

# On Clifford groups in quantum computing

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**Abstract.** The term Clifford group was introduced in 1998 by D. Gottesmann in his investigation of quantum error-correcting codes. The simplest Clifford group in multiqubit quantum computation is generated by a restricted set of unitary Clifford gates - the Hadamard,  $\pi/4$ -phase and controlled-X gates. Because of this restriction the Clifford model of quantum computation can be efficiently simulated on a classical computer (the Gottesmann-Knill theorem). However, this fact does not diminish the importance of the Clifford model, since it may serve as a suitable starting point for a full-fledged quantum computation.

In the general case of a single or composite quantum system with finite-dimensional Hilbert space the finite Weyl-Heisenberg group of unitary operators defines the quantum kinematics and the states of the quantum register. Then the corresponding Clifford group is defined as the group of unitary operators leaving the Weyl-Heisenberg group invariant. The aim of this contribution is to show that our comprehensive results on symmetries of the Pauli gradings of quantum operator algebras – covering any single as well as composite finite quantum systems – directly correspond to Clifford groups defined as quotients with respect to  $U(1)$ .

## 1. Introduction

In quantum mechanics of single  $N$ -level systems in Hilbert spaces of finite dimension  $N$ , the basic operators are the *generalized Pauli matrices*. They generate the finite *Weyl-Heisenberg group* (defining quantum kinematics and the states of the quantum register) as a subgroup of the unitary group  $U(N)$  [1, 2, 3, 4].

Its normalizer within the unitary group  $U(N)$ , or in other words the largest subgroup of the unitary group having the Weyl-Heisenberg group as a normal subgroup, is in the papers on quantum information conventionally called the *Clifford group* [5]. Since this normalizer necessarily contains the continuous group  $U(1)$  of phase factors, some authors adopt an alternative definition of the Clifford group as the quotient of the normalizer with respect to  $U(1)$  [6]. In this paper we call it the *Clifford quotient group*.

Our results on symmetries of the Pauli gradings of quantum operator algebras [7, 8, 9] account

for all possible Clifford quotient groups corresponding to arbitrary single or composite quantum systems. This contribution presents a brief review of their description. In Sect. 2 the Weyl-Heisenberg groups of  $N$ -level quantum systems are defined as subgroups of  $U(N)$  for  $N = 2, 3, \dots$ . In Sect. 3 the corresponding Clifford quotient groups are described and in Sect. 4 the Clifford quotient groups for arbitrary composite quantum systems are shortly introduced.

## 2. Weyl-Heisenberg groups of single $N$ -level systems

In finite-dimensional quantum mechanics of a single  $N$ -level system the  $N$ -dimensional Hilbert space  $\mathcal{H}_N = \mathbb{C}^N$  has an orthonormal basis  $\mathcal{B} = \{|0\rangle, |1\rangle, \dots, |N-1\rangle\}$ . The basic unitary operators  $Q_N, P_N$  are defined by their action on the basis [1, 2]

$$Q_N|j\rangle = \omega_N^j|j\rangle, \quad (1)$$

$$P_N|j\rangle = |j-1 \pmod{N}\rangle, \quad (2)$$

where  $j = 0, 1, \dots, N-1$ ,  $\omega_N = \exp(2\pi i/N)$ . This is the well-known *clock-and-shift representation* of the basic operators  $Q_N, P_N$ . In the canonical or computational basis  $\mathcal{B}$  the operators  $Q_N$  and  $P_N$  are represented by the *generalized Pauli matrices*

$$Q_N = \text{diag}\left(1, \omega_N, \omega_N^2, \dots, \omega_N^{N-1}\right) \quad (3)$$

and

$$P_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (4)$$

Their commutation relation

$$P_N Q_N = \omega_N Q_N P_N \quad (5)$$

expresses the minimal non-commutativity of the operators  $Q_N$  and  $P_N$ . Further, they are of the order  $N$ ,

$$P_N^N = Q_N^N = I, \quad \omega_N^N = 1.$$

The *Pauli group*  $\Pi_N$  of order  $N^3$  is generated by  $\omega_N, Q_N$  and  $P_N$  [7]

$$\Pi_N = \left\{ \omega_N^l Q_N^i P_N^j | l, i, j = 0, 1, 2, \dots, N-1 \right\}. \quad (6)$$

Elements of  $\mathbb{Z}_N = \{0, 1, \dots, N-1\}$  label the vectors of the canonical basis  $\mathcal{B}$  with the physical interpretation that  $|j\rangle$  is the (normalized) eigenvector of position at  $j \in \mathbb{Z}_N$ . In this sense the cyclic group  $\mathbb{Z}_N$  plays the role of the *configuration space for an  $N$ -level quantum system*.

Now the Clifford group to be constructed should contain the corresponding *Weyl-Heisenberg group* as its normal subgroup. The phase factors emerging in the Clifford group lead to the necessity to define the Weyl-Heisenberg groups  $H(N)$  in even dimensions  $N$  by doubling the

Pauli group so that  $H(N)$  for even  $N$  contains the Pauli group  $\Pi_N$  as its subgroup. For this purpose the phase factor  $\tau_N = -e^{\frac{\pi i}{N}}$  is introduced such that  $\tau_N^2 = \omega_N$  [10, 11, 12] (see also [13]). Then the Weyl-Heisenberg groups

$$H(N) = \Pi_N = \left\{ \omega_N^l Q_N^i P_N^j \mid l, i, j = 0, 1, \dots, N-1 \right\} \quad \text{for odd } N,$$

$$H(N) = \left\{ \tau_N^l Q_N^i P_N^j \mid i, j = 0, 1, \dots, N-1, l = 0, 1, \dots, 2N-1 \right\} \quad \text{for even } N,$$

are of orders  $N^3$ ,  $2N^3$ , respectively. Note that for  $N = 2$  we have  $H(2) = \langle \tau_2 I_2, Q_2, P_2 \rangle$ , where  $\tau_2 = -i$ ,  $Q_2 = \sigma_z$ ,  $P_2 = \sigma_x$ ;  $H(2)$  is of order 16.

The *center*  $Z(H(N))$  of the Weyl-Heisenberg group is the set of all those elements of  $H(N)$  which commute with all elements in  $H(N)$ . For odd  $N$  we have

$$Z(H(N)) = \{\omega_N, \omega_N^2, \dots, \omega_N^N = 1\} = \{\tau_N, \tau_N^2, \dots, \tau_N^N = 1\},$$

while for even  $N$

$$Z(H(N)) = \{\tau_N, \tau_N^2, \dots, \tau_N^{2N} = 1\}.$$

Since the center is a normal subgroup, one can go over to the quotient group

$$\mathcal{P}_N = H(N)/Z(H(N)).$$

Its elements are the cosets labeled by pairs of exponents  $(i, j)$ ,  $i, j = 0, 1, \dots, N-1$ . The quotient group  $\mathcal{P}_N$  is usually identified with the *finite phase space*  $\mathbb{Z}_N \times \mathbb{Z}_N$ . Namely, denoting the cosets corresponding to elements  $(i, j)$  of the phase space by  $Q^i P^j = \left\{ \tau_N^l Q_N^i P_N^j \right\}$ , the correspondence

$$\Phi : H(N)/Z(H(N)) \rightarrow \mathbb{Z}_N \times \mathbb{Z}_N : Q^i P^j \mapsto (i, j), \quad (7)$$

is an *isomorphism of Abelian groups*, since

$$\Phi \left( (Q^i P^j) (Q^{i'} P^{j'}) \right) = \Phi \left( (Q^i P^j) \right) \Phi \left( (Q^{i'} P^{j'}) \right) = (i, j) + (i', j') = (i + i', j + j'). \quad (8)$$

### 3. Clifford groups of single $N$ -level quantum systems

As already mentioned, in quantum information the term *Clifford group* means the group of symmetries of the Weyl-Heisenberg group in the following sense [10, 11]:

#### Definition

*The Clifford group comprises all unitary operators  $X \in \mathbf{U}(N)$  for which the Ad-action preserves the subgroup  $H(N)$  in  $\mathbf{U}(N)$ , i.e. the Clifford group is the normalizer  $N_{\mathbf{U}(N)}(H(N))$ .*

In this sense the Clifford group consists of all those  $X \in \mathbf{U}(N)$  such that their Ad-action leaves  $H(N)$  invariant,

$$\text{Ad}_X H(N) = X H(N) X^{-1} = H(N).$$

But  $H(N)$  is generated by  $\tau_N$ ,  $Q_N$  and  $P_N$ , so the Clifford group consists of all  $X \in \mathbf{U}(N)$  such that

$$\text{Ad}_X Q_N = X Q_N X^{-1} \in H(N) \quad \text{and} \quad \text{Ad}_X P_N = X P_N X^{-1} \in H(N).$$

Some authors use the notion of *Clifford operations* [12]. They are the one-step unitary evolution operators acting on the nodes of a quantum register and are physically realized by so-called *Clifford gates*.<sup>1</sup>

Since the subgroup  $H(N)$  is a normal subgroup of the normalizer, one can formally write a short exact sequence of group homomorphisms

$$1 \rightarrow H(N) \rightarrow N_{U(N)}(H(N)) \rightarrow N_{U(N)}(H(N))/H(N) \rightarrow 1.$$


It turns out that the full structure of the normalizer  $N_{U(N)}(H(N))$  for arbitrary  $N$  is complicated by phase factors and rather difficult to describe in general [12]. In order to get insight into the structure of the normalizer *up to arbitrary phase factors* we turn to the definition of the *Clifford quotient group*  $C(N)$  as the quotient [6]


$$C(N) = N_{U(N)}(H(N))/U(1).$$

Its elements are the cosets  $\{e^{i\alpha}X\}$ . The following lemma is crucial for our alternative view of the Clifford quotient group.

**Lemma**

Let  $X, Y \in U(N)$ . Then the equality  $\text{Ad}_X A = \text{Ad}_Y A$  holds for all  $A \in H(N)$  if and only if  $X = e^{i\alpha}Y$ .

The proof of the converse implication is trivial. For the proof of the direct implication one writes down the first identity in the form  $Y^{-1}XA = AY^{-1}X$ , i.e.  $Y^{-1}X$  commutes with all  $A \in H(N)$ .  the elements of  $H(N)$  form an irreducible set, hence by Schur's lemma  $Y^{-1}X$  is proportional to the unit matrix,  $Y^{-1}X = e^{i\alpha}I_N$ .  $\square$

In accordance with the Lemma, instead of the cosets  $\{e^{i\alpha}X\}$  one can equivalently consider Ad-actions induced by the elements  $X$  of the normalizer. Generally, the mappings  $\text{Ad}_X : A \rightarrow XAX^{-1}$ , where  $A \in \text{GL}(N, \mathbb{C})$ , are inner automorphisms of  $\text{GL}(N, \mathbb{C})$  induced by elements  $X \in \text{GL}(N, \mathbb{C})$ . In our case we need the subgroup  $\mathcal{M}_N$  of  $\text{Int}(\text{GL}(N, \mathbb{C}))$  generated by unitary operators,  $\mathcal{M}_N = \{\text{Ad}_X | X \in U(N)\}$ . In fact, if inner automorphisms  $\text{Ad}_X$  transform unitary operators  $A$  in unitary operators  $A' = XAX^{-1}$ , then to each  $X \in \text{GL}(N, \mathbb{C})$  there exists a ary operator  $U \in U(N)$  such that  $\text{Ad}_U = \text{Ad}_X$  and  $U$  is unique up to a phase factor.

In this equivalent approach the Clifford quotient group  $C(N)$  can be studied as a subgroup of  $\mathcal{M}_N$  – the subgroup of those Ad-actions which preserve the Weyl-Heisenberg group  $H(N)$ . But  $C(N)$ , consisting of the cosets  $\{e^{i\alpha}X\}$  leaving  $H(N)$  invariant, contains the cosets  $\{e^{i\alpha}A\}$  of operators  $A \in H(N)$  as a subgroup, and this subgroup is isomorphic to the group of Ad-actions

<sup>1</sup> Note that the original name Clifford group is connected with Clifford algebras. For instance, the Clifford algebra generated by four Dirac matrices  $\gamma^\mu$  is the real Dirac algebra  $D$  of dimension  $2^4 = 16$ . The corresponding Clifford group is the universal covering of the Lorentz group  $O(3, 1)$  of isometries of the real vector space  $\mathbb{R}^4$  with an indefinite inner product, spanned by  $\gamma^0, \gamma^1, \gamma^2$  and  $\gamma^3$  such that

$$S\gamma^\mu S^{-1} = \Lambda_\nu^\mu \gamma^\nu.$$

The real Pauli algebra is the even part of  $D$  of dimension  $2^3 = 8$  and its isometries form the universal covering of the orthogonal group  $O(3)$ .

$\text{Ad}_A$ ,  $A \in H(N)$ . Note that Ad-actions of unitary operators commute  $\text{Ad}_A \text{Ad}_B = \text{Ad}_B \text{Ad}_A$  if and only if there exists  $q \in \mathbb{C}^*$  such that  $AB = qBA$  and  $q^N = 1$ . Hence in this picture the group of Ad-actions  $\text{Ad}_A$ ,  $A \in H(N)$ , is generated by the commuting Ad-actions  $\text{Ad}_{Q_N}$  and  $\text{Ad}_{P_N}$ ,

$$\{\text{Ad}_{Q_N^i P_N^j} | i, j = 0, 1, \dots, N-1\} \cong \mathbb{Z}_N \times \mathbb{Z}_N \cong \mathcal{P}_N.$$

### Proposition

The Clifford quotient group  $C(N)$  is isomorphic to the subgroup of those inner automorphisms  $\mathcal{M}_N$  which preserve  $\mathcal{P}_N$ , i.e.  $C(N)$  is the normalizer of  $\mathcal{P}_N$  in  $\mathcal{M}_N$ ,

$$C(N) \cong N_{\mathcal{M}_N}(\mathcal{P}_N).$$

We will show now that the short exact sequence of homomorphisms of subgroups of  $\mathcal{M}_N$

$$1 \rightarrow \mathcal{P}_N \rightarrow N_{\mathcal{M}_N}(\mathcal{P}_N) \rightarrow N_{\mathcal{M}_N}(\mathcal{P}_N)/\mathcal{P}_N \rightarrow 1 \quad (9)$$

can be fully decoded [7].

Obviously, Ad-actions of elements of  $\mathcal{P}_N$  leave  $\mathcal{P}_N$  invariant. Then the elements of the quotient group  $N_{\mathcal{M}}(\mathcal{P}_N)/\mathcal{P}_N$  are the cosets corresponding to possibly non-trivial transformations of  $\mathcal{P}_N$  forming a symmetry (or Weyl) group. Let us consider the Ad-actions  $\text{Ad}_X(A) = XAX^{-1}$ , where  $X \in U(N)$ , on elements  $A \in H(N)$ , which induce permutations of cosets in  $H(N)/Z(H(N))$ . We consider them to be equivalent if, for each pair  $(i, j) \in \mathbb{Z}_N \times \mathbb{Z}_N$ , they define the same transformation of cosets in  $H(N)/Z(H(N))$ :

$$\text{Ad}_Y \sim \text{Ad}_X \quad \Leftrightarrow \quad YQ^i P^j Y^{-1} = XQ^i P^j X^{-1}.$$

We have seen that the group  $\mathcal{P}_N$  has two generators,  $\text{Ad}_{Q_N}$  and  $\text{Ad}_{P_N}$ , corresponding to cosets  $Q$  and  $P$ . Hence if  $\text{Ad}_Y$  induces a permutation of elements in  $\mathcal{P}_N$ , then there must exist  $a, b, c, d \in \mathbb{Z}_N$  such that

$$YQY^{-1} = Q^a P^b \quad \text{and} \quad YPY^{-1} = Q^c P^d.$$

It follows that to each equivalence class of Ad-actions  $\text{Ad}_Y$  a quadruple  $(a, b, c, d)$  of elements in  $\mathbb{Z}_N$  is assigned. In matrix notation

$$\begin{aligned} \text{Ad}_Y \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \text{Ad}_Y \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Now inserting the relations

$$\text{Ad}_Y Q_N = \mu Q_N^a P_N^b, \quad \text{Ad}_Y P_N = \nu Q_N^c P_N^d,$$

into the basic commutation condition

$$\text{Ad}_Y(P_N Q_N) = \omega_N \text{Ad}_Y(Q_N P_N),$$

we find

$$\omega_N^{ad-1} = \omega_N^{bc} \quad \text{i.e.} \quad ad - bc = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = 1 \pmod{N}.$$

This result can also be stated as the value in  $\mathbb{Z}_N$  of a bilinear alternating non-degenerate form (symplectic form) on the finite phase space  $\mathcal{P}_N = \mathbb{Z}_N \times \mathbb{Z}_N$ . Namely, the symplectic form is the bilinear mapping  $q : \mathcal{P}_N \times \mathcal{P}_N \longrightarrow \mathbb{Z}_N$ , where

$$q : ((i, j)(i', j')) \mapsto i'j - ij' = (i, j) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i' \\ j' \end{pmatrix} = \det \begin{pmatrix} i' & j' \\ i & j \end{pmatrix}.$$

**Theorem [7]**

For integer  $N \geq 2$  there is an isomorphism  $\Phi$  between the set of equivalence classes of  $\text{Ad}$ -actions  $\text{Ad}_Y$  which induce permutations of cosets, and the group  $\text{SL}(2, \mathbb{Z}_N)$  of  $2 \times 2$  matrices with determinant equal to 1  $\pmod{N}$ ,

$$\Phi(\text{Ad}_Y) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}_N. \quad (10)$$

The action of these automorphisms on  $\mathcal{P}_N$  is given by (left) action of  $\text{SL}(2, \mathbb{Z}_N)$  on elements  $(i, j)^T$  of the phase space  $\mathcal{P}_N = \mathbb{Z}_N \times \mathbb{Z}_N$ ,

$$\text{Ad}_Y \begin{pmatrix} i \\ j \end{pmatrix} = \begin{pmatrix} i' \\ j' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix}. \quad (11)$$

It follows from the above results that the groups entering (9) are isomorphic to

$$\mathcal{P}_N \cong \mathbb{Z}_N \times \mathbb{Z}_N, \quad (12)$$

$$N_{\mathcal{M}_N}(\mathcal{P}_N)/\mathcal{P}_N \cong \text{SL}(2, \mathbb{Z}_N), \quad (13)$$

$$C(N) = N_{\mathcal{M}_N}(\mathcal{P}_N) \cong (\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \text{SL}(2, \mathbb{Z}_N), \quad (14)$$

where  $\rtimes$  denotes a semidirect product. Here  $\mathcal{P}_N$  is a normal subgroup of the normalizer  $N_{\mathcal{M}_N}(\mathcal{P}_N)$  with two generators  $\text{Ad}_{Q_N}$ ,  $\text{Ad}_{P_N}$ .

Summarizing, the Clifford quotient group  $C(N)$  is isomorphic to the normalizer of the Abelian subgroup  $\mathcal{P}_N$  in the group of unitary inner automorphisms  $\mathcal{M}_N$ . Since it contains all inner automorphisms transforming the phase space into itself, it necessarily contains  $\mathcal{P}_N$  as an Abelian semidirect factor. The symmetry (or Weyl) group is then isomorphic to the quotient group of the normalizer with respect to  $\mathcal{P}_N$ .

The generators of the normalizer  $N_{\mathcal{M}_N}(\mathcal{P}_N)$  are  $\text{Ad}_{Q_N}$ ,  $\text{Ad}_{P_N}$  and  $\text{Ad}_{S_N}$ ,  $\text{Ad}_{D_N}$  defined below [7]: The unitary Sylvester matrix  $S_N$  is the matrix of the discrete Fourier transformation (for  $N = 2$  the Hadamard gate):

$$(S_N)_{jk} = \frac{\omega_N^{jk}}{\sqrt{N}}. \quad (15)$$

It acts on  $Q_N$  and  $P_N$  according to

$$S_N Q_N S_N^{-1} = P_N^{-1} \quad S_N P_N S_N^{-1} = Q_N, \quad (16)$$

The unitary operator  $D_N$  (for  $N = 2$  the phase gate) is diagonal,

$$D_N = \text{diag} (d_0, d_1, \dots, d_{N-1}),$$

where  $d_j = \tau_N^{j(1-j)}$  if  $N$  is odd,  $d_j = \tau_N^{j(N-j)}$  if  $N$  is even. It acts on  $Q_N$  and  $P_N$  according to

$$D_N Q_N D_N^{-1} = Q_N, \quad D_N P_N D_N^{-1} = \alpha_N Q_N P_N$$

where  $\alpha_N = 1$  for  $N$  odd and  $\alpha_N = \tau_N^{N+1}$  for  $N$  even.

Given the prime decomposition of  $N = \prod_{i=1}^r p_i^{k_i}$ , the general formula for the number of elements of  $\text{SL}(2, \mathbb{Z}_N)$  is the following multiplicative function of number theory:

$$|\text{SL}(2, \mathbb{Z}_N)| = N^3 \prod_{i=1}^r \left(1 - \frac{1}{p_i^2}\right).$$

Some cardinalities are given in the table:

$N$	$ \mathcal{P}_N $	$ \text{SL}(2, \mathbb{Z}_N) $	$ \mathcal{P}_N \rtimes \text{SL}(2, \mathbb{Z}_N) $
2	4	6	24
3	9	24	216
4	16	48	768
5	25	120	3000
6	36	144	5184
7	49	336	16464
8	64	384	24576

**Example**  $N = 2$  [14]: The phase space consists of 4 elements  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ ,  $(1, 1)$ . The group  $\text{SL}(2, \mathbb{Z}_2)$  has 6 elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and acts transitively on the orbit  $\{(1, 0), (0, 1), (1, 1)\}$ . Unitary operators  $S_2$  and  $D_2$  are

$$S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

The finite Clifford group generated by  $S_2$  and  $D_2$  has  $24 \times 8 = 192$  elements, since  $(S_2 D_2)^3 = \eta I_2$  is of order 8:

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \beta \\ \alpha & -\alpha\beta \end{pmatrix}, \eta^\nu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where  $\eta = \exp(i\frac{\pi}{4})$ ,  $\nu = 0, 1, \dots, 7$ ,  $\alpha, \beta \in \{1, i, -1, -i\}$ . Note that  $Q_2 = \sigma_3$ ,  $P_2 = \sigma_1$  as well as their products and powers are generated by  $S_2$  and  $D_2$ .

In this example the finite Clifford group is defined neither as the whole normalizer of the Heisenberg-Weyl group nor as its quotient with respect to  $U(1)$ , but a certain finite subgroup of unitary operators in the normalizer [15].

In mathematical terms the motivation for this construction comes from Cartan's method for simple Lie algebras. There in the first step the maximal commutative subalgebra containing semi-simple elements is identified. This Cartan subalgebra generates a commutative subgroup (complex torus) in the corresponding complex Lie group or the torus in the corresponding compact Lie group. Then the eigenspaces of its Ad-action are the root subspaces of the Lie algebra. They are permuted by the Weyl group of symmetries of the root subspaces.

The construction of the *finite Clifford group* can then be seen as an analogue of the *Demazure-Tits finite group* for simple Lie algebras [16]. In dimension  $N$  the finite Clifford group is a special finite subgroup of the normalizer  $N_{U(N)}(H(N))$  completely defined by its generators  $\tau_N$ ,  $Q_N$ ,  $P_N$  and  $S_N$ ,  $D_N$ . Moreover, it was shown in [15] that the finite Clifford group for odd  $N$  is generated only by  $S_N$  and  $D_N$ .

#### 4. Clifford quotient groups of multipartite systems

Our further results concern detailed description of groups of symmetries of finite Weyl-Heisenberg groups for *finitely composed quantum systems consisting of subsystems with arbitrary dimensions*. We have fully described these symmetries on the level of Ad-actions. In our notation [9] the most general *symmetry (or Weyl) groups* in the sense of [17] are  $\text{Sp}_{[n_1, \dots, n_k]}$ , where the indices denote arbitrary dimensions of the constituent Hilbert spaces.

More in detail, let the Hilbert space of a composite system be the tensor product  $\mathcal{H}_{n_1} \otimes \dots \otimes \mathcal{H}_{n_k}$  of dimension  $N = n_1 \dots n_k$ , where  $n_1, \dots, n_k \in \mathbb{N}$ . For the *composite system*, quantum phase space is the Abelian subgroup of  $\text{Int}(\text{GL}(N, \mathbb{C}))$  defined by

$$\mathcal{P}_{(n_1, \dots, n_k)} = \{\text{Ad}_{M_1 \otimes \dots \otimes M_k} | M_i \in H(n_i)\}.$$

The *Clifford quotient group*, or the normalizer of this Abelian subgroup in the group of unitary inner automorphisms of  $\text{GL}(N, \mathbb{C})$ , contains all unitary inner automorphisms transforming the phase space into itself, hence necessarily *contains*  $\mathcal{P}_{(n_1, \dots, n_k)}$  *as an Abelian semidirect factor*. The symmetry (or Weyl) group is then given by the quotient group of the normalizer with respect to this Abelian subgroup.

The generating elements of  $\mathcal{P}_{(n_1, \dots, n_k)}$  are the inner automorphisms

$$e_j := \text{Ad}_{A_j} \quad \text{for } j = 1, \dots, 2k,$$

where (for  $i = 1, \dots, k$ )

$$A_{2i-1} := I_{n_1 \dots n_{i-1}} \otimes P_{n_i} \otimes I_{n_{i+1} \dots n_k}, A_{2i} := I_{n_1 \dots n_{i-1}} \otimes Q_{n_i} \otimes I_{n_{i+1} \dots n_k}.$$



The normalizer of  $\mathcal{P}_{(n_1, \dots, n_k)}$  in  $\text{Int}(\text{GL}(n_1 \cdots n_k, \mathbb{C}))$  will be denoted

$$\mathcal{N}(\mathcal{P}_{(n_1, \dots, n_k)}) := N_{\text{Int}(\text{GL}(n_1 \cdots n_k, \mathbb{C}))}(\mathcal{P}_{(n_1, \dots, n_k)}),$$

We need also the normalizer of  $\mathcal{P}_n$  in  $\text{Int}(\text{GL}(n, \mathbb{C}))$ ,

$$\mathcal{N}(\mathcal{P}_n) := N_{\text{Int}(\text{GL}(n, \mathbb{C}))}(\mathcal{P}_n),$$

and

$$\mathcal{N}(\mathcal{P}_{n_1}) \times \cdots \times \mathcal{N}(\mathcal{P}_{n_k}) := \{\text{Ad}_{M_1 \otimes \cdots \otimes M_k} \mid M_i \in \mathcal{N}(\mathcal{P}_{n_i})\} \subseteq \text{Int}(\text{GL}(N, \mathbb{C})).$$

Further,

$$\mathcal{N}(\mathcal{P}_{n_1}) \times \cdots \times \mathcal{N}(\mathcal{P}_{n_k}) \subseteq \mathcal{N}(\mathcal{P}_{(n_1, \dots, n_k)}).$$

Now the *symmetry group*  $\text{Sp}_{[n_1, \dots, n_k]}$  is defined in several steps. First let  $\mathcal{S}_{[n_1, \dots, n_k]}$  be a set consisting of  $k \times k$  matrices  $H$  of  $2 \times 2$  blocks

$$H_{ij} = \frac{n_i}{\gcd(n_i, n_j)} A_{ij}$$

where  $2 \times 2$  matrices  $A_{ij} \in \text{M}_2(\mathbb{Z}_{n_i})$  for  $i, j = 1, \dots, k$ . Then  $\mathcal{S}_{[n_1, \dots, n_k]}$  is (with the usual matrix multiplication) a monoid. Next, for a matrix  $H \in \mathcal{S}_{[n_1, \dots, n_k]}$ , we define its adjoint  $H^* \in \mathcal{S}_{[n_1, \dots, n_k]}$  by

$$(H^*)_{ij} = \frac{n_i}{\gcd(n_i, n_j)} A_{ji}^T.$$

Further, we need a skew-symmetric matrix

$$J = \text{diag}(J_2, \dots, J_2) \in \mathcal{S}_{[n_1, \dots, n_k]}$$

where

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the *symmetry group* is defined as

$$\text{Sp}_{[n_1, \dots, n_k]} := \{H \in \mathcal{S}_{[n_1, \dots, n_k]} \mid H^* J H = J\}$$

and is a finite subgroup of the monoid  $\mathcal{S}_{[n_1, \dots, n_k]}$ .

Our first theorem states the group isomorphism:

**Theorem 1** [9]

$$\mathcal{N}(\mathcal{P}_{(n_1, \dots, n_k)}) / \mathcal{P}_{(n_1, \dots, n_k)} \cong \text{Sp}_{[n_1, \dots, n_k]}.$$

Our second theorem describes the generating elements of the normalizer:

**Theorem 2** [9]

The normalizer  $\mathcal{N}(\mathcal{P}_{(n_1, \dots, n_k)})$  is generated by

$$\mathcal{N}(\mathcal{P}_{n_1}) \times \cdots \times \mathcal{N}(\mathcal{P}_{n_k}) \quad \text{and} \quad \{\text{Ad}_{R_{ij}}\},$$

where (for  $1 \leq i < j \leq k$ )

$$R_{ij} = I_{n_1 \dots n_{i-1}} \otimes \text{diag}(I_{n_{i+1} \dots n_j}, T_{ij}, \dots, T_{ij}^{n_i-1}) \otimes I_{n_{j+1} \dots n_k}$$

and

$$T_{ij} = I_{n_{i+1} \dots n_{j-1}} \otimes Q_{n_j}^{\frac{n_j}{\gcd(n_i, n_j)}}.$$

An important special case is

**Corollary** [9]

If  $n_1 = \dots = n_k = n$ , i.e.  $N = n^k$ , the symmetry group is  $\text{Sp}_{[n, \dots, n]} \cong \text{Sp}_{2k}(\mathbb{Z}_n)$ .

These cases are of particular interest, since they uncover *symplectic symmetry* of  $k$ -partite systems composed of subsystems with the same dimensions. This circumstance was found, to our knowledge, first by [18] for  $k = 2$  under additional assumption that  $n = p$  is prime, leading to  $\text{Sp}(4, \mathbb{F}_p)$  over the field  $\mathbb{F}_p$ . We have generalized this result [8] to bipartite systems with arbitrary  $n$  (non-prime) leading to  $\text{Sp}(4, \mathbb{Z}_n)$  over the modular ring, and also to multipartite systems [9]. The corresponding result was independently obtained by [19] who studied symmetries of the tensored Pauli grading of the Lie algebra  $\mathfrak{sl}(n^k, \mathbb{C})$  (see also [20]).

## 5. Illustrative examples

Consider a *bipartite system* created by coupling two single multi-level subsystems with arbitrary dimensions  $n, m$ , i.e.

$$G = \mathbb{Z}_n \times \mathbb{Z}_m \quad \text{and} \quad \mathcal{H} = \mathcal{H}_n \otimes \mathcal{H}_m.$$

The corresponding finite Weyl-Heisenberg group is embedded in  $\text{U}(N)$ ,  $N = nm$ . Via *inner automorphisms* it induces an Abelian subgroup in  $\text{Int}(\text{GL}(N, \mathbb{C}))$ .

The *Clifford quotient group*, or the normalizer of this Abelian subgroup in the group of inner automorphisms of  $\text{GL}(N, \mathbb{C})$ , contains all unitary inner automorphisms transforming the phase space into itself, hence necessarily *contains*  $\mathcal{P}_{(n,m)}$  *as an Abelian semidirect factor*. The symmetry is then given by the quotient group of the normalizer with respect to this Abelian subgroup.

According to the Corollary the case  $n = m$ ,  $N = n^2$ , corresponds to the symmetry group

$$\text{Sp}_{[n,n]} \cong \text{Sp}(4, \mathbb{Z}_n).$$

If  $N = nm$ ,  $n, m$  coprime, the symmetry group is

$$\text{Sp}_{[n,m]} \cong \text{SL}(2, \mathbb{Z}_n) \times \text{SL}(2, \mathbb{Z}_m) \cong \text{SL}(2, \mathbb{Z}_{nm}).$$

Further, if  $d = \gcd(n, m)$ ,  $n = ad$ ,  $m = bd$ , the finite configuration space can be further decomposed under the condition that  $a, b$  are both coprime to  $d$ ,

$$G = \mathbb{Z}_n \times \mathbb{Z}_m = \mathbb{Z}_{ad} \times \mathbb{Z}_{bd} \cong (\mathbb{Z}_a \times \mathbb{Z}_d) \times (\mathbb{Z}_b \times \mathbb{Z}_d).$$

Thus the symmetry group is reduced to the direct product

$$\mathrm{Sp}_{[n,m]} \cong \mathrm{Sp}_{[a,b]} \times \mathrm{Sp}(4, \mathbb{Z}_d)$$

For instance, if  $n = 15$ ,  $m = 12$ , then  $d = 3$  is coprime to both  $a = 5$  and  $b = 4$ , and also  $a$  and  $b$  are coprime, hence the symmetry group is reduced to the standard types SL and Sp,

$$\mathrm{Sp}_{[n,m]} \cong \mathrm{SL}(2, \mathbb{Z}_a) \times \mathrm{SL}(2, \mathbb{Z}_b) \times \mathrm{Sp}(4, \mathbb{Z}_d).$$

Consider a still more general situation, for instance let  $n = 180$  and  $m = 150$ . Then  $d = 30$ ,  $a = 180/30 = 6$  and  $b = 150/30 = 5$ , hence  $a$  divides  $d$  and also  $b$  divides  $d$ . In this case the reduction to solely standard groups Sp and SL is not possible. One has to break down the composite system consisting of two single systems into its *elementary building blocks* [21]. We decompose each of the finite configuration spaces

$$\mathbb{Z}_{180} \times \mathbb{Z}_{150} = (\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5) \times (\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{5^2}),$$

and take notice of coprime factors  $2.2^2$ ,  $3.3^2$  and  $5.5^2$  leading to the factorization of the symmetry group in agreement with the elementary divisor decomposition

$$\mathrm{Sp}_{[180,150]} \cong \mathrm{Sp}_{[2,2^2]} \times \mathrm{Sp}_{[3,3^2]} \times \mathrm{Sp}_{[5,5^2]}.$$

Thus if attention is paid to the elementary building blocks of finite quantum systems (quantal degrees of freedom) [21], then the symmetries can be reduced – in accordance with the elementary divisor decomposition – to direct products of finite groups of the types  $\mathrm{SL}(2, \mathbb{Z}_n)$ ,  $\mathrm{Sp}(2k, \mathbb{Z}_n)$ , and  $\mathrm{Sp}_{[p^k, p^l, \dots]}$ . The last type of symmetry groups corresponds to a *new class of Clifford quotient groups of composite systems*. These symmetry groups like  $\mathrm{Sp}_{[p^k, p^l]}$  with indices given by different powers of the same prime  $p$  deserve to be added to the standard types Sp and SL.

## 6. Conclusion

We have considered three definitions of the Clifford group. As the most interesting seems not the first one – the whole normalizer of the Heisenberg-Weyl group, but the second – its quotient with respect to  $U(1)$  or even the third – a certain finite subgroup of unitary operators in the normalizer like that presented in the example in dimension  $N = 2$ .

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