

# Honors Analysis I

Based on Principles of Mathematical Analysis W. Rudin

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# 1 The real number system

## 1.1 Exercises

**Exercise (1).** If  $r$  is a rational ( $r \neq 0$ ) and  $x$  is irrational, prove that  $r + x$  and  $rx$  are irrational.

*Proof.* If  $r$  is a rational, then  $r = \frac{p}{q}$ , and the claim can be rewritten as  $\frac{p}{q} + x$  is irrational. Lets assume by contradiction that this quantity is rational. Then  $\frac{p}{q} + x = \frac{a}{b}$ . It follows that

$$x = \frac{a}{b} - \frac{p}{q} = \frac{aq - bp}{bq}$$

However, this is a contradiction, since  $x$  is irrational, which means that  $r + x$  must be irrational as well. Using a similar argument, lets assume by contradiction that  $\frac{p}{q}x$  is rational. Then  $\frac{p}{q}x = \frac{a}{b}$ . It follows that

$$x = \frac{aq}{bp}$$

However, this is a contradiction, since  $x$  is irrational, which means that  $rx$  must be irrational as well.  $\square$

**Exercise (2).** Prove that there is no rational number whose square is 12.

*Proof.*  $\sqrt{12}$  is equivalent to  $2\sqrt{3}$ . By exercise 1, a rational times an irrational is also irrational. If  $\sqrt{3}$  is irrational (since  $2 \in \mathbb{N}$ ,  $2 \in \mathbb{Q}$ , then it is true that  $2\sqrt{3}$ . Lets assume by contradiction that  $\sqrt{3}$  is rational. Then

$$\left(\frac{p}{q}\right)^2 = 3$$

It can also be assumed that  $p$  and  $q$  are both relatively prime, since if they are not, an equivalent  $\frac{p_1}{q_1}$  exists that's equivalent to  $\frac{p}{q}$ , but such that  $p_1$  and  $q_1$  are relatively prime. Now, this can be rewritten as

$$p^2 = 3q^2$$

Since  $p^2$  is divisible by 3, and 3 is prime,  $p$  is also divisible by 3. This means there is some  $a$  such that  $3a = p$ . By some algebra:

$$3p^2 = q^2$$

However, this is a contradiction, since both  $q$  and  $p$  are divisible by 3, although we originally claimed the two were relatively prime. Therefore, there is no rational number whose square is 12.  $\square$

**Exercise (3).** Prove the following proposition given that  $x, y, z \in F$ , where  $F$  is a field.

- (a) If  $x \neq 0$  and  $xy = xz$  then  $y = z$
- (b) If  $x \neq 0$  and  $xy = x$  then  $y = 1$

(c) If  $x \neq 0$  and  $xy = 1$  then  $y = \frac{1}{x}$

(d) If  $x \neq 0$  then  $\frac{1}{\frac{1}{x}} = x$

*Proof.* Parts:

(a) Applying axiom (M5)

$$y = \frac{1}{x}xy = \frac{1}{x}xz = z$$

(b) Applying axiom (M5)

$$y = \frac{1}{x}xy = \frac{1}{x}x = 1$$

(c) Applying axiom (M5)

$$y = \frac{1}{x}xy = \frac{1}{x}(1) = \frac{1}{x}$$

(d) Applying axiom (M5)

$$\frac{1}{\frac{1}{x}} = \frac{x\frac{1}{x}}{\frac{1}{x}} = x$$

□

**Exercise (4).** Let  $E$  be a nonempty subset of an ordered set; suppose  $\alpha$  is a lower bound of  $E$  and  $\beta$  is an upper bound of  $E$ . Prove that  $\alpha \leq \beta$ .

*Proof.* If  $\alpha$  is a lower bound of  $E$ , then for every  $x \in E$ ,  $\alpha \leq x$ . Similarly, if  $\beta$  is an upper bound of  $E$ , then for every  $x \in E$ ,  $\beta \geq x$ . Lets assume by contradiction that  $\alpha > \beta$ . Then  $\beta < \alpha$  and  $\alpha \leq x$  for every  $x \in E$ , then  $\beta \leq x$  for every  $x \in E$ . However, this is a contradiction, since  $\beta$  was already defined as  $\beta \geq x$  for every  $x \in E$ , which can only be true if  $\beta = \alpha$ . Therefore,  $\alpha \leq \beta$ . □

**Exercise (5).** Let  $A$  be a nonempty set of real numbers which is bounded below. Let  $-A$  be the set of all numbers  $-x$ , where  $x \in A$ . Prove that

$$\inf A = -\sup(-A)$$

*Proof.* Take some arbitrary  $x \in -A$ . This means that  $-x \in A$ . Now, the infimum of  $A$  is defined as some  $\alpha \leq y$  for every  $y \in A$ . This means that

$$\alpha \leq -x$$

and thus

$$-\alpha > x$$

Since  $x$  was chosen arbitrarily, this means that  $-\alpha$  is greater than  $x$  for every  $x \in -A$ . By definition,  $-\alpha$  is thus the supremum of  $-A$ . Since  $\alpha$  was defined as the infimum of  $A$ ,

$$-\inf A = \sup(-A)$$

which is equivalent to saying

$$\inf A = -\sup(-A)$$

□

**Exercise (6).** Fix  $b > 1$ .

- (a) If  $m, n, p, q$  are integers,  $n > 0, q > 0$ , and  $r = m/n = p/q$ , prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

- (b) Prove that  $b^{r+s} = b^r b^s$  if  $r$  and  $s$  are rational.

- (c) If  $x$  is real, define  $B(x)$  to be the set of all numbers  $b^t$ , where  $t$  is rational and  $t \leq x$ . Prove that

$$b^r = \sup B(r)$$

when  $r$  is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real  $x$ .

- (d) Prove that  $b^{x+y} = b^x b^y$  for all real  $x$  and  $y$ .

*Proof.* Parts:

- (a) Since  $m/n = p/q$ ,  $mq = pn$ , and  $1/n = p/mq$ . Using substitution,  $(b^m)^{1/n} = (b^m)^{p/mq} = (b^{p/q}) = (b^p)^{1/q}$

- (b) Since  $r$  and  $s$  are rationals,  $m/n = r$  and  $p/q = s$ , so

$$b^{r+s} = b^{m/n+p/q} = b^{\frac{mq+pn}{nq}} = (b^{mq}b^{pn})^{1/nq} = b^{m/n}b^{p/q} = b^r b^s$$

- (c) First,  $b^r$  must be an upper bound for the set  $B(r)$ , since  $r \geq t$  for every  $t \leq r$ , and thus  $b^r \geq b^t$ . Now all that is left is to show that  $b^r$  is the least upper bound of  $B(r)$ . Assume by contradiction that it is not. This would mean that there is some  $t$  such that  $b^t > b^r$ , but  $t \geq x$  for every  $x$  in  $B(r)$ . However, this is impossible, since  $b^r$  is in  $B(r)$  as well, and  $r \geq t$  and  $r \geq x$  for every  $x$  in  $B(r)$ . This is a contradiction, and thus  $b^r$  is the supremum of the set.

- (d)

□

**Exercise (7).** Fix  $b > 1, y > 0$ , and prove that there is a unique real  $x$  such that  $b^x = y$ , by completing the following outline.

- (a) For any positive integer  $n$ ,  $b^n - 1 \geq n(b - 1)$ .
- (b) Hence  $b - 1 \geq n(b^{1/n} - 1)$ .
- (c) If  $t > 1$  and  $n > (b - 1)/(t - 1)$ , then  $b^{1/n} < t$ .
- (d) If  $w$  is such that  $b^w < y$ , then  $b^{w+(1/n)} < y$  for sufficiently large  $n$ ; to see this, apply part (c) with  $t = y \cdot b^{-w}$ .
- (e) If  $w$  is such that  $b^w < y$ , then  $b^{w-(1/n)} > y$  for sufficiently large  $n$ .
- (f) Let  $A$  be the set of all  $w$  such that  $b^w < y$ , and show that  $x = \sup A$  satisfies  $b^x = y$ .

(g) Prove that this  $x$  is unique.

*Proof.* (a) **TODO**

□

**Exercise (8).** Prove that no order can be defined in the complex field that turns it into an ordered field.

*Proof.* For the complex field to be an ordered field, it must have the property that  $x^2 > 0$  if  $x \in \mathbb{C}$  and  $x \neq 0$ . However,  $(0, 1) \neq 0$ , but  $(0, 1) * (0, 1) = (-1, 0) = -1$ . This violates the above property, and thus the complex field cannot be ordered. □

**Exercise (9).** Suppose  $z = a + bi$ ,  $w = c + di$ . Define  $z < w$  if  $a < c$  or  $a = c$  and  $b < d$ . Prove that this is an order. Does it have the least upper bound property?

*Proof.* To prove that this is an order, it must satisfy two conditions. First,  $z < w$ ,  $z > w$ , and  $z = w$  must all be distinct. Now, since the set of reals is ordered, if  $a < c$ ,  $z < w$ , and if  $a > c$ ,  $z > w$ . If  $a = c$ , then the check falls to  $b$  and  $d$ . The same logic follows, and thus  $z < w$ ,  $z > w$ , and  $z = w$  are all distinct. Second, if  $x < y$  and  $y < z$ , then  $x < z$ . Using a new  $x = e + fi$ , if  $a < c$  and  $c < e$ , then **TODO** □

**Exercise.** Exercise 10: Suppose  $z = a + ib$ ,  $w = u + iv$ , and

$$a = \left( \frac{|w| + u}{2} \right)^{1/2}, \quad b = \left( \frac{|w| - u}{2} \right)^{1/2}.$$

Prove that  $z^2 = w$  if  $v \geq 0$  and that  $(\bar{z})^2 = w$

*Proof.* **TODO**

□

## 2 Basic Topology

### 2.1 Exercises

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