Solution 1 (Exercise 4.29)**.** *Let $SL(2,5)$ denote the group of 2×2 matrices over $\mathbb{Z}_5$ with determinant 1.*

(i) To compute $|SL(2,5)|$, recall that

$$|\mathrm{GL}_2(\mathbb{F}_5)| = (5^2-1)(5^2-5) = 24 \cdot 20 = 480.$$

The determinant map $\det : \mathrm{GL}_2(\mathbb{F}_5) \to \mathbb{F}_5^*$ is a surjective group homomorphism with kernel $SL(2,5)$. Since $|\mathbb{F}_5^*| = 4$, we get

$$|SL(2,5)| = \frac{|\mathrm{GL}_2(\mathbb{F}_5)|}{|\mathbb{F}_5^*|} = \frac{480}{4} = 120.$$

(ii) We know that $SL(2,5)$ has center $Z(SL(2,5)) = \{\pm I\}$. Note that $-I$ is the only element of order 2 in this group, i.e., the only involution. Indeed, any other matrix with determinant 1 and square equal to the identity must lie in the center, but the center has only two elements. Thus, there is a unique involution.

Let P be a Sylow 2-subgroup of $SL(2,5)$. Then $|P| = 8$, and it must contain the unique involution $-I$. But the quaternion group Q_8 is the only group of order 8 with a unique involution. Hence, $P \cong Q_8$.

(iii) In contrast, consider S_5. Let it act naturally on $\{1,2,3,4,5\}$. The symmetries of a square (say, acting on $\{1,2,3,4\}$) form a subgroup isomorphic to the dihedral group D_4 of order 8. This subgroup lies inside S_5 and is a Sylow 2-subgroup, since 8 divides $120 = |S_5|$, and no larger power of 2 does. All Sylow 2-subgroups are conjugate, so every Sylow 2-subgroup of S_5 is isomorphic to D_4. Therefore, $SL(2,5)$ and S_5 cannot be isomorphic, since one has a Sylow 2-subgroup isomorphic to Q_8 and the other to D_4.

(iv) Since $|SL(2,5)| = 120$ and $|A_5| = 60$, it is natural to ask whether A_5 embeds into $SL(2,5)$. But we already know that

$$SL(2,5)/Z(SL(2,5)) \cong PSL(2,5) \cong A_5.$$

Thus, if A_5 were to embed into $SL(2,5)$, the composition with the projection would give an automorphism of A_5. This would split the short exact sequence

$$1 \to Z(SL(2,5)) \to SL(2,5) \to A_5 \to 1,$$

giving a section $A_5 \hookrightarrow SL(2,5)$. But this sequence does not split, so no such embedding exists. Hence, A_5 cannot be embedded in $SL(2,5)$.