CHUKA UNIVERSITY

UNIVERSITY EXAMINATIONS

EXAMINATION FOR THE AWARD OF DEGREE OF BACHELOR OF SCIENCE IN ELECTRICAL AND ELECTRONICS ENGINEERING

EENG 475: CONTROL ENGINEERING 1

STREAMS: **TIME: 2 HOURS**

DAY/DATE: TUESDAY 15/04/2025 **8.30 A.M – 10.30 A.M**

INSTRUCTIONS
Answer question ONE and any other TWO questions.
Do not write on the question paper.

QUESTION ONE (30 MARKS)

(a) Define the following terms:

- **State Variables:** State variables are a set of variables that describe the smallest possible subset of system variables such that the knowledge of these variables at any time t_0, together with the input for $t \geq t_0$, completely determines the behavior of the system for all future times $t \geq t_0$.
- **State Vector:** A state vector is a column vector that contains all the state variables of a system. It represents the complete state of the system at a given time and is typically denoted as $\mathbf{x}(t)$.
- **State Space:** State space is a mathematical model of a physical system expressed as a set of input, output, and state variables related by first-order differential (or difference) equations. It provides a framework to model and analyze systems using a set of equations in matrix form.

[3 marks]

(b) What are the key advantages of state-space representation over classical control techniques? List and briefly explain three advantages. [3 marks]

(c) Define the following terms in the context of digital control systems:

- Sampling

- Zero-order hold
- Quantization

[2 marks]

(d) What are eigenvalues and eigenvectors, and how are they used in state-space analysis? [3 marks]

(e) Define controllability and observability. Provide the criteria for testing them. [3 marks]

(f) Describe the difference between continuous-time and discrete-time systems in control. [2 marks]

Question Two (20 Marks)

(a) **Test the controllability and observability of the system below: [4 marks]**

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Solution:
The controllability matrix is:

$$\mathcal{C} = [B \quad AB] = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

Since $\text{rank}(\mathcal{C}) = 1 < 2$, the system is **not controllable**.

The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det(\mathcal{O}) = (1)(2) - (1)(1) = 1 \neq 0$$

Therefore, the system is **observable**.

(b) **For the state-space system:**

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 4 & 5 \end{bmatrix} x$$

(i) **Calculate the state transition matrix $\Phi(t)$.** [4 marks]
Solution:

(i) State Transition Matrix $\Phi(t)$

The state transition matrix is:

$$\Phi(t) = e^{At} \quad \text{where} \quad A = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$$

Since A is upper triangular, the eigenvalues are 2 and -3.
Thus:

$$\Phi(t) = \begin{bmatrix} e^{2t} & f(t) \\ 0 & e^{-3t} \end{bmatrix}$$

where $f(t)$ satisfies:

$$\frac{d}{dt} f(t) = 1 \times e^{-3t} + 2f(t)$$

Using integrating factor e^{-2t}:

$$\frac{d}{dt}\left(e^{-2t} f(t)\right) = e^{-5t}$$

Integrating:

$$e^{-2t} f(t) = \frac{e^{-5t}}{-5} + C$$

$$f(t) = e^{2t}\left(\frac{1}{5}(1 - e^{-5t})\right)$$

Thus:

$$\boxed{\Phi(t) = \begin{bmatrix} e^{2t} & \frac{1}{5}e^{2t}(1 - e^{-5t}) \\ 0 & e^{-3t} \end{bmatrix}}$$

(ii) **Find the zero-input response for the initial condition $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T$.** [3 marks]

Solution:
The zero-input response is:

$$x(t) = \Phi(t)x(0) = \begin{bmatrix} \frac{4}{5}e^{2t} + \frac{1}{5}e^{-3t} \\ -e^{-3t} \end{bmatrix}$$

(c) **For the continuous-time system:**

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Design a state feedback controller $u = -Kx$ to place the eigenvalues at $s = -1$ and $s = -2$. [7 marks]

(d) **Explain the concepts of reliability and redundancy in digital control systems.** [2 marks]

Question b

For the state-space system:

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \quad y = \begin{bmatrix} 4 & 5 \end{bmatrix} x$$

(i) Calculate the state transition matrix $\Phi(t)$.

(ii) Find the zero-input response for the initial condition $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

solution

(i) State Transition Matrix $\Phi(t)$

The state transition matrix is:

$$\Phi(t) = e^{At} \quad \text{where} \quad A = \begin{bmatrix} 2 & 1 \\ 0 & -3 \end{bmatrix}$$

Since A is upper triangular, the eigenvalues are 2 and -3.
Thus:

$$\Phi(t) = \begin{bmatrix} e^{2t} & f(t) \\ 0 & e^{-3t} \end{bmatrix}$$

where $f(t)$ satisfies:

$$\frac{d}{dt} f(t) = 1 \times e^{-3t} + 2f(t)$$

Using integrating factor e^{-2t}:

$$\frac{d}{dt}\left(e^{-2t} f(t)\right) = e^{-5t}$$

Integrating:

$$e^{-2t} f(t) = \frac{e^{-5t}}{-5} + C$$

$$f(t) = e^{2t}\left(\frac{1}{5}(1 - e^{-5t})\right)$$

Thus:

$$\boxed{\Phi(t) = \begin{bmatrix} e^{2t} & \frac{1}{5}e^{2t}(1 - e^{-5t}) \\ 0 & e^{-3t} \end{bmatrix}}$$

(ii) Zero-input Response

The zero-input response is:

$$x(t) = \Phi(t)x(0)$$

Substituting:

$$x(t) = \begin{bmatrix} e^{2t} & \frac{1}{5}e^{2t}(1 - e^{-5t}) \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

First component:

$$e^{2t}(1) + \frac{1}{5}e^{2t}(1 - e^{-5t})(-1) = e^{2t}\left(\frac{4}{5} + \frac{1}{5}e^{-5t}\right)$$

Second component:

$$0(1) + e^{-3t}(-1) = -e^{-3t}$$

Thus:

$$\boxed{x(t) = \begin{bmatrix} e^{2t}\left(\frac{4}{5} + \frac{1}{5}e^{-5t}\right) \\ -e^{-3t} \end{bmatrix}}$$

Question c

For the continuous-time system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Design a state feedback controller $u = -Kx$ to place the eigenvalues at $s = -1$ and $s = -2$.

solution

The closed-loop system is:

$$\dot{x} = (A - BK)x$$

where:

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

Thus:

$$A - BK = \begin{bmatrix} 0 & 1 \\ -2 - k_1 & -3 - k_2 \end{bmatrix}$$

The characteristic equation:

$$\det(sI - (A - BK)) = 0$$

Expanding:

$$\det\left(\begin{bmatrix} s & -1 \\ 2 + k_1 & s + 3 + k_2 \end{bmatrix}\right) = 0$$

Thus:

$$s^2 + (3 + k_2)s + (2 + k_1) = 0$$

The desired characteristic equation is:

$$(s + 1)(s + 2) = s^2 + 3s + 2$$

Comparing:

$$3 + k_2 = 3 \quad \Rightarrow \quad k_2 = 0$$
$$2 + k_1 = 2 \quad \Rightarrow \quad k_1 = 0$$

Thus:

$$\boxed{K = \begin{bmatrix} 0 & 0 \end{bmatrix}}$$

Question Three **(20 Marks)**

(a) Convert the continuous transfer function

$$G(s) = \frac{1}{s+2}$$

to z-domain using $T = 0.1$ seconds. [3 marks]

(b) Convert the following transfer function to state-space representation in controllable canonical form:

$$G(s) = \frac{5s+3}{s^2+4s-0.5}$$

[3 marks]

(c) Consider a continuous-time system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

(i) Convert this system to a discrete-time system with a sampling period of $T = 0.1$ seconds. [4 marks]

(ii) Determine whether the discrete-time system is stable. [1 mark]

(d) Consider the discrete-time system:

$$x(k+1) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \quad , \quad y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(k)$$

(i) Design a state feedback controller $u(k) = -Kx(k)$ to place the eigenvalues of the closed-loop system at $z = 0.3$ and $z = 0.4$. [6 marks]

(ii) Verify that the designed controller achieves the desired eigenvalue placement. [3 marks]

Question Four **(20 Marks)**

(a) Consider a temperature control system for an industrial furnace. The continuous-time plant model is:

$$G(s) = \frac{2}{5s+1}$$

You need to implement a digital controller with a sampling time of $T = 0.5$ seconds.

(i) Find the discrete-time plant model $G(z)$. [4 marks]

(ii) Design a digital PID controller in the form:

$$C(z) = K_p + K_i\left(\frac{Tz}{z-1}\right) + \frac{K_d(z-1)}{Tz}$$

with parameters $K_p = 1.2$, $K_i = 0.5$, and $K_d = 0.1$. [4 marks]

(iii) Write the difference equation for the controller implementation. [3 marks]

(b) A discrete-time control system has the following state-space representation:

$$x(k+1) = \begin{bmatrix} 0.8 & 0.1 \\ 0 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \quad , \quad y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

(i) Design a full-order observer to estimate the states with observer poles at $z = 0.2$ and $z = 0.3$. [4 marks]

(ii) Write the observer equations. [3 marks]

(iii) Discuss the trade-off in selecting observer pole locations. [2 marks]