Homework 13

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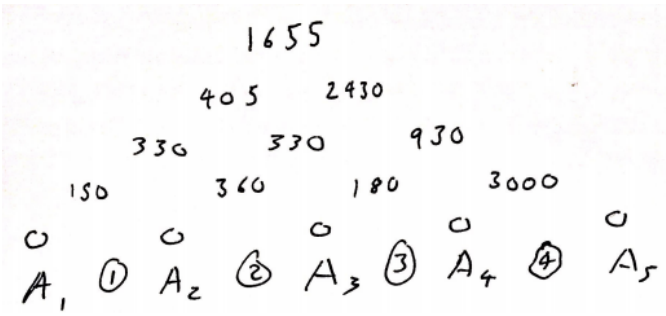
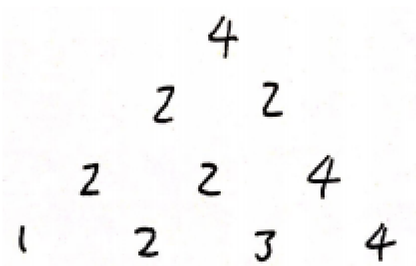
1

Find an optimal parenthesisation of the matrix-chain product $\{5, 10, 3, 12, 5, 50\}$.

The tables produced by the algorithm are below.





So the optimal solution is

$$((A_1A_2)(A_3A_4))A_5$$

2

Consider a variant of the matrix-chain multiplication problem in which the goal is to parenthesise the sequence of matrices so as to maximise, rather than minimise, the number of scalar multiplications. Does this problem exhibit optimal substructure? Justify your answer.

Yes it does. The argument is the same as for the minimisation problem. The worst case way to parenthesise the matrix chain $A_{1,\ldots,n}$ must split the matrix chain at some point, k, to get $A_{1,\ldots,k}$ and $A_{k+1,\ldots,n}$. The worst case way to parenthesise $A_{1,\ldots,k}$ in $A_{1,\ldots,n}$ must be one of the worst case ways to parenthesise $A_{1,\ldots,k}$. If not, we could substitute this even-worse-case-parenthesisation into the optimal parenthesisation of $A_{1,\ldots,n}$ to produce another way to parenthesise $A_{1,\ldots,n}$ whose cost is higher than the worst case: a contradiction. Same argument goes for $A_{k+1,\ldots,n}$. Thus this problem exhibits optimal substructure.

3

Prove the optimal substructure property of the knapsack problem in the case where the optimal subset does not contain a_n.

Let S be the optimal solution of $KS_{n,w}$. We need to prove that if we take away a_n, which is not in the optimal solution, the remaining solution $S - \{a_n\}$ is optimal in $KS_{n-1,w}$. Suppose $S' \neq S - \{a_n\}$ is optimal in $KS_{n-1,w}$ and let $S'' = S' \cup \{a_n\}$. Then

1. Since, a_n is not in the optimal solution, it wont be in S'' either. So

$$\sum_{j:a_j \in S''} w_j = \sum_{j:a_j \in S'} w_j \leq w$$

   so S'' is feasible.

2. Since a_n wont be in S'':

$$\sum_{j:a_j \in S''} v_j = \sum_{j:a_j \in S'} v_j > \sum_{j:a_j \in S-\{a_n\}} v_j = \sum_{j:a_j \in S} v_j$$

   so S'' is better than S. This contradicts the optimality of S, thus our assumption $S' \neq S - \{a_n\}$ must be incorrect, hence $S - \{a_n\}$ is optimal in $KS_{n-1,w}$.