

Honours Diary 2020

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March 16, 2020

Notation

In this diary unless explicitly stated within a section, I have been using the notation specified by Nielsen and Chuang, with the following additions:

- Implicit quantifiers for index variables such as i, j, k . (Nielsen and Chuang seem to do this actually, perhaps dropping more than I do)
 - $\{x_i\} = \{x_i \mid i \in I\}$, $\{|x_i\rangle\} = \{|x_i\rangle \mid i \in I\}$ etc.
 - $(x_i) = (x_1, x_2, \dots, x_n)$
 - \sum_i in place of $\sum_{i \in I}$
 - $\forall i$ in place of $\forall i \in I$

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Exercise 2.3

$$A : V \rightarrow W$$

$$B : W \rightarrow X$$

$$V = \text{span}\{|v_i\rangle\} \text{ etc.}$$

At this point we are already identifying A , B with their matrix representations. We would like to show equality between the function composition $B \circ A$ and the matrix product of the corresponding matrix representations, $B \times A$. Once we have done this we will be able to identify both of these concepts as simply the expression BA , but for now we will use the explicit operators \circ and \times .

Our goal then is to show that the matrix representation of $B \circ A$ is $B \times A$.

$$\begin{aligned}
& \sum_i (B \circ A)_{ij} |x_i\rangle && \text{(arbitrary column of matrix } B \circ A) \\
&= (B \circ A)|v_j\rangle && \text{(Definition of matrix representation)} \\
&= B(A|v_j\rangle) && \text{(Definition of composition)} \\
&= B\left(\sum_k A_{kj} |w_k\rangle\right) && \text{(Matrix representation)} \\
&= \sum_k A_{kj} (B|w_k\rangle) && \text{(Linearity of } B) \\
&= \sum_k A_{kj} \left(\sum_i B_{ik} |x_i\rangle\right) && \text{(Matrix representation)} \\
&= \sum_i \left(\sum_k B_{ik} A_{kj}\right) |x_i\rangle && \text{(distribution)} \\
&= \sum_i (B \times A)_{ij} |x_i\rangle && \text{(matrix product)}
\end{aligned}$$

So by linear independence of $|x_i\rangle$ we know that $(B \circ A)_{ij} = (B \times A)_{ij}$ for arbitrary i, j , i.e. the matrix representation of $B \circ A$ is the matrix product $B \times A$.

Exercise 2.4

$$I : V \rightarrow V$$

$$I|x_i\rangle = |x_i\rangle$$

We would like to show that $I_{ij} = \delta_{ij}$.

$$\begin{aligned}
& \sum_i I_{ij} |x_i\rangle \\
&= I|x_j\rangle \\
&= |x_j\rangle \\
&= \sum_i \delta_{ij} |x_i\rangle
\end{aligned}$$

Again by linear independence of $\{|x_i\rangle\}$ we have $I_{ij} = \delta_{ij}$.

Exercise 2.5

$$((y_i), (z_i)) = \sum_i y_i^* z_i$$

We need to prove 3 properties. First linearity, taking $|v\rangle = (v_j) = (v_1, v_2, \dots, v_n)$ and $|w_i\rangle = (w_{ij}) = (w_{i1}, w_{i2}, \dots, w_{in})$.

For this we define

$$|z\rangle = (z_j) = \sum_i \lambda_i |w_i\rangle$$

Then by the definitions of sum and scalar product in \mathbb{C}^n we observe

$$\begin{aligned} & z_j \\ &= \left(\sum_i \lambda_i |w_i\rangle \right)_j \\ &= \sum_i (\lambda_i |w_i\rangle)_j \\ &= \sum_i \lambda_i w_{ij} \end{aligned}$$

With this linearity falls out.

$$\begin{aligned} & (|v\rangle, |z\rangle) \\ &= ((v_j), (z_j)) \\ &= \sum_j v_j^* z_j \\ &= \sum_j v_j^* \left(\sum_i \lambda_i w_{ij} \right) \\ &= \sum_i \lambda_i \left(\sum_j v_j^* w_{ij} \right) \\ &= \sum_i \lambda_i ((v_j), (w_{ij})) \\ &= \sum_i \lambda_i (|v\rangle, |w_i\rangle) \end{aligned}$$

Next we prove conjugate symmetry.

$$\begin{aligned} & (|w\rangle, |v\rangle)^* \\ &= ((w_i), (v_i))^* \\ &= \left(\sum_i w_i^* v_i \right)^* \\ &= \sum_i w_i^{**} v_i^* \\ &= \sum_i v_i^* w_i \\ &= ((v_i), (w_i)) \\ &= (|v\rangle, |w\rangle) \end{aligned}$$

Finally positivity:

$$\begin{aligned}
 &(|v\rangle, |v\rangle) \\
 &= ((v_i), (v_i)) \\
 &= \sum_i v_i^* v_i \\
 &= \sum_i |v_i|^2
 \end{aligned}$$

Clearly this expression is at least 0, with equality when $v_i = 0 \forall i$, i.e. when $|v\rangle = (0, 0, \dots, 0)$.

Therefore the operator (\cdot, \cdot) is an inner product on the vector space \mathbb{C}^n .

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Set up TeXstudio and basic document structure.

Exercise 2.1

Linear Dependence, show that $(1, -1)$, $(1, 2)$ and $(2, 1)$ are linearly dependent.

$$\begin{aligned}
 &(1, -1) + (1, 2) - (2, 1) \\
 &= (1 + 1 - 2, -1 + 2 - 1) \\
 &= (0, 0)
 \end{aligned}$$

Exercise 2.2

Matrix representations: Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A , with respect to the input basis $|0\rangle, |1\rangle$, and the output basis $|0\rangle, |1\rangle$. Find input and output bases which give rise to a different matrix representation of A .

Equation 2.12 gives us the defining property of matrix representations:

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle$$

This gives us a pair of vector equations:

$$|1\rangle = A|0\rangle = A_{00}|0\rangle + A_{10}|1\rangle$$

$$|0\rangle = A|1\rangle = A_{01}|0\rangle + A_{11}|1\rangle$$

By linear independence of $|0\rangle, |1\rangle$, it follows that

$$\begin{aligned}
 A_{00} &= 0 & A_{01} &= 1 \\
 A_{10} &= 1 & A_{11} &= 0
 \end{aligned}$$

i.e. A has the matrix representation:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$