Combinatorics HW recurrence relations – 1

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1. Please prove the following equation of fibonacci sequence F_i: $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$.

   We will prove this claim via induction on n. Note that we can rewrite the claim as

$$F_{2n} = \sum_{i=1}^{n} F_{2i-1}$$

   Base case: $n = 1$. $F_2 = F_1$ is clearly true since $F_0 = 0$, $F_1 = 1$ and $F_2 = F_1 + F_0 = 1$.
   Inductive hypothesis: assume the claim holds for $n = k$, for some integer $k > 1$. i.e. we assume that $F_{2k} = \sum_{i=1}^{k} F_{2i-1}$.
   Consider the next term: $n = k + 1^{\text{st}}$.

$$F_{2(k+1)} = F_{2k+2} = F_{2k+1} + F_{2k} = F_{2k+1} + \sum_{i=1}^{k} F_{2i-1} = F_{2(k+1)-1} + \sum_{i=1}^{k} F_{2i-1} = \sum_{i=1}^{k+1} F_{2i-1}$$

   If the claim holds for $n = k$, then it also holds for $n = k + 1$. Since it holds for the base case, it holds for all $n \geq 1$ by induction.

2. Please provide the corresponding characteristic equations for the following recurrence relation: $a_n = 2a_{n-1} + 4a_{n-2} - 5a_{n-3}$.

$$x^3 - 2x^2 - 4x + 5 = 0$$

3. Solve the recurrence relation $h_n = 2h_{n-1} + 8h_{n-2}$, $n \geq 2$, $h_1 = 1, h_2 = 10$.

   The characteristic equation is $x^2 - 2x - 8 = (x - 4)(x + 2) = 0$. The roots are -2 and 4. Thus, we can write

$$h_n = A \times 4^n + B \times (-2)^n$$

   Using the initial conditions, we get $1 = 4A - 2B$, $10 = 16A + 4B$. These simulations equations yield $B = \frac{1}{2}$ and $A = \frac{1}{2}$, and so we get

$$h_n = \frac{1}{2}(4^n + (-2)^n)$$