CAT 1: Transient Analysis of RL, RC, and RLC Circuits

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Question 1

(a) Writing the Differential Equation

Using Kirchhoff's Voltage Law (KVL):

$$V = iR + L\frac{di}{dt} \tag{1}$$

$$24 = 10i + 2\frac{di}{dt} \tag{2}$$

Rearranging to standard form:

$$2\frac{di}{dt} + 10i = 24 \tag{3}$$

$$\frac{di}{dt} + 5i = 12 \tag{4}$$

This is our governing first-order differential equation.

(b) Finding the Current $i(t)$ for $t > 0$

The general solution form for this type of equation is:

$$i(t) = A(1 - e^{-t/\tau}) \tag{5}$$

where $\tau = \frac{L}{R}$ is the time constant.

At steady state ($t = \infty$):

$$i = \frac{V}{R} = \frac{24}{10} = 2.4A \tag{6}$$

Therefore, $A = 2.4$.

The complete solution is:

$$i(t) = 2.4(1 - e^{-5t}) \quad \text{amperes, for } t > 0 \tag{7}$$

(c) Calculations

Time constant:

$$\tau = \frac{L}{R} = \frac{2}{10} = 0.2 \text{ seconds} \tag{8}$$

To find $i(0.5s)$:

$$i(0.5) = 2.4(1 - e^{-5(0.5)}) \tag{9}$$

$$i(0.5) = 2.4(1 - e^{-2.5}) \tag{10}$$

$$i(0.5) = 2.4(1 - 0.082) \tag{11}$$

$$i(0.5) = 2.4(0.918) \tag{12}$$

$$i(0.5) = 2.20A \tag{13}$$

Question 2

(a) Deriving the Equation for Capacitor Charging

Using Kirchhoff's Voltage Law (KVL):

$$V = iR + v_c \tag{14}$$

where $V = 20V$ (DC supply).

The current through the capacitor is:

$$i = C\frac{dv_c}{dt} \tag{15}$$

Substituting this into KVL:

$$20 = RC\frac{dv_c}{dt} + v_c \tag{16}$$

Rearranging to standard form:

$$RC\frac{dv_c}{dt} + v_c = 20 \tag{17}$$

Substituting given values:

$$(1000)(100 \times 10^{-6})\frac{dv_c}{dt} + v_c = 20 \tag{18}$$

$$0.1\frac{dv_c}{dt} + v_c = 20 \tag{19}$$

(b) Solving for $v_c(t)$

This is a first-order differential equation with the general solution:

$$v_c(t) = A(1 - e^{-t/\tau}) \tag{20}$$

where $\tau = RC$ is the time constant.

At $t = \infty$, $v_c = 20V$ (capacitor fully charged), so $A = 20$.

Thus, the complete solution is:

$$v_c(t) = 20(1 - e^{-t/0.1}) \quad \text{volts, for } t > 0 \tag{21}$$

(c) Calculations

Time constant:

$$\tau = RC = (1000)(100 \times 10^{-6}) = 0.1 \text{ seconds} = 100 \text{ ms} \tag{22}$$

To find $v_c(10ms)$:

$$v_c(10ms) = 20(1 - e^{-0.01/0.1}) \tag{23}$$

$$v_c(10ms) = 20(1 - e^{-0.1}) \tag{24}$$

$$v_c(10ms) = 20(1 - 0.905) \tag{25}$$

$$v_c(10ms) = 20(0.095) \tag{26}$$

$$v_c(10ms) = 1.9V \tag{27}$$

Question 3

(a) Deriving the Equation for Current Decay

After disconnection, using KVL:

$$0 = iR + L\frac{di}{dt} \tag{28}$$

where i = current, $R = 15\Omega$, $L = 3H$.

Rearranging:

$$L\frac{di}{dt} = -iR \tag{29}$$

$$3\frac{di}{dt} = -15i \tag{30}$$

$$\frac{di}{dt} = -5i \tag{31}$$

This is our governing differential equation for current decay.

(b) Finding $i(t)$ after Disconnection

For a decaying current, the solution form is:

$$i(t) = Ae^{-t/\tau} \tag{32}$$

where $\tau = \frac{L}{R} = \frac{3}{15} = 0.2$ seconds.

At $t = 0$, $i = 5A$ (initial current).

Therefore,

$$5 = A \tag{33}$$

The complete solution is:

$$i(t) = 5e^{-5t} \text{ amperes} \tag{34}$$

(c) Calculating Current at $t = 1s$

$$i(1) = 5e^{-5(1)} \tag{35}$$

$$i(1) = 5e^{-5} \tag{36}$$

$$i(1) = 5(0.0067) \tag{37}$$

$$i(1) = 0.0335A \approx 33.5mA \tag{38}$$

Question 4. Solving the RLC Circuit Problem Step by Step

(a) Writing the Second-Order Differential Equation

For a series RLC circuit with a DC voltage source V, using Kirchhoff's Voltage Law:

$$V = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt \tag{39}$$

Taking the derivative of both sides to eliminate the integral:

$$0 = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i \tag{40}$$

Therefore, the governing differential equation is:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0 \tag{41}$$

Substituting the given values:

$$(1)\frac{d^2i}{dt^2} + (50)\frac{di}{dt} + \left(\frac{1}{100 \times 10^{-6}}\right)i = 0 \tag{42}$$

$$\frac{d^2i}{dt^2} + 50\frac{di}{dt} + 10^4 i = 0 \tag{43}$$

(b) Identifying the Damping Type

Calculate the damping coefficient (α) and natural frequency (ω_0):

$$\alpha = \frac{R}{2L} = \frac{50}{2 \times 1} = 25 \tag{44}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{1 \times 100 \times 10^{-6}}} = 100 \tag{45}$$

Calculate the discriminant ($\alpha - \omega_0$):

$$25 - 100 = -75 \tag{46}$$

Since $\alpha < \omega_0$, this is an underdamped system. The general solution form is:

$$i(t) = e^{-\alpha t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right] \tag{47}$$

where:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{9375} \approx 96.8 \text{ rad/s} \tag{48}$$

$$i(t) = Ae^{-25t}\left[\cos(96.8t) + B\sin(96.8t)\right] \tag{49}$$

(c) Natural Frequencies and Behavior

Damped natural frequency:

$$\omega_d \approx 96.8 \text{ rad/s} \tag{50}$$

Undamped natural frequency:

$$\omega_0 = 100 \text{ rad/s} \tag{51}$$

The circuit behavior over time:

- The current will oscillate with frequency $\omega_d \approx 96.8$ rad/s.
- The amplitude will decay exponentially with time constant $\tau = \frac{1}{\alpha} = \frac{1}{25} = 0.04$ seconds.
- Since it's underdamped, the current will oscillate around zero while gradually decreasing in amplitude.
- The system will eventually settle to zero current ($t \to \infty$) since there's only a DC source and the circuit is initially unenergized.
- The response will effectively reach steady state after approximately $5\tau = 0.2$ seconds.

Question 5. Deriving the Characteristic Equation

(a) Writing the Second-Order Differential Equation

For a series RLC circuit with a step voltage input $V(t)$, using Kirchhoff's Voltage Law:

$$V(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt \tag{52}$$

Taking the derivative to eliminate the integral:

$$0 = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i \tag{53}$$

Substituting the given values:

$$0.5\frac{d^2i}{dt^2} + 5\frac{di}{dt} + \left(\frac{1}{50 \times 10^{-6}}\right) i = 0 \tag{54}$$

Simplifying:

$$\frac{d^2i}{dt^2} + 10\frac{di}{dt} + 40000i = 0 \tag{55}$$

The characteristic equation is:

$$s^2 + 10s + 40000 = 0 \tag{56}$$

(b) Testing for Underdamped Condition and Finding Damped Frequency

Calculate the damping coefficient (α) and natural frequency (ω_0):

$$\alpha = \frac{R}{2L} = \frac{5}{2 \times 0.5} = 5 \tag{57}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{0.5 \times 50 \times 10^{-6}}} = 200 \text{ rad/s} \tag{58}$$

For the underdamped condition: $\alpha < \omega_0$

$$5 < 200, \quad \text{therefore the system is underdamped} \tag{59}$$

Damped frequency:

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{60}$$

$$\omega_d = \sqrt{40000 - 25} \tag{61}$$

$$\omega_d = \sqrt{39975} \approx 199.94 \text{ rad/s} \tag{62}$$

(c) Time for Oscillations to Decay to 10%

For an underdamped system, the envelope of decay is $e^{-\alpha t}$. We want to find t when amplitude reaches 10% of initial:

$$0.1 = e^{-5t} \tag{63}$$

Taking the natural logarithm:

$$\ln(0.1) = -5t \tag{64}$$

$$t = \frac{-\ln(0.1)}{5} \tag{65}$$

$$t \approx 0.46 \text{ seconds} \tag{66}$$

Therefore, it takes approximately 0.46 seconds for the oscillations to decay to 10% of their initial amplitude.

The complete solution can be written as:

$$i(t) = Ae^{-5t}\left[\cos(199.94t) + B\sin(199.94t)\right] \tag{67}$$

where A and B are constants determined by initial conditions.