Anisotropic Variants of the Heisenberg-Pauli-Weyl inequality

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July 2021

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1 Introduction

The Heisenberg-Pauli-Weyl Inequality is an important result in functional analysis with applications in signal processing, information theory, and famously, quantum physics, where it is known simply as the uncertainty principle.

For such a fundamental result, it has a remarkably simple proof using only basic calculus. In this report, we hope to review the proof of the uncertainty principle in an accessible manner and explore the mathematical properties of similar inequalities.

2 The Inequality in $\mathbb{R}^n$

2.1 Compact Support Proof

Now that we have some context, let's prove the inequality. We hope you'll find the technique used quite elegant. First, we need to assume that f has compact support. We will generalize later.

Theorem 1 (Heisenberg-Pauli-Weyl Inequality for nice f)**.** Let f be a function $\mathbb{R}^n \to \mathbb{R}$ such that $f \in C^1(\mathbb{R}^n)$, f has compact support, $f \in L^2(\mathbb{R}^n)$, and $|\nabla f| \in L^2(\mathbb{R}^n)$. Then we have:

$$\left(\int_{\mathbb{R}^n} |x|^2|f(x)|^2\,dx\right)\left(\int_{\mathbb{R}^n} |\nabla f(x)|^2\,dx\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{R}^n} |f(x)|^2\,dx\right)^2$$

Proof. We start with a clever trick. For any real number λ, we can write the following inequality:

$$0 \leq \int_{\mathbb{R}^n} |\lambda\nabla f(x) - xf(x)|^2\,dx$$

Now let's expand the right hand side. We get:

$$\begin{aligned} &= \int_{\mathbb{R}^n} \big\langle \lambda\nabla f(x) - xf(x),\, \lambda\nabla f(x) - xf(x)\big\rangle\,dx \\ 0 &\leq \int_{\mathbb{R}^n} \lambda^2|\nabla f(x)|^2 + |x|^2(f(x))^2 - 2\lambda\big\langle\nabla f(x),\, xf(x)\big\rangle\,dx \qquad (1) \end{aligned}$$

This looks quite hopeless, but in fact we can simplify the rightmost term of the integral: We have:

$$\begin{aligned} 2\int_{\mathbb{R}^n}\big\langle\nabla f(x),\, xf(x)\big\rangle\,dx &= \int_{\mathbb{R}^n} 2f(x)Df(x)x\,dx && \text{(since } f(x) \text{ is a real number)} \\ &= \int_{\mathbb{R}^n} D\big(x \to x^2\big)\big(f(x)\big)Df(x)x\,dx \\ &= \int_{\mathbb{R}^n} D(f^2)(x)x\,dx && \text{(chain rule in reverse)} \\ &= \int_{\mathbb{R}^n} \big\langle\nabla(f^2)(x),\, x\big\rangle\,dx \end{aligned}$$

Now we'd like to use integration by parts, but first we need Fubini:

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \langle \nabla(f^2)(x),\, x\rangle\, dx \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^n} x_i\, \partial_i f^2(x)\, dx \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} x_i\, \partial_i f^2(y, x_i)\, dx_i\, dy \qquad\qquad \text{(where } x = (y, x_i)\text{)} \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^{n-1}} \left(\lim_{a\to\infty} \left| x_i\, f^2(y, x_i) \right|_{x_i=-a}^{a} - \int_{\mathbb{R}} f^2(x)\, dx_i \right) dy \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(integration by parts)}
\end{aligned}
$$

Because f has compact support, the boundary limit goes to zero. We can deduce the following:

$$
\begin{aligned}
2\int_{\mathbb{R}^n} \langle \nabla f(x),\, x f(x)\rangle\, dx &= -\sum_{i=1}^{n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f^2(x)\, dx_i\, dy \\
&= -n \int_{\mathbb{R}^n} f^2(x)\, dx \qquad (2)
\end{aligned}
$$

Now, substituting this result back into (1) gives us:

$$
0 \le \lambda^2 \int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx + n\lambda \int_{\mathbb{R}^n} f^2(x)\, dx + \int_{\mathbb{R}^n} |x|^2 (f(x))^2\, dx
$$

Curiously, this is polynomial in terms of λ.

$$
0 \le \lambda^2 \int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx + n\lambda \int_{\mathbb{R}^n} f^2(x)\, dx + \int_{\mathbb{R}^n} |x|^2 (f(x))^2\, dx
$$

The inequality tells us the the discriminant of the polynomial must be ≤ 0.

$$
\left(n \int_{\mathbb{R}^n} f^2(x)\, dx \right)^2 - 4 \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx \right) \left(\int_{\mathbb{R}^n} |x|^2 (f(x))^2\, dx \right) \le 0
$$

Upon rearranging, this gives us exactly the inequality we were after! $\square$

2.2 Conditions

If the function f has compact support, we can be sure that boundary conditions vanish in the integration by parts, that is, the limit

$$
\lim_{a\to\infty} \left| f^2(y, x_i) x_i \right|_{-a}^{a}
$$

goes to zero. However, for more general functions, this limit is tricky.

Can we find a condition that will guarantee well behaved boundaries for functions without compact support? Is smoothness sufficient?

Unfortunately, no. Consider the following in one dimension (which generalizes easily to $\mathbb{R}^n$ as we can rotate the function):

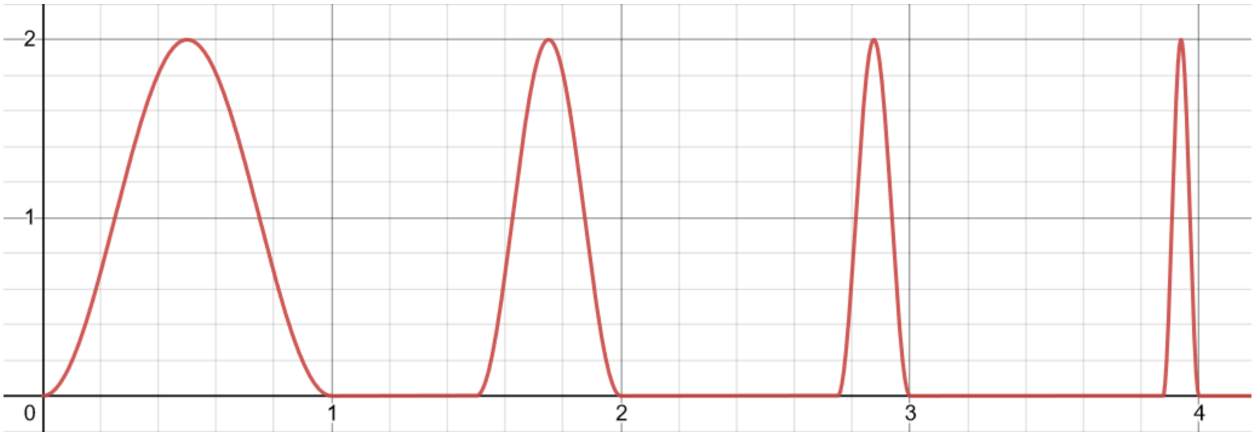

Figure 1: A smooth $f \in L^2(\mathbb{R}^n)$ such that $\lim_{x\to\infty} f(x) \neq 0$

Example 2. The plot shows f^2 where the area of the peaks forms a geometric series: $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$. The function is square integrable since the limit converges, but the limit does not exist. We could even make this graph unbounded.

It turns out that even smooth functions aren't "nice" enough! However, what happens if we also demand $f' \in L^2$?

It turns out that if the derivative if square integrable, we get a nice lemma.

Lemma 3. Let f be a function from $\mathbb{R} \to \mathbb{R}$ and suppose both f and f' are in $L^2(\mathbb{R})$. Then:

$$\lim_{x\to\infty} f(x) = 0$$

This is not quite what we want, but it's very close!

Proof. Let f and f' be square integrable. Also, for now, let's assume f is smooth. First, the proof of $-\infty$ works exactly the same as this proof. WLOG Assume for the sake of contradiction that: $\lim\limits_{x\to\infty} f \neq 0$

So, there exists a sequence of points $\{x_j\}_{j=0}^{\infty}$ such that for each x_j we have $|f(x_j)| > \delta$ for some $\delta > 0$. Moreover, note that $x_{j+1} - x_j$ can be made as large as we wish.

Since f' is square integrable, for any $\epsilon > 0$, we can pick an M large enough such that:

$$\int_M^{\infty} \left(f'(x)\right)^2 dx < \epsilon$$

Now, we can use Cauchy Schwartz to write for $x > M$ that:

$$\begin{aligned}
|f(x+y) - f(x)| &= \left| \int_x^{x+y} 1 \cdot f'(t)\, dt \right| \\
&\leq \left(\int_x^{x+y} 1^2\, dt \right)^{\frac{1}{2}} \left(\int_x^{x+y} \left(f'(t)\right)^2 dt \right)^{\frac{1}{2}} \\
&\leq y^{\frac{1}{2}} \left(\int_M^{\infty} \left(f'(x)\right)^2 dx \right)^{\frac{1}{2}} \\
|f(x+y) - f(x)| &< y^{\frac{1}{2}} \epsilon^{\frac{1}{2}}
\end{aligned}$$

Now we can lower bound the integral of f^2. On $(x_j, x_j + y)$ we have that $|f|$ is bounded below by:

$$|f(x)| \geq \delta - y^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \text{ for } x \in (x_j, x_j + y)$$

Likewise, we can bound the square integral of f:

$$\int_{x_j}^{x_j+y} |f(x)|^2\, dx \geq y(\delta - y^{\frac{1}{2}} \epsilon^{\frac{1}{2}})^2$$

Finally, we can pick $y = 1$ and $\epsilon = \delta^2/4$ to contradict $f \in L^2$:

$$\int_{\mathbb{R}} |f(x)|^2\, dx \geq \sum_{j=0}^{\infty} \int_{x_j}^{x_j+1} |f(x)|^2\, dx \geq \sum_{j=0}^{\infty} \frac{\delta^2}{4} = \infty$$

This works since we took $x_{j+1} - x_j > 1$ in our initial assumption about the sequence. $\square$

2.3 General Function Proof

First, notice that if f had compact support, that is, if there's some B such that for $|x| > B$ we have $f(x) = 0$, the integration by parts is completely fine. We can extend the result from functions with compact support to the general case through approximation. Here's how.

Theorem 4 (Heisenberg-Pauli-Weyl Inequality)**.** Let f be a function $\mathbb{R}^n \to \mathbb{R}$ such that $f \in C^1(\mathbb{R}^n)$, $f \in L^2(\mathbb{R}^n)$, and $|\nabla f| \in L^2(\mathbb{R}^n)$. Then we have:

$$\left(\int_{\mathbb{R}^n} |x|^2 |f(x)|^2\, dx \right) \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2\, dx \right) \geq \frac{n^2}{4} \left(\int_{\mathbb{R}^n} |f(x)|^2\, dx \right)^2$$

Let $\xi_B(x)$ be a smooth bell-shaped function like the one pictured below.

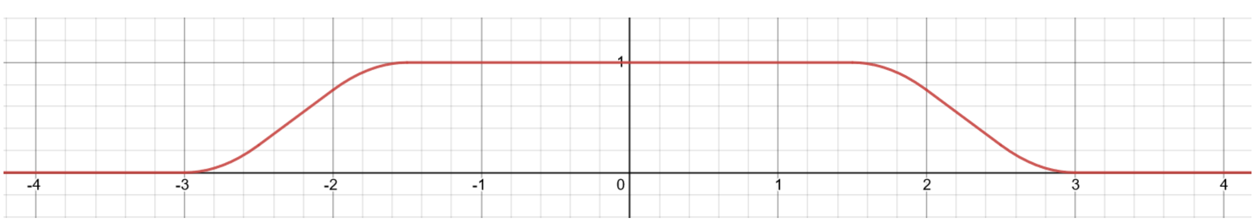

Figure 2: $\xi_2(x) \in R$

Take $\xi_B(x)$ so it is 1 on the origin ball of radius B, it is 0 outside the origin ball of radius $B+1$, and has $\left\|\nabla \xi_B(x)\right\| < C$ for some fixed C.

Lemma 5. The following approach 0:

1. $\lim\limits_{B\to\infty} \int_{\mathbb{R}^n} f^2 \xi_B(x)(1-\xi_B(x))dx$
2. $\lim\limits_{B\to\infty} \int_{\mathbb{R}^n} f^2 (1-\xi_B(x))^2 dx$
3. $\lim\limits_{B\to\infty} \int_{\mathbb{R}^n} f \xi_B(x) \langle \nabla f(x), \nabla \xi_B(x) \rangle dx$
4. $\lim\limits_{B\to\infty} \int_{\mathbb{R}^n} f^2 |\nabla \xi_B(x)|^2$

Proof. Consider the second term. Call $g_B = f^2(1-\xi_1(x))^2 - f^2(1-\xi_B(x))^2$. If $I \geq J$, then $g_I(x) \geq g_J(x)$, and $\lim_{B\to\infty} g_B = f^2(1-\xi_1(x))^2$ pointwise. This limit satisfies

$$\int_{\mathbb{R}^n} f^2(1-\xi_1(x))^2 dx \leq \int_{\mathbb{R}^n} f^2 dx < \infty$$

since

$$f \in L^2(\mathbb{R}^n)$$

Hence by monotone convergence theorem,

$$\lim_{B\to\infty} \int_{\mathbb{R}^n} g_B dx = \int_{\mathbb{R}^n} f^2(1-\xi_1(x))^2 dx$$

so

$$\lim_{B\to\infty} \int_{\mathbb{R}^n} f^2(1-\xi_1(x))^2 - f^2(1-\xi_B(x))^2 dx = \int_{\mathbb{R}^n} f^2(1-\xi_1(x))^2$$

meaning

$$\lim_{B\to\infty} \int_{\mathbb{R}^n} f^2(1-\xi_B(x))^2 dx = 0$$

so it goes to 0.

Consider the first term.

$$f^2 \xi_B(x)(1-\xi_B(x))$$

goes pointwise to 0. Further,

$$|f^2 \xi_B(x)(1-\xi_B(x))| \leq f^2$$

and clearly f^2 is integrable. By dominated convergence theorem, we get that

$$\int_{\mathbb{R}^n} f^2 \xi_B(x)(1-\xi_B(x))dx$$

goes to 0.

By a very similar MCT/DCT argument as we did earlier, it can be shown last two terms go to 0 as B gets large. This works since $\nabla \xi_B(x)$ is bounded. $\square$

Let's prove the actual theorem now.

Proof. Write $f = f\xi_B(x) + f(1-\xi_B(x))$, which means $f\xi_B(x)$ is compactly supported. We have:

$$\int_{\mathbb{R}^n} f^2 dx = \int_{\mathbb{R}^n} (f\xi_B(x) + f(1-\xi_B(x)))^2 dx =$$

$$\int_{\mathbb{R}^n} f^2\xi_B(x)^2 dx + 2\int_{\mathbb{R}^n} f^2\xi_B(x)(1-\xi_B(x))dx + \int_{\mathbb{R}^n} f^2(1-\xi_B(x))^2 dx$$

The above analysis in lemma 6 gives

$$\lim_{B\to\infty} \int_{\mathbb{R}^n} f^2\xi_B(x)^2 dx = \int_{\mathbb{R}^n} f^2 dx$$

Note since $f\xi_B(x)$ has compact support, by our original proof, we have

$$\left(\int_{\mathbb{R}^n} |x|^2|f\xi_B(x)|^2\, dx\right)\left(\int_{\mathbb{R}^n} |\nabla f\xi_B(x)|^2\, dx\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{R}^n} |f\xi_B(x)|^2\, dx\right)^2$$

With $|\xi_B(x)| \leq 1$,

$$\int_{\mathbb{R}^n} |x|^2|f\xi_B(x)|^2\, dx \leq \int_{\mathbb{R}^n} |x|^2|f|^2\, dx$$

Finally, expanding gives that

$$\int_{\mathbb{R}^n} |\nabla(f\xi_B(x))|^2 dx = \int_{\mathbb{R}^n} \xi_B(x)^2|\nabla f|^2 dx + \int_{\mathbb{R}^n} f^2|\nabla\xi_B(x)|^2 dx$$

$$+2\int_{\mathbb{R}^n} f\xi_B(x)\big\langle\nabla f(x),\, \nabla\xi_B(x)\big\rangle dx$$

By lemma 6, the last two terms go to 0. So

$$\lim_{B\to\infty} \int_{\mathbb{R}^n} |\nabla(f\xi_B(x))|^2 dx = \lim_{B\to\infty} \int_{\mathbb{R}^n} \xi_B(x)^2|\nabla f|^2 dx$$

$$\leq \lim_{B\to\infty} \int_{\mathbb{R}^n} |\nabla f|^2 dx = \int_{\mathbb{R}^n} |\nabla f|^2 dx$$

So combining inequalities:

$$\int_{\mathbb{R}^n} f^2 dx = \lim_{B\to\infty} \int_{\mathbb{R}^n} f^2\xi_B(x)^2 dx$$

$$\leq \frac{2}{n} \lim_{B\to\infty} \left(\int_{\mathbb{R}^n} |x|^2|f\xi_B(x)|^2\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla(f\xi_B(x))|^2\, dx\right)^{\frac{1}{2}}$$

$$\leq \frac{2}{n}\left(\int_{\mathbb{R}^n} |x|^2|f|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla f|^2 dx\right)^{\frac{1}{2}}$$

as desired. □

2.4 Equality Case

The equality case is a transformed Gaussian. Recall we started the proof with the inequality:

$$0 \le \int_{\mathbb{R}^n} |\lambda \nabla f(x) - x f(x)|^2 \, dx$$

It suffices to take the term inside the absolute value to be 0.

$$\lambda \nabla f(x) = x f(x)$$

So for each i, we need

$$f(x) x_i = \lambda \partial_i f$$

which means

$$\frac{\partial_i f}{f} = \frac{x_i}{\lambda}$$

Now integrating on both sides and solving everything simultaneously over all i gives

$$f = C e^{\frac{\|x\|^2}{2\lambda}}$$

as the equality case with $\lambda < 0$ and $C \in \mathbb{R}$.

A final note: our proof above essentially repeats the proof of Cauchy Schwarz when we apply the discriminant. We could've applied Cauchy Schwarz directly for a lower bound:

$$\left(\int_{\mathbb{R}^n} |x|^2 |f(x)|^2 \, dx \right) \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx \right) \ge \left(\int_{\mathbb{R}^n} \langle \nabla f(x), \, x f(x) \rangle dx \right)^2$$

At this point, we can repeat the same proof from before.

3 Connection to the Fourier Transform

Before we explore some other inequalities, let's see the connection of the Heisenberg inequality to the Fourier transform.

Recall the definition of the Fourier transform:

Definition 6 (Fourier Transform)**.**

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x) \, dx$$

Also, recall that Parseval's Theorem states:

Fact 7 (Parseval)**.** For nice enough f:

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2 \, d\omega$$

Lemma 8. Given a square integrable f with compact support and in $C^1(\mathbb{R}^n)$, we have:

$$\widehat{f'}(\omega) = i\omega\widehat{f}$$

Proof. Using integration by parts together with lemma 3, we get:

$$\begin{aligned}
\widehat{f'}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega x} f'(x)\, dx \\
&= \lim_{a\to\infty} \left| e^{-i\omega x} f(x) \right|_{-a}^{a} - \int_{-\infty}^{\infty} -i\omega e^{-i\omega x} f(x)\, dx \\
&= i\omega\widehat{f}
\end{aligned}$$

□

Now consider the 1D form of the inequality:

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} |f'(x)|^2\, dx\right) \geq \frac{1}{4}\left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right)^2$$

Using Parseval's theorem and lemma 9, we have:

$$\int_{-\infty}^{\infty} |f'(x)|^2\, dx = \int_{-\infty}^{\infty} |\widehat{f'}(\omega)|^2\, d\omega = \int_{-\infty}^{\infty} \omega^2|\widehat{f}(\omega)|^2\, d\omega$$

Thus, we can rewrite Heisenberg's inequality in terms of the Fourier transform of f:

$$\left(\int_{-\infty}^{\infty} x^2|f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} \omega^2|\widehat{f}(\omega)|^2\, dx\right) \geq \frac{1}{4}\left(\int_{-\infty}^{\infty} |f(x)|^2\, dx\right)^2$$

Our above inequality holds for any pair of conjugate variables $\widehat{f}(\omega)$ and $f(x)$. In practice, the wave functions for pairs of variables are fourier transform duals (and are therefore conjugate): linear momentum and linear position; angular momentum and angular position; energy and time; electric potential and density of free charge.

Plugging in these for f and $\widehat{f}$ gives us a few of the key Heisenberg uncertainty inequalities. In fact, conjugate variables are known as incompatible observables for exactly this reason.

4 Anisotropic Generalization in $\mathbb{R}^n$

4.1 Generalization in $\mathbb{R}^2$

Theorem 9. Let's fix $a, b, c, d > 0$. $C(a,b,c,d) = \frac{(ac+bd)^2}{4}$ is the largest such constant such that the below inequality will be satisfied (for all u already satisfying the conditions for the Heisenberg-Pauli-Weyl inequality):

$$\left(\int_{\mathbb{R}^2} |a^2x_1^2 + b^2x_2^2||u(x)|^2dx\right)\left(\int_{\mathbb{R}^2} c^2(\frac{\partial u}{\partial x_1})^2 + d^2(\frac{\partial u}{\partial x_2})^2dx\right)$$

$$\geq C(a,b,c,d)\left(\int_{\mathbb{R}^2} |u(x)|^2dx\right)^2$$

Proof. By the trivial inequality,

$$0 \leq \int_{\mathbb{R}^2} (c\lambda\frac{\partial u}{\partial x_1} - ax_1u)^2 + (d\lambda\frac{\partial u}{\partial x_2} - bx_2u)^2dx$$

Expanding gives that

$$0 \leq \lambda^2\int_{\mathbb{R}^2}(c^2(\frac{\partial u}{\partial x_1})^2 + d^2(\frac{\partial u}{\partial x_2})^2) - 2\lambda\int_{\mathbb{R}^2}(ac\,ux_1\frac{\partial u}{\partial x_1} + bd\,ux_2\frac{\partial u}{\partial x_2}u)dx + \int_{\mathbb{R}^2}(a^2x_1^2 + b^2x_2^2)u^2dx$$

Let's concentrate on the middle term. By Fubini, we can split up the integral as:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} ac\,ux_1\frac{\partial u}{\partial x_1}dx_1dx_2 + \int_{\mathbb{R}}\int_{\mathbb{R}} bd\,ux_2\frac{\partial u}{\partial x_2}dx_2dx_1$$

We can apply a similar argument as we did for the original inequality with integration by parts since we can assume this works for compactly supported u and generalize.

$$\frac{ac}{2}\int_{\mathbb{R}}\left(\lim_{k\to\infty}|u^2x_1|_{x_1=-k}^{x_1=k} - \int_{\mathbb{R}} u^2\,dx_1\right)dx_2 + \frac{bd}{2}\int_{\mathbb{R}}\left(\lim_{k\to\infty}|u^2x_2|_{x_2=-k}^{x_2=k} - \int_{\mathbb{R}} u^2\,dx_2\right)dx_1$$

$$= -\frac{ac+bd}{2}\left(\int_{\mathbb{R}^2}|u(x)|^2dx\right)$$

The discriminant of the quadratic in λ is less than or equal to 0, so we get the desired inequality. □

Corollary 10. Letting $a = b = c = d = 1$ gives back the original inequality in $\mathbb{R}^2$

Corollary 11. Let $a = \frac{1}{c}$ and $b = \frac{1}{d}$. If a and b get extremely large, then c and d gets correspondingly small which keeps the constant $C(a,b,c,d)$ the same.

4.2 Equality Case in $\mathbb{R}^2$

We claim that the equality case is a skewed Gaussian. In order to have equality, we need both the below equations to hold:

$$ax_1u = \lambda c\frac{\partial u}{\partial x_1}$$

and

$$bx_2u = \lambda d\frac{\partial u}{\partial x_2}$$

Solve simultaneously over $i = 1$ and $i = 2$ to get

$$u = Ce^{\frac{ax_1^2}{2c\lambda}+\frac{bx_2^2}{2d\lambda}}$$

for $C \in \mathbb{R}$ and $\lambda < 0$ as above.

Each level ellipse of the generalization is a stretched version of original 2-dim gaussian's circle level surface.

4.3 Full Generalization

Theorem 12. Fix $p_1, p_2, ..., p_n, d_1, d_2, ..., d_n > 0$. The largest C that satisfies the below inequality (for all u already satisfying the conditions for the Heisenberg-Pauli-Weyl inequality) is

$$C = (\frac{\sum_{i=1}^n p_i d_i}{2})^2$$

And

$$\left(\int_{\mathbb{R}^n} |u(x)|^2 \sum_{i=1}^n p_i^2 x_i^2 dx\right)\left(\int_{\mathbb{R}^n} \sum_{i=1}^n d_i^2 (\frac{\partial u}{\partial x_i})^2 dx\right) \geq C\left(\int_{\mathbb{R}^n} |u(x)|^2 dx\right)^2$$

Proof. The proof is very similar to the $\mathbb{R}^2$ case. Expanding

$$0 \leq \int_{\mathbb{R}^n} \sum_{i=1}^n (d_i \lambda \frac{\partial u}{\partial x_i} - p_i x_i u)^2 dx$$

and applying fubini/integration by parts on the cross terms gives a quadratic in λ. Taking the discriminant in λ to be ≤ 0 gives the desired result. $\square$