Derived adjunctions, translated twice: French to English, symbols to diagrams

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« La parole a été donnée à l'homme
pour cacher sa pensée. »
Stendhal, *Le rouge et le noir* (1830)

This note serves as a translation of the paper 'Le théorème de Quillen, d'adjonction des foncteurs dérivés, revisité' by Maltsiniotis (2007) in two ways. On the one hand, as its title may suggest, the paper is written in French, whereas the note lying before you is not. On the other hand, the proof of the main theorem of the paper (which we will discuss momentarily) is finished by some tedious and hard-to-parse symbol pushing, reasoning about 2-morphisms (natural transformations) in the 2-category of categories. Although it is not impossible to decipher the algebraic manipulations, there is little insight gained from this in our opinion.

String diagrams offer an alternative, more graphical notation for 2-morphisms in 2-categories, which *might* make reasoning about them easier, quicker or more intuitive. For the author, the goal of writing this note is to get acquainted with string diagrams, and to see whether they provide any benefits in a 'real-world' case (if your real world contains e.g. model categories) where it seems *any* alternative notational system should be better than the conventional symbolic notation. To become acquainted with string diagrams, the author used the book *Introducing String Diagrams* by Hinze and Marsden (2023) and a tutorial to make string diagrams in LaTeX using TikZ by the same authors[1].

The theorem we set out to reprove using the language of string diagrams is rather abstract, so let us for now discuss the main application in the context of model categories[2]. Recall that a *Quillen adjunction* is an adjunction $F \dashv U$ of functors $F : \mathscr{M} \rightleftarrows \mathscr{N} : U$ between model categories where F is a *left Quillen functor*, a functor that preserves cofibrations, trivial cofibrations and cofibrant objects (or, equivalently, U is a *right Quillen functor*, which preserves the fibrational data). Given a Quillen adjunction $F \dashv U$, we can construct *total derived functors* $\mathbf{L}F : \mathbf{Ho}(\mathscr{M}) \rightleftarrows \mathbf{Ho}(\mathscr{N}) : \mathbf{R}U$ between the homotopy

[1]Available from https://stringdiagramcom.files.wordpress.com/2023/11/howtodrawv0.4.pdf.

[2]We will not be very precise about our conventions about model categories, such as whether we assume functorial factorisations or not, since we will take for instance the existence of derived functors under the usual hypotheses for granted. For an introduction to model categories, we recommend Dwyer and Spaliński (1995) or Riehl (2022); the latter is more modern and its conventions are closer to ours than the former's. In particular, Riehl refers to Maltsiniotis (2007) for the proof of the derived adjunction.

categories of $\mathscr{M}$ and $\mathscr{N}$, whose universal property is that they are certain Kan extensions of F and U.

The main theorem of Maltsiniotis (2007) shows that these total derived functors again form an adjunction $\mathbf{L}F \dashv \mathbf{R}U$ at the level of homotopy categories. This result is traditionally proven using the details of the specific construction of the homotopy category; this is the case for all of the following standard accounts: Dwyer, Hirschhorn, et al. (2005), Dwyer and Spaliński (1995), Hirschhorn (2003), Hovey (2007), May and Ponto (2012), and Quillen (1967). Using the fact that the total derived functors are *absolute* Kan extensions, however, Maltsiniotis (2007) shows that the derived adjunction can be proven using abstract nonsense, from the universal properties.

For the statement of the theorem, we abstract away from model categories by considering arbitrary localisations and derived functors with respect to those localisations.

This note is organised as follows: § 1 records what adjunctions and Kan extensions look like using string diagrams; § 2 introduces the necessary definitions to be able to speak of total derived functors; and finally, the theorem is proven using string diagrams in § 3.

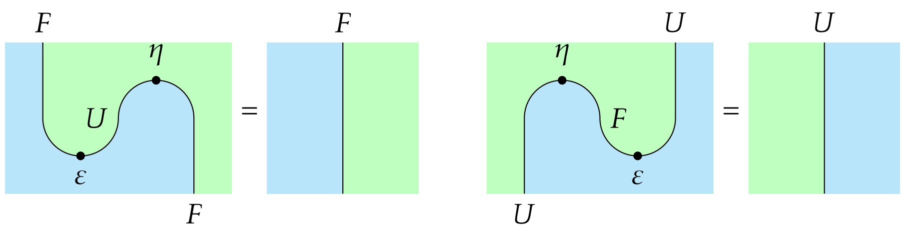
1 Adjunctions and Kan extensions in string diagrams

This note is not meant to be an introduction to string diagrams, but for the unexperienced user of string diagrams (such as the author at the time of writing), we record some basic string-diagrammatic equations that we will need in the proof of Theorem 3.1. Specifically, we will need to reason about adjunctions, Kan extensions, and also *absolute* Kan extensions.

Adjunctions String diagrams for adjunctions are rather intuitive. An adjunction

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, bend left, "U"] \end{tikzcd}$$

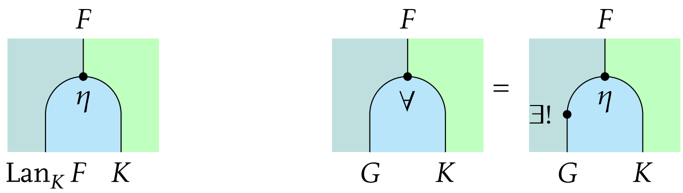
is specified by natural transformations $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow UF$, the *unit*, and $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathscr{D}}$, the *counit*, satisfying the *triangle identities*[3]:



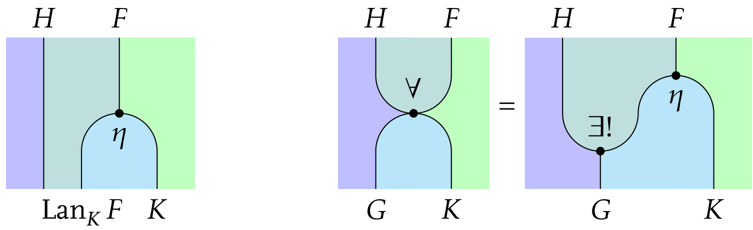
Geometrically, the triangle identities mean we can straighten a wiggly string which passes consecutively the unit and counit.

[3]This name refers to the shape of commutative diagrams, but one would be hard-pressed to recognise a triangle in the corresponding string diagrams.

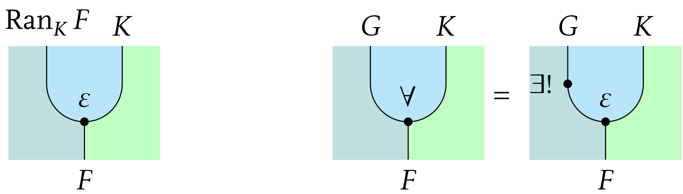
Kan extensions A *left Kan extension* of a functor $F : \mathscr{C} \to \mathscr{E}$ along a functor $K : \mathscr{C} \to \mathscr{D}$ is a functor $\mathrm{Lan}_K F : \mathscr{D} \to \mathscr{E}$ with a natural transformation $\eta : F \Rightarrow \mathrm{Lan}_K F \circ K$ such that for any pair of a functor $G : \mathscr{D} \to \mathscr{E}$ and a natural transformation $\alpha : F \Rightarrow GK$, the transformation α factors uniquely through η as in the following string diagrams:



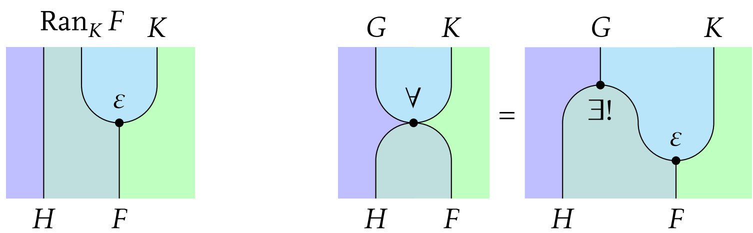
The left Kan extension $\mathrm{Lan}_K F$ is *absolute* if for any functor $H : \mathscr{E} \to \mathscr{F}$, the composite functor $H\,\mathrm{Lan}_K F : \mathscr{D} \to \mathscr{F}$ together with the whiskered transformation $H\eta$ is a left Kan extension of HF along K. Spelling out the definition, this means that for any pair of a functor $G : \mathscr{D} \to \mathscr{F}$ and a natural transformation $\alpha : HF \Rightarrow GK$, the transformation α factors uniquely through $H\eta$:



Dually, the corresponding diagrams for *right Kan extensions* $\mathrm{Ran}_K F : \mathscr{D} \to \mathscr{E}$ of $F : \mathscr{C} \to \mathscr{E}$ along $K : \mathscr{C} \to \mathscr{D}$ are:



and *absolute* right Kan extensions look as follows:



2 Localisations and derived functors

In this section, we introduce the definitions necessary for the statement of Theorem 3.1. Our treatment of localisations differs slightly from Maltsiniotis (2007): all our categories are assumed to be *locally small* (meaning that there is only a *set* of maps between any two objects), so the localisation of a category with respect to a class of maps need not exist in our universe. Maltsiniotis (2007) also restricts to locally small localisations, but using another linguistic trick: by speaking of *localisateurs*, which are defined to be pairs of a category $\mathscr{C}$ and a class of maps S such that the localisation of $\mathscr{C}$ with respect to S is locally small[4]. (Of course, these size problems are addressed by model categories.)

DEFINITION 2.1 · A *localisation* of a category $\mathscr{C}$ with respect to a class of maps S is a category $\mathscr{C}[S^{-1}]$ together with a functor $\gamma : \mathscr{C} \to \mathscr{C}[S^{-1}]$ taking maps in S to isomorphisms in $\mathscr{C}[S^{-1}]$ such that any functor $F : \mathscr{C} \to \mathscr{D}$ taking maps in S to isomorphisms in $\mathscr{D}$ factors uniquely through γ:

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathscr{D} \\ \mathscr{C}[S^{-1}] \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

In other words, such a functor γ defines a localisation of $\mathscr{C}$ with respect to S if it is initial among the functors out of $\mathscr{C}$ taking maps in S to isomorphisms.

EXAMPLES 2.2 · The *homotopy category* $\mathbf{Ho}(\mathscr{M})$ of a model category $\mathscr{M}$ is the localisation of $\mathscr{M}$ with respect to the class of weak equivalences in $\mathscr{M}$. The *derived category* $\mathbf{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$ is the localisation of the category $\mathbf{Ch}(\mathscr{A})$ of chain complexes with respect to the class of quasi-isomorphisms.

DEFINITION 2.3 · Let $\mathscr{C}$ be a category with a class of maps S, suppose $\gamma : \mathscr{C} \to \mathscr{C}[S^{-1}]$ is a localisation of $\mathscr{C}$ with respect to S, and let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then a *left derived functor* of F is a *right* Kan extension of F along γ, denoted $\mathrm{L}F : \mathscr{C}[S^{-1}] \to \mathscr{D}$. Dually, a *right derived functor* of F is a *left* Kan extension of F along γ, denoted $\mathrm{R}F : \mathscr{C}[S^{-1}] \to \mathscr{D}$. A left or right derived functor is *absolute* if it is an absolute Kan extension.

REMARK 2.4 · Especially in the literature on model categories, the terminology of derived functors is unfortunately rather inconsistent: read three different sources, and you will encounter four (subtly) different ideas of what a derived functor is supposed to be. Here we are exclusively interested in what we call absolute derived functors, but we do not incorporate absoluteness in the definition of derived functors to stress the requirement of this hypothesis for the proof of the theorem.

EXAMPLE 2.5 · A functor $F : \mathscr{M} \to \mathscr{C}$ from a model category to any category taking weak equivalences between cofibrant objects to isomorphisms has an *absolute* left derived functor (with respect to the localisation $\mathscr{M} \to \mathbf{Ho}(\mathscr{M})$) given by precomposing F with cofibrant replacement (see for example Riehl 2014, Theorem 2.2.8); there is an obvious dual statement for right derived functors.

DEFINITION 2.6 · Let $\mathscr{C}$ and $\mathscr{D}$ be categories with classes S of maps in $\mathscr{C}$ and T of maps in $\mathscr{D}$, suppose $\gamma : \mathscr{C} \to \mathscr{C}[S^{-1}]$ is a localisation of $\mathscr{C}$ with respect to S and $\delta : \mathscr{D} \to \mathscr{D}[T^{-1}]$ is a localisation of $\mathscr{D}$ with respect to T, and let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then a *total left derived functor* of F is a left derived functor of the composite

[4] Here we should probably be a bit more careful by demanding the localisation to be *essentially* locally small.

δF, denoted $\mathbf{L}F : \mathscr{C}[S^{-1}] \to \mathscr{D}[T^{-1}]$. Dually, a *total right derived functor* of F is a right derived functor of the composite δF, denoted $\mathbf{L}F : \mathscr{C}[S^{-1}] \to \mathscr{D}[T^{-1}]$. A total left or right derived functor is *absolute* if it is an absolute derived functor.

EXAMPLE 2.7 · If $F : \mathscr{M} \to \mathscr{N}$ is a functor between model categories taking weak equivalences between cofibrant objects in $\mathscr{M}$ to weak equivalences in $\mathscr{N}$, then it has an *absolute* total left derived functor, which follows from Example 2.5; again, there is a dual statement for total right derived functors. By Ken Brown's lemma (Riehl 2014, Lemma 11.3.14), it suffices for F to be a *left Quillen functor*, meaning that it preserves cofibrations, trivial cofibrations and cofibrant objects, for the absolute total left derived functor to exist.

3 Derived adjunction

We are now ready to state and prove the theorem.

THEOREM 3.1 · *Let $\mathscr{C}$ and $\mathscr{D}$ be categories with classes S of maps in $\mathscr{C}$ and T of maps in $\mathscr{D}$ and suppose $\gamma : \mathscr{C} \to \mathscr{C}[S^{-1}]$ is a localisation of $\mathscr{C}$ with respect to S and $\delta : \mathscr{D} \to \mathscr{D}[T^{-1}]$ is a localisation of $\mathscr{D}$ with respect to T. Let*

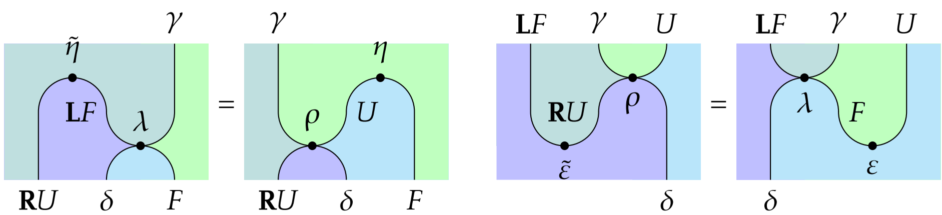
$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, shift left, "U"] \end{tikzcd}$$

be an adjunction between $\mathscr{C}$ and $\mathscr{D}$ with unit η and counit ε. Suppose that F and U admit respectively absolute total left and right derived functors $\mathbf{L}F : \mathscr{C}[S^{-1}] \to \mathscr{D}[T^{-1}]$ and $\mathbf{R}U : \mathscr{D}[T^{-1}] \to \mathscr{C}[S^{-1}]$ with natural transformations

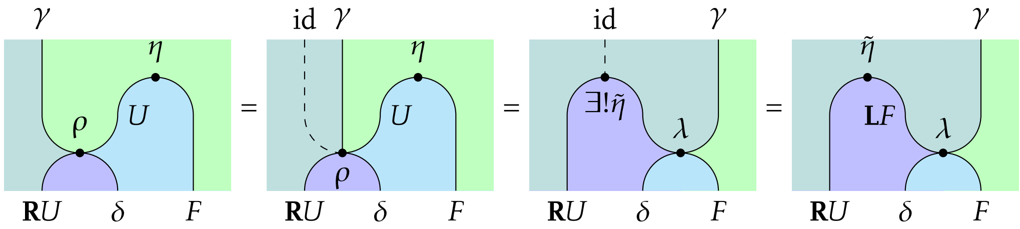
Then the total derived functors from an adjunction

$$\begin{tikzcd} \mathscr{C}[S^{-1}] \arrow[r, shift left, "\mathbf{L}F"] \arrow[r, phantom, "\perp"] & \mathscr{D}[T^{-1}] \arrow[l, shift left, "\mathbf{R}U"] \end{tikzcd}$$

with unit $\tilde{\eta}$ and counit $\tilde{\varepsilon}$ satisfying the following equations:

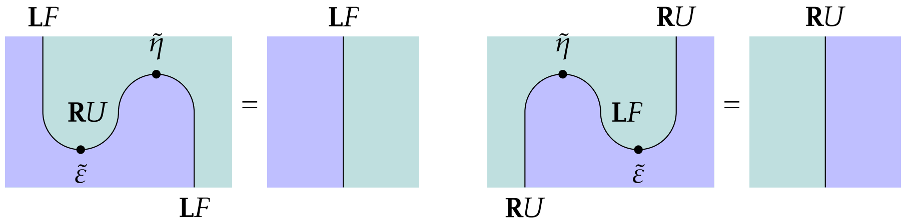


PROOF. Since the absolute total left derived functor $\mathbf{L}F$ is an *absolute* right Kan extension of δF along γ, the composite $\mathbf{R}U \circ \mathbf{L}F$ together with $\mathbf{R}U\lambda$ is a right Kan extension of $\mathbf{R}U\delta F$ along γ. We obtain the unit $\tilde{\eta}$ of the derived adjunction from the universal property of this Kan extension as the unique natural transformation satisfying:

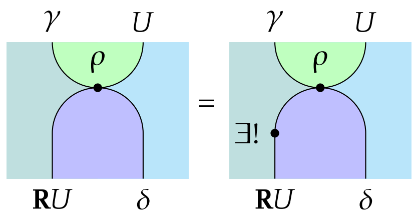


Dually, we obtain the counit $\tilde{\varepsilon}$ satisfying the desired equation.

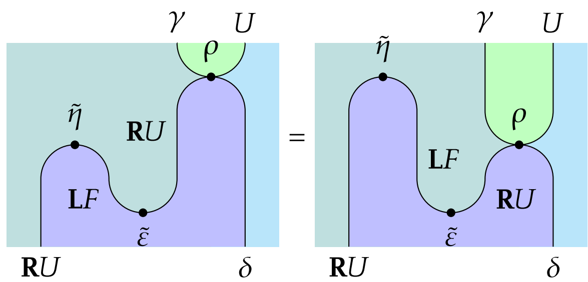
To show that the unit $\tilde{\eta}$ and counit $\tilde{\varepsilon}$ assemble into an adjunction, it remains to show the triangle identities:



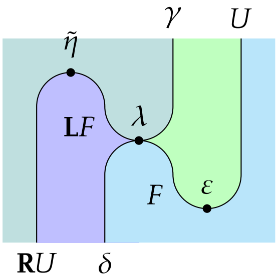
We only prove the latter; the former is dual. By the universal property of $\mathbf{R}U$ as a left Kan extension of γU along δ:



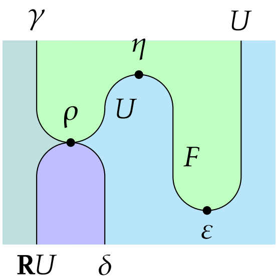
it suffices to show this equation holds when precomposed with ρ (and appropriately whiskered). This is now proven using the established equations. The first step follows from naturality of $\tilde{\eta}$:



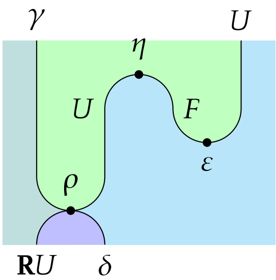
We can now apply the defining equation of $\tilde{\varepsilon}$:

$=$ 

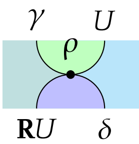
Then we can apply the defining equation of $\tilde{\eta}$:

$=$ 

Using naturality of ρ we can move ε up:

$=$ 

Finally, we apply one of the triangle identities of the adjunction $F \dashv U$:

$=$ 

This finishes the proof of the derived adjunction $\mathbf{L}F \dashv \mathbf{R}U$. □

EXERCISE · Use string diagrams to obtain the counit $\tilde{\varepsilon}$ and prove the other triangle identity.

COROLLARY 3.2 · *Let $F \dashv U$ be a Quillen adjunction of functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ between model categories. Then the total derived functors form an adjunction*

$$\mathbf{Ho}(\mathcal{M}) \underset{\mathbf{R}U}{\overset{\mathbf{L}F}{\underset{\perp}{\rightleftarrows}}} \mathbf{Ho}(\mathcal{N})$$

between the homotopy categories of $\mathcal{M}$ and $\mathcal{N}$.

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