Combinatorics HW Generating Function and Integer Partition

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1. Integer composition: Integer 5 is partitioned into orderly partitions which are made up by numbers 1,2,3,4. Such as $(1+1+3, or\ 1+3+1\ or\ 2+3, 4+1, \ldots)$. How many different ways are there?

   Using $C(n-1, r-1)$ for $r = n, \ldots, 2$, we get
   $$C(4,4) + C(4,3) + C(4,2) + C(4,1) = 15$$

2. Integer partition: How many ways to partition n into several numbers that the order between numbers is ignored. Please write the corresponding generating function.

   $$G(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \cdots (1 + x^n + x^{2n} + \cdots) = \prod_{i=1}^{n} \frac{1}{1 - x^i}$$

3. Provide proof that the partition number for integer n using **different odd numbers** (ordering is ignored), equals to the partition number of n being partitioned into the self-conjugated Ferrers Diagrams.

   Let $p = \{p_1, \ldots, p_m\}$ be a unique odd partition of n into m parts, listed such that $p_1 > p_2 > \cdots > p_m > 0$. To construct the self-conjugating Ferrers Diagram, use the following algorithm.

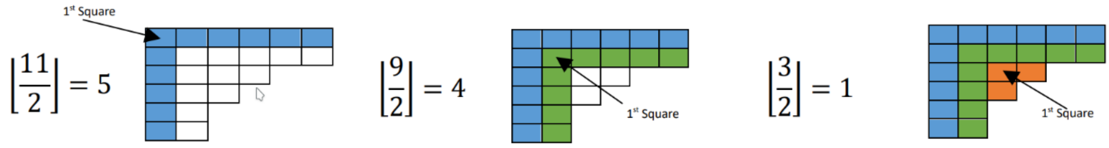
   for $i = 1, \ldots, m$

       put a square in the top left corner

       add $\left\lfloor \frac{p_i}{2} \right\rfloor$ squares below the first square

       add $\left\lfloor \frac{p_i}{2} \right\rfloor$ squares to the left of the first square

   This algorithm works because all p_i are odd, and hence can be expressed as $p_i = 2 \times \left\lfloor \frac{p_i}{2} \right\rfloor + 1$. To see how this algorithm works, consider the following example of $n = 23$, $p = \{11,9,3\}$.



   Now let q be a self-conjugating partition of n. Define the ith "hook", h_i, of q to be the squares that make up the ith row and ith column, excluding the squares to the left and above the corner square. Let $|h_i|$ denote the number of squares in hook h_i. To create a unique odd partition, use the following algorithm.

   let $p = \{\}$

   for each hook h in q

       $p = p \cup |h|$

   The steps of the algorithm on the example above are: $p = \{\}$, add the blue hook: $p = \{11\}$, add the green hook: $p = \{11,9\}$, add the orange hook: $p = \{11,9,3\}$.

   Since there is exactly one result for each input in both algorithms, the number of partitions of n into different odd numbers is the same as the number of self-conjugating partitions That is, we have a bijection between distinct odd partitions and self-conjugating partitions, hence the size of their sets is the same.