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Title:

Pivot versus Interior Point Methods:
Pros and Cons

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Abstract

Linear optimization (LO) is the fundamental problem of mathematical optimization. It admits an enormous number of applications in economics, engineering, science and many other fields. The three most significant classes of algorithms for solving LO problems are: Pivot, Ellipsoid and Interior Point Methods. Because Ellipsoid Methods are not efficient in practice we will concentrate on the computationally successful Simplex and Primal-Dual Interior Point Methods only, and summarize the Pros and Cons of these algorithm classes.

Key words. Linear programming, linear optimization, pivot methods, simplex algorithms, interior point methods, complexity, sensitivity analysis.

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1 The LO problem.

In plain English one can say that a linear optimization problem consists of optimizing, i.e., minimizing or maximizing, a *linear function* over a certain domain. The domain is given by a set of *linear constraints*. The constraints can be either equalities or inequalities. For the first sight, LO problems have a quite simple structure. Only linear functions are involved, however not only a set of linear equations has to be solved, but our task is made more difficult in two ways. Inequalities are involved as well, and an optimization component "find a solution that has the best possible value of the objective function" is present. Before discussing mathematical properties and entering our main theme – pros and cons of the major algorithms – let us first devote some paragraphs to the history of LO. The two major components: optimizing a function and solving a linear system can be traced back for centuries.

Optimization is a natural activity. Optimization elements can be found in the ancient Greek mathematics too. Lagrange [41] considered the minimization of certain functions while a set of – possibly nonlinear – equality constraints are satisfied. Although Lagrange has not considered inequalities, he had laid the foundations of duality theory.

On the side of solving linear systems an epoch making step was made by Gauss by establishing his celebrated *Gauss elimination* algorithm that is still one of the main techniques to solve systems of linear equations. However, besides equations, LO involves inequalities as well. An intellectually appealing, but computationally inefficient algorithm to solve systems of linear inequalities was designed by Fourier and Motzkin [41]. Systems of linear inequalities were thoroughly studied by Farkas [14]. He developed the theory of alternative systems. Farkas' fundamental result is still one of the most frequently cited theorems in the LO literature, it is essentially equivalent to the Strong Duality Theorem of LO.

Linear optimization – an important discipline with ever growing number of applications – was born in the middle of the 20th century as a new turbulent research field. Although some specially structured LO problems were formulated and systematically studied earlier by Kantorovich [41], Dantzig [12] was the first who developed a concise theory: geometrical analysis, duality theory, library of typical practical models and a computationally efficient algorithm, the simplex method which is still the most efficient pivot method to solve LO problems. Illuminating true stories about the birth of LO and the simplex method can be found in [32], including Dantzig's recall of those historic years. Because solving large scale LO problems is a highly computation intensive activity, from the 1950's the algorithmic developments on the area of LO are closely related to the advances in computer technology. As the speed and capacity of computers increases, new avenues open to solve larger and larger LO problems. In the 1950's, problems with hundreds of variables were considered to be large scale problems. Today everyone can solve problems of tens of thousands of variables on his own desktop PC in a couple of minutes and LO problems with millions of variables are already solved. The reader is referred to [2, 3, 7] for surveys of the computational state of the art.

The practical efficiency of the simplex method could not hide the frustrating fact that some variants of the simplex method might require prohibitively long time to solve some particular LO problems. This negative property was made explicit in the early seventies by Klee and Minty [29] as they showed that a variant of the simplex method requires exponentially many steps to solve an LO problem where the set of feasible solutions is a slightly perturbed hypercube. On the other hand, under a probabilistic model, it is proved that the average case behavior of some simplex methods is polynomial [9].

The Klee-Minty example stimulated the research for alternative algorithms for LO. It was an open problem for long time if LO problems are polynomially solvable or not. The positive result was presented by Khachian [28]. Khachian's ellipsoid method solves LO problems in polynomial time. This major result reached the front page of New York Times, but it turned out soon that in practice

the ellipsoid method is computationally not competitive with the simplex method. In practice the simplex method performs much better than its theoretical worst case bound suggests, while the practical behaviour of the ellipsoid method is close to its worst case bound.

The next break-through was reached by Karmarkar [27] in 1984. He presented an interior point method (IPM) whose worst case complexity bound was better than the one for the simplex and the ellipsoid methods. Karmarkar also claimed superior computational performance. Karmarkar's results and claims started the interior point revolution [42, 50, 51, 53]. By now, the superior theoretical properties of IPMs are obvious; highly efficient implementations are available; the implementations are at least competitive with the implementations of simplex methods and they typically have superior performance on large scale problems [3].

In this paper we summarize and compare the advantages and disadvantages of the two computationally highly successful major algorithm classes for solving linear optimization problems. We concentrate on features of Simplex Methods and Primal-Dual Path-following IPMs.

Furthermore, we discuss some important, recent developments in linear optimization, namely, the existence of short admissible pivot sequences for LO problem [20] and the concept of self-regular functions [40, 39] which became a useful tool to obtain improved complexity results for Large Update Primal-Dual IPMs. Some remarks close our survey.

2 The primal and dual LO problems

2.1 Fundamentals.

The primal and dual LO problems in standard form can be defined as

$$\text{minimize } \{ c^T x \ : \ Ax = b, \ x \geq 0 \}, \tag{P-LO}$$

$$\text{maximize } \{ b^T y \ : \ A^T y + s = c, \ s \geq 0 \}, \tag{D-LO}$$

where $c, x, s \in \mathbb{R}^n$, $b, y \in \mathbb{R}^m$ are vectors, and $A \in \mathbb{R}^{m \times n}$ is a given matrix. Without loss of generality it can be assumed that the matrix A has rank m.

2.1.1 Duality.

The *Weak Duality Theorem* of LO can easily be proved.

Theorem 2.1 (Weak Duality) *If both the primal and dual problems admit a feasible solution, let they be denoted by $\bar{x}$ and $(\bar{y}, \bar{s})$ respectively, then*

$$c^T \bar{x} \geq b^T \bar{y},$$

where equality holds if and only if

$$\bar{x}^T \bar{s} = 0.$$

Proof: By using the primal and dual equality constraints, the nonnegativity of $\bar{x}$ and $\bar{s}$ we can write:

$$c^T \bar{x} - b^T \bar{y} = (A^T \bar{y} + \bar{s})^T \bar{x} - (A\bar{x})^T \bar{y} = \bar{s}^T \bar{x} \geq 0,$$

and the theorem is proved. $\qquad \square$

The deep, fundamental result, the so-called *Strong Duality Theorem* of LO [12, 35, 42] states that if the primal and dual problems admit a feasible solution then optimal solutions with zero duality gap exist.

Theorem 2.2 (Strong Duality) *If both the primal and dual problems admit a feasible solution then both admit an optimal solution and for any optimal solutions $\bar{x}$ and $\bar{y}$ we have $c^T\bar{x} = b^T\bar{y}$, i.e., the duality gap is zero.*

Duality results can be used in many different ways. They can be used to check optimality of solutions. If a pair of solutions $\bar{x}$ and $(\bar{y}, \bar{s})$ are given, then we need to check if $\bar{x}$ is primal feasible (if $A\bar{x} = b$, $\bar{x} \geq 0$ holds); if $(\bar{y}, \bar{s})$ is dual feasible (if $A^T\bar{y} + \bar{s} = c$, $\bar{s} \geq 0$ holds) and finally, if the two objective values are equal, i.e., if $c^T\bar{x} = b^T\bar{y}$, or equivalently, if

$$\bar{x}_j(c - A^T\bar{y})_j = \bar{x}_j\bar{s}_j = 0 \quad \forall \; j = 1, \dots, n.$$

The last relation says that optimal solutions are complementary. A stronger result, characteristic only for LO problems, state that not only complementary but *strictly complementary optimal solutions*[1] always exists [23, 42].

Theorem 2.3 (Goldman-Tucker Theorem) *If both the primal and dual problems admit a feasible solution then there are primal and dual optimal solutions $\bar{x}$ and $(\bar{y}, \bar{s})$ with $\bar{x} + \bar{s} > 0$.*

These relations provide the fundamentals of different algorithmic concepts that are fleshed briefly in the sequel.

2.1.2 Fundations of algorithms.

The duality theory says that in order to find optimal solutions for both the primal and dual problems, we need to solve the system

$$
\begin{aligned}
Ax &= b, & x \geq 0, \\
A^Ty + s &= c, & s \geq 0, \\
x_j s_j &= 0, & j = 1, \dots, n.
\end{aligned}
\tag{1}
$$

Here the first line represents primal feasibility, the second one dual feasibility and the last one is the complementarity condition which guarantees optimality, i.e., $c^Tx = b^Ty$.

The specific algorithms for LO approach to solve the system of optimality conditions differently. Figure 1 illustrates the requirements of the different methods, specify which equations and inequalities are preserved by the various algorithm classes.

The complementarity condition represents a strong combinatorial character of LO problems that is featured by pivot algorithms. *Pivot Algorithms* use basic solutions and hence they always keep the primal and the dual equality constraints true. Further, by the construction of the primal and dual basic solutions the complementarity conditions are satisfied as well while nonnegativity of the variables does not necessarily hold,. Thus, pivot algorithms work toward satisfying the nonnegativity constraints.

Contrary to pivot methods where complementarity is maintained throughout the solution process, intermediate solutions generated by *Interior Point Methods* (IPMs) typically satisfy all equality and nonnegativity conditions but complementarity (see Figure 1) is attained just at termination. IPMs do not make use of the combinatorial features of LO.

In the following subsections pivot and interior point methods are reviewed briefly.

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[1] A pair of optimal solutions $\bar{x}$ and $(\bar{y}, \bar{s})$ are called strictly complementary if $\bar{x} + \bar{s} > 0$.
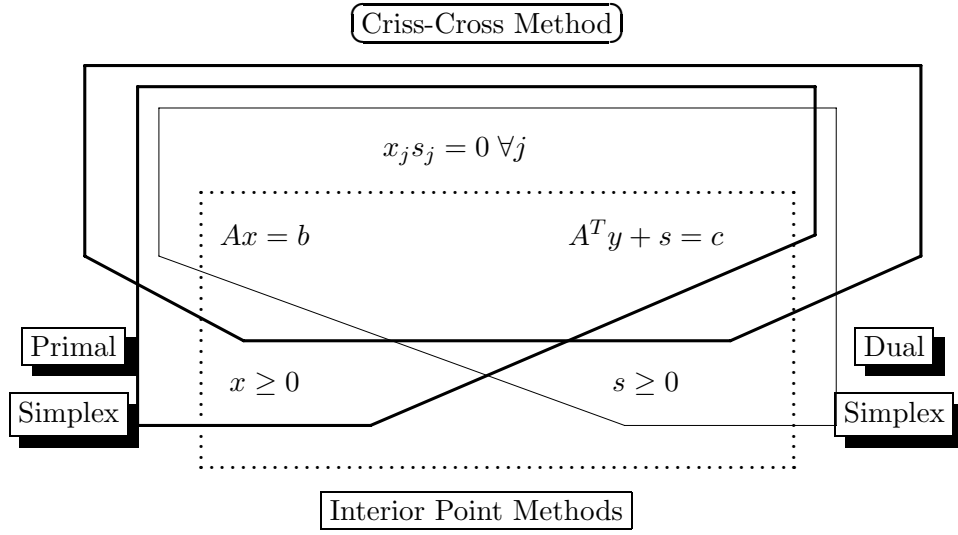
Figure 1: Characteristics of Algorithms for LO

2.2 Pivot Algorithms.

Simplex methods [12] proceed from feasible basic solution to feasible basic solution towards an optimal basic solution. Geometrically the path of basic solutions is a path on the boundary of the set of feasible solutions. It is actually a path from vertex-to-vertex, because the set of feasible solutions is a polyhedron and basic solutions are associated with vertices of this polyhedron. This process could require many steps to go around a multifaceted feasible region.

If no feasibility (nonnegativity) condition is forced to hold or forced to be preserved, then we talk about *criss-cross methods*. A *finite criss-cross method* goes through some (possibly both primal and dual infeasible) basic solutions until either primal infeasibility, dual infeasibility or optimality is detected. This procedure can be started with *any* basic solution and solves the linear programming problem in **one phase**, in a finite number of pivot steps. The least-index criss-cross method [44, 19] is perhaps the simplest possible pivot algorithm and gives rise to a very short algorithmic proof of the Strong Duality Theorem 2.2 (see e.g., [19]).

Simplex pivot methods always require and preserve either primal or dual feasibility of the actual basic solutions. A primal (resp. dual) simplex method is initiated by a primal (dual) feasible basic solution. If neither optimality nor unboundedness is detected, then a new basis and the related basic solution is chosen in such a way that a dual (primal) infeasible variable enters (leaves) the basis and: (a) the new basis is again primal (dual) feasible; (b) the new basis differs exactly by one element, one variable from the old one, i.e., the new basis is a neighbor of the old one; (c) the primal (dual) objective value monotonically improves. Feasibility of the basic solution is preserved throughout and the primal (dual) objective function value is monotonically decreasing (increasing) at each basis exchange. To produce a primal or dual feasible basic solution is a nontrivial task. It requires the solution of another LO problem, the so-called *first phase* problem [12].

2.2.1 Basic solutions.

We have seen on Figure 1 and equations (1) that to solve the primal and dual LO problems one need to solve a system of nonlinear equations. The candidate solutions in all pivot algorithms always satisfy the equality and complementarity constraints. This property follows from the fact that they use basic solutions and in the subsequent iterations they go from basis to basis while approaching optimality. Let us see first what can we gain by using basic solutions.

Definition 2.1 *Let $A_B \in \mathbb{R}^{m \times m}$ be a nonsingular submatrix of A, i.e.,* <mark>*A_B contains m linearly independent columns of the matrix $A = [a_1, \ldots, a_j, \ldots, a_n]$. Then A_B is called a basis of the LO problem.*</mark>

Let $I_B \subseteq \{1, \ldots, n\}$, with $j \in I_B$ if and only if the vector a_j is a column of A_B, be the set of basis indices. Further, let A_N denote the complement matrix, the nonbasic part of the matrix A. Let we also partition the vectors c, x and s analogously, i.e., c_B, x_B and s_B denote the subvectors containing the coordinates whose index is in I_B while the complement vectors are denoted by c_N, x_N and s_N, respectively.

Clearly, by setting all the coordinates of the vector x_N equal to zero a unique solution of the linear equation system

$$Ax = A_B x_B = b$$

can be obtained. The unique solution is given by $x_N = 0$ and $x_B = A_B^{-1} b$. The solution x obtained this way is called a *primal basic solution* and can equivalently be defined as $x_j = 0$ when $j \in I_N = I \setminus I_B$ and $x_j = (A_B^{-1}b)_i$ if a_j is the i-th column of the basis matrix A_B.

Analogously, a *dual basic solution* can be constructed by considering the dual constraints associated with the basis indices I_B and setting all the coordinates of s_B equal to zero. Then the dual basis solution y is the unique solution of the linear equation system

$$A_B^T y = c_B.$$

Thus $y = A_B^{-T} c_B$, where the superscript $^{-T}$ denotes the inverse of the transposed matrix A_B^T. Then the vector s_N can easily be computed as $s_N = c_N - A_N^T y = c_N - A_N^T A_B^{-T} c_B$. We can introduce the *(short) basic tableau* in the following way

$$T = \left(\begin{array}{c|c} -c_B^T A_B^{-1} b & c_N^T - c_B^T A_B^{-1} A_N \\ \hline A_B^{-1} b & A_B^{-1} A_N \end{array} \right)$$

where $T \in \mathbb{R}^{(m+1) \times (n-m+1)}$ and $T = (\tau_{ij})$ with τ_{ij} denoting the coefficient of the basic vector a_i in the basis representation of the non-basic vector a_j, i.e., $aj = \sum_{i \in I_B} \tau_{ij} a_i$ for all $j \in I_N$.

If we take a pair x and (y, s) of primal and dual basic solutions associated with the same basis A_B then by definition not only the primal equality constraints $Ax = b$ and the dual equality constraints $A^T y + s = c$ are satisfied but the complementarity conditions $x_j s_j = 0, \forall j = 1, \ldots, n$ are satisfied as well. This property is obvious because $x_j = 0$ if $j \notin I_B$ while $s_j = 0$ if $j \in I_B$.

The discussions so far demonstrate that basic solutions satisfy all the equality and complementarity constraints. When using basic solutions we have to work to satisfy the inequality constraints. To make our discussions more plausible we need some further terminology. <mark>A basis A_B, and the corresponding basic solutions x and (y, s) are called *primal feasible* if $x_B \geq 0$, it is *dual feasible* if $s_N \geq 0$ and the solutions are *optimal* if they are both primal and dual feasible.</mark>

All pivot algorithms proceed from basic solution to basic solution towards an optimal basic solution. At each step a basis is at hand, optimality i.e., primal and/or dual feasibility is checked and a new basis is selected. We consider three main classes of pivot algorithms:

2.2.2 Criss–cross methods.

Criss–cross methods can be initialized by an arbitrary basic solution. Then optimality is checked. If the basic solution is optimal we are done, else a primal or dual infeasible variable (coordinate) is chosen and we go over to a new basis in which this specific coordinate becomes both primal and dual feasible. The process is repeated as long as optimality is reached or evidence of primal or dual infeasibility is established (see e.g. [44, 19]).

Least–Index Criss–Cross Method

Initialization:

    A_B be an arbitrary initial basis,

    I_B, resp. I_N is the index set of the basis and nonbasis variables.

begin
  while true **do**

    if $x_B \geq 0$ and $s_N \geq 0$ **then**
(I)   **stop**: the current solution solves the LO problem;

    else
      $p := \min\{i \in I_B : x_i < 0\}$;
      $q := \min\{j \in I_N : s_j < 0\}$;
      $r := \min\{p, q\}$;
      if $r = p$ **then**
        if the p-row of the tableau is nonnegative **then**
(II)   **stop**: **(P-LO)** is inconsistent;

        else
          let $q := \min\{j \in I_N \ : \ \tau_{pj} < 0\}$;
        endif
      else (i.e., $r = q$)
        if the q-column of the tableau is nonpositive **then**
(III)   **stop**: **(D-LO)** is inconsistent;

        else
          let $p := \min\{i \in I_B : \tau_{iq} > 0\}$;
        endif
      endif
      perform a pivot: $I_B := I_B \cup \{q\} \setminus \{p\}$;
  endwhile
end.

The following striking result [19, 44] can be proved:

Theorem 2.4 *The least-index criss–cross method is finite.*

The least-index criss–cross rule seems to be completely rigid, there is no any flexibility in pivot selection. However, it is worth to note that the finiteness proofs allow significant flexibility in pivot selection [18]. Exploring this flexibility several variants, e.g., the *last-in-first-out* and the *most-often-selected variable* criss–cross methods were developed [52] (see also [19, 25]). If we alleviate all this consistent selections and apply the criss–cross scheme by making arbitrary pivot choices that respect only the sign requirements at the respective steps then the general criss–cross scheme is obtained. In this generality nothing can be said about finiteness of criss–cross rules.

Finally, note that in spite of their elegance and simplicity, to date criss–cross methods are not efficient in practice.

2.2.3 Primal simplex methods.

Primal simplex algorithms require that at each stage the basic solution is primal feasible. Having a primal feasible basic solution at hand, primal simplex methods check if the basis is dual feasible. If so, it is optimal and we are done. If it is not dual feasible, a dual infeasible coordinate is chosen and we pass over to a new basis which is again primal feasible and the previously dual infeasible variable enters the basis and hence became both primal and dual feasible. The process is repeated as long as optimality is reached or evidence of dual infeasibility is established (see e.g. [12, 36]).

Primal Simplex Method

Initialization:
    A_B be a primal feasible basis, i.e., $x_B \geq 0$;
    I_B, resp. I_N is the index set of the basis and nonbasis variables.

begin
  while true **do**

    if $s_N \geq 0$ **then**
(I)   **stop**: the current solution solves the LO problem;

    else
      let $q \in I_N$ be an index with $s_q < 0$;
      (in the least index version let $q := \min\{j \in I_N : s_j < 0\}$);
        if the q-column of the tableau is nonpositive **then**

(III)   **stop**: (**D–LO**) is inconsistent;

      else
        let $\vartheta := \min\{\frac{x_i}{\tau_{iq}} : i \in I_B \text{ and } \tau_{iq} > 0\}$;
        let $p \in I_B$ be such that $\frac{x_p}{\tau_{pq}} = \vartheta$; (ratio test)
        (in the least index version let $p := \min\{i \in I_B : \frac{x_i}{\tau_{iq}} = \vartheta\}$);
      endif
    endif
    perform a pivot: $I_B := I_B \cup \{q\} \setminus \{p\}$;
  endwhile
end.

Observe, that when dual inconsistency is detected, then one may also conclude that the primal problem is unbounded. This is due to the extra information that a primal feasible solution is at hand. The following result can be proved:

Theorem 2.5 *The primal simplex method produces feasible basic solutions with monotonically non-increasing objective values. Moreover it terminates in a finite number of steps when implemented with the least-index resolution.*

The vast literature of LO contains an extensive family of primal simplex methods. The inherent flexibility of the entering and leaving variable selection allows to develop various pivot selection criteria. Here just the recently most popular *steepest-edge* criteria is mentioned [22]. There are other finite variants like the lexicographic and double lexicographic method, variants that build-up

the solution in some specific way etc. in the literature. The reader can find most classical simplex method variants in [12, 36] and it is advised to consult the recent survey [46] to have an impression about the latest results about simplex methods.

The initialization of primal simplex methods require a primal feasible basis to start with. To get a primal feasible basis is not trivial, a so-called first phase problem needs to be solved to get a primal feasible basic solution. The optimal solution of the first phase problem either provides a primal feasible basic solution, or gives evidence of infeasibility of the problem **(P-LO)**. Later on in Section 3 some first-phase strategies will be discussed.

2.2.4 Dual simplex methods.

Dual simplex algorithms require that at each stage the basic solution is dual feasible. Having a dual feasible basic solution at hand, dual simplex methods check if the basis is primal feasible. If so, it is optimal and we are done. If it is not primal feasible, a primal infeasible coordinate is chosen and we pass over to a new basis which is again dual feasible and the previously primal infeasible variable enters the basis and hence became both primal and dual feasible. The process is repeated as long as optimality is reached or evidence of primal infeasibility is established (see e.g. [12, 31, 36, 47]).

<div align="center">

Dual Simplex Method

</div>

Initialization:
A_B be a dual feasible basis, i.e., $s_N \geq 0$;
I_B, resp. I_N is the index set of the basis and nonbasis variables.

begin
  while true **do**

    if $x_B \geq 0$ **then**
(I)   **stop**: the current solution solves the LO problem;

    else
      let $p \in I_B$ be an index with $x_p < 0$;
      (in the least index version let $p := \min\{i \in I_B : x_i < 0\}$);
        if the p-row of the tableau is nonnegative **then**
(II)   **stop**: **(P-LO)** is inconsistent;

      else
        let $\vartheta := \min\{\frac{s_j}{-\tau_{pj}} : j \in I_N \text{ and } \tau_{pj} < 0\}$;
        let $q \in I_N$ be such that $\frac{s_q}{-\tau_{pq}} = \vartheta$; (ratio test)
        (in the least index version let $q := \min\{j \in I_N : \frac{s_j}{-\tau_{pj}} = \vartheta\}$);
      endif
    endif
    perform a pivot: $I_B := I_B \cup \{q\} \setminus \{p\}$;
  endwhile
end.

The following result can be proved:

Theorem 2.6 *The dual simplex method produces dual feasible basic solutions with monotonically non-decreasing objective values. Moreover, it terminates in a finite number of steps when it is implemented with the least-index resolution.*

Some variants of the dual simplex method can be found in [12, 46] as well.
To get a dual feasible basis is not trivial, a so-called first phase problem needs to be solved to get a dual feasible basic solution. The optimal solution of the first phase problem either provides a dual feasible basic solution, or gives evidence of infeasibility of the problem **(D-LO)**.

2.3   Interior Point Methods for LO

Intuitively, a path through the interior of the region of the LO problem is appealing since there exists the possibility of moving through the polytope just in a few steps. This is what interior point methods do.

IPMs seek to approach the optimal solution through a sequence of points that are always strictly feasible. Such methods have been known for a long time [15] but, because they are more demanding for storage and reliable floating point arithmetic than simplex methods, in the early years of OR they were not considered to be efficient. Contrary to pivot algorithms, IPMs – at least the most popular primal-dual ones – require and preserve both primal and dual feasibility while the complementarity condition (1) is relaxed. In fact, the feasibility requirement is even stronger, not only $x, s \geq 0$ is required but $x, s > 0$. IPMs follow the so-called *central path* [43]. We may assume w.l.g. that rank$(A) = m$ and strictly feasible solutions exist [42, 45, 53, 51]. Then the parameterized nonlinear system

$$
\begin{aligned}
Ax &= b, & x \geq 0, \\
A^T y + \quad s &= c, & s \geq 0, \\
xs &= \mu e,
\end{aligned}
\tag{2}
$$

where $xs \in \mathbb{R}^n$ denotes the coordinatewise (Hadamard) product of the vectors x and s and $e = (1, \ldots, 1)^T \in \mathbb{R}^n$. System (2) has a unique solution $(x(\mu), y(\mu), s(\mu))$ for each $\mu > 0$. The primal central path is defined as $\{x(\mu) : \mu > 0\}$, while the dual central path is $\{(y(\mu), s(\mu)) : \mu > 0\}$. It is evident that as $\mu \to 0$ the central path approaches the set of optimal solutions, because the perturbation of the complementarity condition approaches zero.

Note, that the parameterized system (2) was originally derived as the Karush-Kuhn-Tucker system of the primal, as well as the dual *logarithmic barrier problems*:

$$
\text{minimize}\{c^T x - \mu \sum_{j=1}^{n} \log x_j \; : \; Ax = b, \; x \geq 0\},
$$

$$
\text{maximize}\{b^T y + \mu \sum_{j=1}^{n} \log s_j \; : \; A^T y + s = c, \; s \geq 0\}.
$$

The role of the barrier function $\log(\cdot)$ is to keep the variables x and s strictly positive.

2.3.1   Primal-Dual Newton Methods

Primal-Dual Path-following IPMs use *Newton steps* to follow the central path that leads the iterative process to an optimal solution. The Newton step $(\Delta x, \Delta y, \Delta s)$ is the unique solution of the equation

system

$$A\Delta x \qquad\qquad = 0,$$
$$A^T \Delta y \qquad +\Delta s = 0, \qquad\qquad (3)$$
$$s\Delta x + x\Delta s = \mu e - xs.$$

The reader easily verifies that the displacements Δx and Δs are orthogonal, they are in the null and range spaces of the row vectors of the matrix A, respectively. IPMs take damped Newton steps, such that the new triple $(x + \alpha\Delta x, y + \alpha\Delta y, s + \alpha\Delta s)$, where α represents the step-length, remains in the interior of the feasible set and is 'close' to the target point $(x(\mu), y(\mu), s(\mu))$ on the central path. Closeness is measured by using a proper proximity function.

With subsequent damped Newton steps IPMs reduce the proximity below a threshold value and then the parameter μ is reduced to move the target point on the central path closer to the optimal face. The general scheme of IPMs is as follows:

Large Update Primal-Dual Interior Point Algorithm

Input:
    A proximity measure $\delta(xs, \mu)$;
    a proximity parameter $\tau > 0$;
    an accuracy parameter $\varepsilon > 0$;
    a variable damping factor α;
    a barrier update parameter $0 < \theta < 1$;
    (x^0, s^0) and $\mu^0 = 1$ such that $\delta(x^0 s^0, \mu^0) \leq \tau$.
begin
    $x := x^0$; $s := s^0$; $\mu := \mu^0$;
    while $n\mu \geq \varepsilon$ **do**
    begin
        $\mu := (1 - \theta)\mu$;
        while $\delta(xs, \mu) \geq \tau$ **do**
        begin
            Solve the Newton system for $\Delta x, \Delta y, \Delta s$;
            $x := x + \alpha\Delta x$;
            $s := s + \alpha\Delta s$;
            $y := y + \alpha\Delta y$
        end
    end
end

During this process we need to measure the distance of the iterates to the central path, thus we need a proximity measure that satisfies the following basic requirements: a proximity must be zero for the target point $(x(\mu), s(\mu))$ on the central path and increasing as the point is getting further away from the central path. Various proximity measures were introduced and analyzed in the last decade, e.g.,

$$\delta_c(xs, \mu) = \left\| \frac{xs}{\mu} - e \right\| \quad \text{and} \quad \delta_b(xs, \mu) = \left\| \sqrt{\frac{xs}{\mu}} - \sqrt{\frac{\mu}{xs}} \right\|.$$

Both of $\delta_c(xs, \mu)$ and $\delta_b(xs, \mu)$ satisfy the basic requirement, however $\delta_c(xs, \mu)$ does not go to infinity as the candidate solution (x, y, s) approaches the boundary of the feasible set, i.e., it does not have the barrier property. The proximity $\delta_b(xs, \mu)$ does have the barrier property, thus applicable for global analysis of IPMs.

Large families of primal–dual IPMs were designed in the last years. Most of those algorithms fit somehow in the above frame. They differ in the proximity measure used, how the update of the parameter μ is controlled, how the step-length α is calculated. Practical IPMs that use large neighborhood and large update of μ have weaker worst case complexity bound than IPM that follow the central path closely.

3   Simplex versus Interior Point Methods

In this section the major features, pros and cons of simplex and IPMs are listed. At each feature first the properties/results related to simplex methods, then in italic the same issues regarding IPMs are addressed. Here we take the freedom to consider only Simplex Methods from the class of pivot methods and Primal-Dual Path-Following IPMs. Most results listed below, regarding simplex methods can be found in the books [12, 35] and in the survey [46] in the books [42, 51, 53]. Due to space limitations the list is not complete. Further, we note that some of the statements do not have strong mathematical foundation, they are based on empirical evidence.

Characteristics of the iterative sequences: Simplex methods generate a sequence of feasible basic solutions. Complementarity of the candidate primal-dual solutions holds by definition. This geometrically means jumping from vertex-to-vertex on the boundary of the feasible set. The iterates are primal (or dual) feasible, the primal (or dual) objective improves monotonically; dual (or primal) feasibility is reached at optimality.

IPMs generate a sequence of strictly feasible primal and dual solutions. The iterates are in the (relative) interior of the feasible sets, in a predetermined neighborhood of the central path. Complementarity is reached only at optimality. The duality gap decreases monotonically. IPMs exhibit asymptotic superlinear and quadratic convergence to the optimal face [42, 51, 53].

The generated solutions: Simplex methods generate an optimal basic solution. In case of degeneracy the generated optimal basic solution is not strictly complementary. To find a strictly complementary solution from an optimal basis is not easy, it is just as difficult as solving the original problem.

IPMs generate a sequence of strictly feasible (interior w.r.t. the positive orthant) primal and dual solutions. The duality gap, but not necessarily the primal or dual objective, decreases monotonically. IPMs produce an ϵ optimal solution, i.e., a feasible solution pair for which the duality gap $c^T x - b^T y < \epsilon$ is small. In case of degeneracy IPMs converge to the analytic center[2] *of the optimal face and produce a strictly complementary pair of solutions.*

To produce an exact solution to the LO problem is not the privilege of pivot algorithms. Starting from a sufficiently precise solution an exact strictly complementary solution can be identified [42] in strongly polynomial time. Moreover, if a basic solution is needed then a strongly polynomial basis identification procedure can be applied to identify an optimal basis [3, 42, 53, 51].

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[2]The analytic center x^a of a polyhedron given by $\{x \,:\, Ax = b, \ x \geq 0\}$ that admits an interior solution $x > 0$ is defined as

$$x^a := \underset{Ax=b, x>0}{\arg \max} \{\prod_{j=1}^{n} x_j\}.$$

Complexity: Simplex methods [12] proceed from basic solution to basic solution towards an optimal basic solution. The number of basic solutions might be an exponential function of the problem dimension and in the worst case, simplex methods might visit all those basic solutions. This was made explicit by Klee and Minty [29]. They showed that if the set of feasible solutions is a slightly out of kilter n-dimensional hypercube then all the feasible basic solutions are visited by a simplex method. The actual path is initiated at the origin and visits all 2^n vertices so that the objective function is strictly improved at each step. This result proves that the worst case complexity of some variants of the simplex method is exponential. An interesting result is that the dual simplex method solves the Klee-Minty problem in a polynomial number of iterations [35]. A more challenging exponential example is given by [11] (see also [35]). The main feature of Clausen's example is that the primal simplex method is exponential on the primal problem while the dual simplex method is exponential on the dual problem.

Since Klee and Minty's work has been published, the exponential behavior of several simplex method variants is demonstrated. However, there are still some variants, like Zadeh's rule (see e.g. [46]) whose exponential worst case complexity is not established yet.

An interesting result is proved recently by [20]. They proved that starting from any basis there exists a short admissible pivot[3] sequence – containing at most n-pivot steps – leading to an optimal basis. This result indicates that polynomial pivot algorithms might exist, at least, starting from any basis, there exists always a short admissible pivot path leading to an optimal basis.

There was a huge gap between the theoretical worst case complexity and practical performance of simplex methods. It was obvious that no practically implemented simplex method would use the exponentially long Klee-Minty path. Actually, commercial software solve the Klee-Minty problem in one step. This is primarily due to the implemented, quite sophisticated multiple pricing pivot selection methods. The need to explain the efficiency of practical simplex methods became transparent. It was shown soon that under a probabilistic model the expected (average) number of steps required by the shadow-vertex simplex algorithm is linear [9].

Interior point methods enjoy polynomial time worst case complexity. So far the best known iteration bound is $\mathcal{O}(\sqrt{n}L)$ Newton steps which cost each $\mathcal{O}(n^{2.5})$ arithmetic operations resulting in an $\mathcal{O}(n^3L)$ total complexity [38, 42, 53, 51].[4] The cost of identifying an exact strictly complementary solution is $\mathcal{O}(n^3)$ arithmetic operations, while an optimal basis can be found by using at most n additional pivots, i.e., by $\mathcal{O}(n^3)$ arithmetic operations.

As we mentioned in Subsection 2.3, IPMs that are efficient in practice have a weaker, $\mathcal{O}(nL)$ worst case complexity bound. Thus, a gap occurred between theory and practice. This gap was narrowed by the discovery of Self-Regular proximity based IPMs [40] (see Section 4.2 for a brief review). The complexity $\mathcal{O}(n^{\frac{q+1}{2q}}L)$ of these IPMs primarily depend on the parameter $q \geq 1$. The best result $\mathcal{O}(\sqrt{n}\log n \cdot L)$ is obtained when the Self-Regular proximity $\Upsilon_{pq}(t)$ (for the definition see Section 4.2) is applied with $p = 2$ and $q = \log n$.

Initialization: Simplex methods solve LO problems in two-phases [12, 35] because simplex methods require a feasible basic solution to start with. Such a solution is readily available for the so-called first-phase problem and the optimal basic solution of this first phase problem provides a feasible basic solution for solving the original problem. Criss-cross methods solve LO problems in one phase, but to date no practically efficient variant of the criss-cross method is known.

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[3]For the definition of an *admissible pivot* the reader is referred to [20] or Definition 4.1 later in this paper.

[4]Here L denotes the input length of the given LO problem. The $\mathcal{O}(n^{2.5})$ operation bound to solve the Newton system (3) is reached by using a partial update scheme.

IPMs require a positive vector to start with. When one applies an infeasible IPM *then the Newton procedure makes sure that finally the equality constraints will be satisfied as well. The price of this simple direct approach is some weaker complexity result. The best approach is based on the so-called* self-dual embedding model *[53, 42, 51] which can be initialized by any positive vector. By marginal increase of the computational cost per iteration the most efficient feasible IPMs can be applied for the embedding model and this way the LO problem can be solved in one phase.*

Degeneracy: Let us discuss first what we mean by degeneracy. A basis (basic solution) is called *primal degenerate* if x_B contains a zero coordinate, i.e., if there is an $i \in I_B$ with $x_i = 0$. Analogously, a basis (basic solution) is called *dual degenerate* if s_N contains a zero coordinate, i.e., if there is a $j \in I_N$ with $s_j = 0$. Finally, the basis is called degenerate if it is either primal or dual degenerate.

A sever consequence of degeneracy is that in primal or dual simplex methods the primal, resp. dual objective value remains the same in subsequent iterations. This property opens the possibility for cycling, i.e., starting from a certain basic solution, the same basic solutions are revisited again and again. This problem was recognized and demonstrated early e.g., by [5, 24]. Various tools were developed to avoid cycling. The two most famous one are the lexicographic simplex method [12, 10, 49] which is equivalent to a perturbation technique [13] and the utmost simple least–index rule [8, 44] which was discussed in this paper. Further, various heuristics are developed to circumvent these problems.

Not only theoretically, but also in practice pivot methods suffer from problems – especially stalling – arising from degeneracy. Various heuristics are developed and implemented to circumvent these problems [16, **?**].

Another implication of degeneracy is that multiple optimal solutions occur. This might sound appealing, after all we will have the freedom to choose from the multiple optimal solutions. It turned out that having multiple solutions cause more problems than benefits. Typically we do not know all the multiple optimal solutions and have limited or no control at all which optimal solution will be produced by simplex methods.

Contrary to pivot methods, degeneracy is not an issue in IPMs. Theoretical complexity results hold without any non-degeneracy assumption and, what is at least so important, the practical performance of IPMs is not affected by degeneracy. In case of degeneracy IPMs produce a strictly complementary solution close to the analytic center of the optimal face.

Software: The state-of-the-art implementations of simplex methods, e.g., XPRESS-MP, OSL and CPLEX keep competing with IPM based software. The simplex method is very flexible, it allows the implementation of various heuristics to enhance practical performance. A major development of the last years is the implementation of automatized dualization and dual simplex methods. These allow flexibility in choosing which of the primal or the dual problems allow more efficient solution. Dual simplex methods are very efficient to solve relaxations of discrete optimization problems and are particularly powerful in branch-and cut methods.

One of the secrets behind the computational success of IPMs is highly efficient sparse matrix technology. Symbolic factorization, sparse Cholesky and Bunch-Parlett factorization, [3, 2] are the key tools in calculating the Newton step. As a result, various highly efficient IPM based software were developed in the last years. The reader can try BPMPD, CPLEX-Barrier, LOQO, LIPSOL, MOSEK, OSL, PCx (see e.g. `http://neos.mcs.anl.gov`*).*

High performance packages like XPRESSMP, CPLEX, MOSEK and OSL offer both simplex and

IPM solvers. At the following sites extensive information is available about various optimization software:

NEOS: Optimization Software Guide   `http://www-fp.mcs.anl.gov/otc/Guide`

Decision tree for optimization software   `http://plato.la.asu.edu/guide.html`

Practical performance: Despite the contention that simplex methods could require a large number of steps, they are still – sometimes favorably – compete with polynomial interior point methods [7, 3, 2, 16, 33, 34].

Although no polynomial time version of the simplex method is known and exponential examples are constructed for most variants, the simplex method allows the implementation of various heuristics to enhance practical performance. Implementations of simplex methods keep competing and are still in the core of LO packages. The computational state of the art of simplex-type algorithms can be reviewed by studying the following literature: [7, 16, 37, 34].

The reader can gain updated information at the following internet sites about availability, capability and performance of the state-of-the-art LO packages.

CPLEX   `http://www.cplex.com`

XPRESSMP   `http://www.dash.co.uk`

OSL   `http://www.research.ibm.com/osl`

MOSEK   `http://www.mosek.com`

IPM software is especially efficient for solving very-large scale LO problems. When solving huge problems, possibly involving millions of variables, and solving highly degenerate problems IPMs outperform simplex method based codes. In practice IPMs need about 15-30 Newton steps to solve LO problems [3, 2, 33].

Some comparative performance result can be found at:
Benchmark for optimization software: `http://plato.la.asu.edu/bench.html`

Warm-start: Having the problem solved, the LO problem frequently need to be solved again with slightly changed data. Re-starting a simplex algorithm from the previous optimal solution, in most cases, allows quick solution of the modified problem.

Although many efforts, sometimes with promising results, were invested to design efficient warm-start procedures, IPM codes do not exhibit such efficiency in re-solving LO problems with perturbed data as simplex based software does.

The relative inefficiency of IPM warm-start is due to the high cost of one iteration. An IPM iteration costs $\mathcal{O}(n^3)$ arithmetic operations while a pivot costs only $\mathcal{O}(n^2)$ arithmetic operations. Thus, in case of large-scale problems, thousands of pivots are still cheaper than a single IPM iteration. As a consequence, for slightly modified LO problems simplex warm-start re-solves the LO problem faster than a single IPM iteration.

Efficiency when solving integer linear problems: To date simplex methods are the clear winner in solving (mixed) integer LO problems. One of the reasons is that to generate cuts a basic solution is needed. More importantly, the recently implemented dual simplex methods are incredibly efficient in Branch-cut-and-bound schemes when a large number of slightly modified

subproblems are solved repeatedly. In these schemes the ability of efficient warm-start is a paramount feature [7, 16, 34].

Another significant development is the implementation of the – by now classical – steepest-edge simplex algorithm [22, 7, 16]. The steepest-edge simplex algorithm is proved to be efficient in practice to solve combinatorial optimization problems.

As mentioned earlier, a basic solution can always be obtained by using IPM codes when proper basis identification schemes are implemented. Therefore cutting plane methods are applicable by using IPMs as well. The lack of efficient warm-start procedures make IPMs less efficient for solving integer LO problems.

Sensitivity Analysis Sensitivity analysis has a great impact when LO problems are used in practice. Sometimes sensitivity information is more important for practical use then the solution itself. Traditionally, sensitivity information is provided by most commercial packages. Unfortunately, even today, most packages neglect the traps arising from degeneracy and provide sensitivity information based only on the generated optimal basic solution. In case of degeneracy, such sensitivity information is incomplete and incorrect, thus might lead to erroneous decisions with severe consequences [48]. Recently, easily computable methods were developed to provide correct and complete sensitivity information [1, 26, 42, 30]. The key observation is that for correct and complete sensitivity information not only a single optimal solution, but a representation of the set of optimal solutions is needed. Which representation of the optimal set is available might depend on the algorithm used.

To give an impression how to calculate correct sensitivity information we discuss here briefly how to calculate correct shadow prices and the intervals where these shadow prices are valid. Let us assume that the vector b varies, the actual right-hand-side is $b(\beta) = b + \beta\Delta b$ for some $\beta \in \mathbb{R}$. We are interested in how the optimal value function $f(\beta)$ varies, where

$$f(\beta) := \min_x \{c^T x \mid Ax = b + \beta\Delta b, \ x \geq 0\}.$$

It is well known that $f(\beta)$ is a piecewise linear function, thus it is not everywhere differentiable. At a given β value, its right and the left derivative might be different. The left $f'_-(\beta)$ (right $f'_+(\beta)$) derivative of $f(\beta)$ is called the *left (right) shadow price*. We are interested to calculate these shadow prices e.g. at $\beta = 0$ and the intervals (left and right again) where these derivatives are constant.

Let $\mathcal{P}^*_\beta$ and $\mathcal{D}^*_\beta$ denote the set of optimal solutions for the primal and dual problems, respectively, i.e.,

$$\mathcal{P}^*_\beta := \{x \mid c^T x = f(\beta), \ Ax = b + \beta\Delta b, \ x \geq 0\},$$
$$\mathcal{D}^*_\beta := \{(y, s) \mid (b + \beta\Delta b)^T y = f(\beta), \ A^T y + s = c, \ s \geq 0\}.$$

Then, the left and right shadow prices can be calculated by solving some subsidiary LO problems defined on the dual optimal set.

Theorem 3.1 *Let $f'_-(\beta)$ and $f'_+(\beta)$ be the left and right derivative of $f(\cdot)$ in β. Then*

$$f'_-(\beta) = \min \ \{ \ \Delta b^T y \ : \ y \in \mathcal{D}^*_\beta \ \},$$
$$f'_+(\beta) = \max \ \{ \ \Delta b^T y \ : \ y \in \mathcal{D}^*_\beta \ \}.$$

The corresponding linearity intervals *(ranges)* can similarly be calculated by solving subsidiary LO problems defined on the primal optimal set.

Theorem 3.2 *Let $[\beta_1, \beta_2]$ be a linearity interval of the optimal value function $f(\beta)$. Let $\overline{\beta} \in (\beta_1, \beta_2)$ and define $\overline{\mathcal{D}}^* := \mathcal{D}^*_{\overline{\beta}}$. Then*

$$\beta_1 = \min_{\beta,x}\{\beta : Ax - \beta\Delta b = b, \ x \geq 0, \ x^T s = 0, \ for \ an \ s \in \overline{\mathcal{D}}^*\},$$

$$\beta_2 = \max_{\beta,x}\{\beta : Ax - \beta\Delta b = b, \ x \geq 0, \ x^T s = 0, \ for \ an \ s \in \overline{\mathcal{D}}^*\}.$$

The analogous results for reduced costs and their validity ranges can be found in the already cited literature, however by dualization one might easily derive them. For more details the reader is referred to Chapter 7 in [42].

It seems to be quite costly to solve an LO problem for each shadow price and range, but one must realize, that in the presence of degeneracy there is no cheaper way to get correct sensitivity information. It worth to mention that these subsidiary problems can be solved quite efficiently by appropriately choosing primal or dual methods for which the previous optimal solution provides a feasible solution to start with. The ability of efficient warm start might favor simplex methods to perform sensitivity analysis.

The success of IPMs forced to reconsider the theory of sensitivity and parametric analysis. As it is demonstrated in the foregoing paragraphs, sensitivity analysis can be performed by having any optimal solution pair and, in case of degeneracy, an optimal basis does not contain any useful extra information regarding this question. A maximally complementary solution given by IPMs suits perfectly for making correct sensitivity analysis [42].

Generalizations Simplex methods provide a solid base for various nonlinear optimization algorithms, such as complementary pivot algorithms to solve linear complementary problems, (generalized) reduced gradient methods and decomposition methods to solve nonlinear and semi-infinite optimization problems [4].

All the areas where pivot methods were adapted, IPMs were applied as well [38]. On the top of those, IPMs provide most efficient algorithms to solve conic-linear optimization problems. First of all semi-definite and second-order cone optimization problems are formed and solved with success by using IPMs. Such conic-linear optimization problems provide a rich area of novel applications in engineering sciences [6]. Moreover, semi-definite optimization allows powerful approximation schemes for various hard combinatorial optimization problems. The obtained approximation results are significantly better for many problems than those obtained by using linear relaxations [21].

4   Some important, recent developments in linear optimization

Unfortunately, none of the existing pivot algorithms admits polynomial complexity for LO problems. The most challenging open question in linear optimization is the following: find a polynomial pivot algorithm for LO problem, i.e., one for which the number of pivot operations is bounded by a polynomial function of the input size, or to prove that such pivot algorithm do not exist. Fukuda, Lüthi and Namiki [17] proved that the length of a shortest admissible pivot sequence *from any* (not necessarily feasible) basis to an *optimal* basis is $\min\{m, n\}$, under the assumption that the LO problem is totally (i.e., primally and dually) nondegenerate.

Recently, Fukuda and Terlaky [20] presented an *admissible pivot algorithm* which finds an admissible pivot sequence *from any* (not necessarily feasible) basis to an *optimal* basis. The length of the generated pivot sequence is at most n, and the totally nondegeneracy assumption (which was used

by Fukuda, Lüthi and Namiki [17]) has been removed. The admissible pivot algorithm of Fukuda and Terlaky [20] will be discussed in the next subsection.

Large Update Primal–Dual IPMs perform much more efficiently in practice than small-update methods [42, 39], however the complexity results are much better for the small-update methods. There have been several attempts in the last decade to close the gap between the theoretical and practical performance of Large Update Primal–Dual IPMs. None of these led to a real improvement in complexity analysis.

We do not go in detailed discussion of the algorithmic variants of Large Update Primal–Dual IPMs, just mention the most recent development, the novel family of IPMs by Peng, Roos and Terlaky [40, 39] that are based on *Self-Regular (SR) functions*. This result significantly narrows the gap between the theory and practice of IPMs.
Some characteristics of SR based IPMs are as follows:
– the SR barrier is imposed in a scaled space;
– the SR barrier is a proximity w.r.t. the solution of the perturbed complementarity conditions;
– user controlled, adjustable stronger barrier and growth behavior;
– new search directions;
– complexity of large-update IPMs is almost identical to the small update ones.

A more detailed discussion of the Large Update Primal–Dual IPMs based on self-regular proximity measures, will be in the Subsection 4.2.

4.1 Short admissible pivot sequences

We assume that the LO problem has an optimal solution. Let us denote by x^* and (y^*, s^*) a given pair of optimal solutions of the LO problem. Furthermore, let $I_* = \{i \in I \mid x_i^* > 0\}$ and $J_* = \{j \in I \mid s_j^* > 0\}$ be the support index sets of the given optimal solutions.

Definition 4.1 *A pivot on the (k, ℓ) position, where $k \in I_B$ and $\ell \in I_N$ is said to be admissible if*
  (I) either $x_k < 0$ and $\tau_{k\ell} < 0$
  (II) or $s_\ell < 0$ and $\tau_{k\ell} > 0$.

Before describing the *Admissible Pivot Algorithm* of Fukuda and Terlaky [20] we need to introduce some procedures.

Reduce-F: Let $F := I_* \cup I_B$ and there is a $k \in I_B \setminus I_*$ such that $x_k < 0$. Then there is an $\ell \in I_N \cap I_*$ such that pivot (k, ℓ) is admissible. (This is true, because the problem $A_F\, x_F = b,\ x_F \geq 0$, is feasible.) Make pivot at (k, ℓ); let $I_B := (I_B \cup \{\ell\}) \setminus \{k\}$ and $F := F \setminus \{k\}$.

Reduce-G: Let $G := J_* \cup I_N$ and there is an $\ell \in I_N \setminus J_*$ such that $s_\ell < 0$. Then there is a $k \in I_B \cap J_*$ such that pivot (k, ℓ) is admissible. (This is true, because the problem $A^T\, y + s = c,\ s_G \geq 0,\ s_{I \setminus G} = 0$ is feasible.) Make pivot at (k, ℓ); let $I_B := (I_B \cup \{\ell\}) \setminus \{k\}$ and $G := G \setminus \{\ell\}$.

Reduce solutions: Here $x_i \geq 0$ for all $I_B \setminus I_*$ and $s_j \geq 0$ for all $I_N \setminus J_*$, but

- either there is an $i \in I_B \cap I_*$ such that $x_i < 0$, then **Modify-x^***;
- or there is a $j \in I_N \cap J_*$ such that $s_j < 0$, then **Modify-s^***.

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Admissible Pivot Algorithm

Input:

    A pair of optimal solutions x^* and (y^*, s^*);

    A_B is an arbitrary initial basis;

    I_B resp. I_N are the index sets of the basis/nonbasis variables;

    x and s denotes the corresponding primal/dual basic solutions.

    Let $F = I_* \cup I_B$ and $G = J_* \cup I_N$.

begin

    while true **do**

        if $x_B \geq 0$ and $s_N \geq 0$ **then**

            stop: the current solution solves the LO problem;

          else

            if $\exists k \in I_B \setminus I_* : x_k < 0$ or $\exists l \in I_N \setminus J_* : s_l < 0$

               then *Reduce-F or Reduce -G*

               else *Reduce solutions*

            endif

        endif

      endwhile

    end

Modify-x^*: Here $x_i \geq 0$ for all $i \in I_B \setminus I_*$, but there is an $i \in I_B \cap I_*$ such that $x_i < 0$. By taking an appropriate convex combination of the current basic solution and the optimal solution x^* we eliminate a positive coordinate of x^*. Let

$$\lambda := \min \left\{ \frac{x_i^*}{x_i^* - x_i} \mid x_i < 0 \right\} = \frac{x_k^*}{x_k^* - x_k}.$$

Then, by defining

$$x^* := \lambda\, x + (1 - \lambda)\, x^* \geq 0,$$

we have $A\, x^* = b$ and so a new primal optimal solution with fewer nonzero coordinates is obtained. Let $I_* = \{i \in I \mid x_i^* > 0\}$.

Modify-s^*: Here $s_j \geq 0$ for all $j \in I_N \setminus J_*$, but there is a $j \in I_N \cap J_*$ such that $s_j < 0$. By taking an appropriate convex combination of the current basic solution and the optimal solution s^* we eliminate a positive coordinate of s^*. Let

$$\lambda := \min \left\{ \frac{s_j^*}{s_j^* - s_j} \mid s_j < 0 \right\} = \frac{s_l^*}{s_l^* - s_l}.$$

Then, by defining

$$s^* := \lambda\, s + (1 - \lambda)\, s^* \geq 0,$$

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a new dual optimal solution with fewer nonzero coordinates is obtained. Let $J_* = \{j \in I \mid s_j^* > 0\}$. It can be shown that the new solutions x^* and s^* are optimal.

The presented admissible pivot algorithm produces an admissible pivot path which is initiated by the basis I_B and it stops only if an optimal basis is found. At the step **Reduce-F or -G** either $|G|$ reduces by one, which can happen at most $m = |I_B|$ times, or $|F|$ reduces by one, which can happen at most $n - m = |I_N|$ times. At the step **Reduce solutions** no pivot is performed, and in the following iteration, one of the steps **Reduce-F** or **Reduce-G** will be applied.

Consequently, we need at most $m + (n - m) = n$ pivots to find an optimal basis.

4.2 New interior point methods based on self-regular functions

If we introduce the notation $v = (v_1, \dots, v_n)^T$, where $v_j = \sqrt{\frac{x_j s_j}{\mu}}$ $j = 1, \dots, n$ then we can define a SR proximity as

$$\Psi(v) := \Psi(x, s, \mu) = \sum_{j=1}^{n} \psi(v_j),$$

where $\psi(t) : (0, \infty) \to [0, \infty)$ is a one dimensional SR function satisfying the following definition:

Definition 4.2 *The function $\psi(t) \in \mathcal{C}^2 : (0, \infty) \to [0, \infty)$ is self-regular if it satisfies the following conditions:*

SR1 $\psi(t)$ is strictly convex with respect to $t > 0$ and vanishes at its global minimal point $t = 1$, i.e., $\psi(1) = \psi'(1) = 0$. Further, there exist positive constants $\nu_2 \geq \nu_1 > 0$ and $p \geq 1, q \geq 1$ such that

$$\nu_1(t^{p-1} + t^{-1-q}) \leq \psi''(t) \leq \nu_2(t^{p-1} + t^{-1-q}), \quad \forall t \in (0, \infty); \tag{4}$$

SR2 For any $t_1, t_2 > 0$,

$$\psi(t_1{}^r t_2{}^{1-r}) \leq r\psi(t_1) + (1 - r)\psi(t_2), \quad r \in [0, 1]. \tag{5}$$

As an example, we present the functions

$$\Upsilon_{p,1}(t) = \frac{t^{p+1} - 1}{p(p + 1)} - \frac{\log t}{q} + \frac{p - 1}{p}(t - 1), \quad \text{if} \quad q = 1,$$

$$\Upsilon_{p,q}(t) = \frac{t^{p+1} - 1}{p(p + 1)} + \frac{t^{1-q} - 1}{q(q - 1)} + \frac{p - q}{pq}(t - 1), \quad \text{if} \quad q > 1.$$

that are Self-Regular functions with $p \geq 1$ and with $\nu_1 = \nu_2$. One can easily see that the classical log-barrier function is an SR function as well:

$$\Upsilon_{1,1}(t) := \frac{t^2 - 1}{2} - \log t = \frac{1}{2}\left(\frac{x_j s_j}{\mu} - 1 - \log \frac{x_j s_j}{\mu}\right) \quad \text{with} \quad t = v_j.$$

IPMs based on SR functions have the same scheme as outlined earlier in the diagram of Large–Update Primal–Dual IPMs, only the Newton directions is calculated differently and instead of the traditional proximities δ the SR proximity $\Psi(x, s, \mu)$ is used to measure the distance from the central path. The Newton direction is given by the solution of a modification of system (3), where the last set of equations is replaced by

$$s\Delta x + x\Delta s = -\mu v \nabla_v \Psi(v).$$

It still needs further research to explore what is the practical use and computational power of this novel approach.

5 Conclusions, further research

Based on our discussions we can conclude that there is no clear champion in the race to solve LO problems in practice. Theoretical criteria, such as worst case complexity, the ability to generate strictly complementary solutions, clearly favor IPMs, but in practice pivot methods keep competing. IPMs win for very large, sparse LO problems, while pivot algorithms are favorable for integer linear problems. A key issue in practice is how to utilize the underlying structure of the LO problems, which method is able to make better use of sparsity and computer architecture. Research is going on to find better algorithms both considering the theoretical worst case complexity and practical performance.

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