, \\ 0, & j \text{ odd, } j < n, \\ \mathbb{Z}_2, & j \text{ even, } 0 < j < n, \\ \mathbb{Z}, & j = n, \text{ odd,} \\ \mathbb{Z}_2, & j = n, \text{ even.} \end{cases} \quad (39.34)$$

To get the cohomology with \mathbb{Z}_2 coefficients we tensor the complex (39.33) with \mathbb{Z}_2 . This simply makes all differentials 0:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{0} & \mathbb{Z}_2 & \xrightarrow{2} & \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{2} \dots \\ \text{cohomology:} & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & \dots \end{array} \quad (39.35)$$

and we easily conclude that:

$$H^\bullet(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{n+1}) . \quad (39.36)$$

Next, the group $H^k(M_n; \mathbb{R}/\mathbb{Z})$ will play an important role in our applications of differential cohomology to higher-form fields so it behooves us to understand the structure of this group. By the universal coefficient theorem, we know that $H^k(M_n; \mathbb{R}/\mathbb{Z}) \cong \text{Hom}(H_k(M_n; \mathbb{Z}), \mathbb{R}/\mathbb{Z})$, and this is a compact Abelian group. Accordingly, it has a group of connected components. We now determine that group.

Associated to the short exact sequence of coefficients,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0 , \quad (39.37)$$

is a long exact sequence in cohomology, which we now derive. Referring to the definition of the Bockstein map above, we take $A = C^\bullet(M_n)$ and $B = C^\bullet(M_n) \otimes \mathbb{R}$. Now, $B/A =$

$C^\bullet(M_n) \otimes \mathbb{R}/\mathbb{Z}$. We have

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \uparrow d \\
 0 \longrightarrow C^{k+1}(M_n; \mathbb{Z}) \longrightarrow C^{k+1}(M_n; \mathbb{R}) \xrightarrow{\pi} C^{k+1}(M_n; \mathbb{R}/\mathbb{Z}) \longrightarrow \cdots & & & & & & \\
 & \uparrow d \\
 0 \longrightarrow C^k(M_n; \mathbb{Z}) \longrightarrow C^k(M_n; \mathbb{R}) \xrightarrow{\pi} C^k(M_n; \mathbb{R}/\mathbb{Z}) \longrightarrow \cdots & & & & & & (39.38) \\
 & \uparrow d \\
 0 \longrightarrow C^{k-1}(M_n; \mathbb{Z}) \longrightarrow C^{k-1}(M_n; \mathbb{R}) \xrightarrow{\pi} C^{k-1}(M_n; \mathbb{R}/\mathbb{Z}) \longrightarrow \cdots & & & & & & \\
 & \uparrow d \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{array}$$

For $\bar{a} \in Z^k(M_n; \mathbb{R}/\mathbb{Z})$, lift to $a \in C^k(M_n; \mathbb{R})$, $\pi(da) = 0$, so da has a lift to $b \in C^{k+1}(M_n; \mathbb{Z})$, but $db = 0$, so $b \in Z^{k+1}(M_n; \mathbb{R}/\mathbb{Z})$. With a little thought, one finds that we have a long exact sequence:

$$\begin{array}{ccccccc}
 \cdots \longrightarrow H^k(M_n; \mathbb{Z}) \xrightarrow{\iota} H^k(M_n; \mathbb{R}) \xrightarrow[\beta]{\psi} H^k(M_n; \mathbb{R}/\mathbb{Z}) & & & & & & \\
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
 \xrightarrow{\quad} H^{k+1}(M_n; \mathbb{Z}) \xrightarrow{\iota} H^{k+1}(M_n; \mathbb{R}) \longrightarrow H^{k+1}(M_n; \mathbb{R}/\mathbb{Z}) \longrightarrow \cdots & & & & & & (39.39)
 \end{array}$$

Here β is the Bockstein map. Since the sequence is exact,

$$\text{im}(\beta) \cong \ker(\iota) \cong \text{Tors } H^{k+1}(M_n; \mathbb{Z}) . \quad (39.40)$$

But

$$\begin{aligned}
 \text{im}(\psi) &= H^k(M_n; \mathbb{R}) / \iota(H^k(M_n; \mathbb{Z})) \\
 &= \text{Connected component of } H^k(M_n; \mathbb{R}/\mathbb{Z}) .
 \end{aligned} \quad (39.41)$$

Therefore,

$$\boxed{\pi_0(H^k(M_n; \mathbb{R}/\mathbb{Z})) \cong \text{Tors } H^{k+1}(M_n; \mathbb{Z})} \quad (39.42)$$

Of course, we also have, by (39.3),

$$0 \longrightarrow \text{Tors } H^k(M_n; \mathbb{Z}) \longrightarrow H^k(M_n; \mathbb{Z}) \longrightarrow \overline{H}^k(M_n; \mathbb{Z}) \cong \mathbb{Z}^{b_k} \longrightarrow 0 . \quad (39.43)$$

Note that $H_{dR}^k(M_n) \cong H^k(M_n; \mathbb{Z}) \otimes \mathbb{R} \cong H^k(M_n; \mathbb{R}) \cong \mathbb{R}^{b_k}$.

Hi Exercise 39.2: Künneth theorem

Using the Künneth theorem (look it up), show that

$$0 \longrightarrow \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z} \longrightarrow H^3(L_m \times L_m; \mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{Z}_m \longrightarrow 0 . \quad (39.44)$$

Box 39.3: A Useful Short Exact Sequence

Another useful short-exact sequence that we will encounter below is

$$0 \longrightarrow H^k(M_n; \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z} \longrightarrow H^k(M_n; \mathbb{R}/\mathbb{Z}) \longrightarrow \text{Tors } H^{k+2}(M_n; \mathbb{Z}) \longrightarrow 0 , \quad (39.45)$$

where $H^k(M_n; \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$ is the connected component of the identity of $H^k(M_n; \mathbb{R}/\mathbb{Z})$, and $\text{Tors } H^{k+2}(M_n; \mathbb{Z})$ is the group of connected components of $H^k(M_n; \mathbb{R}/\mathbb{Z})$ as shown above.

39.2 Flat Characters

We now return to the dancing hexagon. Let us begin with the group homomorphism $\chi \mapsto F(\chi)$. The very definition of a differential character assigns to a character χ a form $F \in \Omega_{\mathbb{Z}'}^\ell(M_n)$, such that,

$$\chi(\partial\mathcal{B}) = \exp \left[i \int_{\mathcal{B}} F \right] . \quad (39.46)$$

We will call the group homomorphism,

$$\chi \mapsto F , \quad (39.47)$$

the “fieldstrength map”:

$$\check{H}^\ell(M_n) \xrightarrow[\text{strength}]{\text{field}} \Omega_{\mathbb{Z}'}^\ell(M_n) \longrightarrow 0 . \quad (39.48)$$

We will see (e.g., from the Čech model in [Section 40.1](#)) that this map is surjective, hence the second arrow above.

The homomorphism $\check{H}^\ell(M_n) \rightarrow \Omega_{\mathbb{Z}'}^\ell(M_n)$ has a kernel. The kernel is called the set of *flat differential characters*, i.e., characters with vanishing fieldstrength. Having a zero fieldstrength means that if two cycles $\Sigma_{\ell-1}$ and $\Sigma'_{\ell-1}$ are homologous, i.e., $\Sigma_{\ell-1} - \Sigma'_{\ell-1} = \partial\mathcal{B}$, then

$$\chi(\Sigma_{\ell-1})/\chi(\Sigma'_{\ell-1}) = \exp \left[i \int_{\mathcal{B}} F \right] = 1 , \quad (39.49)$$

so $\chi(\Sigma_{\ell-1}) = \chi(\Sigma'_{\ell-1})$. It follows that flat characters define homomorphisms $\chi : Z_{\ell-1}(M_n) \rightarrow$

$\mathrm{U}(1)$ that only depend on the homology class of $\Sigma_{\ell-1}$. One can show that⁹⁹

$$\mathrm{Hom}(H_{\ell-1}(M_n); \mathrm{U}(1)) \cong H^{\ell-1}(M_n; \mathrm{U}(1)) , \quad (39.50)$$

so we identify the group of flat characters with $H^{\ell-1}(M_n; \mathrm{U}(1))$, or $H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')$, if we are using additive notation. In general, if $\phi \in H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')$, we denote the corresponding flat differential character by $\check{\phi}$.

We will need to understand the structure of the group $H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')$. This is a compact topological Abelian group. Recall that if A is a compact topological Abelian group, then we have a short exact sequence (39.4), where A_0 is the connected component of the identity, and the sequence in general does not split. In the present case, $A = H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')$, and $A_0 = H^{\ell-1}(M_n; \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}'$. The latter group $A_0 = H^{\ell-1}(M_n; \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}'$ should be identified with the group of “Wilson lines” or “Wilson surfaces” that can be continuously deformed to the identity character. In Section 39.3, we will define topologically trivial characters and will see that $H^{\ell-1}(M_n; \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}'$ is the group of topologically trivial flat characters.

To understand the group of components of $H^{\ell-1}(M_n; \mathrm{U}(1))$, we use the exact sequence (39.39) with the Bockstein map,

$$\beta : H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}') \longrightarrow \mathrm{Tors}(H^{\ell}(M_n; \mathbb{Z})) \subset H^{\ell}(M_n; \mathbb{Z}) , \quad (39.51)$$

to identify,

$$\pi_0(H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')) \cong \mathrm{Tors}(H^{\ell}(M_n; \mathbb{Z})) . \quad (39.52)$$

It follows that there can be a nontrivial group of connected components of the group of flat fields, whenever there is torsion in the cohomology. We stress again that the group $H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')$ is in general not a direct product of $H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')_0$ and $\pi_0(H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}'))$. The topological class of $\check{\phi}$ for $\phi \in H^{\ell-1}(M_n; \mathrm{U}(1))$ is:

$$c(\check{\phi}) = -\beta(\phi) . \quad (39.53)$$

This follows immediately from the cochain model of differential characters (40.40) and the definition of the Bockstein map. (The minus sign leads to the slightly awkward property that the arrow in the bottom triangle of (39.1) is the map $\phi \mapsto -\check{\phi}$.)

As an example, consider a Lens space $M = L_m = \mathbb{S}^3/\mathbb{Z}_m$ discussed in Box 39.2, and take $\ell = 2$. In this case, $H^1(M; \mathbb{R}/\mathbb{Z}') \cong H^2(M; \mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z}$, and $\pi_1(L_m, x_0) = \mathbb{Z}/m\mathbb{Z}$. Let γ be a generator of $\pi_1(L_m, x_0)$. Every closed loop in L_m is isotopic to γ^r for some $r \in \mathbb{Z}_m$. For each $k \in \mathbb{Z}/m\mathbb{Z}$, we define a character by saying that if a connected 1-cycle Σ_1 is isotopic

⁹⁹ Applying the UCT (39.10) to a manifold M_n at degree $j = \ell - 1$, we get the noncanonical isomorphism (39.12): $H^{\ell-1}(M_n; \mathrm{U}(1)) \cong \mathrm{Ext}(H_{\ell-2}(M; \mathbb{Z}), \mathrm{U}(1)) \oplus \mathrm{Hom}(H_{\ell-1}(M; \mathbb{Z}), \mathrm{U}(1))$. Since $\mathrm{U}(1)$ is a divisible group (see Box 39.1), $\mathrm{Ext}(-, \mathrm{U}(1))$ vanishes, and this reduces to (39.50).

to γ^r , then

$$\chi_k(\Sigma_1) := \exp\left(\frac{2\pi i r k}{m}\right), \quad k \in \mathbb{Z}_m. \quad (39.54)$$

We then extend to all 1-cycles by demanding that χ be a homomorphism. Clearly, the value only depends on the homology class of the 1-cycle and therefore, the character is a flat character. The group of flat characters is isomorphic to $\mathbb{Z}/m\mathbb{Z}$.

39.3 Topologically Trivial Characters

As we have mentioned, there is another group homomorphism,

$$c : \check{H}^\ell(M_n) \rightarrow H^\ell(M_n; \mathbb{Z}), \quad (39.55)$$

known as the “characteristic class,” or “topological class” of the character. It is best defined using explicit models for differential characters in [Section 40](#).

The kernel of $\chi \mapsto c(\chi)$ is the subgroup of *topologically trivial characters*. These are the characters for which there exists a globally well-defined form $A \in \Omega^{\ell-1}(M_n)$, such that

$$\chi(\mathcal{W}_{\ell-1}) = \exp\left(i \int_{\mathcal{W}_{\ell-1}} A\right). \quad (39.56)$$

We denote such characters as χ_A , where, we stress, A is globally well-defined on M_n .

It is important to recall that a differential character only encodes gauge-invariant information. Accordingly, for topologically trivial characters, there can be different forms $A, A' \in \Omega^{\ell-1}(M_n)$ so that $\chi_A = \chi_{A'}$. Plugging into (39.56), we see that $\omega := A' - A$ has the property that the periods of ω are in \mathbb{Z}' . Conversely, shifting $A \mapsto A' + \omega$, where ω has periods in \mathbb{Z}' , does not change the character. So we can think of these as *gauge transformations*, and here we make contact with the standard literature on higher form gauge fields. We see that the kernel of the topological class homomorphism is:

$$\ker(c) = \Omega^{\ell-1}(M_n)/\Omega_{\mathbb{Z}'}^{\ell-1}(M_n). \quad (39.57)$$

As we have noted before, if $\omega \in \Omega_{\mathbb{Z}'}^{\ell-1}(M_n)$, then it must be closed. Physicists tend to distinguish between two kinds of gauge transformations: The “large gauge transformations” are those which shift A by an ω that has nonzero periods. Within the group of gauge transformations $\Omega_{\mathbb{Z}'}^{\ell-1}(M_n)$, there is an important subgroup of “small gauge transformations” for which the periods of ω all vanish. Therefore, ω is not only closed but exact, and we can

write the gauge transformation as:

$$A \mapsto A + d\Lambda , \quad \text{for} \quad \Lambda \in \Omega^{\ell-2}(M_n) . \quad (39.58)$$

We have now explained the short exact sequences on the south-west to north-east and north-west to south-east diagonals in the dancing hexagon (39.1) above. Next, we discuss the compatibilities of the homomorphisms.

39.4 Compatibilities Of The Maps

The blue dashed arrow at the top of the dancing hexagon (39.1) expresses the evident fact that if we define a topologically trivial character χ_A then, by Stokes' theorem, $F = dA$. Thus, the triangle in the northern part of the diagram commutes. Similarly, the topological class of a flat character $\check{\phi}$ based on $\phi \in H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')$ is expressed in terms of the Bockstein map $c(\check{\phi}) = -\beta(\phi)$, so the diagram in the southern part also commutes. (Recall the arrow pointing to the northeast has the awkward sign $\phi \mapsto -\check{\phi}$.)

Now, let us consider the square in the east. There is a map:

$$H^\ell(M_n; \mathbb{Z}) \rightarrow H^\ell(M_n; \mathbb{Z}) \otimes \mathbb{R} \cong H^\ell(M_n; \mathbb{R}) \cong H_{dR}^\ell(M_n) , \quad (39.59)$$

where the last isomorphism follows from de Rham's theorem. The kernel of $H^\ell(M_n; \mathbb{Z}) \rightarrow H^\ell(M_n; \mathbb{Z}) \otimes \mathbb{R}$ is the torsion subgroup of $H^\ell(M_n; \mathbb{Z})$. The commutativity of the square says that the topological class of the character, reduced modulo torsion, is captured by the de Rham class of $F/2\pi$:

$$\left[\frac{F}{2\pi} \right]_{dR} = c(\chi)_{\mathbb{R}} . \quad (39.60)$$

Finally, we consider the square in the west. A class $\alpha \in H^{\ell-1}(M_n; \mathbb{R})$ can, again by de Rham's theorem, be represented by a closed $\ell - 1$ form $\alpha = [A]$. Once we choose A , we can define a topologically trivial character χ_A , and χ_A is a flat character because, as we just said, A is closed. Note that two different representatives A and A' of α differ by an exact form, and therefore define the same character $\chi_A = \chi_{A'}$, so the map,

$$H^{\ell-1}(M_n; \mathbb{R}) \rightarrow \Omega^{\ell-1}(M_n)/\Omega_{\mathbb{Z}'}^{\ell-1}(M_n) , \quad (39.61)$$

is well-defined. In fact, if we shift $\alpha \mapsto \alpha + \omega$, where $\omega \in H^{\ell-1}(M_n; \mathbb{Z}')$, we also produce the same character, so the map descends to a map,

$$H^{\ell-1}(M_n; \mathbb{R})/H^{\ell-1}(M_n; \mathbb{Z}') \rightarrow \Omega^{\ell-1}(M_n)/\Omega_{\mathbb{Z}'}^{\ell-1}(M_n) . \quad (39.62)$$

On the other hand, there is also a natural map,

$$H^{\ell-1}(M_n; \mathbb{R}) / H^{\ell-1}(M_n; \mathbb{Z}') \rightarrow H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}') , \quad (39.63)$$

which maps into the topologically trivial flat characters. It is easy to check that the square on the western part of the dancing hexagon (39.1) also commutes.

The discussion of equations (37.7) and (37.7) above generalizes immediately to higher degree forms, and so we have the general, noncanonical, decomposition of the group of differential characters into a product of a connected torus, a finitely generated Abelian group, and an infinite-dimensional vector space:

In general, we have a noncanonical decomposition of the infinite-dimensional Abelian group $\check{H}^\ell(M_n)$ as:

$$\check{H}^\ell(M_n) = \mathbb{T}_{\ell-1} \times V_\ell \times \Gamma_\ell , \quad (39.64)$$

where the factors are:

$$\mathbb{T}_{\ell-1} = \text{Connected torus of Wilson lines} = H^{\ell-1}(M_n; \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}' \cong \text{U}(1)^{b_{\ell-1}} , \quad (39.65)$$

$$V_\ell = \infty\text{-dimensional vector space} = \text{im}(d^\dagger : \Omega^{\ell+1} \rightarrow \Omega^\ell) , \quad (39.66)$$

$$\Gamma_\ell = \text{Finitely generated Abelian group} = H^\ell(M_n; \mathbb{Z}) \leftarrow \text{topological classes} . \quad (39.67)$$

This is a natural generalization of (37.9).

Example: $\ell = 1$, $M = \mathbb{S}^1$. We have,

$$\check{H}^1(\mathbb{S}^1) = \text{Map}(\mathbb{S}^1 \rightarrow \text{U}(1)) = \text{loop group } \text{LU}(1) . \quad (39.68)$$

Let us identify the domain $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ with coordinate $\sigma \sim \sigma + n$, $n \in \mathbb{Z}$. An element of $\check{H}^1(\mathbb{S}^1)$ can be expanded as:

$$f(\sigma) = \exp \left[i\phi_0 + 2\pi i w\sigma + \sum_{n \neq 0} \frac{1}{n} \phi_n e^{2\pi i n\sigma} \right] . \quad (39.69)$$

Here $\phi_0 \in \mathbb{R}/\mathbb{Z}'$ represents the degree of freedom in the flat field, and $w \in \mathbb{Z}$, which corresponds to the winding number, denotes the characteristic class. Since f is a pure phase valued in $\text{U}(1)$, we must have $\phi_n^* = -\phi_{-n}$ for $n \neq 0$.

Remark 39.70. In the 2d conformal field theory of a single periodic scalar – sometimes called the Gaussian 2d CFT – the above decomposition corresponds to the mode decomposition of a free field.

40 Models Of Differential Cohomology

As we have stressed, $\check{H}^\ell(M_n)$ is the set of gauge equivalence classes of field configurations of an “ $(\ell-1)$ -form gauge potential.” In gauge theory, working all the time with \mathcal{A}/\mathcal{G} is difficult. One wants to work with gauge potentials, i.e., elements of \mathcal{A} . This is an affine linear space. We can write Lagrangians – it is great for locality. There are a lot of good reasons to find a better local model for gauge equivalence classes. The same goes for differential cohomology. (As we have seen, for $\ell = 2$, \mathcal{A}/\mathcal{G} is indeed the same thing as $\check{H}^2(M_n)$.)

We will discuss two kinds of models:

1. Čech model.
2. Hopkins-Singer model.

40.1 Čech Model

This formulation goes back to P. Deligne, and indeed in the context of complex geometry, Cheeger-Simons cohomology is also known as Deligne cohomology. It was first introduced into physics by O. Alvarez [Alv85] and independently, by K. Gawedzki [Gaw87].

The basic idea is that on a contractible space, all field configurations must be topologically trivial, and there is no room for the extra gauge-invariant information encoded in the group of flat fields or characteristic classes. The extra gauge-invariant information arises from patching local data, but one must include multiple intersections of patches (to a degree determined by the degree of the differential character). To implement this idea, we choose a good cover $\{\mathcal{U}_\alpha\}$ of M_n , that is, a cover such that all $U_{\alpha_1 \dots \alpha_k} = \mathcal{U}_{\alpha_1} \cap \dots \cap \mathcal{U}_{\alpha_k}$ are contractible. One can show (see, for example, [BT82]) that such covers always exist on manifolds.

We describe the first 3 cases, for $\ell = 1, 2, 3$ quite explicitly.

$\ell = 1$: The periodic scalar. We begin with the globally defined fieldstrength $F \in \Omega^1(M_n)$, which is, moreover, closed: $dF = 0$. Restricting F to each patch \mathcal{U}_α , we have locally defined 1-forms $F_\alpha \in \Omega^1(\mathcal{U}_\alpha)$ and, because $dF_\alpha = 0$, and because \mathcal{U}_α is contractible $F_\alpha = d\phi_\alpha = -i f_\alpha^{-1} df_\alpha$, where $f_\alpha : \mathcal{U}_\alpha \rightarrow U(1)$ has a well-defined logarithm $\log f_\alpha = i\phi_\alpha$. Now, on the patch overlaps $\mathcal{U}_{\alpha\beta}$, we have $F_\alpha - F_\beta = 0$ because F is globally well-defined, but this implies that

$$f_\alpha^{-1} df_\alpha - f_\beta^{-1} df_\beta = 0 \implies f_\alpha/f_\beta = \text{const. on } \mathcal{U}_{\alpha\beta}. \quad (40.1)$$

With this line of reasoning, we now come to an important choice. We can denote the constants in $\mathcal{U}_{\alpha\beta}$ as:

$$\phi_\alpha - \phi_\beta = 2\pi ir_{\alpha\beta}, \quad (40.2)$$

with $r_{\alpha\beta} \in \mathbb{R}$. From their definition,

$$\text{On } \mathcal{U}_{\alpha\beta\gamma} : r_{\alpha\beta} + r_{\beta\gamma} + r_{\gamma\alpha} = 0 . \quad (40.3)$$

A collection of real numbers $r_{\alpha\beta}$ on $\mathcal{U}_{\alpha\beta}$ satisfying (40.3) is known as a “Čech 2-cocycle”. It is shown in textbooks (e.g., see Bott-Tu [BT82]) that such a cocycle determines a cohomology class in $H^1(M_n; \mathbb{R})$.

Now, we caution the reader that what we have described at this point is not yet a Cheeger-Simons differential character. To get differential characters, we make the additional assumption that

$$r_{\alpha\beta} = n_{\alpha\beta} \in \mathbb{Z} . \quad (40.4)$$

Note that this implies that on $\mathcal{U}_{\alpha\beta}$, $f_\alpha/f_\beta = 1$, that is $f_\alpha = f_\beta$, but this implies that there is a single globally defined function $f : M_n \rightarrow \mathrm{U}(1)$. We have already noted that $\check{H}^1(M_n)$ can be identified with the Abelian group of continuous maps $M_n \rightarrow \mathrm{U}(1)$, so with the extra quantization condition (40.4), we determine an element of differential cohomology of degree $\ell = 1$. So in the Čech description, an element of $\check{H}^1(M_n)$ is a collection $(F_\alpha, n_{\alpha\beta})$ where $n_{\alpha\beta}$ define an integral Čech 1-cocycle, which in turn determines an integral cohomology class $H^1(M_n; \mathbb{Z})$. This class is the topological class of the differential character.

$\ell = 2$: U(1)-bundle with connection. We now apply the same line of reasoning for the case $\ell = 2$. We begin with a globally defined closed 2-form $F \in \Omega^2(M_n; \mathbb{R})$. Then on each patch, we have the pullback under the inclusion map: $F_\alpha \in \Omega^2(\mathcal{U}_\alpha)$. Since the curvature is globally well-defined:

$$\text{On } \mathcal{U}_{\alpha\beta} : \quad F_\alpha - F_\beta = 0 . \quad (40.5)$$

On the other hand, $dF_\alpha = 0$ implies $F_\alpha = dA_\alpha$, where A_α is a globally well-defined 1-form on \mathcal{U}_α . (Here we are using the properties of the good cover.) We should think of A_α as defining a connection on a line bundle $L_\alpha = \mathcal{U}_\alpha \times \mathbb{C}$. Now,

$$\text{On } \mathcal{U}_{\alpha\beta} : \quad dA_\alpha - dA_\beta = 0 \implies A_\alpha - A_\beta = d\epsilon_{\alpha\beta} = -ig_{\alpha\beta}^{-1}dg_{\alpha\beta} , \quad (40.6)$$

where $g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \rightarrow \mathrm{U}(1)$ again has a well-defined logarithm $\epsilon_{\alpha\beta}$. (Note that $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$.)

$$\text{On } \mathcal{U}_{\alpha\beta\gamma} : \quad (40.6) \implies g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \text{constant}. \quad (40.7)$$

At this stage, we introduce an important quantization condition and we impose the condition that this constant is 1:

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1 \quad \text{on } \mathcal{U}_{\alpha\beta\gamma} . \quad (40.8)$$

Thus the $\{g_{\alpha\beta}\}$ are transition functions for a principal $U(1)$ -bundle over M_n . Note that on $\mathcal{U}_{\alpha\beta\gamma}$,

$$\log g_{\alpha\beta} - \log g_{\alpha\gamma} + \log g_{\beta\gamma} = n_{\alpha\beta\gamma} \quad (40.9)$$

$$\implies n_{\alpha\beta\gamma} - n_{\alpha\beta\gamma\delta} + n_{\alpha\gamma\delta} - n_{\beta\gamma\delta} = 0 \text{ on } \mathcal{U}_{\alpha\beta\gamma\delta}. \quad (40.10)$$

This implies that $\{n_{\alpha\beta\gamma}\}_{\mathcal{U}_{\alpha\beta\gamma}}$ is a Čech 2-cocycle, which defines an element of Čech cohomology: $\{n_{\alpha\beta\gamma}\} \in H_{\text{Čech}}^2(M_n, \{\mathcal{U}_\alpha\})$. As shown in Bott and Tu [BT82], the Čech cohomology is isomorphic to $H^2(M_n; \mathbb{Z})$, and the resulting cohomology class is just the first Chern class of the principal $U(1)$ -bundle (see footnote 90). We have recovered the description of $\check{H}^2(M_n)$ as the space of isomorphism classes of principal $U(1)$ -bundles with connection over M_n .

The nice thing about this approach is that it is straightforward to extend it to $\ell > 2$.

$\ell = 3$: “Gerbe connection”. We begin with a globally well-defined closed 3-form H . On \mathcal{U}_α , we have a local fieldstrength

$$H_\alpha \in \Omega^3(\mathcal{U}_\alpha) \quad \text{with} \quad dH_\alpha = 0. \quad (40.11)$$

Therefore, there exists $B_\alpha \in \Omega^2(\mathcal{U}_\alpha)$ such that $H_\alpha = dB_\alpha$.

$$\text{On } \mathcal{U}_{\alpha\beta}, \quad B_\alpha - B_\beta = d\Lambda_{\alpha\beta} \quad \text{where } \Lambda_{\alpha\beta} \in \Omega^1(\mathcal{U}_{\alpha\beta}) \quad (40.12)$$

$$\implies \text{On } \mathcal{U}_{\alpha\beta\gamma}, \quad d(\Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha}) = 0 \quad (40.13)$$

$$\implies \text{On } \mathcal{U}_{\alpha\beta\gamma}, \quad \Lambda_{\alpha\beta} + \Lambda_{\beta\gamma} + \Lambda_{\gamma\alpha} = -i f_{\alpha\beta\gamma}^{-1} df_{\alpha\beta\gamma} \quad (40.14)$$

$$\implies \text{On } \mathcal{U}_{\alpha\beta\gamma\delta}, \quad f_{\alpha\beta\gamma} f_{\alpha\beta\delta}^{-1} f_{\alpha\gamma\delta} f_{\beta\gamma\delta}^{-1} = \text{constant}. \quad (40.15)$$

Now impose quantization: On $\mathcal{U}_{\alpha\beta\gamma\delta}$,

$$f_{\alpha\beta\gamma} f_{\alpha\beta\delta}^{-1} f_{\alpha\gamma\delta} f_{\beta\gamma\delta}^{-1} = 1. \quad (40.16)$$

Taking a logarithm, we find that

$$n_{\alpha\beta\gamma\delta} := \frac{1}{2\pi i} (\log f_{\alpha\beta\gamma} - \log f_{\alpha\beta\delta} + \log f_{\alpha\gamma\delta} - \log f_{\beta\gamma\delta}), \quad (40.17)$$

is a Čech cocycle, and it defines an integral class in $H^3(M_n; \mathbb{Z})$. This is the topological class of the differential character.

General Values of ℓ . Clearly, one can carry this discussion out for any ℓ . One just has to use more indices. We begin with a globally well-defined closed form $F \in \Omega^\ell(M_n; \mathbb{R})$ from

which we have pullbacks under the inclusion maps $F_\alpha \in \Omega^\ell(\mathcal{U}_\alpha)$. Then, as before

$$F_\alpha := F|_{\mathcal{U}_\alpha} = dA_\alpha^{(\ell-1)} \quad \text{where } A_\alpha^{(\ell-1)} \in \Omega^{\ell-1}(\mathcal{U}_\alpha) . \quad (40.18)$$

On $\mathcal{U}_{\alpha\beta}$,

$$F_\alpha - F_\beta = 0 \iff d(A_\alpha^{(\ell-1)} - A_\beta^{(\ell-1)}) = 0 \implies A_\alpha^{(\ell-1)} - A_\beta^{(\ell-1)} = dA_{\alpha\beta}^{(\ell-2)} , \quad (40.19)$$

because on the contractible subspace $\mathcal{U}_{\alpha\beta}$, any closed form is exact. Therefore, in the Čech model, we have a collection of forms:

$$\{F_\alpha, A_\alpha^{(\ell-1)}, A_{\alpha\beta}^{(\ell-2)}, A_{\alpha\beta\gamma}^{(\ell-3)}, \dots, A_{\alpha_1\dots\alpha_\ell}^{(0)}, n_{\alpha_1,\dots,\alpha_{\ell+1}}\} , \quad (40.20)$$

where $A_{\alpha_1\dots\alpha_\ell}^{(0)}$ are $U(1)$ -valued and $n_{\alpha_1,\dots,\alpha_{\ell+1}}$ are integral-valued, and define a cohomology class in $H^\ell(M_n; \mathbb{Z})$. That cohomology class is the topological class of the character.

Gauge Redundancy: It should be clear that the above description has a lot of gauge redundancy. We could make “gauge transformations” by a series of forms $(\epsilon_\alpha^{(\ell-1)}, \epsilon_{\alpha\beta}^{(\ell-2)}, \dots)$, so that

$$\begin{aligned} \tilde{A}_\alpha^{(\ell-1)} &= A_\alpha^{(\ell-1)} + d\epsilon_\alpha^{(\ell-2)} , \\ \tilde{A}_{\alpha\beta}^{(\ell-2)} &= A_{\alpha\beta}^{(\ell-2)} - (\epsilon_\alpha^{(\ell-2)} - \epsilon_\beta^{(\ell-2)}) + d\epsilon_{\alpha\beta}^{(\ell-3)} , \\ \vdots &\qquad \vdots \end{aligned} \quad (40.21)$$

Remark 40.22.

1. We will sometimes denote the collection of data (40.20) by \check{A} , and its gauge equivalence class by $[\check{A}]$.

2. **Pros And Cons Of The Čech Formulation.** The Čech model is very good for physicists who like to think in terms of differential forms, and honest-to-God computations with towers of fields. The model is very explicit. It also makes clear that there is extra gauge-invariant data beyond the globally well-defined fieldstrength $F \in \Omega^\ell(M; \mathbb{R})$. Most notably, it gives a clear description of the topological class $c(\chi) \in H^\ell(M_n; \mathbb{Z})$ of the character. A disadvantage is that it is much harder to describe the holonomy of the character. (There is a formula for that, see (41.8) below.) A related disadvantage is that it is hard to see the global structure of the differential cohomology group as the product of a finitely-generated Abelian group, a connected torus, and an infinite-dimensional vector space. A final disadvantage is that it does not readily generalize to arbitrary cohomology theories. These disadvantages are nicely addressed by a complementary model of differential characters, due to Hopkins and Singer [HS02], to which we turn next.

Exercise 40.1: Čech Model

- (a) Show how the periods of F can be expressed in terms of the integers $n_{\alpha_1, \dots, \alpha_{\ell+1}}$ in (40.20).
- (b) A graded ring structure on differential characters will be defined in Section 41.1 below. Show how to define it using the Čech model for differential characters described above.

This exercise was inspired by an unpublished note by D. Harlow.

40.1.1 Gerbes And Gerbe Connections

As we have noted, the case of $\ell = 1$ and $\ell = 2$ in the above constructions have very commonly encountered physical manifestations: The periodic scalar field and an Abelian gauge field, respectively. It turns out that the case $\ell = 3$ also has diverse applications and manifestations in physics.

Equivalence classes of degree three characters are equivalent to “gerbe connections.” Mathematically, a “gerbe” can be defined in many different ways. One way, described for example in [Hit99], simply generalizes the description of $U(1)$ line bundles via $U(1)$ -valued transition functions $g_{\alpha\beta}$ by considering the data $\{f_{\alpha\beta\gamma}\}$ satisfying the cocycle condition as a gerbe. (One then imposes suitable equivalence relations.) Just as complex line bundles over M_n are classified by $H^2(M_n; \mathbb{Z})$, $U(1)$ gerbes are classified by an element of $H^3(M_n; \mathbb{Z})$. Another approach to defining a gerbe is to consider a family of complex line bundles $\mathcal{L}_{\alpha\beta}$ on patch overlaps together with isomorphisms on the triple overlaps

$$\theta_{\alpha\beta\gamma} : \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\alpha} \rightarrow \mathcal{U}_{\alpha\beta\gamma} \times \mathbb{C}. \quad (40.23)$$

On quadruple overlaps, the isomorphisms $\theta_{\alpha\beta\gamma}$ satisfy a cocycle condition.¹⁰⁰

Now, turning to a *gerbe connection*, the local data for \check{A} for $\ell = 2$ defines a principal $U(1)$ -bundle with connection. The local data for \check{A} for $\ell = 3$ with specified $\{f_{\alpha\beta\gamma}\}$ (defining a gerbe) defines a gerbe connection.

One manifestation of a gerbe connection in physics is via the “Neveu-Schwarz B -field” of string theory, which is mathematically modeled as differential character of degree $\ell = 3$. In the WZW model of 2d conformal field theory of maps from spacetime to a compact Lie group G , there is a family (parametrized by a “level”) of canonical B -fields. The associated differential cohomology class is described in more detail in Section 51.2 below. The precise nature of the differential cohomology group that models the B -field is different in different string models. See [DFM10] for a summary.

¹⁰⁰There are many other mathematical definitions of “gerbes.” For other accounts, see [Bry93, Mur07].

Another way in which gerbes and gerbe connections appear in physics is in the context of reduction of structure group. In general, given a principal G -bundle $P \rightarrow M_n$ and a homomorphism $\varphi : H \rightarrow G$, we have a *reduction of structure group from G to H along φ* if there is a principal H -bundle $Q \rightarrow M_n$, and an isomorphism of principal G -bundles,

$$Q \times_{\varphi} G \cong P . \quad (40.24)$$

This terminology applies even when the homomorphism $\varphi : H \rightarrow G$ is a covering map, such as the covering of $\text{Spin}(n)$ over $\text{SO}(n)$. When φ is a covering map, a reduction of structure group is also commonly referred to as a “lifting” of structure group.

Let G be a topological group and Z a subgroup of its center. Gerbes naturally arise when we consider obstructions to lifting the structure group of a G/Z bundle over M_n to a G -bundle over M_n , i.e., to the reduction of structure group along the natural quotient homomorphism $\varphi : G \rightarrow G/Z$. The transition functions of a principal G/Z bundle satisfy

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1_{G/Z} , \quad (40.25)$$

on triple overlaps $\mathcal{U}_{\alpha\beta\gamma}$. Note especially that this is an equation in the quotient group G/Z . On a good cover, one can choose lifts $\tilde{g}_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \rightarrow G$ such that $\pi(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta}$, then we conclude that,

$$\tilde{g}_{\alpha\beta} \tilde{g}_{\beta\gamma} \tilde{g}_{\gamma\alpha} = \zeta_{\alpha\beta\gamma} \quad \text{on } \mathcal{U}_{\alpha\beta\gamma} , \quad (40.26)$$

where $\zeta_{\alpha\beta} : \mathcal{U}_{\alpha\beta\gamma} \rightarrow Z$. The Čech cocycle $\{\zeta_{\alpha\beta\gamma}\}$ determines a class in $H^2(M_n; \underline{\mathbb{Z}})$ where $\underline{\mathbb{Z}}$ is a sheaf. If $Z \subset \text{U}(1)$, this is equivalent to a class in $H^3(M_n; \mathbb{Z})$.

If G is a compact Lie group with a finite center then the class in $H^3(M_n; \mathbb{Z})$. Then we can think of the obstruction to lifting a G/Z bundle to a G -bundle as a flat degree three differential cohomology class. Its characteristic class is:

$$\beta(\zeta) \in H^3(M_n; \mathbb{Z}) , \quad (40.27)$$

where β is the Bockstein map. Here are some significant special cases of the above construction:

1. Suppose that M_n is an oriented Riemannian n -manifold. Then its oriented frame bundle is a principal $\text{SO}(n)$ -bundle. But $\text{SO}(n) \cong \text{Spin}(n)/\mathbb{Z}_2$ for a suitable \mathbb{Z}_2 subgroup of the center. A reduction of the bundle of oriented frames to a principal $\text{Spin}(n)$ bundle is a spin structure. Evidently, from the above discussion, that reduction is obstructed by a degree two cohomology class with \mathbb{Z}_2 coefficients. That obstruction is the second Stiefel-Whitney class of the tangent bundle $w_2(TM)$. It is a \mathbb{Z}_2 class that depends continuously on the metric and hence is metric-independent.

2. In applications to Yang-Mills theory for a compact Lie group of the form \tilde{G}/Z , where \tilde{G} is a compact semisimple and simply connected gauge group the obstruction to lifting a principal \tilde{G}/Z to a principal \tilde{G} -bundle is known as an 't Hooft flux in honor of a very deep paper that studied the case of $\mathrm{PSU}(N)$ bundles on the torus [tH78]. In more modern language, background gerbe connections associated with this flux are called "background 1-form gauge fields."

A very closely related construction appears when we consider bundles of simple algebras. These are sometimes referred to as *Azumaya bundles* or *Dixmier-Douady bundles*. Suppose A is a simple algebra and $\pi : \mathcal{A} \rightarrow M_n$ is a bundle whose fibers are isomorphic as vector spaces to A and whose transition functions are inner: $(x, a_\alpha) \in \mathcal{U}_\alpha \times A$ is glued to $(x, a_\beta) \in \mathcal{U}_\beta \times A$, via

$$a_\beta = g_{\alpha\beta} a_\alpha g_{\alpha\beta}^{-1}, \quad (40.28)$$

where $g_{\alpha\beta}$ is a continuous map from the overlap $\mathcal{U}_{\alpha\beta}$ to the group $G(A)$ of invertible elements of A . Note that if $\rho_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \rightarrow G(A) \cap Z(A)$ is in the center, then the transition functions are unchanged. Therefore, the structure group of the bundle of algebras is:

$$PG(A) := G(A)/(G(A) \cap Z(A)). \quad (40.29)$$

Therefore, on triple overlaps, our bundle of algebras will satisfy:

$$[g_{\alpha\beta}][g_{\beta\gamma}][g_{\gamma\alpha}] = 1_{PG(A)}. \quad (40.30)$$

We find ourselves precisely in the setting described previously: If we make a specific choice of maps $g_{\alpha\beta}$, then,

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \zeta_{\alpha\beta\gamma}. \quad (40.31)$$

For a simple algebra, $Z(A) \cong \mathbb{C}$, and we obtain a class in $H^3(M_n; \mathbb{Z})$. In this setting, the class in $H^3(M_n; \mathbb{Z})$ is called the *Dixmier-Douady class*. For a nice description in terms of algebraic geometry, see [Sch09].

Example 1: If we have a bundle $\mathcal{A} \rightarrow M_n$ of algebras with fiber $A = \mathrm{Mat}_N(\mathbb{C})$, their transition functions will be in $\mathrm{PGL}(N, \mathbb{C})$, and the $g_{\alpha\beta}$ above will be valued in $\mathrm{GL}(N, \mathbb{C})$. We can choose representatives of $g_{\alpha\beta}$ with unit determinant. Then, taking the determinant of equation (40.31), we see that for finite-dimensional algebras, the Dixmier-Douady class must be torsion. However, if one considers infinite-dimensional C^* algebras, it is entirely possible to achieve non-torsion classes [Ros89, Bry93].

A natural way to obtain a bundle of algebras is to start with a vector bundle $E \rightarrow M_n$, and consider the bundle of endomorphisms $\mathrm{End}(E) \rightarrow M_n$. It is natural to ask if the converse is true. Given a bundle of algebras $\mathcal{A} \rightarrow M_n$, is there a bundle of A -modules $E \rightarrow M_n$, so

that $\text{End}(E) \cong \mathcal{A}$? It turns out that the Dixmier-Douady class is precisely the obstruction to realizing such a bundle of A -modules.

Example 2: An elegant example is provided by considering a Riemannian manifold $(M, g_{\mu\nu})$. There is then a canonical bundle of Clifford algebras, where the fibers are the algebras generated by the Clifford relations on the tangent bundle:

$$\{e, e'\} = 2g(e, e') . \quad (40.32)$$

For complex Clifford algebras, the Dixmier-Douady class is the Stiefel-Whitney class $W_3(TM)$ and if it is trivial, we can write the bundle of algebras as $\text{End}(\mathcal{S})$ where \mathcal{S} is the spinor bundle of a spin^c structure [CS84, Ply86].

An important physical application of the Dixmier-Douady class is in the Hamiltonian formulation of anomalies [Fad84, FS84, AGN85, CM96, CMM97, MS99, Seg23]. The standard setting for the discussion of the Hamiltonian formulation of anomalies is the case of gauge fields coupled to fermions. For a fixed gauge field, the quantization of the Clifford algebra canonical quantization rules for the fermions produces a projective Hilbert space. One way to see that it is projective is that one must make a choice of Clifford vacuum. For a nonanomalous theory, we wish to have a bundle of Hilbert spaces over the space of gauge fields on the spatial manifold. For a more general formulation of Hamiltonian anomalies see the discussion in Sections 4.1 and 5.4 of [FHT03].

Closely related to the above consideration is the topic of twisted K -theory which plays an important role in the study of D-branes in the presence of B -fields [Wit98c, Kap99, Wit00, BM00, HM00, MS06, BMRS07].

The mathematics of twisted K -theory goes back to [DK70, Kar07, Kar08]. It was also studied in [MS00, BCM⁺01, AS04, AS05]. A remarkable theorem in the subject is the Freed-Hopkins-Teleman theorem identifying twisted G -equivariant K -theory of G (with the action by conjugation) with the Verlinde algebra [FHT02, FHT03, FHT05, FHT07], which can be interpreted physically in terms of the D-brane boundary conditions of the topological field theory obtained by gauging the G -WZW theory by G [Moo03]. Twisted equivariant K -theory is also the proper setting for discussing RR fields and charges in type II string theory orientifolds [Wit98c, Guk99, DFM09].

Gerbes and gerbe connections have also played some role in condensed matter physics in recent years. Twisted equivariant K -theory is needed when discussing topological insulators in the framework of band structure [FM12, Thi14, Kel15, KdBvW⁺16, Gom17, BEC⁺17, SdBKP18].

The higher Berry curvatures studied in [KS20a, KS20b, HKT20, AKS23] bear a strong resemblance to the curvatures of a gerbe connection. A recent paper making a more direct connection to the gerbe connection is [Gei24].

40.2 Hopkins-Singer Models

The Hopkins-Singer model of differential cohomology is motivated by a “homotopy pushout” construction from homotopy theory. It has some advantages over the Čech description:

1. It can be applied to any generalized cohomology theory. This is especially important in string theory which makes the use of “differential K -theory.” (See below.)
2. It is easier to view the fields as forming a groupoid. Then the automorphism group of an object with isomorphism class in $\check{H}^\ell(M_n)$ is $H^{\ell-2}(M; \mathbb{U}(1))$. These automorphism groups play important roles in physics in formulating Gauss laws, defining charges, etc.
3. The holonomy of the character is built into the description.

There are other models described in [HS02] as well as in the book by Amabel, Debray, and Haine [ADH21].

One way to motivate the Hopkins-Singer model is to consider a formula for the holonomy of a differential character in \check{H}^ℓ on an $(\ell - 1)$ -cycle Σ which is n -torsion. Thus $n\Sigma = \partial\mathcal{B}$ for some integral n -chain \mathcal{B} . Therefore,

$$(\chi(\Sigma))^n = \exp\left(i \int_{\mathcal{B}} F\right) . \quad (40.33)$$

It would be wrong to conclude that

$$\chi(\Sigma) \stackrel{?}{=} \exp\left(\frac{i}{n} \int_{\mathcal{B}} F\right) . \quad (40.34)$$

For one thing, $F \in \Omega_{2\pi\mathbb{Z}}(M_n)$ could have periods $\neq 0 \pmod{2\pi\mathbb{Z}}$. Then the above formula becomes ill-defined. However, one can show that there exists an integral cochain $a \in C^\ell(M_n; \mathbb{Z})$ such that,

$$\chi(\Sigma) = \exp\left[\frac{i}{n} \left(\int_{\mathcal{B}} F - 2\pi\langle a, \mathcal{B}\rangle\right)\right] , \quad (40.35)$$

is well-defined. Here δ is the coboundary operator and $\langle \cdot, \cdot \rangle$ denotes the natural pairing between cochains and chains on M_n . One can show that the class a has $\delta a = 0$ and hence, $[a] \in H^\ell(M_n; \mathbb{Z})$. This is the characteristic class (a.k.a. topological class) of the character $\chi \in \check{H}^\ell(M_n)$. We can write (40.35) heuristically by writing “ $\Sigma = \frac{1}{n}\partial\mathcal{B}$,” taking the log of (40.35), and dividing by $2\pi i$ to get:

$$\left\langle \frac{\log \chi}{2\pi i}, \partial\mathcal{B} \right\rangle = \left\langle \frac{F}{2\pi} - a, \mathcal{B} \right\rangle , \quad (40.36)$$

which we rewrite (still heuristically) as:

$$\delta \left(\frac{\log \chi}{2\pi i} \right) = \frac{F}{2\pi} - a_{\mathbb{R}} . \quad (40.37)$$

This is a motivating equation for the definition of Hopkins-Singer cocycles.

40.2.1 The Cochain Model

Definition 40.38 (Hopkins-Singer cochain and cocycle). A Hopkins-Singer cochain is a triple,

$$\chi = (a, h, \omega) \in C^\ell(M_n; \mathbb{Z}) \times C^{\ell-1}(M_n; \mathbb{R}) \times \Omega^\ell(M_n) . \quad (40.39)$$

Hopkins and Singer define a *cocycle* to be a triple such that,

$$\begin{aligned} \delta a &= 0 , \\ \delta h &= \omega - a_{\mathbb{R}} , \\ d\omega &= 0 , \end{aligned} \quad (40.40)$$

where $a_{\mathbb{R}}$ is the image of a under $C^\ell(M_n; \mathbb{Z}) \rightarrow C^\ell(M_n; \mathbb{R})$, and ω is likewise embedded as $\Omega^\ell(M_n) \rightarrow C^\ell(M_n; \mathbb{R})$. The set of all Hopkins-Singer cocycles is denoted $\check{Z}^\ell(M_n)$.

Above, we used the language of a cochain complex. On the space of Hopkins-Singer cochains, one can define a differential d_{HS} , which acts on a Hopkins-Singer cochain (a, h, ω) as:

$$d_{HS}(a, h, \omega) := (\delta a, \omega - a_{\mathbb{R}} - \delta h, d\omega) . \quad (40.41)$$

The definition (40.40) of a Hopkins-Singer cocycle is equivalent to $d_{HS}(a, h, \omega) = 0$. It is straightforward to verify that $d_{HS}^2 = 0$. However, if we just compute the cohomology of the differential d_{HS} , we will not obtain differential cohomology. In order to obtain differential cohomology, we need to generalize to a double complex as follows.

We define a collection of cochain complexes $\check{C}(\ell)^\bullet(M_n)$ as:

$$\check{C}(\ell)^k(M_n) = \begin{cases} C^k(M_n; \mathbb{Z}) \times C^{k-1}(M_n; \mathbb{R}) \times \Omega^k(M_n) , & \text{for } k \geq \ell , \\ C^k(M_n; \mathbb{Z}) \times C^{k-1}(M_n; \mathbb{R}) , & \text{for } k < \ell , \end{cases} \quad (40.42)$$

with the differential acting as:

$$\begin{aligned} d_{HS}(a, h, \omega) &= (\delta a, \omega - a_{\mathbb{R}} - \delta h, d\omega) , \\ d_{HS}(a, h) &= \begin{cases} (\delta a, -a_{\mathbb{R}} - \delta h, 0) , & \text{or } (a, h) \in \check{C}(\ell)^{\ell-1} , \\ (\delta a, -a_{\mathbb{R}} - \delta h) , & \text{otherwise.} \end{cases} \end{aligned} \quad (40.43)$$

The k^{th} cohomology group of the cochain complex $\check{C}(\ell)^\bullet(M_n)$ is denoted by

$$\check{H}(\ell)^k(M_n) := H^k(\check{C}(\ell)^k(M_n), d_{\text{HS}}) . \quad (40.44)$$

For $k > \ell$, the cohomology class is an integral class and a de Rham class with the constraint that they are the same in real cohomology, so we just get a copy of $H^k(M; \mathbb{Z})$. It is thus evident that [HS02, Sec 3.2]:

$$\check{H}(\ell)^k(M_n) = \begin{cases} H^k(M_n; \mathbb{Z}) , & \text{for } k > \ell , \\ H^{k-1}(M_n; \mathbb{R}/\mathbb{Z}') , & \text{for } k < \ell . \end{cases} \quad (40.45)$$

The Cheeger-Simons group of differential characters of degree $(\ell - 1)$ is identified as

$$\check{H}^\ell(M_n) \cong \check{H}(\ell)^\ell(M_n) . \quad (40.46)$$

Remark 40.47. We have chosen a different grading of the differential cohomology group from that used by Cheeger-Simons and Hopkins-Singer. For many purposes, it is more convenient to grade it by the degree of the fieldstrength and characteristic class.

Remark 40.48. Hopkins-Singer cocycles were also described by P. Deligne and D. Freed in [DF99a, Sec. 6.3]. Indeed, in the holomorphic context, a very similar construction is known as *Deligne cohomology*.

40.2.2 The Groupoid Model

Another approach to modeling differential cohomology (also described in [HS02, Sec. 2.3]) is to define a groupoid whose objects are Hopkins-Singer cocycles and the isomorphism classes of these objects correspond to differential characters.

We can motivate the groupoid approach of Hopkins and Singer by interpreting the standard cohomology group $H^\ell(M_n; \mathbb{Z})$ as the set of isomorphism classes of a groupoid $\mathcal{H}^\ell(M_n)$ as follows: The objects of $\mathcal{H}^\ell(M_n)$ are smooth cocycles $c \in Z^\ell(M_n; \mathbb{Z})$. The morphism set between two objects $\text{Hom}(c_2, c_1)$ is identified with the set of elements $[b] \in C^{\ell-1}(M_n; \mathbb{Z})/\delta C^{\ell-2}(M_n; \mathbb{Z})$, such that,

$$c_2 = c_1 + \delta b . \quad (40.49)$$

Put differently: $\text{Hom}(c_2, c_1)$ consists of $b \in C^{\ell-1}(M_n; \mathbb{Z})$, with an equivalence relation on b defined by $b' \sim b + \delta v$, $v \in C^{\ell-2}(M_n; \mathbb{Z})$. This is a groupoid because we can bring δb to the LHS of equation (40.49). The isomorphism classes of objects in $\mathcal{H}^\ell(M_n)$ are in 1–1 correspondence with cohomology classes in $H^\ell(M_n; \mathbb{Z})$. What do we gain from this viewpoint? Now, we can understand that a cohomology class has an automorphism group.

The automorphism group of c is the equivalence class of $b \in C^{\ell-1}(M_n; \mathbb{Z})$, such that $\delta b = 0$. Thus, the automorphism group of any object in $\check{\mathcal{H}}^\ell(M_n)$ is isomorphic to the cohomology group $H^{\ell-1}(M_n; \mathbb{Z})$.

The viewpoint of the previous section generalizes nicely to the differential setting. One can define a groupoid, $\check{\mathcal{H}}^\ell(M)$ whose objects are Hopkins-Singer cocycles $(a, h, \omega) \in \check{Z}^\ell(M)$ (and thus satisfy (40.40)). The morphism space $\text{Hom}(x, x')$ for $x = (a, h, \omega)$ and $x' = (a', h', \omega')$ consists of equivalence classes of pairs,

$$(b, g) \in C^{\ell-1}(M_n; \mathbb{Z}) \times C^{\ell-2}(M_n; \mathbb{R}) , \quad (40.50)$$

such that,

$$\begin{aligned} a' &= a - \delta b , \\ h' &= h + \delta g + b . \end{aligned} \quad (40.51)$$

(Note that since we are working with differential cocycles it follows that $\omega' = \omega$.) The equivalence relation on pairs (b, g) is:

$$(b, g) \sim (b - \delta e, g + \delta v + e) , \quad (40.52)$$

for all $(e, v) \in C^{\ell-2}(M; \mathbb{Z}) \times C^{\ell-3}(M; \mathbb{R})$. See [HS02], Definition 2.5.

As we have said, the isomorphism class of $x \in \check{\mathcal{H}}^\ell(M_n)$ can be identified with a differential character. The relation to the direct product description of $\check{H}^\ell(M_n)$ is rather direct: ω is the fieldstrength $F/2\pi$, h defines the holonomy, and $[a]$ is the topological class. Once again, the advantage of this viewpoint is that objects have a nontrivial automorphism group:

Exercise 40.2:

Show that the automorphism group of an object $x \in \check{\mathcal{H}}^\ell(M_n)$ can be identified with $H^{\ell-2}(M_n; \mathbb{R}/\mathbb{Z})$.

Remark 40.53.

1. When describing the C-field of M-theory, it is very natural to work with a Hopkins-Singer model since, as we will see in Section 53, the isomorphism classes of C-fields form a torsor for $\check{H}^4(M_{11})$, and the C-field itself must be defined using a differential cochain. Another model for the differential cocycles, used in [DFM03], uses E_8 bundles with connection.
2. The automorphism group of a differential character plays an important role in some physical applications. For example, it is useful to give a quantum definition of electric charge. See (43.13) below. This viewpoint is very useful in giving a precise definition of C-field electric charge in M-theory [DFM03].

3. There is a natural functor $F : \check{\mathcal{H}}^\ell(M_n) \rightarrow \mathcal{H}^\ell(M_n)$. On objects, we simply have $F(a, h, \omega) = a$. The mapping of automorphism groups $F : \text{Aut}(a, h, \omega) \rightarrow \text{Aut}(a)$ is the Bockstein map $H^{\ell-2}(M_n; \mathbb{R}/\mathbb{Z}') \rightarrow H^{\ell-1}(M_n; \mathbb{Z})$.

40.2.3 Higher Groupoids

It is possible and useful to promote the groupoid $\check{\mathcal{H}}^\ell(M_n)$ to a higher groupoid. We explain that here, following the nice discussion in [FN22, App. A.3].

Given a chain complex (A^\bullet, d) , one can define an ℓ -groupoid whose objects, or 0-morphisms, are the closed ℓ -forms, $A_{\text{closed}}^\ell := \ker(d : A^\ell \rightarrow A^{\ell+1})$. The 1-morphisms are defined to be:

$$\mathsf{Hom}_1(\omega_0^\ell, \omega_1^\ell) := \{\epsilon^{\ell-1} \in A^{\ell-1} \mid \omega_1^\ell = \omega_0^\ell + d\eta^{\ell-1}\} . \quad (40.54)$$

Note that the morphisms are all invertible. The 2-morphisms are:

$$\mathsf{Hom}_2(\eta_0^\ell, \eta_1^\ell) := \{\zeta^{\ell-2} \in A^{\ell-2} \mid \eta_1^\ell = \eta_0^\ell + d\zeta^{\ell-2}\} , \quad (40.55)$$

and so on. Note that Hom_k between two $(k-1)$ -morphisms is an A_{closed}^{k-3} -torsor.

We apply the above construction using the Hopkins-Singer model to enhance $\check{\mathcal{H}}^\ell(M)$ to an ℓ -groupoid. The objects are $\check{Z}^\ell(M)$ as before. Now define:

$$\check{C}^q(M) := \check{C}(q)^q(M) , \quad (40.56)$$

and note that there is a natural inclusion of $\check{C}^q(M) \hookrightarrow \check{C}(q-1)^q(M)$. Now, the 1-morphisms between \check{A}_0 and \check{A}_1 are:

$$\mathsf{Hom}_1(\check{A}_0, \check{A}_1) := \{\check{\epsilon}^{\ell-1} \in \check{C}^{\ell-1}(M) \mid \check{A}_1 = \check{A}_0 + d_{\text{HS}}\check{\epsilon}^{\ell-1}\} . \quad (40.57)$$

Note that the defining condition here takes place in $\check{C}(\ell-1)^\ell(M)$. We can now proceed to define the 2-morphisms as:

$$\mathsf{Hom}_2(\check{\epsilon}_0^{\ell-1}, \check{\epsilon}_1^{\ell-1}) := \{\check{\eta}^{\ell-2} \in \check{C}^{\ell-2}(M) \mid \check{\epsilon}_1^{\ell-1} = \check{\epsilon}_0^{\ell-1} + d_{\text{HS}}\check{\eta}^{\ell-2}\} , \quad (40.58)$$

and so on. From this higher groupoid, one can form a simplicial set. The generalized gauge field as a field in the sense of Freed-Hopkins-Teleman is a sheaf valued in this simplicial set.

41 Graded Ring Structure, Integration, And Perfect Pairing Of Differential Cohomology

No matter what model we use, the gauge-invariant information in differential characters satisfies some important properties. We summarize them briefly here:

41.1 Graded Ring Structure

There is a graded associative product:

$$\begin{aligned} \check{H}^{\ell_1} \times \check{H}^{\ell_2} &\longrightarrow \check{H}^{\ell_1 + \ell_2} \\ (\chi_1, \chi_2) &\longmapsto \chi_1 \odot \chi_2 := (-1)^{\ell_1 \ell_2} \chi_2 \odot \chi_1 . \end{aligned} \tag{41.1}$$

This satisfies two nice properties:

$$F(\chi_1 \odot \chi_2) = F(\chi_1) \wedge F(\chi_2) , \quad \text{wedge product} , \tag{41.2}$$

$$c(\chi_1 \odot \chi_2) = c(\chi_1) \cup c(\chi_2) , \quad \text{cup product} . \tag{41.3}$$

The formula for the holonomy is more complicated. One way to express it is to give the product in terms of Hopkins-Singer cocycles (see [Section 40.2](#)):

$$(a_1, h_1, \omega_1) \odot (a_2, h_2, \omega_2) := (a_1 \cup a_2, \pm a_1 \cup h_2 + h_1 \cup \omega_2 + H(\omega_1, \omega_2), \omega_1 \wedge \omega_2) , \tag{41.4}$$

where H is a homotopy between the cup product \cup and the wedge product \wedge on Ω^ℓ , considered as defining smooth \mathbb{R} -valued ℓ -cochains.

41.2 Integration

If Σ_ℓ is an ℓ -cycle, then the holonomy on Σ_ℓ can be considered as an integration:

$$\int_{\Sigma_\ell}^{\check{H}} : \check{H}^{\ell+1}(\Sigma_\ell) \longrightarrow \check{H}^1(\text{pt}) . \tag{41.5}$$

If we think of the group $\check{H}^1(\text{pt})$ additively, we would write it as \mathbb{R}/\mathbb{Z}' , and if we think of it multiplicatively, we would write it as $U(1)$.

In the Čech model, there is an explicit formula for (41.5): Choose a triangulation of Σ_ℓ such that ℓ -simplices sit in a definite \mathcal{U}_α . We write these as: $\Sigma_{\ell,\alpha}^{(i)} \subset \mathcal{U}_\alpha$. Then, arrange that the faces are in patch overlaps, so we write: $\Sigma_{\ell,\alpha\beta}^{(i)} \subset \mathcal{U}_{\alpha\beta}$, etc. Then, denoting the data of the Čech model by,

$$\widehat{A} = (A_\alpha, A_{\alpha\beta}, A_{\alpha\beta\gamma}, \dots) , \tag{41.6}$$

and the corresponding character by,

$$\chi = [\check{A}] , \tag{41.7}$$

we have,

$$\exp \left(i \int_{\Sigma_\ell} \check{H} [\check{A}] \right) := \prod_{\alpha, j} \exp \left(i \int_{\Sigma_{\ell, \alpha}^j} A_\alpha \right) \cdot \prod_{\alpha < \beta, j} \exp \left(i \int_{\Sigma_{\ell, \alpha\beta}^j} A_{\alpha\beta} \right) \dots \quad (41.8)$$

The integration generalizes to families. If we have a family \mathcal{X} of closed n -manifolds M_n fibered over a manifold \mathcal{S} of control parameters:

$$\begin{array}{ccc} M_n & \longrightarrow & \mathcal{X} \\ & & \downarrow \\ & & \mathcal{S} \end{array} \quad (41.9)$$

then the notion of integration along fibers defines a homomorphism (for any d):

$$\int_{\mathcal{X}/\mathcal{S}}^{\check{H}} : \check{H}^d(\mathcal{X}) \longrightarrow \check{H}^{d-n}(\mathcal{S}) . \quad (41.10)$$

For generalizations of (41.8) to families, see Bär-Becker [BB13].

41.3 Pairing

Putting together the graded ring structure and the integration, we derive an important pairing on differential cohomology: multiply and integrate:

$$\begin{aligned} \check{H}^\ell(M_n) \times \check{H}^{n-\ell+1}(M_n) &\longrightarrow \check{H}^1(\text{pt}) = \mathbb{R}/\mathbb{Z}' \\ (\chi_1, \chi_2) &\longmapsto \langle \chi_1, \chi_2 \rangle := \int^{\check{H}} \chi_1 \odot \chi_2 . \end{aligned} \quad (41.11)$$

It follows from (41.1) that

$$\langle \chi_1, \chi_2 \rangle = (-1)^{\ell(n+1-\ell)} \langle \chi_2, \chi_1 \rangle . \quad (41.12)$$

There are two important and useful special cases of the pairing (41.11):

1. If χ_1 is topologically trivial, i.e., $\chi_1 = \chi_{A_1}$ for $A_1 \in \Omega^{\ell-1}(M_n)$, then,

$$\langle \chi_1, \chi_2 \rangle = \int_{M_n} A_1 \wedge F(\chi_2) , \quad (41.13)$$

which is an ordinary integral of differential forms. If χ_2 is also topologically trivial, i.e., $\chi_2 = \chi_{A_2}$ for $A_2 \in \Omega^{n-\ell}(M_n)$, then,

$$\langle \chi_1, \chi_2 \rangle = \int A_1 \wedge dA_2 , \quad (41.14)$$

2. If $\chi_1 = \check{\phi}_1$ is a flat field associated to $\phi_1 \in H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')$, then,

$$\exp(i\langle \chi_1, \chi_2 \rangle) = \exp\left(i \int_{M_n} \phi_1 \cup c(\chi_2)\right), \quad (41.15)$$

where we use the cup product on $H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}')$ and $H^{n-\ell+1}(M_n; \mathbb{Z})$ to get an element of $H^n(M_n; \mathbb{R}/\mathbb{Z}')$. So in the pairing, if χ_1 is a flat field, it only sees the topological class of χ_2 and nothing else. This will be a useful observation for the definition of flux sectors in (45.7) below.

A good example of the pairing is the simple case of $\ell = 1$ in two dimensions. We can use the mode decomposition of equation (39.69) to separate out the winding mode. So we write $f = \exp(2\pi i\phi)$, and then,

$$\langle \chi_{f^1}, \chi_{f^2} \rangle = -w^1 \phi^2(0) + \int_0^1 \phi^1 \frac{d}{d\sigma} \phi^2 d\sigma. \quad (41.16)$$

For a proof, see [FMS06b].

Remark 41.17. Note that the pairing (41.14) is none other than the BF or Chern-Simons action. Therefore the pairing can be viewed as one way of generalizing the actions, such as (27.22) of BF theory to topologically nontrivial situations. In particular, the 3d Chern-Simons action for Abelian gauge groups can be expressed as a pairing. A natural way to define an action for BF theory is via the pairing on differential cohomology. This point was first made in [MMS01b, App. A]. Note that without a quantization condition on the periods, the level k of the BF theory can always be scaled to one.

41.4 Poincaré-Pontryagin Duality

The pairings we have defined above satisfy an important property related to both Poincaré and Pontryagin duality.

In general, a pairing of two Abelian groups A and B to $\text{U}(1)$,

$$\langle \cdot, \cdot \rangle : A \times B \rightarrow \mathbb{R}/\mathbb{Z}, \quad (41.18)$$

is *perfect*, if,

- $\langle a, x \rangle = 0 \forall x \in B \implies a = 0$, and,
- $\langle y, b \rangle = 0 \forall y \in A \implies b = 0$,

where we are working additively. A pairing $\langle a, \cdot \rangle$ is a homomorphism $B \rightarrow \mathbb{R}/\mathbb{Z}$. Being perfect means $a \mapsto \langle a, \cdot \rangle$ is an isomorphism of $\text{Hom}(B, \mathbb{R}/\mathbb{Z}) \cong A$.

In general, given any Abelian group A , the *Pontryagin dual of A* is by definition the Abelian group:

$$\tilde{A} = \text{Hom}(A, \mathbb{R}/\mathbb{Z}) . \quad (41.19)$$

We will also regard the Pontryagin dual as $\text{Hom}(A, \text{U}(1))$, when working multiplicatively. For locally compact groups, the Pontryagin duality theorem says that the dual of the dual is isomorphic to the original group: $\tilde{\tilde{A}} \cong A$. For more on all this, see [Section 42](#).

Using the diagram of the dancing hexagon (39.1) and standard Poincaré duality for compact oriented manifolds, one can show that the pairing (41.11),

$$\check{H}^\ell(M_n) \times \check{H}^{n-\ell+1}(M_n) \rightarrow \mathbb{R}/\mathbb{Z}' , \quad (41.20)$$

is a perfect pairing. That is,

$$\text{Hom}(\check{H}^\ell(M_n); \mathbb{R}/\mathbb{Z}') \cong \check{H}^{n-\ell+1}(M_n) , \quad (41.21)$$

and therefore, the Poincaré dual cohomology group is also the Pontryagin dual group. We call this property Pontryagin-Poincaré duality of differential cohomology. The following diagram provides a proof:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z}') & \longrightarrow & \check{H}^\ell(M_n) & \longrightarrow & \Omega_{\mathbb{Z}}^\ell(M_n) \longrightarrow 0 \\ & & \swarrow \text{perfect pairing} & & \nearrow \text{perfect pairing} & & \\ 0 & \longrightarrow & \Omega^{n-\ell}(M_n)/\Omega_{\mathbb{Z}'}^{n-\ell}(M_n) & \longrightarrow & \check{H}^{n-\ell+1}(M_n) & \longrightarrow & H^{n-\ell+1}(M_n; \mathbb{Z}) \longrightarrow 0 \end{array} \quad (41.22)$$

Remark 41.23.

1. Poincaré-Pontryagin duality is essential for the manifestly S -duality invariant formulation of the Hilbert space of generalized Maxwell theory.
2. If Σ is an $(n-\ell)$ -cycle, then the evaluation map gives a homomorphism $\check{H}^{n+1-\ell}(M_n) \rightarrow U(1)$ and hence by Pontryagin-Poincaré duality, defines a differential character $\check{\delta}(\Sigma) \in \check{H}^\ell(M_n)$, via

$$\langle \check{\delta}(\Sigma), \chi \rangle = \chi(\Sigma) . \quad (41.24)$$

The fieldstrength of this map is not a smooth differential form. It is a δ -function supported representative of the Poincaré dual of Σ , also known as a “current” in the subject of geometric measure theory [dR84].

42 The Stone-von Neumann Representation Of Heisenberg Groups

Before returning to the physics of the generalized Abelian gauge field, we need to review one more mathematical topic.

Let A be an Abelian group with a Haar measure. We can then define the Hilbert space:

$$\mathcal{H} = L^2(A) . \quad (42.1)$$

We will also assume in the following discussion that the measure is translationally invariant. That is, if $a \in A$ and S is a subset of A , then the measure of S is the same as the measure of the set, where we shift all elements of S by a . Simple examples of such Hilbert spaces are:

1. If $A = \mathbb{R}$, $L^2(\mathbb{R})$ is the space of square-integrable functions over the reals, familiar to us from quantum mechanics.
2. If $A = \mathbb{Z}$, $L^2(\mathbb{Z})$ is the set of wavefunctions on a lattice, e.g., a spin chain.
3. Finally, if $A = \mathrm{U}(1)$, then $L^2(\mathrm{U}(1))$ enters the quantum mechanics of a particle on a circle.
4. If A is a cyclic group \mathbb{Z}_n , then $L^2(A)$ is the Hilbert space for discrete quantum mechanics on the circle. This is the canonical example where there are clock and shift operators, as we will see below.
5. We can take finite products of all the above groups.

Note that $L^2(A)$ is a representation of A by translation operators. So for each $a_0 \in A$, there is a translation operator T_{a_0} , such that,

$$(T_{a_0}\psi)(a) := \psi(a + a_0) , \quad (42.2)$$

$$T_{a_0} \circ T_{a'_0} = T_{a_0 + a'_0} , \quad (42.3)$$

Also, $L^2(A)$ is a representation of the Pontryagin dual \tilde{A} defined in (41.19). Thinking of the Pontryagin dual multiplicatively, for a character $\chi \in \tilde{A}$, there is a multiplication operator M_χ , such that,

$$(M_\chi\psi)(a) := \chi(a)\psi(a) , \quad (42.4)$$

$$M_{\chi_1} \circ M_{\chi_2} = M_{\chi_1 \cdot \chi_2} . \quad (42.5)$$

However, famously, $L^2(A)$ is not a representation of the direct product group $A \times \tilde{A}$, because the group law on the direct product says the two factors commute. On the other hand, a simple computation shows that:

$$T_{a_0} M_\chi = \chi(a_0) M_\chi T_{a_0} . \quad (42.6)$$

Rather, $L^2(A)$ is a representation of the Heisenberg extension, which, as a set, is $\mathsf{U}(1) \times A \times \tilde{A}$, but has a group law,

$$(z_1, (a_1, \chi_1)) \cdot (z_2, (a_2, \chi_2)) := (z_1 z_2 (\chi_1(a_2))^{-1}, (a_1 + a_2, \chi_1 \chi_2)) . \quad (42.7)$$

(Here we are writing A additively, and \tilde{A} multiplicatively.) Therefore the Heisenberg group sits in a short exact sequence,

$$1 \longrightarrow \mathsf{U}(1) \longrightarrow \mathsf{Heis}(A \times \tilde{A}) \longrightarrow A \times \tilde{A} \longrightarrow 1 . \quad (42.8)$$

This is a *central extension*, which means that the image of $\mathsf{U}(1)$ in $\mathsf{U}(1) \rightarrow \mathsf{Heis}(A \times \tilde{A})$ is central. This is an example of an anomaly. One can say that there is an anomaly in $A \times \tilde{A}$ and what is actually represented on the Hilbert space is the Heisenberg group.

Because of (42.6),

$$\rho(z, (a, \chi)) = z T_a M_\chi , \quad (42.9)$$

is an honest representation of the Heisenberg group. These operators are composed via the group law (42.7).

Now the key theorem on representations of $\mathsf{Heis}(A \times \tilde{A})$ is:

Theorem 42.10 (Stone-von Neumann-Mackey). *Up to isomorphism there is a unique, unitary irrep of $\mathsf{Heis}(A \times \tilde{A})$, such that the central $\mathsf{U}(1)$ acts by scalars:*

$$\rho(z) = z \cdot \mathbb{1}_{\mathcal{H}} \quad \forall z \in \mathsf{U}(1) . \quad (42.11)$$

Proof. The proof can be found in many places. One of them (with further references) is [Moo23a, Sec. 15.5.5]. \square

One model for the SvN representation is $L^2(A)$ with A and \tilde{A} acting as translation + multiplication operators. If A is locally compact, then Pontryagin duality says $\tilde{A} \cong A$, so

$$\mathsf{Heis}(A \times \tilde{A}) = \mathsf{Heis}(\tilde{A} \times A) . \quad (42.12)$$

Therefore another equivalent representation is $L^2(\tilde{A})$, where \tilde{A} acts by translation and A acts by multiplication. The isomorphism $L^2(A) = L^2(\tilde{A})$ is Fourier transformation.

1. We stress that although A and \tilde{A} are Abelian groups, the Heisenberg group $\text{Heis}(A \times \tilde{A})$ is non-Abelian. It fits in the short exact sequence (42.8) as a central extension. But the feature of the Heisenberg extension that distinguishes it from all the other central extensions of $A \times \tilde{A}$ is that this $\text{U}(1)$ is the entire center: $Z(\text{Heis}(A \times \tilde{A})) \cong \text{U}(1)$, so it is as non-Abelian as it can possibly be, given the fact that it is a central extension by $\text{U}(1)$.
2. **Generalizations Of The Pontryagin dual for non-Abelian groups.** What happens if we try to generalize the definition of the Pontryagin dual group to the case where G is non-Abelian. First, we show that $\text{Hom}(G, \text{U}(1))$ factors through the Abelianization of G . Suppose $\chi \in \text{Hom}(G, \text{U}(1))$. Then $\chi : G \rightarrow \text{U}(1)$ is a group homomorphism. Then,

$$\begin{aligned} \chi(g_1 g_2) &= \chi(g_1)\chi(g_2) \\ &= \chi(g_2)\chi(g_1) \\ &= \chi(g_2 g_1), \end{aligned} \tag{42.13}$$

which, of course, means that $\chi([g_1, g_2]) = 1$, where $[g_1, g_2]$ denotes the group commutator. Therefore, χ factors through the Abelianization:

$$\begin{array}{ccc} G & \xrightarrow{\quad} & \text{U}(1) \\ & \searrow & \nearrow \tilde{\chi} \\ & G/[G, G] & \\ & =: \text{Ab}(G) & \end{array} \tag{42.14}$$

This takes us back to the theory of Abelian groups. So the definition of Pontryagin dual of G as $\text{Hom}(G, \text{U}(1))$ does not give anything new. In fact, for some groups, the Abelianization is trivial, so in general, it does not give anything interesting.

There is, however, another definition that does better and goes further. First, recall that for Abelian A , the Pontryagin dual \tilde{A} can be identified with the group of isomorphism classes of irreducible unitary representations of A . So if we have an Abelian group, one can diagonalize the operators simultaneously. So one might as well make them one-dimensional, and the irreducible ones are in $\text{U}(1)$. So one can equivalently think of \tilde{A} as the set of irreducible unitary representations of A . For non-Abelian G , we can identify \tilde{G} as the set of unitary irreps of G :

$$\tilde{G} = \{\text{set of unitary irreps of } G\}. \tag{42.15}$$

Now we do not have a group anymore. But we do have a category. In fact, we

have a monoidal category, which is called $\text{Rep}(G)$. The objects are unitary irreps, the morphisms are G -equivariant maps, and the tensor product is the usual tensor product of representations. What replaces Pontryagin duality is something called Tannaka-Krein duality [JS91]. Given a fusion category with a fiber functor (i.e., a monoidal functor to the category of vector spaces), one can actually reconstruct a group. The construction reproduces the group G from the fusion category $\text{Rep}(G)$.

43 The Hilbert Space Of A Generalized Abelian Gauge Theory

The set of gauge equivalence classes of Generalized Abelian Gauge Theory (GAGT) on spacetime M_n is $\check{H}^\ell(M_n)$. We now assume a spacetime splitting,

$$M_n = N_{n-1} \times \mathbb{R} , \quad (43.1)$$

with \mathbb{R} parametrized by the time coordinate, and an action (31.5):

$$\pi \int_{M_n} \lambda F \wedge \star F . \quad (43.2)$$

As we have noted before, since F has quantized periods, the coupling λ is physically relevant. For example, the periodic scalar λ is related to the radius R of the target circle by $\lambda = R^2$.

The classical momentum is the $(n - \ell)$ -form,

$$\pi = 2\pi\lambda (\star_{M_n} F)|_{N_{n-1}} . \quad (43.3)$$

The classical phase space is the cotangent space of field configurations on the spatial slice:

$$T^* \check{H}^\ell(N_{n-1}) = \check{H}^\ell(N_{n-1}) \times \Omega^{n-\ell}(N_{n-1}) / \text{im } d^\dagger . \quad (43.4)$$

This follows from the noncanonical decomposition:

$$\check{H}^\ell(N_{n-1}) \approx \Omega^{\ell-1}(N_{n-1}) / \Omega_{\mathbb{Z}'}^{\ell-1}(N_{n-1}) \times \Gamma_\ell . \quad (43.5)$$

We will use standard quantization, so we have Heisenberg relations for the quantum fields F_q, Π_q :

$$\left[\int_{N_{n-1}} \omega_1 F_q, \int_{N_{n-1}} \omega_2 \Pi_q \right] = i\hbar \int_{N_{n-1}} \omega_1 d\omega_2 , \quad (43.6)$$

where $\omega_1 \in \Omega^{n-1-\ell}(N_{n-1})$ and $\omega_2 \in \Omega^{\ell-1}(M_n)$ are classical and the subscript q indicates a

quantum operator. Equation (43.6) is a nice clean implementation of the heuristic relation

$$\Pi_q \sim -i\hbar \frac{\delta}{\delta A_q}, \quad (43.7)$$

In particular, Π_q is the generator of translations along $\check{H}^\ell(N_{n-1})$. So in $L^2(\check{H}^\ell(N_{n-1}))$, we have

$$\left(\exp \left[\frac{i}{\hbar} \int_{N_{n-1}} \omega_2 \Pi_q \right] \Psi \right) ([\check{A}]) = \Psi([\check{A}] + [\check{\omega}_2]), \quad (43.8)$$

where $[\check{\omega}_2]$ is the additive notation for the topologically trivial character χ_{ω_2} .

For a cotangent space, the quantization is completely standard. Now we use heavily the property that $\check{H}^\ell(N_{n-1})$ is an Abelian group. Therefore, at least formally, it has a translationally-invariant measure so we can formulate the quantum Hilbert space of the theory:

$$\mathcal{H}(N_{n-1}) := L^2(\check{H}^\ell(N_{n-1})). \quad (43.9)$$

We are imagining that $\check{H}^\ell(N_{n-1})$ has a translationally-invariant Haar measure. As we have seen, $\check{H}^\ell(N_{n-1})$ is an infinite-dimensional group, so some analysis is needed to give this formula meaning. From the noncanonical decomposition (39.64),

$$\check{H}^\ell(N_{n-1}) = \mathbb{T}_{\ell-1} \times V_\ell \times \Gamma_\ell, \quad (43.10)$$

we see that the infinite-dimensional part comes from V_ℓ . The issues are the same as in the quantization of the free scalar field. The oscillator modes of A with $d^\dagger A = 0$ are quantized as in standard QFT. In these lectures, we are more concerned with the subtleties arising from the first two finite-dimensional factors. Hence, we are somewhat cavalier about the functional-analytic aspects. Roughly speaking, the allowed wavefunctions should have Gaussian decay:

$$\psi([\check{A}]) \sim \mathcal{P} \exp \left(- \int_{N_{n-1}} k F \star F \right), \quad (43.11)$$

where k is a constant and an ON basis for $L^2(\check{H}^\ell(N_{n-1}))$ would involve expressions where \mathcal{P} is polynomial in the oscillators from V_ℓ . It is possible to make this much more precise, but we will refrain from doing so here.

Remark 43.12. We have worked with the wavefunction as a function on the space of gauge equivalence classes. One can also work with a wavefunction as a section of a line bundle over the set of differential cocycles \check{Z}^ℓ , and impose invariance under gauge equivalences (such as the gauge equivalences described in Section 40) as a Gauss law. This latter viewpoint is useful because it gives a clear definition of quantum electric charge. For example, the automorphism group described in Exercise 40.2 must act as a trivial character on the wavefunction. In

general, the action of the automorphism group defines the electric charge:

$$\alpha \cdot \Psi(\check{A}) = e^{2\pi i \int \alpha Q_e} \Psi(\check{A}) , \quad (43.13)$$

where $\alpha \in H^{\ell-2}(N_{n-1}; \mathbb{R}/\mathbb{Z})$ and $Q_e \in H^{n-\ell+1}(N_{n-1}; \mathbb{Z})$. When N_{n-1} is compact, the Gauss law says that if the wavefunction is nonzero then the total charge Q_e must vanish.

44 Manifest Electromagnetic Duality

We can now apply the remarks of [Section 42](#) to Generalized Abelian Gauge Theory following the papers [[FMS06a](#), [FMS06b](#)]. We observed in [Section 43](#) that the Hilbert space of a theory based on \check{H}^ℓ on $N_{n-1} \times \mathbb{R}$ should be $L^2(\check{H}^\ell(N_{n-1}))$. Since $\check{H}^\ell(N_{n-1})$ is an Abelian group, we can regard this as a Stone-von Neumann (SvN) representation of

$$\text{Heis}(\check{H}^\ell(N_{n-1}) \times \check{H}^\ell(N_{n-1})^\sim) . \quad (44.1)$$

But then Pontryagin-Poincaré duality (41.21) of the differential cohomology says that this is the unique Stone-von Neumann representation of¹⁰¹

$$\text{Heis}(\check{H}^\ell(N_{n-1}) \times \check{H}^{n-\ell}(N_{n-1})) . \quad (44.2)$$

But the Stone-von Neumann representation is unique up to isomorphism! We could switch the factors, use that $\tilde{\check{A}} \cong A$ for an Abelian group A , and equally well say the Hilbert space is:

$$L^2(\check{H}^{n-\ell}(N_{n-1})) . \quad (44.3)$$

Recall that we earlier argued that EM duality is just exchanging ℓ for $(n - \ell)$.

Box 44.1: Important Remark

Identifying the Hilbert space of the ℓ -form gauge theory as the Stone-von Neumann representation of (44.1) is a manifestly electric-magnetically dual formulation of the Hilbert space.

In fact, there is a natural and specific isomorphism between the “electric” and “magnetic” formulations. If $\Psi \in L^2(\check{H}^\ell(N_{n-1}))$, then we can – at least formally – define an associated $\Psi^\vee \in L^2(\check{H}^\ell(N_{n-1}))$ by the Fourier transform,

$$\Psi^\vee([\check{A}_{n-\ell}]) := \int_{\check{H}^\ell(N_{n-1})} e^{i\langle [\check{A}_{n-\ell}], [\check{A}_\ell] \rangle} \Psi([\check{A}_\ell]) d\mu , \quad (44.4)$$

¹⁰¹One uses (41.21) with $\dim(N_{n-1}) = n - 1$.

where $d\mu$ is a (unique up to overall scale) translation-invariant measure on the Abelian group $\check{H}^\ell(N_{n-1})$.

Remark 44.5. The off-the-shelf mathematical theorems regarding Pontryagin duality apply to locally compact Abelian groups, but $\check{H}^\ell(N_{n-1})$ is not locally compact. It is, however, a direct product (noncanonically) of a finitely-generated Abelian group and a compact torus and an infinite-dimensional vector space. The rigorous theorems apply to the first two factors, but not the third. The infinite-dimensional vector space is a topological vector space describing the oscillator modes of the field. One can impose an energy cutoff on this space to make it finite-dimensional and then, the Pontryagin duality theorem applies because finite-dimensional vector spaces are locally compact. The energy is moreover an invariant of electromagnetic duality. To make our statements fully rigorous, one would need to discuss the limit as the energy cutoff is taken to infinity. We will not discuss that here, but we do not expect any difficulties from this limit.

45 The Definition And Noncommutativity Of Quantum Electric And Magnetic Fluxes

Let us start with a puzzle. We have seen that the periods of the fieldstrength F of a differential character in $\check{H}^\ell(N_{n-1})$ are quantized:

$$\int_{\Sigma_\ell} F \in 2\pi\mathbb{Z} \quad \text{for} \quad \partial\Sigma_\ell = 0 . \quad (45.1)$$

However, the periods $\int_{\Sigma_{n-\ell}} \star F$ are definitely not quantized! In general, continuous changes of the metric alter the result.¹⁰² This might seem confusing since the electromagnetic dual field \tilde{F} should have quantized periods and we usually write the dual field as $\tilde{F} = \star F$. There is thus some tension with electromagnetic duality. We can resolve this puzzle by thinking more carefully about the quantum definition of electric flux.

As we have noted, if $M_n = N_{n-1} \times \mathbb{R}$ and $\mathcal{H} = L^2(\check{H}^\ell(N_{n-1}))$, then the generator of translations is:

$$\Pi_q \sim \frac{\delta S}{\delta A} \longrightarrow \Pi_q = \lambda (\star F_q)|_{N_{n-1}} . \quad (45.2)$$

¹⁰²If $\ell \neq \frac{1}{2}n$, a simple conformal scaling of the metric will continuously change the periods of $\star F$ thanks to the property (29.9). When $\ell = \frac{1}{2}n$, the periods of $\star F$ will be conformally invariant, but nevertheless, under general deformations of the metric, the periods of $\star F$ will change. Here is a simple example to illustrate the point: Consider the four-manifold X given by the product of two two-dimensional spheres. We write $X = \mathbb{S}_1^2 \times \mathbb{S}_2^2$, and endow X with a direct sum metric which is a sum of standard round sphere metrics of radii R_1 and R_2 , respectively. Choosing a standard orientation $d\phi_i \wedge d(\cos \theta_i)$, $i = 1, 2$ on each sphere, let ω_i , $i = 1, 2$ be the unit volume form for the two spheres. One easily computes that, with the product orientation, $\int_{\mathbb{S}_2^2} \star \omega_1 = \frac{R_2^2}{R_1^2}$. We confirm the conformal invariance, but note that independent changes of R_1, R_2 change the period of $\star \omega_1$.

Let us look at translation eigenstates $\Psi(\check{A}) \in L^2(\check{H}^\ell(N_{n-1}))$. A translation eigenstate would satisfy,

$$\Psi(\check{A} + \check{\phi}) = \exp\left(2\pi i \int_{N_{n-1}}^{\check{H}} \check{\varepsilon} \cdot \check{\phi}\right) \Psi(\check{A}), \quad (45.3)$$

where the phase is a homomorphism $\check{H}^\ell(N_{n-1}) \rightarrow \mathbf{U}(1)$, and its form is fixed by Pontryagin duality to be $\exp(2\pi i \langle \check{\varepsilon}, \check{\phi} \rangle)$, where $\check{\varepsilon}$ is some differential character of degree $(n - \ell)$.

It is natural to regard the eigenvalue $\check{\varepsilon} \in \check{H}^{n-\ell}(N_{n-1})$ as the quantum definition of definite ‘‘electric flux’’ (we will soon alter the meaning of this term). Eigenstates of the translation operator are plane waves on fieldspace.¹⁰³ Such ‘‘plane wave’’ functionals are definitely not normalizable so this notion of electric flux is of limited utility. But we could make wavepackets using $\check{\varepsilon}$ in the same connected component. In this way, we will get a wavefunction of definite electric flux. So a much more useful definition of electric flux is the topological class of $\check{\varepsilon}$, a class in $H^{n-\ell}(N_{n-1}; \mathbb{Z})$. We note that $\check{\varepsilon}_1, \check{\varepsilon}_2$ are in the same path component of $\check{H}^{n-\ell}(N_{n-1})$ if and only if for all flat fields $\phi_f \in H^{\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}')$, we have,

$$\int_{N_{n-1}}^{\check{H}} \check{\phi}_f \cdot \check{\varepsilon}_1 = \int_{N_{n-1}}^{\check{H}} \check{\phi}_f \cdot \check{\varepsilon}_2. \quad (45.5)$$

This leads us to the crucial definition:

Definition 45.6 (State of definite electric flux [FMS06a, FMS06b]). A state $\psi \in L^2(\check{H}^\ell(N_{n-1}))$ is in a state of definite electric flux if it is a translation eigenstate under the subgroup of **flat fields**. That is, for all $\phi_f \in H^{\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}')$,

$$\psi(\check{A} + \check{\phi}_f) = \exp\left(i \int_{N_{n-1}} e \cup \phi_f\right) \psi(\check{A}), \quad (45.7)$$

for some $e \in H^{n-\ell}(N_{n-1}; \mathbb{Z})$. This follows from Section 39.2 and Poincaré-Pontryagin duality. Here e is the quantized electric flux and is the proper quantization of $[\star F]$. We denote the ‘‘electric’’ translation operator by $\phi_f \in H^{\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}')$, as,

$$(\mathcal{U}_e(\phi_f) \cdot \Psi)(\check{A}) := \Psi(\check{A} + \check{\phi}_f). \quad (45.8)$$

Recall that the energy-momentum tensor of the generalized Abelian gauge field is given by (31.12), and therefore only involves the fieldstrength. It is therefore invariant under

¹⁰³In the theory of a real-valued scalar field ϕ , we could consider a quantum wavefunctional

$$\Psi(\phi(x)) = \exp\left(i \int_{N_{n-1}} \phi(x) j(x)\right). \quad (45.4)$$

Note that it is an eigenstate of the group of translations on the scalar field $\phi(x) \mapsto \phi(x) + a(x)$. Such a wavefunctional exists as a complex-valued function on fieldspace, but it is obviously very far from being normalizable – there is no Gaussian falloff in fieldspace. Electric flux eigenstates, with the naive definition of electric flux, are the analogs of such plane waves in the scalar field.

translation by flat fields. Thus, we could say that the translations by flat fields are a symmetry of the theory.

Now it is natural to ask how to formulate magnetic flux sectors. Of course, we could formulate the Hilbert space as $L^2(\check{H}^{n-\ell}(N_{n-1}))$, and define a flux sector of definite magnetic flux to be an eigenstate under the action of the dual flat fields of $\check{\phi}_f^m \in \check{H}^{n-\ell}(N_{n-1})$, where $\phi_f^m \in H^{n-\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}')$. This is a valid definition, but how do the magnetic flux sectors appear if we continue to work with the Hilbert space as modeled by $L^2(\check{H}^\ell(N_{n-1}))$? Recall that in $L^2(A)$, the group A acts on wavefunctions by translation, and the group \check{A} acts by multiplication by a character. In our case, the action of the dual flat fields is:

$$\begin{aligned} (\mathcal{U}_m(\phi_f^m) \cdot \Psi)(\check{A}) &= e^{i\langle \check{\phi}_f^m, \check{A} \rangle} \Psi(\check{A}) \\ &= e^{i \int_{N_{n-1}} \phi_f^m c(\check{A})} \Psi(\check{A}) . \end{aligned} \quad (45.9)$$

The quantity $\int_{N_{n-1}} \phi_f^m c(\check{A})$ is a character on the group of components of $\check{H}^\ell(N_{n-1})$. Recall that $\check{H}^\ell(N_{n-1})$ has connected components:

$$\check{H}^\ell(N_{n-1}) = \coprod_{m \in H^\ell(N_{n-1}; \mathbb{Z})} \check{H}^\ell(N_{n-1})_m . \quad (45.10)$$

We conclude that a state Ψ is an eigenstate of definite *magnetic flux* if $\text{supp}(\Psi)$ is contained in exactly one connected component of $\check{H}^\ell(N_{n-1})$. This turns out to be the eigenstate of translations by the dual flat fields and so we define:

Definition 45.11 (State of definite magnetic flux [FMS06a, FMS06b]). A state of definite magnetic flux $m \in H^\ell(N_{n-1}; \mathbb{Z})$ is a wavefunction ψ with support in the component $\check{H}^\ell(N_{n-1})_m$.

We can get from $\mathcal{U}_m(\phi_f^m)$ to $\mathcal{U}_e(\phi_f^e)$ by a Fourier transform. Now, recall that the compact Abelian group $H^{\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}')$ might be disconnected, and, recalling (39.42),

$$\pi_0(H^{\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}')) = \text{Tors}(H^\ell(N_{n-1}; \mathbb{Z})) . \quad (45.12)$$

Suppose a wavefunction Ψ has definite magnetic flux – so it is supported in a definite connected component of $\check{H}^\ell(N_{n-1})$ – and consider a topologically nontrivial flat electric field ϕ_f^e . Translating Ψ by such a flat field cannot preserve the original component of support of Ψ . This means that, in general, $\mathcal{U}_e(\phi_f^e)$ and $\mathcal{U}_m(\phi_f^m)$ will not commute if there is torsion in the cohomology of N_{n-1} . In fact, $\mathcal{U}_e(\phi_f^e)$ and $\mathcal{U}_m(\phi_f^m)$ are simply special cases of translation and multiplication operators, respectively, on the Stone-von Neumann representation

$L^2(\check{H}^\ell(N_{n-1}))$. In the language of (42.3) and (42.5), we have,

$$\mathcal{U}_e(\phi_f^e) = T_{\check{\phi}_f^e}, \quad \mathcal{U}_m(\phi_f^m) = M_{\chi(\check{\phi}_f^m)}, \quad (45.13)$$

where $\chi(\check{\phi}_f^m)$ is the character of $\check{H}^\ell(N_{n-1})$ defined by $\chi(\check{\phi}_f^m)([\check{A}]) = e^{i\langle \check{\phi}_f^m, \check{A} \rangle}$. It therefore follows from the general exchange relation (42.6), that we have the exchange relation:

$$\mathcal{U}_e(\phi_f^e)\mathcal{U}_m(\phi_f^m) = \chi(\check{\phi}_f^m)(\check{\phi}_f^e)\mathcal{U}_m(\phi_f^m)\mathcal{U}_e(\phi_f^e). \quad (45.14)$$

But we can write,

$$\begin{aligned} \chi(\check{\phi}_f^m)(\check{\phi}_f^e) &= e^{\frac{i}{2\pi}\langle \check{\phi}_f^m, \check{\phi}_f^e \rangle} \\ &= e^{-\frac{i}{2\pi}\int_N \phi_f^m \beta(\phi_f^e)}, \end{aligned} \quad (45.15)$$

where in the second line, we used (41.15), and the fact that $c(\check{\phi}_f^e) = -\beta(\phi_f^e)$, as mentioned in equation (39.53).

From the previous paragraph, we conclude that:

$$\mathcal{U}_e(\phi_f^e)\mathcal{U}_m(\phi_f^m) = \mathsf{T}(\phi_f^e, \phi_f^m)\mathcal{U}_m(\phi_f^m)\mathcal{U}_e(\phi_f^e), \quad (45.16)$$

where T is the torsion pairing (defined below):

$$\mathsf{T} : H^{\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}') \times H^{n-\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}') \longrightarrow \mathsf{U}(1). \quad (45.17)$$

The torsion pairing on these cohomology groups is defined using the Bockstein map by:

$$\mathsf{T}(\phi_f^e, \phi_f^m) := \exp\left(-\frac{i}{2\pi} \int_{N_{n-1}} \phi_f^e \beta(\phi_f^m)\right) = \exp\left(-(-1)^{\ell(n-\ell)} \frac{i}{2\pi} \int_{N_{n-1}} \phi_f^m \beta(\phi_f^e)\right). \quad (45.18)$$

It descends to a pairing on the connected components of:

$$H^{\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}') \times H^{n-\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}'). \quad (45.19)$$

That is, it is a pairing on $\mathsf{Tors}(H^\ell(N_{n-1}; \mathbb{Z})) \times \mathsf{Tors}(H^{n-\ell}(N_{n-1}; \mathbb{Z}))$, hence the name. In fact, it is a perfect pairing on these torsion groups. Because the pairing is perfect, the noncommutativity on the electric and magnetic fluxes is maximal, and in fact defines a Heisenberg group extension of $\mathsf{Tors}(H^\ell(N_{n-1}; \mathbb{Z})) \times \mathsf{Tors}(H^{n-\ell}(N_{n-1}; \mathbb{Z}))$.

Finally, recall that the energy-momentum tensor $T_F = T_{\tilde{F}}$ only involves the fieldstrength. Therefore, shifting by a flat field commutes with the energy-momentum tensor. Therefore, the group generated by shifts by electric and magnetic flat fields is a symmetry of the standard dynamics of the generalized Abelian gauge field in \check{H}^ℓ .

The groups of translations by flat electric and flat magnetic fields have an interpretation as a group of symmetries of the theory. Since (31.12) only depends on the fieldstrength, translations by flat electric fields – implemented by the operators $\mathcal{U}_e(\phi_f^e)$ – is a group of symmetries of the generalized Abelian gauge theory, with symmetry group $H^{\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}')$. Similarly, since $T_F = T_{\tilde{F}}$ only involves the dual fieldstrength, the group of translations by flat magnetic fields – implemented by the operators $\mathcal{U}_e(\phi_f^e)$ – is a group of symmetries, with symmetry group $H^{n-\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}')$. Together these groups generate a noncommutative group of symmetries, which fits in a central extension,

$$1 \rightarrow \mathrm{U}(1) \rightarrow \mathcal{S} \rightarrow H^{\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}') \times H^{n-\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z}') \rightarrow 1 , \quad (45.20)$$

where the cocycle is defined by the torsion pairing.

Example: The above remarks apply to the case of four dimensions with $\ell = 2$, that is, to the ordinary Maxwell field as studied in standard physics textbooks. Since the symmetry group (45.20) is, in general, non-Abelian, this has the interesting implication that on manifolds where the cocycle is nontrivial, the degeneracy of the eigenspaces of the Hamiltonian of the Maxwell field is nontrivial. A simple example is the case of $d = 3 + 1$ Maxwell Theory where the spatial manifold is a Lens space $X = L_m \times \mathbb{R}_t$. See Box 39.2 for more on Lens spaces. Recall that $L_m = \mathbb{S}^3/\mathbb{Z}_m$ and $H^1(L_m; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_m$. In this case, the group generated by electric and magnetic translation operators \mathcal{U}_e and \mathcal{U}_m is a Heisenberg group:

$$0 \longrightarrow \mathbb{Z}_m \longrightarrow \mathrm{Heis}(\mathbb{Z}_m \times \mathbb{Z}_m) \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_m \longrightarrow 0 . \quad (45.21)$$

The translation by flat fields is clearly a symmetry of the Hamiltonian, so we conclude that the energy eigenstates of the Maxwell theory must be degenerate: $\mathcal{U}_e, \mathcal{U}_m$ commute with the Hamiltonian, so the group generated by $\mathcal{U}_e, \mathcal{U}_m$ commutes with the Hamiltonian. These operators generate the Heisenberg group $\mathrm{Heis}(\mathbb{Z}_m \times \mathbb{Z}_m)$, which has a unique irreducible representation of dimension m (where a phase ζ in the $\mathrm{U}(1)$ center acts as scalar multiplication by ζ). Therefore, all the energy eigenspaces must have a degeneracy divisible by m . In particular, the ground state has a degeneracy divisible by m .¹⁰⁴

One might wonder if one could use this result to get a topological qubit or qudit in the real world. The catch is that the theory must be formulated on topologically nontrivial spaces. An attempt was sketched in [KMW07]. The difficulty one must overcome is that one must convince our experimentalist friends to make a Lens space in their laboratory and put electromagnetic fields in that Lens space. At first sight, this appears to be impossible because is a mathematical theorem that says that one cannot *embed* a Lens space into \mathbb{R}^3 (where

¹⁰⁴Essentially this kind of phenomenon was used in the old string compactification literature where vacua were constructed by turning on flat Wilson lines. An important difference is that the argument for the exactness of the degeneracy uses supersymmetry.

we live). Nevertheless, one can *immerse* a Lens space into \mathbb{R}^3 (so that self-intersections are allowed). The basic idea of [KMW07] is to use Josephson junctions to mimic the behavior of an electromagnetic field in a nontrivial 3-manifold, at least in the low-energy regime. It is argued there that the electromagnetic field in a suitable configuration of such Josephson junctions will indeed have a groundstate which forms a qubit. A notable feature of this phenomenon, though, is that it is a long-distance, macroscopic, quantum phenomenon. One might ask if the phenomenon could have applications in cosmology. The topology of the universe is not known and might be nontrivial [CSS98, COM25] and topologies like the Poincaré sphere, for which our phenomenon applies, have been considered. The question is complex, since one must ask whether the topology is visible to an observer so we leave this remark as a tantalizing possibility.¹⁰⁵

46 Wilson And 't Hooft Operators

We now define *Wilson operators* and *'t Hooft operators* in the Hamiltonian formalism. We observe that, given an $(\ell - 1)$ -cycle $\Sigma_{\ell-1} \subset N$, the holonomy along $\Sigma_{\ell-1}$ defines a character on $\check{H}^\ell(N_{n-1})$:

$$h_{\Sigma_{\ell-1}} : \check{H}^\ell(N_{n-1}) \rightarrow \mathrm{U}(1) , \quad (46.1)$$

just given by the evaluation map. Therefore, we can define the *Wilson operator associated to $\Sigma_{\ell-1}$* to be the multiplication of the wavefunction by this holonomy function:

$$(\mathcal{W}(\Sigma_{\ell-1}) \cdot \Psi)[\check{A}] := h_{\Sigma_{\ell-1}}[\check{A}] \Psi[\check{A}] . \quad (46.2)$$

Now, if $\Sigma'_{(n-1)-\ell} \subset N_{n-1}$ is a closed $(n - 1 - \ell)$ -cycle, then the holonomy function:

$$h_{\Sigma'_{n-1-\ell}} : \check{H}^{n-\ell}(N_{n-1}) \rightarrow \mathrm{U}(1) , \quad (46.3)$$

defines a character $\check{\delta}(\Sigma'_{n-1-\ell}) \in \check{H}^\ell(N_{n-1})$ by Poincaré-Pontryagin duality:

$$h_{\Sigma'_{n-1-\ell}}([\check{A}_1^{n-\ell}]) = \exp[2\pi i \langle [\check{A}_1^{n-\ell}], \check{\delta}(\Sigma'_{n-1-\ell}) \rangle] . \quad (46.4)$$

Here, $\check{\delta}(\Sigma'_{n-1-\ell})$ is the non-smooth differential character introduced in equation (41.24) above. We can define the *'t Hooft operator associated to $\Sigma'_{\ell-1}$* , to be the translation operator along $\check{\delta}(\Sigma'_{n-1-\ell})$:

$$(\mathcal{H}(\Sigma'_{n-1-\ell}) \cdot \Psi)([\check{A}]) := \Psi([\check{A}] + [\check{\delta}(\Sigma'_{n-1-\ell})]) . \quad (46.5)$$

Since $\check{\delta}(\Sigma'_{n-1-\ell})$ has delta-function support, this is very close to the original definition of 't Hooft.

¹⁰⁵G.M. thanks T. Banks, D. Friedan, and J. Maldacena for useful remarks on the relevant issues one must consider.

Just like the operators \mathcal{U}_e and \mathcal{U}_f , the Wilson and 't Hooft operators are multiplication and translation operators in the Stone-von Neumann representation of the Heisenberg group (44.1). In the representation $L^2(\check{H}^\ell(N_{n-1}))$, we have (using the general notation (42.3) and (42.5) for Heisenberg representations):

$$\begin{aligned}\mathcal{W}(\Sigma_{\ell-1}) &= M_{h_{\Sigma_{\ell-1}}} , \\ \mathcal{H}(\Sigma'_{n-1-\ell}) &= T_{\check{\delta}(\Sigma'_{n-1-\ell})} .\end{aligned}\tag{46.6}$$

According to the general relation (42.6), we might expect the Wilson and 't Hooft operators to have a nontrivial exchange relation, but here we find a surprise. The phase in the exchange relation is just

$$h_{\Sigma_{\ell-1}}(\check{\delta}(\Sigma'_{n-1-\ell})) = \exp \left[2\pi i \int_{\Sigma_{\ell-1}} \check{\delta}(\Sigma'_{n-1-\ell}) \right] = \exp [2\pi i \langle \check{\delta}(\Sigma_{n-\ell}), \check{\delta}(\Sigma'_{n-1-\ell}) \rangle] .\tag{46.7}$$

Now the support of $\check{\delta}(\Sigma_{\ell-1})$ (say, in terms of the Hopkins-Singer cochains (a, h, ω)) can be restricted to an arbitrarily small tubular neighborhood of $\Sigma_{\ell-1}$, so it follows from the pairing formula (41.3) that, so long as $\Sigma_{\ell-1} \subset N_{n-1}$ and $\Sigma'_{n-\ell-1} \subset N_{n-1}$ are in general position, the phase (46.7) is equal to 1. We conclude – somewhat surprisingly – that (for generic non-intersecting cycles), the Wilson and 't Hooft operators actually commute:

$$\mathcal{W}(\Sigma_{\ell-1}) \mathcal{H}(\Sigma'_{n-1-\ell}) = \mathcal{H}(\Sigma'_{n-1-\ell}) \mathcal{W}(\Sigma_{\ell-1}) .\tag{46.8}$$

Remark 46.9.

1. Of course, if we apply the S -duality isomorphism to the dual description of the Hilbert space $L^2(\check{H}^{n-\ell}(N_{n-1}))$ using the Fourier transform (44.4), then the Wilson operator becomes a shift operator and the 't Hooft operator becomes a multiplication operator.
2. There has been some confusion in the literature about the relation of (46.8) and (45.16) to the results of Gukov-Rangamani-Witten [GRW98], especially since these authors made use of fluxes, Heisenberg groups, and Lens spaces. Briefly, [GRW98] concerns exchange relations for 't Hooft and Wilson defects. These are different from the operators that appear in (45.16). Indeed, equation (46.8) states that the 't Hooft and Wilson operators commute, and do not satisfy a nontrivial exchange relation.¹⁰⁶
3. Another interesting paradox associated with (46.8) has been raised by I. García-Etxebarria. In a renowned paper [tH78], 't Hooft introduced non-Abelian exchange

¹⁰⁶Also, [GRW98] claimed that there is a Heisenberg exchange relation for the Wilson and 't Hooft operators for Maxwell theory on a Lens space, thus contradicting (46.8). There is a subtlety in the argument in [GRW98]: It does not take into account that $\check{\delta}$ is not flat, so that restricting the domain of the wave function to flat fields does not commute with the action by an 't Hooft operator.

relations for Wilson and 't Hooft operators associated to 1-cycles in four-dimensional non-Abelian gauge theory. If we consider a Yang-Mills-Higgs theory so that there is spontaneous symmetry breaking to a Coulomb branch, we might expect the IR limit of the Wilson/'t Hooft operators to be those of the Wilson/'t Hooft operators in the Abelian low-energy effective theory (LEET). How are 't Hooft's nontrivial exchange relations consistent with (46.8)? The answer can be found from a careful reading of [KS14, Sec. 6]. When the global form of the gauge group is carefully taken into account, either the Wilson or the 't Hooft operator will not be a true line operator. Rather one of them will involve a spanning surface (which can be continuously deformed without changing results). So the naive expectation that the IR limit of both the non-Abelian Wilson and 't Hooft operators are those of the low energy Abelian effective theory is not correct. (Nevertheless, in theories such as $d = 4$, $\mathcal{N} = 2$ theories with a nontrivial Coulomb branch, one can recover the “noncommuting flux relations” such as (45.16), just by using the LEET IR theory [DZGE22].)

4. Note that the operation of shifting by a flat field $\check{\phi}_f$ multiplies the Wilson operator by a c-number phase:

$$\mathcal{U}_e(\check{\phi}_f)^{-1}\mathcal{W}(\Sigma_{\ell-1})\mathcal{U}_e(\check{\phi}_f) = e^{i\langle \Sigma_{\ell-1}, \phi_f \rangle} \mathcal{W}(\Sigma_{\ell-1}) . \quad (46.10)$$

47 Relation To Generalized Symmetries And Their Anomalies

The discussion of Section 45 and Section 46 is very closely related to the popular subject of *generalized symmetries*. In modern language, the group of symmetries obtained by shifting by a flat field $\check{\phi}_f^e \in \check{H}^\ell(N_{n-1})$ in the representation $L^2(\check{H}^\ell(N_{n-1}))$ is known as the group of “ $(\ell - 1)$ -form symmetries.”¹⁰⁷ Similarly, the symmetry group given by the action of $\mathcal{U}_m(\phi_f^m)$ for $\phi_f^m \in H^{n-\ell-1}(N_{n-1}; \mathbb{R}/\mathbb{Z})$ is commonly referred to as the group of “magnetic $(n - \ell - 1)$ -form symmetries.” As we have seen in Section 45, the symmetry groups generated by \mathcal{U}_e and \mathcal{U}_m do not commute and, in fact, generate, in general, a nontrivial extension (45.20). A common terminology used to describe this is that there is an “'t Hooft anomaly between

¹⁰⁷The terminology was introduced by Kapustin and Seiberg in [KS14], and adopted and generalized in the renowned paper [GKSW14]. (In an earlier paper [KT13b], Kapustin and Thorngren used the related term “2-group symmetry.”) The terminology is not optimal because the important role played by torsion in many applications of “ p -form symmetries” means one cannot really express the symmetry group in terms of differential forms. For example, consider a $U(1)$ gauge field. The group of flat differential characters is $H^1(N_{n-1}; \mathbb{R}/\mathbb{Z})$. Shifts by the elements of the connected component of the identity can indeed be described as shifts of the gauge field $A \mapsto A + \alpha$, where $d\alpha = 0$. However, shifts of the differential character by the non-identity components of $H^1(N_{n-1}; \mathbb{R}/\mathbb{Z})$ cannot be described in this way. Nevertheless, the term “ p -form symmetry” is a convenient shorthand and has been almost universally adopted.

the electric $(\ell - 1)$ -form and magnetic $(n - \ell - 1)$ -form symmetries.”

In modern language, our definition of a definite electric flux sector is:

Box 47.1: Definite Electric Flux Sector

The eigenstates of the “ $(\ell - 1)$ -form symmetry” are the states of definite electric flux. They have definite value of $e \in H^{n-\ell}(N_{n-1}; \mathbb{Z})$.

The work of Gaiotto, Kapustin, Seiberg, and Willett [GKSW14] generalizes the ideas of the previous two sections in several ways:

1. First, the Wilson and ’t Hooft operators make sense in a non-Hamiltonian framework as defects. Thus, if we consider cycles $\Sigma_{\ell-1}$ and $\Sigma'_{n-\ell-1}$ in a spacetime M_n , rather than in a spatial slice N_{n-1} (as we did in Section 46), then the defects can be “inserted” into the path integral over the spacetime, or, more generally, in the computation of correlation functions associated with the spacetime. This leads to the important idea that symmetry operations can be implemented by topological defects. Such topological defects link Wilson and ’t Hooft defects so that, for example, the relation (46.10) can be reinterpreted as the effect of linking the support $\Sigma_{\ell-1}$ by a topological defect operator on the linking $(n - \ell - 1)$ -sphere. Heuristically, this is to be thought of as the exponential of the integral of a conserved $(n - \ell - 1)$ -form current integrated over the linking sphere.
2. An old idea going back to the work of ’t Hooft and Polyakov is that non-Abelian gauge theories have a “center symmetry.” This can be viewed as a generalization of the translation by a flat gauge in the Abelian gauge theories to non-Abelian gauge theories. The center symmetry is defined by a group homomorphism $\chi : \pi_1(M_n, x_0) \rightarrow Z(G)$, where $Z(G)$ is the center of the non-Abelian gauge group G . The proper definition of the action of the center symmetry on the isomorphism classes of gauge fields is a nice application of Theorem 36.4. We can modify the holonomy function using χ as follows. Assume the gauge bundle has been trivialized at x_0 . If we characterize a connection ∇ by its holonomy function $\text{Hol}_{\nabla, x_0} : \Omega_{x_0}(M_n) \rightarrow G$, then the 1-form symmetry transformation χ takes ∇ to the new connection ∇' , such that,

$$\text{Hol}_{\nabla', x_0}(\gamma) = \text{Hol}_{\nabla, x_0}(\gamma)\chi([\gamma]) , \quad (47.1)$$

where $\gamma \in \Omega_{x_0}(M_n)$, and $[\gamma]$ is its homotopy class. In the case of an Abelian gauge group $U(1)^r$, such transformations can be obtained by a shift of the gauge field $\nabla \mapsto \nabla + \alpha$, where $\alpha \in \Omega^1(M_n; \mathfrak{t})$. However, in contrast to the Abelian case, there is no way to define the non-Abelian center symmetry as a shift of the gauge field by a 1-form. (This is another reason the term “1-form symmetry” is a misnomer. See footnote 107.)

3. It has been known since the work of 't Hooft and Polyakov that the behavior of line defect expectation values under transformation by center symmetries carries important information about the dynamics of non-Abelian gauge theories. These older works were clarified in an important way in [KS14] and [GKSW14]. An important contribution of [GKSW14] is the insight that these results can be reinterpreted, and generalized, using the idea that higher-form symmetries can be spontaneously broken. For example, the photon of electromagnetism is interpreted as a Goldstone boson of a broken 1-form symmetry. Confinement is interpreted as a phase of unbroken 1-form symmetry. Partial confinement means the 1-form symmetry is broken to a subgroup. The Coulomb phase is a phase in which the 1-form symmetry is entirely broken (spontaneously). These ideas, and further developments, along with other aspects of applications to particle physics, are reviewed in [BH23].
4. As we have repeatedly stressed in these notes, whenever a theory has a symmetry, one should introduce background fields that “couple” to the symmetry. This gives a fresh interpretation to the well-studied question of the behavior of the partition function in the presence of background electric and magnetic currents. Section 48.3 expands on this point.
5. Closely related to the previous point is the remark, stressed in [GKSW14], that higher-form symmetries can have anomalies. This is closely related to the anomaly associated with simultaneous coupling to electric and magnetic current. Again, see Section 48.4 below. The idea that higher-form symmetries have can have anomalies gave an elegant conceptual explanation to a result of B. Acharya and C. Vafa, derived from M-theory [AV01]. The domain wall with topological field theory found by these authors is needed to cancel anomalies in 1-form symmetries.

48 Partition Functions For The Generalized Maxwell Field

48.1 Coupling To Gravity

We now consider the partition function of the generalized Maxwell field whose set of gauge inequivalent field configurations is $\check{H}^\ell(M_n)$. We will at first take the background fields to be a Riemannian metric g on M_n and a coupling constant $\lambda > 0$. Formally, we may (naively) write the partition function as a path integral over gauge inequivalent field configurations:

$$Z_\ell(M_n; \lambda, g) = \int_{\check{H}^\ell(M_n)} \mu(\check{\mathbf{A}}) e^{-S} , \quad (48.1)$$

where S is the action (31.5).¹⁰⁸ To give this meaning, we need to define a measure $\mu(\check{A})$ on $\check{H}^\ell(M_n)$. As far as we know, there is no fully adequate description of the partition function of the generalized Maxwell field in the literature. In particular, a completely satisfactory description of the ghost zeromodes that leads to a local definition of the field theory is lacking. “Local” means that the theory satisfies the kinds of gluing properties described in Part I of these lectures. We will make a proposal, but we will not demonstrate the desired gluing properties. A guiding principle will be electromagnetic duality: If the field theory is viewed projectively – that is, is defined up to tensor product with a local invertible field theory – then the projective theory of the ℓ -form fieldstrength with coupling λ should be isomorphic to that of the $(n-\ell)$ -form fieldstrength with coupling $1/\lambda$. Of course, in a proper treatment, electromagnetic duality should be a theorem, not an axiom.

Recall the decomposition (39.64) of Abelian groups. We will use a measure that respects this product structure, where we take the counting measure on the set of connected components, so,

$$Z_\ell(M_n; \lambda, g) = \sum_{x \in H^\ell(M_n; \mathbb{Z})} \int_{\check{H}^\ell(M_n)_x} \mu(\check{A}) e^{-S}, \quad (48.2)$$

where $\check{H}^\ell(M_n)_x$ is the connected component labeled by x . The Riemannian metric on M_n defines a natural metric on $\Omega^\bullet(M_n)$, and this metric descends to a translation-invariant metric on $\Omega^{\ell-1}(M_n)/\Omega_{\mathbb{Z}}^{\ell-1}(M_n)$. That metric formally defines a measure on $\check{H}^\ell(M_n)_x$, and we take this to be the measure $\mu(\check{A})$.

In each connected component labeled by x , the classical solutions $\check{H}^\ell(M_n)_x^{\text{class}}$ form a torsor for the harmonic torus,

$$\mathbb{T}_{\ell-1} \cong \mathcal{H}^{\ell-1}(M_n)/\mathcal{H}_{\mathbb{Z}}^{\ell-1}(M_n), \quad (48.3)$$

and every element of $\check{H}^\ell(M_n)_x^{\text{class}}$ has a field strength which is a harmonic representative of the image of x in de Rham cohomology.

The harmonic torus inherits a translation-invariant measure from the Riemannian metric.¹⁰⁹ The standard procedure for defining Gaussian integrals in field theory is to integrate over the classical solutions and evaluate a determinant of nonzeromodes in the orthogonal directions to $\check{H}^\ell(M_n)_x^{\text{class}}$ inside $\check{H}^\ell(M_n)_x$.

Since the measure on the topologically trivial modes is independent of x , we can factorize:¹¹⁰

$$Z_\ell(M_n; \lambda, g) = \mathcal{N}_\ell Z_\ell^{\text{flux sectors}}, \quad (48.4)$$

¹⁰⁸In this section, the period of the Maxwell fieldstrength is valued in \mathbb{Z} and not $2\pi\mathbb{Z}$. To emphasize this, we will use the symbols A and F for the Maxwell connection and curvature respectively (and \check{A} for the corresponding differential character), as opposed to symbols A and F used elsewhere. We hope no confusion will arise by this local change of notation.

¹⁰⁹If we choose an integral basis $\{f_i^{(\ell-1)}\}$ for $\mathcal{H}_{\mathbb{Z}}^{\ell-1}$, then $\text{vol}(\mathbb{T}_{\ell-1}) = \sqrt{\det_{i,j} \int_{M_n} f_i^{(\ell-1)} \wedge \star f_j^{(\ell-1)}}.$

¹¹⁰Such a factorization typically only holds in free field theories.

where

$$Z_\ell^{\text{flux sectors}} = T_\ell \times \Theta_\ell(\lambda) . \quad (48.5)$$

Here $T_\ell := |\text{Tors } H^\ell(M_n; \mathbb{Z})|$ is the order of the torsion group, and the theta function is:

$$\Theta_\ell(\lambda) := \sum_{f \in \mathcal{H}_\mathbb{Z}^\ell(M_n)} e^{-\int_{M_n} \pi \lambda f \wedge \star f} . \quad (48.6)$$

The sum is over the lattice of harmonic forms with integral periods. The order of the torsion group enters as an overall factor, since shifting the field by a topologically nontrivial flat field does not change the action. Alternatively, we could have expanded the partition function as a sum over fluxes $\bar{x} \in \overline{H}_\mathbb{Z}^\ell(M_n)$ modulo torsion. Then for fixed \bar{x} , the space of classical solutions is a torsor for the group of flat fields $H^{\ell-1}(M_n; \mathbb{R}/\mathbb{Z})$.

The factor \mathcal{N}_ℓ in equation (48.4) is the Gaussian integral on fields in the topologically trivial sector. It is slightly elaborate to evaluate because there are gauge invariances for gauge invariances (“ghosts for ghosts”) [Sie80, BV81, BV83, BT91, GK93, HKS20, BKT22].¹¹¹ We can write,

$$\mathcal{N}_\ell = \mathcal{N}_\ell^{\text{nzm}} \mathcal{N}_\ell^{\text{zm}} , \quad (48.7)$$

corresponding to the contributions of the zeromodes and nonzeromodes of the fields. Below, we find what some readers might regard as a surprise: *The partition function is best viewed as an element of a real line.* We can express it as a function if we write it relative to a nonzero element of the same line (which should be considered to be an extra background field). We first describe the contribution of the nonzeromodes. The path integral over these modes will define a functional of the metric $g_{\mu\nu}$ on M_n .

The contribution of the determinants for nonzeromodes must take account of the fact that the gauge transformation $A_{\ell-1} \mapsto A_{\ell-1} + d\eta_{\ell-2}$ is redundant. A simple gauge-fixing condition such as $d^\dagger A_{\ell-1} = 0$ does not completely fix this redundancy in the sense that if we shift $\eta_{\ell-2}$ by an exact form, then the gauge transformation on $A_{\ell-1}$ is unchanged. One is led to a BRST quantization procedure involving ghosts-for-ghosts or, better, to BV quantization. In the simplest incarnation, the ghost for the (bosonic) gauge field in $\Omega^{\ell-1}$ is a fermionic (i.e., anticommuting) ghost in $\Omega^{\ell-2}$, its commuting ghost-for-ghost is in $\Omega^{\ell-3}$, and so on, with alternating statistics. Thus, for each $0 \leq k < \ell - 1$, we have a ghost field which is a k -form with kinetic operator,

$$\mathcal{O}_k := d^\dagger d|_{\Omega^k(M_n) \cap \text{im } d^\dagger} . \quad (48.8)$$

The field of degree $\ell - 1 - j$ is even (bosonic) if j is even and odd (fermionic) if j is odd.

¹¹¹This is closely related to the fact that for higher-form fields, the sheaf \mathcal{F} on Man_n must be valued in sSET .

The path integral over the nonzeromodes is then,

$$\log \mathcal{N}_\ell^{\text{nzm}} = \frac{1}{2} \sum_{k=0}^{\ell-1} (-1)^{\ell-k} \log (\det'(\mathcal{O}_k)) . \quad (48.9)$$

The determinants of the operators \mathcal{O}_k can be defined in many ways. Two different ways that both define a local quantum field theory will differ by local counterterms – that is – the two theories thus defined will differ by tensor product with an invertible field theory. One standard way to define the determinants for free field path integrals is through the use of ζ -function regularization (an idea going back to D. Ray and I. Singer [RS71, RS73a], and discussed in detail in [BD82]).¹¹² Briefly, the spectrum of \mathcal{O}_k is discrete, countable, real, and nonnegative, and is denoted $\{\mu_i^{(k)}\}_{i \in \mathbb{Z}_+}$. Ray and Singer define a ζ -function,

$$\zeta_{\mathcal{O}_k}(s) = \sum_i \left(\mu_i^{(k)} \right)^{-s} , \quad (48.10)$$

and prove that on a compact manifold, $\zeta_{\mathcal{O}_k}(s)$ is absolutely convergent for sufficiently large $\text{Re}(s)$, and admits a meromorphic continuation to the complex s -plane that is analytic in the neighborhood of $s = 0$. Then they define:

$$\log \det'(\mathcal{O}_k) := - \left. \frac{d}{ds} \right|_0 \zeta_{\mathcal{O}_k}(s) . \quad (48.11)$$

We will assume that the path integral, defined with determinants defined by ζ -function localization is a local quantum field theory, although to our knowledge, that has not been explicitly demonstrated.¹¹³ Referring back to [Section 29](#) (see [Exercise 29.2](#)), we have:

$$\log \mathcal{N}_\ell^{\text{nzm}} = -\frac{1}{2} \left(\sum_i \log \mu_i^{(\ell-1)} - \sum_i \log \mu_i^{(\ell-2)} + \cdots + (-1)^{\ell-1} \sum_i \log \mu_i^{(0)} \right) . \quad (48.12)$$

Rewriting (48.12) in terms of determinants of Laplacians on spaces orthogonal to the spaces of harmonic forms, we get:

$$\log \mathcal{N}_\ell^{\text{nzm}} = -\frac{1}{2} \left[\sum_{k=1}^{\ell} k(-1)^{k+1} \log (\det' \Delta_{\ell-k}) \right] . \quad (48.13)$$

Thus far, we have ignored the coupling constant λ . As we remarked in [Section 31](#), if we do not impose a quantization condition on the periods of F , then λ can be scaled away. This applies as well to the nonzeromodes of $\mathbf{A}_{\ell-1}$ since these modes are always noncompact.

¹¹²It would be interesting to see if the considerations of [PS24] can be usefully combined with ζ -function regularization.

¹¹³Some positive results in two dimensions along these lines have been obtained in [Wen10].

The path integral over these modes has no λ -dependence. The way this comes about from the path integral viewpoint is that the proper definition of the measure on phase space is $dp dq/(2\pi\hbar)$, which turns into $dq/\sqrt{2\pi\hbar}$ in the configuration space representation of the path integral. Then, a coefficient α_n for the eigenmode of eigenvalue μ_n enters the path integral as,

$$\int_{\mathbb{R}} \frac{d\alpha_n}{\sqrt{2\pi\hbar}} e^{-\frac{\pi}{\hbar}\mu_n\alpha_n^2}, \quad (48.14)$$

where, in our case, we identify \hbar with $1/\lambda$. Consequently, the integral over the nonzeromodes does not give rise to λ -dependence.

Finally, the most subtle part of the computation has to do with the treatment of the ghost (and ghost-for-ghost) zeromodes in the topologically trivial sector. See [BT91, MS02, BM06a, Kel07, Mne14, HKS20, BKT22, BKT24] for some discussions of ghost zeromodes. (Several of these papers actually discuss the related case of BF theory rather than the theory addressed in this section, but the considerations are closely related.) The ghosts and ghosts-for-ghosts will not be valued in differential cohomology, since they should have nonzero flux sectors, but the considerations of [Section 40.2.3](#) strongly hint that they should be treated within the framework of differential cohomology. One of the key issues is whether ghost zeromodes should be valued in the torus:

$$\mathbb{T}_k = \mathcal{H}^k(M_n)/\mathcal{H}_{\mathbb{Z}}^k(M_n) ? \quad \text{or} \quad H^k(M_n; \mathbb{R}/\mathbb{Z}) ? \quad (48.15)$$

We will refer to this question as the *ghost zeremode dichotomy*, and refer to them as cases *A* and *B* respectively.

Whether or not the periods of the ghosts fields are quantized, the presence of fermionic zeromodes formally sets the path integral to zero. If the bosonic ghost zeromodes are not periodic, the integral over these modes will be infinite.¹¹⁴ Therefore the path integral should not be regarded as a number but rather $\mathcal{N}_{\ell}^{\text{zm}}$ should really be considered as an element of the Berezinian line on the superspace of zeromodes of ghost fields. We recall that the ghosts of degree $(\ell - 2), (\ell - 4), \dots$ are anticommuting and the ghosts of degree $(\ell - 3), (\ell - 5), \dots$ are commuting. We also include the zeromodes of $A_{\ell-1}$, which are commuting. By a manipulation explained in [CMR15, Cat16], we can regard the measure as an element of the

¹¹⁴Even in ordinary Maxwell theory in four dimensions the fermionic ghost field c is a scalar field, which has a zeremode corresponding to global gauge transformations. In this case the zeremode is generally ignored and most authors simply work with ghost fields orthogonal to the zeremode. Such a procedure is indeed suggested by the original Faddeev-Popov derivation. However, it will not generalize, it will not fit well with electromagnetic duality in general dimensions, and it will violate the expected topological invariance when similar arguments are applied to the quantization of BF theory as in [Section 52](#).

inverse of the determinant of a complex:¹¹⁵

$$\text{DET}\left(\bigoplus_{k=0}^{\ell-1} H^k(M_n)\right) \cong \text{DET}(H^{\ell-1}(M_n)) \otimes \text{DET}(H^{\ell-2}(M_n))^{-1} \otimes \cdots. \quad (48.16)$$

We can understand why we take the inverse of (48.16) as follows: A measure on $\mathbb{T}_{\ell-1}$ is a nonzero element of $\Lambda^{\max} T^* \mathbb{T}_{\ell-1} \cong \text{DET}(H^{\ell-1}(M_n))^{-1}$. The rest of the signs are determined since we have an alternating product.

Now we need to describe the specific element $\mathcal{N}_\ell^{\text{zm}}$ of the inverse line to (48.16). We propose (modulo a subtlety explained in the next paragraph) that it is:

$$\mathcal{N}_\ell^{\text{zm}} = \xi_\ell \left(\omega_1^{(\ell-1)} \wedge \cdots \wedge \omega_{b_{\ell-1}}^{(\ell-1)} \right)^{-1} \otimes \left(\omega_1^{(\ell-2)} \wedge \cdots \wedge \omega_{b_{\ell-2}}^{(\ell-2)} \right)^{+1} \otimes \cdots, \quad (48.17)$$

where, for fixed k , $\{\omega_i^{(k)}\}_{i=1,\dots,b_k}$ is an ON basis of harmonic forms on $\mathcal{H}^k(M_n)$. Here ξ_ℓ is a positive rational number determined by how we handle the ghost zeromode dichotomy (48.15). If they are valued in $\mathcal{H}^k(M_n)/\mathcal{H}_{\mathbb{Z}}^k(M_n)$, then $\xi_\ell = 1$, but if they are in $H^\bullet(M_n; \mathbb{R}/\mathbb{Z})$, then,

$$\xi_\ell = \text{Alt}_{<\ell}(T) := \prod_{k=0}^{\ell-1-k} T_k^{(-1)^{\ell-k}}. \quad (48.18)$$

The subtlety alluded to above is the overall sign. There is no canonical orientation of the cohomology groups in question and without such a choice of orientation, (48.17) is only defined up to sign. Specifying a nonzero vector in a real line L only defined up to sign, that is, choosing $\pm v \in L$, is equivalent to specifying a norm on L . Indeed, for any other vector $w \in L$, we define the norm $p(w) := \text{sign}(\frac{w}{v}) \frac{w}{v} \in \mathbb{R}_+$. Conversely, given a norm on L , the unit vector for this norm is only determined up to sign. Thus, more properly, the partition function of the Maxwell field is a norm on the line inverse to (48.16).

A partial justification of (48.17) may be given as follows. Note that the integral over the topologically trivial harmonic zeromodes of $\mathbf{A}_{\ell-1}$ is $\text{vol}(\mathbb{T}_{\ell-1})$. A good way to write this, which will then extend to the ghosts-for-ghosts, is as follows. Let $\{f_i^{(k)}\}_{i=1,\dots,b_k}$ be an integral basis of harmonic forms with $\{f_i^{(k)}\}_{i=1,\dots,b_k}$ oriented compatibly with the ON basis $\{\omega_i^{(k)}\}_{i=1,\dots,b_k}$. We have,

$$\text{vol}(\mathbb{T}_k) = \frac{f_1^{(k)} \wedge \cdots \wedge f_{b_k}^{(k)}}{\omega_1^{(k)} \wedge \cdots \wedge \omega_{b_k}^{(k)}}. \quad (48.19)$$

Now, using the metric-independent integral basis $\{f_i^{(k)}\}_{i=1,\dots,b_k}$ for the harmonic forms,

¹¹⁵For a bounded complex of vector bundles $E^\bullet : \cdots \rightarrow E^n \xrightarrow{d^n} E^{n+1} \rightarrow \cdots$ over a topological space, the determinant line bundle of the complex is $\text{DET}(E^\bullet) := \bigotimes_k (\text{DET} E^k)^{(-1)^k}$.

we refer the partition function to the metric-independent element,

$$\mu_{\mathbb{Z}}^{(\ell)} := \prod_{k=0}^{\ell-1} \left(f_1^{(k)} \wedge \cdots \wedge f_{b_k}^{(k)} \right)^{(-1)^{\ell-1-k}}. \quad (48.20)$$

The element (48.20) is independent of the integral basis up to sign. The sign can be fixed by a cohomology orientation (or one can interpret the results in terms of norms). We can therefore write the zeromode contribution as:

$$\begin{aligned} \mathcal{N}_\ell^{\text{zm}} &= \xi_\ell \prod_{k=0}^{\ell-1} \left(\frac{f_1^{(k)} \wedge \cdots \wedge f_{b_k}^{(k)}}{\omega_1^{(k)} \wedge \cdots \wedge \omega_{b_k}^{(k)}} \right)^{(-1)^{\ell-1-k}} \left(\mu_{\mathbb{Z}}^{(\ell)} \right)^{-1} \\ &= \begin{cases} \prod_{k=0}^{\ell-1} (\text{vol}(\mathbb{T}_k))^{(-1)^{\ell-1-k}} \left(\mu_{\mathbb{Z}}^{(\ell)} \right)^{-1}, & \text{Case A}, \\ \prod_{k=0}^{\ell-1} (\text{vol}(H^k(M_n; \mathbb{R}/\mathbb{Z})))^{(-1)^{\ell-1-k}} \left(\mu_{\mathbb{Z}}^{(\ell)} \right)^{-1}, & \text{Case B}, \end{cases} \end{aligned} \quad (48.21)$$

where we have used $\text{vol}(H^k(M_n; \mathbb{R}/\mathbb{Z})) = T_{k+1} \text{vol}(\mathbb{T}_k)$.

Remark 48.22.

1. In our discussion above, and in the discussion of continuum BF theories below, we have taken some important shortcuts. A proper discussion would involve BV quantization [BV81, BV83, Sch92, Los07, Cat16] and [Cos11a, Ch. 5]. In particular, the Berezinian measure on the ghost zero-modes should be expressed in terms of the “BV pushforward”. See [Los07, BPT10] and [CMR15, Sec. 2.2.2] for descriptions of the BV pushforward. One can then convert a half-density on the finite BV space given by a shifted cotangent bundle into a density on the base (see [Cat16], example 2.5 and section 3), thus making contact with our formulation in terms of a determinant line. While the BV approach is the proper approach, it is beyond the scope of these lectures.
2. Some discussions of p -form electrodynamics, notably [Sie80, BBRT91] use a triangular system of ghosts and antighosts. However, the determinants and zero-mode measures for the ghosts that are not on the “right-hand side” of the BV triangle cancel, and can be ignored in this problem. This follows from the general theory of “trivial BV pairs.” See, for example, [BT91].
3. We must now include the λ -dependence in $\mathcal{N}_\ell^{\text{zm}}$. While λ does not affect the integral over nonzeromodes, it does affect the integral over the zeromodes provided the periods are quantized. Since we model the gauge equivalence class of our field to be $\check{H}^\ell(M_n)$, the periods of F are quantized. Likewise, the periods of the flat fields are quantized, so in particular, the periods of the zeromodes of $A_{\ell-1}$ are quantized. The question then

arises whether the ghost fields likewise have quantized periods. If we assume that they are, following the hint of [Section 40.2.3](#), the metric on the space of zeromodes is scaled by a factor of λ and hence the zeremode dependence of $\mathcal{N}_\ell^{\text{zm}}$ is obtained by the rescaling:

$$\mathcal{N}_\ell^{\text{zm}} \mapsto \lambda^{\frac{1}{2}(-1)^{\ell-1}\chi_{<\ell}} \mathcal{N}_\ell^{\text{zm}}, \quad (48.23)$$

where b_k are the Betti numbers $b_k := \dim_{\mathbb{R}} H^k(M_n; \mathbb{R})$, and

$$\chi_{<\ell} := b_0 - b_1 + \cdots + (-1)^{\ell-1} b_{\ell-1} = \sum_{k=0}^{\ell-1} (-1)^k b_k. \quad (48.24)$$

4. Our main justification for equation (48.21) is that it is the only formula which is consistent with the Berezinian measure on the space of zeromodes and with electromagnetic duality (as demonstrated in the next section). The alternating product of volumes of tori of harmonic forms is justified in a different way in [Kel07]. Electromagnetic duality also determines the subtle factor ξ_ℓ . The factor T_ℓ in (48.4) will not lead to an electromagnetically dual partition function unless we interpret the volumes of the ghost zeromodes to be volumes of $H^k(M_n; \text{U}(1))$ rather than the volumes of \mathbb{T}_k . That is, we take case B in the zero mode dichotomy. With this understood our final answer for the partition function is:

$$Z_\ell(M_n; \lambda, g) = \lambda^{\frac{1}{2}(-1)^{\ell-1}\chi_{<\ell}} \prod_{k=0}^{\ell-1} [\text{vol } H^k(M_n; \text{U}(1))]^{(-1)^{\ell-1-k}} \cdot \mathcal{N}_\ell^{\text{nzm}} \cdot \Theta_\ell(\lambda) (\mu_{\mathbb{Z}}^{(\ell)})^{-1}. \quad (48.25)$$

5. Finally, we note that the line (48.16) has a rank-one \mathbb{Z} -submodule provided by the integral cohomology. It is natural to define a second norm on the line by defining the norm of a generator to be one. The ratio of two norms on the real line is a positive real number. A generator can be taken to be $(\mu_{\mathbb{Z}}^{(\ell)})^{-1}$. The net effect is to define the partition function as a real number by dropping the last factor on the RHS of (48.25).

48.2 Checking Electromagnetic Duality

It is interesting to check electromagnetic duality of the partition function $Z_\ell(M_n; \lambda, g)$. By Poisson resummation, one can prove¹¹⁶

$$\Theta_\ell(\lambda) = \left(\frac{1}{\lambda}\right)^{b_\ell/2} (\text{vol } \mathbb{T}_\ell)^{-1} \Theta_{n-\ell}\left(\frac{1}{\lambda}\right). \quad (48.26)$$

We next compare the nonzeromode contributions $\mathcal{N}_\ell^{\text{nzm}}$ and $\mathcal{N}_{n-\ell}^{\text{nzm}}$. Referring back to the expression (48.12), we must compare the spectra of the Laplacians. Note that both $\mathcal{N}_\ell^{\text{nzm}}$ and $\mathcal{N}_{n-\ell}^{\text{nzm}}$ involve Laplacians with degrees descending to zero. These factors in general will not be equal. In order to relate them, we must use Hodge duality – as expected when checking electromagnetic duality. The Hodge dual may be refined to an isometry,

$$\star : V_k \rightarrow \tilde{V}_{n-k}, \quad (48.27)$$

from which we conclude an equality of spectra: $\{\mu_i^{(k)}\} = \{\mu_i^{(n-k-1)}\}$ (see Exercises 29.2 and 48.1). Thus, the Hodge dual of a spectrum of degrees descending to zero is a spectrum of degrees ascending to n . In order to relate $\mathcal{N}_\ell^{\text{nzm}}$ and $\mathcal{N}_{n-\ell}^{\text{nzm}}$, we must introduce the *analytic torsion* of Ray and Singer [RS71, RS73a, RS73b, Che77, Lot24]. We now briefly explain the definition of Ray-Singer torsion.

An important component of analytic torsion is:

$$\begin{aligned} \log \tau'_{\text{RS}}(M_n) &:= \frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Tr}'_{\Omega^\bullet(M_n)} F(-1)^F e^{-t\Delta} = -\frac{1}{2} \sum_{k=0}^n (-1)^k k \log \det' \Delta_k \\ &= \frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \left(\sum_i \log \mu_i^{(k)} \right), \end{aligned} \quad (48.28)$$

where the prime indicates the space orthogonal to the space of harmonic forms, and F is the degree of the form. (It is called “F” because in the supersymmetric quantum mechanics interpretation, the degree is the “fermion number.”) Because of the identity $\{\mu_i^{(k)}\} = \{\mu_i^{(n-k-1)}\}$, $\tau'_{\text{RS}}(M_n) = 1$ when n is even, but it is typically not 1 when n is odd. In any case, we have,

$$\log \mathcal{N}_\ell^{\text{nzm}} = \log \mathcal{N}_{n-\ell}^{\text{nzm}} + (-1)^\ell \log \tau'_{\text{RS}}(M_n). \quad (48.29)$$

The quantity $\tau'_{\text{RS}}(M_n)$ is not yet the Ray-Singer torsion. It is not the most natural

¹¹⁶The only nontrivial thing to check here is that if one defines integral bases of harmonic forms $f_i^{(\ell)}$, and $f_i^{(n-\ell)}$, $i = 1, \dots, b_\ell$, then the matrix $\int f_i^{(\ell)} \star f_j^{(\ell)}$ is conjugate to the inverse of $\int f_i^{(n-\ell)} \star f_j^{(n-\ell)}$. This follows by noting two facts. First, if we define $\star f_i^{(\ell)} = S_{ji}^{(\ell)} f_j^{(n-\ell)}$ and $\star f_i^{(n-\ell)} = S_{ji}^{(n-\ell)} f_j^{(\ell)}$ then $S^{(n-\ell)} S^{(\ell)} = (-1)^{\ell(n-\ell)}$. Second, $A_{ij} = \int f_i^{(\ell)} \wedge f_j^{(n-\ell)}$ is an integral unimodular matrix, by Poincaré duality. The quadratic form for Θ_ℓ is proportional to $AS^{(\ell)}$ and that for $\Theta_{n-\ell}$ is proportional to $(-1)^{\ell(n-\ell)}(A^{\text{tr}} S^{(n-\ell)})$. But the inverse of the latter is $A^{-1}(AS^{(\ell)})A^{\text{tr}, -1}$.

quantity to consider because it does not properly take into account the metric dependence of the space of zeromodes. Ray and Singer observed that we use the metric to define a canonical element,

$$\mu_g \in \text{DET} \left(\bigoplus_{k=0}^n H^k(M_n) \right), \quad (48.30)$$

up to a sign: We simply take the wedge product of an ON basis of harmonic forms. The sign can be fixed by choosing an orientation on the space of harmonic forms. Here, the degree of the complex is given by the degree of the form. Then, according to Ray and Singer [RS73b], the torsion should be defined as:

$$\tau_{\text{RS}} := \tau'_{\text{RS}}(M_n) \mu_g \in \text{DET} \left(\bigoplus_{k=0}^n H^k(M_n) \right), \quad (48.31)$$

up to a sign. The quantity τ_{RS} is known as the Ray-Singer torsion. The main assertion of Ray and Singer is that (48.31) is in fact metric-independent, see [RS73b, Mne14].

Recall that a nonzero vector in a line up to sign is equivalent to a norm on that line as explained under equation (48.18), so one can interpret RS torsion without the choice of cohomology orientation as a norm on the determinant line. The observation that Ray-Singer torsion should be viewed conceptually as a norm on the line was made in [Fre92a].

Putting all the above together, we find that:

$$Z_\ell(M_n; \lambda, g) = \mathcal{I}_\ell Z_{n-\ell}(M_n; 1/\lambda, g), \quad (48.32)$$

with

$$\mathcal{I}_\ell = \frac{T_\ell}{T_{n-\ell}} \cdot \left(\frac{\mathcal{N}_\ell^{\text{zm}}}{\mathcal{N}_{n-\ell}^{\text{zm}}} (\text{vol } \mathbb{T}_\ell)^{-1} e^{(-1)^\ell \log \tau'_{\text{RS}}(M_n)} \right) \cdot \lambda^{(-1)^{\ell-1}\chi}, \quad (48.33)$$

where χ is the Euler characteristic of M_n , i.e., $\chi = \sum_{k=0}^n (-1)^k b_k$.

Now, choose a fixed oriented integral basis for $\mathcal{H}_{\mathbb{Z}}^\ell(M_n)$, namely, f_1, \dots, f_{b_ℓ} , and let $\omega_1, \dots, \omega_{b_\ell}$ be an oriented ON basis determined by a metric. Then, we use (48.19) again with $k = \ell$, and conclude that,

$$\frac{\mathcal{N}_\ell^{\text{zm}}}{\mathcal{N}_{n-\ell}^{\text{zm}}} (\text{vol } \mathbb{T}_\ell)^{-1} = \frac{\xi_\ell}{\xi_{n-\ell}} \cdot \left(\frac{\mu_g}{\mu_{\mathbb{Z}}} \right)^{(-1)^\ell} \cdot \frac{(\mu_{\mathbb{Z}})^{(-1)^\ell}}{f_1^{(\ell)} \wedge \dots \wedge f_{b_\ell}^{(\ell)}}, \quad (48.34)$$

where,

$$\mu_{\mathbb{Z}} = \bigotimes_{k=0}^n \left(f_1^{(k)} \wedge \dots \wedge f_{b_k}^{(k)} \right)^{(-1)^k}, \quad (48.35)$$

is metric-independent. (We have here used Poincaré duality to relate the integral basis

$f_i^{(n-\ell-k)}$ to the integral basis $f_i^{(\ell+k)}$.) Finally, we consider the factor,

$$\frac{T_\ell}{T_{n-\ell}} \frac{\xi_\ell}{\xi_{n-\ell}} . \quad (48.36)$$

Recall that the torsion pairing gives a perfect pairing, so $T_k = T_{n-k+1}$. Using this, we identify,

$$\frac{T_\ell}{T_{n-\ell}} \frac{\xi_\ell}{\xi_{n-\ell}} = \left(\prod_{k=0}^n T_k^{(-1)^k} \right)^{(-1)^\ell} := (\text{Alt}(T))^{(-1)^\ell} . \quad (48.37)$$

Thus, the factor relating the partition functions for the dual theories is:

$$\mathcal{I}_\ell = \left(\text{Alt}(T) \cdot \frac{\tau_{\text{RS}}(M_n)}{\mu_{\mathbb{Z}}} \right)^{(-1)^\ell} \cdot \lambda^{(-1)^{\ell-1}\chi} \cdot \frac{(\mu_{\mathbb{Z}})^{(-1)^\ell}}{f_1^{(\ell)} \wedge \cdots \wedge f_{b_\ell}^{(\ell)}} . \quad (48.38)$$

Again, the last factor is only defined up to sign, and should be viewed as a norm on the relevant real line.

Remark 48.39. The Cheeger-Müller theorem equates $\tau_{\text{RS}}(M_n)$ with Reidemeister torsion. Using the integral structures on the determinants of complexes related to cohomology, one can define two norms on the determinant line and interpret the torsion as the ratio of norms – hence identifying it as a real number $\tilde{\tau}_{\text{RS}}(M_n)$. Using this interpretation, Dan Freed has shown that $\tilde{\tau}_{\text{RS}}(M_n)\text{Alt}(T) = 1$, a very satisfying answer. See [Appendix A](#), written by Dan Freed, for the detailed proof.

Exercise 48.1: Isometry from Hodge star

Show that Hodge star maps

$$\star : V_\ell \rightarrow \tilde{V}_{n-\ell} \cong V_{n-\ell-1} , \quad (48.40)$$

isomorphically, and that,

$$\star d^\dagger d \star^{-1} = dd^\dagger , \quad (48.41)$$

and conclude that we can identify $\mu_i^{(k)}$ with $\mu_i^{(n-k-1)}$.

48.2.1 A Path Integral Derivation Of Electromagnetic Duality

There is a simple (formal) path-integral derivation of the electromagnetic duality of the partition functions going back to [Sie84b, Bus88, RV91, SW94, Wit95]. Here is a version that uses differential cohomology.

We begin by considering a path integral over fields in $\check{H}^\ell \times \check{H}^{n-\ell}$ and in addition, inte-

gration of an auxiliary ℓ -form $G \in \Omega^\ell$:

$$\mathcal{Z} = \frac{1}{\text{vol } \check{H}^\ell} \int_{\check{H}^\ell} \mu(\check{A}) \int_{\check{H}^{n-\ell}} \mu(\check{A}_D) \int_{\Omega^\ell} \mu(G) e^{-\pi \int \lambda F \star F + 2\pi i \int GF(\check{A}_D)} , \quad (48.42)$$

where

$$F := F(\check{A}) - G . \quad (48.43)$$

Note that \check{A} and \check{A}_D only enter the action through their fieldstrengths. Moreover, there is a “gauge invariance,” where we shift \check{A} by an arbitrary element of \check{H}^ℓ , and then shift G by the fieldstrength of that element, so we have divided by the volume of this “gauge group” as an overall factor in front of the integral.

The basic idea of [RV91, Wit95] is to evaluate \mathcal{Z} in two different ways and equate the result of the two computations. That equation becomes a statement of electromagnetic duality of partition functions for fields in \check{H}^ℓ and $\check{H}^{n-\ell}$, respectively.

The first way to do the integral is by starting with the Gaussian integral over $G \in \Omega^\ell$. We can shift away $F(\check{A})$, so the G -integral becomes:

$$\int_{\Omega^\ell} \mu(G) e^{-\int \pi \lambda G \star G + 2\pi i \int GF(\check{A}_D) + 2\pi i \int F(\check{A}) F(\check{A}_D)} . \quad (48.44)$$

Note that $e^{2\pi i \int F(\check{A}) F(\check{A}_D)} = 1$, because the periods of the fieldstrengths in differential cohomology are integers. The remaining Gaussian integral is straightforward: The saddle point is $G_0 = \frac{i}{\lambda} (-1)^{\ell(n-\ell)} \star F(\check{A}_D)$, so we get:

$$\exp \left[-\frac{\pi}{\lambda} \int F(\check{A}_D) \star F(\check{A}_D) \right] , \quad (48.45)$$

where the measure $\mu(G)$ is normalized so that:

$$\int_{\Omega^\ell} \mu(G) e^{-\int \pi \lambda G \star G} = 1 . \quad (48.46)$$

(Note that the Gaussian integral formally depends on λ as $\lambda^{-\dim \Omega^\ell / 2}$, an expression which clearly must be regularized and defined. In any case, a measure normalized so that (48.46) holds is λ -dependent.) Next, one integrates over \check{A}_D and the remaining integral is just:

$$\mathcal{Z} = Z_{n-\ell}(M; \lambda^{-1}, g) . \quad (48.47)$$

On the other hand, if we first integrate over \check{A}_D then we get a “periodic delta function” of G , which constrains G to have support on $\Omega_\mathbb{Z}^\ell(M_n)$:

$$\int_{\check{H}^{n-\ell}} \mu(\check{A}_D) e^{2\pi i \int GF(\check{A}_D)} = \text{vol}(H^{n-\ell-1}(M; \mathbb{U}(1))) \int_{\Omega_\mathbb{Z}^\ell} \mu(G_0) \delta(G - G_0) . \quad (48.48)$$

Next, we do the \mathbf{G} integral to replace \mathbf{G} by \mathbf{G}_0 in \mathcal{F} . However, now we can use the translation invariance of the measure $\mu(\check{\mathbf{A}})$ to shift away any $\mathbf{G}_0 \in \Omega_{\mathbb{Z}}^\ell(M_n)$. Therefore, after shifting away \mathbf{G}_0 we are left with:

$$\mathcal{Z} = \frac{\mathcal{T}_{n-\ell} \text{vol}(\mathbb{T}_{n-\ell})}{\mathcal{T}_\ell \text{vol}(\mathbb{T}_\ell)} \int_{\widetilde{H}^\ell} \mu(\check{\mathbf{A}}) e^{-\pi\lambda \int \mathbf{F}(\check{\mathbf{A}}) \star \mathbf{F}(\check{\mathbf{A}})} . \quad (48.49)$$

So, once again, up to an invertible field theory, associated with the volumes of zeromodes, we verify electromagnetic duality.¹¹⁷

48.3 Including A Coupling To Background Electric And Magnetic Currents

Thus far, we have considered the partition function as a function of the background fields λ , the coupling constant, and the metric on M_n . It is natural to introduce electric and magnetic currents into the classical equations of motion as in (33.4) and hence these should also be considered as background fields. It turns out that doing so raises many issues, and we will only consider some of them briefly here.

48.3.1 Coupling To Electric Currents

To begin, let us set the magnetic current \mathbf{J}_m to zero. Then $\mathbf{F} = d\mathbf{A}$ locally, and (assuming momentarily that \mathbf{F} is globally exact) we can derive the equation of motion from the classical action principle (in Lorentzian signature):

$$\int \pi\lambda \mathbf{F} \star \mathbf{F} + 2\pi \int \mathbf{J}_e \mathbf{A} . \quad (48.50)$$

(We have slightly rescaled \mathbf{J}_e from that in equation (33.4) for convenience in the following discussion.)

This immediately raises some issues:

1. The coupling to \mathbf{J}_e is not gauge-invariant unless $d\mathbf{J}_e = 0$, that is, unless we have a conserved current. Of course, the existence of a solution to the equations of motion implies $\mathbf{J}_e = 0$, so we will henceforth assume \mathbf{J}_e is indeed closed.
2. However, if we shift $\mathbf{A} \mapsto \mathbf{A} + \omega$ where ω is closed with (nonzero) integral periods, the action will not be gauge-invariant. In the (Lorentz signature) quantum theory, the

¹¹⁷A very careful accounting of the volumes of zeromodes, and the normalization of the delta-functions and Gaussian integrals in the case of two dimensions can be found in [Moo15, Sec. 12.9.4, p. 195]. In the case of four dimensions, it is done carefully in [Wit95]. It would be interesting to recover the Ray-Singer torsion in odd dimensions using this style of argument. To our knowledge, that has not been done.

measure in the path integral would have a factor:

$$\exp \left[i\pi \int \lambda F \star F + 2\pi i \int J_e A \right]. \quad (48.51)$$

Still, this will not be gauge invariant unless J_e has integral periods. So we will assume that J_e has integral periods.

3. If the cohomology class of F is nontrivial, then A is not globally defined, and the formula for coupling to the current is ambiguous.

In these notes, we are working under the assumption that the quantum theory should be formulated so that the gauge equivalence class of the generalized Maxwell field is a class in $\check{H}^\ell(M_n)$. In view of this, the three issues above are neatly put to rest if we view J_e as the *fieldstrength* of a differential cohomology class:

$$[\check{\kappa}_e] \in \check{H}^{n-\ell+1}(M_n). \quad (48.52)$$

That is, $F([\check{\kappa}_e]) = J_e$. Once we make this assumption, the coupling to the background electric current is neatly written in terms of the pairing in differential cohomology, so that the path integral measure (in Euclidean signature) has a factor:

$$\exp \left[- \int \pi \lambda F \star F + 2\pi i \langle [\check{A}], [\check{\kappa}_e] \rangle \right]. \quad (48.53)$$

Just as the conceptual leap from the standard formulation of (generalized) Maxwell theory to differential cohomology extended the domain of validity of the action, so too, this generalization of the standard electric current allows one to generalize to topologically more interesting situations.

Remark 48.54.

1. In the case when the current is sourced by an electrically charged brane \mathcal{W}_e of (34.2), the electric couplings are related by:

$$\exp \left(2\pi i \int_M J_e A \right) = \exp \left(2\pi i \int_{\mathcal{W}_e} q_e A \right). \quad (48.55)$$

2. As an example showing that equation (48.53) is a nontrivial extension of the standard electric coupling (48.50), consider $\varphi \in \check{H}^1(M_n)$. We take $n = \dim M_n = 2$, so that the electric current defines a class $[\check{\kappa}_e] \in \check{H}^2(M_2)$, and can be considered to be the isomorphism class of a line bundle with connection ∇ , where J_e is the curvature $F(\nabla)$ of the connection ∇ . For topologically trivial field configurations, $\varphi = e^{2\pi i \phi}$ with a

well-defined logarithm $\phi : M_2 \rightarrow \mathbb{R}$ we can write the electric coupling as:

$$\exp \left[2\pi i \int_{M_2} \phi F(\nabla) \right] , \quad (48.56)$$

and we therefore recognize a form of “background charge” familiar in conformal field theory. Note that integration over the translation by a flat field $\phi_0 \in \mathbb{R}/\mathbb{Z}$ shows that the path integral is zero unless $\int_{M_2} F(\nabla) = 0$. This is the quantum-mechanical implementation of the classical statement that J_e must be trivialized by the on-shell fields.

3. Continuing with our two-dimensional example, formulating the electric coupling in terms of differential cohomology has consequences not visible in the usual naive formulation of the coupling. For example, if the electric current is such that ∇ is a flat connection, then the pairing only depends on the characteristic class of φ which gives $c(\check{\varphi}) \in H^1(M_2; \mathbb{Z})$, measuring the winding number. Choosing a basis of cycles on M_2 , the electric coupling then becomes $\prod_i h_i^{w_i}$, where h_i are the holonomies of ∇ around the basis cycles and w_i are the winding numbers. Such factors are not present in the standard formulation of coupling to electric current.
4. In the path integral,

$$Z_\ell(M; \lambda, g, \check{k}_e) = \int_{\check{H}^\ell} \mu(\check{A}) e^{-\pi \int \lambda F \star F + 2\pi i \int^{\check{H}} \check{A} \cdot \check{k}_e} , \quad (48.57)$$

we can shift \check{A} by a flat field: The kinetic term does not change, but the second term changes under $\check{A} \mapsto \check{A} + \check{\phi}_f$ by $\exp(2\pi i \int \check{\phi}_f c(\check{k}_e))$. Therefore, the integral over the compact Abelian group $H^{\ell-1}(M; \mathbb{R}/\mathbb{Z})$ makes the partition function vanish unless $c(\check{k}_e) = 0$. This is the quantum-mechanical version of the statement that there is no solution of the equation of motion unless the electric current is trivializable.

5. The path integral (48.57) is still Gaussian. It is an interesting exercise to work it out exactly (see [Moo11]), but we will not work it out here.

48.3.2 Coupling To Magnetic Currents

Once we have interpreted background electric currents in terms of differential cohomology, electromagnetic duality obliges us to interpret the magnetic current as a differential cohomology class,

$$[\check{k}_m] \in \check{H}^{\ell+1}(M_n) . \quad (48.58)$$

However, once we do so, something very profound happens: The equation,

$$dF = J_m , \quad (48.59)$$

shows that on the support of J_m we cannot interpret $[\check{A}]$ as a differential cohomology class! In fact, differential cohomology provides an elegant way out of this conundrum. We should, rather, interpret \check{A} as a differential cochain that trivializes the differential magnetic current:

$$\delta \check{A} = \check{\kappa}_m , \quad (48.60)$$

where a precise meaning to these equations can be given in terms of Hopkins-Singer cochains and cocycles (see [Section 40.2](#)). The gauge equivalence class of the fields then lives in a torsor for differential cohomology [[HS02](#), [BM06a](#), [MM18a](#), [FH19](#)].

Finally, we come to the interesting case of the partition function in the presence of simultaneous electric and magnetic currents. (The ideas we are about to mention go back to [[Fre00](#)] but have yet to be completely worked out.) Now the partition function must be considered as a section of a line bundle with connection. We consider a family of theories:

$$\mathcal{X} \rightarrow \mathcal{S} , \quad (48.61)$$

with fiber M_n . For example, \mathcal{S} could be the space of metrics and electric and magnetic currents. The line bundle with connection is determined by the pairing in differential cohomology

$$\int_{\mathcal{X}/\mathcal{S}}^{\check{H}} : \check{H}^{n-\ell+1}(\mathcal{X}) \times \check{H}^{\ell+1}(\mathcal{X}) \rightarrow \check{H}^2(\mathcal{S}) , \quad (48.62)$$

where we use the pairing of differential cohomology on the electric and magnetic currents. That is, the path integral is a section of a line bundle with connection determined by,

$$\int_{\mathcal{X}/\mathcal{S}}^{\check{H}} [\check{\kappa}_e] \odot [\check{\kappa}_m] \in \check{H}^2(\mathcal{S}) . \quad (48.63)$$

A Hamiltonian version of this construction is based on a family of spatial slices,

$$\mathcal{X}^{\text{sp}} \rightarrow \mathcal{S} . \quad (48.64)$$

Now one finds that there is a bundle of projective Hilbert spaces with nontrivial Dixmier-Douady (DD) class^{[118](#)} determined by the pairing of electric and magnetic currents:

$$\int_{\mathcal{X}^{\text{sp}}/\mathcal{S}}^{\check{H}} [\check{\kappa}_e] \odot [\check{\kappa}_m] \in \check{H}^3(\mathcal{S}) . \quad (48.65)$$

The fact that the partition function is a section, and the bundle of projective Hilbert spaces has a nontrivial DD class, is a manifestation of the general fact that there will be an anomaly in the presence of simultaneous electric and magnetic currents.

¹¹⁸The DD class was introduced in [Section 40.1.1](#).

48.4 Relation To Generalized Symmetries

The coupling to background electric and magnetic currents can be viewed as a version of coupling to background gauge fields of “higher-form symmetries.” To make this evident, let us first consider the coupling to background electric current, and let us suppose that $[\check{\kappa}_e]$ is topologically trivial, so that:

$$\mathbf{J}_e = d\kappa_e , \quad (48.66)$$

with $\kappa_e \in \Omega^{n-\ell}(M_n)$ globally well-defined. Define $b_e = \frac{i}{\lambda} \star \kappa_e \in \Omega^\ell(M_n)$. Then after completing the square, we can write the exponentiated action (48.57) as:

$$\exp \left[-\pi \int \lambda(\mathbf{F} - b_e) \wedge \star(\mathbf{F} - b_e) + \pi \lambda b_e \wedge \star b_e \right] . \quad (48.67)$$

If we shift $[\check{\mathbf{A}}]$ by flat fields, then \mathbf{F} is invariant, as we have utilized extensively above. If we gauge this symmetry, then we shift by “spacetime-dependent flat fields.” That is, we shift by arbitrary gauge fields, we can consider b_e as a gauge field gauging such a symmetry so that (dropping the quadratic term in b_e) the action (48.67) has a symmetry under:

$$\begin{aligned} \mathbf{A} &\mapsto \mathbf{A} + \alpha , \\ b_e &\mapsto b_e + d\alpha . \end{aligned} \quad (48.68)$$

Note that this symmetry multiplies Wilson operators by a phase, as expected for an electric higher-form symmetry.

Similarly, if we consider a topologically trivial magnetic current with $\mathbf{J}_m = d\kappa_m$, where $\kappa_m \in \Omega^{\ell-1}$, then we have another term in the action:

$$\exp \left[i\pi \int b_m \wedge \mathbf{F} \right] , \quad (48.69)$$

where $b_m \sim \star \kappa_m$. This term has a “magnetic higher-form symmetry” $b_m \mapsto b_m + \omega_m$, where $\omega_m \in \Omega_Z^{n-\ell}$, since \mathbf{F} has quantized periods. Adding such a term to the action accounts for a phase shift of ’t Hooft operators under the magnetic shift symmetry.

Now, if we try to consider simultaneous electric and magnetic currents (in the topologically trivial case), we must consider an action:

$$\exp \left[-\pi \int \lambda(\mathbf{F} - b_e) \star (\mathbf{F} - b_e) + i\pi \int b_m \wedge \mathbf{F} \right] , \quad (48.70)$$

we find a “mixed anomaly” under the simultaneous electric and magnetic higher-form sym-

metry:

$$\begin{aligned} F &\mapsto F + d\alpha_e , \\ b_e &\mapsto b_e + \alpha_e , \\ b_m &\mapsto b_m + \alpha_m . \end{aligned} \tag{48.71}$$

The anomaly is:

$$\exp \left[i\pi \int \alpha_e \wedge d\alpha_m \right] , \tag{48.72}$$

and cannot be eliminated by a local counterterm in n dimensions, although it can be canceled by inflow from an invertible BF theory in $(n+1)$ dimensions. In this way, we make contact with some of the ideas in T. Dumitrescu's TASI 2023 lectures [[Dum23a](#), [Dum23b](#), [Dum23c](#), [Dum23d](#)].

49 A Hilbert Space For The Self-Dual Field

We can apply these ideas to the self-dual field in $n = 2 \bmod 4$ dimensions with classical equations of motion,

$$F = \star F , \quad dF = 0 , \tag{49.1}$$

where $F \in \Omega^\ell(M_n)$, and $n = 2\ell$. Since $n = 2 \bmod 4$, we can write $n = 4s + 2$, so that $\ell = 2s + 1$.

The non-self-dual field with $n = 2\ell$ provides an SvN representation of $\text{Heis}(\check{H}^\ell \times \check{H}^\ell)$. The self-dual case should involve half the number of degrees of freedom. The idea is to make sense of $\text{Heis}(\check{H}^\ell(N_{n-1}))$ by viewing $\check{H}^\ell(N_{n-1})$ itself as a group with a symplectic form. We need another theorem from group theory:

Theorem 49.2. *Let G be a topological Abelian group. The isomorphism classes of central extensions,*

$$1 \longrightarrow \mathbb{U}(1) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1 , \tag{49.3}$$

are in one-to-one correspondence with continuous, alternating, bihomomorphisms,

$$s : G \times G \rightarrow \mathbb{U}(1) . \tag{49.4}$$

*Alternating means $s(x, x) = 1$ for all $x \in G$, and implies skew: $s(x, y) = s(y, x)^{-1}$ for all $x, y \in G$.*¹¹⁹

For more technical details as well as a proof, see [[FMS06a](#), App. A].

¹¹⁹Proof: $s(x, y)s(y, x) = s(xy, yx) = s(xy, xy) = 1 \implies s(y, x) = s(x, y)^{-1}$.

Here $s(x, y)$ is the commutator function. If,

$$(z_1, x_1) \cdot (z_2, x_2) = (z_1 z_2 c(x_1, x_2), x_1 + x_2) , \quad (49.5)$$

is a central extension, then,

$$s(x, y) = \frac{c(x, y)}{c(y, x)} . \quad (49.6)$$

Note that the commutator function is “gauge-invariant” in the sense that if we change the cocycle $c(x, y)$ by a coboundary (thereby defining an isomorphic Heisenberg group), the function $s(x, y)$ does not change.

The point of the theorem is that, if one works at the level of the isomorphism class, then we can use the cruder knowledge of the commutator function. Put differently, from a commutator function one can deduce (nonuniquely) a cocycle, and thereby a central extension. So, to find,

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Heis}(\check{H}^\ell(N_{n-1})) \longrightarrow \check{H}^\ell(N_{n-1}) \longrightarrow 1 , \quad (49.7)$$

it will suffice to find a suitable s -function on $\check{H}^\ell(N_{n-1})$. There is a canonical guess, given by the pairing,

$$s_{\text{trial}}(\chi_1, \chi_2) = \exp(i\langle \chi_1, \chi_2 \rangle) . \quad (49.8)$$

Surprisingly, this is not exactly the right answer! But it is close. The function s_{trial} is certainly a bihomomorphism because ℓ is odd. It is also skew, but it is not alternating! Rather one can show that:

$$s_{\text{trial}}(\chi, \chi) = (-1)^{\int_{N_{n-1}} \nu_{2s} \cup c(\chi)} . \quad (49.9)$$

Here ν_{2s} is the degree $2s$ Wu class of N . One can write the Wu class ν_{2s} of a manifold as a polynomial in the Stiefel-Whitney classes of the tangent bundle of the manifold. For example, in the case where the manifold is oriented, so $w_1(TM_n) = 0$, and we have:

$$\nu_0 = 1 , \quad \nu_2 = w_2 , \quad \nu_4 = w_4 + w_2^2 , \dots \quad (49.10)$$

Of course, the topological class of the differential character χ is:

$$c(\chi) \in H^\ell(N_{n-1}; \mathbb{Z}) = H^{2s+1}(N_{n-1}; \mathbb{Z}) . \quad (49.11)$$

We therefore introduce a \mathbb{Z}_2 -grading of $\check{H}^\ell(N_{n-1})$:

$$\epsilon(\chi) = \begin{cases} 0 , & \int \nu_{2s} \cup c(\chi) = 0 \pmod{2} , \\ 1 , & \int \nu_{2s} \cup c(\chi) = 1 \pmod{2} . \end{cases} \quad (49.12)$$

We find that,

$$s(\chi, \chi') := \exp(i\langle \chi, \chi' \rangle - i\pi\epsilon(\chi)\epsilon(\chi')) , \quad (49.13)$$

is now a bihomomorphism which is alternating, so $\text{Heis}(\check{H}^\ell(N_{n-1}))$ is a \mathbb{Z}_2 -graded Heisenberg group.

There are parallel theorems for \mathbb{Z}_2 -graded Heisenberg groups to those we quoted above. In particular, there is a unique \mathbb{Z}_2 -graded unitary Stone-von Neumann \mathbb{Z}_2 -graded irreducible representation. A central proposal of [FMS06a, FMS06b] is that if we define (up to isomorphism) a \mathbb{Z}_2 -graded Heisenberg group $\text{Heis}(\check{H}^\ell(N_{n-1}))$ using the commutator function (49.13), then the SvN representation is the Hilbert space of the self-dual field (up to isomorphism). Note that it would be of interest to refine this so it is not just a description up to isomorphism. That is, it would be of interest to provide an explicit cocycle that produces the commutator function (49.13).

A striking consequence of the above proposal is that the Hilbert space $\mathcal{H}(N_{n-1})$ of the self-dual field must be \mathbb{Z}_2 -graded. It is natural to expect that the \mathbb{Z}_2 grading (49.12) corresponds to a boson/fermion grading based on statistics. This has not been derived carefully in the literature, and it would be desirable to have a rigorous proof that this is the case. We can check the claim in some special cases: The chiral scalar in $\mathbb{M}^{1,1}$ is related to the chiral fermion. In this case, $s = 0$, so $\nu_0 = 1$, and the grading (49.12) is determined by the winding number of the field. Indeed vertex operators creating states with odd winding number are fermionic. Moreover, if we reduce a self-dual field in $4s + 1$ dimensions from $\mathbb{M}^{1,1} \times Y$, where $\dim Y = 4s$ to $\mathbb{M}^{1,1}$ via Kaluza-Klein reduction then we get a theory of chiral and antichiral scalars. Explicitly, if we choose bases ω_a^+ and ω_i^- of self-dual and anti-self-dual Harmonic forms of degree $2s$ with integral periods then we can write,

$$F = \sum_{a=1}^{b_{2s}^+(Y)} d\phi^a \wedge \omega_a^+ + \sum_{i=1}^{b_{2s}^-(Y)} d\phi^i \wedge \omega_i^- . \quad (49.14)$$

The ϕ^a are chiral and the ϕ^i are anti-chiral periodic scalars. Vertex operators will depend on a “momentum” $x \in H^{2s}(Y; \mathbb{Z})$. The fermionic parity of vertex operators in this $(1+1)$ -dimensional theory is related to the parity of x^2 , where $x \in H^{2s}(Y; \mathbb{Z})$.

Remark 49.15.

1. As we remark briefly in Section 50 below, the total RR field in type II string theory is a self-dual field. Considerations analogous to those above apply to the definition of the Hilbert space of the RR field on nine-dimensional manifolds. In particular, they will be \mathbb{Z}_2 -graded and have fermionic sectors. This striking fact has not yet found an application in the literature.
2. For more about differential cohomology and the formulation of self-dual fields, see [Wit96b, Fre00, HS02, Moo04, BM06a, BM06b, Wit07, HTY20]. In particular, these

references describe the partition “function” of the self-dual field, which is a nontrivial generalization of the discussion of [Section 48](#). The partition function again has the general form,

$$Z = \mathcal{N}\Theta , \quad (49.16)$$

where \mathcal{N} is the contribution from the topologically trivial modes, analogous to what we described in [Section 48](#). It is related to Cheeger’s half-torsion [[Mon10](#)]. An important and novel feature of the partition function of the self-dual field is that Θ is a holomorphic section of a line bundle that in general admits a multi-dimensional space of holomorphic sections. So, in general, the partition function is not an element of a line, but a vector in a higher dimensional vector space. The formalization of this mathematical structure led to the notion of a *relative field theory* [[Seg87](#), [ST11](#), [FT12](#)].

50 Spectra, Generalized Cohomology Theories, And Their Differential Counterparts

In these notes we have emphasized cohomology groups such as singular and de Rham cohomology groups. It turns out that there is a broader notion of “cohomology theory,” and that several of these generalized cohomology theories are playing a prominent role in modern physical mathematics.

50.1 Generalized Cohomology Theories

The cohomology groups $H^k(X)$ that we have used thus far can be defined and computed in many ways – one can use singular cochains, Čech cochains, finite cellular or simplicial decompositions or, (with suitable coefficients), differential forms. In a famous work [[ES45](#)], S. Eilenberg and N. Steenrod described cohomology axiomatically and proved a uniqueness theorem, thereby explaining why all the above techniques give “the same result.” Eilenberg and Steenrod axiomatized cohomology as follows:

We form a category **CW** of pairs of topological spaces (X, S) where $S \subset X$ (with morphisms as continuous maps of pairs)^{[120](#)} and defines a *cohomology theory* to be a contravariant functor H to the category **GrAb** of \mathbb{Z} -graded Abelian groups:

$$H : \mathbf{CW} \rightarrow \mathbf{GrAb} . \quad (50.1)$$

We denote $H^k(X, S)$ to be the subgroup with grading k . We also denote $H^k(X) = H^k(X, \emptyset)$. If $f : (X, S) \rightarrow (Y, T)$ is a morphism in **CW**, i.e., a continuous map of pairs, then the functor

¹²⁰Technically one takes them to be CW complexes with CW subcomplexes, hence the name.

maps this to homomorphisms of Abelian groups:

$$f^* : H^k(Y, T) \rightarrow H^k(X, S) . \quad (50.2)$$

The functor H must be such that for any pair (X, S) there exist group homomorphisms for all $k \in \mathbb{Z}$:

$$\delta^k : H^k(S) \rightarrow H^{k+1}(X, S) , \quad (50.3)$$

such that we have:

1. **Homotopy Invariance:** If two morphisms in **CW** are homotopic, that is, if $f : (X, S) \rightarrow (Y, T)$ is homotopic to $g : (X, S) \rightarrow (Y, T)$, then the induced maps are equal: $f^* = g^*$.
2. **Long Exact Sequence:** If $i : S \hookrightarrow X$, and $j : (X, \emptyset) \hookrightarrow (X, S)$, then we have a long exact sequence,

$$\cdots \longrightarrow H^k(X, S) \xrightarrow{j^*} H^k(X) \xrightarrow{i^*} H^k(S) \xrightarrow{\delta^*} H^{k+1}(X, S) \longrightarrow \cdots \quad (50.4)$$

3. **Excision:**

$$\text{Int}(U) \subset \text{Int}(S) \implies H^k(X, S) \cong H^k(X - U, S - U) . \quad (50.5)$$

(This can also be phrased in terms of two subspaces $S_1, S_2 \subset X$ as,

$$f^* : H^k(X, S_2) \rightarrow H^k(S_1, S_1 \cap S_2) , \quad (50.6)$$

where $f : (S_1, S_1 \cap S_2) \hookrightarrow (X, S_2)$ is the inclusion.)

4. **Additivity:** If (X, S) is a disjoint union of pairs (X_α, S_α) , then the inclusions induce an isomorphism:

$$H^k\left(\coprod_\alpha X_\alpha, S_\alpha\right) \cong \bigoplus_\alpha H^k(X_\alpha, S_\alpha) . \quad (50.7)$$

5. **Dimension Axiom:**

$$H^k(\text{pt}, \emptyset) = \begin{cases} \mathbb{Z} , & k = 0 , \\ 0 , & k \neq 0 . \end{cases} \quad (50.8)$$

Axioms (1) through (4) imply the Mayer-Vietoris sequence (see, for example, [May99, Ch. 14], [Swi75, Ch. 9], and [Ada78, Str23]).

Eilenberg and Steenrod showed that these axioms uniquely characterize cohomology in the sense that if H_1 and H_2 are two such functors and there is a natural transformation $\tau : H_1 \rightarrow H_2$ such that $H_1^\bullet(\text{pt}, \emptyset) \cong H_2^\bullet(\text{pt}, \emptyset)$ then there is an isomorphism $H_1^k(X, S) \cong H_2^k(X, S)$, for all k and all pairs (X, S) .

In the 1950's and 1960's, bordism theory and K -theory were being investigated, and it was recognized that they satisfy many of the properties of cohomology theory, namely, they satisfy Axioms (1) through (4) but not Axiom (5), where the value on a point is a more general Abelian group. For example, for complex K -theory, the cohomology of a point is:

$$K^j(\text{pt}) = \begin{cases} \mathbb{Z}, & j \text{ even}, \\ 0, & j \text{ odd}. \end{cases} \quad (50.9)$$

In fact, K -theory groups form a graded ring. The ring associated with a point is $\mathbb{Z}[u, u^{-1}]$, where u is a degree two element associated with Bott periodicity.

In analogy to the theory of non-Euclidean geometry, one can drop Axiom 5 and still have an interesting mathematically consistent framework. Such functors are known as “exotic cohomology theories” or “extraordinary cohomology theories” or “generalized cohomology theories.” They have proven to be of tremendous importance.

The axioms characterize the theory uniquely in the following sense: Suppose h_1, h_2 are two functors satisfying Axioms 1 – 4, and there is a natural transformation shifting degree by s :

$$\tau_{X,S} : h_1^k(X, S) \rightarrow h_2^{k+s}(X, S), \quad (50.10)$$

such that,

$$\tau_{\text{pt}, \emptyset} : h_1^k(\text{pt}) \rightarrow h_2^{k+s}(\text{pt}), \quad (50.11)$$

are isomorphisms, then all the $\tau_{X,S}$ are isomorphisms.

Examples of generalized cohomology theories include the various flavors of K -theory [Bot69, Kar78, Ati18], bordism theory [Sto16], and elliptic cohomology [Och87, Lan88, LRS95, Wit87, Wit88a, Lur09a, BE24], all of which have proven to be of interest in the context of physics.

50.2 Spectra

We have already mentioned that the Eilenberg-MacLane spaces $K(A, p)$ (where A is an Abelian group and p is a positive integer) satisfy the property that,

$$H^p(X; A) \cong [X, K(A, p)]. \quad (50.12)$$

It turns out (thanks to the Brown representability theorem [Swi75]) that to any generalized cohomology theory h^p , one can associate a collection of topological spaces $\{E_p\}_{p \in \mathbb{Z}}$ such that

$$h^p(X) \cong [X, E_p]. \quad (50.13)$$

The properties of a generalized cohomology theory imply that a set of spaces $\{E_p\}$ such that (50.13) holds, must be related to each other in a way that defines a *spectrum*.

Formally, a spectrum E is a sequence $\{E_n, s_n\}_{n=-\infty}^{\infty}$ of pointed CW spaces $\{E_n\}$ and pointed CW embeddings $s_n : \Sigma E_n \rightarrow E_{n+1}$ where Σ denotes the pointed suspension (a.k.a. reduced suspension).¹²¹ An Ω spectrum is a spectrum E where the adjoint maps $S_n : E_n \rightarrow \Omega E_{n+1}$ are weak homotopy equivalences.¹²² Here $\Omega_{x_0} X$ denotes the (based) loop space of a pointed space (X, x_0) . The S_n are called “adjoint” maps because of the 1–1 correspondence $[X, \Omega Y] = [\Sigma X, Y]$ of homotopy classes of maps of pointed spaces.

Example: The Eilenberg-MacLane spectrum is an Ω spectrum, but the sphere spectrum is not.

The sequence of spaces $\{E_p\}$ representing a generalized cohomology theory form an Ω spectrum. To see why, apply (50.4) to the pair,

$$(X \times I, X \times \{0\} \cup X \times \{1\} \cup x_0 \times I) , \quad (50.14)$$

where x_0 is the basedpoint of X , to obtain an isomorphism,

$$\tilde{h}^n(X) \cong \tilde{h}^{n+1}(\Sigma X) , \quad (50.15)$$

where $\tilde{h}^n(X)$ is the reduced cohomology (i.e., the kernel of the map $h^n(X) \rightarrow h^n(\text{pt})$ from the inclusion of a point into X). But now we have,

$$[X, E_n] \cong \tilde{h}^n(X) \cong \tilde{h}^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}] . \quad (50.16)$$

Since this holds for every X , we must have a homotopy equivalence between E_n and ΩE_{n+1} .

Example: Complex K -theory. The spectrum of complex K -theory is $\mathbb{Z} \times \mathbf{BU}$ (with the discrete topology on \mathbb{Z}), where \mathbf{BU} is the colimit of the classifying space of unitary groups $\mathbf{BU}(n) \cong \mathbf{Gr}(n, \mathbb{C}^\infty)$. This means that the K -theory spectrum of a topological space X is given by $K(X) \cong [X_+, \mathbb{Z} \times \mathbf{BU}]$, where X_+ is X adjoined with a disjoint basepoint. In particular, this means that $K_n(X) := \pi_n(K(X))$. There is a homotopy equivalence $\mathbb{Z} \times \mathbf{BU} \cong \Omega^2 \mathbf{BU}$, due to which the groups $\pi_q(\mathbf{BU}, *)$ ($*$ denotes a basepoint) have periodicity 2. This is an instance of *Bott periodicity*¹²³ which implies that $K_{n+2}(X) \cong K_n(X)$.

¹²¹The *reduced suspension* ΣX of a based topological space (X, x_0) is defined as $\Sigma X = (X \times I)/(X \times \{0, 1\} \cup \{x_0\} \times I)$. It can be obtained from the *unreduced suspension* $SX = (X \times I)/\{(x, 0) \sim (x', 0), (x, 1) \sim (x', 1) \text{ for all } x, x' \in X\}$ by shrinking $\{x_0\} \times I$ to a point, so there is a natural quotient map $SX \rightarrow \Sigma X$. We remark that the symbols for reduced (Σ) and unreduced (S) suspensions are sometimes interchanged in the literature. Our convention is opposite to the one in [Swi75, Rud98, DK01].

¹²²Some authors (e.g., [Swi75, Rud98]) define the notion of spectrum for pointed CW spaces and pointed CW embeddings, while others (e.g., [DK01]) define it more generally for pointed topological spaces and pointed maps. One can make a more careful distinction between a *prespectrum* defined as above but for pointed spaces and maps and a *spectrum* which has the added requirement that the adjoint maps are homeomorphisms (see, for example, [Fre12a, p. 86]).

¹²³For a proof, see [Swi75, LM90].

50.3 Differential Generalized Cohomology

Hopkins and Singer showed that any generalized cohomology theory can be generalized to a differential cohomology theory [HS02]. See also [Fre00, FMS06a, BNVar, ADH21, Deb23] for other discussions.

Given a generalized cohomology theory E , there is a differential analog \check{E} defining a functor from smooth manifolds to graded Abelian groups (in general, they will be infinite dimensional).

To describe \check{E} , we note that from E , one can derive cohomology theories with coefficients in \mathbb{R} and \mathbb{R}/\mathbb{Z} , denoted $E_{\mathbb{R}}$ and $E_{\mathbb{R}/\mathbb{Z}}$ respectively. (See [FMS06a, App. B] for a short summary.) In particular, $E_{\mathbb{R}/\mathbb{Z}}^k(M)$ will be compact Abelian groups, and we have:

$$0 \rightarrow E^k(M_n) \otimes \mathbb{R} \rightarrow E_{\mathbb{R}/\mathbb{Z}}^k(M_n) \rightarrow \text{Tors } E^{k+1}(M_n) \rightarrow 0 . \quad (50.17)$$

Define a real \mathbb{Z} -graded vector space:

$$V_E = E_{\mathbb{R}}^\bullet(\text{pt}) . \quad (50.18)$$

There will be a natural map,

$$\check{E}^\ell(M_n) \rightarrow \bigoplus_{p+q=\ell} H^p(M_n; V_E^q) . \quad (50.19)$$

The differential theory will have a characteristic class map,

$$c : \check{E}^\ell(M) \rightarrow E^\ell(M_n) , \quad (50.20)$$

and a fieldstrength map,

$$F : \check{E}^\ell(M_n) \rightarrow \bigoplus_{p+q=\ell} \Omega_E^p(M_n; V_E^q) , \quad (50.21)$$

where the subscript E on Ω^p indicates that the image of the map is only to forms with a quantization condition compatible with (50.19). That is, (50.21) and (50.20) are compatible when mapped into cohomology using (50.19).

In general, there is an analog of the Simons-Sullivan dancing hexagon (39.1):

(50.22)

It is interesting to see how the above diagram works in the important case of complex K -theory. The \mathbb{Z} -grading is \mathbb{Z}_2 -periodic, i.e., all groups with even degree are isomorphic, and all groups with odd degree are isomorphic. One of many models for $\widetilde{K}^0(M)$ is the group of isomorphism classes of objects in a groupoid whose objects are triples (E, ∇, c) , where $E \rightarrow M_n$ is a finite rank complex vector bundle, ∇ is a connection on E , and c is a globally defined real differential form of odd degree: $c \in \Omega^{\text{odd}}(M_n)$. Morphisms are given by such triples on $M_n \times [0, 1]$, restricting to source and target on the ends of the interval. This implies a number of equivalence relations, an important one being that (E, ∇, c) is equivalent to (E', ∇', c') if,

$$c' - c = \text{CS}(\nabla, \nabla') , \quad (50.23)$$

where on the RHS, we have the relative Chern-Simons form described in equation (51.10) below. The fieldstrength is:

$$F(E, \nabla, c) := \text{ch}(\nabla) + dc , \quad (50.24)$$

and the characteristic class is just the class $[E] \in K^0(M)$. In this model, the crucial pairing on differential cohomology on an odd-dimensional Riemannian spin c manifold (which gives an orientation for differential K -theory) is given by [FMS06b, App. A], [Klo08]:

$$\begin{aligned} & \langle (E_1, \nabla_1, c_1), (E_2, \nabla_2, c_2) \rangle \\ &= \left\{ \eta(D_{\nabla_1 \otimes 1 + 1 \otimes \bar{\nabla}_2}) + \int c_1 \text{Tr} e^{-F(\nabla_2)} + \int \bar{c}_2 \text{Tr} e^{F(\nabla_1)} + \int c_1 d\bar{c}_2 \right\} \bmod \mathbb{Z} . \end{aligned} \quad (50.25)$$

(Bar means complex conjugation, and we consider general complex objects, as is natural in

complex K -theory.)

50.4 Some Physical Applications

In general, if the dynamical fields of a theory have gauge equivalence classes living in differential generalized cohomology, then that theory should be multiplicative and oriented so that there is a pairing. If the physical fields should satisfy some kind of self-duality condition, then the generalized cohomology theory should satisfy some kind of self-duality (such as definition 2.9 of [FMS06a]).

For example, in type II string theory, differential K -theory is thought to be the proper framework for describing Ramond-Ramond fields [MM97, Wit98c, Wit00, MMS01a, FMS06a, FMS06b, Evs06, MS06, DFM09]. In particular, in type II string theory, the gauge equivalence class of the RR field is identified with an element of $\widetilde{K}^0(M_{10})$ for type IIA, and $\widetilde{K}^1(M_{10})$ for type IIB string theory on a 10-dimensional spacetime M_{10} . The groupoid described above for the RR field in type IIA fits in well with the theory of D-branes and tachyon condensation. The equivalence relation (50.23) implies that there is a gauge equivalence, where we can shift $\nabla \mapsto \nabla + \alpha$ for an arbitrary $\alpha \in \Omega^1(M_{10}; \text{End}(E))$, provided we make a compensating change in c . Thus, the “degrees of freedom” in ∇ are “topological,” and the local degrees of freedom in the RR field are captured by c . In the type II string interpretation, the fieldstrength encodes the total RR fieldstrength,

$$F(E, \nabla, c) = G = G_0 + u^{-1}G_2 + u^{-2}G_4 + u^{-3}G_6 + u^{-4}G_8 + u^{-5}G_{10}, \quad (50.26)$$

where G_0 is the Romans mass and the independent propagating degrees of freedom have fieldstrengths G_2 and G_4 . Clearly, we need to impose a self-duality condition, and thus the proper mathematical formulation involves a theory of self-dual fields in the context of differential cohomology. This has been explored to some extend in [MW99, FMS06a, FMS06b, BM06a], but more work remains to be done. In other variations of string theory, such as orientifolds, other versions of differential cohomology such as twisted equivariant differential cohomology are used [DFM09, DFM10].

Briefly, some other significant examples that have shown up in physics are:

1. There is generalized cohomology theory called E -theory in [Fre06] and “supercohomology” in [Deb23], which fits in a long exact sequence

$$\cdots H^k(M_n) \rightarrow E^k(M_n) \rightarrow H^{k-2}(M_n; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{k+1}(M_n) \rightarrow \cdots, \quad (50.27)$$

which is useful in describing theories that depend on spin structure. This includes spin-Chern-Simons theories [BM05, Jen06], and the effective theory of pions in QCD [Fre06]. Closely related are the Wu-Chern-Simons theories investigated by S. Monnier

[Mon14, Mon16, Mon17], which find useful applications to anomaly-cancellation in 6d supergravity [MMP17, MM18a, MM18b].

2. There is a very large literature on the application of bordism theory to the classification of topological terms in actions, and to phases of matter in condensed matter physics. Some key papers are [FH16] and [KT13a, KT13b, KT14a, Kap14, KT14b, KTTW14, KT17, HKT20]. Closely related to this is the application of K -theory to the classification of theories of phases of noninteracting electrons [Hoř05, Kit09, SRFL08, SRF⁺09, RSFL09, FM12].

50.4.1 Brief Remarks On Elliptic Cohomology

In this section, we give a highly condensed introduction to elliptic cohomology and topological modular forms in the spirit of the preceding discussions on spectra. For detailed expositions, see for example, [Hop95, Hop02, Goe09, DFHH14, BE24] and references therein.

Elliptic cohomology – as a generalized cohomology theory – arose from the discovery of the *elliptic genus* by S. Ochanine, P. Landweber, D. Ravenel and R. Stong [Och87, Lan88, LRS95]. Given a commutative ring R , an R -valued *genus* is a ring homomorphism $\sigma : \Omega_{\bullet}^{\mathcal{B}} \rightarrow R$, from a cobordism ring for cobordisms¹²⁴ of manifolds equipped with some structure \mathcal{B} . The ring R relevant to the elliptic genus is the ring of integral holomorphic modular forms, which we will denote as \mathbf{MF}_{\bullet} . The elliptic genus was interpreted in terms of two-dimensional quantum field theory by Witten [Wit87, Wit88a]. (See also [AKMW87a, AKMW87b].)

We first recall Witten’s interpretation and then comment on some of the subsequent mathematical development. To a manifold M satisfying suitable conditions (we will comment below on the meaning of “suitable”) one can define a 2d $\mathcal{N} = (0, 1)$ supersymmetric sigma model σ_M with target M . Witten considered the analog of the Witten index in this context, namely, the partition function of the theory on a circle at finite temperature with Ramond boundary conditions around the Euclidean time circle:

$$Z_{\text{ell}}(\sigma_M; q) := \text{Tr}_{\mathcal{H}_R}(-1)^F q^{(H+P)/2} \bar{q}^{(H-P)/2} = \text{Tr}_{\mathcal{H}_R}(-1)^F e^{-\beta H} e^{i\theta P}. \quad (50.28)$$

Here H is the Hamiltonian, P is the generator of translations around the spatial circle, and F is the fermion number (defined modulo 2). By supersymmetry, this will be an expansion in a parameter $q = e^{-\beta+i\theta}$, independent of \bar{q} .¹²⁵ Moreover, the Hilbert space \mathcal{H}_R is notionally the space of sections of a spin bundle over loop space and the supersymmetry operator is the Dirac-Ramond operator, a generalization of the Dirac operator to a Dirac-like operator on loop space. From this interpretation, $Z_{\text{ell}}(\sigma_M; q)$ is the character-valued index and gives

¹²⁴We use “cobordism” in deference to mathematical literature, but at slight variance with Remark 4.10.

¹²⁵The right-moving sector has $\mathcal{N} = 1$ supersymmetry and the supercharge \mathcal{Q} satisfies $\mathcal{Q}^2 = H - P$, so by standard arguments, only states with $H = P$ contribute to the trace over \mathcal{H}_R .

the elliptic genus, where q is conjugate to the symmetry operation of rotation around the spatial circle.

As is standard in physics, the trace (50.28) has an interpretation as the partition function of the theory on a torus with modular parameter τ , where $q = e^{2\pi i\tau}$. (In terms of (50.28), $\tau = (\theta + i\beta)/2\pi$.) On the other hand, by localization around the constant loops in the sigma model, one can argue that the fields of the sigma model are valued in the normal bundle to the constant loops in the entire loop space and from this interpretation derive

$$Z_{\text{ell}}(\sigma_M; q) = q^{-\nu} \int_M \widehat{A}(R) \operatorname{tr} \left(\prod_{\ell=1}^{\infty} (1 - q^\ell e^{iR/2\pi})^{-1} \right), \quad (50.29)$$

where R is the curvature 2-form¹²⁶ and $\widehat{A}(R)$ is the A-hat roof genus (a.k.a. \widehat{A} -class).¹²⁷ Expanding in q , we could thus interpret the elliptic genus as the generating function for the index of Dirac operators coupled to various tensor powers of the tangent and cotangent bundle. The overall power $q^{-\nu}$ depends on a parameter ν and is essential for modularity. For CFTs, $\nu = -2(c_L - c_R)$ is minus twice the chiral central charge, which is an integer for modular-invariant spin CFTs (see, for example, [LS19, CL20]). It is sometimes called the “anomaly coefficient.” Being an RG invariant, it is a well-defined grading for the space of theories obtained from a given 2d $\mathcal{N} = (0, 1)$ SCFT by supersymmetry-preserving deformations. (The deformations will, in general, break conformal symmetry.) In the mathematical interpretations defined below, the “anomaly coefficient” is an element $\nu \in (I_{\mathbb{Z}}\Omega^{\text{Spin}})^4(\text{pt}) \cong \mathbb{Z}$, in terms of the Anderson dual of the spin bordism spectrum.

We now turn to the meaning of “suitable manifold” M in the definition of σ_M . The more precise meaning is that the sigma model should be anomaly-free. Sigma model anomalies were first studied in [MN84, MN85, MMN85], and in order for them to cancel, we must couple the sigma model to a gerbe whose field strength H satisfies $dH = \frac{1}{2}p_1(TM)_{\mathbb{R}}$, where we use the Chern-Weil representative of the characteristic class. At the integral level, we require a trivialization of a certain canonical cohomology class usually denoted λ , which can be defined for spin manifolds. This class satisfies $2\lambda = p_1(M) := p_1(TM)$.¹²⁸

¹²⁶In local coordinates, we can write the curvature 2-form as $R^a{}_b = \frac{1}{2}R^a{}_{b\mu\nu}dx^\mu \wedge dx^\nu$, where a, b denote frame indices or tangent space indices (a.k.a. “flat indices”) and μ, ν denote coordinate indices (a.k.a. “curved indices”). The traces are understood to be taken over the a, b indices.

¹²⁷See, for example, [Fre87, Sec. 1.1] for a review of the \widehat{A} -class.

¹²⁸Suppose M is n -dimensional, for some $n \in \mathbb{Z}_{>0}$. A lift \tilde{f} of the classifying map $f : M \rightarrow BSO(n)$ to $B\text{Spin}(n)$ exists if and only if M is spin, which is when the triangle on the left in the diagram

$$\begin{array}{ccc} B\text{Spin}(n) & \xrightarrow{\lambda} & K(\mathbb{Z}, 4) \\ \tilde{f} \nearrow & \downarrow \pi_1 & \downarrow \pi_2 \\ M & \xrightarrow{f} & BSO(n) \xrightarrow{p_1} K(\mathbb{Z}, 4) \end{array}$$

commutes. The double cover π_2 induces multiplication by 2 on $H_{\mathbb{Z}}^4$.

Importantly, the real dimension of M is equal to ν .¹²⁹ Now, there is a hierarchy of manifolds: Orientable manifolds have $w_1(M) = 0$. Spin manifolds have $w_1(M) = w_2(M) = 0$. The paper [Kil87] introduced the notion of a *string structure*, where, in addition, $\lambda = 0$. For a more recent discussion, see [Wal23]. A string structure can be interpreted as a lift of the spin structure to the homotopy fiber of the map $B\text{Spin}(n) \rightarrow K(\mathbb{Z}, 4)$.¹³⁰ The cobordism ring of string manifolds is denoted by $\Omega_\bullet^{\text{String}}(\text{pt})$. Sigma model anomaly cancellation therefore requires that M be a string manifold of dimension ν .

Sigma models can be generalized to include a target space gauge bundle with field strength F . In this case, there is a modification of the anomaly cancellation condition: There is also a contribution from the second Chern class of the gauge bundle, which modifies the condition to

$$dH = c_2(F)_\mathbb{R} - \frac{1}{2}p_1(TM)_\mathbb{R}. \quad (50.30)$$

(This condition plays a central role in string theory.) This amounts to the data of a trivialization of $c_2(F) - \lambda$, and following the above line of reasoning, leads to the mathematical notion of a twisted string structure on M . For a recent discussion, see [TY25, Sec. 2].

Witten's construction above can be interpreted as describing a ring homomorphism,

$$\begin{aligned} \sigma_\nu : \Omega_\nu^{\text{String}}(\text{pt}) &\rightarrow \mathbf{MF}_{\nu/2} \\ [M] &\mapsto \sigma_\nu([M]) = \eta(q)^\nu Z_{\text{ell}}(\sigma_M; q), \end{aligned} \quad (50.31)$$

where $\eta(q)$ is the Dedekind eta function, $Z_{\text{ell}}(\sigma_M; q)$ is given by (50.29), and we restrict to ν even. Mathematicians refer to this homomorphism as the *Witten genus*.

After Witten's work [Wit87], mathematicians began to look for a spectrum-level understanding of the Witten genus (50.31).¹³¹ The task is analogous to starting with the observation that the genus $M \mapsto \int_M \widehat{A}(TM)$ is an integer-valued bordism invariant of spin manifolds, and deducing the existence of BU and K -theory. We can introduce a spectrum $\mathbf{MString}$ of string manifolds so that $\mathbf{MString}_\bullet = \pi_\bullet(\mathbf{MString}) = \Omega_\bullet^{\text{String}}(\text{pt})$.¹³² M. Ando, M.

¹²⁹For a 2d $\mathcal{N} = (0, 1)$ sigma model with anomaly coefficient $\nu = -2(c_L - c_R)$, the perturbative anomaly polynomial is $I_4 = \frac{\nu}{48}p_1$. Since ν is an RG invariant, one can compute it in the free or weak coupling limit where the target is flat, and the 2d field theory is modeled by free fields. For a target space M of dimension $n \in \mathbb{Z}_{>0}$, the free field formulation gives $c_L = n$ (n bosonic embedding supercoordinates) and $c_R = n + n/2 = 3n/2$ (n bosonic and n fermionic supercoordinates). This implies that $-2(c_R - c_L) = n$, and therefore, $n = \nu$.

¹³⁰See Section 24.2 for a definition of the homotopy fiber of a map between two topological spaces.

¹³¹There exists an independent geometric construction of the Witten genus due to K. Costello [Cos10, Cos11b] based on factorization algebras [CG16, CG21]. In this construction, the Witten genus of a complex manifold X arises from the quantization of a commutative factorization algebra associated to a classical field theory with fields given by maps from a Riemann surface to the total space of the cotangent bundle T^*X .

¹³²See Section 50.2 for an introduction to spectra. The spectrum corresponding to cobordisms of manifolds equipped with some structure \mathcal{B} is called the *Thom spectrum* \mathbf{MB} [Ati61, Ada74, Fre12a]. The string spectrum $\mathbf{MString}$ is a special case of the Thom spectrum for manifolds admitting a string structure $\mathcal{B} = \text{String}$.

Hopkins, C. Rezk, and N. Strickland [AHS01, AHR10] showed that the Witten genus can be realized as a morphism of commutative ring spectra,

$$\sigma : \mathbf{MString} \rightarrow \mathbf{tmf} := \mathbf{tmf}(\text{pt}) , \quad (50.32)$$

where σ is surjective on homotopy in non-negative degrees. Here, \mathbf{tmf} stands for Topological Modular Forms, a generalized cohomology theory. Physically, the existence of a spectrum version (50.32) of the Witten genus implies the following: an ν -dimensional string manifold M determines a class $[M] \in \mathbf{tmf}_\nu := \pi_\nu(\mathbf{tmf})$, that depends only on the string bordism class of M in $\Omega_\nu^{\text{String}}$, and therefore, there is a homomorphism $\pi_\nu(\mathbf{MString}) = \Omega_\nu^{\text{String}}(\text{pt}) \rightarrow \pi_\nu(\mathbf{tmf}) = \mathbf{tmf}_\nu$, which is known to be surjective [Hop02, Thm. 6.25].¹³³

Thus we can reinterpret equation (50.31) as a map,

$$\begin{aligned} \varphi_W : \pi_\nu \mathbf{tmf} &\rightarrow \mathbf{MF}_{\nu/2} \\ [M] &\mapsto \varphi_W([M]) = \eta(q)^\nu Z_{\text{ell}}(\sigma_M; q) , \end{aligned} \quad (50.33)$$

and we restrict to ν even. The map φ_W is neither injective nor surjective. (We will suppress ‘pt’ for brevity as is conventional.) The spectrum \mathbf{tmf} owes its name to the fact that its ring of homotopy groups is rationally isomorphic to the ring of holomorphic modular forms. The spectrum is *connective*, which means $\pi_{<0}(\mathbf{tmf}) = 0$.

To go further we need to recall a bit more about the structure of the ring of integral holomorphic modular forms \mathbf{MF}_\bullet . For some references, see [Ser73, Zag91, Kob93, DS05, Mum06, BvdGHZ08]. Any modular form in \mathbf{MF}_\bullet can be written as a power series in a formal variable $q = e^{2\pi i\tau}$ with \mathbb{Z} -valued coefficients, leading to the familiar *q-expansion*. This defines an injective map from the ring of modular forms to the ring of formal power series in q over \mathbb{Z} :

$$q_{\text{exp}} : \mathbf{MF}_\bullet \rightarrow \mathbb{Z}[[q]] , \quad (50.34)$$

but this map is far from surjective.¹³⁴ The ring of modular forms can be presented as

$$\mathbf{MF}_\bullet \cong \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta) . \quad (50.35)$$

Here c_4 and c_6 denote the Eisenstein series (the subscript denotes the modular weight), and Δ denotes the modular discriminant, which equals $\Delta = (c_4^3 - c_6^2)/1728$. The modular discriminant plays a central role in the theory of modular forms.

Returning now to the theory of elliptic cohomology, it can be shown that while Δ is

¹³³In [Hop02], surjectivity was stated but not proved, and in [HM02], a limited sketch of a proof was given. Surjectivity was proved recently in [Dev19].

¹³⁴An more intrinsic definition of the map (50.34) is obtained by evaluation on the Tate curve, which is the generalized elliptic curve over $\mathbb{Z}[[q]]$.

not in the image of φ_W , the modular form Δ^{24} of weight $288 = \frac{1}{2} \cdot 576$ – is in the image. Moreover, the kernel of $\varphi_W : \mathbf{tmf}_{576} \rightarrow \mathbf{MF}_{288}$ is trivial, so there is a unique element in \mathbf{tmf}_{576} that maps to Δ^{24} . It is conventional to refer to this element also as Δ^{24} . We can define the non-connective spectrum:¹³⁵

$$\mathbf{TMF}_\bullet := \mathbf{TMF}_\bullet(\text{pt}) := \mathbf{tmf}[\Delta^{-24}]_\bullet(\text{pt}) , \quad (50.36)$$

which is now 576-periodic.¹³⁶ Moreover, for \mathbf{TMF} , there is a map,

$$\varphi_W : \pi_\bullet \mathbf{TMF} \rightarrow \mathbf{MF}_{\bullet/2}[\Delta^{-1}] := \mathbb{Z}[c_4, c_6, \Delta, \Delta^{-1}] / (c_4^3 - c_6^2 - 1728\Delta) , \quad (50.37)$$

to the ring of *weakly holomorphic integral modular forms*.

As hinted in Section 23, there is a physical connection to \mathbf{KO} theory: take a 2d $\mathcal{N} = (0, 1)$ SQFT and compactify it on a circle – this turns it into an SQM with time-reversal symmetry that is equivariant with respect to rotations around \mathbb{S}^1 . Such systems are classified by \mathbf{KO} theory, as proved by Stolz and Teichner [ST04, ST11]. The power of q in the q -expansion of the partition function specifies the weight under this $U(1) \cong \mathbb{S}^1$ action [Wit87, YY23]. The essence of this mathematical construction is captured by the following commutative diagram:¹³⁷

$$\begin{array}{ccccc} \Omega_\nu^{\text{String}}(\text{pt}) & \xrightarrow{\text{Wit}_{\text{String}}} & \mathbf{TMF}_\nu(\text{pt}) & \xrightarrow{\varphi_W} & \mathbf{MF}[\Delta^{-1}]_{\nu/2} \\ \downarrow & & \downarrow & & \downarrow \text{q-exp} \\ \Omega_\nu^{\text{Spin}}(\text{pt}) & \xrightarrow{\text{Wit}_{\text{Spin}}} & \mathbf{KO}((q))_\nu(\text{pt}) & \longrightarrow & \mathbb{Z}((q)) \end{array} \quad (50.38)$$

The vertical map $\Omega_\nu^{\text{String}}(\text{pt}) \rightarrow \Omega_\nu^{\text{Spin}}(\text{pt})$ is induced by the natural forgetful map from the spectrum of string manifolds to the spectrum of spin manifolds, given by forgetting the string structure. Witten's construction realized the composite map $\varphi_W \circ \text{Wit}_{\text{String}}$, and Witten's map factors through the \mathbf{TMF} spectrum.

¹³⁵There are three distinct versions of topological modular forms, namely, \mathbf{tmf} (connective), \mathbf{Tmf} (non-connective and non-periodic), and \mathbf{TMF} (non-connective and periodic). The latter two can be defined more intrinsically in terms of sheaves of spectra over the moduli stack of elliptic curves or its Deligne-Mumford compactification. For details, see, for example, [DFHH14]. In our discussion, we define \mathbf{TMF} in terms of \mathbf{tmf} . However, one can alternatively start from \mathbf{TMF} and define \mathbf{Tmf} via $\mathbf{Tmf}[\Delta^{-1}] := \mathbf{TMF}$, and then $\mathbf{tmf} := \mathbf{Tmf}\langle 0 \rangle$ as the 0-connected cover. Here the notation $E\langle n \rangle$ for a spectrum E and an integer n denotes the n^{th} connective cover (a.k.a. Postnikov truncation) defined such that $\pi_k(X\langle n \rangle) = 0$ for $k < n$ and for $k \geq n$, the map $X\langle n \rangle \rightarrow X$ induces an isomorphism on π_k .

¹³⁶The physical ramifications of this periodicity are not fully understood. However, see [JFY24] for a recent interesting discussion.

¹³⁷For further technical details of this commutative diagram, see, for example, [GPPV18, TY21, BE24]. There are generalizations where one considers \mathbf{TMF} classes not of a point but of a topological space X . Crucially, the commutative diagram as stated here is true in degrees $\nu = 0 \pmod{8}$ and $\nu = 4 \pmod{8}$. In other degrees, there are subtleties: for example, there can be 2-torsion from \mathbf{KO} that is not present in \mathbf{MF} . In degree $\nu = 4k - 1$, there is a novel torsional invariant [BN14, GJF19, BE23] for $(4k - 1)$ -dimensional string manifolds in the image of the map $\Omega_{4k-1}^{\text{String}}(\text{pt}) \rightarrow \mathbb{C}((q))/(\mathbb{Z}((q)) + \mathbf{MF}_{2k})$. This naturally modifies the diagram (50.38). For a careful discussion, see [BE23], and for some physical consequences, see [TY23].

An important development that connects TMF to 2d $\mathcal{N} = (0, 1)$ SQFTs is the Segal-Stolz-Teichner conjecture, which is based on work by Stolz and Teichner [ST04, ST11] building upon earlier work of Segal [Seg88b, Seg07]. As we have noted, the space of 2d $\mathcal{N} = (0, 1)$ SQFTs is graded by $\nu \in \mathbb{Z}$ and we denote the graded component by $\mathcal{S}_{-\nu}$. The Segal-Stolz-Teichner conjecture states that there is an isomorphism between deformation classes of 2d $\mathcal{N} = (0, 1)$ SQFTs and TMF classes. Specifically, the conjecture says that deformation classes of families of 2d $\mathcal{N} = (0, 1)$ SQFTs with a fixed anomaly coefficient $\nu \in \mathbb{Z}$ which are parametrized by a topological space X are in one-to-one correspondence with classes in $\text{TMF}^{-\nu}(X)$. Physically, this means that the deformation class of a theory in $\mathcal{S}_{-\nu}$ parametrized by X is a cocycle representative for a class in $\text{TMF}^{-\nu}(X)$, an Abelian group.¹³⁸ Mathematically, this means $[X, S_{-\nu}] \cong \text{TMF}^{-\nu}(X)$, where $[X, S_{-\nu}]$ denotes the set of homotopy classes of maps from X to the space $S_{-\nu}$. In the simple case when X is a point (which we restricted to in the preceding paragraphs), this conjectured isomorphism reads $[\text{pt}, \mathcal{S}_{-\nu}] = \pi_0(\mathcal{S}_{-\nu}) \cong \text{TMF}^{-\nu}(\text{pt})$. So in this case, TMF classes probe the path connected component of the space of theories.

Formally, TMF can be viewed as a categorification of K -theory, measuring a 2-category of vector bundles. K -theory studies a space X by looking at vector bundles over X . One can equip any such vector bundle with a connection, which is required to discuss parallel transport over a curve in X . Categorifying this setup, one would get “2-vector bundles” over X , involving a study of parallel transport over 2d surfaces in X . Intuitively, one could think of an “elliptic object” as a 2d CFT where the surfaces are mapped into X . For such a perspective, see [BDR03]. It would be interesting to investigate the physical implications of this viewpoint.

For some recent work on elliptic cohomology/topological modular forms and applications to quantum field theory and string theory, see [BN09, BE13, BE14, Kit14, BET21, GPPV18, BET18, GJF18, GJFW19, GJF19, BE20, GM20, JF20b, LP21, BE23, Yon22, TY21, Tac21, AKL22, TYY23, TY23, JFY24, TZ24, Sax24, DY24, LY24, TY25, Tac25, GKMP25]. The subject of equivariant topological modular forms is under rapid development and it is likely to have important implications for string theory.

51 Other Applications I: Chern-Simons And WZW Terms

Chern-Simons and WZW terms are terms that can be added to the actions of physical theories, the addition of which changes the theory in profound ways. From the mathematical viewpoint, they are particularly important examples of invertible field theories. They are

¹³⁸It is worth noting that for the classifying space of a family of QFTs to form a spectrum in the first place, one needs extra structure which may not generically exist, namely, a suspension isomorphism. Moreover, for the genus associated with a QFT to be well-defined as a map out of a cobordism spectrum, the partition function of the QFT must be a cobordism invariant. V.S. thanks D. Berwick-Evans for discussions on this point.

also closely related, and both are nicely described in the context of differential cohomology.

We will be following [ADH21, Deb23]. Related, but slightly different discussions of these topological terms from a modern viewpoint can be found in [DGRW20, LOT20].

51.1 Chern-Simons Terms

We continue the discussion from [Section 27.2](#) above, where we presented the standard discussion of 3-dimensional Chern-Simons terms one typically finds in the physics literature. As noted there, the discussion has limitations. A better definition – that applies to all groups – makes use of differential refinements of Chern-Weil theory.

51.1.1 Chern-Weil Theory

Let G be a compact Lie group and \wp an invariant polynomial of degree ℓ on the Lie algebra \mathfrak{g} of G .¹³⁹ Suppose $P \rightarrow M_n$ is a principal G -bundle with connection ∇ . The curvature is $F(\nabla) \in \Omega^2(M_n; \text{ad } P)$, so $\wp(F(\nabla)) \in \Omega^{2\ell}(M_n)$ is a globally well-defined 2ℓ -form on M_n . The Bianchi identity shows that this form is closed. One then naturally wonders how this form depends on the choice of connection ∇ . We will show, presently, that when we deform ∇ , the form $\wp(F(\nabla))$ changes by a globally defined exact form, and therefore, the de Rham cohomology class $[\wp(F(\nabla))]$ only depends on the topology of the bundle P . Here is the proof:

Suppose $\nabla_{0,1}$ are two connections on P . Then, since the space of connections on P is an affine space, we have $\nabla_1 = \nabla_0 + \alpha$ where $\alpha \in \Omega^1(M_n; \text{ad } P)$. We can form the path of connections,

$$\nabla_t := (1-t)\nabla_0 + t\nabla_1 = \nabla_0 + t\alpha . \quad (51.1)$$

Let $F_t := F(\nabla_t)$. Then one easily computes:

$$\frac{d}{dt}F_t = D_{A_0}\alpha + 2t\alpha^2 = D_{A_t}\alpha , \quad (51.2)$$

where for a gauge field A , $D_A\alpha = d\alpha + A\alpha + \alpha A$ is the covariant derivative with respect A , and we write $\nabla_t = d + A_t$. It follows that,

$$\frac{d}{dt}\wp(F_t) = d\omega(\alpha, F_t) , \quad (51.3)$$

where $\omega(\alpha, F_t)$ is a globally well-defined $2k - 1$ form on M . For example, if $\wp(x) = \text{Tr } x^k$, then,

$$\frac{d}{dt}\text{Tr } F_t^k = d(k\text{Tr } \alpha F_t^{k-1}) . \quad (51.4)$$

¹³⁹It might help some readers to think of $\wp(x) = \text{Tr } \rho(x)^k$, where $x \in \mathfrak{g}$ and ρ is a finite-dimensional representation of \mathfrak{g} . Henceforth, when we use this example, we will just write $\text{Tr } x^k$ to keep the notation simple.

Let G be a compact and connected Lie group. A theorem of Borel [Bor55, Dup78] identifies the invariant polynomials on \mathfrak{g} with the real cohomology of G ,¹⁴⁰

$$I(\mathfrak{g}) \cong H^\bullet(BG; \mathbb{R}) . \quad (51.5)$$

Here $I(\mathfrak{g}) = S^\bullet(\mathfrak{g}^\vee)^G$ is the ring of invariant polynomials under the adjoint action, and \mathfrak{g}^\vee denotes the dual Lie algebra of \mathfrak{g} . It can be identified with the ring of Weyl-invariant polynomials on a Cartan subalgebra $S^\bullet(\mathfrak{t}^\vee)^W$. For a simple Lie group, it is a polynomial ring in the r independent Casimirs, where r is the rank of G .¹⁴¹

Under this correspondence, if $c \in H^{2k}(BG; \mathbb{R})$ corresponds to φ and $\bar{\gamma} : M_n \rightarrow BG$ is the classifying map for P , then,

$$\bar{\gamma}^*(c) = [\varphi(F(\nabla))] . \quad (51.6)$$

Now if c is normalized so that it is the image of an integral class, then $\varphi(F(\nabla))$ will represent an integral cohomology class.

Example: Let G be a compact, connected, simply connected, simple Lie group. Then,

$$\varphi(x) = \frac{1}{16\pi^2 h^\vee} \text{Tr}_{\mathfrak{g}} x^2 , \quad (51.7)$$

where h^\vee is the dual Coxeter number, then the corresponding c is the image of a generator of $H^4(BG; \mathbb{Z})$ embedded into $H^4(BG; \mathbb{R})$. See [Bot56, MT91].

Specializing to $\text{SU}(N)$, this implies that,

$$\frac{1}{8\pi^2} \text{Tr}_N F^2 , \quad (51.8)$$

represents the pullback of a generator of $H^4(B\text{SU}(N); \mathbb{Z})$. For the generalization to quotients of simply connected semisimple compact groups, see [Tod86] and [MMP17, App. A].

Remark 51.9. It is possible to extend Chern-Weil theory to include noncompact Lie groups and even some infinite-dimensional Lie groups, but this is beyond the scope of these notes.

¹⁴⁰The proof proceeds by identifying $H^\bullet(BG; \mathbb{R}) \cong H^\bullet(BT; \mathbb{R})^W$, where W is the action of the Weyl group.

The latter is identified with the ring of invariant polynomials where the generators of $H^\bullet(BT; \mathbb{R})$ have degree 2. Note that a corollary is that the odd degree cohomology of BG is torsion.

¹⁴¹When G is not connected, we make use of the exact sequence (5.19) relating G to the connected component of the identity and the group of components π_0 . One has a fibration $BG_0 \rightarrow BG \rightarrow B\pi_0$ from which one can relate the cohomology of BG to that of BG_0 by a spectral sequence. There is an action of π_0 on $H^\bullet(BG_0)$. To define it, for every element $\gamma \in \pi_0$, we choose a lift $\tilde{\gamma}$ to G , and then consider the conjugation action of $\tilde{\gamma}$ on G_0 . This induces an action on the cohomology that does not depend on the choice of lift. Then, $H^\bullet(BG)$ is the π_0 -invariant subring of $H^\bullet(BG_0)$. As an example, consider the group $G = \widetilde{U}(r+1)$ of unitary and anti-unitary operators on \mathbb{C}^{r+1} . Then $\widetilde{U}(r+1)$ has two connected components, and $G_0 = U(r+1)$. The cohomology $H^\bullet(BG_0; \mathbb{R})$ is generated by invariant polynomials $\varphi_1, \dots, \varphi_r$, where $\varphi_k(x) = \text{Tr}(x^k)$. The action of the nontrivial connected component is complex conjugation, so the cohomology $H^\bullet(BG; \mathbb{R})$ is the polynomial subring generated by $\varphi_2, \varphi_4, \dots$.

51.1.2 Relative Chern-Simons And Absolute Chern-Simons Forms And The Chern-Simons Invariant

We now pull (P, ∇_t) back to $M_n \times [0, 1]$ with $t \in [0, 1]$, so that now the covariant derivative is $\nabla_t + dt \frac{d}{dt}$, and the fieldstrength is $F(\nabla_t) + dt \wedge \alpha$, and consider the integral,

$$\text{CS}_\varphi(\nabla_2, \nabla_1) := \int_0^1 \varphi(F(\nabla_t)) dt . \quad (51.10)$$

This a globally well-defined $2k - 1$ form on M . It is called the *relative Chern-Simons form* and satisfies:

$$d_M \text{CS}_\varphi(\nabla_2, \nabla_1) = \varphi(F_2) - \varphi(F_1) . \quad (51.11)$$

Exercise 51.1: Relative Chern-Simons forms at low degrees

Suppose we can express $\varphi_k(x) = \text{Tr}(x^k)$, and denote the corresponding relative Chern-Simons form by CS_{2k-1} . Show that,

$$\text{CS}_{2k-1}(\nabla + \alpha, \nabla) = k \int_0^1 dt \text{Tr} \alpha (F + tD\alpha + t^2\alpha^2)^{n-1} , \quad (51.12)$$

and in particular:

$$\text{CS}_1(\nabla + \alpha, \nabla) = \text{Tr}(\alpha) , \quad (51.13)$$

$$\text{CS}_3(\nabla + \alpha, \nabla) = \text{Tr} \left(2\alpha F + \alpha D\alpha + \frac{2}{3}\alpha^3 \right) , \quad (51.14)$$

$$\text{CS}_5(\nabla + \alpha, \nabla) = \text{Tr} \left(3\alpha F^2 + \frac{3}{2}\alpha \{F, D\alpha\} + F\alpha^3 + \alpha(D\alpha)^2 + \frac{3}{4}\alpha^3 D\alpha + \frac{3}{5}\alpha^5 \right) , \quad (51.15)$$

where $F = F(\nabla)$.

We would now like to define a Chern-Simons form and invariant for a single connection on a principal G -bundle $\pi : P \rightarrow M_n$. Here, we follow [Fre08, App. A]. Recall that for any map $f : N \rightarrow M_n$, we define the pullback bundle $f^*(P) \rightarrow N$ as a fiber product:

$$f^*(P) = \{(n, p) \in N \times P \mid f(n) = \pi(p)\} , \quad (51.16)$$

then $\pi' : f^*(P) \rightarrow N$ given by projection on the first factor is a principal G -bundle with right G -action $(n, p) \cdot g := (n, p \cdot g)$. Apply this construction to the case where $N = P$ and $f = \pi$:

$$\pi^*(P) = \{(p_1, p_2) \in P \times P \mid \pi(p_1) = \pi(p_2)\} , \quad (51.17)$$

and observe that there is a canonical globally defined section $s : P \rightarrow \pi^*(P)$, given by $s(p) = (p, p)$. Therefore, the principal G -bundle $\pi' : \pi^*(P) \rightarrow P$ is trivializable. Moreover,

there is a canonical connection ∇_0 on $\pi^*(P)$. Suppose $\gamma : [0, 1] \rightarrow P$ is a curve, and let $p_0 = \gamma(0)$. Choose any point $s(p_0) \cdot g_0$ in the fiber above $\gamma(0)$. Then, the connection ∇_0 is defined by declaring that the lifted path with initial point $s(p_0) \cdot g_0$ is simply $s(\gamma(t)) \cdot g_0$. There is, rather trivially, no holonomy, and hence the connection is trivial. We now define the Chern-Simons form of a connection ∇ on $\pi : P \rightarrow M_n$ by,

$$\text{CS}_\varphi(\nabla) := \text{CS}_\varphi(\pi^*(\nabla), \nabla_0) . \quad (51.18)$$

If φ has degree k , this is a globally defined $(2k - 1)$ -form on P . We stress that it is well-defined on P , and not on the base space M_n . Indeed, in the special case where we consider the group G itself to be a principal G -bundle over a point, there is a canonical connection, the *Maurer-Cartan form*: $\theta_{\text{MC}} = g^{-1}dg$, and we get a globally well-defined form $\text{CS}_\varphi(\theta_{\text{MC}})$ on G which is, in general, nonzero.

Exercise 51.2: Polyakov-Wiegmann formula

- (1) Show that

$$d(\text{CS}_\varphi(\nabla)) = \pi^*(\varphi(F)) . \quad (51.19)$$

- (2) Derive $\text{CS}_\varphi(\nabla)$ for the case of a trivial bundle in the case $\varphi(x) = \text{Tr}(x^2)$ as follows: Let the globally defined connection form Θ on $P = M_n \times G$ be:

$$\Theta = g_1^{-1}dg_1 + g_1^{-1}Ag_1 , \quad (51.20)$$

where $g_1 \in G$ and $A \in \Omega^1(M; \mathfrak{g})$ is globally well-defined.

- (2a) Show that $\pi^*(P)$ is naturally identified as the space of triples,

$$(m, g_1, g_2) \in M \times G \times G , \quad (51.21)$$

and the connection ∇_0 has connection form $g_2^{-1}dg_2$ on $\pi^*(P)$.

- (2b) Show that $\pi^*(\nabla)$ has connection from $g_2^{-1}dg_2 + g_2^{-1}\Theta g_2$ on $\pi^*(P)$.

- (2c) Show that,

$$\text{CS}_\varphi(\nabla) = -\frac{1}{3}\text{Tr}(g_1^{-1}dg_1)^3 + \text{Tr}(AdA + \frac{2}{3}A^3) - d(\text{Tr}(dg_1g_1^{-1}A)) . \quad (51.22)$$

- (3) Now take $M = \text{pt}$ in the previous exercise, and show that the globally well-defined connection form on $\pi^*(P)$ is,

$$\tilde{\Theta} = g_2^{-1}dg_2 + g_2^{-1}(g_1^{-1}dg_1)g_2 = (g_1g_2)^{-1}d(g_1g_2) . \quad (51.23)$$

Moreover, show that,

$$\text{Tr}(\tilde{\Theta})^3 = \text{Tr}(g_1^{-1}dg_1)^3 + \text{Tr}(g_2^{-1}dg_2) + d(3\text{Tr}(dg_2g_2^{-1})(g_1^{-1}dg_1)) . \quad (51.24)$$

Equation (51.24) is known as the *Polyakov-Wiegmann* formula in the physics literature and is extremely useful when working with the WZW model.

We are now ready to define the Chern-Simons invariant of a connection ∇ on $\pi : P \rightarrow M_n$. Suppose first that P is trivializable. Then there is a globally well-defined section $s : M_n \rightarrow P$, and we can consider the well-defined integral:

$$\int_M s^*(\text{CS}_\wp(\nabla)) . \quad (51.25)$$

Two different sections will differ by a gauge transformation: $s_1(x) = s_2(x)g(x)$, for some function $g : M_n \rightarrow G$, and this will change the value of the integral. (See the exercise below.) However, if \wp is normalized so that $\text{CS}_\wp(\theta_{\text{MC}})$ has integral periods then the value of (51.25) only changes by an integer under change of section. Therefore, we can define the Chern-Simons invariant to be:

$$S_{\text{CS},\wp}(\nabla) := \int_{M_n} s^*(\text{CS}_\wp(\nabla)) \bmod \mathbb{Z} . \quad (51.26)$$

As the notation suggests, we can consider $S_{\text{CS},\wp}$ to be an action, and the exponentiated action,

$$\exp[2\pi i S_{\text{CS},\wp}(\nabla)] , \quad (51.27)$$

is well-defined. Of course, if \wp has degree k , this will only be interesting on manifolds of dimension $n = 2k - 1$.

Exercise 51.3: More on the Chern-Simons 3-form

- (1) In the example where $\wp(x) = \text{Tr}x^2$ and $P = M_3 \times G$, is the trivial bundle a natural section is $s(x) = (x, g_0)$ for some fixed g_0 . Using (51.22), recover the standard expression:

$$S_{\text{CS}}(A) = \int_{M_3} \text{Tr} \left(A dA + \frac{2}{3} A^3 \right) , \quad (51.28)$$

in the case where $A \in \Omega^1(M_3; \mathfrak{g})$ is globally well-defined.

- (2) Show that a general change of section corresponds to $\tilde{s}(x) = (x, \bar{g}(x)g_0)$ where $\bar{g} : M_3 \rightarrow G$ defines a gauge transformation. Using (51.22) and the (51.24),

show that

$$\int_{M_3} \tilde{s}^*(\text{CS}_\varphi(\nabla)) = \int_{M_3} s^*(\text{CS}_\varphi(\nabla)) - \int_{M_3} \frac{1}{3} \text{Tr} (\bar{g}(x)^{-1} d\bar{g}(x))^3 . \quad (51.29)$$

If G is compact, connected, simple and simply connected, we normalize Tr according to (51.7), then $-\frac{1}{3} \text{Tr} \theta_{\text{MC}}^3$ represents an integral generator of $H^3(G; \mathbb{Z})$. In particular, for $\text{SU}(N)$, a normalized Maurer-Cartan form that represents a generator is:

$$\frac{1}{24\pi^2} \text{Tr}_N (g^{-1} dg)^3 . \quad (51.30)$$

When the bundle $\pi : P \rightarrow M_n$ is nontrivial, the definition of the Chern-Simons invariant is more elaborate. Again, we follow [Fre08, App. A]. Suppose M_n is of dimension $n = 2k - 1$. Let $\bar{\gamma} : M_n \rightarrow BG$ be a classifying map, and $\gamma : P \rightarrow EG$ be a G -equivariant lift. Since the odd degree cohomology of BG is torsion, the image $\bar{\gamma}(M_n) \subset BG$ represents a torsion homology class. So there exists an integer N and a $2k$ -chain $W \subset BG$ so that,

$$N\bar{\gamma}(M_n) = \partial W , \quad (51.31)$$

then, $[\frac{1}{N}W] \in H_{2k}(BG; \mathbb{R}/\mathbb{Z})$. Now, choose any connection ∇^u on $EG \rightarrow BG$, and normalize φ so that $\varphi(F(\nabla^u))$ has integral periods on BG . Then there exists an integral class $\lambda \in H^{2k}(BG; \mathbb{Z})$ so that $\varphi(F(\nabla^u))$ is an integral representative of λ . In general, there can be many choices of λ , and the Chern-Simons invariant will depend on the choice of λ . Then, we define,

$$S_{\text{CS}, \varphi, \lambda}(\nabla) := \frac{1}{N} \int_W \varphi(F(\nabla^u)) + \int_{M_n} \text{CS}_\varphi(\gamma^*(\nabla^u), \nabla) + \lambda \left(\left[\frac{1}{N}W \right] \right) \bmod \mathbb{Z} . \quad (51.32)$$

One can show that the right hand side actually does not depend on the choices N, W, γ, ∇^u that we made in the above construction. Comparing this formula with equation (40.35) above suggests that the Chern-Simons form and invariant can be formulated in terms of differential cohomology. This is indeed the case, as we discuss next.

51.1.3 Differential Cohomology Refinement Of Chern-Weil Classes

The Chern-Simons invariant can be nicely formulated in terms of differential cohomology. See [ADH21, Deb23] and references therein for details. Here we will be brief.

Suppose we have a principal G -bundle with connection $\pi : P \rightarrow M_n$ with connection ∇ . As we have seen, given a suitably normalized degree k invariant polynomial $\varphi \in I(\mathfrak{g})$, we have,

$$\varphi(F(\nabla)) \in \Omega_{\mathbb{Z}}^{2k}(M_n) . \quad (51.33)$$

Moreover, there exists an integral class $\lambda \in H^{2k}(BG; \mathbb{Z})$ such that, if $\gamma : M_n \rightarrow BG$ is a classifying map for P , then $\wp(F(\nabla))$ is a representative in real cohomology of $\gamma^*(\lambda) \in H^{2k}(M_n; \mathbb{Z})$:

$$[\wp(F(\nabla))] = \gamma^*(\lambda)_{\mathbb{R}} . \quad (51.34)$$

A comparison of (51.34) with (39.60) raises the natural question of whether there is a differential cohomology class $\chi_{\nabla, \lambda} \in \check{H}^{2k}(M_n)$, such that,

$$F(\chi_{\nabla, \lambda}) = \wp(F(\nabla)) , \quad \text{and} \quad c(\chi_{\nabla, \lambda}) = \gamma^*(\lambda) . \quad (51.35)$$

Theorem 2.2 in the original paper of Cheeger and Simons [CS85] states that indeed such a differential class exists, and moreover, if it is gauge-invariant, then it is unique.

In more modern terms, there is a space $B_{\nabla}G$ analogous to BG which classifies principal G -bundles with connection. (See Chapter 13 of [ADH21] for a description of $B_{\nabla}G$.) Then, given $\lambda \in H^{2k}(BG; \mathbb{Z})$ and $\wp \in I(\mathfrak{g})$, such that,

$$\lambda_{\mathbb{R}} = [\wp(F(\nabla^u))] \in H^{2k}(BG; \mathbb{R}) , \quad (51.36)$$

(where ∇^u is any connection on $\pi : EG \rightarrow BG$), there is a unique differential cohomology class,

$$\check{\lambda} \in \check{H}^{2k}(B_{\nabla}G) , \quad (51.37)$$

such that if the map $\gamma : M \rightarrow B_{\nabla}G$ classifies (P, ∇) , then $\gamma^*(\check{\lambda}) = \chi_{\nabla, \lambda}$. See [Deb23], and Theorem 14.1.1 and Section 19.1 of [ADH21] for more details.

The Chern-Simons invariant of a connection ∇ on $\pi : P \rightarrow M_n$ can now be described very nicely in terms of differential cohomology, without the need to extend to a bounding manifold. (We remind the reader that, in general, such extensions do not exist.) If M_n has dimension $n = 2k - 1$, then,

$$\int_{M_n}^{\check{H}} \chi_{\nabla, \lambda} \in \check{H}^1(\text{pt}) \cong \mathbb{R}/\mathbb{Z}' , \quad (51.38)$$

is the Chern-Simons invariant. We note that, if M_n has dimension $n = 2k$, then,

$$\int_{M_n}^{\check{H}} \chi_{\nabla, \lambda} = \langle [M_n], \lambda \rangle \in \mathbb{Z} , \quad (51.39)$$

and, importantly, if we have a family $\mathcal{X} \rightarrow S$ of manifolds diffeomorphic to M_n , of dimension $2k$, then,

$$\int_{\mathcal{X}/S}^{\check{H}} \chi_{\nabla, \lambda} \in \check{H}^2(S) . \quad (51.40)$$

Recall that $\check{H}^2(S)$ is the group of isomorphism classes of line bundles with connection. This

fact is quite relevant to the study of anomalies.

As a particularly important case, suppose we have a family of 2-manifolds: $M_2 \rightarrow \mathcal{X} \rightarrow S$. (The base space could be, for example, the space of connections on M_2 .) Then we obtain a bundle with connection as in (51.40). This line bundle provides meaning to the value of the Chern-Simons invariant on a 3-manifold with boundary. Suppose that $\partial M_3 = M_2$. We wish to make sense of $\exp[2\pi i \int_{M_3} CS_3(\nabla)]$ as a “function” of the connection ∇ on the boundary. Consider the set $\mathcal{E}(\nabla)$ of all extensions $(P^{\text{ext}}, \nabla^{\text{ext}})$ of the gauge field ∇_{M_2} on M_2 to a gauge field ∇^{ext} on some principal bundle $P \rightarrow M_3$ on some bounding manifold M_3 . Consider the set $\mathcal{L}_{\nabla_{M_2}}$ of functions $F : \mathcal{E}(\nabla) \rightarrow U(1)$ such that, for two different extensions $(P^{\text{ext}}, \nabla^{\text{ext}})$ and $(P^{\text{ext}}, \nabla^{\text{ext}})'$, we have,

$$\frac{F((P^{\text{ext}}, \nabla^{\text{ext}}))}{F((P^{\text{ext}}, \nabla^{\text{ext}})')} = \exp[2\pi i S_{\text{CS}}((P^{\text{ext}}, \nabla^{\text{ext}})) \cup_{M_2} (P^{\text{ext}}, \nabla^{\text{ext}})'] , \quad (51.41)$$

where on the RHS, we have the Chern-Simons invariant of the connection on the closed manifold obtained by gluing the two extensions. Observe that $\mathcal{L}_{\nabla_{M_2}}$ is in fact a $U(1)$ -torsor. These $U(1)$ -torsors patch together to define a principal $U(1)$ -bundle over the set of connections ∇_{M_2} on M_2 . If we are given a specific family of extensions $(P^{\text{ext}}, \nabla^{\text{ext}})$, we obtain (locally) a section of that line bundle. The line bundle descends to the gauge equivalence classes of connections on M_2 and is nontrivial there, reflecting the fact that quantum Chern-Simons theory on a manifold with boundary is anomalous.

51.2 WZW Terms

The Wess-Zumino term was originally introduced into the low-energy pion effective action of QCD to account for the flavor group anomalies. It was derived by J. Wess and B. Zumino [WZ71].

The effective theory of pions is a special case of what is known as the principal chiral model. To define this model consider a sigma model in $2n$ dimensions with target space G , where G is a Lie group. The action for a field $\phi : M_{2n} \rightarrow G$ is, (before the addition of the WZ term),

$$S_{\text{pcp}}[\phi] := \int_{M_{2n}} f^2 \text{Tr} \phi^*(\theta_{\text{MC}}) \star \phi^*(\theta_{\text{MC}}) , \quad (51.42)$$

where f^2 is a constant (dimensionful except in two dimensions) and $\text{Tr}(x^2)$ is an invariant degree 2 polynomial on \mathfrak{g} . Since the Maurer-Cartan form is usually written as $\theta_{\text{MC}} = g^{-1}dg$ for $g \in G$, if we instead write $g : M_{2n} \rightarrow G$ as a field, then the action takes a form that is commonly written in the physics literature:

$$S_{\text{pcp}}[\phi] := \int_{M_{2n}} f^2 \text{Tr} g^{-1}dg \wedge \star g^{-1}dg . \quad (51.43)$$

This sigma model is often referred to as the *principal chiral model*. When coupled to external gauge fields for the global $G \times G$ symmetry, the model is not anomalous, and therefore cannot be an accurate LEET for QCD, which has $G \times G$ anomalies. Wess and Zumino remedied this defect by adding their famous term Γ to the action (51.42).

A beautiful re-interpretation of Γ was given by Witten in [Wit83a, Wit84]. Moreover, Witten showed that the addition of the WZ term to the action of the principal chiral model in both two and four dimensions has dramatic physical consequences [Wit83a, Wit83b, Wit84]. A related construction was given by Novikov in [Nov82] (see p. 40 et. seq.), so the model is sometimes referred to as the WZNW model.

The description of Witten [Wit83a, Wit84] and Novikov [Nov82] proceeds along the following lines. Choose a cohomology class Θ_{2n+1} of degree $2n+1$ on G , and suppose that Θ_{2n+1} has integral periods. Let $\phi : M_{2n} \rightarrow G$ be the sigma model map. Suppose that the image $\phi(M_{2n})$ is homologous to zero in G , so that there is a bounding chain: $\phi(M_{2n}) = \partial W_{2n+1}$. Then Witten's definition of the WZ term is:

$$\text{WZ}(\phi) = 2\pi i \int_{W_{2n+1}} \Theta_{2n+1}. \quad (51.44)$$

As usual, there can be multiple ways of bounding the image $\phi(M_{2n})$ so the action (51.44) is ambiguous. When Θ_{2n+1} is suitably normalized, the exponentiated action will be a well-defined functional of ϕ . While the action appears to be an action in $2n+1$ dimensions, under a small variation of ϕ , the WZ term changes by an integral of a local density on M_{2n} .

As mentioned, the addition of these topological terms to the two- and four-dimensional sigma model with target a Lie group G has dramatic physical consequences. In the case of two-dimensions, when the kinetic and topological terms are suitably normalized one obtains a two-dimensional conformal field theory – the renowned WZW conformal field theory [Wit84]. For textbook treatments, see [Fuc95, DFMS97].

In four dimensions, the skyrmion field configuration can (with suitable normalization of the WZ term) obey fermionic statistics [Wit83b].

It turns out that Witten's definition of the WZ term (51.44) is problematic because there are examples where the image of spacetime under ϕ is not a boundary.¹⁴² For a rather trivial example, consider the case where M_2 is a torus and the group G is a torus of dimension greater than or equal to 2, and the image of ϕ is a nontrivial subtorus of G . For this reason, it is desirable to have a definition of the WZ term that applies to all finite dimensional Lie groups.

The two-dimensional WZ term for a general group is provided by a gerbe connection $\check{H} \in \check{H}^3(G)$. The WZ term for $\phi : M_2 \rightarrow G$ is then the holonomy of the gerbe around the

¹⁴²For the 2d WZ term, Witten only considers the spacetime \mathbb{S}^2 and uses the fact that $\pi_2(G) = 0$ for all compact groups. However, we most definitely would want to consider other spacetimes, and in any case, the relevant topological invariants are the homology, not the homotopy groups of G .

image $\phi(M_2) \subset G$. For example, if $c \in H^3(G; \mathbb{Z})$, and φ is such that,

$$[\varphi(\theta_{\mathbf{MC}})] = c_{\mathbb{R}}, \quad (51.45)$$

there is a differential character $\check{H} \in \check{H}^3(G)$ that can be used to define the WZ term. If G is a compact connected simply connected Lie group, then \check{H} is unique. We can prove this by showing that the group of flat fields $H^2(G; \mathbb{R}/\mathbb{Z})$ vanishes. To see this group vanishes, use the universal coefficient theorem (39.10) and the fact that $H_1(G; \mathbb{Z}) = 0$ (since G is simply connected) and $H_2(G; \mathbb{Z}) = 0$ (by the Hurewicz theorem¹⁴³ combined with the fact that $\pi_2(G) = 0$).

The WZ and Chern-Simons topological terms in actions are closely related. Mathematically, the relation involves lift of the *transgression map* on cohomology to differential cohomology.

We first recall the transgression map on cohomology. This is a map

$$\tau : H^k(BG; \mathbb{Z}) \rightarrow H^{k-1}(G; \mathbb{Z}), \quad (51.46)$$

defined for a compact group [Bor67].

It is defined as follows: Suppose $\lambda \in H^{2k}(BG; \mathbb{Z})$. We choose a representative c of λ , $[c] = \lambda$ and pull it back under $\pi : EG \rightarrow BG$. Since EG is contractible, $\pi^*(c)$ must be trivializable, so there exists a cochain c' on EG with $c = \delta c'$. Now, because BG has a natural basepoint, there is a natural fiber inclusion $\iota : G \rightarrow EG$, the fiber over the basedpoint. Then $\iota^*(c) = 0$, and hence, $\iota^*(c')$ is a closed integral class on G . The transgression map is defined to be:

$$\tau(\lambda) = [\iota^*(c')]. \quad (51.47)$$

It can be shown to be independent of the choices we have made, and is hence a well-defined map. It is not a group homomorphism.

Now, equation (51.19) above strongly hints that there should be a differential refinement of the transgression map, and this indeed proves to be the case. For a careful treatment in degree three, see [CJM⁺05], which refines the transgression map to a map,

$$\check{\tau} : \check{H}^4(B_{\nabla} G) \rightarrow \check{H}^3(G; \mathbb{Z}). \quad (51.48)$$

This can be interpreted as a map relating Chern-Simons in 3-dimensions and Wess-Zumino

¹⁴³The Hurewicz theorem says that for a path-connected space X and for $n \in \mathbb{Z}_{>0}$, there is a group homomorphism $h_n : \pi_n(X) \rightarrow H_n(X)$ – called the Hurewicz homomorphism – which is defined as follows: one picks a canonical generator of $H_n(\mathbb{S}^n) \cong \mathbb{Z}$, call it x_n . A based map $f : \mathbb{S}^n \rightarrow X$ induces a homomorphism $f_* : H_n(\mathbb{S}^n) \rightarrow H_n(X)$. Denoting by $[f]$ the homotopy class of f in $\pi_n(X)$, one defines $h_n([f]) = f_*(x_n) \in H_n(X)$. In general, h_n is neither injective nor surjective. For $n \geq 2$, if $\pi_i(X) = 0$ for all $i < n$ (we say X is $(n-1)$ -connected), then the Hurewicz homomorphism becomes an isomorphism. In the present case, since G is connected and simply connected, it follows that $\pi_2(G) \cong H_2(G; \mathbb{Z})$.

terms in 2-dimensions. It is a version of anomaly inflow, and was used to explain the Chern-Simons/RCFT correspondence in [MS89a, MS89c, EMSS89].

Moving on to four-dimensions, we encounter an interesting puzzle. As noted above, for suitable normalization of the WZ term, the skyrmion of the principal chiral model obeys fermionic statistics. This suggests that the principal chiral model with such a WZ term should be a spin theory, but it is not evident how the spin structure is used to construct the WZ term. A resolution was provided by D. Freed in [Fre06], which constructed the term using generalized differential cohomology based on a cohomology theory (E-theory) that refines singular cohomology. For a related discussion, see [LOT20].

52 Other Applications II: Quantization Of BF Theory

An important example of a topological field theory is “BF theory.” The study of this theory goes back at least to [Sch79, Hor89]. We will consider the theory where the fields are formulated using differential cohomology, as introduced in [MMS01b].¹⁴⁴

The BF theory can be defined in n spacetime dimensions with fields whose gauge equivalence classes are:¹⁴⁵

$$[\check{B}_{n-p-1}] \in \check{H}^{n-p}(M_n) , \quad [\check{A}_p] \in \check{H}^{p+1}(M_n) . \quad (52.1)$$

To formulate the action, we need to introduce a nonzero integer k , and an orientation σ . Then, we have,

$$S = 2\pi i k \langle [\check{B}_{n-p-1}], [\check{A}_p] \rangle . \quad (52.2)$$

Our fields are normalized so that the periods of the fieldstrengths are integers. In the case of topologically trivial fields, we can write,

$$[\check{B}_{n-p-1}] = \chi_{B_{n-p-1}} , \quad [\check{A}_p] = \chi_{A_p} . \quad (52.3)$$

For globally defined fields $B_{n-p-1} \in \Omega^{n-p-1}(M_n)$ and $A_p \in \Omega^p(M_n)$, the action simplifies to:

$$S = 2\pi i k \int_{M_n} B_{n-p-1} dA_p . \quad (52.4)$$

We denote the resulting field theory by $T_{\text{BF}}^{n,p,k,\sigma}$. A change of orientation is equivalent to a flip of the sign of k , so, without loss of generality, we can take k to be positive. Moreover,

¹⁴⁴In [Sch79, Hor89, BT91, BBRT91], no quantization of the periods of the fieldstrengths was imposed, resulting in rather different answers from those discussed here. In [Wit98d], a special case was studied and the quantization of the periods played an important role in Witten’s discussion.

¹⁴⁵In this section, the periods of fieldstrengths are valued in \mathbb{Z} and not $2\pi\mathbb{Z}$, hence the use of the notation F as opposed to F . See footnote 108.

the pairing on differential cohomology classes is graded symmetric:

$$\langle [\check{B}_{n-p-1}], [\check{A}_p] \rangle = (-1)^{n(p+1)} \langle [\check{A}_p], [\check{B}_{n-p-1}] \rangle . \quad (52.5)$$

The exchange of \mathbf{A} and \mathbf{B} is a form of electric-magnetic duality, and so we expect, at least naively, that the statement of electric-magnetic duality for these theories is:

$$T_{\text{BF}}^{n,p,k,\mathfrak{o}} = \begin{cases} (T_{\text{BF}}^{n,n-p-1,k,\mathfrak{o}}) , & \text{for } n(p+1) = 0 \pmod{2} , \\ (T_{\text{BF}}^{n,n-p-1,k,\mathfrak{o}})^* , & \text{for } n(p+1) = 1 \pmod{2} . \end{cases} \quad (52.6)$$

Remark 52.7. The fields can be twisted to take values in flat bundles based on representations ρ_1, ρ_2 of the fundamental group, and indeed this generalization was studied in [Sch79]. In particular, if we twist one of the fields by the orientation line, then we can extend the domain of definition from oriented manifolds to unoriented manifolds.

The quantization of the theories was studied in [Sch79, Hor89, BT91, BBRT91, Wit98d, MMS01b, Sza12, BH23] and in many other works. Despite all this work, as far as we are aware, the theory has not been constructed as a fully local topological field theory in complete detail.¹⁴⁶ Doing so is highly desirable and should be quite achievable given our current understanding of field theory – it is “just” free field theory! We briefly comment on some highlights of the quantization.

We remark that the path integral of the fields $[\check{A}_p]$ with the action (52.2) shows that the path integral is localized on fields $[\check{B}_{n-p-1}]$ which are k -torsion in differential cohomology. Similarly, the integral over $[\check{B}_{n-p-1}]$ shows that the path integral is localized on fields $[\check{A}_p]$ which are k -torsion. Since the fields are k -torsion, they are certainly flat:

$$F(\check{A}) = 0 , \quad F(\check{B}) = 0 , \quad (52.8)$$

(as follows easily from (52.4)) but the extended defects and operators defined by the holonomies of the fields are k -torsion. For topologically trivial field configurations, these holonomies are written as integrals over cycles as:

$$T([\Sigma_{n-p-1}]) := \exp \left[2\pi i \int_{\Sigma_{n-p-1}} \mathbf{B}_{n-p-1} \right] , \quad (52.9)$$

and

$$W([\Sigma_p]) := \exp \left[2\pi i \int_{\Sigma_p} \mathbf{A}_p \right] . \quad (52.10)$$

Since the path integral localizes to flat fields, the operators T and W only depend on

¹⁴⁶Of course, full locality would follow once one establishes the equivalence with the homotopy sigma model $\sigma_{B^p \mathbb{Z}_k, \mathcal{C}}^{(n)}$, but as we will describe below, there are some gaps in the literature establishing that equivalence.

homology classes. In particular:

$$T([\Sigma^1] + [\Sigma^2]) = T(\Sigma^1)T(\Sigma^2) , \quad (52.11)$$

just like the classical character. A similar formula holds for W . Moreover, since the path integral localizes to differential cohomology classes which are k -torsion, we have,

$$(T([\Sigma_{n-p-1}]))^k = 1 , \quad (W([\Sigma_p]))^k = 1 . \quad (52.12)$$

An elementary Gaussian integral computation shows that the path integral with defects inserted satisfies:

$$\langle T([\Sigma_{n-p-1}])W([\Sigma_p]) \rangle = Z \exp \left[\frac{2\pi i}{k} L([\Sigma_{n-p-1}], [\Sigma_p]) \right] , \quad (52.13)$$

where Z is the partition function without insertions, and L is the linking number of the oriented homology cycles. This is certainly consistent with (52.12), and moreover shows that the operators on the Hilbert space $\mathcal{H}(N_{n-1})$, where N_{n-1} is a spatial manifold, must satisfy the basic relations defining a Heisenberg extension of:

$$H_{n-p-1}(N_{n-1}; \mathbb{Z}/k\mathbb{Z}) \times H_p(N_{n-1}; \mathbb{Z}/k\mathbb{Z}) , \quad (52.14)$$

namely,

$$T([\Sigma_{n-p-1}])W([\Sigma_p]) = \exp \left[\frac{2\pi i}{k} I([\Sigma_{n-p-1}], [\Sigma_p]) \right] W([\Sigma_p])T([\Sigma_{n-p-1}]) , \quad (52.15)$$

where I is the oriented intersection number. Call this Heisenberg group $\mathcal{H}_{n,p,k}$.

In the quantization of the BF theory, we note that the action (52.4) is first order in derivatives and should be viewed as an action on a phase space. The action is degenerate, and there will be ghosts-for-ghosts. One can quantize, and then impose Gauss law constraints, or impose constraints and then quantize. Proceeding with the latter method to formulate the Hilbert space, we impose the Gauss law in steps. Let $M_n = N_{n-1} \times \mathbb{R}$, let t be a time coordinate, and let $A_0 \in \Omega^p(N_{n-1} \times \mathbb{R})$ be proportional to dt . A shift of \check{A} by χ_{A_0} changes the action as $S \mapsto S + 2\pi ik \int_{N_{n-1} \times \mathbb{R}} F(\check{B}) A_0$. Integration over A_0 localizes the path integral on the flat fields $F(\check{B}) = 0$. Similar remarks apply with A and B exchanged. The result is that we have a phase space of flat fields,

$$H^{n-p-1}(N_{n-1}; U(1)) \times H^p(N_{n-1}; U(1)) , \quad (52.16)$$

equipped with a prequantum line bundle¹⁴⁷ which comes from the integration:

$$\int_{N_{n-1}}^{\check{H}} : \check{H}^{n-p} \times \check{H}^{p+1} \rightarrow \check{H}^2. \quad (52.17)$$

(Recall that \check{H}^2 describes gauge equivalences classes of line bundles with connection.) The phase space (52.16) has many components, indexed by

$$\text{Tors}(H^{n-p}(N_{n-1}; \mathbb{Z})) \times \text{Tors}(H^{p+1}(N_{n-1}; \mathbb{Z})), \quad (52.18)$$

but the automorphism group $\text{Aut}(\check{H}^{n-p}(N_{n-1})) \cong H^{n-p-2}(N_{n-1}; \text{U}(1))$ acts nontrivially on the prequantum line bundle via the pairing with $\text{Tors}(H^{p+1}(N_{n-1}; \mathbb{Z}))$, and hence, the support of the wavefunction will be concentrated in the topologically trivial component of $H^p(N_{n-1}; \text{U}(1))$.¹⁴⁸ Similarly, the automorphisms of \mathbf{A} act to concentrate the wavefunction on the topologically trivial component of $H^{n-p-1}(N_{n-1}; \text{U}(1))$.¹⁴⁹ The quantization of the trivial component gives an irreducible representation of the Heisenberg group $\mathcal{H}_{n,p,k}$. A Stone-von Neumann representation is provided by the vector space of complex-valued functions on the cohomology group:

$$\mathcal{H}(N_{n-1}) = \text{Fun}(H^p(N_{n-1}; \mathbb{Z}_k) \rightarrow \mathbb{C}). \quad (52.19)$$

Comparison with equation (24.28) raises the issue whether $T_{\text{BF}}^{n,p,k,\sigma}$ is related to the finite homotopy field theory $\sigma_{\mathcal{X}, \lambda, C}^{(n)}$ studied in Section 24, for the case $\mathcal{X} = B^p A = K(A, p)$, with $A = \mathbb{Z}/k\mathbb{Z}$. We discuss to what extent that might be the case.

It is certainly true that the BF theory $T_{\text{BF}}^{n,p,k,\sigma}$ is closely related to a $\mathbb{Z}/k\mathbb{Z}$ p -form gauge theory. Indeed, in [MMS01b], that relation was observed by noting that a Higgs mechanism breaking a $\text{U}(1)$ gauge group to $\mathbb{Z}/k\mathbb{Z}$ has an IR description given by the BF theory (52.2). Nevertheless, the precise relation between $T_{\text{BF}}^{n,p,k,\sigma}$ and $\sigma_{B^p \mathbb{Z}_k, \lambda, C}^{(n)}$ has not been spelled out in the literature. Roughly speaking, they should be “essentially the same,” up to an invertible TQFT. Here are a few issues which should be clarified to make this completely precise:

1. First of all, the domain and codomain of the theories should be equal if we are to compare them. Let us begin with the domain. The domain of $T_{\text{BF}}^{n,p,k,\sigma}$ consists of the oriented bordism category, although if we twist one of the fields by the orientation line bundle, then the domain can be extended to the unoriented bordism category.

¹⁴⁷Given a symplectic manifold X and a closed symplectic form ω with integer periods, a prequantum line bundle on X with connection is a line bundle $L \rightarrow X$ with connection ∇ such that $\omega = F_\nabla$ is the curvature 2-form of the connection ∇ .

¹⁴⁸G.M. thanks E. Witten for a useful discussion on this point.

¹⁴⁹There is a subtle point here. In the closely related case of the partition function of the self-dual field, it turns out that the wavefunction in general is not supported on the topologically trivial component of the group of flat fields. This was first pointed out in [Wit99b, Sec. 5.1]. See also [BM06a, Sec. 5.2] for further explanation.

On the other hand, $\sigma_{B^p\mathbb{Z}_k, \lambda, \mathcal{C}}^{(n)}$ is defined on the unoriented bordism category for $\lambda = 1$. For nontrivial Dijkgraaf-Witten cocycles, it is only defined on the oriented bordism category.

2. The theory $T_{\text{BF}}^{n,p,k,\mathfrak{o}}$ does not make use of a Dijkgraaf-Witten cocycle λ . A continuum field theory analog of the Dijkgraaf-Witten cocycle has been proposed in [KS14] in three dimensions. As far as we know, the generalization to all (n, p) has not appeared. Without such an additional term, the comparison with $\sigma_{B^p\mathbb{Z}_k, \lambda, \mathcal{C}}^{(n)}$ should be carried out with a trivial Dijkgraaf-Witten cocycle. The simplest comparison, then, is that of $T_{\text{BF}}^{n,p,k,\mathfrak{o}}$ with $\sigma_{B^p\mathbb{Z}_k, \mathcal{C}}^{(n)}$, where the domain is the oriented bordism category.
3. Let us now consider the codomain. As we have discussed at length, the choice of the n -category \mathcal{C} is an essential choice of data in defining the theory $\sigma_{B^p\mathbb{Z}_k, \mathcal{C}}^{(n)}$. While we may require $\Omega^n \mathcal{C} = \mathbb{C}$ and $\Omega^{n-1} \mathcal{C} = \text{VECT}$, different choices of \mathcal{C} at the next level produce different theories. The corresponding choice is not evident in the definition of the theory $T_{\text{BF}}^{n,p,k,\mathfrak{o}}$, although, given the relevance of higher categories to defects, it is natural to expect that such a choice should become apparent with careful studies of defects and defects within defects.
4. Turning now to the partition function, the partition function of $T_{\text{BF}}^{n,p,k,\mathfrak{o}}$ on a compact n -manifold M_n takes the form,

$$T_{\text{BF}}^{n,p,k,\mathfrak{o}}(M_n) = \mathcal{N}_{n,p,k} \times \Theta_{n,p,k} . \quad (52.20)$$

Here $\Theta_{n,p,k}$ is a “sum over the classical solutions.” The counting of the classical solutions turns out to be extremely subtle. One interpretation of how to do this leads to precisely the expression,

$$\begin{aligned} \Theta_{n,p,k} &:= |H^p(M_n; \mathbb{Z}/k\mathbb{Z})| \cdot |H^{p-1}(M_n; \mathbb{Z}/k\mathbb{Z})|^{-1} \cdots \\ &= \prod_{j=0}^p |H^{p-j}(M_n; \mathbb{Z}_k)|^{(-1)^j} . \end{aligned} \quad (52.21)$$

We thank Mike Hopkins for essential discussions on this point.¹⁵⁰ In this way, the “classical” contribution recovers the expression (24.41) appearing in the partition function of the theory $\sigma_{B^p\mathbb{Z}_k, \mathcal{C}}^{(n)}$. The one-loop term $\mathcal{N}_{n,p,k}$ is closely related to the Ray-Singer

¹⁵⁰A naive way to perform the path integral with action (52.2) is to say that

$$\int_{\widetilde{H}^{n-p} \times \widetilde{H}^{p+1}} d\mu(\check{A}) d\mu(\check{B}) e^{2\pi i k \langle [\check{B}], [\check{A}] \rangle} = \int_{\widetilde{H}^{p+1}} d\mu(\check{A}) \delta(k[\check{A}]) = \#\ker \left(\times k : \widetilde{H}^{p+1} \rightarrow \widetilde{H}^{p+1} \right) , \quad (52.22)$$

and therefore say the path integral is the number of k -torsion points in $\widetilde{H}^{p+1}(M_n)$. While this gives the desired triviality for $k = 1$ in general it gives the wrong answer. The quantity $\#\ker \left(\times k : \widetilde{H}^{p+1} \rightarrow \widetilde{H}^{p+1} \right)$ is, in general, not equal to $\Theta_{n,p,k}$. One need look no further than spheres to find counterexamples.

analytic torsion [Sch79, BBRT91, BKT22]. The contributions of the nonzeromodes of the fields (and ghosts and ghosts-for-ghosts) produces $\tau'_{\text{RS}}(M_n)$ defined in (48.28). The contribution of the zeromodes must be dictated by the desired physical properties of the answer, and is part of the definition of the theory. We conjecture, based on electromagnetic duality of the generalized Maxwell field, that the treatment which defines a fully local field theory has,

$$\mathcal{N}_{n,p,k} = \left(\bigotimes_{j=0}^n \left(\omega_1^{(j)} \wedge \cdots \wedge \omega_{b_j}^{(j)} \right)^{(-1)^j} \cdot \text{Alt}(T) \right)^{(-1)^p}, \quad (52.23)$$

where,

$$\text{Alt}(T) = \prod_{j=0}^n T_j^{(-1)^j}. \quad (52.24)$$

Recall that T_j is the order of the torsion subgroup of $H^j(M_n; \mathbb{Z})$. As we have discussed above in (48.31), there is no canonical orientation of the cohomology, so the RHS of (52.23) is defined up to sign, and \mathcal{N} should be interpreted as a norm on a determinant line. Taking into account equation (48.19), we understand that we are interpreting the ghost zeromodes to be in $H^j(M_n; \text{U}(1))$, rather than in $H^j(M_n; \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$. The partition function is therefore:

$$(\tau_{\text{RS}}(M_n) \text{Alt}(T))^{(-1)^p} \Theta_{n,p,N}. \quad (52.25)$$

As shown in [Appendix A](#) by Dan Freed, when the Reidemeister torsion is suitably interpreted we have $\tau_{\text{Reidemeister}} \text{Alt}(T) = 1$. On the other hand, the Cheeger-Müller theorem identifies $\tau_{\text{Reidemeister}}$ with τ_{RS} . In this way we can align the partition function of the BF theory with that of the finite homotopy sigma model.

5. Note that for $k = 1$, the homotopy sigma model is trivial. The fact that BF theory with fields in differential cohomology should be trivial for $k = 1$ is a corollary of the claim of [MMS01b]. An important special case of this statement in the case $n = 3$ and $p = 1$ was independently argued by E. Witten in [Wit03]. In that case the BF theory is an Abelian Chern-Simons theory, and the triviality of the $k = 1$ theory played an important role in Witten's $\text{SL}(2, \mathbb{Z})$ action on 3d superconformal theories.
6. The statements of electric-magnetic duality of BF theory (equation (52.6) above) and finite homotopy sigma model (recall equation (24.59)) are slightly different.
7. Turning now to defects, we recall that some discrepancies in the standard descriptions of defects in the two theories were already noted in [Section 25.2](#).
8. *Relation to the lattice description.* There is a “lattice gauge theory” version of BF

theory introduced by Kapustin and Seiberg in [KS14] and studied further in [GLSS21, CCH⁺21, Kap25, KS25]. In particular, [GKSW14, App. B] or [CCH⁺21, App. A] derives the partition function (24.41) from the lattice version. In principle, ghosts should be included in the derivation, although a careful treatment of ghosts in this context has not appeared in the literature.¹⁵¹ It would be nice to extend the discussion to higher codimension manifolds. Granting that the lattice version of BF theory agrees with the finite homotopy sigma model, the above list of discrepancies then applies to the relation between the lattice and continuum formulations of BF theory.

Of course, we expect that all the above discrepancies have satisfactory resolutions and that $T_{\text{BF}}^{n,p,k,\sigma}$ is “essentially” the same as the theory $\sigma_{B^p \mathbb{Z}_k, \mathcal{C}}^{(n)}$, up to tensor product with an invertible theory. Given this, a natural question occurs: Find a continuum field-theoretic model for the general finite homotopy theory $\sigma_{\mathcal{X}, \lambda, \mathcal{C}}^{(n)}$. Some relevant remarks extending from the case of $\mathcal{X} = B^p A$ to the case of a two-group can be found in [KT13b].

53 Other Applications III: The M-Theory Abelian Gauge Field

One very striking application of differential cohomology is to M-theory. We sketch that here in the briefest terms.

Eleven-dimensional supergravity, as formulated in [CJS78], involves a Lorentzian 11-dimensional oriented spin manifold M_{11} . The bosonic fields in the supergravity multiplet consist of a metric $g_{\mu\nu} \in \text{Met}(M_{11})$ and a “3-form gauge potential,” often denoted by C . After Wick rotation to Euclidean signature, and in the case that $C \in \Omega^3(M_{11})$ is globally defined, the part of the bosonic action involving the C -field is¹⁵²

$$\exp \left(-\frac{\pi}{\ell^3} \int_{M_{11}} G \wedge \star G + 2\pi i \int_{M_{11}} \left(\frac{1}{6} C \wedge G \wedge G - CI_8(g) \right) \right), \quad (53.1)$$

where $G = dC$, ℓ is the eleven-dimensional Planck length¹⁵³ and $I_8(g)$ is the Chern-Weil

¹⁵¹Thanks to the Cheeger-Müller theorem, the Ray-Singer torsion can be identified with the Reidemeister torsion [Che77, Che79, M78]. It would be quite interesting if the ghosts in the lattice BF theory led directly to Reidemeister torsion.

¹⁵²In this section, the periods of fieldstrengths are valued in \mathbb{Z} and not $2\pi\mathbb{Z}$, hence we use C and G rather than C and G . See footnote 108.

¹⁵³Normalized so that the path integral weighting from the Einstein-Hilbert term is:

$$\exp \left[-\frac{2\pi}{\ell^9} \int \mathcal{R}(g) \text{vol}(g) \right].$$

representative¹⁵⁴ of:

$$\frac{4p_2 - p_1^2}{4 \cdot 48} . \quad (53.2)$$

Naively, one would expect the gauge-invariant information of the C-field to be encoded in a holonomy function,

$$\chi_C : Z_3(M_{11}) \rightarrow U(1) , \quad (53.3)$$

written heuristically as $\chi_C(\Sigma) = e^{2\pi i \int_\Sigma C}$. This is only heuristic because in general, there will be no globally well-defined 3-form we can call C. One would then expect G to be the fieldstrength such that, if $\Sigma = \partial W_4$ is the boundary of an integral 4-chain, then,

$$\chi_C(\Sigma) = \exp \left[2\pi i \int_{W_4} G \right] . \quad (53.4)$$

That is, one would expect that the set of gauge equivalence classes of M-theory C-fields is the differential cohomology group $\check{H}^4(M_{11})$. This still turns out to be too naive for the following reason: The physical meaning of the holonomy χ_C is that it is a term in the action of an M2-brane, very much as in the discussion of Section 35 and near equation (38.3). However, the worldvolume of the M2-brane contains fermions valued in $S(\Sigma) \times S(N\Sigma)^-$, where $S(\Sigma)$ is the spin bundle, $S(N\Sigma)$ is the spin bundle of the normal bundle $N\Sigma$ of Σ in M_{11} and the superscript $-$ is a chirality projection.¹⁵⁵ The path integral of these fermions is a Pfaffian, which has a sign ambiguity. It is a section of a nontrivial (but two-torsion) real line bundle $\mathcal{L}_{\text{memb}}$ over the space of 3-dimensional membrane worldvolumes Σ and metrics on M_{11} . Accordingly, χ_C must likewise be twisted by the line bundle $\mathcal{L}_{\text{memb}}$ and hence if we try to write the C-field holonomy as a function as in (53.3), it will be ambiguous by a sign. This subtlety was pointed out by Witten in [Wit96a]. He showed that the topological class of $\mathcal{L}_{\text{memb}}$ is such that the fieldstrength must satisfy the following quantization law on its periods:

$$\int_{\Sigma_4} \left(G - \frac{1}{4} p_1(TM_{11}) \right) \in \mathbb{Z} . \quad (53.5)$$

On a spin manifold, p_1 has a canonical division by 2, so there exists a canonical integral

¹⁵⁴ The term involving I_8 was first discussed by Vafa and Witten in [VW95] and further studied in [DLM95, Wit96a]. Explicitly, $I_8(g) = \frac{1}{(2\pi)^3 \cdot 4!} \left(\frac{1}{8} \text{tr } R^4 - \frac{1}{32} (\text{tr } R^2)^2 \right)$ which can also be written as $I_8(g) = -\frac{\pi}{24} (p_2(M_{11}) - \lambda^2) = -\frac{2\pi}{6} \frac{1}{8} (p_2(M_{11}) - \lambda^2)$, where $\lambda = \frac{1}{2} p_1(M_{11})$ is an integral class and we are here using the canonical Chern-Weil representatives defined by a Riemannian metric. λ defines an integral de Rham class which is a de Rham representative of an integral lift of the 4th Wu class $\nu_4(M_{11})$. Since M_{11} is spin, $\nu_4(M_{11}) = w_4(M_{11})$ and furthermore its first Pontryagin class is canonically divisible by 2. The canonical quotient is λ . See footnote 126 for a description of the curvature 2-form in local coordinates.

¹⁵⁵When we include the fermionic degrees of freedom, we should view eleven-dimensional supergravity as defined on a superspace of real dimension (11|32). The fermionic modes of the membrane theory are obtained from the restriction $S(TM_{11})|_\Sigma$ of the spin bundle of M_{11} to the worldvolume Σ . There is a local supersymmetry known as κ -supersymmetry that makes one chirality pure gauge. See [Sie83, Sie84a, BST87, BBS95] for details about κ -symmetry.

class $\lambda \in H^4(M_{11}; \mathbb{Z})$ (see footnote 128), with $p_1 = 2\lambda$. Therefore, the periods of \mathbf{G} can be half-integral.

While χ_C cannot be written as a character on $Z_3(M_{11})$, the ratio of the holonomies for two C -fields does make sense as a function:

$$\frac{\chi_{C_1}}{\chi_{C_2}} : Z_3(M_{11}) \rightarrow \mathbf{U}(1) . \quad (53.6)$$

It follows that the gauge equivalence class of the C -field, which we will write as $[\check{C}]$, forms a torsor for $\check{H}^4(M_{11})$.

If we want to pin down the C -field more precisely, then it can be formulated as a differential chain that trivializes a degree 5 background magnetic current $\check{j}_m = \check{w}_4(M_{11})$ in $\check{H}^5(M_{11})$. In the Hopkins-Singer cochain model (40.39), we can represent $\check{w}_4(M_{11})$ by the cocycle $(0, 0, \frac{1}{2}\lambda(g))$, where $\lambda(g)$ is the usual Chern-Weil representative of $\lambda = p_1/2$ defined by a Riemannian metric.

Differential cohomology turns out to be very useful in making sense of the extremely subtle “topological term” in the action for the low-energy effective theory of M-theory:

$$\exp \left[2\pi i \int_{M_{11}} \left(\frac{1}{6} C \wedge G \wedge G - CI_8(g) \right) \right] , \quad (53.7)$$

But before using differential cohomology, we note that if (M_{11}, C) admits an extension to a 12-manifold M_{12} , then we can write (53.7) as:

$$\exp \left[2\pi i \int_{M_{12}} \left(\frac{1}{6} G \wedge G \wedge G - GI_8(g) \right) \right] , \quad (53.8)$$

As usual, to check that this gives a sensible definition of (53.8) one compares two extensions, so the question of whether the term is well-defined comes down to whether (53.8) is $+1$ on all closed 12-dimensional manifolds equipped with a C -field. This is far from obvious! Since G has half-integral periods, $\exp[2\pi i \int_{M_{12}} \frac{G^3}{6}]$ can be a 48^{th} root of unity. Similarly, the term involving I_8 can be a 96^{th} root of unity. In the case that M_{12} is a spin manifold, Witten gave an ingenious argument to show that, actually, the expression takes values $\{\pm 1\}$ [Wit96a]. In the case that no spin manifold M_{12} bounds M_{11} but there exists a bounding Pin^+ manifold, Corollary 4.54 of [FH19] assures us that it still takes values in $\{\pm 1\}$. Taking values in $\{\pm 1\}$ is still not good enough to avert a catastrophe for M-theory, but it turns out that the Rarita-Schwinger partition function has a canceling sign ambiguity, so that the product can be well-defined. The first indication that this is so was given in [Wit96a]. The argument was subsequently refined in [DMW00a, Sec. 2] and in [FM04] for spin manifolds, and was extended to the case of Pin^+ manifolds in [FH19].

We now explain Witten’s ingenious argument alluded to above. Due to some mathematical “coincidences,” first used in [Wit85, Wit86, Wit96a], there is a remarkable relation of the

C -field to E_8 gauge theory. The first remarkable “coincidence” is the topological equivalence of $K(\mathbb{Z}, 4)$ with $B\mathbf{E}_8$, up to the 15-skeleton. As we have seen from equation (53.5) the periods of G form a torsor for $H^4(M_{11}; \mathbb{Z})$. Recall that this cohomology group is just the group of homotopy classes of maps of M_{11} into $K(\mathbb{Z}, 4)$. Of course, the homotopy groups $\pi_k(K(\mathbb{Z}, 4))$ vanish for all $k \neq 4$, and $\pi_4(K(\mathbb{Z}, 4)) \cong \mathbb{Z}$. On the other hand, the low-dimension homotopy groups of E_8 are:

$$\pi_3(E_8) \cong \mathbb{Z} , \quad (53.9)$$

$$\pi_k(E_8) = 0 , \quad 0 \leq k < 15 , \quad \text{and} \quad k \neq 3 , \quad (53.10)$$

while $\pi_{15}(E_8) \cong \mathbb{Z}$. This means there is a homotopy equivalence of E_8 and $K(\mathbb{Z}, 3)$ up to the 14-skeleton, and hence an equivalence of $B\mathbf{E}_8$ and $K(\mathbb{Z}, 4)$ up to the 15-skeleton.¹⁵⁶ In the current application, this means that the topological class of a shift of an M-theory C -field can be uniquely identified with an isomorphism class of an E_8 bundle on 11-manifolds, and even on families of 11 manifolds, provided the base of the family is at most 4-dimensional.

The second remarkable coincidence is that the expression (53.8) is closely related to the index density of a Dirac operator coupled to an E_8 bundle with connection over M_{11} ! This extraordinary observation goes back to Witten [Wit96a]. One version of Witten’s observation can be explained by giving a groupoid model for the M-theory C -field using E_8 bundles with connection [DFM03]. In this model, the objects (the “ C -fields”) are triples (P, ∇, c) , where $P \rightarrow M_{11}$ is a principal E_8 bundle with connection ∇ and $c \in \Omega^3(M_{11})$ is a globally defined 3-form. The morphisms (“gauge transformations”) are given by equivalences:

$$(P, \nabla, c) \sim (P', \nabla', c') \quad \text{if} \quad c' - c = \text{CS}(\nabla, \nabla') , \quad (53.11)$$

along with gauge transformations like $c \mapsto c + d\rho$, where ρ is a globally well-defined 1-form on M_{11} . Note that the fieldstrength,

$$G = \text{Tr } F^2 - \frac{1}{2} \text{Tr } R^2 + dc . \quad (53.12)$$

is gauge-invariant under these equivalences.¹⁵⁷

Next, let $i(\mathcal{D}_{V, \nabla})$ denote the index density of Atiyah and Singer for a Dirac operator coupled to a bundle V with connection ∇ . Then we have the astonishing result that:

$$\left[\frac{1}{2} i(\mathcal{D}_{(\text{ad } P, \nabla)}) + \frac{1}{4} i(\mathcal{D}_{\text{RS}}) \right]^{(12)} = \frac{1}{6} G^3 - GI_8 + d(\iota_{\text{local}}) , \quad (53.13)$$

¹⁵⁶Recall footnote 50. One must exhibit a map inducing the isomorphism of homotopy groups. This gap can be filled in. See Proposition 5.2.122 (and 5.1.41) of [Sch13], so in fact $B\mathbf{E}_8$ and $K(\mathbb{Z}, 4)$ are homotopic up to the 15-skeleton.

¹⁵⁷Note that in these terms the WZW term in the membrane action (i.e., the holonomy of the C -field) is $\exp[2\pi i \int_{\Sigma} (\text{CS}(\nabla) - \frac{1}{2} \text{CS}(g) + c)]$. Once again, it is the division by 2 in the Chern-Simons invariant for the gravitational field that leads to the interpretation of the holonomy as valued in $\mathcal{L}_{\text{memb}}$.

where ι_{local} is a globally defined 11-form depending on G , c , and Chern-Weil representatives for $p_1(TM_{11})$ and $p_2(TM_{11})$. (The explicit formula is given below.) Finally, \not{D}_{RS} is the index density for the Rarita-Schwinger (RS) operator. It is the Dirac operator coupled to $TM_{11} - 3\mathcal{O}$ in 11-dimensions and $TM_{11} - 4\mathcal{O}$ in 12-dimensions, where \mathcal{O} is a trivial real line bundle.¹⁵⁸

Using the Atiyah-Patodi-Singer (APS) index theorem, we can give an intrinsic 11d definition of the phase of the C-field partition function:

$$\Phi([\check{C}]) = \exp \left\{ 2\pi i \left(\frac{\xi(\not{D}_{P,\nabla})}{2} + \frac{\xi(\not{D}_{\text{RS}})}{4} \right) + 2\pi i I_{\text{local}} \right\}, \quad (53.14)$$

$$\xi(\not{D}) := \frac{1}{2} (\eta(\not{D}) + h(\not{D})), \quad (53.15)$$

$$I_{\text{local}} = \int \left(\frac{1}{2} c G^2 - \frac{1}{2} c d c G + \frac{1}{6} c (dc)^2 - c I_8 \right). \quad (53.16)$$

Here $\eta(\not{D})$ is the Atiyah-Patodi-Singer invariant and $h(\not{D})$ is the dimension of the kernel of \not{D} . Note the strange fact that, although the C-field is a bosonic field, this term in the action actually depends on the choice of a spin structure on M_{11} . The division of the RS density in the above formulae by four introduces an extremely subtle sign – it is, once again, the \pm sign ambiguity of (53.8). Put differently, the topological term in the action for the C-field is only defined as a section of a line bundle \mathcal{L}_C with \mathbb{Z}_2 -holonomy. On the other hand, the Rarita-Schwinger partition function is also defined as a section of a line bundle \mathcal{L}_{RS} which is 2-torsion. The product,

$$\text{Pfaff}(\not{D}_{\text{RS}})\Phi(X), \quad (53.17)$$

is therefore a section of a line bundle $\mathcal{L}_{\text{RS}} \otimes \mathcal{L}_C$. The line bundle is trivializable, meaning that there is a choice of nowhere-zero section s . Using that section, one can define an function on the space of metrics and C-fields modulo gauge equivalence. The function is just the ratio of (53.17) divided by s . The choice of section can be normalized and is known as a “setting of the quantum integrand” [FM04]. The theories defined by two different settings s_1, s_2 differ by an invertible TQFT whose partition function is s_1/s_2 . As noted above, this was demonstrated, with increasing degree of rigor in [Wit96a, DMW00a, FM04]. The discussion was extended to Pin^+ manifolds in [FH19].

There are numerous murky points about the topology of M-theory which should be clarified. Six of them are listed below:

1. *Ten-dimensional E_8 super-Yang-Mills (SYM) multiplets as “edge modes.”* Note that

¹⁵⁸The Rarita-Schwinger field is more naturally valued in $T^*M_{11} \otimes S$, but the degree 12 component of the index density does not change if we use TM_{11} instead of T^*M_{11} , and the former is more convenient for some topological arguments. The subtraction by multiples of \mathcal{O} accounts for ghost fields. See Appendix A of [FM04] for an attempt at a careful account.

the equivalence relation (53.11) means that we can shift the E_8 gauge field ∇ by an arbitrary $\alpha \in \Omega^1(M_{11}; \text{ad } P)$, so there is a “topological symmetry” of the type familiar from the Baulieu-Singer discussion of Donaldson-Witten theory: The dependence of physical quantities on the E_8 gauge field only depends on the topology of P . Thanks to the work of Hořava and Witten [HW95, HW96], we expect this to change dramatically in the presence of boundaries, where we expect to find propagating E_8 gauge fields in a 10d E_8 SYM multiplet on each component of the boundary. According to [FM04], anomalies will cancel on 11-manifolds with arbitrary numbers of boundary components with E_8 SYM multiplets of either chirality. The argument is most easily given using the above E_8 -model for the C field, and suggests that the propagating E_8 gauge fields on the boundary components should be regarded as “edge modes” of the theory with topological symmetry $\nabla \mapsto \nabla + \alpha$ in the bulk. This is still a murky point, and the nature of the edge mode picture should be clarified.¹⁵⁹

2. *Cubic Refinements.* A crucial aspect of the term $\Phi([\check{C}])$ is that it is a cubic-refinement of a trilinear form on \check{H}^4 in 11 dimensions. This is more or less obvious given the heuristic expression,

$$\exp \left[2\pi i \left\{ \int_{M_{12}} \left(\frac{G^3}{6} + GI_8 \right) \right\} \right]. \quad (53.18)$$

More precisely, it is shown in [DFM03] that for any 3 elements $\check{a}_i \in \check{H}^4(M_{11})$,

$$\frac{\Phi([\check{C}] + \check{a}_1 + \check{a}_2 + \check{a}_3)\Phi([\check{C}] + \check{a}_1)\Phi([\check{C}] + \check{a}_2)\Phi([\check{C}] + \check{a}_3)}{\Phi([\check{C}] + \check{a}_1 + \check{a}_2)\Phi([\check{C}] + \check{a}_2 + \check{a}_3)\Phi([\check{C}] + \check{a}_3 + \check{a}_1)\Phi([\check{C}])} = \int_{M_{11}}^{\check{H}} \check{a}_1 \odot \check{a}_2 \odot \check{a}_3 \in U(1). \quad (53.19)$$

It is interesting to consider the implications of (53.19) for the vanishing of the M-theory partition function on closed 11-manifolds. This was studied in [DMW00a, DMW00b, DFM03]. Since the kinetic energy $\sim \int_{M_{11}} G \star G$ of the C -field is invariant under a shift of \check{C} by a flat field $\check{\phi}$ for $\phi \in H^3(M_{11}; U(1))$, we can first average over flat C -fields. (In modern language, we are studying the consequences of a “3-form generalized symmetry” for the vanishing of the partition function.) The partition function is proportional to,

$$\int_{\phi \in H^3(M_{11}; U(1))} \Phi([\check{C}] + \check{\phi}). \quad (53.20)$$

The integral over flat fields can be written as an integrated integral over the topologically trivial flat fields, followed by a sum over $T := \text{Tors}(H^4(M_{11}; \mathbb{Z}))$. The integral

¹⁵⁹Non-supersymmetric extensions of the Hořava-Witten setup have been studied. See, for example, [FH00, HK07, MZ25].

over the topologically flat fields is easily seen to enforce the cohomological condition,

$$\left[\frac{1}{2} \mathbf{G}^2 - I_8 \right] \in H_{\text{dR}, \mathbb{Z}}^8(M_{11}) . \quad (53.21)$$

This is certainly compatible with the equation of motion for the C-field:

$$\ell^{-3} d \star \mathbf{G} = \frac{1}{2} \mathbf{G}^2 - I_8 . \quad (53.22)$$

(In Hamiltonian quantization, the condition (53.21) can be refined considerably to the Gauss law $\Theta(a) = 0$, where $\Theta(a)$ is a degree 8 integral class defined in [DFM03].)

When (53.21) is satisfied the integral over topologically trivial fields descends to a $\text{U}(1)$ -valued function $\tilde{\Phi}_{[\check{C}]} : T \rightarrow \text{U}(1)$,

$$\tilde{\Phi}_{[\check{C}]}(a_T) = \Phi([\check{C}] + \check{a}_T) , \quad (53.23)$$

where \check{a}_T is any lift of $a_T \in T$ to a flat differential character. (The choice of lift does not matter when (53.21) is satisfied.) $\tilde{\Phi}_{[\check{C}]}$ is a cubic refinement of the natural trilinear function on T . Specializing further to the case where $M_{11} = M_{10} \times \mathbb{S}^1$, we can decompose the degree four torsion classes in T as $a = h \cup [d\theta] + \tilde{a}$, where \tilde{a} is pulled back from M_{10} . At fixed h , as a function of \tilde{a} , the cubic refinement becomes a quadratic refinement, and it follows from [DMW00a, DMW00b, DFM03], that,

$$\frac{\tilde{\Phi}_{[\check{C}]}(h[d\theta] + \tilde{a}_1 + \tilde{a}_2) \tilde{\Phi}_{[\check{C}]}(h[d\theta])}{\tilde{\Phi}_{[\check{C}]}(h[d\theta] + \tilde{a}_1) \tilde{\Phi}_{[\check{C}]}(h[d\theta] + \tilde{a}_2)} = \exp \left[i\pi \int_{M_{10}} \tilde{a}_1 (\text{Sq}^3 + h) \tilde{a}_2 \right] , \quad (53.24)$$

where Sq^3 is a Steenrod square operation.¹⁶⁰ A key step in the proof of this result uses an observation from [DMW00a]: Let $a \in H^4(M_{10}; \mathbb{Z})$ and consider the E_8 -adjoint bundle $V(a)$, and let $f(a)$ be the mod-two index of the Dirac operator on M_{10} coupled to $V(a)$. Then $f(a)$ is a quadratic function:

$$f(a_1 + a_2) - f(a_1) - f(a_2) + f(0) = \int a_1 \text{Sq}^2 a_2 . \quad (53.25)$$

The significance of (53.24) will be explained in the next remark. Further discussion of the M-theory phase as a cubic refinement can be found in [FH19]. While Gauss sums – sums over finite groups of quadratic refinements – are extremely well-studied, the analogous sums over cubic refinements are much less studied. It might be interesting to understand better when the sum over the torsion group of the cubic refinements does and does not vanish.

¹⁶⁰See [DMW00a, Sec. 3] for a crash course on Steenrod squares.

3. *Comparison Of Dirac Quantization Conditions In M-theory And In IIA String Theory.*

The Dirac quantization conditions we have described for RR fields in Section 50.4 (using K -theory) and for the Abelian gauge field of M-theory described in this section leads to a puzzle, first raised in [DMW00a]: How can this be consistent with the duality of M-theory on a product manifold $M_{10} \times \mathbb{S}^1$ and type IIA string theory on M_{10} ? Reference [DMW00a] shows that the partition functions of the RR field on M_{10} and of the C-field on $M_{10} \times \mathbb{S}^1$ are in fact equal, but in a highly nontrivial way, which makes use of (53.24). The basic idea of the proof is to use the comparison of K -theory and $H^{\text{even}}(X_{10}; \mathbb{Z})$ based on the Atiyah-Hirzebruch spectral sequence. The first nontrivial differential is Sq^3 . Thus, if a class $a \in H^4(M_{10}; \mathbb{Z})$ has a K -theory lift $x \in K(M_{10})$ with $\text{ch}(x) = -a + \dots$, then it must be that $\text{Sq}^3(a) = 0$. Putting $h = 0$, it follows from (53.24) that two C-fields with characteristic classes a_1, a_2 will only contribute if $\text{Sq}^3(a_1 - a_2) = 0$. For details of the argument, see [DFM03, Sec. 9.3]. The proof there allows the inclusion of $h \neq 0$, in which case $\text{Sq}^3 + h$ is the AHSS differential for twisted K -theory. Next, one needs to compare the phases in the sum over fluxes. The equality of phases is highly nontrivial. See [DMW00a] for details. (A summary is given in [DMW00b].) Thus, with some work, the compatibility of the known quantization conditions for the RR field and the M-theory Abelian gauge field can be demonstrated at the level of the comparison of partition functions in the long-distance, weak-coupling limit.¹⁶¹ Much remains to be understood. For example, one would wish to include G_0 flux in the comparison. More importantly, a better conceptual understanding of the compatibility that applies to all physical quantities, not just the partition functions, is missing and would be a real step forward. Other quantization conditions have been proposed in [FSS20b, FSS20a, SS20], and references therein.

4. *The W_7 “Anomaly.”* In the course of studying the sum over torsion C-fields, reference [DMW00a] noted that if $W_7(TM_{11})$ is nonzero, the sum will vanish. We have seen in Exercise 7.9 and Remark 7.52 that partition functions can vanish without having an anomaly. The proper interpretation of the W_7 “anomaly” is that one must insert membranes to obtain nonvanishing amplitudes. It remains to be seen if this has interesting physical consequences in, say, M-theory compactifications and phenomenology.
5. *Compatibility Of K -theory Quantization With S-Duality.* Perhaps the most important unsolved problem in this corner of string theory is the compatibility of the differential- K -theoretic formulation of the RR field with S-duality. The problem is that, in type IIB string theory, the NS B-field is modeled using differential cohomol-

¹⁶¹Reference [DMW00a] generalizes the discussion to include nontrivial G_2 flux in type IIA, which corresponds to a sum over circle bundles in M-theory. Reference [MS02] demonstrates compatibility with T -duality.

ogy $[\check{B}] \in \check{H}^3(M_{10})$, while the 3-form component of the RR fieldstrength is part of the data of a differential K -theory class $\check{K}^{1,\check{\tau}}(M_{10})$. To make matters worse, the differential K -theory is twisted by a differential cocycle $\check{\tau}$ whose isomorphism class is $[\check{B}]$. Thus, the “RR 2-form potential C_2 ” and the “NS 2-form potential B_2 ” appear on a very different footing in the framework of differential K -theory. On the other hand, they are supposed to form a doublet of the S -duality group $\text{SL}(2, \mathbb{Z})$. No convincing framework has been presented that explains how the formulation of RR fields in terms of differential K -theory is compatible with the S -duality invariance of type IIB string theory. One might attempt to explain this away by noting that all the evidence for the K -theoretic quantization of RR fields applies in the long-distance and weak-coupling limit of string theory. Since S -duality exchanges strong and weak coupling, there is no direct contradiction. Such an explanation is not very satisfactory because there are subgroups of the $\text{SL}(2, \mathbb{Z})$ symmetry that preserve weak coupling, and yet, the incompatibility remains. For example, one can compare the K -theoretic classification of orientifold planes with the more naive description using singular cohomology in a weak-coupling regime. In particular, the discussion in [Wit98b, Sec. 3.2] classifies $O3$ planes based on torsion RR and NS fluxes in the linking real projective space. These are permuted by a subgroup of the $\text{SL}(2, \mathbb{Z})$ duality group that preserves weak coupling. It would be extremely interesting to demonstrate that Witten’s classification is compatible with the K -theoretic description of $O3$ planes described in [DFM09, DFM10].

6. M-theory is supposed to be parity invariant, but this is not obvious from the above description of the C-field action. Parity invariance means that the (Wick rotated) theory should be defined on unorientable manifolds. Because of the Rarita-Schwinger field, the proper statement is that the Wick rotated theory should be defined on Pin^+ 11-manifolds.¹⁶² A parity transform of the C-field $C \rightarrow C^P$ must satisfy,

$$\chi_{C^P}(\Sigma) = \chi_C(\Sigma)^* , \quad (53.26)$$

(which makes sense, even when χ_C is valued in $\mathcal{L}_{\text{memb}}$). Therefore,

$$G \mapsto -G . \quad (53.27)$$

so G is a “ w_1 -twisted differential form,” which means that when pulled back to the oriented double cover, it is odd under the \mathbb{Z}_2 deck transformation. Compatibility

¹⁶²One quick way to see that the domain should be Pin^+ and not Pin^- is that, thanks to the AdS/CFT correspondence, the theory should make sense on $\mathbb{RP}^4 \times S^7$ (from stacks of 5-branes on orientifold planes). The obstruction to a Pin^+ structure is $w_2(TM)$ and the obstruction to defining a Pin^- structure is $w_2(TM) + w_1(TM)^2$. The total Stiefel-Whitney class for \mathbb{RP}^n is $(1+x)^{n+1}$. So \mathbb{RP}^4 is Pin^+ but not Pin^- . A more systematic argument is in [FH16].

with the quantization condition for the C field means that the characteristic class a transforms under parity by:

$$a \mapsto \lambda - a . \quad (53.28)$$

The use of η -invariants to discuss parity anomalies in M-theory goes back to [ÁGDP85]. Some discussion of parity anomalies can be found in [DMW00a] and [Wit16]. All these references assumed that the manifold is spin. In [FH19], D. Freed and M. Hopkins extend the discussion to Pin^+ manifolds. In this case, there is no known E_8 -model of the C -field. The heart of the problem is that there is no natural isomorphism between a pair of principal E_8 bundles with characteristic classes a and $\lambda - a$.¹⁶³ Nevertheless, the expression (53.8) makes sense on Pin^+ manifolds, and the main result of [FH19] is that $\mathcal{L}_{\text{RS}} \otimes \mathcal{L}_C$ is trivializable. (The result follows from computations using specific generators of the relevant bordism group and is not conceptual, but computational, as the authors lament). Interestingly, there are two possible trivializations of $\mathcal{L}_{\text{RS}} \otimes \mathcal{L}_C$. Put differently, there is exactly one new topological term one can add to M-theory: The mod-two index of the Pin^+ Dirac operator. When M_{11} is spin, there is a canonical setting defined by the E_8 model of the C -field, as explained in [FM04]. For general Pin^+ manifolds, there is no known E_8 model of the C -field, and no known canonical setting. There thus appear to be two M-theories depending on whether or not the topological term is included in the action. An interesting open problem is whether this topological term is consistent with other physical constraints: Does it make sense when we include boundaries (and what are the edge modes)? Is it consistent with anomaly freedom for M2 and M5 branes? Is it consistent with duality to the type II string on circle bundles? Etc.

¹⁶³The end of [Moo04] suggests an approach to this difficulty, but it needs further development.

A Torsion And Torsion

Daniel S. Freed[†]

In Remark 48.39 and in the discussion below equation (52.25), we made use of a result relating Reidemeister torsion to an alternating product of the orders of torsion groups. This Appendix gives a proof of the relevant mathematical result. It is possible that Theorem A.19 is known, but the authors have not found a reference. If someone knows a reference they would be happy to hear about it.

A.1 Conventions, Terminology, Exterior Powers, Determinants

Throughout, all groups F are finitely generated abelian, and we don't mention these properties further. Some groups are also free, which is equivalent to torsionfree. A subgroup of a free group is free. An element $f \in F$ is *indivisible* if whenever $f = n \cdot f'$ for $n \in \mathbb{Z}$, $f' \in F$, then $n = \pm 1$. If F is free of rank-one, then an indivisible element is a *generator*; there are precisely two generators. A general (finitely generated Abelian) F sits in a group extension

$$0 \longrightarrow \text{Tors } F \longrightarrow F \longrightarrow \text{Free } F \longrightarrow 0 \quad (\text{A.1})$$

with the torsion subgroup and free quotient. If F is free of rank r , then the exterior powers $\Lambda^k F$, $k = 0, \dots, r$, are defined and are free. The top exterior power $\text{Det } F = \Lambda^r F$ is free of rank 1. We denote by ' π_F ' a choice of generator of $\text{Det } F$. In Theorem A.19 below, the sign choice will cancel out.

Let

$$C^\bullet : 0 \longrightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} \dots \xrightarrow{d} C^n \longrightarrow 0 \quad (\text{A.2})$$

be a bounded cochain complex of free groups. Its determinant line is defined as

$$\text{Det } C^\bullet = \bigotimes_{i=0}^n (\text{Det } C^i)^{\otimes (-1)^i}. \quad (\text{A.3})$$

Introduce the notations

$$\begin{aligned} C_{\mathbb{Q}}^i &= C^i \otimes_{\mathbb{Z}} \mathbb{Q}, \\ H^i &= H^i(C), \\ H_{\mathbb{Q}}^i &= H^i(C_{\mathbb{Q}}), \\ N^i &= \#\text{Tors } H^i \\ N^\bullet &= \prod_{i=0}^n (N^i)^{(-1)^i}. \end{aligned} \quad (\text{A.4})$$

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and similar notations with \mathbb{R} replacing \mathbb{Q} . Note that $\text{Free } H^\bullet$ is a complex of free groups with zero differential, and so $\text{Det Free } H^\bullet$ is defined; it is a lattice in the one-dimensional \mathbb{Q} -vector space $\text{Det } H_{\mathbb{Q}}^\bullet$, as well as in the one-dimensional \mathbb{R} -vector space $\text{Det } H_{\mathbb{R}}^\bullet$.

A.2 Plücker Elements And Quotients

Suppose F is free and $A \subset F$ is a subgroup of rank r . We call a choice of generator $\pi_A \in \text{Det } A \subset \bigwedge^r F$ a *Plücker element* for A . Fix this choice.

Lemma A.5.

- (1) Suppose F/A is free, $\pi_{F/A} \in \text{Det } F/A$ is a choice of generator, and $\tilde{\pi}_{F/A} \in \bigwedge^\bullet F$ is a lift of $\pi_{F/A}$ to a decomposable vector. Then $\tilde{\pi}_{F/A} \wedge \pi_A \in \text{Det } F$ is a generator.
- (2) For general F/A , choose a generator $\pi_{F/A} \in \text{Det Free}(F/A)$. Then $\tilde{\pi}_{F/A} \wedge \pi_A \in \text{Det } F$ is N times a generator, where $N = \#\text{Tors } F/A$.

Observe that the wedge product $\tilde{\pi}_{F/A} \wedge \pi_A$ is independent of the lift of $\pi_{F/A}$.

Proof. Assume F/A is free. Then the short exact sequence

$$0 \longrightarrow A \longrightarrow F \longrightarrow F/A \longrightarrow 0 \quad (\text{A.6})$$

of free abelian groups leads to the isomorphism

$$\begin{aligned} \text{Det } F/A \otimes \text{Det } A &\longrightarrow \text{Det } F \\ \pi_{F/A} \otimes \pi_A &\longmapsto \tilde{\pi}_{F/A} \wedge \pi_A \end{aligned} \quad (\text{A.7})$$

of determinant lines. Assertion (1) follows immediately.

In general, define $B \subset F$ by the diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & A & \xlongequal{\quad} & A & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & F & \longrightarrow & F/B \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & \text{Tors } F/A & \longrightarrow & F/A & \longrightarrow & \text{Free } F/A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \quad (\text{A.8})$$

in which the rows and columns are exact. Noting that F/B is free, apply (A.7) to the middle row of (A.8) to deduce the isomorphism

$$\begin{aligned} \text{Det } F/B \otimes \text{Det } B &\longrightarrow \text{Det } F \\ \pi_{F/B} \otimes \pi_B &\longmapsto \tilde{\pi}_{F/B} \wedge \pi_B \end{aligned} \tag{A.9}$$

It remains to observe that A and B are equirank and that the map $\text{Det } A \longrightarrow \text{Det } B$ induced by the inclusion takes a generator to N times a generator.

□

A.3 Coboundaries And Cocycles

Given a chain complex (A.2), define as usual the (free) subgroups of *coboundaries* and *cocycles*:

$$B^i \subset Z^i \subset C^i. \tag{A.10}$$

The *cohomology* is $H^i = Z^i/B^i$.

Lemma A.11. *The quotient C^i/Z^i is free.*

Proof. If $c \in C^i$, $n \in \mathbb{Z}^{\neq 0}$, and $0 = d(nc) = n dc$, then $dc = 0$ since C^{i+1} is free. Hence $c \in Z^i$. □

A.4 Plücker Elements

Recalling (A.10), choose generators

$$\begin{aligned} \pi_{B^i} &\in \text{Det } B^i, \\ \pi_{H^i} &\in \text{Det Free } H^i, \\ \pi_{C^i/Z^i} &\in \text{Det } C^i/Z^i. \end{aligned} \tag{A.12}$$

and choose lifts (Plücker elements)

$$\begin{aligned} \tilde{\pi}_{H^i} &\in \Lambda^\bullet C^i, \\ \tilde{\pi}_{C^i/Z^i} &\in \Lambda^\bullet C^i. \end{aligned} \tag{A.13}$$

Since the differential induces an isomorphism $C^{i-1}/Z^{i-1} \rightarrow B^i$, we have the following.

Lemma A.14. *The relation*

$$d(\tilde{\pi}_{C^{i-1}/Z^{i-1}}) = \pm \pi_{B^i}, \tag{A.15}$$

holds.

Fix the sign choices of π_{B^i} in (A.12) so that (A.15) holds with + for each i .

A.5 The Determinant And The Main Theorem

Definition A.16. The **determinant** of the rationalization of the cochain complex (A.2) is the isomorphism

$$\begin{aligned} \det d_{\mathbb{Q}} : \text{Det } H_{\mathbb{Q}}^{\bullet} &\longrightarrow \text{Det } C_{\mathbb{Q}}^{\bullet} \\ \bigotimes_{i=0}^n (\pi_{H^i})^{(-1)^i} &\longmapsto \bigotimes_{i=0}^n (\tilde{\pi}_{C^i/Z^i} \wedge \tilde{\pi}_{H^i} \wedge \pi_{B^i})^{(-1)^i}. \end{aligned} \quad (\text{A.17})$$

The reader should check that this map is well-defined independent of all choices (subject to the convention after the statement of [Theorem A.14](#)).

Remark A.18. This determinant is called the *torsion* of the cochain complex $(C_{\mathbb{Q}}^{\bullet}, d_{\mathbb{Q}})$.

Theorem A.19. *The determinant $\det d_{\mathbb{Q}}$ maps a generator of $\text{Det Free } H^{\bullet}$ to N^{\bullet} times a generator of $\text{Det } C_{\mathbb{Q}}^{\bullet}$.*

Proof. Apply [Theorem A.5\(2\)](#) to $B^i \subset Z^i$ to deduce that $\tilde{\pi}_{H^i} \wedge \pi_{B^i}$ is N^i times a generator of $\text{Det } Z^i$. Then apply [Theorem A.5\(1\)](#) to the inclusion $Z^i \subset C^i$ to conclude. \square

We obtain an equivalent statement by modifying the determinant to the map

$$\begin{aligned} \widetilde{\det} d_{\mathbb{Q}} : \text{Det } H_{\mathbb{Q}}^{\bullet} &\longrightarrow \text{Det } C_{\mathbb{Q}}^{\bullet} \\ \bigotimes_{i=0}^n (N^i \pi_{H^i})^{(-1)^i} &\longmapsto \bigotimes_{i=0}^n (\tilde{\pi}_{C^i/Z^i} \wedge \tilde{\pi}_{H^i} \wedge \pi_{B^i})^{(-1)^i}. \end{aligned} \quad (\text{A.20})$$

Corollary A.21. $\widetilde{\det} d_{\mathbb{Q}}$ maps a generator of $\text{Det Free } H^{\bullet}$ to a generator of $\text{Det } C_{\mathbb{Q}}^{\bullet}$.

A.6 The Real Determinant

The two generators of $\text{Det } C^{\bullet}$, which differ by a sign, determine a norm on the real determinant line $\text{Det } C_{\mathbb{R}}^{\bullet}$. The real version $\det d_{\mathbb{R}}$ of (A.17) transports it to a norm on $\text{Det } H_{\mathbb{R}}^{\bullet}$. On the other hand, the integral lattice $\text{Det Free } H^{\bullet}$ also determines a norm on $\text{Det } H_{\mathbb{R}}^{\bullet}$. [Theorem A.19](#) asserts that the ratio of these norms is N^{\bullet} .

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