Complexity of Pivot Algorithms

Qi Wang, Sean Kelley

December 16, 2020

Overview

1. Pivot Rules
   - S-monotone index selection rules
2. Complexity Analysis
   - Worst Case - Klee Minty Cube
   - Kitahara and Mizuno Analysis

Other Pivot Rules

- Best improvement rule: choose entering $j := \arg\max_{j \in N}(c_B^T A_B^{-1} A_j - c_j) \times t_j$ where $t_j = \min_{i \in \{B : (A_B^{-1} A_j)_i > 0\}} \frac{x_i}{(A_B^{-1} A_j)_i}$
- Steepest edge rule: choose entering $j := \arg\max_{j \in N} \frac{c_B^T A_B^{-1} A_j - c_j}{\|A_B^{-1} A_j\|}$
- Last-in-first-out rule (LIFO): choose the most recently moved variable.
- Most-often-selected-variable (MOSV): choose the variable that has been selected the largest amount of times before.

S-monotone index selection rules[1]

What is **s**?

- At iteration k, $s_k \in \mathbb{N}^n$. Generate s_k at each iteration. And choose the candidate of entering basis indices $j := \arg\max_j s_k^j$.

How to generate s_k?

- Bland's rule: $s_k = (n, n-1, \cdots, 1)^T$ for all k.
- LIFO.

$$s_{k+1}^i = \begin{cases} k, & \text{if } i \in \{i_k, o_k\}, \\ s_k^i & \text{otherwise.} \end{cases}$$

- MOSV.

$$s_{k+1}^i = \begin{cases} s_k^i + 1, & \text{if } i \in \{i_k, o_k\}, \\ s_k^i & \text{otherwise.} \end{cases}$$

S-monotone index selection rules I

What is "s-monotone"?

- For iteration $k-1$ and k, $s_{k-1}^{i} \leq s_{k}^{i}$ for all $i = 1, \cdots, n$.

Theorem 1

The simplex algorithms and the criss-cross algorithm with s-monotone index selection rules are finite for linear programming problems.

S-monotone index selection rules II

Revise steepest-edge rule to be s-monotone version.

1. At the beginning, define a strictly increasing sequence $\{p_k\}$.
2. At iteration k, let

$$\gamma = \max_{j \in N} \frac{c_{B^k}^T A_{B^k}^{-1} A_j - c_j}{\|A_{B^k}^{-1} A_j\|},$$

   and adjust p_k by

$$p_k = \begin{cases} p_k + \delta, & \text{if } p_{k-1} \geq \gamma, \\ \gamma, & \text{otherwise.} \end{cases}$$

   where $\delta > 0$ is a given number.
3. Update the s_k as

$$s_k^i = \begin{cases} p_k, & \text{if } i \in \{i_k, o_k\}, \\ s_{k-1}^i & \text{otherwise.} \end{cases}$$

Worst Case - Klee Minty Cube

Klee-Minty Cube LO problem

$$
\begin{aligned}
\max \sum_{j=1}^{n} 2^{n-j} x_j & \\
\text{s.t. } 2\sum_{j=1}^{i-1} 2^{i-j} x_j + x_i \leq 5^i, & \quad i = 1, \cdots, n \\
x_j \geq 0, \quad j = 1, \cdots, n &
\end{aligned}
$$

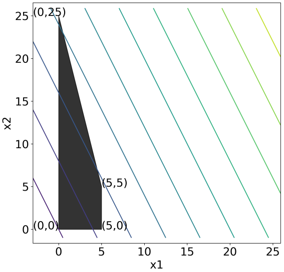
Primal simplex with Dantzig's rule may visit all vertices before finally finding the optimal solution ($2^n - 1$ iterations).

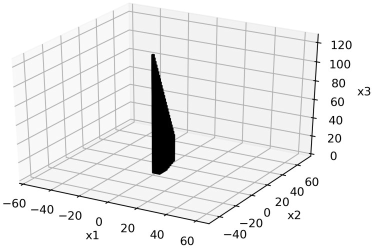
Worst Case - Klee Minty Cube

Klee-Minty Cube LO problem ($n = 2$ and $n = 3$)

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 \le 5 \\ & 4x_1 + x_2 \le 25 \\ & x_1 \ge 0,\ x_2 \ge 0. \end{aligned}$$

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 \le 5 \\ & 4x_1 + x_2 \le 25 \\ & 8x_1 + 4x_2 + x_3 \le 125 \\ & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0. \end{aligned}$$





Solving 2-d Klee Minty Cube using Simplex method with Dantzig's rule

	rhs	x_1	x_2	s_1	s_2	
z	0	2	1	0	0	at vetex (0,0), x_1 enter, s_1 leave
s_1	5	1^*	0	1	0	
s_2	25	4	1	0	1	
z	-10	0	1	-2	0	at vetex (5,0), x_2 enter, s_2 leave
x_1	5	1	0	1	0	
s_2	5	0	1^*	-4	1	
z	-15	0	0	2	-1	at vetex (5,5), s_1 enter, x_1 leave
x_1	5	1	0	1^*	0	
x_2	5	0	1	-4	1	
z	-25	-2	0	0	-1	at vetex (0,25), optimal
s_1	5	1	0	1	0	
x_2	25	4	1	0	1	

Kitahara and Mizuno Analysis[2]

Notations:

- BFS: a basic feasible solution for primal standard LO problem.
- δ and γ: the minimum and the maximum values of all the positive elements of all BFSs. That is, for any BFS $\hat{x} \in \mathbb{R}^n$, if $\hat{x}_j \neq 0,\ j \in \{1, \cdots, n\}$ where j represents the jth entry of $\hat{x}$, we have

$$\delta \leq \hat{x}_j \leq \gamma$$

Theorem 2

When applying simplex method with the Dantzig's rule or the best improvement rule for LO having optimal solutions, we encounter at most

$$n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil$$

different basic feasible solutions.

Kitahara and Mizuno Analysis

Corollary 3

If the primal problem is nondegenerate, the simplex method finds an optimal solution in at most $n\lceil m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})\rceil$ *iterations.*

In practice, the ratio of $\frac{\gamma}{\delta}$ is not easy to get prior to solving the problem. However, for some specific LP, $\frac{\gamma}{\delta}$ can be bounded by LP coefficients.

Definition 4

A matrix A is totally unimodular if every square submatrix has determinant 0, -1 or 1. In particular, this implies that all entries are 0, -1 or 1.

E.g., the node-arc incidence matrix of a directed graph is a totally unimodular matrix.

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & +1 \\ +1 & 0 & -1 & -1 & 0 & 0 \\ 0 & +1 & +1 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & +1 & -1 \end{pmatrix}$$

Kitahara and Mizuno Analysis

With totally unimodular A and integral b, all the elements of any BFS are integers, so

$$\delta \geq 1.$$

For a BFS $x = (x_B, x_N)$, $x_N = 0$ and $x_B = A_B^{-1} b \geq 0$, and A_B^{-1} are all 0,-1,1. Thus for any $j \in B$, we have $x_j \leq \|b\|_1$, so

$$\gamma \leq \|b\|_1.$$

Thus $\frac{\gamma}{\delta} \leq \frac{\|b\|_1}{1}$.

Corollary 5

Assume that the constraint matrix A of an LO is totally unimodular and vector b is integral. When we apply the simplex method with the Dantzig's rule or the best improvement rule for LO, we encounter at most $n\lceil m\|b\|_1 \log(m\|b\|_1)\rceil$ different basic feasible solutions. Moreover if the LO is nondegenerate, this is the most iterations to find optimal solution.

References

Zsolt Csizmadia, Tibor Illés, and Adrienn Nagy. "The s-monotone index selection rules for pivot algorithms of linear programming". In: *European journal of operational research* 221.3 (2012), pp. 491–500.

Tomonari Kitahara and Shinji Mizuno. "A bound for the number of different basic solutions generated by the simplex method". In: *Mathematical Programming* 137.1-2 (2013), pp. 579–586.