Short simplex paths in lattice polytopes

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Polyhedra and Polytopes

$P \subseteq \mathbb{R}^n$ is a polyhedron if, for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

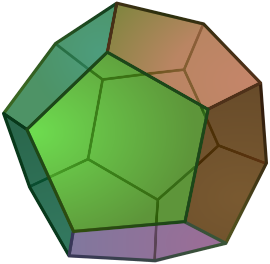
$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

A bounded polyhedron is called polytope

A vertex is a 0-dimensional face

An edge is a 1-dimensional face

Two vertices are adjacent if they lie on a common edge



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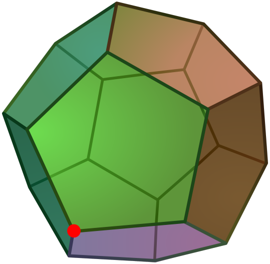

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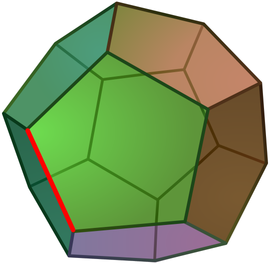

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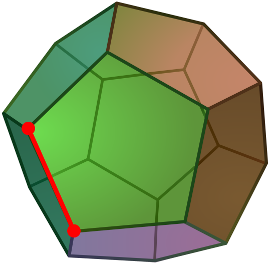
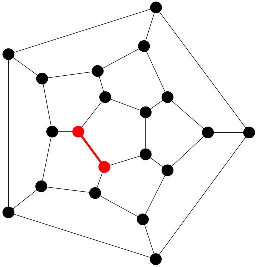
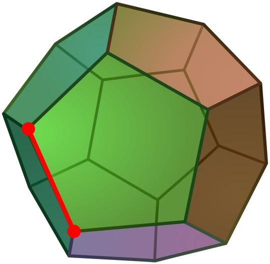


1-skeleton

The 1-skeleton of P is a graph having:

- a node for each vertex of P
- an edge joining two nodes $\Leftrightarrow$ the corresponding vertices of P are adjacent





Linear Programming (LP)

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & x \in P \end{array}$$

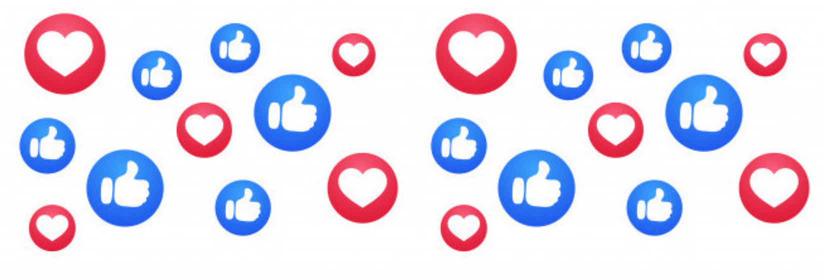
- $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$
- P is a polyhedron in $\mathbb{R}^n$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$
- $c \in \mathbb{R}^n$

A powerful tool for tackling wide classes of optimization problems

The simplex method

The most popular algorithm for LP is

the simplex method



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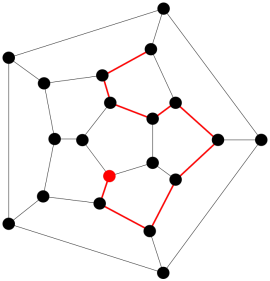
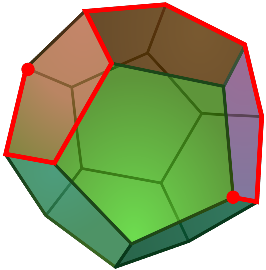
The simplex method

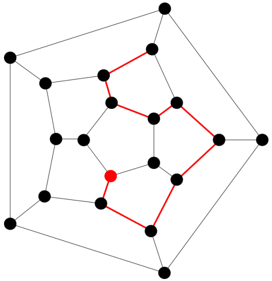
- Start at an initial vertex of P
- Move along an (improving) edge to an adjacent vertex
- Repeat until an optimal vertex is found or unboundedness is detected

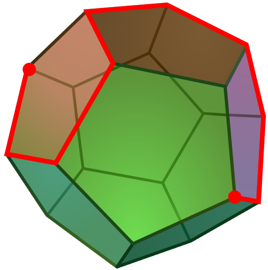




The simplex method

- Start at an initial vertex of P
- Move along an (improving) edge to an adjacent vertex [pivot]
- Repeat until an optimal vertex is found or unboundedness is detected





The choice of the next vertex depends on a pivoting rule

The simplex method

MAJOR OPEN QUESTION $\#1$:

pivoting rule that performs a polynomial nb of steps?

The simplex method

MAJOR OPEN QUESTION #1:

pivoting rule that performs a polynomial nb of steps?

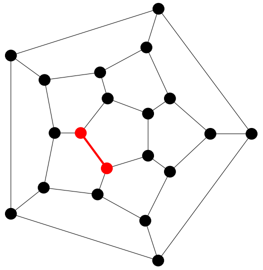
MAJOR OPEN QUESTION #2:

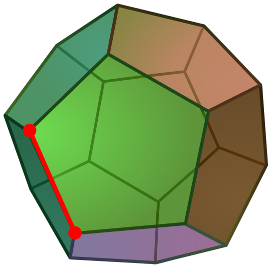
'best possible' pivoting rule performs a polynomial nb of steps?

Diameter

Distance between two vertices:

length of shortest path between two vertices on 1-skeleton of P





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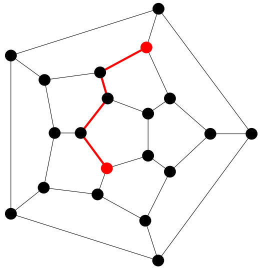
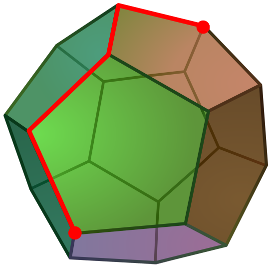
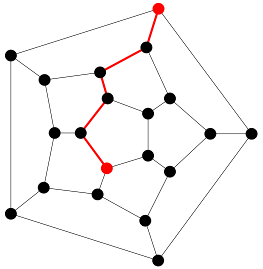


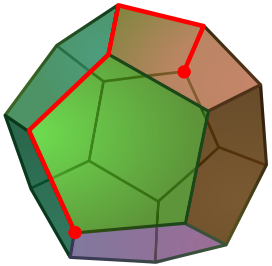


Diameter

Diameter:

largest value among the distances between any pair of vertices of P





The diameter of P is a LOWER BOUND on the worst-case simplex path length for a LP problem on P

The simplex method

MAJOR OPEN QUESTION #1:

pivoting rule that performs a polynomial nb of steps?

MAJOR OPEN QUESTION #2: (Polynomial Hirsch conjecture)

'best possible' pivoting rule performs polynomial nb of steps?
$\Leftrightarrow$ diameter bounded by a polynomial?

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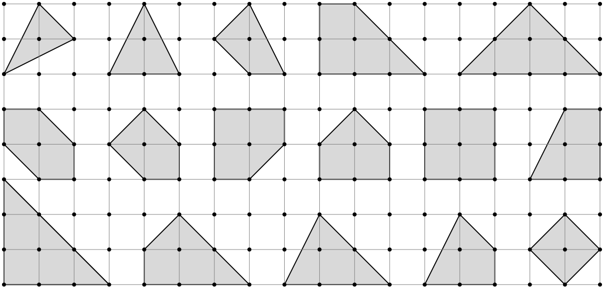
Lattice polytopes

Lattice polytopes



Lattice polytope: polytope whose vertices are integral

LP on lattice polytopes

We study the LP problem:

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & x \in P \end{array}$$

- P is a lattice polytope in $[0, k]^n$
- $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$
- $c \in \mathbb{Z}^n$

Applications in: polyhedral combinatorics, integer programming, combinatorial optimization

LP on lattice polytopes

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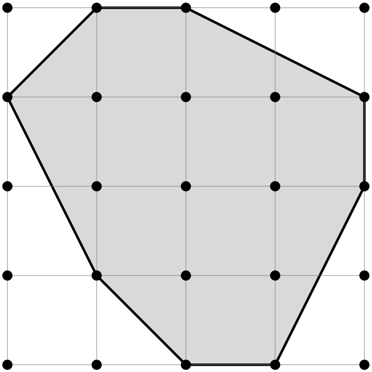
GOAL: Simplex algorithm that traces "short" simplex paths on P from given vertex x^0 to optimal vertex x^*

...possibly, polynomially far from the worst-case diameter

How "short" can a simplex path be on a lattice polytope?

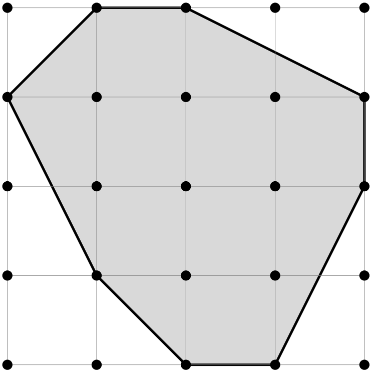
Diameter of lattice polytopes in $[0, k]^n$



Upper bounds:

- n if $k = 1$ [Naddef 89]
- nk [Kleinschmidt Onn 92]
- $\left\lfloor n(k - \frac{1}{2}) \right\rfloor$ if $k \geq 2$ [Del Pia Michini 16]
- $nk - \left\lceil \frac{2}{3} n \right\rceil - (k - 3)$ if $k \geq 3$ [Deza Pournin 18]

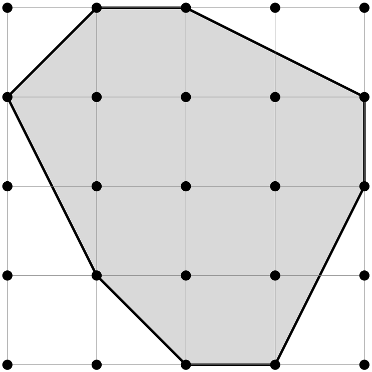
Diameter of lattice polytopes in $[0, k]^n$



Lower bounds:

- n if $k = 1$
- $\left\lfloor \frac{3}{2} n \right\rfloor$ if $k = 2$ [Del Pia Michini 16]
- $\left\lfloor \frac{1}{2} n(k+1) \right\rfloor$ if $k < 2n$ [Deza Manoussakis Onn 18]
- $ck^{\frac{2}{3}}$ if $n = 2$, $k \to \infty$ [Balog Bárány 91]
- $c(n)k^{\frac{n}{n+1}}$ if n fixed, $k \to \infty$ [Deza Pournin Sukegawa 19]

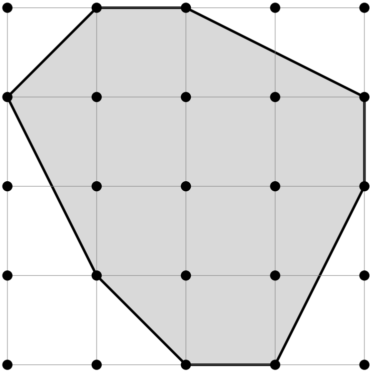
Diameter of lattice polytopes in $[0, k]^n$



Take-home message:

Worst-case diameter $\approx nk$.

Diameter of lattice polytopes in $[0, k]^n$



Take-home message:
Worst-case diameter $\approx nk$.

GOAL: Simplex algorithm that traces simplex paths of length polynomially far from nk

Other simplex algorithms

Dantzig, **best improvement** [Kitahara Mizuno '12], and **steepest edge** [Blanchard, De Loera, Louveaux '20] pivoting rules:

$\Rightarrow$ simplex path length

$$\leq n^2 \max\{k, S\} \log(n \max\{k, S\})$$

where $S = \max\{\|b - Ax\|_\infty \mid x \in P\}$.

- However even for $k = 1$, S can be $\frac{(n-1)^{\frac{n-1}{2}}}{2^{2n+o(n)}}$
- This bound is not polynomially bounded in nk.

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Can we eliminate dependence on S?

Short simplex paths in lattice polytopes

Theorem 1: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length in $O(n^4 k \log(nk))$

The simplex path length is polynomially far from optimal $\approx nk$

Short simplex paths in lattice polytopes

Theorem 1: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length in $O(n^4 k \log(nk))$

Independent on:

- cost vector c
- description $Ax \leq b$ of P
- number of inequalities m

Short simplex paths in lattice polytopes

Theorem 1: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length in $O(n^4 k \log(nk))$

Most known lattice polytopes are defined via $0, \pm 1$ constraint matrices

Can we exploit the largest absolute value α of the entries in the constraint matrix?

Short simplex paths in lattice polytopes

Theorem 2: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length in $O(n^2 k \log(nk\alpha))$

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Short simplex paths in lattice polytopes

Theorem 2: Simplex algorithm for lattice polytopes in $[0, k]^n$ s.t. simplex path length in $O(n^2 k \log(nk\alpha))$

- If $\alpha \leq \text{poly}(n, k)$, then simplex path length in $O(n^2 k \log(nk))$
- If also $k = 1$, then simplex path length in $O(n^2 \log n)$

How does it work?

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We move to an adjacent vertex by calling:

Oracle

Input: Polytope P, $c \in \mathbb{Z}^n$, vertex $\bar{x}$ of P

Output:

- Either a statement that $\bar{x}$ maximizes $c^\top x$ over P
- or a vertex adjacent to $\bar{x}$ with strictly larger cost

---

The input to the oracle is key to compute a short simplex path...

How does it work?

1. **Basic algorithm** length $\leq nk\,\|c\|_\infty$

2. **Scaling algorithm** length $O(nk\log\|c\|_\infty)$

3. **Preprocessing & scaling algorithm** length $O(n^4 k\log(nk))$

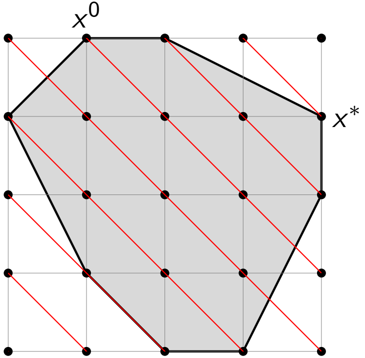
4. **Face-fixing algorithm** length $O(n^2 k\log(nk\alpha))$

Basic algorithm

Basic algorithm

Input: Lattice polytope P in $[0, k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P
Output: A vertex x^* of P maximizing $c^\top x$.

for $t = 0, 1, 2, \ldots$ **do**
Invoke **oracle**(P, c, x^t)
If the **oracle** states that x^t is optimal, return x^t
Otherwise, $x^{t+1} :=$ oracle(P, c, x^t)



Observation: The length of the simplex path generated is at most $c^\top x^* - c^\top x^0 \leq nk \|c\|_\infty$

Example: $c = (1, 1)$

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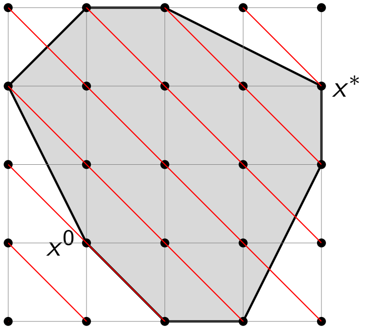

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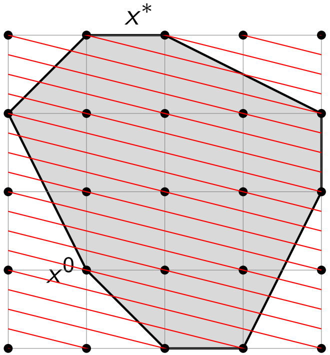

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Example: $c = (1, 4)$

Scaling algorithm

Let $\ell := \lceil \log \|c\|_\infty \rceil$
For $t = 0, \ldots, \ell$, define the integral approximations of c:

$$c^t := \left\lceil \frac{c}{2^{\ell - t}} \right\rceil \qquad \text{(Note: } c^\ell = c\text{)}$$

Example:

$$\begin{aligned} c &= (1,2,3,4,5,6,7) \\ c^0 &= (1,1,1,1,1,1,1) \\ c^1 &= (1,1,1,1,2,2,2) \\ c^2 &= (1,1,2,2,3,3,4) \\ c^3 &= (1,2,3,4,5,6,7) \end{aligned}$$

- $\|c^t\|_\infty \leq 2^t$ for $t = 0, \ldots, \ell$
- $c^\ell = c$
- $2c^{t-1} - c^t \in \{0,1\}^n$ for $t = 1, \ldots, \ell$

Scaling algorithm

For $t = 0, \ldots, \ell$: $\quad c^t := \left\lceil \frac{c}{2^{\ell - t}} \right\rceil$

Scaling algorithm

Input: Lattice polytope P in $[0, k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P
Output: A vertex x^* of P maximizing $c^\top x$

for $t = 0, \ldots, \ell$ **do**
  $x^{t+1} :=$ **basic algorithm**(P, c^t, x^t)
Return $x^{\ell+1}$

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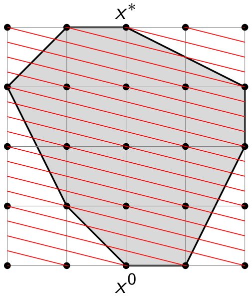

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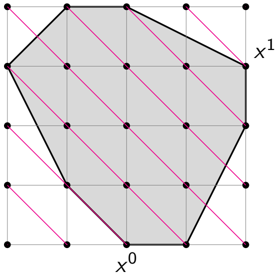
Example: $c = (1, 4)$ $c^0 = (1, 1)$



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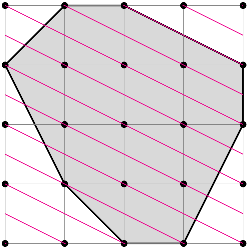
Example: $c = (1, 4)$ $c^1 = (1, 2)$



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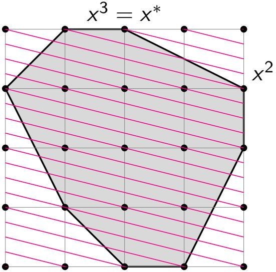
Example: $c = (1, 4)$ $c^2 = (1, 4)$



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Scaling algorithm

Proposition: Simplex path length is in $O(nk \log \|c\|_\infty)$

Scaling algorithm

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Scaling algorithm

Proposition: Simplex path length is in $O(nk \log \|c\|_\infty)$
Proof:

- **Basic algorithm** is called $\ell + 1 := \lceil \log \|c\|_\infty \rceil + 1$ times.
- Each time **basic algorithm** generates path of length $\leq nk$:
  - $2c^{t-1} - c^t \in \{0,1\}^n$
  - ${c^t}^\top x^{t+1} \leq 2{c^{t-1}}^\top x^{t+1} \leq 2{c^{t-1}}^\top x^t$
  - ${c^t}^\top x^{t+1} - {c^t}^\top x^t \leq 2{c^{t-1}}^\top x^t - {c^t}^\top x^t = (2c^{t-1} - c^t)^\top x^t \leq nk$

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Proof:

- **Basic algorithm** is called $\ell + 1 := \lceil \log \|c\|_\infty \rceil + 1$ times.
- Each time **basic algorithm** generates path of length $\leq nk$:
  - $2c^{t-1} - c^t \in \{0, 1\}^n$
  - ${c^t}^\top x^{t+1} \leq 2{c^{t-1}}^\top x^{t+1} \leq 2{c^{t-1}}^\top x^t$
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Scaling algorithm

Input: Lattice polytope P in $[0, k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P

Output: A vertex x^* of P maximizing $c^\top x$

for $t = 0, \ldots, \ell$ **do**

  $x^{t+1} :=$ **basic algorithm**(P, c^t, x^t)

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for $t = 0, \ldots, \ell$ **do**
  $x^{t+1} :=$ **basic algorithm**(P, c^t, x^t)
Return $x^{\ell+1}$

Preprocessing algorithm

Preprocessing algorithm
Input: $c \in \mathbb{Q}^n$, positive integer N
Output: $\tilde{c} \in \mathbb{Z}^n$ such that
▶ $\|\tilde{c}\|_\infty \leq 2^{4n^3} N^{n(n+2)}$
▶ $\text{sign}(c^\top z) = \text{sign}(\tilde{c}^\top z)\ \forall z \in \mathbb{Z}^n$ with $\|z\|_1 \leq N-1$

- Due to [Frank Tardos 87]
- Relies on the simultaneous approximation algorithm of [Lenstra Lenstra Lovász 82]

Setting $N := nk + 1$, x^* optimal for $\tilde{c} \Rightarrow$ optimal for c:

- $\forall x \in P \cap \mathbb{Z}^n$:
- $x - x^* \in \mathbb{Z}^n$ and $\|x - x^*\|_1 \leq nk = N - 1$
- $\tilde{c}^\top (x - x^*) \leq 0 \quad \Rightarrow \quad c^\top (x - x^*) \leq 0$

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- $x - x^* \in \mathbb{Z}^n$ and $\|x - x^*\|_1 \leq nk = N - 1$
- $\tilde{c}^\top x^* \geq \tilde{c}^\top x \quad \Rightarrow \quad c^\top x^* \geq c^\top x$

Preprocessing & scaling algorithm

Preprocessing & scaling algorithm

Input: Lattice polytope P in $[0, k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P
Output: A vertex x^* of P maximizing $c^\top x$

$\tilde{c} :=$ preprocessing algorithm$(c, N := nk + 1)$
$x^* :=$ **scaling algorithm**$(P, \tilde{c}, x^0)$
Return x^*

Theorem 1: Simplex path length in $O(nk \log \|\tilde{c}\|_\infty)$

$\log \|\tilde{c}\|_\infty$ in $O(n^3 \log(nk))$

Preprocessing & scaling algorithm

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Return x^*

Theorem 1: Simplex path length in $O(n^4 k \log(nk))$

Face-fixing algorithm

GOAL: shorter simplex path length, dependent on α

$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$
$\alpha :=$ largest absolute value of the entries of A

IDEA: Identify at each iteration one constraint of $Ax \leq b$ that is active at each optimal solution of $\max\{c^\top x \mid x \in P\}$

Inspired by [Tardos '86]

Face-fixing algorithm

Face-fixing algorithm

Input: Lattice polytope P in $[0,k]^n$, $c \in \mathbb{Z}^n$, vertex x^0 of P
Output: A vertex x^* of P maximizing $c^\top x$

0: Let $\mathcal{E} := \emptyset$ and $x^* := x^0$

1: Let $\bar{c}$ be the projection of c onto $\{x \in \mathbb{R}^n \mid a_i^\top x = 0 \ \forall i \in \mathcal{E}\}$.
If $\bar{c} = 0$ return x^*

2: Let $\tilde{c} \in \mathbb{Z}^n$ be defined by $\tilde{c}_i := \left\lfloor \frac{n^3 k \alpha}{\|\bar{c}\|_\infty} \bar{c}_i \right\rfloor$ for $i = 1, \dots, n$

3: Consider the following pair of primal and dual LP problems:

$$\begin{array}{lll} \max & \tilde{c}^\top x & \\ \text{s.t.} & a_i^\top x = b_i & i \in \mathcal{E} \\ & a_i^\top x \le b_i & i \in [m] \setminus \mathcal{E} \end{array} \quad (\tilde{P}) \qquad \begin{array}{lll} \min & b^\top y & \\ \text{s.t.} & A^\top y = \tilde{c} & \\ & y_i \ge 0 & i \in [m] \setminus \mathcal{E} \end{array} \quad (\tilde{D})$$

Compute optimal vertex $\tilde{x}$ of $(\tilde{P})$ with **scaling alg** from x^*
Compute an optimal solution $\tilde{y}$ to $(\tilde{D})$ s.t. $[\dots]$
Let $\mathcal{H} := \{i \mid \tilde{y}_i > nk\}$
$\mathcal{E} \leftarrow \mathcal{E} \cup \mathcal{H}$, $x^* \leftarrow \tilde{x}$ and go back to step 1

Main Results

(correctness)
Proposition: Vector x^* returned maximizes $c^\top x$ over P.

(short simplex paths)
Theorem 2: Simplex path length in $O(n^2 k \log(nk\alpha))$

(polynomial runtime)
Proposition: The number of operations to construct the next vertex in the simplex path is bounded by $\mathrm{poly}(n, m, \log \alpha, \log k)$. If P is 'well-described' by $Ax \leq b$, then it is bounded by $\mathrm{poly}(n, m, \log k)$.

Correctness – key lemma

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At each iteration, we restrict to a face F of P defined as

$F := \{x \in \mathbb{R}^n \mid a_i^\top x \le b_i \text{ for } i \in [m] \setminus \mathcal{E},\ a_i^\top x = b_i \text{ for } i \in \mathcal{E}\}$

Lemma: Each optimal solution of $\max\{c^\top x \mid x \in P\}$ lies in F

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Lemma: Each optimal solution of $\max\{c^\top x \mid x \in P\}$ lies in F

- Beginning of the algorithm: $\mathcal{E} = \emptyset \Rightarrow F = P$.
- Prove claim is true when we update $\mathcal{E}$ in step 3:
  Let $\mathcal{H} := \{i \mid \tilde{y}_i > nk\}$. Set $\mathcal{E} \leftarrow \mathcal{E} \cup \mathcal{H}$

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$$a_i^\top x = b_i \text{ for } i \in \mathcal{H} := \{i \mid \tilde{y}_i > nk\}$$

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Complementary slackness conditions for $(\hat{P})/(\tilde{D})$:
If $\tilde{y}$ optimal for $(\tilde{D})$ then $\forall\ \hat{x}$ optimal for $(\hat{P})$:

$$\tilde{y}_i > nk \quad \Rightarrow \quad a_i^\top \hat{x} = b_i \qquad\qquad i \in [m] \setminus \mathcal{E} \quad (*)$$

Short Simplex Paths – key lemma

At each iteration, we restrict to a face F of P defined as

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Lemma: At the end of iteration j: $\dim(F) \leq n - j$
$\Rightarrow$ at most n iterations

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Lemma: At the end of iteration j: $\dim(F) \leq n - j$
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At each iteration, we run the **scaling algorithm** to solve $(\tilde{P})$
Obs: F is a lattice polytope in $[0, k]^n$ and $\|\tilde{c}\|_\infty \leq n^3 k \alpha$.

At each iteration the **scaling algorithm** constructs a simplex path of length in $O(nk \log \|\tilde{c}\|_\infty) \in O(nk \log(nk\alpha))$

Theorem 2: Simplex path length in $O(n^2 k \log(nk\alpha))$

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In step 3: Compute an optimal solution $\tilde{y}$ to $(\tilde{D})$ s.t. $[\dots]$
Let $\mathcal{H} := \{i \mid \tilde{y}_i > nk\}$, and set $\mathcal{E} \leftarrow \mathcal{E} \cup \mathcal{H}$

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$\Uparrow$

We prove that at each iteration the rank of the row submatrix of A indexed by $\mathcal{E}$ increases by at least 1

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(i) $\tilde{y}_j = 0 \; \forall \, j \in [m] \setminus \mathcal{E}$ s.t. a_j is linear combination of a_i, $i \in \mathcal{E}$
(ii) $\tilde{y}$ has at most n nonzero components

We prove that at each iteration $\mathcal{H} \setminus \mathcal{E} \neq \emptyset$

$$\begin{array}{lll} \min & b^\top y & \\ \text{s.t.} & A^\top y = \tilde{c} & \\ & y_i \geq 0 & i \in [m] \setminus \mathcal{E} \end{array} \qquad (\tilde{D})$$

$\tilde{y}$ optimal solution of $(\tilde{D})$ with at most n nonzero components

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$\tilde{y}$ optimal solution of $(\tilde{D})$ with at most n nonzero components

Let $\mathcal{B} = \{i \in \{1, \ldots, m\} \mid \tilde{y}_i \neq 0\}$

$$\tilde{c} = \sum_{i \in [m]} a_i \tilde{y}_i = \sum_{i \in \mathcal{B}} a_i \tilde{y}_i$$

We prove that at each iteration $\mathcal{H} \setminus \mathcal{E} \neq \emptyset$

$$\begin{aligned} \min \quad & b^\top y \\ \text{s.t.} \quad & A^\top y = \tilde{c} \\ & y_i \geq 0 \qquad i \in [m] \setminus \mathcal{E} \end{aligned} \tag{$\tilde{D}$}$$

$\tilde{y}$ optimal solution of $(\tilde{D})$ with at most n nonzero components

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By contradiction, $\mathcal{H} \setminus \mathcal{E} = \emptyset$

$$\mathcal{H} := \{i \mid \tilde{y}_i > nk\}$$

$\Rightarrow |\tilde{y}_j| \leq nk \; \forall\, j \in \mathcal{B} \setminus \mathcal{E}$

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By contradiction, $\mathcal{H} \setminus \mathcal{E} = \emptyset$

$$\mathcal{H} := \{i \mid \tilde{y}_i > nk\}$$

$\Rightarrow |\tilde{y}_j| \leq nk \ \forall\, j \in \mathcal{B} \setminus \mathcal{E}$

$$\|\tilde{c}\|_\infty \leq \sum_{j \in \mathcal{B}} \|a_j \tilde{y}_j\|_\infty = \sum_{j \in \mathcal{B}} \left(|\tilde{y}_j| \, \|a_j\|_\infty\right)$$

We prove that at each iteration $\mathcal{H} \setminus \mathcal{E} \neq \emptyset$

$$\begin{array}{lll} \min & b^\top y & \\ \text{s.t.} & A^\top y = \tilde{c} & \\ & y_i \geq 0 & i \in [m] \setminus \mathcal{E} \end{array} \qquad (\tilde{D})$$

$\tilde{y}$ optimal solution of $(\tilde{D})$ with at most n nonzero components

Let $\mathcal{B} = \{i \in \{1, \ldots, m\} \mid \tilde{y}_i \neq 0\}$

$$\tilde{c} = \sum_{i \in [m]} a_i \tilde{y}_i = \sum_{i \in \mathcal{B}} a_i \tilde{y}_i$$

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Case 1: $\mathcal{B} \setminus \mathcal{E} = \mathcal{B}$

$$\|\tilde{c}\|_\infty \leq \sum_{j \in \mathcal{B}} \|a_j \tilde{y}_j\|_\infty = \sum_{j \in \mathcal{B}} \left(|\tilde{y}_j| \, \|a_j\|_\infty\right) \leq \sum_{j \in \mathcal{B}} (nk \cdot \alpha) \leq n^2 k \alpha$$

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But $\tilde{c} = \left\lfloor \frac{n^3 k\alpha}{\|\bar{c}\|_\infty} \bar{c} \right\rfloor \Rightarrow \|\tilde{c}\|_\infty = n^3 k\alpha$

contradiction

We prove that at each iteration $\mathcal{H} \setminus \mathcal{E} \neq \emptyset$

$$\begin{aligned} \min \quad & b^\top y \\ \text{s.t.} \quad & A^\top y = \tilde{c} \\ & y_i \geq 0 \qquad i \in [m] \setminus \mathcal{E} \end{aligned} \tag{$\tilde{D}$}$$

$\tilde{y}$ optimal solution of $(\tilde{D})$ with at most n nonzero components

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Case 2: $\mathcal{B} \setminus \mathcal{E} \neq \mathcal{B}$

$\|\tilde{c}\|_\infty \leq \sum_{j \in \mathcal{B}} \|a_j \tilde{y}_j\|_\infty = \sum_{j \in \mathcal{B}} \left(|\tilde{y}_j| \, \|a_j\|_\infty\right)$

more involved...

Thank you!