Honours Diary 2020

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Notation

In this diary unless explicitly stated within a section, I have been using the notation specified by Nielsen and Chuang, with the following additions:

- Implicit quantifiers for index variables such as i, j, k. (Nielson and Chuang seem to do this actually, perhaps dropping more than I do)
  - $\{x_i\} = \{x_i \mid i \in I\}$, $\{|x_i\rangle\} = \{|x_i\rangle \mid i \in I\}$ etc.
  - $(x_i) = (x_1, x_2, \ldots, x_n)$
  - $\sum_i$ in place of $\sum_{i \in I}$
  - $\forall i$ in place of $\forall i \in I$

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Set up TeXstudio and basic document structure.

Exercise 2.1

Linear Dependence, show that $(1, -1)$, $(1, 2)$ and $(2, 1)$ are linearly dependent.

$$\begin{aligned} & (1, -1) + (1, 2) - (2, 1) \\ = & (1 + 1 - 2, -1 + 2 - 1) \\ = & (0, 0) \end{aligned}$$

Exercise 2.2

Matrix representations: Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle$, $|1\rangle$, and the output basis $|0\rangle$, $|1\rangle$. Find input and output bases which give rise to a different matrix representation of A.

Equation 2.12 gives us the defining property of matrix representations:

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle$$

This gives us a pair of vector equations:

$$|1\rangle = A|0\rangle = A_{00}|0\rangle + A_{10}|1\rangle$$

$$|0\rangle = A|1\rangle = A_{01}|0\rangle + A_{11}|1\rangle$$

By linear independence of $|0\rangle$, $|1\rangle$, it follows that

$$\begin{aligned} A_{00} &= 0 & \qquad A_{01} &= 1 \\ A_{10} &= 1 & \qquad A_{11} &= 0 \end{aligned}$$

i.e. A has the matrix representation:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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Exercise 2.3

$$A : V \to W$$

$$B : W \to X$$

$$V = span\{|v_i\rangle\} \ etc.$$

At this point we are already identifying A, B with their matrix representations. We would like to show equality between the function composition $B \circ A$ and the matrix product of the corresponding matrix representations, $B \times A$. Once we have done this we will be able to identify both of these concepts as simply the expression BA, but for now we will use the explicit operators $\circ$ and $\times$.

Our goal then is to show that the matrix representation of $B \circ A$ is $B \times A$.

$$
\begin{aligned}
&\sum_i (B \circ A)_{ij} |x_i\rangle && \text{(arbitrary column of matrix } B \circ A\text{)} \\
&= (B \circ A)|v_j\rangle && \text{(Definition of matrix representation)} \\
&= B(A|v_j\rangle) && \text{(Definition of composition)} \\
&= B\left(\sum_k A_{kj}|w_k\rangle\right) && \text{(Matrix representation)} \\
&= \sum_k A_{kj}(B|w_k\rangle) && \text{(Linearity of B)} \\
&= \sum_k A_{kj}\left(\sum_i B_{ik}|x_i\rangle\right) && \text{(Matrix representation)} \\
&= \sum_i \left(\sum_k B_{ik}A_{kj}\right)|x_i\rangle && \text{(distribution)} \\
&= \sum_i (B \times A)_{ij}|x_i\rangle && \text{(matrix product)}
\end{aligned}
$$

So by linear independence of $|x_i\rangle$ we know that $(B \circ A)_{ij} = (B \times A)_{ij}$ for arbitrary i, j, i.e. the matrix representation of $B \circ A$ is the matrix product $B \times A$.

Exercise 2.4

$$I : V \to V$$

$$I|x_i\rangle = |x_i\rangle$$

We would like to show that $I_{ij} = \delta_{ij}$.

$$
\begin{aligned}
&\sum_i I_{ij}|x_i\rangle \\
=&I|x_j\rangle \\
=&|x_j\rangle \\
=&\sum_i \delta_{ij}|x_i\rangle
\end{aligned}
$$

Again by linear independence of $\{|x_i\rangle\}$ we have $I_{ij} = \delta_{ij}$.

Exercise 2.5

$$((y_i), (z_i)) = \sum_i y_i^* z_i$$

We need to prove 3 properties. First linearity, taking $|v\rangle = (v_j) = (v_1, v_2, \ldots, v_n)$ and $|w_i\rangle = (w_{ij}) = (w_{i1}, w_{i2}, \ldots, w_{in})$.

For this we define

$$|z\rangle = (z_j) = \sum_i \lambda_i |w_i\rangle$$

Then by the definitions of sum and scalar product in $\mathbb{C}^n$ we observe

$$\begin{aligned}
&z_j \\
=& \left(\sum_i \lambda_i |w_i\rangle\right)_j \\
=& \sum_i \left(\lambda_i |w_i\rangle\right)_j \\
=& \sum_i \lambda_i w_{ij}
\end{aligned}$$

With this linearity falls out.

$$\begin{aligned}
&(|v\rangle, |z\rangle) \\
=& ((v_j), (z_j)) \\
=& \sum_j y_j^* z_j \\
=& \sum_j y_j^* \left(\sum_i \lambda_i w_{ij}\right) \\
=& \sum_i \lambda_i \left(\sum_j v_j^* w_{ij}\right) \\
=& \sum_i \lambda_i \left((v_j), (w_{ij})\right) \\
=& \sum_i \lambda_i \left(|v\rangle, |w_i\rangle\right)
\end{aligned}$$

Next we prove conjugate symmetry.

$$\begin{aligned}
&(|w\rangle, |v\rangle)^* \\
&= ((w_i), (v_i))^* \\
&= \left(\sum_i w_i^* v_i\right)^* \\
&= \sum_i w_i^{**} v_i^* \\
&= \sum_i v_i^* w_i \\
&= ((v_i), (w_i)) \\
&= (|v\rangle, |w\rangle)
\end{aligned}$$

Finally positivity:

$$\begin{aligned}
&(|v\rangle, |v\rangle) \\
&= ((v_i), (v_i)) \\
&= \sum_i v_i^* v_i \\
&= \sum_i |v_i|^2
\end{aligned}$$

Clearly this expression is at least 0, with equality when $v_i = 0 \ \forall i$, i.e. when $|v\rangle = (0, 0, \dots, 0)$.

Therefore the operator $(\cdot, \cdot)$ is an inner product on the vector space $\mathbb{C}^n$.

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Exercise 2.6

Combines second argument linearity with conjugate symmetry, as you would expect.

$$\begin{aligned}
\left(\sum_i \lambda_i |w_i\rangle, |v\rangle\right) &= \left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right)^* \\
&= \left(\sum_i \lambda_i \left(|v\rangle, |w_i\rangle\right)\right)^* \\
&= \sum_i \lambda_i^* \left(|v\rangle, |w_i\rangle\right)^* \\
&= \sum_i \lambda_i^* \left(|w_i\rangle, |v\rangle\right)
\end{aligned}$$

Exercise 2.7

$$\langle w|v\rangle = (1)(1) + (1)(-1) = 0$$

$$|w\rangle, |v\rangle = \left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$$

Exercise 2.8

Suppose that $j < k$, then

$$\langle v_j|v_k\rangle = \langle v_j|w_k\rangle - \sum_i^{k-1} \langle v_i|w_k\rangle\langle v_j|v_i\rangle$$