Combinatorics IEP HW

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1. how many integer numbers from 1 to 10000 are not squares of integers or cubes of integers ?

Let $I = \{1, \dots, 10000\}$. Let S denote the set of square integers and C the set of cubed integers, less than equal to 10000. Then the result we want is simply

$$|I| - |S| - |C| + |S \cap C|$$

Since $100^2 = 10{,}000$, clearly $|S| = 100$. Next, note that $9261 = 21^3 < 10{,}000 < 22^3 = 10648$, hence $|C| = 21$. Finally, out of the 21 cubed integers less than $10{,}000$, only $1^3, 4^3, 9^3$ and 16^3 are also square numbers. Hence the result is $10{,}000 - 100 - 21 + 4 = 9883$.

2. How many permutations of 1, 2, 3， …， 9 have at least one odd number in its natural position?

The total number of permutations of the numbers 1 to 9 is 9!. Now let A_i denote the set of permutations who's ith element is in the natural position. Then the result is

$$|A_1 \cup A_3 \cup A_5 \cup A_7 \cup A_9|$$

Since $|A_i| = 8!$ for all i, $|A_i \cap A_j| = 7!$ for all i, j, … and $|A_1 \cap A_3 \cap A_5 \cap A_7 \cap A_9| = 4!$, using the inclusion exclusion principle we get

$$C(5,1)8! - C(5,2)7! + C(5,3)6! - C(5,4)5! + C(5,5)4! = 157824$$

3. $x_1 + x_2 + x_3 + x_4 = 20, where\ 1 \le x_1 \le 6, 0 \le x_2 \le 7, 4 \le x_3 \le 8, 2 \le x_4 \le 6$
please calculate the number of integral solutions.

Let $y_1 = x_1 - 1$, $y_2 = x_2$, $y_3 = x_3 - 4$, $y_4 = x_4 - 2$. Then the equation becomes

$$y_1 + y_2 + y_3 + y_4 = 13, \qquad 0 \le y_1 \le 5, 0 \le y_2 \le 7, 0 \le y_3 \le 4, 0 \le y_4 \le 4$$

Which has $C(16,3)$ solutions. The upper bound means we must take away the following.

y_1	$C(13-6+3,3) = C(10,3)$
y_3	$C(13-8+3,3) = C(8,3)$
y_3	$C(13-5+3,3) = C(11,3)$
y_4	$C(13-5+3,3) = C(11,3)$

Then add the following combinations.

y_1, y_2	$C(13-6-8+3,3) = 0$
y_1, y_3	$C(13-6-5+3,3) = C(5,3)$
y_1, y_4	$C(13-6-5+3,3) = C(5,3)$
y_2, y_3	$C(13-8-5+3,3) = 1$
y_2, y_4	$C(13-8-5+3,3) = 1$
y_3, y_4	$C(13-5-5+3,3) = C(6,3)$

Then take away the following combinations

y_1, y_2, y_3	$C(13-6-8-5+3,3) = 0$
y_1, y_2, y_4	$C(13-6-8-5+3,3) = 0$

y_1, y_3, y_4	$C(13-6-5-5+3,3)=0$
y_2, y_3, y_4	$C(13-8-5-5+3,3)=0$

And finally, add the combination

y_1, y_2, y_3, y_4	$C(13-6-8-5-5+3,3)=0$

This gives us a total of

$$C(16,3)-C(10,3)-C(8,3)-C(11,3)-C(11,3)+C(5,3)+C(5,3)+1+1+C(6,3)=96$$