Honors Analysis I

Based on Principles of Mathematical Analysis W. Rudin

Notes/Solutions by Eashan Garg

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Contents

1 The real number system

1.1 Exercises

Exercise (1)**.** If r is a rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Proof. If r is a rational, then $r = \frac{p}{q}$, and the claim can be rewritten as $\frac{p}{q} + x$ is irrational. Lets assume by contradiction that this quantity is rational. Then $\frac{p}{q} + x = \frac{a}{b}$. It follows that

$$x = \frac{a}{b} - \frac{p}{q} = \frac{aq - bp}{bq}$$

However, this is a contradiction, since x is irrational, which means that $r+x$ must be irrational as well. Using a similar argument, lets assume by contradiction that $\frac{p}{q}x$ is rational. Then $\frac{p}{q}x = \frac{a}{b}$. It follows that

$$x = \frac{aq}{bp}$$

However, this is a contradiction, since x is irrational, which means that rx must be irrational as well. □

Exercise (2)**.** Prove that there is no rational number whose square is 12.

Proof. $\sqrt{12}$ is equivalent to $2\sqrt{3}$. By exercise 1, a rational times an irrational is also irrational. If $\sqrt{3}$ is irrational (since $2 \in \mathbb{N}$, $2 \in \mathbb{Q}$, then it is true that $2\sqrt{3}$. Lets assume by contradiction that $\sqrt{3}$ is rational. Then

$$(\frac{p}{q})^2 = 3$$

It can also be assumed that p and q are both relatively prime, since if they are not, an equivalent $\frac{p_1}{q_1}$ exists that's equivalent to $\frac{p}{q}$, but such that p_1 and q_1 are relatively prime. Now, this can be rewritten as

$$p^2 = 3q^2$$

Since p^2 is divisible by 3, and 3 is prime, p is also divisible by 3. This means there is some a such that $3a = p$. By some algebra:

$$3p^2 = q^2$$

However, this is a contradiction, since both q and p are divisible by 3, although we originally claimed the two were relatively prime. Therefore, there is no rational number whose square is 12. □

Exercise (3)**.** Prove the following proposition given that $x, y, z \in F$, where F is a field.

(a) If $x \neq 0$ and $xy = xz$ then $y = z$

(b) If $x \neq 0$ and $xy = x$ then $y = 1$

(c) If $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$

(d) If $x \neq 0$ then $\frac{1}{\frac{1}{x}} = x$

Proof. Parts:

(a) Applying axiom (M5)

$$y = \frac{1}{x}xy = \frac{1}{x}xz = z$$

(b) Applying axiom (M5)

$$y = \frac{1}{x}xy = \frac{1}{x}x = 1$$

(c) Applying axiom (M5)

$$y = \frac{1}{x}xy = \frac{1}{x}(1) = \frac{1}{x}$$

(d) Applying axiom (M5)

$$\frac{1}{\frac{1}{x}} = \frac{x\frac{1}{x}}{\frac{1}{x}} = x$$

□

Exercise (4)**.** Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. If α is a lower bound of E, then for every $x \in E$, $\alpha \leq x$. Similarly, if β is an upper bound of E, then for every $x \in E$, $\beta \geq x$. Lets assume by contradiction that $\alpha > \beta$. Then $\beta < \alpha$ and $\alpha \leq x$ for every $x \in E$, then $\beta \leq x$ for every $x \in E$. However, this is a contradiction, since β was already defined as $\beta \geq x$ for every $x \in E$, which can only be true if $\beta = \alpha$. Therefore, $\alpha \leq \beta$. □

Exercise (5)**.** Let A be a nonempty set of real numbers which is bounded below. Let -A be the set of all numbers -x, where $x \in A$. Prove that

$$\inf A = -\sup(-A)$$

Proof. Take some arbitrary $x \in -A$. This means that $-x \in A$. Now, the infimum of A is defined as some $\alpha \leq y$ for every $y \in A$. This means that

$$\alpha \leq -x$$

and thus

$$-\alpha > x$$

Since x was chosen arbitrarily, this means that $-\alpha$ is greater than x for every $x \in -A$. By definition, $-\alpha$ is thus the supremum of A. Since α was defined as the infimum of A,

$$-\inf A = \sup(-A)$$

which is equivalent to saying

$$\inf A = -\sup(-A)$$

□

Exercise (6)**.** Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

Proof. Parts:

(a) Since $m/n = p/q$, $mq = pn$, and $1/n = p/mq$. Using substitution, $(b^m)^{1/n} = (b^m)^{p/mq} = (b^{p/q}) = (b^p)^{1/q}$

(b) Since r and s are rationals, $m/n = r$ and $p/q = s$, so

$$b^{r+s} = b^{m/n+p/q} = b^{\frac{mq+pn}{nq}} = (b^{mq}b^{pn})^{1/nq} = b^{m/n}b^{p/q} = b^r b^s$$

(c) First, b^r must be an upper bound for the set $B(r)$, since $r \geq t$ for every $t \leq r$, and thus $b^r \geq b^t$. Now all that is left is to show that b^r is the least upper bound of $B(r)$. Assume by contradiction that it is not. This would mean that there is some t such that $b^t > b^r$, but $t \geq x$ for every x in $B(r)$. However, this is impossible, since b^r is in $B(r)$ as well, and $r \geq t$ and $r \geq x$ for every x in $B(r)$. This is a contradiction, and thus b^r is the supremum of the set.

(d)

□

Exercise (7)**.** Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline.

(a) For any positive integer n, $b^n - 1 \geq n(b-1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this, apply part (c) with $t = y \cdot b^{-w}$.

(e) If w is such that $b^w < y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this x is unique.

Proof. (a) **TODO**

□

Exercise (8)**.** Prove that no order can be defined in the complex field that turns it into an ordered field.

Proof. For the complex field to be an ordered field, it must have the property that $x^2 > 0$ if $x \in \mathbb{C}$ and $x \neq 0$. However, $(0,1) \neq 0$, but $(0,1) * (0,1) = (-1,0) = -1$. This violates the above property, and thus the complex field cannot be ordered. □

Exercise (9)**.** Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$ or $a = c$ and $b < d$. Prove that this is an order. Does it have the least upper bound property?

Proof. To prove that this is an order, it must satisfy two conditions. First, $z < w$, $z > w$, and $z = w$ must all be distinct. Now, since the set of reals is ordered, if $a < c$, $z < w$, and if $a > c$, $z > w$. If $a = c$, then the check falls to b and d. The same logic follows, and thus $z < w$, $z > w$, and $z = w$ are all distinct. Second, if $x < y$ and $y < z$, then $x < z$. Using a new $x = e + fi$, if $a < c$ and $c < e$, then **TODO** □

Exercise. Exercise 10: Suppose $z = a + ib$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2}\right)^{1/2}, \qquad b = \left(\frac{|w| - u}{2}\right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\overline{z})^2 = w$

Proof. **TODO** □