Combinatorics HW – Pigeon Hole Principle

Student ID: 2020280261 Name: Samuel Pegg Score:

1. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruits that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?

Worst case: 7 apples, 5 bananas, 8 oranges. The next fruit picked will complete either of the requirements, hence the smallest number of pieces of fruit is

$$7+5+8+1=21$$

2. Show that for any given 52 integers there exists two of them whose sum, or else whose difference, is divisible by 100.

Define $r=\{(0,0),(1,99),(2,98),\dots,(49,51),(50,50)\}$. Clearly, for all $(p,q)\in r$, $p+q=0$ mod 100 and $q=100-p$. Let the 52 integers modulo 100 be denoted $a=\{a_1,\dots,a_{52}\}$. If we try to put elements from set a into set r, since there are 51 pairs in r, by the Pigeonhole Principal there must be at least one of these pairs filled by two elements from set a. Call these two elements a_i and a_j. Since there are no constraints on the 52 integers, the pair in r can be filled by a_i and a_j in two different ways:

1. $a_i=a_j$. In this case, $a_i-a_j=0$ modulo 100 and is hence divisible by 100.
2. $a_i\neq a_j$. In this case, by construction of r, we have $a_j=100-a_i$. Thus $a_i+a_j=a_i+100-a_i=100=0$ modulo 100 and is hence divisible by 100.

Clearly, in cases where there are three or more elements from set a in set r, we can just choose any two to add or subtract, and the claim still holds.

What follows is an alternate proof that took me so long I don't have the heart to delete it. I would be very interested to find out if it is valid or not, but as it is quite long please do not feel pressured to check through it all.

Let $a_1, \dots, a_{52}$ be the 52 given integers. Take these integers modulo 50. There are 50 possible unique remainders: $0, \dots, 49$. Because there are 52 integers, there must be two pairs of integers with the same remainder, or three integers with the same remainder. Consider a triple:

$$a_p = 50x_p + c$$
$$a_q = 50x_q + c$$
$$a_r = 50x_r + c$$

There are three x values, all of which are either odd or even. Thus by the Pigeon Hole theorem there must be a combination of two of the x values that yields an even result ($odd + odd$ or $even + even$). Sine an even multiple of 50 is just a multiple of 100, in this case the statement holds, so we need only consider the case with two pairs.

Let (a_p, a_q) where $a_p = 50x_p + c$ and $a_q = 50x_q + c$ be one such pair. If x_p and x_q have the same parity, then the claim also holds in this case too since $a_p - a_q = 50(x_p - x_q)$ is an even multiple of 50.

So let us suppose the x_p and x_q in both pairs have opposite parity. Of the remaining 52 – (2 pairs of integers)=48 integers, if there is an integer $a_k = 50x_k + d$ where $d = 50 - c$, then we can simply consider the following two sums:

$$a_p + a_k = 50x_p + c + 50x_k + d = 50\left(x_p + x_k + 1\right)$$

$$a_q + a_k = 50x_q + c + 50x_k + d = 50(x_q + x_k + 1)$$

Since x_p and x_q have opposite parity, either $x_p + x_k + 1$ or $x_q + x_k + 1$ is even. Either way, the result holds.

Hence, we now suppose the case when such a d does not exist in our selection of 48 integers. We have two pairs which take up two separate remainder values, then we also remove the two corresponding d remainders. This reduces the size of possible remainders from 50 to 50-4=46. As a result, we have 48 integers (52-2 pairs), and 46 remainders. This creates the exact same situation as we started with, but 46 remainders instead of 50, and 48 integers rather than 52. Repeating this process recursively, either one of the above cases occurs and the statement holds, or we eventually end up with 2 remainders left, and 4 integers.

Clearly, we either have a triple and a single, a quadruple (which contains a triple) or two pairs. As described above, a triple means the statement holds, so the one last case to consider is that of two pairs.

Suppose we had not gotten rid of the remainder value of 25 until now. Then one of the pairs must look like

$$a_p = 50x_p + 25$$
$$a_q = 50x_q + 25$$

If x_p and x_q have the same parity, we can do $a_p - a_q = 50(x_p - x_q)$ which is divisible by 100. However, if they have different parity, we can compute $a_p + q_q = 50(x_p + x_q + 1) = 50(odd + even + odd = even)$ which is also divisible by 100. So a remainder of 25 always means the statement holds.

If 25 is not in the remaining two remainders, then 25 occurred somewhere in one of the previous iterations, and we can select the two integers with remainder 25 and either add or subtract them to get the statement to hold. Hence the statement holds for all possible cases, and we can therefore conclude that for any given 52 integers, there exists a pair whose sum or difference is divisible by 100.