Honours Diary 2020

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Notation

In this diary unless explicitly stated within a section, I have been using the notation specified by Nielsen and Chuang, with the following additions:

- Implicit quantifiers for index variables such as i, j, k. (Nielson and Chuang seem to do this actually, perhaps dropping more than I do)
  - $\{x_i\} = \{x_i \mid i \in I\}$, $\{|x_i\rangle\} = \{|x_i\rangle \mid i \in I\}$ etc.
  - $(x_i) = (x_1, x_2, \ldots, x_n)$
  - $\sum_i$ in place of $\sum_{i \in I}$
  - $\forall i$ in place of $\forall i \in I$

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Set up TeXstudio and basic document structure.

Exercise 2.1

Linear Dependence, show that $(1, -1)$, $(1, 2)$ and $(2, 1)$ are linearly dependent.

$$\begin{aligned} & (1, -1) + (1, 2) - (2, 1) \\ =\ & (1 + 1 - 2, -1 + 2 - 1) \\ =\ & (0, 0) \end{aligned}$$

Exercise 2.2

Matrix representations: Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle$, $|1\rangle$, and the output basis $|0\rangle$, $|1\rangle$. Find input and output bases which give rise to a different matrix representation of A.

Equation 2.12 gives us the defining property of matrix representations:

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle$$

This gives us a pair of vector equations:

$$|1\rangle = A|0\rangle = A_{00}|0\rangle + A_{10}|1\rangle$$

$$|0\rangle = A|1\rangle = A_{01}|0\rangle + A_{11}|1\rangle$$

By linear independence of $|0\rangle$, $|1\rangle$, it follows that

$$\begin{aligned} A_{00} &= 0 & A_{01} &= 1 \\ A_{10} &= 1 & A_{11} &= 0 \end{aligned}$$

i.e. A has the matrix representation:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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Exercise 2.3

$$A : V \to W$$

$$B : W \to X$$

$$V = span\{|v_i\rangle\} \ etc.$$

At this point we are already identifying A, B with their matrix representations. We would like to show equality between the function composition $B \circ A$ and the matrix product of the corresponding matrix representations, $B \times A$. Once we have done this we will be able to identify both of these concepts as simply the expression BA, but for now we will use the explicit operators $\circ$ and $\times$.

Our goal then is to show that the matrix representation of $B \circ A$ is $B \times A$.

$$\begin{aligned}
&\sum_i (B \circ A)_{ij} |x_i\rangle && \text{(arbitrary column of matrix } B \circ A) \\
&= (B \circ A)|v_j\rangle && \text{(Definition of matrix representation)} \\
&= B(A|v_j\rangle) && \text{(Definition of composition)} \\
&= B\left(\sum_k A_{kj}|w_k\rangle\right) && \text{(Matrix representation)} \\
&= \sum_k A_{kj}(B|w_k\rangle) && \text{(Linearity of B)} \\
&= \sum_k A_{kj}\left(\sum_i B_{ik}|x_i\rangle\right) && \text{(Matrix representation)} \\
&= \sum_i \left(\sum_k B_{ik}A_{kj}\right)|x_i\rangle && \text{(distribution)} \\
&= \sum_i (B \times A)_{ij}|x_i\rangle && \text{(matrix product)}
\end{aligned}$$

So by linear independence of $|x_i\rangle$ we know that $(B \circ A)_{ij} = (B \times A)_{ij}$ for arbitrary i, j, i.e. the matrix representation of $B \circ A$ is the matrix product $B \times A$.

Exercise 2.4

$$I : V \to V$$

$$I|x_i\rangle = |x_i\rangle$$

We would like to show that $I_{ij} = \delta_{ij}$.

$$\begin{aligned}
&\sum_i I_{ij}|x_i\rangle \\
=&I|x_j\rangle \\
=&|x_j\rangle \\
=&\sum_i \delta_{ij}|x_i\rangle
\end{aligned}$$

Again by linear independence of $\{|x_i\rangle\}$ we have $I_{ij} = \delta_{ij}$.

Exercise 2.5

$$((y_i), (z_i)) = \sum_i y_i^* z_i$$

We need to prove 3 properties. First linearity, taking $|v\rangle = (v_j) = (v_1, v_2, \ldots, v_n)$ and $|w_i\rangle = (w_{ij}) = (w_{i1}, w_{i2}, \ldots, w_{in})$.

For this we define

$$|z\rangle = (z_j) = \sum_i \lambda_i |w_i\rangle$$

Then by the definitions of sum and scalar product in $\mathbb{C}^n$ we observe

$$\begin{aligned}
& z_j \\
= & \left(\sum_i \lambda_i |w_i\rangle\right)_j \\
= & \sum_i \left(\lambda_i |w_i\rangle\right)_j \\
= & \sum_i \lambda_i w_{ij}
\end{aligned}$$

With this linearity falls out.

$$\begin{aligned}
& (|v\rangle, |z\rangle) \\
= & ((v_j), (z_j)) \\
= & \sum_j y_j^* z_j \\
= & \sum_j y_j^* \left(\sum_i \lambda_i w_{ij}\right) \\
= & \sum_i \lambda_i \left(\sum_j v_j^* w_{ij}\right) \\
= & \sum_i \lambda_i \left((v_j), (w_{ij})\right) \\
= & \sum_i \lambda_i \left(|v\rangle, |w_i\rangle\right)
\end{aligned}$$

Next we prove conjugate symmetry.

$$\begin{aligned}
& (|w\rangle, |v\rangle)^* \\
&= ((w_i), (v_i))^* \\
&= \left(\sum_i w_i^* v_i\right)^* \\
&= \sum_i w_i^{**} v_i^* \\
&= \sum_i v_i^* w_i \\
&= ((v_i), (w_i)) \\
&= (|v\rangle, |w\rangle)
\end{aligned}$$

Finally positivity:

$$\begin{aligned}
&(|v\rangle, |v\rangle) \\
&= ((v_i), (v_i)) \\
&= \sum_i v_i^* v_i \\
&= \sum_i |v_i|^2
\end{aligned}$$

Clearly this expression is at least 0, with equality when $v_i = 0\ \forall i$, i.e. when $|v\rangle = (0, 0, \ldots, 0)$.

Therefore the operator $(\cdot, \cdot)$ is an inner product on the vector space $\mathbb{C}^n$.

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Exercise 2.6

Combines second argument linearity with conjugate symmetry, as you would expect.

$$\begin{aligned}
\left(\sum_i \lambda_i |w_i\rangle, |v\rangle\right) &= \left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right)^* \\
&= \left(\sum_i \lambda_i \left(|v\rangle, |w_i\rangle\right)\right)^* \\
&= \sum_i \lambda_i^* \left(|v\rangle, |w_i\rangle\right)^* \\
&= \sum_i \lambda_i^* \left(|w_i\rangle, |v\rangle\right)
\end{aligned}$$

Exercise 2.7

$$\begin{aligned}
\langle w|v\rangle &= (1)(1) + (1)(-1) = 0 \\
|w\rangle, |v\rangle &= \left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)
\end{aligned}$$

Exercise 2.8

Suppose that $j < k$, and that the first $k-1$ vectors are orthonormal,

$$\begin{aligned}
\langle v_j|v_k\rangle &= \langle v_j|w_k\rangle - \sum_i^{k-1} \langle v_i|w_k\rangle\langle v_j|v_i\rangle \\
&= \langle v_j|w_k\rangle - \langle v_j|w_k\rangle\langle v_j|v_j\rangle \\
&= \langle v_j|w_k\rangle - \langle v_j|w_k\rangle\langle v_j|v_j\rangle \\
&= 0
\end{aligned}$$

Obviously $|v_i\rangle$ are explicitly normalized so we are done.

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Feel like generalizing Ex 2.9 based on Ex 2.10.

Exercise 2.10

Suppose $|v_i\rangle$ is an orthonormal basis for an inner product space V.What is the matrix representation for the operator $|v_j\rangle\langle v_k|$, with respect to the $|v_i\rangle$ basis?

Let $A = |v_j\rangle\langle v_k|$, and $A|v_n\rangle = A_{mn}|v_m\rangle$, then

$$\begin{aligned}
A_{mn}|v_m\rangle &= |v_j\rangle\langle v_k|v_n\rangle \\
&= \delta_{kn}|v_j\rangle \\
\implies A_{mn} &= \begin{cases} \delta_{kn} & m = j \\ 0 & else \end{cases} \\
&= \begin{cases} 1 & m = j, n = k \\ 0 & else \end{cases}
\end{aligned}$$

e.g. in a 2d space,

$$|0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Exercise 2.9

From 2.10 it becomes clear that

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Matrix Reps and Outer Products

Visually one can see that we can use $|v_i\rangle\langle v_j|$ as a basis for describing linear maps directly in terms of their matrix representation, e.g.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$$

Stated more generally

$$A = \sum_{ij} A_{ij} |v_i\rangle\langle v_j|$$

Which connects to equation 2.25 via the equality

$$\langle v_j|A|v_i\rangle = A_{ij}$$

Each of these things can be shown, especially once linear maps are themselves understood as a vector space, but I shall take these statements as given.

Exercise 2.11

Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y, and Z.

Z is already diagonal.

X has the familiar and intuitive eigenvectors $|+\rangle$ and $|-\rangle$:

$$X|+\rangle = |+\rangle$$
$$X|-\rangle = -|-\rangle$$

Diagonal representation then is $X = |+\rangle\langle+| \ - \ |-\rangle\langle-|$

Y has the same shape as X so we could guess the eigenvectors are the same, but $Y|+\rangle = -i|-\rangle$.

Solving properly we get $c(\lambda) = \lambda^2 - 1 = 0$, with eigenvectors in the null-spaces of:

$$\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Second row is $-i$ times first row:

$$Y(|0\rangle - i|1\rangle) = -|0\rangle + i|1\rangle$$

Taking the conjugate we unsurprisingly get

$$Y(|0\rangle + i|1\rangle) = |0\rangle + i|1\rangle$$

Their diagonal representation would be just what you'd expect, $|v_0\rangle\langle v_0| - |v_1\rangle\langle v_1|$ where $|v_0\rangle = 2^{-\frac{1}{2}}\left(|0\rangle + i|1\rangle\right)$, $|v_1\rangle = 2^{-\frac{1}{2}}\left(|0\rangle - i|1\rangle\right)$.