Model categories

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This note is meant to give a quick introduction to the theory of Quillen model categories, a framework for abstract homotopy theory. It is mostly meant as a reference for myself; the main goal with writing it was to clarify the theory of derived functors, on which the literature is not very unified. Part of the note is based on material from my Bachelor's thesis [Sui23]; the rest is written as preparation for a talk for the seminar course *Advanced Seminar on Derived Categories* at Radboud University in the spring of 2024.[1] The reader might also want to take a look at [DS95; Hov07; Rie22].

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Outline

This note is structures as follows. In § 1, we will give the definition of a model category, prove some elementary properties of model categories and give some first examples of model categories. The structure of a model category $\mathcal{M}$ will contain three distinguished classes of maps of $\mathcal{M}$, called *weak equivalences*, *fibrations* and *cofibrations*, which should satisfy a number of axioms.

In § 2, we give some nontrivial but motivating examples of model categories, notable the model structures on the categories of topological spaces and simplicial sets. The proofs that these model structures satisfy the required axioms are very technical and do not fit in these notes, however, so we only define the distinguished classes of maps in these model categories.

[1]The handwritten notes for the talk can be found at https://splintersuidman.github.io/files/2024-05-24-homotopy-category-and-Quillen-functors.pdf.

In § 3, we use the structure of a model category to introduce a notion of homotopy between maps in a model category and a corresponding notion of homotopy equivalences. We obtain a generalised version of the Whitehead Theorem (stating that a continuous map between CW-complexes is a homotopy equivalence if and only if it is a weak homotopy equivalence) for homotopy equivalences and weak equivalences in a model category, specialising to the usual result for topological spaces.

The notion of homotopy will allow us to construct the *homotopy category* associated to a model category in § 4, which is a localisation of the model category with respect to the weak equivalences; that is, the homotopy category is obtained by 'freely' inverting the weak equivalences.

Finally, in § 5, we discuss conditions for functors between model categories to induce functors between the corresponding homotopy categories, and specifically how adjunctions between model categories induce adjunctions or even equivalences between homotopy categories. Perhaps the most important result we discuss (but do not prove) is that the model categories of topological spaces and simplicial sets 'encode the same homotopy theory'; in particular, their homotopy categories are equivalent.

1 Model structures

Before we state the definition of a model structure, we introduce some auxiliary definitions that feature in this definition.

Recall that the *walking arrow* $\mathbb{2}$ is the category that contains two objects and a single non-identity map the objects. If $\mathscr{C}$ is a category, then the objects of the functor category $\mathbf{Fun}(\mathbb{2}, \mathscr{C})$, called the *arrow category* of $\mathscr{C}$, are the maps of $\mathscr{C}$, and a map from $f : A \to B$ to $g : X \to Y$ in $\mathbf{Fun}(\mathbb{2}, \mathscr{C})$ is a commutative square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd} \tag{1}$$

An object A in a category $\mathscr{C}$ is a *retract* of B if there are maps $s : A \to B$ and $r : B \to A$ such that $rs = \mathrm{id}_A$; the map r is called a *retraction*.

DEFINITION 1.1 · A map $f : A \to B$ in a category $\mathscr{C}$ is a *retract* of a map $g : X \to Y$ if f is a retract of g in $\mathbf{Fun}(\mathbb{2}, \mathscr{C})$. Explicitly, this means that there is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, equal] & X \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, equal] & Y \arrow[r] & B \end{tikzcd}$$

DEFINITION 1.2 · A *functorial factorisation* in a category $\mathscr{C}$ is a functor $\mathbf{Fun}(\mathbb{2}, \mathscr{C}) \to \mathbf{Fun}(\mathbb{3}, \mathscr{C})$ from the arrow category of $\mathscr{C}$ to the category of pairs of composeable arrows in $\mathscr{C}$ that defines a section to the composition functor $\circ : \mathbf{Fun}(\mathbb{3}, \mathscr{C}) \to \mathbf{Fun}(\mathbb{2}, \mathscr{C})$. Such a functor sends an object $f : X \to Y$ in the arrow category $\mathbf{Fun}(\mathbb{2}, \mathscr{C})$ to a pair of maps $Lf : X \to Ef$ and $Rf : Ef \to Y$ such that $f = Rf \circ Lf$. On maps, a functorial factorisation takes a commutative diagram (which is a map in the arrow category) on the left to a diagram like on the right:

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r, "v"'] & Y
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
A \arrow[r] \arrow[d, "Lf"] \arrow[dd, bend right, "f"'] & X \arrow[d, "Lg"'] \arrow[dd, bend left, "g"] \\
Ef \arrow[r, "E(u,v)"] \arrow[d, "Rf"] & Eg \arrow[d, "Rg"'] \\
B \arrow[r] & Y
\end{tikzcd}
$$

DEFINITION 1.3 · A map $i : A \to B$ has the *left lifting property* with respect to $p : X \to Y$ if in all solid commutative squares of the form

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}
$$

there is a *lift* $h : B \to X$ (dashed) making the triangles commute. In this case, we also say that p has the *right lifting property* with respect to i.

We are now ready to state the definition of a model structure.

DEFINITION 1.4 · A *model structure* on a category $\mathcal{M}$ consists of three distinguished classes of maps of $\mathcal{M}$, *weak equivalences* (sometimes denoted $\xrightarrow{\sim}$), *fibrations* (denoted $\twoheadrightarrow$) and *cofibrations* (denoted $\rightarrowtail$), and two functorial factorisations in $\mathcal{M}$. Each of these classes of maps should be closed under composition and contain all identity maps. A map which is both a fibration and a weak equivalence is called an *acyclic* or *trivial fibration*, and a map which is both a cofibration and a weak equivalence is called an *acyclic* or *trivial cofibration*.

These classes of maps and factorisations should satisfy the following axioms:

(MC1) *Two-out-of-three*: For all maps $f : A \to B$ and $g : B \to C$, if two of the three maps f, g and $g \circ f$ are weak equivalences, then so is the third.

(MC2) *Retracts*: If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.

(MC3) *Lifting*: Cofibrations have the left lifting property with respect to trivial fibrations, and trivial cofibrations have the left lifting property with respect to fibrations.

(MC4) *Factorisation*: Every map f factors functorially as a trivial cofibration followed by a fibration, and as a cofibration followed by a trivial fibration.

DEFINITION 1.5 · A *model category* is a complete and cocomplete category $\mathcal{M}$ with a model structure on $\mathcal{M}$.

LEMMA 1.6 · *The weak equivalences in a model category are precisely the maps that can be factored as a trivial cofibration followed by a trivial fibration.*

PROOF. Since the class of weak equivalences is closed under composition, the composition of a trivial cofibration and a trivial fibration is a weak equivalence. Conversely, use (MC4) to factor a weak equivalence as a trivial cofibration followed by a fibration. By the two-out-of-three property (MC1), the fibration is a trivial fibration. □

LEMMA 1.7 · *The classes of weak equivalences, fibrations and cofibrations in a model category contain all isomorphisms.*

PROOF. If $f : X \to Y$ is an isomorphism, then the retract diagram

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[d, "f"'] \arrow[rr, bend left, equal] & Y \arrow[r, "f^{-1}"'] \arrow[d, equal] & X \arrow[d, "f"] \\
Y \arrow[r, equal] \arrow[rr, bend right, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

shows using (MC2) that f is a weak equivalence, fibration and cofibration since the identity $\mathrm{id}_Y : Y \to Y$ is. □

DEFINITION 1.8 · An object X of a model category $\mathcal{M}$ is *cofibrant* if the unique map $\varnothing \to X$ from the initial object of $\mathcal{M}$ to X is a cofibration, and X is called *fibrant* if the unique map $X \to *$ from X to the terminal object of $\mathcal{M}$ is a fibration.

As we will see, the fibrant and cofibrant objects of a model category are often better behaved than arbitrary objects. In important examples of model categories, we will sometimes see that the objects are all fibrant (topological spaces) or all cofibrant (simplicial sets).

DEFINITION 1.9 · By factoring the unique map $\varnothing \to X$ for any object X of a model category $\mathcal{M}$ as a cofibration $\varnothing \rightarrowtail QX$ followed by a trivial fibration $q_X : QX \overset{\sim}{\twoheadrightarrow} X$, we obtain an endofunctor Q on $\mathcal{M}$ that sends an object to a *cofibrant replacement* QX, together with a natural weak equivalence $q : Q \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathcal{M}}$. Dually, by factoring the unique map $X \to *$ as a trivial cofibration $r_X : X \overset{\sim}{\rightarrowtail} RX$ followed by a fibration $RX \twoheadrightarrow *$, we find an endofunctor R on $\mathcal{M}$ sending an object to its *fibrant replacement*, together with a natural weak equivalence $r : \mathrm{id}_{\mathcal{M}} \overset{\sim}{\Rightarrow} R$.

In particular, every object in a model category is weakly equivalent to a cofibrant and a fibrant object. In the homotopy category of a model category, which we will discuss in § 4, weak equivalences become isomorphisms, so a cofibrant or fibrant replacement of an object becomes isomorphic to that object in the homotopy category.

LEMMA 1.10 · *Let $f : X \to Y$ be a map in a model category. Then the following are equivalent:*

① $f : X \to Y$ *is a weak equivalence;*

② $Qf : QX \to QY$ *is a weak equivalence;*

③ $Rf : RX \to RY$ *is a weak equivalence.*

PROOF. By naturality of q, the diagram

$$\begin{tikzcd} QX \arrow[r, "q_X", "\sim"'] \arrow[d, "Qf"'] & X \arrow[d, "f"] \\ QY \arrow[r, "\sim", "q_Y"'] & Y \end{tikzcd}$$

commutes, so the result follows by the two-out-of-three property; similarly for R. $\square$

Phrased differently, the last lemma says that the endofunctors Q and R on $\mathcal{M}$ *create* weak equivalences.

We will now discuss some simple examples of model categories.

EXAMPLE 1.11 (model structures on **Set**) · There are exactly *nine* model structures on the category of sets (see [Bal21, § 17.3]). Here we discuss one of them. Take the epimorphisms (surjective maps) as cofibrations, the monomorphisms (injective maps) as fibrations, and all maps as weak equivalences. The two-out-of-three property (MC1) is then trivial.

To check (MC2), if $f : A \to B$ is a retract of $g : X \to Y$, then we have a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, tail, "i"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & X \arrow[r, two heads, "r"'] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, tail, "i'"] \arrow[rr, bend right, "\mathrm{id}_B"'] & Y \arrow[r, two heads, "r'"] & B \end{tikzcd}$$

where i and i' are injective, and r and r' are surjective. In the case that g is a weak equivalence, there is nothing to check. If g is a cofibration, then $fr = r'g$ is an epimorphism, and hence f is an epimorphism and thus a cofibration. Finally, if g is a fibration, then $i'f = gi$ is a monomorphism, and hence f is a monomorphism, thus a fibration.

For (MC3), given a lifting problem

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "i"'] & X \arrow[d, tail, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration (both are necessarily weak equivalences), we can define a lift $h : B \to X$ either as the section of i composed with f, or as g composed with the retraction of p. From the commutativity of the square, it follows that these definitions are in fact equal; commutativity of the triangles with side h follows directly from the properties of the section of i and retraction of p.

Finally, for (MC4), as a factorisation of a map $f : A \to B$ (in both cases since all maps are weak equivalences), we can take

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, twoheadrightarrow, "f"'] & & B \\
& f(A) \arrow[ur, hook] &
\end{tikzcd}$$

where $f : A \to f(A)$ is cofibration since any map is surjective on its image, and the inclusion $f(A) \hookrightarrow B$ is injective and thus a fibration. Checking that this factorisation is functorial is straightforward.

The homotopy category of this model structure on **Set**, in which the weak equivalences – in this case, all maps – are inverted, is equivalent to the terminal category $\mathbb{1}$.

EXAMPLE 1.12 · Let $\mathscr{M}$ be any complete and cocomplete category. There is a model structure on $\mathscr{M}$ where all maps are fibrations and cofibrations, and where the weak equivalences are the isomorphisms. With the goal of formally inverting weak equivalences in mind, this model structure is not very interesting: the isomorphisms are already invertible, so the resulting homotopy category will be isomorphic to $\mathscr{M}$.

REMARK 1.13 (duality) · If $\mathscr{M}$ is a model category, then there is a model structure on $\mathscr{M}^{\mathrm{op}}$ where the cofibrations of $\mathscr{M}^{\mathrm{op}}$ are the fibrations of $\mathscr{M}$, the fibrations of $\mathscr{M}^{\mathrm{op}}$ are the cofibrations of $\mathscr{M}$, and the weak equivalences of $\mathscr{M}^{\mathrm{op}}$ are the weak equivalences of $\mathscr{M}$. As a consequence, claims about model categories have dual versions, where cofibrations become fibrations and *vice versa*. This observation is very often used when proving results about model categories.

EXAMPLE 1.14 · Let $\mathscr{C}$ be a category with an object A. Then the *slice category* $\mathscr{C}_{/A}$ of $\mathscr{C}$ *over* A has as objects the maps $x : X \to A$ into A, and a map in $\mathscr{C}_{/A}$ from $x : X \to A$ to $y : Y \to A$ is a map $f : X \to Y$ in $\mathscr{C}$ such that the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "x"'] & & Y \arrow[dl, "y"] \\
& A &
\end{tikzcd}$$

commutes. If $\mathscr{M}$ is a model category with an object A, then there is a model structure on $\mathscr{M}_{/A}$ (which is also complete and cocomplete if $\mathscr{M}$ is) where a map f from $x : X \to A$ to $y : Y \to A$ is a weak equivalence, cofibration or fibration if $f : X \to Y$ is in $\mathscr{M}$. The model category axioms follow directly from those of $\mathscr{M}$. Dually, the slice category $\mathscr{M}_{A/}$ of $\mathscr{M}$ *under* A, whose objects are maps $x : A \to X$ out of A, admits a model structure in a similar way.

The following proposition is a useful characterisation of the (trivial) fibrations and (trivial) cofibrations. It shows that either of the classes of fibrations and cofibrations is determined by the other together with the class of weak equivalences. The proof uses all model structure axioms, except for the two-out-of-three property.

PROPOSITION 1.15 · *Let $\mathscr{M}$ be a model category.*

① *The cofibrations in $\mathscr{M}$ are precisely the maps that have the left lifting property with respect to trivial fibrations.*

② *The trivial cofibrations in $\mathcal{M}$ are precisely the maps that have the left lifting property with respect to fibrations.*

③ *The fibrations in $\mathcal{M}$ are precisely the maps that have the right lifting property with respect to trivial cofibrations.*

④ *The trivial fibrations in $\mathcal{M}$ are precisely the maps that have the right lifting property with respect to cofibrations.*

PROOF. We only prove the first statement; the proof of the second is similar, and the third and fourth follow by duality (Remark 1.13) from the first two. Axiom (MC3) says that cofibrations have the left lifting property with respect to trivial fibrations. Conversely, let $f : A \to B$ be a map with the left lifting property with respect to trivial fibrations. Factor f using (MC4) as a cofibration $i : A \rightarrowtail C$ followed by a trivial fibration $p : C \overset{\sim}{\twoheadrightarrow} B$. Since f has the left lifting property with respect to p, there is a lift $s : B \to C$ in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & C \arrow[d, "p"] \\
B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dashed, "r"] & B
\end{tikzcd}$$

Commutativity of the bottom triangle means that s is a section of p. Recognising f as a retract of i in the diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & A \arrow[r, "\mathrm{id}_A"'] \arrow[d, "i"] & A \arrow[d, "f"] \\
B \arrow[r, "s"] \arrow[rr, bend right, "\mathrm{id}_B"'] & Y \arrow[r, "p"] & B
\end{tikzcd}$$

it follows from (MC2) that f is a cofibration. □

Combined with Lemma 1.6, this shows that the cofibrations and trivial cofibrations (or, dually, the fibrations and trivial fibrations) entirely determine the model structure.

An example of a property of the classes of cofibrations and fibrations that is easy to prove using the characterisation of Proposition 1.15 is the following lemma.

LEMMA 1.16 · *In a model category, the cofibrations and trivial cofibrations are stable under pushouts, and dually, the fibrations and trivial fibrations are stable under pullbacks.*

PROOF. We have to show that a pushout of a cofibration along any map is again a cofibration. Suppose $f : X \rightarrowtail Y$ is a cofibration in the pushout square on the left in the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, tail, "f"'] & Z \arrow[r] \arrow[d, "g"] & A \arrow[d, two heads, "q", "\wr"'] \\
Y \arrow[r] \arrow[urr, dashed, bend left] & Y \coprod_X Z \arrow[r] \arrow[ur, dotted] \arrow[ul, phantom, "\ulcorner"] & B
\end{tikzcd}$$

By Proposition 1.15, to show that g is a cofibration, it suffices to show that g has the right lifting property with respect to trivial fibrations. Attaching a lifting problem given by a trivial fibration $q : A \xrightarrow{\sim} B$ on the right, we find a lift $Y \to A$ (dashed) in the composite diagram. Applying the universal property of the pushout $Y \amalg_X Z$ to this lift and the map $Z \to A$, we find the desired lift $Y \amalg_X Z \to A$ (dotted). The proof for trivial fibrations is analogous, lifting against fibrations instead of trivial fibrations. The proofs of the dual statements are dual. □

Although the map obtained from the universal property of the pushout $Y \amalg_X Z$ in the above proof is unique, the lift $Y \amalg_X Z \to A$ need not be unique since the original lift $Y \to A$ may not be.

The class of (trivial) cofibrations is in fact closed under more operations: it is closed under transfinite composition and retracts, stable under pushouts, and contains all isomorphisms. This follows from the fact that the (trivial) cofibrations are characterised by a left lifting property. A dual result holds for (trivial) fibrations, being characterised by a right lifting property.

2 Examples of model categories

We should discuss some more significant examples of model categories now. The proofs of the theorems in this section are all quite some work; you might have seen some of it in courses on algebraic topology or ∞-categories.

Topological spaces The motivating category for the abstract theory of model categories is the category **Top** of topological spaces. The most important model structure on **Top** is the Quillen model structure. Recall that a *weak homotopy equivalence* is a continuous map $f : X \to Y$ that induces a bijection $\pi_0(X) \to \pi_0(Y)$ on path components and group isomorphisms $\pi_n(X) \to \pi_n(Y)$ on all higher homotopy groups for $n \geqslant 1$.

THEOREM 2.1 (Quillen [Qui67]) · *There is a model structure on the category* **Top** *of topological spaces where a map is:*

- *a weak equivalence if it is a weak homotopy equivalences,*
- *a* (Serre) *fibration if it has the right lifting property against inclusions $D^n \times \{0\} \hookrightarrow D^n \times [0,1]$ for $n \geqslant 0$, and*
- *a trivial fibration if it has the right lifting property against boundary inclusions $\partial D^n \hookrightarrow D^n$ for $n \geqslant 0$:*[2]

$$\begin{tikzcd} D^n \times \{0\} \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ D^n \times [0,1] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} \partial D^n \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "\wr"] \\ D^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

[2]In many examples of model categories, the following happens: there are *sets* of maps called *generating cofibrations* and *generating trivial cofibrations*, and the trivial fibrations and fibrations are defined by the right lifting property with respect to these sets. The (trivial) cofibrations are then defined by the left lifting property with respect to (trivial) fibrations, and it turns out that these classes can also be described as retracts of transfinite compositions of pushouts of the generating (trivial) cofibrations; this is the sense in which they *generate* the (trivial) cofibrations. With some more 'smallness' assumptions on the sets of generating (trivial) cofibrations, such a model category is called *cofibrantly generated*.

We call this the Quillen model structure *and denote it* $\mathbf{Top}_{\text{Quillen}}$*. With respect to this model structure, every space is fibrant, and the cofibrant spaces are the retracts of relative cell complexes, in particular the CW-complexes.*

We will see that the homotopy category of this model category is equivalent to the usual homotopy category of CW-complexes, with CW-complexes as objects and homotopy classes of continuous maps as morphisms.

There is another model structure on the category of topological spaces, due to Strøm in his appropriately titled article 'The Homotopy Category Is a Homotopy Category' [Str72], where the weak equivalences are the homotopy equivalences.

THEOREM 2.2 (Strøm [Str72]) · *There is a model structure on the category* $\mathbf{Top}$ *of topological spaces where a map is:*

- *a weak equivalence if it is a homotopy equivalence, and*
- *a* (Hurewicz) *fibration if it has the right lifting property against all inclusions* $X \times \{0\} \hookrightarrow X \times [0,1]$ *for* X *any space:*

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ X \times [0,1] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

We call this the Strøm model structure *and denote it* $\mathbf{Top}_{\text{Strøm}}$.[3]

Simplicial sets Another important model category is the Kan–Quillen model structure on the category $\mathbf{sSet}$ of simplicial sets. Simplicial sets are a combinatorial model for 'spaces', and one incarnation of the *homotopy hypothesis* says that the Quillen model structure on $\mathbf{Top}$ and the Kan–Quillen model structure on $\mathbf{sSet}$ encode the same homotopy theory; their homotopy categories are equivalent. We will discuss this in Theorem 5.16.

THEOREM 2.3 (Quillen [Qui67]) · *There is a model structure on the category* $\mathbf{sSet}$ *of simplicial sets where a map* f *is:*

- *a weak equivalence if the geometric realisation* $|f|$ *is a weak homotopy equivalence of topological spaces,*
- *a* (Kan) *fibration if it has the the right lifting property with respect to horn inclusions* $\Lambda^n_k \hookrightarrow \Delta^n$ *for* $n \geqslant 1$ *and* $0 \leqslant k \leqslant n$ *(called* Kan fibrations*), and*
- *a trivial* (Kan) *fibration if it has the right lifting property with respect to boundary inclusions* $\partial\Delta^n \hookrightarrow \Delta^n$ *for* $n \geqslant 0$*:*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "\wr"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

We call this the Kan–Quillen model structure *and denote it* $\mathbf{sSet}_{\text{Kan}}$*. The cofibrations are precisely the levelwise injective simplicial maps, and every simplicial set is cofibrant.*

[3] Interestingly, this model structure is not cofibrantly generated [Bal21, Proposition 7.2.5].

The fibrant objects in this model category are called *Kan complexes*. With the model of quasicategories (simplicial sets with the right lifting property with respect to *inner* horn inclusions) for ∞-categories, the Kan complexes are exactly the ∞-groupoids.

Categories There is also a model structure on the category $\mathbf{Cat}$ of small categories. One reason to suspect that $\mathbf{Cat}$ might carry some homotopical structure, is that it is a 2-category, where natural transformations between functors may be seen as some sort of homotopy between maps. To see this more explicitly, note that a natural transformation between functors $\mathscr{C} \to \mathscr{D}$ is a morphism in the category $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$, so a functor $\mathbb{2} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$ where $\mathbb{2}$ is the walking arrow. Since $\mathbf{Cat}$ is cartesian closed, this is the same thing as a functor $\mathscr{C} \times \mathbb{2} \to \mathscr{D}$, which looks like a homotopy where the interval object is the walking arrow $\mathbb{2}$.

Correspondingly, a functor is an *equivalence* of categories if it has an inverse up to natural isomorphism. To discuss the *natural model structure* on $\mathbf{Cat}$, we also need to recall the notion of *isofibrations*: a functor $F : \mathscr{C} \to \mathscr{D}$ is an isofibration if for all objects $x \in \mathscr{C}$ and all isomorphisms $f : Fx \to y$ in $\mathscr{D}$, there is an isomorphism $\bar{f} : x \to \bar{y}$ in $\mathscr{C}$ such that $F\bar{f} = f$.[4]

THEOREM 2.4 ([Rez00]) · *There is a model structure on the category* $\mathbf{Cat}$ *of small categories where a functor is:*

- *a weak equivalence if it is an equivalence of categories,*
- *a fibration if it is an isofibration, and*
- *a cofibration if it is injective on objects.*

We call this the natural model structure *and denote it* $\mathbf{Cat}_{\mathrm{nat}}$. *All categories in this model category are bifibrant.*

Interestingly, this is the only model structure on $\mathbf{Cat}$ where the weak equivalences are the equivalences (see [Bal21, Proposition 9.1.6]); it is therefore also called the *canonical model structure*.

Chain complexes The final example we discuss in this section is the *projective model structure* on the category $\mathbf{Ch}_{\geqslant 0}(R)$ of nonnegatively graded chain complexes of R-modules, where R is a ring.

THEOREM 2.5 ([DS95]) · *There is a model structure on the category* $\mathbf{Ch}_{\geqslant 0}(R)$ *of nonnegatively graded chain complexes of* R*-modules where a map is:*

- *a weak equivalence if it is a quasi-isomorphism,*
- *a fibration if it is an epimorphism in positive degrees, and*
- *a cofibration if it is a monomorphism with projective cokernel in all degrees.*

We call this the projective model structure *and denote it* $\mathbf{Ch}_{\geqslant 0}(R)_{\mathrm{proj}}$. *A cofibrant replacement in this model category is exactly a projective resolution in the sense of homological algebra.*

Dually, there is also an *injective model structure* on the category $\mathbf{Ch}^{\geqslant 0}(R)$ of nonnegatively graded cochain complexes where fibrant replacement corresponds to injective resolution.

[4]As an exercise, try to state this definition in terms of lifting properties of functors.

3 Homotopy in model categories

Using the structure of a model category, we can introduce a notion of *homotopy* between maps in a model category, which in turn will allow us to define the *homotopy category* associated to a model category. The presentation here follows [Rie22], working out some of the details.

DEFINITION 3.1 · Let A be an object in a model category. A *cylinder object* for A is an object $\mathrm{cyl}(A)$ with a factorisation

$$\begin{tikzcd} A \sqcup A \arrow[rr, "(\mathrm{id}_A,\mathrm{id}_A)"] \arrow[dr, tail, "(i_0,i_1)"'] & & A \\ & \mathrm{cyl}(A) \arrow[ur, two heads, "q"', "\sim"] & \end{tikzcd}$$

of the fold map $A \sqcup A \to A$ into a cofibration followed by a trivial fibration. Dually, a *path object* for A is an object $\mathrm{path}(A)$ with a factorisation

$$\begin{tikzcd} A \arrow[rr, "(\mathrm{id}_A,\mathrm{id}_A)"] \arrow[dr, tail, "j"', "\sim"] & & A \times A \\ & \mathrm{path}(A) \arrow[ur, two heads, "(p_0,p_q)"'] & \end{tikzcd}$$

of the diagonal map $A \to A \times A$ into a trivial cofibration followed by a fibration.

By the factorisation axiom, any object in a model category admits a cylinder and a path object, which can even be chosen in a functorial manner (but this is not essential for the theory, and we will not need to make use of this fact).

EXAMPLE 3.2 · A cylinder object for a topological space X is the usual cylinder

$$X \sqcup X \xrightarrow{((-,0),(-,1))} X \times [0,1] \xrightarrow[\sim]{\mathrm{pr}_X} X$$

and a path object for X is

$$X \xrightarrow[\sim]{\mathrm{const}} \mathrm{Map}([0,1],X) \xrightarrow{(\mathrm{ev}_0,\mathrm{ev}_1)} X.$$

For a simplicial set K, cylinder and path objects are similarly given by $K \times \Delta^1$ and $\mathrm{Map}(\Delta^1, K)$. Cylinder and path objects for a category $\mathscr{C}$ are given by $\mathscr{C} \times \mathbb{2}$ and $\mathbf{Fun}(\mathbb{2}, \mathscr{C})$, where $\mathbb{2}$ is the walking arrow category.

Using cylinder and path objects, we may introduce a notion of homotopy between maps in a model category.

DEFINITION 3.3 · Let $f, g : A \to X$ be two maps in a model category. A *left homotopy* from f to g is a map $H :\ \mathrm{cyl}(A) \to B$ making the diagram

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{cyl}(A) \arrow[d, "H"] & A \arrow[l, "i_1"'] \arrow[dl, "g"] \\ & X & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} A \sqcup A \arrow[r, "(f,g)"] \arrow[d, tail, "(i_0,i_1)"'] & X \\ \mathrm{cyl}(A) \arrow[ur, "H"'] & \end{tikzcd}$$

commute. In other words, a left homotopy is an extension of $(f,g) : A \amalg A \to X$ along the 'inclusion at the endoints' $(i_0, i_1) : A \amalg A \to \mathrm{cyl}(A)$. We say f and g are *left homotopic* and write $f \sim_\ell g$ if there exists a left homotopy from f to g.

Dually, a *right homotopy* from f to g is a map $K : A \to \mathrm{path}(X)$ making the diagram

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "K"] \arrow[dr, "g"] & \\ X & \mathrm{path}(X) \arrow[l, "p_0"] \arrow[r, "p_1"'] & X \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} & \mathrm{path}(X) \arrow[d, "(p_0,p_1)"] \\ A \arrow[ur, "K"] \arrow[r, "(f,g)"'] & X \times X \end{tikzcd}$$

commute. We say f and g are *right homotopic* and write $f \sim_r g$ if there exists a right homotopy from f to g.

LEMMA 3.4 · *The endpoint inclusions $i_0, i_1 : A \to \mathrm{cyl}(A)$ into the cylinder object are weak equivalences, and also cofibrations if A is cofibrant. In particular, if $f \sim_\ell g$, then f is a weak equivalence if and only if g is. Dually, the endpoint projections $p_0, p_1 : \mathrm{path}(X) \to X$ are weak equivalences, and also fibrations if X is fibrant. In particular, if $f \sim_r g$, then f is a weak equivalence if and only if g is.*

PROOF. The identity on A factors as i_0 followed by the weak equivalence q, so i_0 is a weak equivalence by the two-out-of-three-property; similarly for i_1. If H is a left homotopy from f to g and f is a weak equivalence, then so is H since $Hi_0 = f$, and then g is also a weak equivalence. If A is cofibrant, then the coproduct inclusions $A \to A \amalg A$ are too on account of the pushout diagram

$$\begin{tikzcd} \varnothing \arrow[r, tail] \arrow[d, tail] & A \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & A \amalg A \end{tikzcd}$$

and Lemma 1.16. The endpoint inclusion i_0 then factors as the cofibrations $A \rightarrowtail A \amalg A \to \mathrm{cyl}(A)$, and is thus itself a cofibration, and so is i_1. $\square$

EXAMPLE 3.5 (based path objects are weakly contractible) · Let X be a fibrant object in a model category and let $x : * \to X$ be a 'point' of X. Then the fibre of the endpoint projection $p :\ \mathrm{path}(X) \overset{\sim}{\twoheadrightarrow} X$ over the point x can be thought of as the object $\mathrm{path}_x(X)$ of paths in X starting at x. The pullback diagram

$$\begin{tikzcd} \mathrm{path}_x(X) \arrow[r] \arrow[d, two heads, "\wr"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{path}(X) \arrow[d, two heads, "\wr"', "p_0"] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

shows that $\mathrm{path}_x(X)$ is *weakly contractible*, meaning that the map $\mathrm{path}_x(X) \to *$ is a weak equivalence, since p_0 is a trivial fibration by Lemma 3.4 and trivial fibrations are stable under pullback by Lemma 1.16.

On the other hand, if we define the object $\mathrm{path}_X(x, y)$ of paths from x to $y : * \to X$ in an analogous manner, we see that $\mathrm{path}_X(x, y)$ is fibrant but not necessarily weakly contractible.

This example illustrates that the intuition you might have from based path objects in concrete homotopical categories, such as topological spaces or simplicial sets, carries over to abstract homotopical categories. If you know some homotopy type theory, you might also have seen that the based path type $\sum_{x' : X} x = x'$ is contractible.

PROPOSITION 3.6 · *If A is a cofibrant and X is a fibrant object in a model category $\mathcal{M}$, then left and right homotopy coincide and define an equivalence relation on* $\mathrm{Hom}_{\mathcal{M}}(A, X)$.

PROOF. For $f : A \to X$, the commuting diagram

$$\begin{tikzcd} A \arrow[r, tail, "i_0", "\sim"'] \arrow[dr, equal] \arrow[ddr, bend right, "f"'] & \mathrm{cyl}(A) \arrow[d, "q", "\wr"'] & A \arrow[l, tail, "i_1"', "\sim"] \arrow[dl, equal] \arrow[ddl, bend left, "f"] \\ & A \arrow[d, "f"] & \\ & X & \end{tikzcd}$$

shows that left homotopy is reflexive.

If $H : \mathrm{cyl}(A) \to X$ is a left homotopy from f to g, then we can use the automorphism $s : A \amalg A \to A \amalg A$ which switches factors to make H into a left homotopy from g to f:

$$\begin{tikzcd} A \amalg A \arrow[r, tail, "s"] \arrow[rr, tail, bend left, "(i_1, i_0)"] \arrow[drr, bend right, "(g,f)"'] & A \amalg A \arrow[r, tail, "(i_0, i_1)"] \arrow[dr, "(f,g)"'] & \mathrm{cyl}(A) \arrow[r, two heads, "q", "\sim"'] \arrow[d, "H"] & A \\ & & X & \end{tikzcd}$$

Since s is an isomorphism, it is indeed a cofibration, which follows from the fact that the class of cofibrations satisfies a lifting property. Hence left homotopy is symmetric.

For transitivity, suppose we have left homotopies $H : \mathrm{cyl}(A) \to X$ from f to g and $K : \mathrm{cyl}'(A) \to X$ from g to h, possibly via different cylinder objects for A. Form the pushout $\mathrm{cyl}(A) \amalg_A \mathrm{cyl}'(A)$ of i_1 and i'_0 as in the following diagram and factor the induced map $\mathrm{cyl}(A) \amalg_A \mathrm{cyl}'(A) \to A$ to obtain a new cylinder object $\mathrm{cyl}''(A)$[5]:

$$\begin{tikzcd}
A \arrow[dr, tail, "i_0", "\sim"'] & & A \arrow[dl, tail, "i_1"', "\sim"] \arrow[dr, tail, "i'_0", "\sim"'] & & A \arrow[dl, tail, "i'_1"', "\sim"] \\
& \mathrm{cyl}(A) \arrow[dr, tail, "\sim", "j_0"'] \arrow[dddr, bend right, two heads, "q"', "\sim"] & & \mathrm{cyl}''(A) \arrow[dl, tail, "\sim"', "j_1"] \arrow[dddl, bend left, two heads, "q'", "\sim"'] & \\
& & \mathrm{cyl}(A) \coprod\limits_A \mathrm{cyl}'(A) \arrow[d, tail, "k", "\wr"'] & & \\
& & \mathrm{cyl}''(A) \arrow[d, two heads, "q''", "\wr"'] & & \\
& & A & &
\end{tikzcd}$$

The endpoint inclusions into the new cylinder $\mathrm{cyl}''(A)$ are the composites kj_0i_0 and $kj_1i'_1$. (Intuitively, $\mathrm{cyl}''(A)$ is obtained by gluing the top of $\mathrm{cyl}(A)$ to the bottom of $\mathrm{cyl}'(A)$, and the endpoint inclusions are given by the inclusion into the bottom of $\mathrm{cyl}(A)$ and into the top of $\mathrm{cyl}'(A)$.) Here we use the fact that trivial cofibrations are stable under pushout to see that j_0 and j_1 are trivial cofibrations and the two-out-of-three property of weak equivalences to see that both k and q'' are weak equivalences. Since the left homotopies H and K are composeable, in the sense that H is given on the top of $\mathrm{cyl}(A)$ by g and K on the bottom of $\mathrm{cyl}'(A)$, they induce a map $H \amalg_A K : \mathrm{cyl}(A) \amalg_A \mathrm{cyl}'(A) \to X$. By fibrancy of X, we can extend $H \amalg_A K$ along the trivial cofibration k as in the diagram

$$\begin{tikzcd}
\mathrm{cyl}(A) \coprod\limits_A \mathrm{cyl}'(A) \arrow[r, "H \amalg_A K"] \arrow[d, tail, "k"', "\wr"] & X \arrow[d, two heads] \\
\mathrm{cyl}''(A) \arrow[r] \arrow[ur, dashed] & *
\end{tikzcd}$$

and we find a left homotopy $\mathrm{cyl}''(A) \to X$ from f to h. We conclude that left homotopy is transitive and hence an equivalence relation.

Finally, let $H : \mathrm{cyl}(A) \to B$ be a left homotopy from f to g. We construct a right homotopy from f to g as the lift

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, tail, "i_0"', "\wr"] & X \arrow[r, tail, "j", "\sim"'] & \mathrm{path}(X) \arrow[d, two heads, "(p_0, p_1)"] \\
A \arrow[r, tail, "i_1", "\sim"'] & \mathrm{cyl}(A) \arrow[rr, "(fq, H)"'] \arrow[urr, dashed] & & X \times X
\end{tikzcd}$$

restricted along i_1.

By duality, we see that right homotopy is also an equivalence relation and that it coincides with left homotopy. □

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[5] If we do not require the map $\mathrm{cyl}''(A) \to A$ to be a fibration, which is usually only demanded for 'very good' cylinder objects, we can already use the pushout $\mathrm{cyl}(A) \amalg_A \mathrm{cyl}'(A)$ itself as a new cylinder object in the rest of the argument.

We will write $\simeq$ for the coinciding equivalence relations $\sim_\ell$ and $\sim_r$ on $\mathrm{Hom}(A, X)$ when A is cofibrant and X is fibrant.

PROPOSITION 3.7 · *Consider maps*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] & Z \arrow[r, "h"] & W \end{tikzcd}$$

in a model category and suppose that g and g' are left or right homotopic. Then the composites hgf and $hg'f$ are respectively left or right homotopic.

PROOF. Let $H : \mathrm{cyl}(Y) \to C$ be a left homotopy from g to g'. By lifting the endpoint inclusion $X \amalg X \rightarrowtail \mathrm{cyl}(X)$ against the projection $\mathrm{cyl}(Y) \overset{\sim}{\twoheadrightarrow} Y$, we find a map $\mathrm{cyl}(f) : \mathrm{cyl}(X) \to \mathrm{cyl}(Y)$ making the diagram

$$\begin{tikzcd} & X \amalg X \arrow[dl, tail] \arrow[rr, "f \amalg f"] & & Y \amalg Y \arrow[dl, tail] \arrow[dr, "{(g,g')}"] & & \\ \mathrm{cyl}(X) \arrow[rr, dashed, "\mathrm{cyl}(f)"] \arrow[d, two heads, "\wr"'] & & \mathrm{cyl}(Y) \arrow[rr, "H"] \arrow[d, two heads, "\wr"'] & & Z \arrow[r, "h"] & W \\ X \arrow[rr, "f"] & & Y & & & \end{tikzcd}$$

commute. Composing $\mathrm{cyl}(f)$ with H and h now gives the desired left homotopy. □

DEFINITION 3.8 · A map $f : X \to Y$ between fibrant–cofibrant objects in a model category is a *homotopy equivalence* if there is a map $g : Y \to X$ (a *homotopy inverse*) such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$.

DEFINITION 3.9 · Let A and X be fibrant–cofibrant objects in a model category. Then A is a *deformation retract* of X if there exist maps

$$A \xrightarrow{i} X \xrightarrow{r} A$$

such that r is a retraction of i and a section up to homotopy, that is, $ri = \mathrm{id}_A$ and $ir \simeq \mathrm{id}_B$.

Note that the maps in a deformation retract are in particular homotopy equivalences.

LEMMA 3.10 · *Every trivial cofibration between fibrant–cofibrant objects is the section in a deformation retract. Dually, every trivial fibration between fibrant–cofibrant objects is the retraction in a deformation retract.*

PROOF. Let $p : A \overset{\sim}{\twoheadrightarrow} B$ be a trivial fibration between fibrant–cofibrant objects. Lifts in the following diagrams show that p has an on-the-nose section i and that i is a retraction up to homotopy:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, tail] & A \arrow[d, two heads, "\wr"', "p"] \\ B \arrow[r, equal] \arrow[ur, dashed, "i"] & B \end{tikzcd} \qquad \begin{tikzcd} A \amalg A \arrow[rr, "{(ip, \mathrm{id}_A)}"] \arrow[d, tail] & & A \arrow[d, two heads, "\wr"', "p"] \\ \mathrm{cyl}(A) \arrow[r, two heads, "\sim"] \arrow[urr, dashed, "H"] & A \arrow[r, two heads, "\sim", "p"'] & B \end{tikzcd}$$

The lift in the left diagram exists by cofibrancy of B, and commutativity of the right diagram follows from i being a section on the nose. □

PROPOSITION 3.11 ('Whitehead') · *Let $f : X \to Y$ be a map between fibrant–cofibrant objects in a model category. Then f is a weak equivalence if and only if f is a homotopy equivalence.*

PROOF. Factor f as a trivial cofibration followed by a fibration:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "j"', "\sim"] & & Y \\ & Z \arrow[ur, two heads, "p"'] & \end{tikzcd}$$

By cofibrancy of X and fibrancy of Y, we see Z is also both fibrant and cofibrant.

If f is a weak equivalence, then p is a trivial fibration, so by Lemma 3.10, both j and q have homotopy inverses. Composing these homotopy inverses gives a homotopy inverse for f on account of Proposition 3.7.

Conversely, let g be a homotopy inverse for f. It suffices to show that p is a weak equivalence. By lifting the homotopy $H : fg \simeq \mathrm{id}_B$ in the diagram

$$\begin{tikzcd} & Y \arrow[r, "g"] \arrow[d, tail, "i_0"', "\wr"] & X \arrow[r, "j"] & Z \arrow[d, two heads, "p"] \\ Y \arrow[r, tail, "i_1", "\sim"'] \arrow[rrr, equal, bend right] & \mathrm{cyl}(Y) \arrow[rr, "H"'] \arrow[urr, dashed] & & Y \end{tikzcd}$$

and restricting along i_1, we find a section i of p which is homotopic to jg. Moreover, Lemma 3.10 gives us an on-the-nose retraction q for j which is a section up to homotopy. Now we have

$$ip \simeq ipjq = ifq \simeq jgfq \simeq jq \simeq \mathrm{id}_P .$$

By Lemma 3.4, ip is a weak equivalence. Then the retract diagram

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, two heads, "p"'] & Z \arrow[r, equal] \arrow[d, "\wr"', "ip"] & Z \arrow[d, two heads, "p"] \\ Y \arrow[r, "i"'] & Z \arrow[r, "p"'] & Y \end{tikzcd}$$

shows that p is a weak equivalence. □

We note that the familiar Whitehead Theorem for weak homotopy equivalences between CW-complexes falls out of the abstract theory:

COROLLARY 3.12 · *A map between CW-complexes is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

PROOF. In the Quillen model structure on **Top** where the weak equivalences are the weak homotopy equivalences, all spaces are fibrant and CW-complexes are cofibrant.

□

4 Homotopy category

Abstractly, the homotopy category $\mathbf{Ho}\,\mathscr{M}$ of a model category $\mathscr{M}$ is the localisation of $\mathscr{M}$ with respect to the class of weak equivalences. In general, we might construct a localisation of any category with respect to a class of maps, but it will not be 'small' in any sense. We might solve this problem by passing to a higher universe or something akin to that, but for model categories we can use the homotopy relation to construct a localisation with respect to weak equivalences in a more controlled way.

On account of Propositions 3.6 and 3.7, we can define the homotopy category $\mathbf{Ho}\,\mathcal{M}$ of a model category $\mathcal{M}$ as follows: as objects, we take all objects of $\mathcal{M}$. The set of maps $X \to Y$ in $\mathbf{Ho}\,\mathcal{M}$ is defined to be

$$\mathrm{Hom}_{\mathbf{Ho}\,\mathcal{M}}(X,Y) := \mathrm{Hom}_{\mathcal{M}}(RQX, RQY)/\simeq,$$

the quotient of the set of maps $RQX \to RQY$ in $\mathcal{M}$ between fibrant–cofibrant replacements[6] of X and Y by the homotopy relation (which is an equivalence relation on the maps between fibrant–cofibrant objects). Composition is defined by $[g] \circ [f] = [gf]$ and the identities are $\mathrm{id}_X = [\mathrm{id}_{RQX}]$. That this indeed defines a category follows from functoriality of Q and R.[7]

There is a canonical identity-on-objects functor $\gamma : \mathcal{M} \to \mathbf{Ho}\,\mathcal{M}$ which sends a map f in $\mathcal{M}$ to the homotopy class of RQf. In fact, this functor γ defines a localisation of $\mathcal{M}$ with respect to weak equivalences. Since we will be discussing functors taking weak equivalences to isomorphisms or weak equivalences a lot, we give them a special name:

DEFINITION 4.1 · A functor from a model category to any category is *homotopical* if it sends weak equivalences to isomorphisms. A functor between model categories is *homotopical* if it preserves weak equivalences, that is, if it sends weak equivalences in the domain to weak equivalences in the codomain.[8]

LEMMA 4.2 · *A homotopical functor from a model category to any category identifies left or right homotopic maps.*

PROOF. Let $H :\ \mathrm{cyl}(X) \to Y$ be a left homotopy from f to g. The endpoint inclusions $i_0, i_1 : X \xrightarrow{\sim} \mathrm{cyl}(X)$ are both sections to the cylinder projection $q :\ \mathrm{cyl}(X) \xrightarrow{\sim}\!\!\!\!\to X$, which is a weak equivalence, so Fi_0 and Fi_1 are both sections to the isomorphism Fq, and thus equal. Then $Ff = FH \circ Fi_0 = FH \circ Fi_1 = Fg$. □

THEOREM 4.3 (Quillen) · *Let $\mathcal{M}$ be a model category. The canonical functor $\gamma : \mathcal{M} \to \mathbf{Ho}\,\mathcal{M}$ is a localisation of $\mathcal{M}$ with respect to the class of weak equivalences.*

PROOF. Using the language of homotopical functors, γ is a localisation of $\mathcal{M}$ with respect to the weak equivalences if it is homotopical and if any homotopical functor

[6] Alternatively, we might commute Q and R and obtain an equivalent definition of $\mathbf{Ho}\,\mathcal{M}$.

[7] If we do not assume that our model category $\mathcal{M}$ has *functorial* factorisations, we might not have functorial cofibrant and fibrant replacements of objects in $\mathcal{M}$. In that case, however, cofibrant and fibrant replacement are functorial 'up to homotopy', so quotienting out by the homotopy relation in the codomain (after restricting to (co)fibrant objects) does give an on-the-nose functor, but this is no longer an endofunctor on $\mathcal{M}$. The rest of the theory goes through, but the details are rather subtle; we refer to [DS95, § 5].

[8] The latter notion almost subsumes the former: this is true if we see the codomain category as a model category where the weak equivalences are the isomorphisms, but then we need this category to be complete and cocomplete.

$F : \mathscr{M} \to \mathscr{C}$ to any category $\mathscr{C}$ factors uniquely through γ:

$$\begin{tikzcd} \mathscr{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathscr{C} \\ \mathbf{Ho}\,\mathscr{M} \arrow[ur, dashed, "\exists!", "\mathbf{Ho}\,F"'] & \end{tikzcd} \tag{2}$$

We first check that γ is homotopical. If f is a weak equivalence in γ, then RQf is a weak equivalence between fibrant–cofibrant objects by virtue of the natural weak equivalences $Q \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathscr{M}}$ and $\mathrm{id}_{\mathscr{M}} \overset{\sim}{\Rightarrow} R$ (Lemma 1.10). Hence RQf has a homotopy inverse by the Whitehead Theorem (Proposition 3.11), so its homotopy class $[RQf] = \gamma f$ is an isomorphism in $\mathbf{Ho}\,\mathscr{M}$.

Now assume $F : \mathscr{M} \to \mathscr{C}$ is a homotopical functor. To make the diagram (2) commute, we should define $\mathbf{Ho}\,F$ to agree with F on objects since γ is the identity on objects. The natural weak equivalences $q : Q \overset{\sim}{\Rightarrow} \mathrm{id}_{\mathscr{M}}$ and $r :\ \mathrm{id}_{\mathscr{M}} \overset{\sim}{\Rightarrow} R$ give rise to a natural isomorphism

$$\alpha :\ F \underset{\cong}{\overset{(Fq)^{-1}}{\Longrightarrow}} FQ \underset{\cong}{\overset{FrQ}{\Longrightarrow}} FRQ$$

between functors $\mathscr{M} \to \mathscr{C}$ since F inverts weak equivalences. Let f represent a map $X \to Y$ in $\mathbf{Ho}\,\mathscr{M}$, that is, f is a map $RQX \to RQY$ in $\mathscr{M}$. Define $\mathbf{Ho}\,F([f])$ to be the composite

$$\mathbf{Ho}\,F([f]) :\ FX \underset{\cong}{\xrightarrow{\alpha_X}} FRQX \xrightarrow{Ff} FRQY \underset{\cong}{\xrightarrow{\alpha_Y^{-1}}} FY.$$

That $\mathbf{Ho}\,F$ is well-defined follows directly from Lemma 4.2. Functoriality of $\mathbf{Ho}\,F$ follows directly from functoriality of F and commutativity of (2) follows from naturality of α.

To see that $\mathbf{Ho}\,F$ is unique, let $f : RQX \to RQY$ represent a map $X \to Y$ in $\mathbf{Ho}\,\mathscr{M}$. Consider the commutative diagram

$$\begin{tikzcd} RQX \arrow[d, "f"'] & QRQX \arrow[l, "q_{RQX}"', "\sim"] \arrow[r, "r_{QRQX}", "\sim"'] \arrow[d, "Qf"] & RQRQX \arrow[d, "RQf"] \\ RQY & QRQY \arrow[l, "\sim", "q_{RQY}"'] \arrow[r, "\sim", "r_{QRQY}"'] & RQRQY \end{tikzcd}$$

in $\mathscr{M}$. Since the image of the homotopy class of the right vertical map under $\mathbf{Ho}\,F$ is uniquely determined by commutativity of (2) to be Ff and the horizontal maps become isomorphisms in $\mathscr{C}$, the image of f is also uniquely determined. □

COROLLARY 4.4 · *The homotopy group functors* $\pi_n : \mathbf{Top}_* \to \mathbf{Grp}$ *factor through* $\mathbf{Ho}\,\mathbf{Top}_{\text{Quillen}}$. *The homology functors* $H_n(-;\mathbb{Z}) : \mathbf{Top} \to \mathbf{Ab}$ *factor through* $\mathbf{Ho}\,\mathbf{Top}_{\text{Strøm}}$.

COROLLARY 4.5 · *For a ring R, the homotopy category* $\mathbf{Ho}(\mathbf{Ch}_{\geqslant 0}(R)_{\text{proj}})$ *of the category of non-negatively graded chain complexes of R-modules with the projective model structure is equivalent to the derived category* $\mathbf{D}_{\geqslant 0}(\mathbf{Mod}_R)$.

Another useful way to phrase Theorem 4.3 is as follows: precomposition with $\gamma : \mathscr{M} \to \mathbf{Ho}\,\mathscr{M}$ induces a bijective correspondence between functors $\mathbf{Ho}\,\mathscr{M} \to \mathscr{C}$ and homotopical functors $\mathscr{M} \to \mathscr{C}$. Using this insight, we will generally conflate functors

$\mathbf{Ho}\,\mathscr{M} \to \mathscr{C}$ and homotopical functors $\mathscr{M} \to \mathscr{C}$. Writing $\mathbf{Fun}^{\mathrm{ho}}(\mathscr{M}, \mathscr{C})$ for the full subcategory of $\mathbf{Fun}(\mathscr{M}, \mathscr{C})$ on the homotopical functors, this bijection extends to an isomorphism of functor categories, showing that it has a 2-categorical nature:

COROLLARY 4.6 · *Let $\mathscr{M}$ be a model category and let $\mathscr{C}$ be any category. Then the localisation $\gamma : \mathscr{M} \to \mathbf{Ho}\,\mathscr{M}$ induces an isomorphism*

$$\gamma^* : \mathbf{Fun}(\mathbf{Ho}\,\mathscr{M}, \mathscr{C}) \xrightarrow{\cong} \mathbf{Fun}^{\mathrm{ho}}(\mathscr{M}, \mathscr{C})$$

of categories.

PROOF. Theorem 4.3 implies that γ^* is a bijection on objects. The inverse sends a homotopical functor $F : \mathscr{M} \to \mathscr{C}$ to the unique functor $\mathbf{Ho}\,\mathscr{F} : \mathbf{Ho}\,\mathscr{M} \to \mathscr{C}$ with $F = \mathbf{Ho}\,F \circ \gamma$.

It remains to check that we can extend this to an inverse functor of γ^* by defining it on morphisms, that is, natural transformations $\alpha : F \Rightarrow G$ between homotopical functors $F, G : \mathscr{M} \to \mathscr{C}$. We define $\mathbf{Ho}\,\alpha : \mathbf{Ho}\,F \Rightarrow \mathbf{Ho}\,G$ by setting

$$(\mathbf{Ho}\,\alpha)_X := \alpha_X : \mathbf{Ho}\,F(X) \to \mathbf{Ho}\,G(X)$$

where we use $\mathbf{Ho}\,F(X) = \mathbf{Ho}\,F \circ \gamma(X) = F(X)$ since γ is the identity on objects, and similarly for G. To check that $\mathbf{Ho}\,\alpha$ is natural, it suffices to check the naturality square for maps f in $\mathbf{Ho}\,\mathscr{M}$ that lie in the image of the endofunctor RQ, and then it follows from naturality of α. Functoriality follows directly from the definitions.

Finally, to see that this functor $\mathbf{Fun}^{\mathrm{ho}}(\mathscr{M}, \mathscr{C}) \to \mathbf{Fun}(\mathbf{Ho}\,\mathscr{M}, \mathscr{C})$ is indeed inverse to γ^* on morphisms, note that γ^* sends $\alpha : F \Rightarrow G$ to $\alpha\gamma$ with $\alpha\gamma_X = \alpha_{\gamma X} = \alpha_X$. □

REMARK 4.7 · The homotopy category $\mathbf{Ho}\,\mathscr{M}$ as constructed above is equivalent to the category commonly denoted $h\mathscr{M}_{\mathrm{cf}}$ of fibrant–cofibrant objects of $\mathscr{M}$ and homotopy classes of maps. However, $h\mathscr{M}_{\mathrm{cf}}$ satisfies a weaker universal property: the isomorphism of Corollary 4.6 becomes an equivalence $\mathbf{Fun}(h\mathscr{M}_{\mathrm{cf}}, \mathscr{C}) \simeq \mathbf{Fun}^{\mathrm{ho}}(\mathscr{M}, \mathscr{C})$; a homotopical functor need not factor strictly through $h\mathscr{M}_{\mathrm{cf}}$, but only up to natural isomorphism.

On the other hand, the category $h\mathscr{M}_{\mathrm{cf}}$ is often conceptually easier to understand. For example, using this equivalent description of $\mathbf{Ho}\,\mathbf{Top}_{\mathrm{Quillen}}$, we can see that this homotopy category can be described (up to equivalence) as the category of CW-complexes and homotopy classes of maps.[9]

A homotopical functor $F : \mathscr{M} \to \mathscr{N}$ between model categories induces a unique functor making

$$\begin{tikzcd} \mathscr{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathscr{N} \arrow[d, "\delta"] \\ \mathbf{Ho}\,\mathscr{M} \arrow[r, dashed, "F"'] & \mathbf{Ho}\,\mathscr{N} \end{tikzcd}$$

by the universal property of localisation, which we will for convenience also call F but is actually $\mathbf{Ho}\,\delta F$, but this should not be read as a generalisation of Corollary 4.6: a functor $\mathbf{Ho}\,\mathscr{M} \to \mathbf{Ho}\,\mathscr{N}$ can be lifted to a homotopical functor $\mathscr{M} \to \mathbf{Ho}\,\mathscr{N}$, but not necessarily to a homotopical functor $\mathscr{M} \to \mathscr{N}$.

[9]The fibrant objects in **Top** are a bit more complicated than CW-complexes (they are retracts of relative cell complexes), but CW-approximation or Corollary 5.17 gives this result.

THEOREM 4.8 · *A map in a model category $\mathscr{M}$ is inverted by the canonical functor $\gamma : \mathscr{M} \to$ $\mathbf{Ho}\,\mathscr{M}$ if and only if it is a weak equivalence.*

PROOF. By Lemma 1.10, f is a weak equivalence in $\mathscr{M}$ if and only if RQf is. Since RQf is a map between fibrant–cofibrant objects, it is a weak equivalence if and only if it is a homotopy equivalence by the Whitehead Theorem (Proposition 3.11). This is in turn equivalent to $[RQf] = \gamma f$ being an isomorphism in $\mathbf{Ho}\,\mathscr{M}$. □

5 Derived functors

We have seen that a model structure on a category allows us to introduce a notion of homotopy between parallel maps in that category. Model categories may hence be seen as 'abstract homotopy theories'. In this section, we will consider functors between model categories to compare different homotopy theories.

If $F : \mathscr{M} \to \mathscr{C}$ is a homotopical functor, meaning that it sends weak equivalences in $\mathscr{M}$ to isomorphisms in $\mathscr{C}$, then the universal property of the localisation $\gamma : \mathscr{M} \to$ $\mathbf{Ho}\,\mathscr{M}$ of Theorem 4.3 implies there is a unique functor $\mathbf{Ho}\,F : \mathbf{Ho}\,\mathscr{M} \to \mathscr{C}$ such that $F = \mathbf{Ho}\,F \circ \gamma$. When F is not homotopical, however, it does not necessarily factor through $\mathbf{Ho}\,\mathscr{M}$ on the nose. We will (under some conditions, more general than F being homotopical) construct approximations of a factorisation of F through γ, which will be the *derived functors* of F.

When $F : \mathscr{M} \to \mathscr{N}$ is a homotopical functor between model categories, preserving the weak equivalences, the univeral property of the localisation $\gamma : \mathscr{M} \to \mathbf{Ho}\,\mathscr{M}$ does not imply that F factors on-the-nose through γ, since the universal property concerns functors taking weak equivalences to isomorphisms. Postcomposing F with $\delta : \mathscr{N} \to$ $\mathbf{Ho}\,\mathscr{N}$ does give such a functor, however, so δF factors uniquely through γ in the case that F is homotopical. If F is not homotopical, we will look at derived functors of the composite δF, which will be the *total derived functors*.

The problem of approximating an extension of a functor $F : \mathscr{C} \to \mathscr{E}$ between arbitrary categories along another functor $K : \mathscr{C} \to \mathscr{D}$ shows up more generally in category theory, and is studied using the concept of *Kan extensions*. We will not need much general theory of Kan extensions here, but we will at least use the language of Kan extensions to introduce derived functors. To give a formal construction of the *derived adjunction* of a Quillen adjunction, we use the stronger concept of *absolute* Kan extensions.

DEFINITION 5.1 · A *left Kan extension* of a functor $F : \mathscr{C} \to \mathscr{E}$ along a functor $K : \mathscr{C} \to \mathscr{D}$ is a functor $\mathrm{Lan}_K F : \mathscr{D} \to \mathscr{E}$ with a natural transformation $\eta : F \Rightarrow \mathrm{Lan}_K F \circ K$ such that for any pair of a functor $G : \mathscr{D} \to \mathscr{E}$ and a natural transformation $\alpha : F \Rightarrow GK$, α factors uniquely through η as in the following diagram[10]:

$$\begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathscr{E} \\ & \mathscr{D} \arrow[ur, dashed, "\mathrm{Lan}_K F"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow\alpha"] & \mathscr{E} \\ & \mathscr{D} \arrow[ur, "G"'] & \end{tikzcd} = \begin{tikzcd} \mathscr{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\eta\Downarrow"] & \mathscr{E} \\ & \mathscr{D} \arrow[ur, "\mathrm{Lan}_K F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\exists!"', shift right=2] & \end{tikzcd}$$

[10]In formulas: there is a unique natural transformation $\tilde{\alpha} : \mathrm{Lan}_K F \Rightarrow G$ with $\alpha = \tilde{\alpha}K \circ \eta$.

The left Kan extension $\operatorname{Lan}_K F$ is *absolute* if for any functor $H : \mathscr{E} \to \mathscr{F}$, the composite functor $H \operatorname{Lan}_K F : \mathscr{D} \to \mathscr{F}$ together with the whiskered transformation $H\eta$ is a left Kan extension of HF along K.

Dually, a *right Kan extension* of F along K is a functor $\operatorname{Ran}_K F : \mathscr{D} \to \mathscr{E}$ with a natural transformation $\varepsilon : \operatorname{Ran}_K F \circ K \Rightarrow F$ such that for any pair of a functor $G : \mathscr{D} \to \mathscr{E}$ and a natural transformation $\beta : GK \Rightarrow F$, β factors uniquely through ε:

$$
\begin{tikzcd}
\mathscr{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow\varepsilon"] & \mathscr{E} \\
& \mathscr{D} \arrow[ur, dashed, "\operatorname{Lan}_K F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Uparrow\beta"] & \mathscr{E} \\
& \mathscr{D} \arrow[ur, "G"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathscr{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\varepsilon\Uparrow"] & \mathscr{E} \\
& \mathscr{D} \arrow[ur, "\operatorname{Ran}_K F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\exists!"] &
\end{tikzcd}
$$

The right Kan extension $\operatorname{Ran}_K F$ is *absolute* if whiskered postcomposition with any functor $H : \mathscr{E} \to \mathscr{F}$ gives a right Kan extension of HF along K.

DEFINITION 5.2 · A *left derived functor* of a functor $F : \mathscr{M} \to \mathscr{C}$ from a model category to any category is an absolute[11] *right* Kan extension of F along the localisation $\gamma : \mathscr{M} \to \mathbf{Ho}\,\mathscr{M}$, denoted $\mathrm{L}F : \mathbf{Ho}\,\mathscr{M} \to \mathscr{C}$.

Dually, a *right derived functor* of F is an absolute *left* Kan extension of F along the localisation $\gamma : \mathscr{M} \to \mathbf{Ho}\,\mathscr{M}$, denoted $\mathrm{R}F : \mathbf{Ho}\,\mathscr{M} \to \mathscr{C}$.

Being characterised by a universal property, derived functors are unique up to natural isomorphism when they exist, allowing us to speak of *the* derived functors.

The remarks at the begin of this section imply that the left and right Kan extensions of a *homotopical* functor $F : \mathscr{M} \to \mathscr{N}$ between model categories along the localisation $\gamma : \mathscr{M} \to \mathbf{Ho}\,\mathscr{M}$ – so the right and left derived functors – always exist, and that these can even be taken to be on-the-nose extensions, that is, via the identity natural transformations.

The following theorem guarantees the existence of the left derived functor of $F : \mathscr{M} \to \mathscr{C}$ under the more general assumption that F is homotopical on the full subcategory of cofibrant objects.

THEOREM 5.3 · *Let $F : \mathscr{M} \to \mathscr{C}$ be a functor from a model category to any category taking weak equivalences between cofibrant objects to isomorphisms. Then* $\mathrm{L}F := \mathbf{Ho}\,FQ$ *is a left derived functor of F. Dually, if $U : \mathscr{M} \to \mathscr{C}$ takes weak equivalences between fibrant objects to isomorphisms, then* $\mathrm{R}U := \mathbf{Ho}\,UR$ *is a right derived functor of U.*

PROOF. The assumption ensures that the composite $FQ : \mathscr{M} \to \mathscr{C}$ is homotopical, so it induces a unique functor $\mathbf{Ho}\,FQ$ commuting with the localisation $\mathscr{M} \to \mathbf{Ho}\,\mathscr{M}$. We first show $\mathbf{Ho}\,FQ$ is the right Kan extension of F along γ. A natural transformation

$$
\begin{tikzcd}
\mathscr{M} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Uparrow"] & \mathscr{C} \\
& \mathbf{Ho}\,\mathscr{M} \arrow[ur, "\mathbf{Ho}\,FQ"'] &
\end{tikzcd}
$$

[11]This condition is not standard in the literature – we follow [Rie14, Proposition 2.2.13] –, but the usual construction of derived functors does satisfy it (at least under the assumption of functorial factorisations) as we will see in Theorem 5.3, so there is no harm in adding it. The benefit will be a completely formal proof the total derived adjunction of a Quillen adjunction in Theorem 5.13, whose proof otherwise depends on details of the construction of the homotopy category.

is given by the whiskered composite $Fq : \mathbf{Ho}\,FQ \circ \gamma = FQ \Rightarrow F$. To verify that this indeed gives the left Kan extension of F along γ, we should consider a functor $G' : \mathbf{Ho}\,\mathscr{M} \to \mathscr{C}$ with a natural transformation $\alpha' : G'\gamma \Rightarrow F$, or equivalently by Corollary 4.6, a homotopical functor $G : \mathscr{M} \to \mathscr{C}$ and a natural transformation $\alpha : G \Rightarrow F$. Since G is homotopical, $Gq : GQ \Rightarrow G$ is a natural isomorphism. By naturality of α, it factors through Fq as

$$\alpha : \ G \xRightarrow{(Gq)^{-1}} GQ \xRightarrow{\alpha Q} FQ \xRightarrow{Fq} F.$$

For uniqueness, suppose α also factors as

$$\alpha : \ G \xRightarrow{\beta} FQ \xRightarrow{Fq} F.$$

By the assumption on F, the natural transformation $FqQ : FQ^2 \Rightarrow FQ$ is a natural isomorphism, so $\alpha Q \circ (Gq)^{-1}$ and β must agree on cofibrant replacements (that is, their components are equal on objects in the image of Q). Furthermore, naturality of β implies that the diagram

$$\begin{tikzcd} GQ \arrow[r, Rightarrow, "\beta Q"] \arrow[d, Rightarrow, "Gq"', "\cong"] & FQ^2 \arrow[d, Rightarrow, "FQq", "\cong"'] \\ G \arrow[r, Rightarrow, "\beta"'] & FQ \end{tikzcd}$$

commutes, and the vertical transformations are natural isomorphisms since G and FQ are homotopical. Hence, β is fully determined by βQ, its values on cofibrant replacements, and there it agrees with $\alpha Q \circ (Gq)^{-1}$.

Finally, to show that $\mathbf{Ho}\,FQ$ is an *absolute* Kan extension of F along γ, let $H : \mathscr{C} \to \mathscr{D}$ be any functor. Then we need to prove that $H \circ \mathbf{Ho}\,FQ$ together with the natural transformation HFq defines a right Kan extension of HF along γ. Since FQ is homotopical, so is HFQ since any functor preserves isomorphisms. Hence we see

$$H \circ \mathbf{Ho}\,FQ \circ \gamma = HFQ = \mathbf{Ho}\,HFQ \circ \gamma,$$

and thus $H \circ \mathbf{Ho}\,FQ = \mathbf{Ho}\,HFQ$ by the universal property of the localisation. Then the argument above shows that the pair $(H \circ \mathbf{Ho}\,FQ = \mathbf{Ho}\,HFQ, HFq)$ is a right Kan extension of HF along γ. $\square$

DEFINITION 5.4 · A *total left derived functor* of a functor $F : \mathscr{M} \to \mathscr{N}$ between model categories is a left derived functor $\mathbf{L}F : \mathbf{Ho}\,\mathscr{M} \to \mathbf{Ho}\,\mathscr{N}$ of the composite $\delta F : \mathscr{M} \to \mathbf{Ho}\,\mathscr{N}$, where $\delta : \mathscr{N} \to \mathbf{Ho}\,\mathscr{N}$ is the localisation:

$$\begin{tikzcd} \mathscr{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Uparrow"] & \mathscr{N} \arrow[d, "\delta"] \\ \mathbf{Ho}\,\mathscr{M} \arrow[r, dashed, "\mathrm{L}F"'] & \mathbf{Ho}\,\mathscr{N} \end{tikzcd}$$

Dually, a *total left derived functor* of a functor $F : \mathscr{M} \to \mathscr{N}$ between model categories is a right derived functor $\mathbf{R}F : \mathbf{Ho}\,\mathscr{M} \to \mathbf{Ho}\,\mathscr{N}$ of the composite $\delta F : \mathscr{M} \to \mathbf{Ho}\,\mathscr{N}$, where $\delta : \mathscr{N} \to \mathbf{Ho}\,\mathscr{N}$ is the localisation.

Expanding the definitions, a total left derived functor of F is an absolute right Kan extension of δF along $\gamma : \mathscr{M} \to \mathbf{Ho}\,\mathscr{M}$, and similarly for the total right derived functor.

From the existence theorem for derived functors (Theorem 5.3) we obtain the following existence criterion for total derived functors.

COROLLARY 5.5 · *Let $F : \mathscr{M} \to \mathscr{N}$ be a functor between model categories taking weak equivalences between cofibrant objects in $\mathscr{M}$ to weak equivalences in $\mathscr{N}$. Then $\mathbf{L}F := \mathbf{Ho}\,\delta F Q$ is a total left derived functor of F. Dually, if $U : \mathscr{N} \to \mathscr{M}$ takes weak equivalences between fibrant objects in $\mathscr{N}$ to weak equivalences in $\mathscr{M}$, then $\mathbf{R}U := \mathbf{Ho}\,\gamma U R$ is a total right derived functor of F.*

EXAMPLE 5.6 · The model categorical notion of derived functors subsumes the classical notion of derived functors in homological algebra. Recall the projective model structure on the category of nonnegatively graded chain complexes of modules over a ring from Theorem 2.5, where the weak equivalences are quasi-isomorphisms and the cofibrant objects are levelwise projective chain complexes. Let $F : \mathbf{Mod}_S \to \mathbf{Mod}_R$ be an additive functor. Applying F levelwise and to the differentials gives a functor $F_\bullet : \mathbf{Ch}_{\geqslant 0}(S) \to \mathbf{Ch}_{\geqslant 0}(R)$ which preserves chain homotopies and thus chain homotopy equivalences. Since any quasi-isomorphism between levelwise projective chain complexes is a chain homotopy equivalence, it is sent by $F_\bullet$ to a quasi-isomorphism; in model categorical terms, $F_\bullet$ takes weak equivalences between cofibrant objects to weak equivalences. Hence $F_\bullet$ has a total left derived functor $\mathbf{D}_{\geqslant 0}(\mathbf{Mod}_S) \to \mathbf{D}_{\geqslant 0}(\mathbf{Mod}_R)$. Dually, the total right derived functor of F is obtained using the injective model structure on the category of nonnegatively graded cochain complexes.

The reason that one usually considers left derived functors of right exact functors is to obtain the long exact sequence of derived functors associated to a short exact sequence in $\mathbf{Mod}_S$; this exactness assumption is not necessary for the existence of the derived functor.

We will discuss weaker conditions under which a functor admits a total derived functor. We will see that an adjoint pair $F \dashv U$ between model categories satisfying certain model categorical properties will induce an adjoint pair $\mathbf{L}F \dashv \mathbf{R}U$ of total derived functors on the level of homotopy categories, and under stronger assumptions, this will even be an adjoint equivalence.

Recall that a pair of functors

$$\begin{tikzcd} \mathscr{C} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathscr{D} \arrow[l, shift left=1.5, "U"] \end{tikzcd}$$

is an *adjoint pair*, where F is the *left adjoint* and U the *right adjoint*, denoted $F \dashv U$, if there is a bijection

$$\mathrm{Hom}_{\mathscr{D}}(FX, Y) \cong \mathrm{Hom}_{\mathscr{C}}(X, UY)$$

natural in $X \in \mathrm{ob}\,\mathscr{C}$ and $Y \in \mathrm{ob}\,\mathscr{D}$. The typical examples of adjunctions are the free–forgetful adjunctions, for instance where the left adjoint is the free group functor $\mathbf{Set} \to \mathbf{Grp}$ or the functor $\mathbf{Set} \to \mathbf{Top}$ that equips a set with the discrete topology. An adjunction can equivalently be specified by natural transformations $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow UF$, the *unit*, and $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathscr{D}}$, the *counit*, which are inverse to each other when appropriately whiskered by F and U. Moreover, the adjoint functors F and U are both equivalences when the unit and counit are natural isomorphisms (in this case, the adjoint pair $F \dashv U$ is called an *adjoint equivalence*).

DEFINITION 5.7 · A functor between model categories is *left Quillen* if it preserves cofibrations, trivial cofibrations and cofibrant objects. Dually, it is *right Quillen* if it preserves fibrations, trivial fibrations and fibrant objects.

Left Quillen functors often preserve colimits; in that case, the cofibrant objects are automatically preserved whenever the cofibrations and trivial cofibrations are. Of course, the dual is true for right Quillen functors that preserve limits.

LEMMA 5.8 (Ken Brown) · *Let $F : \mathcal{M} \to \mathcal{N}$ be a functor between model categories. If F takes trivial cofibrations between cofibrant objects to weak equivalences, then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes trivial fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

PROOF. Let $f : X \xrightarrow{\sim} Y$ be a weak equivalence between cofibrant objects. Factor the 'cograph' map $(f, \mathrm{id}_Y) : X \sqcup Y \to Y$ as a cofibration i followed by a trivial fibration p:

$$
\begin{tikzcd}
 & X \arrow[dr, tail, "\iota_X"] \arrow[drr, tail, "j_X", "\sim"'] \arrow[drrr, bend left, "f", "\sim"'] & & & \\
\varnothing \arrow[ur, tail] \arrow[dr, tail] & & X \sqcup Y \arrow[r, tail, "i"] & Z \arrow[r, two heads, "p", "\sim"'] & Y \\
 & Y \arrow[ur, tail, "\iota_Y"'] \arrow[urr, tail, "j_Y"', "\sim"] \arrow[urrr, bend right, equal] & & &
\end{tikzcd}
$$

Recognising the coproduct $X \sqcup Y$ as the pushout of two cofibrations as in the diagram, it follows that the coproduct inclusions ι_X and ι_Y are cofibrations, so $X \sqcup Y$ and Z are cofibrant. Then the composites j_X and j_Y are also cofibrations, and they are weak equivalences by the two-out-of-three property; to sum up, they are trivial cofibrations between cofibrant objects. Applying F to the diagram, we obtain

$$
\begin{tikzcd}
FX \arrow[dr, "Fj_X", "\sim"'] \arrow[drr, bend left, "Ff"] & & \\
 & FZ \arrow[r, "Fp"] & FY \\
FY \arrow[ur, "Fj_Y", "\sim"'] \arrow[urr, bend right, equal] & &
\end{tikzcd}
$$

so it follows from the two-out-of-three property that Fp and Ff are weak equivalences. Hence F sends any weak equivalence f between cofibrant objects to a weak equivalence. $\square$

A consequence of Ken Brown's lemma and the existence theorem for derived functors, Theorem 5.3, is the existence of total left and right derived functors for respectively left and right Quillen functors:

COROLLARY 5.9 · *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories, the latter with localisation $\delta : \mathcal{N} \to \mathbf{Ho}\,\mathcal{N}$. A left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ has a total left derived functor given by $\mathbf{L}F := \mathbf{Ho}\,\delta F Q$, and a right Quillen functor $U : \mathcal{M} \to \mathcal{N}$ has a total right derived functor given by $\mathbf{R}U := \mathbf{Ho}\,\delta U R$.*

Left and right Quillen functors often occur in adjoint pairs, which we will call *Quillen adjunctions*:

DEFINITION 5.10 · An adjoint pair $F \dashv U$ of functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ between model categories is a *Quillen adjunction* if F is a left Quillen functor (or, equivalently by the following lemma, if U is a right Quillen functor).

LEMMA 5.11 · *Let $F \dashv U$ be an adjunction of functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ between model categories. Then the following are equivalent:*

① *F preserves cofibrations and trivial cofibrations;*

② *U preserves fibrations and trivial fibrations;*

③ *F preserves cofibrations and U preserves fibrations;*

④ *F preserves trivial cofibrations and U preserves trivial fibrations.*

The proof follows directly from the lifting properties satisfied by the maps in a model category and the following general observation about the interaction between adjunctions and lifting properties:

LEMMA 5.12 · *Let $F \dashv U$ be an adjunction of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between arbitrary categories. If i is a map in $\mathcal{C}$ and p is a map in $\mathcal{D}$, then i has the left lifting property with respect to Up if and only if Fi has the left lifting property with respect to p.*

PROOF. We only show one direction, the other follows by duality. Suppose that the map $Fi : FA \to FB$ has the left lifting property with respect to $p : X \to Y$. Given the lifting problem in $\mathcal{C}$ of the outer square of the right-hand diagram below, applying the adjunction $F \dashv U$, we find a lift $h^\sharp : FB \to X$ in the left-hand commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd} FA \arrow[r, "f^\sharp"] \arrow[d, "Fi"'] & X \arrow[d, "p"] \\ FB \arrow[r, "g^\sharp"'] \arrow[ur, dashed, "h^\sharp"] & Y \end{tikzcd} \qquad \leftrightsquigarrow \qquad \begin{tikzcd} A \arrow[r, "f^\flat"] \arrow[d, "i"'] & UX \arrow[d, "Up"] \\ B \arrow[r, "g^\flat"'] \arrow[ur, dashed, "h^\flat"] & UY \end{tikzcd}$$

Applying the adjunction $F \dashv U$ again, we obtain the commutative diagram on the right, showing that $h^\flat : B \to UX$ is a lift. Hence, i has the left lifting property with respect to Up. □

The importance of Quillen adjunctions is the following result, showing that a Quillen adjunction induces an adjunction on the level of homotopy categories. The classical proofs of the derived adjunction depend on the specific construction of the homotopy category. One usually shows that the total derived functors preserve homotopies between fibrant–cofibrant objects (see [Hov07, Lemma 1.3.10]). Using the assumption that our derived functors be *absolute* Kan extensions, we can give a formal proof, however, which is due to Maltsiniotis in [Mal07]; although we defined the derived functors using (co)fibrant replacement, these will not be mentioned in the proof.

THEOREM 5.13 · *Let $F \dashv U$ be a Quillen adjunction of functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ between model categories. Then the total derived functors form an adjunction*

$$\begin{tikzcd} \mathbf{Ho}\,\mathcal{M} \arrow[r, shift left, "\mathbf{L}F"] \arrow[r, phantom, "\perp"] & \mathbf{Ho}\,\mathcal{N} \arrow[l, shift left, "\mathbf{R}U"] \end{tikzcd}$$

between the homotopy categories of $\mathcal{M}$ and $\mathcal{N}$.

PROOF. Let $\eta : \mathrm{id}_{\mathcal{M}} \Rightarrow UF$ and $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{N}}$ denote the unit and counit of the adjunction $F \dashv U$ and denote the natural transformations that make $\mathbf{L}F$ and $\mathbf{R}U$ into total derived functors by λ and ρ:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Uparrow\lambda"] & \mathcal{N} \arrow[d, "\delta"] \\ \mathbf{Ho}\,\mathcal{M} \arrow[r, dashed, "\mathbf{L}F"'] & \mathbf{Ho}\,\mathcal{N} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{N} \arrow[r, "U"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Downarrow\rho"] & \mathcal{M} \arrow[d, "\gamma"] \\ \mathbf{Ho}\,\mathcal{N} \arrow[r, dashed, "\mathbf{R}U"'] & \mathbf{Ho}\,\mathcal{M} \end{tikzcd}$$

Since $\mathbf{L}F$ is an *absolute* right Kan extension of δF along γ, the composite $\mathbf{R}U \circ \mathbf{L}F$ together with $\mathbf{R}U\lambda$ is a right Kan extension of $\mathbf{R}U \circ \delta F$ along γ. By the universal property of this Kan extension, the natural transformation

$$\gamma \xRightarrow{\gamma\eta} \gamma UF \xRightarrow{\rho F} \mathbf{R}U \circ \delta F$$

induces a unique natural transformation $\tilde{\eta} : \mathrm{id}_{\mathbf{Ho}\,\mathcal{M}} \Rightarrow \mathbf{R}U \circ \mathbf{L}F$ satisfying the following:

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "\mathbf{R}U\circ\delta F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Uparrow\rho F\circ\gamma\eta"] & \mathbf{Ho}\,\mathcal{M} \\ & \mathbf{Ho}\,\mathcal{M} \arrow[ur, "\mathrm{id}_{\mathbf{Ho}\,\mathcal{M}}"'] & \end{tikzcd} = \begin{tikzcd} \mathcal{M} \arrow[rr, "\mathbf{R}U\circ\delta F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\mathbf{R}U\lambda\Uparrow"] & \mathbf{Ho}\,\mathcal{M} \\ & \mathbf{Ho}\,\mathcal{M} \arrow[ur, "\mathbf{R}U\circ\mathbf{L}F"] \arrow[ur, bend right=50, "\mathrm{id}_{\mathbf{Ho}\,\mathcal{M}}"'] \arrow[ur, phantom, bend right=25, "\underset{\exists!}{\overset{\tilde{\eta}}{\Nwarrow}}"] & \end{tikzcd}$$

This transformation $\tilde{\eta}$ will the unit of the derived adjunction $\mathbf{L}F \dashv \mathbf{R}U$. Dually, we obtain a natural transformation $\tilde{\varepsilon} : \mathbf{L}F \circ \mathbf{R}U \Rightarrow \mathrm{id}_{\mathbf{Ho}\,\mathcal{N}}$, which will be the derived counit, from the fact that $\mathbf{R}U$ is an absolute left Kan extension. In formulas, the defining properties of the derived unit and counit are:

$$\mathbf{R}U\lambda \circ \tilde{\eta}\gamma = \rho F \circ \gamma\eta, \qquad \tilde{\varepsilon}\delta \circ \mathbf{L}F\rho = \delta\varepsilon \circ \lambda U. \tag{3}$$

To show that the unit and counit assemble into an adjunction, it remains to show the triangle identities:

$$\mathbf{R}U\tilde{\varepsilon} \circ \tilde{\eta}\mathbf{R}U = \mathrm{id}_{\mathbf{R}U}\,, \qquad \tilde{\varepsilon}\mathbf{L}F \circ \mathbf{L}F\tilde{\eta} = \mathrm{id}_{\mathbf{L}F}\,.$$

We only prove the first, since the second proof is dual. By the universal property of $\mathbf{R}U$:

$$\begin{tikzcd} \mathcal{N} \arrow[rr, "\gamma U"] \arrow[dr, "\delta"'] & \arrow[d, phantom, "\Downarrow\rho"] & \mathbf{Ho}\,\mathcal{M} \\ & \mathbf{Ho}\,\mathcal{N} \arrow[ur, "\mathbf{R}U"'] & \end{tikzcd} = \begin{tikzcd} \mathcal{N} \arrow[rr, "\gamma U"] \arrow[dr, "\delta"'] & \arrow[d, phantom, "\rho\Downarrow"] & \mathbf{Ho}\,\mathcal{M} \\ & \mathbf{Ho}\,\mathcal{N} \arrow[ur, "\mathbf{R}U"] \arrow[ur, bend right=50, "\mathbf{R}U"'] \arrow[ur, phantom, bend right=25, "\underset{\exists!}{\Searrow}"] & \end{tikzcd}$$

it suffices to show $(\mathbf{R}U\tilde{\varepsilon} \circ \tilde{\eta}\mathbf{R}U)\delta \circ \rho = \rho$. This is obtained by some persistent symbol pushing:

$$\begin{aligned}
(\mathbf{R}U\tilde{\varepsilon} \circ \tilde{\eta}\mathbf{R}U)\delta \circ \rho &= \mathbf{R}U\tilde{\varepsilon}\delta \circ \tilde{\eta}\mathbf{R}U\delta \circ \rho \\
&= \mathbf{R}U\tilde{\varepsilon}\delta \circ \mathbf{R}U\mathbf{L}F\rho \circ \tilde{\eta}\gamma U && (\tilde{\eta} \text{ natural}) \\
&= \mathbf{R}U(\tilde{\varepsilon}\delta \circ \mathbf{L}F\rho) \circ \tilde{\eta}\gamma U \\
&= \mathbf{R}U(\delta\varepsilon \circ \lambda U) \circ \tilde{\eta}\gamma U && (\text{by } (3)) \\
&= \mathbf{R}U\delta\varepsilon \circ \mathbf{R}U\lambda U \circ \tilde{\eta}\gamma U \\
&= \mathbf{R}U\delta\varepsilon \circ (\mathbf{R}U\lambda \circ \tilde{\eta}\gamma)U \\
&= \mathbf{R}U\delta\varepsilon \circ (\rho F \circ \gamma\eta)U && (\text{by } (3)) \\
&= \mathbf{R}U\delta\varepsilon \circ \rho FU \circ \gamma\eta U \\
&= \rho \circ \gamma U\varepsilon \circ \gamma\eta U && (\rho \text{ natural}) \\
&= \rho \circ \gamma(U\varepsilon \circ \eta U) \\
&= \rho \circ \gamma\, \mathrm{id}_U && (F \dashv U) \\
&= \rho.
\end{aligned}$$

This finishes the proof of the derived adjunction $\mathbf{L}F \dashv \mathbf{R}U$. □

Interestingly, we can give a model categorical criterion, meaning some statement about the functors F and U on the level of model categories, for when the derived adjunction $\mathbf{L}F \dashv \mathbf{R}U$ is an adjoint equivalence.

DEFINITION 5.14 · A Quillen adjunction $F \dashv U$ of functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ between model categories is a *Quillen equivalence* if for all cofibrant objects A of $\mathcal{M}$ and all fibrant objects X of $\mathcal{N}$, a map $f^\sharp : FA \to X$ is a weak equivalence in $\mathcal{N}$ if and only if its adjoint $f^\flat : A \to UX$ is a weak equivalence in $\mathcal{M}$.

PROPOSITION 5.15 · *Let $F \dashv U$ be a Quillen adjunction of functors $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$ between model categories. Then the following are equivalent:*

1. *$F \dashv U$ is a Quillen equivalence;*
2. *for every cofibrant object A of $\mathcal{M}$, the composite $A \to UFA \to URFA$ of the unit and fibrant replacement is a weak equivalence in $\mathcal{M}$, and, dually, for every fibrant object X of $\mathcal{N}$, the composite $FQUX \to FUX \to X$ of cofibrant replacement and the counit is a weak equivalence in $\mathcal{N}$;*
3. *the derived adjunction $\mathbf{L}F \dashv \mathbf{R}U$ is an adjoint equivalence between $\mathbf{Ho}\,\mathcal{M}$ and $\mathbf{Ho}\,\mathcal{N}$.*

PROOF. Write η and ε for the unit and counit of the adjunction $F \dashv U$ and decorate them with a tilde for the unit and counit of the derived adjunction.

To see that ① implies ②, let $F \dashv U$ be a Quillen equivalence and let A be a cofibrant object in $\mathcal{M}$ and X a fibrant object in $\mathcal{N}$. Then the composite

$$A \xrightarrow{\eta_A} UFA \xrightarrow{Ur_{FA}} URFA$$

is the adjoint of the weak equivalence $r_{FA} : FA \xrightarrow{\sim} RFA$ in $\mathcal{N}$ into a fibrant object, and so by assumption itself a weak equivalence. The proof of the dual case is dual.

Conversely, assume that ② holds with the goal to show ①. Let A be a cofibrant object of $\mathscr{M}$ and X a fibrant object of $\mathscr{N}$. If $f^\sharp : FA \xrightarrow{\sim} X$ is a weak equivalence, we want to show that its adjoint $f^\flat = Uf^\sharp \circ \eta_A$ is also a weak equivalence. The diagram

$$\begin{tikzcd}
A \arrow[r, "\eta_A"'] \arrow[rr, bend left, "f^\flat"] \arrow[dr, "\sim"'] & UFA \arrow[r, "Uf^\sharp"'] \arrow[d, "UrF_A"] & UX \arrow[d, "Ur_X", "\wr"'] \\
& URFA \arrow[r, "\sim", "URf^\sharp"'] & URX
\end{tikzcd}$$

commutes. The maps Ur_X and $Urf^\sharp$ are weak equivalences since U preserves all weak equivalences between fibrant objects by Ken Brown's lemma, and the map $A \xrightarrow{\sim} URFA$ is a weak equivalence by assumption. Hence $f^\flat$ is a weak equivalence by the two-out-of-three property. A dual argument shows that $f^\sharp$ is a weak equivalence when $f^\flat$ is.

We now show that ② is equivalent to ③. The adjunction $\mathbf{L}F \dashv \mathbf{R}U$ is an adjoint equivalence exactly when the derived unit $\tilde{\eta} : \mathrm{id}_{\mathbf{Ho}\,\mathscr{M}} \Rightarrow \mathbf{R}U \circ \mathbf{L}F$ and the derived counit $\tilde{\varepsilon} : \mathbf{L}F \circ \mathbf{R}U \Rightarrow \mathrm{id}_{\mathbf{Ho}\,\mathscr{N}}$ are natural isomorphisms. Let us focus on the unit $\tilde{\eta}$. Since every object in $\mathbf{Ho}\,\mathscr{M}$ is isomorphic to a cofibrant object, $\tilde{\eta}$ is a natural isomorphism if and only if the component $\tilde{\eta}_A$ is an isomorphism if A is cofibrant. Expanding the definitions in the equation (3) which defines $\tilde{\eta}$, we see that the diagram

$$\begin{tikzcd}
\gamma A \arrow[r, "\gamma\eta_A"] \arrow[d, "\tilde{\eta}\gamma_A"'] \arrow[dr] & \gamma UFA \arrow[d, "\gamma UrF_A"] \\
\gamma URFQA \arrow[r, "\cong", "\gamma URFq_A"'] & \gamma URFA
\end{tikzcd}$$

commutes for every A. For cofibrant A, the bottom map is an isomorphism since F preserves all weak equivalences between cofibrant objects and U preserves all weak equivalences between fibrant objects by Ken Brown's lemma. Hence if A is cofibrant, the unit $\tilde{\eta}\gamma_A$, which is $\tilde{\eta}_A$ since γ is the identity on objects, is an isomorphism if and only if the diagonal map in the diagram is. This diagonal map is precisely the image under γ of the unit followed by fibrant replacement, which is a weak equivalence if and only if γ takes it to an isomorphism. This establishes the equivalence of the 'unit parts' of the conditions ② and ③; a dual argument shows the equivalence of the 'counit parts'. □

We will now discuss some applications of the abstract theory of model categories and Quillen equivalences.

Spaces The motiviating examples of Quillen's model category approach to abstract homotopy theory are the model structure on the categories of topological spaces and simplicial sets. It turns out that these model categories encode the same homotopy theory, as witnessed by the following, perhaps the best-known, Quillen equivalence:

THEOREM 5.16 (Quillen) · *The adjunction*

$$\begin{tikzcd}
\mathbf{sSet}_{\mathrm{Kan}} \arrow[r, shift left, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{Top}_{\mathrm{Quillen}} \arrow[l, shift left, "\mathrm{Sing}"]
\end{tikzcd}$$

of the geometric realisation and the singular complex is a Quillen equivalence.

In particular, the homotopy categories (which encode the homotopy theories) of topological spaces and simplicial sets are equivalent. We obtain a quick proof of the CW-approximation theorem[12]:

COROLLARY 5.17 (CW-approximation) · *Every topological space is weakly equivalent to a CW-complex.*

PROOF. Any topological space X is fibrant and any simplicial set is cofibrant, so in particular $\operatorname{Sing} X$. By the Quillen equivalence of the theorem, the counit $\varepsilon_X : |\operatorname{Sing} X| \to X$, which is the adjoint of the identity on $\operatorname{Sing} X$, is a weak equivalence, and the geometric realisation of any simplicial set is a CW-complex. □

Infinity-categories Model categories also show up in the study of $(\infty, 1)$-categories (higher categories where all n-morphisms are invertible for $n > 1$). There are many different models of $(\infty, 1)$-categories; two well-known examples are quasicategories (simplicial sets with fillers for inner horn inclusions) and categories enriched in spaces[13] (simplicially enriched categories whose hom-spaces are Kan complexes). There are model structures for these notions of $(\infty, 1)$-categories: the Joyal model structure on **sSet** where the fibrant objects are the quasicategories, and the Bergner model structure on **sCat** where the fibrant objects are the categories enriched in Kan complexes. To justify the statement that these objects model $(\infty, 1)$-categories, there is a Quillen equivalence between the model categories:

THEOREM 5.18 · *The adjunction*

$$\mathbf{sSet}_{\text{Joyal}} \underset{\tilde{N}}{\overset{\mathfrak{C}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{sCat}_{\text{Bergner}}$$

of the rigidification functor and the homotopy-coherent nerve is a Quillen equivalence.

More generally, the known models of $(\infty, 1)$-categories come equipped with a model structure, and all these model categories are Quillen equivalent. We refer to [Bal21, § 5.2] for a list of models of $(\infty, 1)$-categories with Quillen equivalent model structures.

Homological and homotopical algebra The Dold–Kan correspondence states that there is an equivalence of categories $\mathbf{sAb} \simeq \mathbf{Ch}_{\geqslant 0}(\mathbb{Z})$ between simplicial abelian groups and non-negatively graded chain complexes of abelian groups ($\mathbb{Z}$-modules) via functors

$$\mathbf{Ch}_{\geqslant 0}(\mathbb{Z}) \underset{N}{\overset{\Gamma}{\rightleftarrows}} \mathbf{sAb}$$

Using the projective model structure on $\mathbf{Ch}_{\geqslant 0}(\mathbb{Z})$ (where the weak equivalences are quasi-isomorphisms, the fibrations are degreewise epimorphisms in positive degrees,

[12]The reader might rightly object that this we are putting the cart before the horse, since to prove Theorem 5.16 we would probably show that the counit is a weak equivalence.

[13]Here we use Theorem 5.16 to think of spaces as Kan complexes (the fibrant–cofibrant objects in $\mathbf{sSet}_{\text{Kan}}$), which also model ∞-groupoids (being the class of quasicategories whose 1-morphisms are all invertible).

and the cofibrations are degreewise monomorphisms with projective cokernel) and a model structure on $\mathbf{sAb}_{\mathrm{Kan}}$ lifted from $\mathbf{sSet}_{\mathrm{Kan}}$ with the Kan–Quillen model structure (that is, the weak equivalences and fibrations in $\mathbf{sAb}_{\mathrm{Kan}}$ are created by the forgetful functor to $\mathbf{sSet}_{\mathrm{Kan}}$), we obtain the following:

THEOREM 5.19 (Schwede–Shipley after Dold–Kan) · *The functors Γ and N are both left and right Quillen equivalences between $\mathbf{Ch}_{\geqslant 0}(\mathbb{Z})_{\mathrm{proj}}$ and $\mathbf{sAb}_{\mathrm{Kan}}$.*

These functors also induce isomorphisms between homology groups and simplicial homotopy groups.

The derived functors as introduced above subsume the notion of derived functors from homological algebra via the projective model structure on non-negatively graded chain complexes.

Equivariant homotopy theory In equivariant homotopy theory, one studies spaces with a group action. (More generally, we can consider objects in a model category with a group action.) If G is a finite group, then the category $\mathbf{Top}^G := \mathbf{Fun}(G, \mathbf{Top})$ of functors $G \to \mathbf{Top}$ (where G is seen as a one-object category) is the category of *G-spaces*. There is a model structure on $\mathbf{Top}^G$ where the weak equivalences and fibrations are created by the fixed-point functors $(-)^H : \mathbf{Top}^G \to \mathbf{Top}$ for all subgroups H of G (that is, an equivariant map $f : X \to Y$ is a weak equivalence or fibration if and only if the induced map $f^H : X^H \to Y^H$ is such for all subgroups H); we denote this model category by $\mathbf{Top}^G_{\mathrm{f.p.}}$ where 'f.p.' stands for 'fixed points'. There is an adjunction between $\mathbf{Top}^G$ and the category $\mathbf{Fun}(\mathbf{Orb}_G^{\mathrm{op}}, \mathbf{Top})$ of *contravariant orbit diagrams*, where the indexing category is the full subcategory of $\mathbf{Set}^G$ on the cosets G/H with the canonical G-action for all subgroups H. On this category, there is a *projective* model structure where the weak equivalences and fibrations are pointwise weak equivalences and fibrations in $\mathbf{Top}$.

The following result, known as Elmendorf's theorem, relates these two model categories via a Quillen equivalence:

THEOREM 5.20 (Elmendorf [Elm83; Ste16, Theorem 2.10]) · *The adjunction*

$$\begin{tikzcd} \mathbf{Fun}(\mathbf{Orb}_G^{\mathrm{op}}, \mathbf{Top})_{\mathrm{proj}} \arrow[r, shift left, "\mathrm{ev}_{G/e}"] \arrow[r, phantom, "\perp"] & \mathbf{Top}^G_{\mathrm{f.p.}} \arrow[l, shift left, "\Phi"] \end{tikzcd}$$

with $\mathrm{ev}_{G/e} : F \mapsto F(G/e)$ and $\Phi : X \mapsto (G/H \mapsto X^H)$ is a Quillen equivalence.

Stephan's paper [Ste16] shows that the Quillen equivalence of the this theorem generalises to cofibrantly generated model categories for which the fixed-point functors satisfy certain conditions.

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