# thesis

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The following equalities are of use, all of which can be verified through calculation and using the property $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \ldots$

$$\hat{D}^*(\alpha)\hat{D}(\alpha) = 1 \tag{1}$$

$$\begin{aligned} \hat{D}^*(\alpha)\hat{a}\hat{D}(\alpha) &= \hat{a} + \alpha \\ \hat{D}(\alpha)\hat{a}\hat{D}^*(\alpha) &= \hat{a} - \alpha \end{aligned} \tag{2}$$

$$\hat{S}^*(\xi)\hat{S}(\xi) = \hat{S}(-\xi)\hat{S}(\xi) = 1 \tag{3}$$

$$\begin{aligned} \hat{S}^*(\xi)\hat{a}\hat{S}(\xi) &= \hat{a}\cosh r - \hat{a}^* e^{i\theta}\sinh r \\ \hat{S}(\xi)\hat{a}\hat{S}^*(\xi) &= \hat{a}^*\cosh r - \hat{a}e^{-i\theta}\sinh r \end{aligned} \tag{4}$$

Let us write $\hat{a} = A + i\lambda B$, $\lambda \in \mathbb{R}$, [1]. Let us also call $[\hat{A},\hat{B}] = i\hat{C}$ and $\hat{A},\hat{B} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle$.
Using $\hat{F},\hat{C}$, we can rewrite the uncertainty relation as $(\Delta\hat{A})^2(\Delta\hat{B})^2 = \frac{1}{4}\left(\langle\hat{F}\rangle^2 + \langle\hat{C}\rangle^2\right)^2$.
Applying this formula to $\hat{a}$, we see

$$\begin{aligned} \Delta\hat{A}^2 &= \frac{1}{2}\left(\lambda_i\langle\hat{F}\rangle + \lambda_r\langle\hat{C}\rangle\right) \\ \Delta\hat{B}^2 &= \frac{1}{|\lambda|^2}\Delta\hat{A}^2 \\ \lambda_i\langle\hat{C}\rangle - \lambda_r\langle\hat{F}\rangle &= 0 \end{aligned} \tag{5}$$

Combining we see our familiar uncertainty relation

$$\Delta\hat{A}^2\Delta\hat{B}^2 \geq \frac{1}{4}\left(\langle\hat{F}\rangle^2 + \langle\hat{C}\rangle^2\right) \tag{6}$$

[1] This can be thought of of breaking $\hat{a}$ into hermitian and anti-hermitian components. For our usual operators $\hat{q},\hat{p}$, the hermitian component is $\hat{q}$ and the antihermitian is $\frac{\partial}{\partial x}$

[2] As a sanity check, our usual operators for position and momentum $\hat{q},\hat{p}$ follow the commutation relation $[\hat{q},\hat{p}] = i\hbar \implies (\Delta\hat{q})^2(\Delta\hat{p})^2 \geq \frac{\hbar^2}{4}$ following our formula.