2) 9) $X \in \mathbb{R}^2$ vector aleat.

G Discreta con $R_G = \{1, 2\}$ $\quad P(G=1) = \frac{3}{4}$

$P(G=2) = \frac{1}{4}$

$X \mid G=1 \sim N(\mu_1, \Sigma)$ $\quad \mu_1 = (1 \;\; 1/2)^t$

$X \mid G=2 \sim N(\mu_2, \Sigma)$ $\quad \mu_2 = (-1/2 \;\; 1)^t$

$$\Sigma = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

a) $\underset{\substack{\downarrow \\ \text{vector}}}{f_X}(\overset{\substack{\text{vector} \\ \uparrow}}{x}) = f_{X|G=1}(x) \cdot P(G=1) + f_{X|G=2}(x) \; P(G=2)$

$$= \frac{1}{2\pi \left[\det(\Sigma)\right]^{1/2}} \cdot \exp\left\{-\frac{1}{2}(x-\mu_1)^t \Sigma^{-1}(x-\mu_1)\right\} \cdot \frac{3}{4}$$

$$+ \frac{1}{2\pi \left[\det(\Sigma)\right]^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu_2)^t \Sigma^{-1}(x-\mu_2)\right\} \frac{1}{4}$$

$$= \frac{1}{8\pi \left[\det(\Sigma)\right]^{1/2}} \left[3 \exp\left\{-\frac{1}{2}(x-\mu_1)^t \Sigma^{-1}(x-\mu_1)\right\} + \exp\left\{-\frac{1}{2}(x-\mu_2)^t \Sigma^{-1}(x-\mu_2)\right\}\right]$$

$$\Sigma = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \Rightarrow \det(\Sigma) = 1 - 1/4 = 3/4$$

$$\Sigma^{-1} = \begin{pmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{pmatrix}$$

$$\Rightarrow f_x(x) = \underbrace{\frac{1}{8\pi\sqrt{3/4}}}_{\frac{1}{4\pi\sqrt{3}} = \frac{\sqrt{3}}{12\pi}} \left[3\exp\left\{-\frac{1}{2}(x-\mu_1)^t \Sigma^{-1}(x-\mu_1)\right\} + \exp\left\{-\frac{1}{2}(x-\mu_2)^t \Sigma^{-1}(x-\mu_2)\right\}\right]$$

<u>GENTAI.</u>

- $\mu_x = \pi_1 \cdot \mu_1 + \pi_2 \cdot \mu_2 = \frac{3}{4}(1 \;\; 1/2)^t + \frac{1}{4}(-1/2 \;\; 1)^t = \begin{pmatrix} 3/4 \\ 3/8 \end{pmatrix} + \begin{pmatrix} -1/8 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 5/8 \\ 5/8 \end{pmatrix}$

- Σ_x

- $\Sigma_W = \pi_1 \Sigma_1 + \pi_2 \Sigma_2 = \frac{3}{4}\Sigma + \frac{1}{4}\Sigma = \Sigma$.
  $\Sigma_1 = \Sigma_2 = \Sigma$

- $\Sigma_B = \pi_1(\mu_1 - \bar{\mu})(\mu_1 - \bar{\mu})^t + \pi_2(\mu_2 - \bar{\mu})(\mu_2 - \bar{\mu})^t$

$$= \frac{3}{4}\begin{pmatrix} 3/8 \\ -1/8 \end{pmatrix}\begin{pmatrix} 3/8 & -1/8 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -9/8 \\ 3/8 \end{pmatrix}\begin{pmatrix} -\frac{9}{8} & \frac{3}{8} \end{pmatrix}$$

$$= \frac{3}{4}\begin{pmatrix} 9/64 & -3/64 \\ -3/64 & 1/64 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 81/64 & -\frac{27}{64} \\ -\frac{27}{64} & 9/64 \end{pmatrix}$$

b)

$$z = (z_1 \;\; z_2)^t = A^t(x - \mu_x)$$

con $A = (\alpha_1 \;\; \alpha_2)$ $\qquad \alpha_j = c^{-1} B_j$

$$\Sigma_W = \Sigma = \left(\Sigma^{1/2}\right)^t \cdot \Sigma^{1/2}$$

$$\Sigma^{1/2} = U \Lambda^{1/2} U^t$$

$$\left(\Sigma^{1/2}\right)^t \cdot \Sigma^{1/2} =$$

$$U \Lambda^{1/2} \underbrace{U^t \cdot U}_{Id} \Lambda^{1/2} U^t = U \Lambda U^t$$

$$= \Sigma$$

$$B = \left(\Sigma^{-1/2}\right)^t \Sigma_B \left(\Sigma^{1/2}\right)^t$$

$\downarrow$

β_1, β_2 autovect. de B con $\lambda_1 \geq \lambda_2$.

$$\Rightarrow \alpha_j = \Sigma^{-1/2} \cdot \beta_j$$

- $\Sigma_z = Var\left(A^t (x - \mu_x)\right) = A^t \, Var(X) \, A$

$$= A^t \, \Sigma_x \, A$$

$$\Sigma_x = \Sigma_W + \Sigma_B$$

$$\Sigma_z = \underbrace{A^t \Sigma_W A}_{\Sigma_{z,W}} + \underbrace{A^t \Sigma_B A}_{\Sigma_{z,B}}.$$