lecture

October 15, 2025

1 HC

beat greedy baseline of $\frac{1}{3} - \alpha px$. this is average linkage or random

this alg will give $\geq .336OPT$

$$\sum_{ij \in E} w_{ij} \left(n - |T_{ij}|\right) = \sum_{ij \in E} w_{ij} \sum_{k \neq i,j} \mathbb{1}\{\text{k not aleaf of } T_{ij}\} \tag{1}$$

consider

$$y_{ijk} = w_{ij} \mathbb{1}\{\text{k not aleaf of } T_{ij}\} \tag{2}$$

it only atkes on values of $w_{ij}, 0$

$$y_{ij} = \sum_{k \neq i,j} y_{ijk} \tag{3}$$

2 ALG1

random always

for a given triplet i, j, k, the probability $P[ij|k]$[1] is $\frac{1}{3}$.

3 ALG 2

SDP first, random next

solve sdp for hc, look for vectors at mid level, $t^* = \lfloor n/2$, then we set $x_{ij}^t = x_{ij}^{t^*}$.

do hyperplane rounding to partition into $S, \bar{S}$

randomalways S, and sdp $\bar{S}$.

recursive max uncut bisection gives 0.585.

[1] see lec 5

remember x_{ij}^t, at level t each cluster is at most size t.

sdp objective,

$$\max \sum_{t=1}^{n} \sum_{ij \in E} w_{ij}(1 - x_{ij}^t) \tag{4}$$

constraints:

$$\sum_{i \neq j} x_{ij}^t \geq n - t, \forall i, \forall t \tag{5}$$

$$x_{ij}^{t+1} \leq x_{ij}^t \forall ij \in E, \forall t, x_{ij}^{(0)} = 1 \tag{6}$$

to convert to sdp, $x^t + ij \frac{1}{2} \left| v_i^t - v_j^t \right|_2^2$, $v_i^{((t))} \in \mathbb{R}^n$

4 case 1

$$OPT < (1 - \epsilon_1)(n - 2) \sum_{ij \in E} w_{ij} \tag{7}$$

if ϵ_1 is large then just use random always

5 case2

if SDP≥OPT $\geq (1 - \epsilon_1)W$,

consider 3 cases

ϵ_{ij} ij is together, ϵ_{ijk} ijk all together $\epsilon_{ij|k}$ij together but k split

the final one is $\frac{\theta_{ik} + \theta_{jk} - \theta_{ij}}{2\pi}$

the expectation of 3 is

$$E[y_{ijk}] = \frac{w_{ij}}{3} p[\epsilon_{ijk}] + w_{ij} P[\epsilon_{ij|k}] \tag{8}$$

$$E[y_{ij}] = \sum_{k \neq i,j} E[y_{ijk}] \tag{9}$$

because

$$P[\epsilon_{ij}] = 1 - P[\epsilon_{ijk}] - P[\epsilon_{ij|k}] \tag{10}$$

we get that 8 is also equal to

$$\frac{w_{ij}}{3} P[\epsilon_{ij} + \frac{2w_{ij}}{3} P[\epsilon_{ij|k}]] \tag{11}$$

and 9 is

$$E[y_{ij}] = \frac{w_{ij}}{3}\left((n-2)P[\epsilon_{ij}] + 2\sum_{k\neq i,j} P[\epsilon_{ij|k}]\right) \tag{12}$$