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SHORT COMMUNICATION

A bound for the number of different basic solutions generated by the simplex method

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Abstract In this short paper, we give an upper bound for the number of different basic feasible solutions generated by the simplex method for linear programming problems (LP) having optimal solutions. The bound is polynomial of the number of constraints, the number of variables, and the ratio between the minimum and the maximum values of all the positive elements of primal basic feasible solutions. When the problem is primal nondegenerate, it becomes a bound for the number of iterations. The result includes strong polynomiality for Markov Decision Problem by Ye (http://www.stanford.edu/~yyye/simplexmdp1.pdf, 2010) and utilize its analysis. We also apply our result to an LP whose constraint matrix is totally unimodular and a constant vector b of constraints is integral.

Keywords Simplex method · Linear programming · Number of iterations · Strong polynomiality · Basic feasible solutions

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1 Introduction

The simplex method for solving linear programming problems (LP) is originally developed by Dantzig [1]. The simplex method works very efficiently in practice and it has

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been widely used for years. In spite of the practical efficiency of the simplex method, we do not have any good bound for the number of iterations. Klee and Minty [2] show that the simplex method needs an exponential number of iterations for an elaborately designed problem.

We analyze the primal simplex method with the most negative pivoting rule (Dantzig's rule) or the best improvement pivoting rule under the condition that the primal problem has an optimal solution. We give an upper bound for the number of different basic feasible solutions (BFSs) generated by the simplex method. The bound is

$$n\left\lceil m\frac{\gamma}{\delta}\log\left(m\frac{\gamma}{\delta}\right)\right\rceil,$$

where m is the number of constraints, n is the number of variables, δ and γ are the minimum and the maximum values of all the positive elements of primal BFSs, respectively, and $\lceil a\rceil$ is the smallest integer bigger than $a \in \Re$. When the primal problem is nondegenerate, it becomes a bound for the number of iterations. Note that the bound depends only on the constraints of LP, but not on the objective function.

Our work is motivated by a recent research by Ye [3]. He shows that the simplex method is strongly polynomial for the Markov Decision Problem. We apply the analysis in [3] to general LPs and obtain the upper bound. Our results include his strong polynomiality.

When we apply our result to an LP where a constraint matrix is totally unimodular and a constant vector b of constraints is integral, the number of different solutions generated by the simplex method is at most

$$n\lceil m\|b\|_1 \log(m\|b\|_1)\rceil.$$

The paper is organized as follows. In Sect. 2, we explain basic notions of an LP and we briefly review the simplex method. In Sect. 3, analyses of the simplex method are conducted to show our results. In Sect. 4, applications of our results to special LPs are discussed.

2 The simplex method for LP

In this paper, we consider the linear programming problem of the standard form

$$\begin{array}{ll} \min & c^T x, \\ \text{subject to} & Ax = b,\ x \geq 0, \end{array} \tag{1}$$

where $A \in \Re^{m\times n}$, $b \in \Re^m$ and $c \in \Re^n$ are given data, and $x \in \Re^n$ is a variable vector. The dual problem of (1) is

$$\begin{array}{ll} \max & b^T y, \\ \text{subject to} & A^T y + s = c,\ s \geq 0, \end{array} \tag{2}$$

where $y \in \Re^m$ and $s \in \Re^n$ are variable vectors.

We assume that $\mathrm{rank}(A) = m$, the primal problem (1) has an optimal solution and an initial BFS x^0 is available. Let x^* be an optimal basic feasible solution of (1), (y^*, s^*) be an optimal solution of (2), and z^* be the optimal value of (1) and (2).

Given a set of indices $B \subset \{1, 2, \ldots, n\}$, we split the constraint matrix A, the objective vector c, and the variable vector x according to B and $N = \{1, 2, \ldots, n\} - B$ like

$$A = [A_B, A_N], \quad c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}, \quad x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}.$$

Define the set of bases

$$\mathcal{B} = \{B \subset \{1, 2, \ldots, n\} \mid |B| = m, \ \det(A_B) \neq 0\}.$$

Then a primal basic feasible solution for $B \in \mathcal{B}$ and $N = \{1, 2, \ldots, n\} - B$ is written as

$$x_B = A_B^{-1} b \geq 0, \ x_N = 0.$$

Let δ and γ be the minimum and the maximum values of all the positive elements of BFSs. Hence for any BFS $\hat{x}$ and any $j \in \{1, 2, \ldots, n\}$, if $\hat{x}_j \neq 0$, we have

$$\delta \leq \hat{x}_j \leq \gamma. \tag{3}$$

Note that these values depend only on A and b, but not on c.

Let $B^t \in \mathcal{B}$ be the basis of the t-th iteration of the simplex method and set $N^t = \{1, 2, \ldots, n\} - B^t$. Problem (1) can be written as

$$\begin{array}{ll} \min & c_{B^t}^T A_{B^t}^{-1} b + (c_{N^t} - A_{N^t}^T (A_{B^t}^{-1})^T c_{B^t})^T x_{N^t}, \\ \text{subject to} & x_{B^t} = A_{B^t}^{-1} b - A_{B^t}^{-1} A_{N^t} x_{N^t}, \\ & x_{B^t} \geq 0, \ x_{N^t} \geq 0. \end{array}$$

The coefficient vector $\bar{c}_{N^t} = c_{N^t} - A_{N^t}^T (A_{B^t}^{-1})^T c_{B^t}$ is called a reduced cost vector. When $\bar{c}_{N^t} \geq 0$, the current solution is optimal. Otherwise we conduct a pivot. That is, we choose one nonbasic variable (entering variable) and increase the variable until one basic variable (leaving variable) becomes zero. Then we exchange the two variables. Several rules for choosing the entering variable have been proposed. For example, the most negative rule, the best improvement rule, and the minimum index rule are well known. Under the most negative rule, we choose a nonbasic variable whose reduced cost is minimum. To put it precisely, we choose an index

$$\hat{j}^t = \arg \min_{j \in N_t} \bar{c}_j.$$

We set $\Delta^t = -\bar{c}_{\hat{j}^t}$.

In the case of the best improvement rule, we choose an entering variable so that the objective value at the next iterate is minimum. We summarize the notations in Table 1.

Table 1 Notations

x^*	An optimal basic feasible solution of (1)
(y^*, s^*)	An optimal solution of (2)
z^*	The optimal value of (1)
x^t	The t-th iterate of the simplex method
B^t	The basis of x^t
N^t	The nonbasis of x^t
$\bar{c}_{N^t}$	The reduced cost vector at t-th iteration
Δ^t	$-\min_{j \in N_t} \bar{c}_j$
$\hat{j}^t$	An index chosen by the most negative rule at t-th iteration

3 Analysis of the simplex method

Our analysis is motivated by a recent work by Ye [3], where he shows a strongly polynomial result of the simplex method for the Markov Decision Problem, a special class of LP. We apply his analysis to general LPs and obtain an upper bound for the number of different basic feasible solutions generated by the simplex method. Later we confirm that our results include his strong polynomiality.

In the next lemma, we get a lower bound of the optimal value at each iteration of the simplex method.

Lemma 1 *Let z^* be the optimal value of Problem (1) and x^t be the t-th iterate generated by the simplex method with the most negative rule. Then we have*

$$z^* \geq c^T x^t - \Delta^t m\gamma. \tag{4}$$

Proof Let x^* be a basic optimal solution of Problem (1). Then we have

$$\begin{aligned} z^* &= c^T x^* \\ &= c_{B^t}^T A_{B^t}^{-1} b + \bar{c}_{N^t}^T x_{N^t}^* \\ &\geq c^T x^t - \Delta^t e^T x_{N^t}^* \\ &\geq c^T x^t - \Delta^t m\gamma, \end{aligned}$$

where the second inequality follows since x^* has at most m positive elements and each element is bounded above by γ. Thus we get the inequality (4). □

Next we show a constant reduction of the gap between the objective function value and the optimal value at each iteration when an iterate is updated. The result is interesting becasuse the reduction rate $\left(1 - \frac{\delta}{m\gamma}\right)$ does not depend on the objective vector c.

Theorem 1 *Let x^t and x^{t+1} be the t-th and $(t+1)$-th iterates generated by the simplex method with the most negative rule. If $x^{t+1} \neq x^t$, then we have*

$$c^T x^{t+1} - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)(c^T x^t - z^*). \tag{5}$$

Proof Let $x^t_{\hat{j}^t}$ be the entering variable chosen at the t-th iteration. If $x^{t+1}_{\hat{j}^t} = 0$, then we have $x^{t+1} = x^t$, a contradiction occurs. Thus $x^{t+1}_{\hat{j}^t} \neq 0$, and we have $x^{t+1}_{\hat{j}^t} \geq \delta$ from (3). Then we have

$$\begin{aligned} c^T x^t - c^T x^{t+1} &= \Delta^t x^{t+1}_{\hat{j}^t} \\ &\geq \Delta^t \delta \\ &\geq \frac{\delta}{m\gamma}(c^T x^t - z^*), \end{aligned}$$

where the last inequality comes from (4). The desired inequality readily follows from the above inequality. □

Note that under the best improvement pivoting rule, the objective function reduces at least as much as that with the most negative rule. Thus the next corollary easily follows.

Corollary 1 *Let x^t and x^{t+1} be the t-th and $(t+1)$-th iterates generated by the simplex method with the best improvement rule. If $x^{t+1} \neq x^t$, then we also have (5).*

From Theorem 1 and Corollary 1, we can easily get an upper bound for the number of different BFSs generated by the simplex method.

Corollary 2 *Let $\bar{x}$ be a second optimal BFS of LP (1), that is, a minimal BFS except for optimal BFSs. When we apply the simplex method with the most negative rule or the best improvement rule for LP (1) from an initial BFS x^0, we encounter at most*

$$\left\lceil m\frac{\gamma}{\delta}\log\left(\frac{c^T x^0 - z^*}{c^T \bar{x} - z^*}\right)\right\rceil$$

different BFSs.

Proof Let x^t be the t-th iterates generated by the simplex method and let $\tilde{t}$ be the number of different BFSs appearing up to this iterate. Then we have

$$c^T x^t - z^* \leq \left(1 - \frac{\delta}{m\gamma}\right)^{\tilde{t}} (c^T x^0 - z^*)$$

from (5). If $\tilde{t}$ is bigger than or equal to the number in the corollary, we easily get

$$c^T x^t - z^* < c^T \bar{x} - z^*.$$

Since $\bar{x}$ is a second optimal BFS of LP (1), x^t must be an optimal BFS from the inequality above. □

Note that the bound in the corollary above depends on the objective function. In the succeeding discussion, we will get another bound which is independent of the objective function.

The next Lemma states that if the current solution is not optimal, there is a basic variable which has an upper bound proportional to the gap between the objective value and the optimal value.

Lemma 2 *Let x^t be the t-th iterate generated by the simplex method. If x^t is not optimal, there exists $\bar{j} \in B^t$ such that $x^t_{\bar{j}} > 0$ and*

$$s^*_{\bar{j}} \geq \frac{1}{m x^t_{\bar{j}}}(c^T x^t - z^*),$$

where s^* is an optimal slack vector of (2). Then for any k, the k-th iterate x^k satisfies

$$x^k_{\bar{j}} \leq \frac{m(c^T x^k - z^*)}{c^T x^t - z^*} x^t_{\bar{j}}.$$

Proof Since

$$c^T x^t - z^* = (x^t)^T s^* = \sum_{j \in B^t} x^t_j s^*_j,$$

there exists $\bar{j} \in B^t$ which satisfies

$$s^*_{\bar{j}} x^t_{\bar{j}} \geq \frac{1}{m}(c^T x^t - z^*),$$

or equivalently, $x^t_{\bar{j}} > 0$ and

$$s^*_{\bar{j}} \geq \frac{1}{m x^t_{\bar{j}}}(c^T x^t - z^*).$$

Moreover, for any k, the k-th iterate x^k satisfies

$$c^T x^k - z^* = (x^k)^T s^* = \sum_{j=1}^{n} x^k_j s^*_j \geq x^k_{\bar{j}} s^*_{\bar{j}},$$

which implies

$$x^k_{\bar{j}} \leq \frac{c^T x^k - z^*}{s^*_{\bar{j}}} \leq \frac{m(c^T x^k - z^*)}{c^T x^t - z^*} x^t_{\bar{j}}.$$

□

Lemma 3 *Let x^t be the t-th iterate generated by the simplex method with the most negative rule or the best improvement rule. Assume that x^t is not an optimal solution. Then there exists $\bar{j} \in B^t$ satisfying the following two conditions.*

1. $x^t_{\bar{j}} > 0$.
2. *If the simplex method generates $\left\lceil m\frac{\gamma}{\delta} \log\left(m\frac{\gamma}{\delta}\right)\right\rceil$ different basic feasible solutions after t-th iterate, then $x_{\bar{j}}$ becomes zero and stays zero.*

Proof For $k \geq t+1$, let $\tilde{k}$ be the number of different basic feasible solutions appearing between the $(t+1)$-th and k-th iterations. Then from Theorem 1 and Lemma 2, there exists $\bar{j} \in B_t$ which satisfies

$$x_{\bar{j}}^k \leq m\left(1 - \frac{\delta}{m\gamma}\right)^{\tilde{k}} x_{\bar{j}}^t \leq m\gamma\left(1 - \frac{\delta}{m\gamma}\right)^{\tilde{k}}.$$

The second inequality follows from (3). Therefore, if $\tilde{k} > m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})$, we have $x_{\bar{j}}^k < \delta$, which implies $x_{\bar{j}}^k = 0$ from the definition of δ. □

The event described in Lemma 3 can occur at most once for each variable. Thus we get the following result.

Theorem 2 *When we apply the simplex method with the most negative rule or the best improvement rule for LP* (1) *having optimal solutions, we encounter at most* $n\left\lceil m\frac{\gamma}{\delta}\log\left(m\frac{\gamma}{\delta}\right)\right\rceil$ *different basic feasible solutions.*

Note that the result is valid even if the simplex method fails to find an optimal solution because of a cycling.

If the primal problem is nondegenerate, we have $x^{t+1} \neq x^t$ for all t. This observation leads to a bound for the number of iterations of the simplex method.

Corollary 3 *If the primal problem is nondegenerate, the simplex method finds an optimal solution in at most* $n\left\lceil m\frac{\gamma}{\delta}\log\left(m\frac{\gamma}{\delta}\right)\right\rceil$ *iterations.*

4 Applications to special LPs

4.1 LP with a totally unimodular matrix

In this subsection, we consider an LP (1) whose constraint matrix A is totally unimodular and all the elements of b are integers. Recall that the matrix A is totally unimodular if the determinant of every nonsingular square submatrix of A is 1 or -1. Then all the elements of any BFS are integers, so $\delta \geq 1$. Let us bound γ. Let $(x_B, 0) \in \Re^m \times \Re^{n-m}$ be a basic feasible solution of (1). Then we have $x_B = A_B^{-1}b$. Since A is totally unimodular, all the elements of A_B^{-1} are ± 1 or 0. Thus for any $j \in B$ we have $x_j \leq \|b\|_1$, which implies that $\gamma \leq \|b\|_1$. By Theorem 3, we obtain the following result.

Corollary 4 *Assume that the constraint matrix A of* (1) *is totally unimodular and the constraint vector b is integral. When we apply the simplex method with the most negative rule or the best improvement rule for* (1)*, we encounter at most* $n\lceil m\|b\|_1\log(m\|b\|_1)\rceil$ *different basic feasible solutions.*

4.2 Markov decision problem

The Markov Decision Problem (MDP), where the number of possible actions is two, is formulated as

$$\begin{array}{ll} \min & c_1^T x_1 + c_2^T x_2, \\ \text{subject to} & (I - \theta P_1)x_1 + (I - \theta P_2)x_2 = e, \\ & x_1, x_2 \geq 0, \end{array} \tag{6}$$

where I is the $m \times m$ identity matrix, P_1 and P_2 are $m \times m$ Markov matrices, θ is a discount rate, and e is the vector of all ones. MDP(6) has the following properties.

1. MDP(6) is nondegenerate.
2. The minimum value of all the positive elements of BFSs is greater than or equal to 1, or equivalently, $\delta \geq 1$.
3. The maximum value of all the positive elements of BFSs is less than or equal to $\frac{m}{1-\theta}$, or equivalnetly, $\gamma \leq \frac{m}{1-\theta}$.

Therefore we can apply Corollary 3 and obtain a similar result to Ye [3].

Corollary 5 *The simplex method for solving MDP* (6) *finds an optimal solution in at most* $n \left\lceil \frac{m^2}{1-\theta} \log \frac{m^2}{1-\theta} \right\rceil$ *iterations, where* $n = 2m$.

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