April 10

Reading J Tolar, "On Clifford groups in quantum computing"

An N state system corresponds to a hilbert space $\mathbb{C}^N$.

"Generalized Pauli Matrices" generate a group, "Weyl-Heisenberg group", semantics.

The "normalizer" of this is called the Clifford group. So I guess WH is not normal in U(N), but in Clifford group it is. That's surprising to me, but I guess it makes sense given the normal property becomes weaker the less group elements you are conjugating against. So Clifford group is the set $\{g \mid g^{-1}Xg \in WH, \forall x \in WH\}$.

"Clifford quotient group" sounds like Clifford group without scalar multiplication, which sounds good to me. $U(N)$ seems so redundant/free I will take every quotient I can get.

"Symmetries of Pauli gradings" of an algebra apparently describe some detail of clifford quotient groups, and this paper will describe something more detailed than that? No idea what a Pauli grading is.

$Q_N|j\rangle = \omega_N^j|j\rangle$, $P_N|j\rangle = |j+1\rangle$, so in 2d:

$$Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = Z$$
$$P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = X$$

These elements along with ω_N are order N, and are nearly commutative.

$$\Pi_N = \{\omega_N^i P_N^j Q_N^k\}$$

This is not $H(N)$ apparently? Do we need a generalized version of Y before this becomes the Weyl Heisenberg group? Or am I missing something.

ω_N and Q_N clearly have computational basis as their eigenvectors, being diagonal, and P_N will have $|v_i\rangle = \sum_j \omega_N^{ij}|j\rangle$ as eigenvectors, eigenvalues ω_N^i.... transforming into this basis is the discrete fourier transform! Aha! Ok back to the text. I don't know what a configuration space is or what "eigenvector of position means".

Ah yes $\tau_N = \omega_N^{\frac{1}{2}}$ lets us define Y.

$$\tau_2 P_2 Q_2 = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y$$

Then $H(N) = \{\tau_N^h \omega_N^i Q_N^j P_N^k\}$, good. $|H(N)| = 2\,|\Pi_N| = 2N^3$.

Oh this phase factor is just for even N. Fascinating. Naively that sounds like a novel thing to attack in a mixed level system?

Apparently $\tau_2 = -i$, so the equation is actually $Y = \tau_2 Q_2 P_2$

Then the centralizer is just the set of scalars $\{\tau^i\}$, and since Q_N and P_N commute, up to phase shift ω_N, quotienting by the centralizer gives the abelian group $\mathbb{Z}_N^2$. Easy.

Next we move on to the clifford group. Indeed the clifford group is the set of terms against which $H(N)$ is closed under conjugation. Since $H(N)$ is finitely generated, and τ_N, ω_N are scalar, we can simply check $XQ_NX^{-1} \in H(N)$ and $XP_NX^{-1} \in H(N)$.

Apparently these "Clifford operations" are one-step evolutions of "Clifford Gates", which makes sense.

I don't follow what the $1 \to$ and $\to 1$ have to do with the statement of how $H(N)$ maps into the clifford group and quotient group, seems like it has significance in generalized abstract nonsense world. (not meant in a derogatory way)

We don't quotient clifford by $H(N)$, however, though we could. We quotient by $U(1)$ to get a simpler space without phase factors.

Lemma: $XAX^{-1} = YAY^{-1} \iff X \propto Y$

The proof is an application of "Schur's Lemma" which I will intuit as related to the observation before that the centralizer of $H(N)$ is exactly the set of scalars. Here $Y^{-1}X$ is in the centralizer of the clifford group, and turns out to be a scalar, so $X \propto Y$.

The next paragraph at least, is very representation heavy, so I will try to understand the significance of this in my own terms first.

Two matrices conjugate any element of $H(N)$ the same way if and only if they are proportional to eachother. Since the clifford algebra is exactly the set of actions that conjugate $H(N)$ to other elements of $H(N)$, this statement can be refined to the statement that the conjugation action on $H(N)$, that is the automorphism $A \mapsto XAX^{-1}$, is equal only to the actions of scalar multiples of X. So then if we quotient the clifford group, we will end up with some group of automorphisms on $H(N)$. Wonderful.