HW Magic Sequences
Name: Samuel Pegg Student ID: 2020280261

1. Using 5 numbers 1, 2, 3, 4, 5 to fill in $1 \times n$ grids, each grid is filled with one digit. If there are odd number of grids that have 1 written on them, and an even number of grids with 2, please write the corresponding exponential generating function and figure out how many arrangements there for 1×6 grids?

An odd number of 1s corresponds to $\left(\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}\ldots\right)=\frac{e^x-e^{-x}}{2}$ (since 0 is an even number)
An even number of 2s corresponds to $\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)=\frac{e^x+e^{-x}}{2}$
And the other three digits correspond to $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)=e^x$
Thus the generating function is $G(x)=\frac{e^{3x}}{4}(e^x+e^{-x})(e^x-e^{-x})=\frac{e^{5x}+e^x}{4}$
But we can write e^x as $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$, hence we can write $G(x)$ as

$$G(x)=\sum_{k=0}^{\infty}\frac{5^k-1}{4}\frac{x^k}{k!}$$

And hence the coefficient of the $k=6$ term is $\frac{5^6-1}{4}=3906.$

2. There are six people in a library queuing up, three of them want to return the book "Interviewing Skills", and 3 of them want to borrow the same book. If at the beginning, all the books of "Interviewing Skills" are out of stock in the library, how many ways can these people line up?

This is simply the lattice problem where the paths are not allowed to cross or touch the line $y=x-1$. $(0,0)$ reflected in this line is $(1,-1)$, hence the valid paths are all the paths from $(1,-1)$ to $(3,3)$ which is $C(3+1+3-1,3-1)=C(6,2)$. The total valid paths are just $C(3+3,3)=C(6,3)$, so the total is

$$C(6,3)-C(6,2)=5$$

However, the people lending are separate people, and can be permuted in 3! ways. The same goes for people borrowing. Hence the result is

$$5\times 3!\times 3!=180$$