# Complexity of Pivot Algorithms

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**Abstract**

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## 1 Introduction

Simplex method was invented by Dantzig in 1947 [3] to solve the Linear Optimization (LP) problem. It was tableau pivoting based method and pivoting under certain rules. The rules can be flexible so many simplex variants have been developed afterwards. Another framework of pivot algorithm is criss-cross method, which was proposed by Zionts at 1969 [4] and then [5], [1] present finite criss-cross version independently around 1980s. Some variant pivot methods appeared by proposing different pivots rules. We will describe them in the following section. **Mention Short simplex paths** Although simplex method was generally efficient in practical, theoretically, Klee and Minty [2] in 1970s showed the worst case that, a specific type of LO problem, a variant of simplex method visited all vertices (exponential steps) until solve the problem to be optimal. The feasible region of such problems was call Klee-Minty cube whose corners have been "squashed". We will demonstrate some examples in the following section. For now, it is still an unaccomplished problem that to design a pivot algorithm and prove that the number of pivot steps is bounded by polynomial of number of variables and constraints. The paper is structured as follows: in Section 2 we define the LO problem, list the notation and (**might present some algorithms**); in section 3 we discuss the complexity of pivot algorithms and give a brief conclusion in the final section.

## 2 The pivot algorithm for LO

In this paper, we consider the Linear optimization problem of the standard form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \\ x \in \mathbb{R}^n,\ c \in \mathbb{R}^n,\ & A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m \end{aligned} \tag{1}$$

Table 1: Notation

| | |
|---|---|
| $n$ | number of variables |
| $m$ | number of constraints (assume $m \leq n$) |

## 3 Complexity - worst case

Given $m$ constraints and $n$ variables, there are at most $\binom{n}{m}$ possible basis, which is a upper bound for the number of iterations for pivot algorithms. When $m = \frac{n}{2}$, $\binom{n}{m}$ get its maximum $\binom{n}{\frac{n}{2}} \approx \sqrt{\frac{2}{\pi n}} 2^n$. Klee and Minty defined a type of LO problems which many pivot algorithms will visit all the vertices to find the optimal solution. Detailed introduction of Klee Minty cube can be referred to [6]. Here we demonstrate some simple examples. For such problem

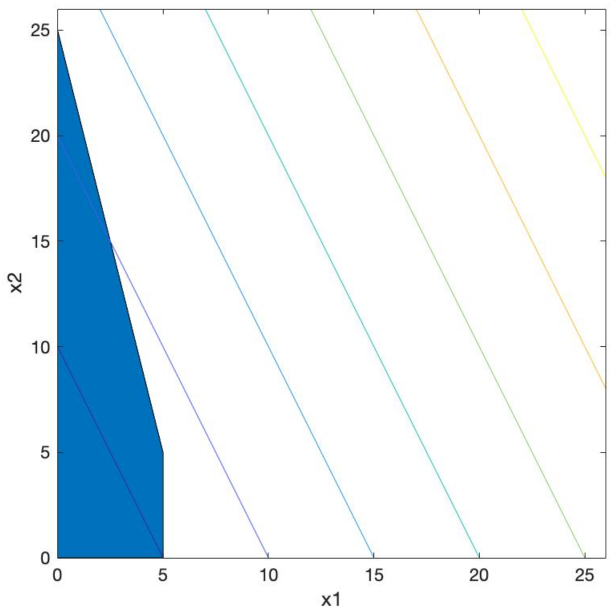
$$\begin{aligned} & \max \sum_{j=1}^{n} 2^{n-j} x_j \\ & \text{s.t. } 2 \sum_{j=1}^{i-1} 2^{i-j} x_j + x_i \leq 5^i & i = 1, \cdots, n \\ & \quad x_j \geq 0 & j = 1, \cdots, n \end{aligned}$$

When $n = 2$ and $n = 3$ the 2-d and 3-d Klee Minty LO problems are

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 \leq 5 \\ & 4x_1 + x_2 \leq 25 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 \leq 5 \\ & 4x_1 + x_2 \leq 25 \\ & 8x_1 + 4x_2 + x_3 \leq 125 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0. \end{aligned}$$

Figure 1: Feasible region of 2-d Klee Minty cube with contour lines.




The feasible region of 2-d Klee Minty problem is showed in Figure 1. And the feasible region of 3-d problem is shown in **TO be added**. Let's apply primal simplex method with least-index and Dantzig (choose the biggest reduced cost) pivot rules to demonstrate the number of pivot steps. **To be nicer table**

# 4 Conclusion

Conclusion...

# References


[1] Yow-Yieh Chang. Least-index resolution of degeneracy in linear complementarity problems. Technical report, STANFORD UNIV CA DEPT OF OPERATIONS RESEARCH, 1979.

[2] Wikipedia Contributors. Klee–minty cube, 11 2020.

| | rhs | x1 | x2 | s1 | s2 | |
|---|---|---|---|---|---|---|
| | z=0 | 2 | 1 | 0 | 0 | at vertex (0,0) |
| s1 | 5 | 1 | 0 | 1 | 0 | 5/1=5 this row out |
| s2 | 25 | 4 | 1 | 0 | 1 | 25/4=6.25 |

this col enter

least index version

pivot x1 enter, s1 leave

| | rhs | x1 | x2 | s1 | s2 | |
|---|---|---|---|---|---|---|
| | z=-10 | 0 | 1 | -2 | 0 | at vertex (5,0) |
| x1 | 5 | 1 | 0 | 1 | 0 | |
| s2 | 5 | 0 | 1 | -4 | 1 | |

pivot x2 enter, s2 leave

| | rhs | x1 | x2 | s1 | s2 | |
|---|---|---|---|---|---|---|
| | z=-15 | 0 | 0 | 2 | -1 | at vertex (5,5) |
| x1 | 5 | 1 | 0 | 1 | 0 | |
| x2 | 5 | 0 | 1 | -4 | 1 | |

pivot s1 enter, x1 leave

| | rhs | x1 | x2 | s1 | s2 | |
|---|---|---|---|---|---|---|
| | z=-25 | -2 | 0 | 0 | -1 | at vertex (0,25) |
| s1 | 5 | 1 | 0 | 1 | 0 | |
| x2 | 25 | 4 | 1 | 0 | 1 | |

Dantzig rule (steepest descent, the nonbasic variable with biggest reduced cost enter)

pivot x1 enter, s1 leave

| | rhs | x1 | x2 | s1 | s2 |
|---|---|---|---|---|---|
| | z=-10 | 0 | 1 | -2 | 0 |
| x1 | 5 | 1 | 0 | 1 | 0 |
| s2 | 5 | 0 | 1 | -4 | 1 |

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[3] George B Dantzig. Maximization of a linear function of variables subject to linear inequalities. *Activity analysis of production and allocation*, 13:339–347, 1951.

[4] K. Fukuda and T. Terlaky. Criss-cross methods: A fresh view on pivot algorithms. *Mathematical programming*, 79(1-3):369–395, 1997.

[5] T. Terlaky. A finite crisscross method for oriented matroids. *Journal of Combinatorial Theory, Series B*, 42(3):319–327, 1987.

[6] Robert J Vanderbei. *Linear programming: foundations and extensions*, volume 285. Springer Nature, 2020.