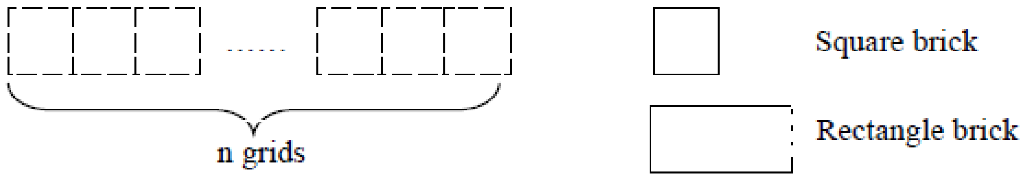
HW W5-2:
Name : Samuel Pegg Student ID: 2020280261

1. A worker is tiling a road with n square grids, there are two different kinds of bricks available, one is the square brick and the other is the rectangle brick which covers 2 grids. Please count the total different ways the road can be tiled.



Let a_n bet the number of ways the road can be tiled for n tiles. There are two possibilities for the last tile, a one square brick or a two square brick. If it is a one square brick, then there are a_{n-1} ways to get to a_n. If it is a two square brick, there are a_{n-2} ways to get a_n. Combining these results, since it is one of these two options, we obtain the recurrence relation

$$a_n = a_{n-1} + a_{n-2}$$

i.e. the Fibonacci sequence. There is one way to pave 0 grids, and 1 way to pave 1 gride (fill with a square brick), hence $a_0 = 1$ and $a_1 = 1$. The characteristic equation is

$$0 = x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$$

We have

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$a_0 = a_1 = 1$. Clearly the final result is $a_n = F_{n+1}$, but let us go through the analysis anyway. These initial conditions yield $A + B = 1$, $A\frac{1+\sqrt{5}}{2} + B\frac{1-\sqrt{5}}{2} = 1$ which solve to give $A = \frac{\sqrt{5}+1}{2\sqrt{5}}$, $B = \frac{\sqrt{5}-1}{2\sqrt{5}}$. Thus the number of ways to lay n brings is

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

2. How many different ways to colour n grids in a line with red, white or blue colours but no two adjacent grids are coloured with red?

Let a_n denote the number of ways to colour the n grids. If the n^{th} grid is coloured white, there are a_{n-1} ways to colour the previous $n-1$ grids. If the n^{th} grid is coloured blue, there are again a_{n-1} ways to colour the previous $n-1$ grids. If the n^{th} grid is coloured red, the previous grid must be either white or blue, and each colour has a_{n-2} ways of being that colour. Hence the recurrence relation is

$$a_n = 2a_{n-1} + 2a_{n-2}$$

There are three ways to colour a single grid, and $2 \times 3 + 1 \times 2 = 8$ ways of colouring two grids. Hence $a_1 = 3$ and $a_2 = 8$. The characteristic polynomial is

$$0 = x^2 - 2x - 2 = \left(x - 1 + \sqrt{3}\right)\left(x - 1 - \sqrt{3}\right)$$

Thus we can write a_n as

$$a_n = A\left(1+\sqrt{3}\right)^n + B\left(1-\sqrt{3}\right)^n$$

The initial conditions yield the equations $A\left(1+\sqrt{3}\right) + B\left(1-\sqrt{3}\right) = 3$, and $A\left(1+\sqrt{3}\right)^2 + B\left(1-\sqrt{3}\right)^2 = 8$. These solve to give $A = \frac{3+2\sqrt{3}}{6}$, $B = \frac{3-2\sqrt{3}}{6}$. Hence we deduce that

$$a_n = \frac{3+2\sqrt{3}}{6}\left(1+\sqrt{3}\right)^n + \frac{3-2\sqrt{3}}{6}\left(1-\sqrt{3}\right)^n$$