Honours Diary 2020

Jarvis Carroll

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Notation

In this diary unless explicitly stated within a section, I have been using the notation specified by Nielsen and Chuang, with the following additions:

- Implicit quantifiers for index variables such as i, j, k. (Nielson and Chuang seem to do this actually, perhaps dropping more than I do)
  - $\{x_i\} = \{x_i \mid i \in I\}$, $\{|x_i\rangle\} = \{|x_i\rangle \mid i \in I\}$ etc.
  - $(x_i) = (x_1, x_2, \ldots, x_n)$
  - $\sum_i$ in place of $\sum_{i \in I}$
  - $\forall i$ in place of $\forall i \in I$

March 13

Set up TeXstudio and basic document structure.

Exercise 2.1

Linear Dependence, show that $(1, -1)$, $(1, 2)$ and $(2, 1)$ are linearly dependent.

$$\begin{aligned} &(1, -1) + (1, 2) - (2, 1) \\ =\ &(1 + 1 - 2, -1 + 2 - 1) \\ =\ &(0, 0) \end{aligned}$$

Exercise 2.2

Matrix representations: Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle$, $|1\rangle$, and the output basis $|0\rangle$, $|1\rangle$. Find input and output bases which give rise to a different matrix representation of A.

Equation 2.12 gives us the defining property of matrix representations:

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle$$

This gives us a pair of vector equations:

$$|1\rangle = A|0\rangle = A_{00}|0\rangle + A_{10}|1\rangle$$

$$|0\rangle = A|1\rangle = A_{01}|0\rangle + A_{11}|1\rangle$$

By linear independence of $|0\rangle$, $|1\rangle$, it follows that

$$\begin{aligned} A_{00} &= 0 & A_{01} &= 1 \\ A_{10} &= 1 & A_{11} &= 0 \end{aligned}$$

i.e. A has the matrix representation:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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Exercise 2.3

$$A : V \to W$$

$$B : W \to X$$

$$V = span\{|v_i\rangle\} \ etc.$$

At this point we are already identifying A, B with their matrix representations. We would like to show equality between the function composition $B \circ A$ and the matrix product of the corresponding matrix representations, $B \times A$. Once we have done this we will be able to identify both of these concepts as simply the expression BA, but for now we will use the explicit operators $\circ$ and $\times$.

Our goal then is to show that the matrix representation of $B \circ A$ is $B \times A$.

$$
\begin{aligned}
& \sum_i (B \circ A)_{ij} |x_i\rangle && \text{(arbitrary column of matrix } B \circ A\text{)} \\
&= (B \circ A)|v_j\rangle && \text{(Definition of matrix representation)} \\
&= B(A|v_j\rangle) && \text{(Definition of composition)} \\
&= B\left(\sum_k A_{kj}|w_k\rangle\right) && \text{(Matrix representation)} \\
&= \sum_k A_{kj}(B|w_k\rangle) && \text{(Linearity of B)} \\
&= \sum_k A_{kj}\left(\sum_i B_{ik}|x_i\rangle\right) && \text{(Matrix representation)} \\
&= \sum_i \left(\sum_k B_{ik}A_{kj}\right)|x_i\rangle && \text{(distribution)} \\
&= \sum_i (B \times A)_{ij}|x_i\rangle && \text{(matrix product)}
\end{aligned}
$$

So by linear independence of $|x_i\rangle$ we know that $(B \circ A)_{ij} = (B \times A)_{ij}$ for arbitrary i, j, i.e. the matrix representation of $B \circ A$ is the matrix product $B \times A$.

Exercise 2.4

$$I : V \to V$$

$$I|x_i\rangle = |x_i\rangle$$

We would like to show that $I_{ij} = \delta_{ij}$.

$$
\begin{aligned}
& \sum_i I_{ij}|x_i\rangle \\
=& I|x_j\rangle \\
=& |x_j\rangle \\
=& \sum_i \delta_{ij}|x_i\rangle
\end{aligned}
$$

Again by linear independence of $\{|x_i\rangle\}$ we have $I_{ij} = \delta_{ij}$.

Exercise 2.5

$$((y_i), (z_i)) = \sum_i y_i^* z_i$$

We need to prove 3 properties. First linearity, taking $|v\rangle = (v_j) = (v_1, v_2, \ldots, v_n)$ and $|w_i\rangle = (w_{ij}) = (w_{i1}, w_{i2}, \ldots, w_{in})$.

For this we define

$$|z\rangle = (z_j) = \sum_i \lambda_i |w_i\rangle$$

Then by the definitions of sum and scalar product in $\mathbb{C}^n$ we observe

$$\begin{aligned}
&z_j \\
&= \left(\sum_i \lambda_i |w_i\rangle\right)_j \\
&= \sum_i \left(\lambda_i |w_i\rangle\right)_j \\
&= \sum_i \lambda_i w_{ij}
\end{aligned}$$

With this linearity falls out.

$$\begin{aligned}
&(|v\rangle, |z\rangle) \\
&= ((v_j), (z_j)) \\
&= \sum_j y_j^* z_j \\
&= \sum_j y_j^* \left(\sum_i \lambda_i w_{ij}\right) \\
&= \sum_i \lambda_i \left(\sum_j v_j^* w_{ij}\right) \\
&= \sum_i \lambda_i \left((v_j), (w_{ij})\right) \\
&= \sum_i \lambda_i \left(|v\rangle, |w_i\rangle\right)
\end{aligned}$$

Next we prove conjugate symmetry.

$$\begin{aligned}
&(|w\rangle, |v\rangle)^* \\
&= ((w_i), (v_i))^* \\
&= \left(\sum_i w_i^* v_i\right)^* \\
&= \sum_i w_i^{**} v_i^* \\
&= \sum_i v_i^* w_i \\
&= ((v_i), (w_i)) \\
&= (|v\rangle, |w\rangle)
\end{aligned}$$

Finally positivity:

$$\begin{aligned}
&(|v\rangle, |v\rangle) \\
&= ((v_i), (v_i)) \\
&= \sum_i v_i^* v_i \\
&= \sum_i |v_i|^2
\end{aligned}$$

Clearly this expression is at least 0, with equality when $v_i = 0 \ \forall i$, i.e. when $|v\rangle = (0, 0, \dots, 0)$.

Therefore the operator $(\cdot, \cdot)$ is an inner product on the vector space $\mathbb{C}^n$.

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Exercise 2.6

Combines second argument linearity with conjugate symmetry, as you would expect.

$$\begin{aligned}
\left(\sum_i \lambda_i |w_i\rangle, |v\rangle\right) &= \left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right)^* \\
&= \left(\sum_i \lambda_i \left(|v\rangle, |w_i\rangle\right)\right)^* \\
&= \sum_i \lambda_i^* (|v\rangle, |w_i\rangle)^* \\
&= \sum_i \lambda_i^* (|w_i\rangle, |v\rangle)
\end{aligned}$$

Exercise 2.7

$$\langle w|v\rangle = (1)(1) + (1)(-1) = 0$$
$$|w\rangle, |v\rangle = \left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$$

Exercise 2.8

Suppose that $j < k$, and that the first $k-1$ vectors are orthonormal,

$$\begin{aligned}
\langle v_j|v_k\rangle &= \langle v_j|w_k\rangle - \sum_i^{k-1} \langle v_i|w_k\rangle\langle v_j|v_i\rangle \\
&= \langle v_j|w_k\rangle - \langle v_j|w_k\rangle\langle v_j|v_j\rangle \\
&= \langle v_j|w_k\rangle - \langle v_j|w_k\rangle\langle v_j|v_j\rangle \\
&= 0
\end{aligned}$$

Obviously $|v_i\rangle$ are explicitly normalized so we are done.

March 27

Feel like generalizing Ex 2.9 based on Ex 2.10.

Exercise 2.10

Suppose $|v_i\rangle$ is an orthonormal basis for an inner product space V.What is the matrix representation for the operator $|v_j\rangle\langle v_k|$, with respect to the $|v_i\rangle$ basis?

Let $A = |v_j\rangle\langle v_k|$, and $A|v_n\rangle = A_{mn}|v_m\rangle$, then

$$\begin{aligned} A_{mn}|v_m\rangle &= |v_j\rangle\langle v_k|v_n\rangle \\ &= \delta_{kn}|v_j\rangle \\ \implies A_{mn} &= \begin{cases} \delta_{kn} & m = j \\ 0 & else \end{cases} \\ &= \begin{cases} 1 & m = j, n = k \\ 0 & else \end{cases} \end{aligned}$$

e.g. in a 2d space,

$$|0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Exercise 2.9

From 2.10 it becomes clear that

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

Matrix Reps and Outer Products

Visually one can see that we can use $|v_i\rangle\langle v_j|$ as a basis for describing linear maps directly in terms of their matrix representation, e.g.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$$

Stated more generally

$$A = \sum_{ij} A_{ij} |v_i\rangle\langle v_j|$$

Which connects to equation 2.25 via the equality

$$\langle v_j|A|v_i\rangle = A_{ij}$$

Each of these things can be shown, especially once linear maps are themselves understood as a vector space, but I shall take these statements as given.

Exercise 2.11

Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y, and Z.

Z is already diagonal.

X has the familiar and intuitive eigenvectors $|+\rangle$ and $|-\rangle$:

$$X|+\rangle = |+\rangle$$
$$X|-\rangle = -|-\rangle$$

Diagonal representation then is $X = |+\rangle\langle+| \ - \ |-\rangle\langle-|$

Y has the same shape as X so we could guess the eigenvectors are the same, but $Y|+\rangle = -i|-\rangle$.

Solving properly we get $c(\lambda) = \lambda^2 - 1 = 0$, with eigenvectors in the null-spaces of:

$$\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Second row is $-i$ times first row:

$$Y(|0\rangle - i|1\rangle) = -|0\rangle + i|1\rangle$$

Taking the conjugate we unsurprisingly get

$$Y(|0\rangle + i|1\rangle) = |0\rangle + i|1\rangle$$

Their diagonal representation would be just what you'd expect, $|v_0\rangle\langle v_0| - |v_1\rangle\langle v_1|$ where $|v_0\rangle = 2^{-\frac{1}{2}}(|0\rangle + i|1\rangle)$, $|v_1\rangle = 2^{-\frac{1}{2}}(|0\rangle - i|1\rangle)$.

March 30

Exercise 2.12

Prove that the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

is not diagonalizable.

The equation $c(\lambda) = (1-\lambda)^2 = 0$ has repeated root $\lambda = 1$, so it either has one eigenvector, or a eigenspace of 2 degenerate eigenvectors.

Clearly if it had 2 degenerate eigenvectors it would simply be the identity operation, mapping all vectors to themselves (scaled by the eigenvalue 1), so we know there is no eigenvector basis, let alone the weaker result that there is no orthonormal eigenvector basis.

To apply the formal structure of QM we can prove this weaker result explicitly: If M above is diagonalizable, then $M = |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|$ which by the completeness relation 2.22 gives $M = I$, therefore M is not diagonalizable.

Exercise 2.13

If $|w\rangle$ and $|v\rangle$ are any two vectors, show that $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

Since we are told that the adjoint is unique, it is sufficient to show equation 2.32:

$$(|v'\rangle, |w\rangle\langle v|w'\rangle) = (|v\rangle\langle w|v'\rangle, |w'\rangle)$$

Then by linearity of the inner product this is equivalent to

$$\langle v|w'\rangle(|v'\rangle, |w\rangle) = \langle w|v'\rangle^*(|v\rangle, |w'\rangle)$$

Apply conjugate symmetry and note that $\langle v|w'\rangle = (|v\rangle, |w'\rangle)$ by definition, and the result clearly follows.

Exercise 2.14

Show that the adjoint operation is anti-linear

Straight forward, again by uniqueness we simply need equation 2.32 to hold, which after linearity becomes:

$$\sum_i a_i(|v\rangle, A_i|w\rangle) = \sum_i a_i^{**}(A_i^\dagger|v\rangle, |w\rangle)$$

By double-conjugate elimination and the adjointness property, this is clearly true.

Exercise 2.15

Show that $(A^\dagger)^\dagger = A$.

$$\begin{aligned}
(|v\rangle, A^\dagger|w\rangle) &= (A^\dagger|w\rangle, |v\rangle)^* \\
&= (|w\rangle, A|v\rangle)^* \\
&= (A|v\rangle, |w\rangle)
\end{aligned}$$

So A is the adjoint of $A^\dagger$, i.e. by uniqueness of adjoints $(A^\dagger)^\dagger = A$.

Equation 2.16

Show that any projector P satisfies the equation $P^2 = P$.

$$(\sum_i |i\rangle\langle i|)^2 = \sum_{ij} |i\rangle\langle i|j\rangle\langle j| \qquad = \sum_{ij} \delta_{ij}|i\rangle\langle j| = \sum_i |i\rangle\langle i|$$

Exercise 2.17

Show that a normal matrix is Hermitian if and only if it has real eigenvalues.

First show that a hermitian matrix has real eigenvalues, i.e. if $A|v\rangle = v|v\rangle$ then v is real.

Consider $\langle v|A|v\rangle$:

$$\begin{aligned}\langle v|A|v\rangle &= (|v\rangle, A|v\rangle)\\ &= (|v\rangle, v|v\rangle)\\ &= v(|v\rangle, |v\rangle)\\ &= v\langle v|v\rangle\end{aligned}$$

But by adjointness this can also be shown for v^*:

$$\begin{aligned}\langle v|A|v\rangle &= (|v\rangle, A|v\rangle)\\ &= (A|v\rangle, |v\rangle)\\ &= (v|v\rangle, |v\rangle)\\ &= v^*(|v\rangle, |v\rangle)\\ &= v^*\langle v|v\rangle\end{aligned}$$

So $v = v^*$, and necessarily v is real.

Next show that if A is normal and has all real eigenvalues, then A is hermitian.

By spectral decomposition:

$$A = \sum_i v_i|v_i\rangle\langle v_i|$$

Then taking the adjoint the result becomes obvious:

$$\begin{aligned}A^\dagger &= \sum_i v_i^*|v_i\rangle\langle v_i|\\ &= \sum_i v_i|v_i\rangle\langle v_i|\\ &= A\end{aligned}$$

April 3

Exercise 2.18

Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

Suppose that $U|v\rangle = v|v\rangle$, then take the squared norm of both sides

$$\begin{aligned}\langle v|U^\dagger U|v\rangle &= \langle v|v^* v|v\rangle \\ \implies \langle v|v\rangle &= |v|^2 \langle v|v\rangle \\ \implies |v| &= 1\end{aligned}$$

Exercise 2.19

Show that the Pauli matrices are Hermitian and unitary.

A matrix is hermitian *and* unitary precisely when $U^\dagger = U$. I, X, and Z are real symmetric matrices, so this is clearly true, and Y has i opposite $-i$ in its matrix representation, so it will satisfy this as well.

Exercise 2.20

Suppose A' and A'' are matrix representations of an operator A on a vector space V with respect to two different orthonormal bases, $|v_i\rangle$ and $|w_i\rangle$. Then the elements of A' and A'' are $A'_{ij} = \langle v_i|A|v_j\rangle$ and $A''_{ij} = \langle w_i|A|w_j\rangle$. Characterize the relationship between A' and A''.

Change of basis takes the form of a similarity/conjugation, and we have an intuition that orthonormal bases are related somehow to unitary operators, so a guess would be that $A'' = U^{-1}A'U$ for some unitary U. Take the ijth entry:

$$A'_{ij} = \sum_{lm} U^{-1}_{il} A''_{lm} U^{-1}_{mj}$$

At the same time we know that A' and A'' represent the same operator, i.e.

$$\sum_{ij} A'_{ij}|v_i\rangle\langle v_j| = \sum_{ij} A''_{ij}|w_i\rangle\langle w_j| \tag{1}$$

Combine these

$$\sum_{ilmj} U^{-1}_{il} A''_{lm} U^{-1}_{mj}|v_i\rangle\langle v_j| = \sum_{ij} A''_{ij}|w_i\rangle\langle w_j|$$

Didn't work, apply both sides of (1) to $|v_k\rangle$

$$\sum_{ij} A'_{ij}|v_i\rangle\langle v_j|v_k\rangle = \sum_{ij} A''_{ij}|w_i\rangle\langle w_j|v_k\rangle$$

$$\sum_i A'_{ik}|v_i\rangle = \sum_{ij} A''_{ij}|w_i\rangle\langle w_j|v_k\rangle$$

Didn't work, apply $\langle v_l| \cdot |m\rangle$ to both sides of (1):

$$A'_{lm} = \sum_{ij} A''_{ij}\langle v_l|w_i\rangle\langle w_j|v_m\rangle$$

stumped... can't get any δ_{ij} magic to happen, the indices just keep growing!

April 6

Exercise 2.21

Repeat the proof of the spectral decomposition in Box 2.2 for the case when M is Hermitian, simplifying the proof wherever possible

Let P be a projector onto the λ eigenspace of a hermitian matrix M.

Then $M = (P+Q)M(P+Q) = PMP + PMQ + QMP + QMQ$.

Note $MP = \lambda P$ and hence $PMP = \lambda P^2 = \lambda P$.

Then $QMP = (1-P)MP = MP - PMP = \lambda P - \lambda P = 0$.

Next $PMQ = (QM^\dagger P)^\dagger = (QMP)^\dagger = 0^\dagger = 0$.

Thus $M = PMP + QMQ$.

Next QMQ is obviously hermitian since $(QMQ)^\dagger = QM^\dagger Q = QMQ$, so by induction is diagonalizable, so it follows that M is diagonalizable.

Exercise 2.22

Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

$M = M^\dagger$, $M|v_1\rangle = \lambda_1|v_1\rangle$, $M|v_2\rangle = \lambda_2|v_2\rangle$.

Observe the following equality:

$$\begin{aligned}
\lambda_2\langle v_1|v_2\rangle &= \langle v_1|\left(M|v_2\rangle\right) \\
&= \left(\langle v_1|M^\dagger\right)|v_2\rangle \\
&= \lambda_1\langle v_1|v_2\rangle
\end{aligned}$$

$$\implies (\lambda_2 - \lambda_1)\langle v_1|v_2\rangle = 0$$

$$\implies \lambda_1 = \lambda_2 \text{ or } \langle v_1|v_2\rangle = 0$$

So if the eigenvalues of $|v_1\rangle$ and $|v_2\rangle$ are different, then $|v_1\rangle$ and $|v_2\rangle$ must be orthogonal.

Exercise 2.23

Show that the eigenvalues of a projector are all either 0 or 1.

Suppose $P|w\rangle = \lambda|w\rangle$, then by Ex. 2.16, $P^2 = P$, so

$$\begin{aligned} \lambda|w\rangle = P|w\rangle = P^2|w\rangle = \lambda^2|w\rangle \\ \implies \lambda(1-\lambda)|w\rangle = 0 \end{aligned}$$

So λ is either 0 or 1.

Exercise 2.24

Show that a positive operator is necessarily Hermitian. (Hint: Show that an arbitrary operator A can be written $A = B + iC$ where B and C are Hermitian.)

Similar to the matrix as a sum of symmetric and antisymmetric matrices, it's easy to see:

$$\begin{aligned} A &= \frac{A + A^\dagger}{2} + \frac{A - A^\dagger}{2} \\ &= \frac{A + A^\dagger}{2} + i\frac{iA^\dagger - iA}{2} \\ &= B + iC \end{aligned}$$

Then suppose A is positive, i.e. $\langle v|A|v\rangle \geq 0$.

$$\begin{aligned} \langle v|A|v\rangle &= \langle v|(B + iC)|v\rangle \\ &= \langle v|B|v\rangle + i\langle v|C|v\rangle \end{aligned}$$

Note that since B and C are Hermitian, they are diagonalizable, with real eigenvalues, so $\langle v|B|v\rangle$ must be real, and thus $\langle v|C|v\rangle = 0$. Since v is arbitrary, we have $C = 0$, so $A = B$ and therefore $A = A^\dagger$.

Note we don't require positivity, it is enough that $\langle v|A|v\rangle$ is always real. This means that we have effectively generalized Ex 2.17 to the third equivalent condition that $\langle v|A|v\rangle$ is always real.

Exercise 2.25

Show that for any operator A, $A^\dagger A$ is positive.

$\langle v|A^\dagger A|v\rangle = (A|v\rangle, A|v\rangle) \geq 0$.

April 13

Exercise 2.26

Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Write out $|psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly, both in terms of tensor products like $|0\rangle|1\rangle$, and using the Kronecker product.

$$
\begin{aligned}
|\psi\rangle^{\otimes 2} &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
&= \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \\
&= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)
\end{aligned}
$$

Similarly

$$
|\psi\rangle^{\otimes 3} = \frac{1}{2\sqrt{2}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)
$$

Then in Kronecker product terms:

$$
\left(2^{-1/2}\begin{bmatrix}1\\1\end{bmatrix}\right)^{\otimes 2} = 2^{-1}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}
$$

Skipping $|\psi\rangle^{\otimes 3}$

Diversion: Non-normal Diagonalization

This textbook deals with diagonalization of normal operators in the form $\sum_i \lambda_i |v_i\rangle\langle v_i|$, but diagonalization of non-normal operators is also possible, and I would like to see how it looks in bra-ket notation.

$$
\begin{aligned}
A &= V\Lambda V^{-1} \\
&= \left(\sum_i |v_i\rangle\langle i|\right)\left(\sum_i \lambda_i |i\rangle\langle i|\right)\left(\sum_i |v_i\rangle\langle i|\right)^{-1}
\end{aligned}
$$

In the case that V is unitary, i.e. $|v_i\rangle$ are orthogonal, then $V^{-1} = V^\dagger$ and we can simplify from there. Otherwise we have a general matrix inversion to do, which doesn't seem helpful at all.

Diversion: Generalized Diagonalization

Suppose instead that something only has generalized eigenvectors, but that these generalized eigenvectors are orthogonal. For example $A = |0\rangle\langle 1|$ with eigenvector $A|0\rangle = 0|0\rangle$ and generalized eigenvector $A|1\rangle = 1|0\rangle$.

This appears to motivate the form $A = \sum_i \lambda_i |v_{\pi(i)}\rangle\langle v_i|$ so that $A|v_i\rangle = \lambda_i |v_{\pi(i)}\rangle$, but generalized eigenvectors seem more complicated than this, since the rule $(A+I)|v_i\rangle = (\lambda_i+1)|v_i\rangle$ doesn't directly hold, and whatever its analogue might be, would be too much of a diversion for now.

Exercise 2.27

Calculate the matrix representation of the tensor products of the Pauli operators (a) X and Z; (b) I and X; (c) X and I. Is the tensor product commutative?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Exercise 2.28

Show that the transpose, complex conjugation, and adjoint operations distribute over the tensor product,

$$\left(\sum_{ij} A_{ij}|j\rangle\langle i|\right) \otimes \left(\sum_{kl} B_{kl}|l\rangle\langle k|\right)|mn\rangle$$
$$= \sum_{jl} A_{mj}B_{nl}|jl\rangle$$

So clearly by linearity

$$\left(\sum_{ij} A_{ij}|j\rangle\langle i|\right) \otimes \left(\sum_{kl} B_{kl}|l\rangle\langle k|\right) = \sum_{ijkl} A_{ij}B_{kl}|jl\rangle\langle ik|$$

From this all 3 results are obvious.

April 20

Phase and Tensors

Reading ahead to chapter 5, the Fourier transform, the algorithm for eigenvalue measurement first relies on a conditional U^j, which is used to modify the *condition* rather than the vector to which it is applied.

Until now I have been ignoring phase, but this appears to be a direct application of it, so I have concocted a simpler example of phase shift through conditional operations on eigenvectors.

I believe that $\text{CNOT}\,(|+\rangle \otimes |-\rangle) = |-\rangle \otimes |-\rangle$.

The matrix representation of CNOT is as follows:

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $|+\rangle \otimes |-\rangle$ expanded in the computational basis is:

$$\begin{aligned} |+\rangle \otimes |-\rangle &= (|0\rangle + |1\rangle) \otimes (|0\rangle - |1\rangle) \\ &= |00\rangle - |01\rangle + |10\rangle - |11\rangle \end{aligned}$$

and similarly $|-\rangle \otimes |-\rangle$ is:

$$\begin{aligned} |-\rangle \otimes |-\rangle &= (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \\ &= |00\rangle - |01\rangle - |10\rangle + |11\rangle \end{aligned}$$

Clearly the signs of $|10\rangle$ and $|01\rangle$ have been permuted, so the equation makes sense.

I still find this consequence bizarre, and might have to wait until chapter 4 to better understand.

This example was constructed based on $|-\rangle$ being an eigenvector of X, so that performing X on it would have the effect of performing Z on the first bit instead! I guess really this is a curiosity of conditional operations and ultimately of the physical process that represents such an operation.

Or maybe this is best understood in terms of measurement? Although these two states are indistinguishable by measurement alone, so really it's H that might present the phase as state and give us information. And since H is an example of a Fourier transform, this is ultimately a curiosity of Fourier transform *plus* conditional operations, so I haven't reduced the problem at all, just applied the simplest Fourier transform to the simplest conditional operation and gotten the same behavior.

This appears to be related to Exercise 4.34, and surrounding theory, which I shall get to eventually.

April 27

Exercise 2.29, 2.30, 2.31, 2.32

Show that the tensor product of two unitary/hermitian/positive/projector operators is unitary/hermitian/positive/a projector.

In the previous exercise we derived the following formula for the tensor product of two matrices:

$$\left(\sum_{ij} A_{ij}|j\rangle\langle i|\right) \otimes \left(\sum_{kl} B_{kl}|l\rangle\langle k|\right) = \sum_{ijkl} A_{ij}B_{kl}|jl\rangle\langle ik|$$

If A and B are normal, then we can use their eigenvectors as the basis, and this formula becomes the following:

$$\left(\sum_{i} v_i|v_i\rangle\langle v_i|\right) \otimes \left(\sum_{j} u_j|u_j\rangle\langle u_j|\right) = \sum_{ij} v_i u_j \left(|v_i\rangle \otimes |u_j\rangle\right)\left(\langle v_i| \otimes \langle u_j|\right)$$

Clearly in this formula if v_i and u_j are all in some semigroup S, then their products will be as well, so the tensor of any operators with S eigenvalues will be also have S eigenvalues.

Setting $S = \mathrm{U}(1)$ gives products of unitary operators are unitary.

Setting $S = \mathbb{R}$ gives products of Hermitian operators Hermitian.

Setting $S = [0, \infty)$ gives products of positive operators positive (and $(0, \infty)$ for positive definite)

Setting $S = \{0, 1\}$ gives products of projectors are projectors.

Exercise 2.33

The Hadamard operator on one qubit may be written as

$$H = \frac{1}{\sqrt{2}}\left[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|\right]$$

Show explicitly that the Hadamard transform on n qubits, $H^{\otimes n}$, may be written as

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle\langle y|$$

Clearly the formula holds for $n = 1$, so we can take this as the base case of an induction.

Then if the formula holds for $n = k$, we can simply tensor this with the

$n = 1$ formula to get

$$\begin{aligned}
H^{\otimes k+1} &= \left(\frac{1}{\sqrt{2^k}} \sum_{x=0}^{2^k-1} \sum_{y=0}^{2^k-1} (-1)^{x\cdot y} |x\rangle\langle y| \right) \otimes \left(\frac{1}{\sqrt{2}} \sum_{x=0}^{1} \sum_{y=0}^{1} (-1)^{x\cdot y} |x\rangle\langle y| \right) \\
&= \frac{1}{\sqrt{2^{k+1}}} \sum_{x_1=0}^{2^k-1} \sum_{y_1=0}^{2^k-1} \sum_{x_2=0}^{1} \sum_{y_2=0}^{1} (-1)^{x_1\cdot y_1 + x_2\cdot y_2} |x_1\rangle|x_2\rangle\langle y_1|\langle y_2| \\
&= \frac{1}{\sqrt{2^{k+1}}} \sum_{x_1=0}^{2^k-1} \sum_{x_2=0}^{1} \sum_{y_1=0}^{2^k-1} \sum_{y_2=0}^{1} (-1)^{(x_1\otimes x_2)\cdot(y_1\otimes y_2)} |x_1 \otimes x_2\rangle\langle y_1 \otimes y_2| \\
&= \frac{1}{\sqrt{2^{k+1}}} \sum_{x=0}^{2^{k+1}-1} \sum_{y=0}^{2^{k+1}-1} (-1)^{x\cdot y} |x\rangle\langle y|
\end{aligned}$$

Where $x_1 \otimes x_2$ is just $2x_1 + x_2$

Write out an explicit matrix representation for $H^{\otimes 2}$.

$$H^{\otimes 2} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

May 1

Exercise 2.34

Find the square root and logarithm of the matrix

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

Note first the eigenvectors:

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So we diagonalize as follows:

$$\begin{aligned}
A &= 7|+\rangle\langle+| \;+\; |-\rangle\langle-| \\
\sqrt{A} &= \sqrt{7}|+\rangle\langle+| \;+\; |-\rangle\langle-| \\
&= \frac{1}{2} \begin{bmatrix} \sqrt{7}+1 & \sqrt{7}-1 \\ \sqrt{7}-1 & \sqrt{7}+1 \end{bmatrix}
\end{aligned}$$

Exercise 2.35

$$\exp(i\theta v \cdot \sigma) = \cos(\theta)I + i\sin(\theta)v \cdot \sigma$$

First let's write out $v \cdot \sigma$ in matrix form.

$$\begin{aligned} v \cdot \sigma &= v_1 X + v_2 Y + v_3 Z \\ &= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \end{aligned}$$

Then the eigenvalue problem becomes

$$\begin{aligned} & v \cdot \sigma |u\rangle = \lambda |u\rangle \\ \implies & (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) = 0 \\ \implies & \lambda^2 - (v_1^2 + v_2^2 + v_3^2) = 0 \\ \implies & \lambda = \pm \|v\| \end{aligned}$$

And the eigenvectors will satisfy

$$\begin{aligned} & \begin{bmatrix} v_3 \mp \|v\| & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \mp \|v\| \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \implies & \begin{bmatrix} (v_1 + iv_2)(v_3 \mp \|v\|) & v_1^2 + v_2^2 \\ (v_1 + iv_2)(v_3 \mp \|v\|) & v_1^2 + v_2^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Clearly eigenvectors will be proportional to $(v_1^2 + v_2^2, (v_1 + iv_2)(v_3 \mp \|v\|))$, but this isn't really important, we can observe the matrix is Hermitian and so will be diagonalizable with these real eigenvalues we found:

$$\begin{aligned} v \cdot \sigma &= \|v\| \, |u_1\rangle\langle u_1| - \|v\| \, |u_2\rangle\langle u_2| \\ \implies \exp(i\theta v \cdot \sigma) &= \exp(i\theta \|v\|)|u_1\rangle\langle u_1| + \exp(-i\theta \|v\|)|u_2\rangle\langle u_2| \\ &= (\cos(\theta \|v\|) + i\sin(\theta \|v\|))|u_1\rangle\langle u_1| + (\cos(\theta \|v\|) - i\sin(\theta \|v\|))|u_2\rangle\langle u_2| \\ &= \cos(\theta \|v\|)(|u_1\rangle\langle u_1| + |u_2\rangle\langle u_2|) + i\sin(\theta \|v\|)(|u_1\rangle\langle u_1| - |u_2\rangle\langle u_2|) \\ &= \cos(\theta \|v\|)I + i\sin(\theta \|v\|)\frac{1}{\|v\|} v \cdot \sigma \end{aligned}$$

Then since $\|v\| = 1$ this gives the result.

May 4

It seems to be time to move on from chapter 2, even though I never got to the end of 2.1! The chapter will always be available as a reference, and I will see with time what parts deserve doing in full detail.

It currently feels as though my primary education is in "Quantum Algorithm Implementations for Beginners", so rather than read through the same summary of algorithms and operations that I have covered many times, and am currently covering again, I will try moving on to interesting questions of decomposition, and see both through these exercises and through reading what requires more detailed practice.

Exercise 4.37

Provide a decomposition of the level 4 Fourier transform:

$$S_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

Define

$$\alpha = \frac{1}{\sqrt{2}} \implies 2\alpha^2 = 1$$

$$U_{10} = \begin{bmatrix} \alpha & \alpha & 0 & 0 \\ \alpha & -\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then

$$U_{10}S_4 = \frac{1}{2}\begin{bmatrix} 2\alpha & (1+i)\alpha & 0 & (1-i)\alpha \\ 0 & (1-i)\alpha & 2\alpha & (1+i)\alpha \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

next repeat

$$\beta = \frac{1}{\sqrt{3}} \implies 3\beta^2 = 1$$

$$U_{20} = \begin{bmatrix} 2\alpha\beta & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & -2\alpha\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$U_{20}U_{10}S_4 = \frac{1}{2}\begin{bmatrix} 3\beta & i\beta & \beta & -i\beta \\ 0 & (1-i)\alpha & 2\alpha & (1+i)\alpha \\ 0 & (3+i)\alpha\beta & -2\alpha\beta & (3-i)\alpha\beta \\ 1 & -i & -1 & i \end{bmatrix}$$

and again

$$U_{30} = \begin{bmatrix} 3\alpha^2\beta & 0 & 0 & \alpha^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha^2 & 0 & 0 & -3\alpha^2\beta \end{bmatrix}$$

$$U_{30}U_{20}U_{10}S_4 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & (1-i)\alpha & 2\alpha & (1+i)\alpha \\ 0 & (3+i)\alpha\beta & -2\alpha\beta & (3-i)\alpha\beta \\ 0 & 2i\beta & 2\beta & -2i\beta \end{bmatrix}$$

This looks like it might not be unitary, but I am out of time for today. This is obviously taking a lot of time but it feels valuable to me.

May 9

Having written some C code to calculate Christoffel symbols yesterday, I'm thinking this might be a good place to write some C as well. The idea was to do it by hand to get a more intimate understanding of the process, but I think writing code will be equally intimate without the holdups from arithmetic error.

The code gave the same results as me, and seems to have reduced the fourier transform all the way down!

`https://github.com/spiveeworks/Archives/blob/c/unitary.c`