Certifiable Robustness

A. M. Sadeghzadeh, Ph.D.

Sharif University of Technology
Computer Engineering Department (CE)
Data and Network Security Lab (DNSL)

May 2, 2023

The proof of Theorem 1 is based on *Sebastien Bubeck* lecture

Today's Agenda

Recap

Certifiable Robustness

A classifier is **certifiably robust** if for any input x, **there exist a guarantee** that the **classifier's prediction is constant within some set around** x, often an ℓ_2 or ℓ_∞ ball.

Certifiable Robustness

A classifier is **certifiably robust** if for any input x, **there exist a guarantee** that the **classifier's prediction is constant within some set around** x, often an ℓ_2 or ℓ_∞ ball.

- Classifier $f : \mathbb{R}^d \to [0,1]^K$ is **ϵ-robust** at x, if

$$\forall \ \|\delta\|_p \leq \epsilon, \quad \underset{i \in [K]}{argmax} \ f_i(x+\delta) = \underset{i \in [K]}{argmax} \ f_i(x)$$

  where K is the number of classes.

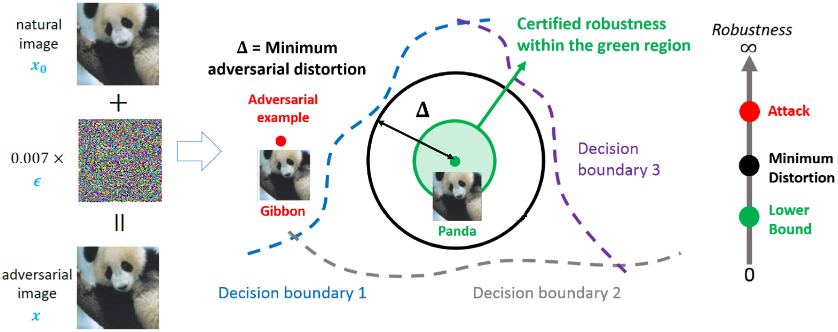
Certifiable Robustness

A classifier is **certifiably robust** if for any input x, **there exist a guarantee** that the **classifier's prediction is constant within some set around** x, often an ℓ_2 or ℓ_∞ ball.

- Classifier $f : \mathbb{R}^d \rightarrow [0,1]^K$ is **ϵ-robust** at x, if

$$\forall \, \|\delta\|_p \leq \epsilon, \quad \underset{i \in [K]}{argmax} \, f_i(x+\delta) = \underset{i \in [K]}{argmax} \, f_i(x)$$

where K is the number of classes.



[Source]

Certifiable Robustness for L-Lipschitz classifier

Theorem 0: If $f : \mathbb{R}^d \to [0,1]^K$ is L-lipschitz, then f is ϵ-robust at x with $\epsilon = \frac{1}{2L}(P_A - P_B)$, where $P_A = \underset{i}{max}\ f_i(x)$, $P_B = \underset{j \neq i}{max}\ f_j(x)$, and $f_k(x)$ is the k-th element of the probability vector $f(x)$.

Certifiable Robustness for L-Lipschitz classifier

Theorem 0: If $f : \mathbb{R}^d \to [0,1]^K$ is L-lipschitz, then f is ϵ-robust at x with $\epsilon = \frac{1}{2L}(P_A - P_B)$, where $P_A = \underset{i}{max}\ f_i(x)$, $P_B = \underset{j \neq i}{max}\ f_j(x)$, and $f_k(x)$ is the k-th element of the probability vector $f(x)$.

Proof.

Since f is L-lipschitz, we have

$$\forall x, y \in \mathbb{R}^d, \|f(y) - f(x)\|_2 \leq L\|y - x\|_2$$

Denote $x' = x + \delta$ and assume that $\|\delta\| \leq \epsilon$. we get

$$\|f(x') - f(x)\|_2 \leq L\|x' - x\|_2 \rightarrow \|f(x') - f(x)\|_2 \leq L\|\delta\|_2 \leq L\epsilon$$

Hence, P_A can be reduced at most by $L\epsilon$ and P_B can be increased at most by $L\epsilon$. We have ($P_i' = f_i(x')$)

$$P_A' \geq P_A - L\epsilon \quad \text{and} \quad P_B' \leq P_B + L\epsilon$$

Since we want that the label of x' be the same as x, P_A' must be greater than P_B' ($P_A' \geq P_B'$). We have

$$P_A - L\epsilon \geq P_B + L\epsilon \ \rightarrow\ 2L\epsilon \leq P_A - P_B \ \rightarrow\ \epsilon \leq \frac{1}{2L}(P_A - P_B)$$

□

Smoothed Classifier

If we compute the (upper bound of) Lipschitz constant of the classifier, we can determine the radius (ϵ) of the robustness for each sample.

- However, **the lipschitz constant of deep neural networks** is very large and it **grows exponentially** with the number of layers. Hence, the certification is useless ($\epsilon \longrightarrow 0$).

Smoothed Classifier

If we compute the (upper bound of) Lipschitz constant of the classifier, we can determine the radius (ϵ) of the robustness for each sample.

- However, **the lipschitz constant of deep neural networks** is very large and it **grows exponentially** with the number of layers. Hence, the certification is useless ($\epsilon \longrightarrow 0$).

We can **transform classifier f to a smoothed version** in order to bound the Lipschitz constant of the classifier.

Smoothed Classifier

If we compute the (upper bound of) Lipschitz constant of the classifier, we can determine the radius (ϵ) of the robustness for each sample.

- However, **the lipschitz constant of deep neural networks** is very large and it **grows exponentially** with the number of layers. Hence, the certification is useless ($\epsilon \to 0$).

We can **transform classifier f to a smoothed version** in order to bound the Lipschitz constant of the classifier.

- To smooth classifier f, we convolve it with a **Gaussian kernel**.

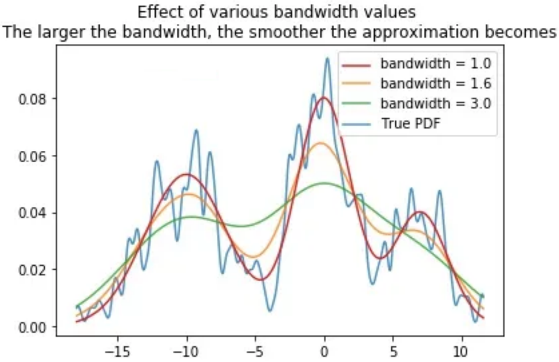
Smoothed Classifier

If we compute the (upper bound of) Lipschitz constant of the classifier, we can determine the radius (ϵ) of the robustness for each sample.

- However, **the lipschitz constant of deep neural networks** is very large and it **grows exponentially** with the number of layers. Hence, the certification is useless ($\epsilon \to 0$).

We can **transform classifier f to a smoothed version** in order to bound the Lipschitz constant of the classifier.

- To smooth classifier f, we convolve it with a **Gaussian kernel**.



[source]

Randomized Smoothing

Consider a classification problem from $\mathbb{R}^d$ to classes $\mathcal{Y}$. Randomized smoothing is a method for constructing a new, **smoothed classifier** $\hat{f}$ from an arbitrary base classifier f.

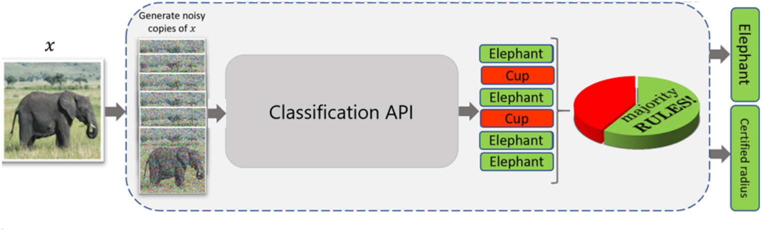
Randomized Smoothing

Consider a classification problem from $\mathbb{R}^d$ to classes $\mathcal{Y}$. Randomized smoothing is a method for constructing a new, **smoothed classifier** $\hat{f}$ from an arbitrary base classifier f.

- When queried at x, the smoothed classifier $\hat{f}$ returns whichever class the base classifier f is most likely to return when x is perturbed by isotropic Gaussian noise:

$$\begin{gathered}\hat{f}(x) = \underset{c \in \mathcal{Y}}{argmax}\ \mathbb{P}(f(x+\epsilon) = c) \\ \text{where} \quad \epsilon \sim \mathcal{N}(0, \sigma^2 I)\end{gathered} \tag{1}$$

The noise level σ is a hyperparameter of the smoothed classifier $\hat{f}$ which controls a robustness/accuracy tradeoff.



[source]

Notation

Suppose that when the base classifier f classifies $\mathcal{N}(x, \sigma^2 I)$, the most probable class c_A is returned with probability P_A, and the "runner-up" class is returned with probability P_B.

- $\underline{P_A}$ is a lower bound for P_A and $\overline{P_B}$ is a lower bound for P_B.

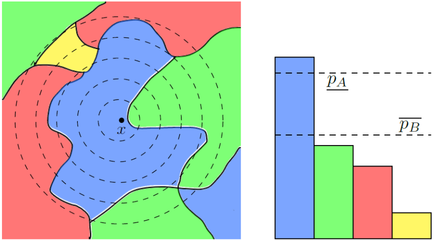


Figure 1. Evaluating the smoothed classifier at an input x. **Left**: the decision regions of the base classifier f are drawn in different colors. The dotted lines are the level sets of the distribution $\mathcal{N}(x, \sigma^2 I)$. **Right**: the distribution $f(\mathcal{N}(x, \sigma^2 I))$. As discussed below, $\underline{p_A}$ is a lower bound on the probability of the top class and $\overline{p_B}$ is an upper bound on the probability of each other class. Here, $g(x)$ is "blue."

Robustness guarantee

Theorem 1. Let $f : \mathbb{R}^d \to \mathcal{Y}$ be any deterministic or random function, and let $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$. Let $\hat{f}$ be defined as in (1). Suppose $C_A \in \mathcal{Y}$ and $\underline{P_A}, \overline{P_B} \in [0, 1]$ satisfy:

$$\mathbb{P}(f(x+\epsilon) = C_A) \geq \underline{P_A} \geq \overline{P_B} \geq \max_{C \neq C_A} \mathbb{P}(f(x+\epsilon) = C)$$

Then $\hat{f}(x+\delta) = C_A$ for all $\|\delta\|_2 \leq R$, where

$$R = \frac{\sigma}{2}(\Phi^{-1}(\underline{P_A}) - \Phi^{-1}(\overline{P_B}))$$

where Φ^{-1} is the inverse of the standard Gaussian CDF.

Robustness guarantee

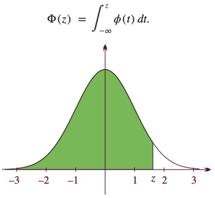
Theorem 1. Let $f : \mathbb{R}^d \to \mathcal{Y}$ be any deterministic or random function, and let $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$. Let $\hat{f}$ be defined as in (1). Suppose $C_A \in \mathcal{Y}$ and $\underline{P_A}, \overline{P_B} \in [0, 1]$ satisfy:

$$\mathbb{P}(f(x+\epsilon) = C_A) \geq \underline{P_A} \geq \overline{P_B} \geq \max_{C \neq C_A} \mathbb{P}(f(x+\epsilon) = C)$$

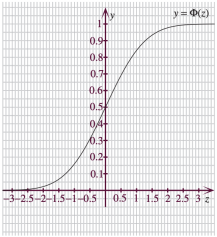
Then $\hat{f}(x+\delta) = C_A$ for all $\|\delta\|_2 \leq R$, where

$$R = \frac{\sigma}{2}(\Phi^{-1}(\underline{P_A}) - \Phi^{-1}(\overline{P_B}))$$

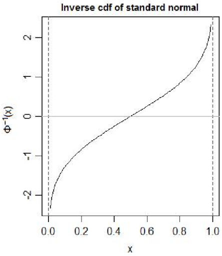
where Φ^{-1} is the inverse of the standard Gaussian CDF.



This is the graph of the standard normal probability density function $\phi(z)$.



This is the graph of the standard normal cumulative distribution function $\Phi(z)$.



Randomized Smoothing

Lipschitz Constant of Randomized Smoothed Classifier

To prove Theorem 1, we need to find the Lipschitz constant of the smoothed classifier $\hat{f}$.

Lipschitz Constant of Randomized Smoothed Classifier

To prove Theorem 1, we need to find the Lipschitz constant of the smoothed classifier $\hat{f}$.

Let

$$f : \mathbb{R}^d \to [0, 1]$$
$$\hat{f}(x) = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x + \sigma z)]$$

It is well-known that $\hat{f}$ is Lipschitz (it has uniform bound on the Lipschitz constant). In practice, we can approximate $\hat{f}$ by empirical average

$$y^{(k)} = \sum_{i=1}^{k} f(x + \sigma z), \quad \text{where } z \sim \mathcal{N}(0, I_d)$$

It can be shown that if $k \to \infty$, $y^{(k)}$ almost surely converges to $\hat{f}$.

Lipschitz Constant of Randomized Smoothed Classifier

Recall: Expected Value

The expectation, or expected value, of some function $f(x)$ with respect to a probability distribution $P_X(x)$ is the average, or mean value, that f takes on when x is drawn from P.

For discrete random variable X, $P_X(x)$ is **Probability Mass Function (PMF)** and expected value can be computed with a summation:

$$\mathbb{E}_{X\sim P}[f(x)] = \sum_x f(x) P_X(x)$$

For continuous random variable X, $P_X(x)$ is **Probability Density Function (PDF)** and expected value is computed with an integral:

$$\mathbb{E}_{X\sim P}[f(x)] = \int f(x) P_X(x) dx$$

Lipschitz Constant of Randomized Smoothed Classifier

Recall

- If $z \sim \mathcal{N}(0, I_d)$, then $\mu + \sigma z \sim \mathcal{N}(\mu, \sigma^2 I_d)$.

- The identity matrix is often denoted by I_n, where n is the dimension. The determinant of the identity matrix is 1.

- The Probability Density Function (PDF) of multivariate normal distribution

$$f_X(x) = \frac{1}{(2\pi)^{d/2} det(\Sigma)^{1/2}} exp\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\}$$

  where X is normally distributed random variable with mean $\mu \in \mathbb{R}^d$ and covariance $\Sigma \in \mathbb{R}^{d \times d}$, and d is the dimension of x.

  The Cumulative Distribution Function (CDF) of multivariate normal distribution

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(t) dt$$

Lipschitz Constant of Randomized Smoothed Classifier

To prove Theorem 1, we need to find the Lipschitz constant of the smoothed classifier $\hat{f}$.

Let

$$f : \mathbb{R}^d \to [0, 1]$$
$$\hat{f}(x) = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x + \sigma z)]$$

It is well-known that $\hat{f}$ is Lipschitz (it has uniform bound on the Lipschitz constant). In practice, we can approximate $\hat{f}$ by empirical average

$$y^{(k)} = \sum_{i=1}^{k} f(x + \sigma z), \quad \text{where } z \sim \mathcal{N}(0, I_d)$$

It can be shown that if $k \to \infty$, $y^{(k)}$ almost surely converges to $\hat{f}$.

We have

$$\hat{f}(x) = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x + \sigma z)] = \frac{1}{(2\pi)^{d/2}} \int f(x + \sigma z) \exp\{-\frac{\|z\|_2^2}{2}\}dz$$

$\hat{f}(x)$ is the weighted average of $f(x)$ in the vicinity of x.

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int f(x + \sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz$$

Change of variable: $w = -\sigma z$.

Lipschitz Constant of Randomized Smoothed Classifier

Recall: change of variable

Double Integral.
Suppose that we want to integrate $f(x,y)$ over the region R. Under the transformation $x = g(u,v)\,, y = h(u,v)$ the region becomes S and the integral becomes

$$\iint\limits_R f(x,y)\,dxdy = \iint\limits_S f\left(g(u,v)\,, h(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ is the absolute value of inverse Jacobian determinant of the transformation.

Vector Integral.
Suppose $x, y \in \mathbb{R}^d$ that we want to integrate $f(x)$ over the region R. Under the transformation $y = g(x)$ the region becomes S and the integral becomes

$$\int\limits_R f(x)\,dx = \int\limits_S f(g(x))\left|\frac{\partial x}{\partial y}\right| dy$$

where $\left|\frac{\partial x}{\partial y}\right|$ is the absolute value of inverse Jacobian determinant of the transformation.

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = \frac{1}{(2\pi)^d} \int f(x + \sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz$$

Change of variable: $w = -\sigma z$. Hence, $dz = \frac{1}{\sigma^d} dw$. We get

$$\hat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int f(x+\sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz = \int f(x-w) \frac{1}{(2\pi)^{d/2}\sigma^d} \exp\{-\frac{\|w\|_2^2}{2\sigma^2}\} dw$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = \frac{1}{(2\pi)^d} \int f(x + \sigma z) \exp\{-\frac{\|z\|_2^2}{2}\}dz$$

Change of variable: $w = -\sigma z$. Hence, $dz = \frac{1}{\sigma^d} dw$. We get

$$\hat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int f(x+\sigma z) \exp\{-\frac{\|z\|_2^2}{2}\}dz = \int f(x-w) \frac{1}{(2\pi)^{d/2}\sigma^d} \exp\{-\frac{\|w\|_2^2}{2\sigma^2}\}dw$$

Recall: Convolution

Convolution of two functions f and g over a infinite range $(-\infty, +\infty)$ is given by

$$f * g = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)d\tau = g * f$$

where the symbol $[f * g](t)$ denotes convolution of f and g.

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = \frac{1}{(2\pi)^d} \int f(x + \sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz$$

Change of variable: $w = -\sigma z$. Hence, $dz = \frac{1}{\sigma^d} dw$. We get

$$\hat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int f(x+\sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz = \int f(x-w) \underbrace{\frac{1}{(2\pi)^{d/2}\sigma^d} \exp\{-\frac{\|w\|_2^2}{2\sigma^2}\}}_{g_\sigma(w)} dw$$

$$\hat{f}(x) = \int f(x - w) g_\sigma(w) dw = \int f(w) g_\sigma(x - w) dw = f * g_\sigma$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = \frac{1}{(2\pi)^d} \int f(x+\sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz$$

Change of variable: $w = -\sigma z$. Hence, $dz = \frac{1}{\sigma^d} dw$. We get

$$\hat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int f(x+\sigma z) \exp\{-\frac{\|z\|_2^2}{2}\} dz = \int f(x-w) \underbrace{\frac{1}{(2\pi)^{d/2}\sigma^d} \exp\{-\frac{\|w\|_2^2}{2\sigma^2}\}}_{g_\sigma(w)} dw$$

$$\hat{f}(x) = \int f(x-w) g_\sigma(w) dw = \int f(w) g_\sigma(x-w) dw = f * g_\sigma$$

Recall

Let $f : I \to R$ be a continuous and differentiable function over some set $I \subseteq \mathbb{R}^d$, if we have $\|f'(x)\|_2 \leq m$ for all $x \in I$, then m is the upper Lipschitz constant of f ($L \leq m$).

Hence, We calculate $\|\nabla_x \hat{f}(x)\|_2$ in order to find the **upper bound on Lipschitz constant** of $\hat{f}$.

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = f * g_\sigma$$
$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma)$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = f * g_\sigma$$
$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma)$$

Generally speaking, since convolution is a linear operator and since only g depends on x, $\nabla_x (f * g_\sigma) = f * \nabla_x g_\sigma$. Therefore

$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma) = f * \nabla_x g_\sigma$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = f * g_\sigma$$
$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma)$$

Generally speaking, since convolution is a linear operator and since only g depends on x, $\nabla_x (f * g_\sigma) = f * \nabla_x g_\sigma$. Therefore

$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma) = f * \nabla_x g_\sigma$$

Thus, we should compute $\nabla_x g_\sigma$

$$\nabla_x g_\sigma = \nabla_x \left(\frac{1}{(2\pi)^{d/2}\sigma^d} \exp\{-\frac{\|x-w\|_2^2}{2\sigma^2}\}\right) = \frac{1}{(2\pi)^{d/2}\sigma^d} \frac{-2(x-w)}{2\sigma^2} \exp\{-\frac{\|x-w\|_2^2}{2\sigma^2}\}$$
$$= \frac{-(x-w)}{\sigma^2} g_\sigma$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\hat{f}(x) = f * g_\sigma$$
$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma)$$

Generally speaking, since convolution is a linear operator and since only g depends on x, $\nabla_x (f * g_\sigma) = f * \nabla_x g_\sigma$. Therefore

$$\nabla_x \hat{f}(x) = \nabla_x (f * g_\sigma) = f * \nabla_x g_\sigma$$

Thus, we should compute $\nabla_x g_\sigma$

$$\nabla_x g_\sigma = \nabla_x \Big(\frac{1}{(2\pi)^{d/2}\sigma^d} \exp\{-\frac{\|x-w\|_2^2}{2\sigma^2}\}\Big) = \frac{1}{(2\pi)^{d/2}\sigma^d} \frac{-2(x-w)}{2\sigma^2} \exp\{-\frac{\|x-w\|_2^2}{2\sigma^2}\}$$
$$= \frac{-(x-w)}{\sigma^2} g_\sigma$$

We have

$$\nabla_x \hat{f}(x) = \int f(w) \frac{-(x-w)}{\sigma^2} g_\sigma(x-w) dw = \int f(x-w) \frac{-w}{\sigma^2} g_\sigma(w) dw$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\nabla_x \hat{f}(x) = \int f(x-w)\frac{-w}{\sigma^2} g_\sigma(w)dw$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\nabla_x \hat{f}(x) = \int f(x-w)\frac{-w}{\sigma^2} g_\sigma(w)dw$$

Change of variable: $w = -\sigma z$. We get

$$\nabla_x \hat{f}(x) = \int f(x+\sigma z)\frac{\sigma z}{\sigma^2} g_\sigma(-\sigma z)\sigma^d dz = \int f(x+\sigma z)\frac{z}{\sigma}\frac{\sigma^d}{(2\pi)^{d/2}\sigma^d}\exp\{-\frac{\|-\sigma z\|_2^2}{2\sigma^2}\}dz$$

$$= \int f(x+\sigma z)\frac{z}{\sigma}\underbrace{\frac{1}{(2\pi)^{d/2}}\exp\{-\frac{\|z\|_2^2}{2}\}}_{\mathcal{N}(0,I_d)}dz = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x+\sigma z)\frac{z}{\sigma}]$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\nabla_x \hat{f}(x) = \int f(x-w)\frac{-w}{\sigma^2} g_\sigma(w)dw$$

Change of variable: $w = -\sigma z$. We get

$$\nabla_x \hat{f}(x) = \int f(x+\sigma z)\frac{\sigma z}{\sigma^2} g_\sigma(-\sigma z)\sigma^d dz = \int f(x+\sigma z)\frac{z}{\sigma}\frac{\sigma^d}{(2\pi)^{d/2}\sigma^d}\exp\{-\frac{\|-\sigma z\|_2^2}{2\sigma^2}\}dz$$

$$= \int f(x+\sigma z)\frac{z}{\sigma}\underbrace{\frac{1}{(2\pi)^{d/2}}\exp\{-\frac{\|z\|_2^2}{2}\}}_{\mathcal{N}(0,I_d)}dz = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x+\sigma z)\frac{z}{\sigma}]$$

Therefore, for Lipschitz constant of $\hat{f}$, we have

$$L_{\hat{f}} \leq \|\nabla_x \hat{f}(x)\|_2 \Rightarrow L_{\hat{f}} \leq \|\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\underbrace{f(x+\sigma z)}_{\in[0,1]}\frac{z}{\sigma}]\|_2 \leq \|\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma}]\|_2$$

Lipschitz Constant of Randomized Smoothed Classifier

Recal: Triangel inequality

Triangle Inequality. Let $a_k \in \mathbb{R}^d$,

$$|\sum_{k=1}^{N} a_k| \leq \sum_{k=1}^{N} |a_k|.$$

Corollary: For random variable X, if X has a finite expectation, then

$$|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$$

Proof sketch:

$$|\mathbb{E}[X]| = |\sum_{x} x P_X(x)| \underset{\text{Triangle Inequality}}{\leq} \sum_{x} |x| P_X(x) = \mathbb{E}[|X|]$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\nabla_x \hat{f}(x) = \int f(x-w)\frac{-w}{\sigma^2} g_\sigma(w)dw$$

Change of variable: $w = -\sigma z$. We get

$$\nabla_x \hat{f}(x) = \int f(x+\sigma z)\frac{\sigma z}{\sigma^2} g_\sigma(-\sigma z)\sigma^d dz = \int f(x+\sigma z)\frac{z}{\sigma}\frac{\sigma^d}{(2\pi)^{d/2}\sigma^d}\exp\{-\frac{\|-\sigma z\|_2^2}{2\sigma^2}\}dz$$

$$= \int f(x+\sigma z)\frac{z}{\sigma}\underbrace{\frac{1}{(2\pi)^{d/2}}\exp\{-\frac{\|z\|_2^2}{2}\}}_{\mathcal{N}(0,I_d)}dz = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x+\sigma z)\frac{z}{\sigma}]$$

Therefore, for Lipschitz constant of $\hat{f}$, we have

$$L_{\hat{f}} \leq \|\nabla_x \hat{f}(x)\|_2 \Rightarrow L_{\hat{f}} \leq \|\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\underbrace{f(x+\sigma z)}_{\in[0,1]}\frac{z}{\sigma}]\|_2 \leq \|\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma}]\|_2$$

$$\leq \frac{1}{\sigma}\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2]$$

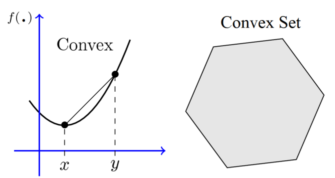
Lipschitz Constant of Randomized Smoothed Classifier

Recall: Convex Function and Jensen's Inequality

Convex set. A set C is convex if the line segment between any two points in C lies in C, i.e., if for any $x_1, x_2 \in C$ and any $\lambda \in [0,1]$, we have: $\lambda x_1 + (1-\lambda)x_2 \in C$

Convex Function. Consider a function $f : I \rightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is a convex set. We say that f is a convex function if, for any two points x and y in I and any $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$



Jensen's Inequality. If $f(x)$ is a convex function on I , and $\mathbb{E}[f(X)]$ and $f(\mathbb{E}[X])$ are finite, then

$$f(E[X]) \leq E[f(X)].$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$\nabla_x \hat{f}(x) = \int f(x-w)\frac{-w}{\sigma^2} g_\sigma(w)dw$$

Change of variable: $w = -\sigma z$. We get

$$\nabla_x \hat{f}(x) = \int f(x+\sigma z)\frac{\sigma z}{\sigma^2} g_\sigma(-\sigma z)\sigma^d dz = \int f(x+\sigma z)\frac{z}{\sigma}\frac{\sigma^d}{(2\pi)^{d/2}\sigma^d}\exp\{-\frac{\|-\sigma z\|_2^2}{2\sigma^2}\}dz$$

$$= \int f(x+\sigma z)\frac{z}{\sigma}\underbrace{\frac{1}{(2\pi)^{d/2}}\exp\{-\frac{\|z\|_2^2}{2}\}}_{\mathcal{N}(0,I_d)}dz = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x+\sigma z)\frac{z}{\sigma}]$$

Therefore, for Lipschitz constant of $\hat{f}$, we have

$$L_{\hat{f}} \leq \|\nabla_x \hat{f}(x)\|_2 \Rightarrow L_{\hat{f}} \leq \|\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\underbrace{f(x+\sigma z)}_{\in[0,1]}\frac{z}{\sigma}]\|_2 \leq \|\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma}]\|_2$$

$$\leq \frac{1}{\sigma}\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2] \leq \frac{1}{\sigma}\sqrt{\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2]}$$

Note: Since $f(x) = x^2$ is a convex function, for random variable X, we have
$f(\mathbb{E}[g(X)]) \underset{\text{Jensen's Inequality}}{\leq} \mathbb{E}[f(g(X))] \Rightarrow \mathbb{E}^2[g(X)] \leq \mathbb{E}[g^2(X)] \Rightarrow \mathbb{E}[g(X)] \leq \sqrt{\mathbb{E}[g^2(X)]}$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$L_{\hat{f}} \leq \frac{1}{\sigma}\sqrt{\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2]}$$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$L_{\hat{f}} \leq \frac{1}{\sigma}\sqrt{\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2]}$$

We know

$$\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2] = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_1^2 + z_2^2 + ... + z_d^2]$$
$$= \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_1^2] + \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_2^2] + ... + \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_d^2] = 1 + 1 + ... + 1 = d$$

Recall: For random variable X, $var(X) = E(X^2) - E[X]^2$

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$L_{\hat{f}} \leq \frac{1}{\sigma}\sqrt{\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2]}$$

We know

$$\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2] = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_1^2 + z_2^2 + ... + z_d^2]$$
$$= \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_1^2] + \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_2^2] + ... + \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_d^2] = 1 + 1 + ... + 1 = d$$

Recall: For random variable X, $var(X) = E(X^2) - E[X]^2$

Finally, we have

$$L_{\hat{f}} \leq \frac{\sqrt{d}}{\sigma}$$

However, since the upper bound depends on the dimension of x, it increases as d grows. As $L_{\hat{f}}$ increases, the radios of our certify bound decreases.

Lipschitz Constant of Randomized Smoothed Classifier

We had

$$L_{\hat{f}} \leq \frac{1}{\sigma}\sqrt{\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2]}$$

We know

$$\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\|z\|_2^2] = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_1^2 + z_2^2 + ... + z_d^2]$$
$$= \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_1^2] + \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_2^2] + ... + \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[z_d^2] = 1 + 1 + ... + 1 = d$$

Recall: For random variable X, $var(X) = E(X^2) - E[X]^2$

Finally, we have

$$L_{\hat{f}} \leq \frac{\sqrt{d}}{\sigma}$$

However, since the upper bound depends on the dimension of x, it increases as d grows. As $L_{\hat{f}}$ increases, the radios of our certify bound decreases.

The main issue is that we calculated the Lipschitz constant working for all x in the input space. We know that the robustness of various inputs is different. Hence, we should compute the **local Lipschitz constant** that depends on specific input x.

Local Lipschitz Constant

Lipschitz Constant

$$\exists L \;\; \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

Lipschitz Constant

$$\exists L \ \ \forall x, y: \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \ \ \exists L_x \ \ \forall y: \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

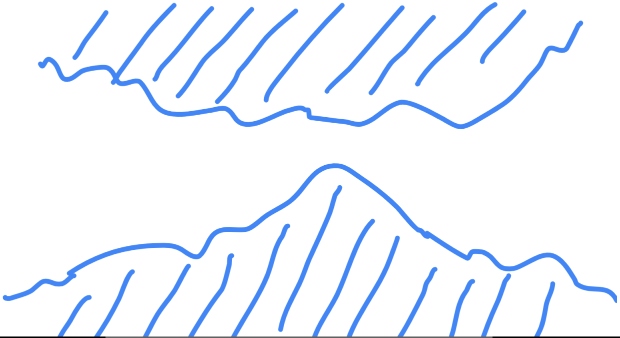
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \ \ \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \ \ \exists L_x \ \ \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



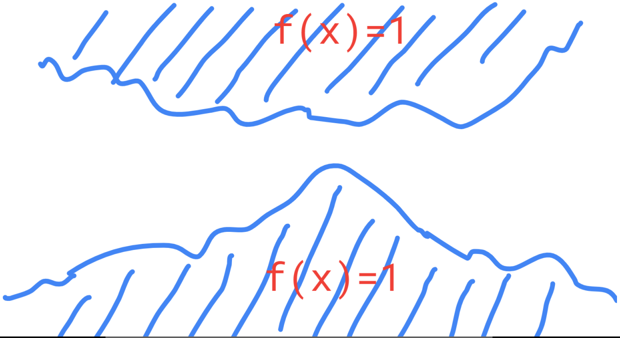
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \ \ \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \ \ \exists L_x \ \ \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



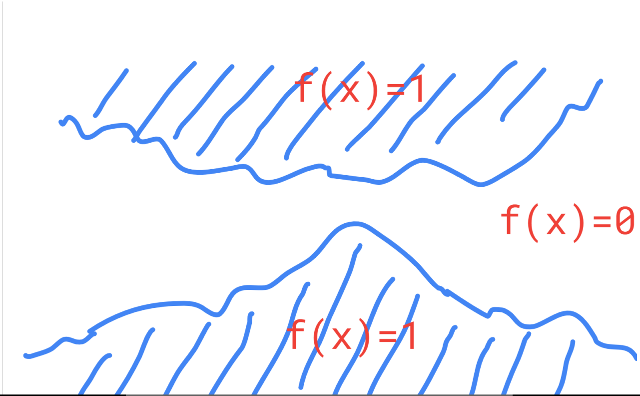
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \;\; \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \;\; \exists L_x \;\; \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



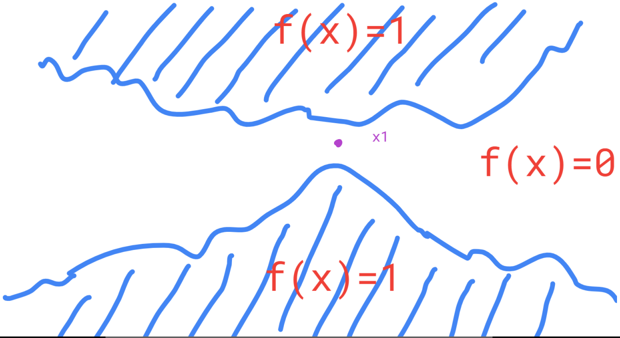
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \ \ \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \ \ \exists L_x \ \ \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



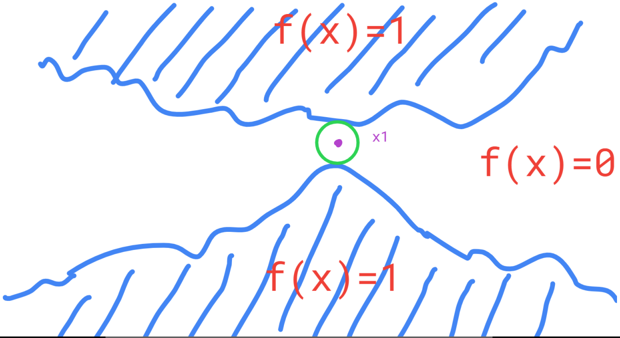
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \ \ \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \ \ \exists L_x \ \ \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



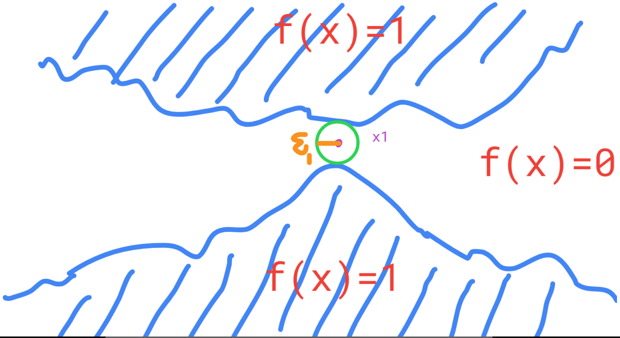
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \;\; \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \;\; \exists L_x \;\; \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



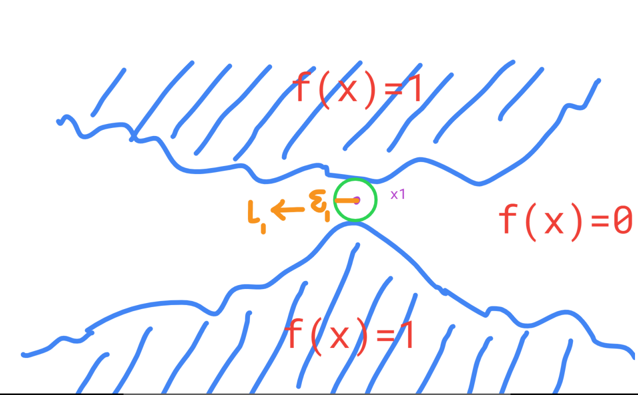
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \quad \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \quad \exists L_x \quad \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



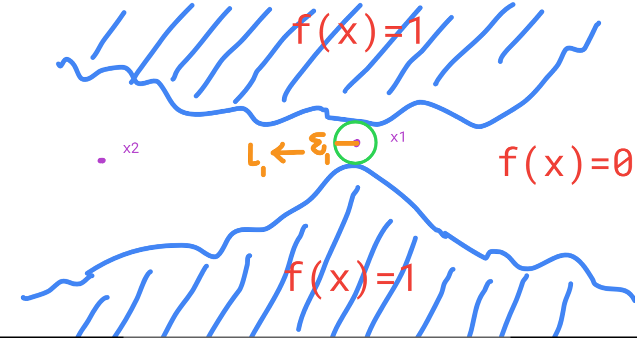
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \ \ \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \ \ \exists L_x \ \ \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



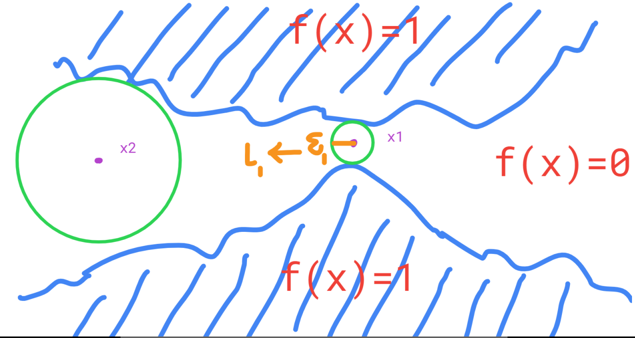
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \ \ \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \ \ \exists L_x \ \ \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



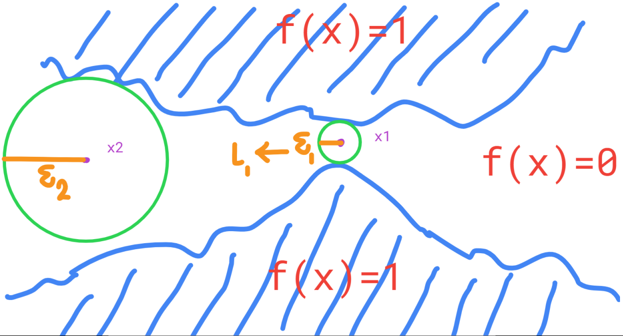
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \ \ \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \ \ \exists L_x \ \ \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



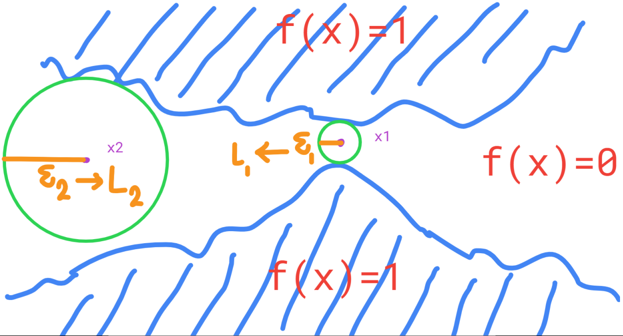
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \;\; \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \;\; \exists L_x \;\; \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



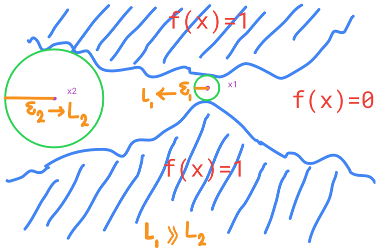
Local Lipschitz Constant

Lipschitz Constant

$$\exists L \;\; \forall x, y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$

Local Lipschitz Constant

$$\forall x \;\; \exists L_x \;\; \forall y : \quad \|f(y) - f(x)\|_2 \leq L_x \|y - x\|_2$$



Local Lipschitz Constant

We can write L_2 as the supremum of an inner product

$$\|\nabla_x \hat{f}\|_2 = \sup_{\|u\|_2=1} u.\nabla_x \hat{f} \quad (\text{where } u^* = \frac{\nabla_x \hat{f}}{\|\nabla_x \hat{f}\|_2})$$

Local Lipschitz Constant

We can write L_2 as the supremum of an inner product

$$\|\nabla_x \hat{f}\|_2 = \underset{\|u\|_2=1}{sup}\ u.\nabla_x \hat{f} \quad (\text{where } u^* = \frac{\nabla_x \hat{f}}{\|\nabla_x \hat{f}\|_2})$$

We know $\nabla_x \hat{f}(x) = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma} f(x+\sigma z)]$. Therefor, we have

$$\|\nabla_x \hat{f}\|_2 = \underset{\|u\|_2=1}{sup}\ u.\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma} f(x+\sigma z)]$$

Local Lipschitz Constant

We can write L_2 as the supremum of an inner product

$$\|\nabla_x \hat{f}\|_2 = \underset{\|u\|_2=1}{sup} \; u.\nabla_x \hat{f} \quad (\text{where } u^* = \frac{\nabla_x \hat{f}}{\|\nabla_x \hat{f}\|_2})$$

We know $\nabla_x \hat{f}(x) = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma} f(x+\sigma z)]$. Therefor, we have

$$\|\nabla_x \hat{f}\|_2 = \underset{\|u\|_2=1}{sup} \; u.\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma} f(x+\sigma z)]$$

Since expectation and inner product are linear operators, we have

$$\|\nabla_x \hat{f}\|_2 = \underset{\|u\|_2=1}{sup} \; \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z.u}{\sigma} f(x+\sigma z)]$$

Local Lipschitz Constant

We can write L_2 as the supremum of an inner product

$$\|\nabla_x \hat{f}\|_2 = \sup_{\|u\|_2=1} u.\nabla_x \hat{f} \quad (\text{where } u^* = \frac{\nabla_x \hat{f}}{\|\nabla_x \hat{f}\|_2})$$

We know $\nabla_x \hat{f}(x) = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma} f(x+\sigma z)]$. Therefor, we have

$$\|\nabla_x \hat{f}\|_2 = \sup_{\|u\|_2=1} u.\mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z}{\sigma} f(x+\sigma z)]$$

Since expectation and inner product are linear operators, we have

$$\|\nabla_x \hat{f}\|_2 = \sup_{\|u\|_2=1} \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z.u}{\sigma} f(x+\sigma z)]$$

We want to $\|\nabla_x \hat{f}\|_2$ becomes input-dependent. We assume $p_x = \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[f(x+\sigma z)]$. Thus, we have

$$\|\nabla_x \hat{f}\|_2 = \sup_{\|u\|_2=1} \mathbb{E}_{z\sim\mathcal{N}(0,I_d)}[\frac{z.u}{\sigma} f(x+\sigma z)]$$

$$\text{such that} \quad \mathbb{E}_z[f(x+\sigma z)] = p_x$$

(See **this video** for the rest of proof.)

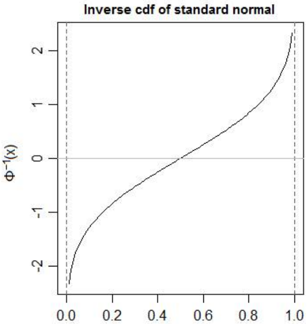
Local Lipschitz Constant

By taking supremum over u and f, the local Lipschitz constant L_x for input x is as follows

$$L_x \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$$

Where Φ is the Cumulative Density Function (CDF) of standard Gaussian distribution $\mathcal{N}(0, I_d)$.

- The bound is input-dependent.
- The bound does not depend on the input dimension d.
- L_x decreases as p_x rises.
- L_x decreases as σ rises.



Local Lipschitz Constant

We introduce $\psi(t)$ so that the Lipschitz constant of $\psi(\hat{f}(x))$ is $\frac{1}{\sigma}$. We have

$$\nabla_x \psi(\hat{f}(x)) = \nabla_x \hat{f}(x).\nabla_{\hat{f}(x)} \psi(\hat{f}(x))$$
$$\|\nabla_x \psi(\hat{f}(x))\|_2 = \|\nabla_x \hat{f}(x)\|_2 |\nabla_{p_x} \psi(p_x)|$$

where $\nabla_{p_x} \psi(p_x)$ is a scalar. We know $\|\nabla_x \hat{f}(x)\|_2 \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$. In order to the Lipschitz constant of $\psi(\hat{f}(x))$ be $\frac{1}{\sigma}$, $\nabla_{p_x} \psi(p_x)$ should be

$$\frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Therefore, we have

$$\psi'(p_x) = \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Local Lipschitz Constant

Recall: The Derivative of the CDF of Standard Normal Distribution

The derivative of the CDF of standard normal distribution $\mathcal{N}(0, I)$

$$\Phi'(t) = \frac{d}{dt}\int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-s^2/2} ds = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

Recall: Fundamental Theorem of Calculus

If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

$$F(x) = \int_{a}^{x} f(t)dt,$$

then $F'(x) = f(x)$ over $[a, b]$.

Another way of stating the conclusion of the fundamental theorem of calculus is:

$$\frac{d}{dx}\int_{a}^{x} f(t)dt = f(x)$$

The conclusion of the fundamental theorem of calculus can be loosely expressed in words as: "the derivative of an integral of a function is that original function", or "differentiation undoes the result of integration".

Local Lipschitz Constant

We introduce $\psi(t)$ so that the Lipschitz constant of $\psi(\hat{f}(x))$ is $\frac{1}{\sigma}$. We have

$$\nabla_x \psi(\hat{f}(x)) = \nabla_x \hat{f}(x).\nabla_{\hat{f}(x)} \psi(\hat{f}(x))$$
$$\|\nabla_x \psi(\hat{f}(x))\|_2 = \|\nabla_x \hat{f}(x)\|_2 |\nabla_{p_x} \psi(p_x)|$$

where $\nabla_{p_x} \psi(p_x)$ is a scalar. We know $\|\nabla_x \hat{f}(x)\|_2 \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$. In order to the Lipschitz constant of $\psi(\hat{f}(x))$ be $\frac{1}{\sigma}$, $\nabla_{p_x} \psi(p_x)$ should be

$$\frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Therefore, we have

$$\psi'(p_x) = \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}} = \left[\Phi'(\Phi^{-1}(p_x))\right]^{-1}$$

Local Lipschitz Constant

Recall:The Derivative of Inverse Function

Given an invertible function $y = f(x)$, the derivative of its inverse function $f^{-1}(y)$ is

$$\left[f^{-1}\right]'(y) = \frac{1}{f'(f^{-1}(y))} = \left[f'(f^{-1}(y))\right]^{-1}$$

Proof

To see why this is true, start with the function $x = f^{-1}(y)$. Write this as $y = f(x)$ and differentiate both sides implicitly with respect to y using the Chain Rule

$$1 = f'(x).\frac{dx}{dy}$$

Thus

$$\frac{dx}{dy} = \frac{1}{f'(x)}$$

but $x = f^{-1}(y)$. Thus,

$$\left[f^{-1}\right]'(y) = \frac{1}{f'\left[f^{-1}(y)\right]}$$

Local Lipschitz Constant

We introduce $\psi(t)$ so that the Lipschitz constant of $\psi(\hat{f}(x))$ is $\frac{1}{\sigma}$. We have

$$\nabla_x \psi(\hat{f}(x)) = \nabla_x \hat{f}(x).\nabla_{\hat{f}(x)} \psi(\hat{f}(x))$$
$$\|\nabla_x \psi(\hat{f}(x))\|_2 = \|\nabla_x \hat{f}(x)\|_2 |\nabla_{p_x} \psi(p_x)|$$

where $\nabla_{p_x} \psi(p_x)$ is a scalar. We know $\|\nabla_x \hat{f}(x)\|_2 \leq \frac{1}{\sqrt{2\pi}\sigma} e^{-\{\Phi^{-1}(p_x)\}^2/2}$. In order to the Lipschitz constant of $\psi(\hat{f}(x))$ be $\frac{1}{\sigma}$, $\nabla_{p_x} \psi(p_x)$ should be

$$\frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}}$$

Therefore, we have

$$\psi'(p_x) = \frac{1}{\frac{1}{\sqrt{2\pi}} e^{-\{\Phi^{-1}(p_x)\}^2/2}} = \left[\Phi'(\Phi^{-1}(p_x))\right]^{-1} = \left[\Phi^{-1}\right]'(p_x)$$

Thus

$$\psi(p_x) = \Phi^{-1}(p_x) + c$$

where c is a constant (we set $c = 0$).

Certifiable Robustness for $1/\sigma$-Lipschitz smoothed classifier

Since, the Lipschitz constant of $\Phi^{-1}(\hat{f}(x))$ is $\frac{1}{\sigma}$, similar to **Theorem 0**, we can show that

$$\underset{i}{\text{argmax}}\ \hat{f}_i(x+\delta) = \underset{i}{\text{argmax}}\ \hat{f}_i(x)$$

for all $||\delta||_2 \leq R$, where

$$R = \frac{1}{2L}(\Phi^{-1}(P_A) - \Phi^{-1}(P_B)) = \frac{\sigma}{2}(\Phi^{-1}(P_A) - \Phi^{-1}(P_B)),$$

$P_A = \underset{i}{max}\ \hat{f}_i(x)$, $P_B = \underset{j \neq i}{max}\ \hat{f}_j(x)$, and $\hat{f}_k(x)$ is the k-th element of the probability vector $\hat{f}(x)$.

Training the Base Classifier

Theorem 1 holds regardless of how the base classifier f is trained.

- If the base classifier f is trained via standard supervised learning on the data distribution, it will **see no noisy images during training**, and hence **will not necessarily learn to classify** $\mathcal{N}(x, \sigma^2 I)$ with x's true label.
- Therefore, we **train the base classifier with Gaussian data augmentation** at variance σ^2.

Experiments

Evaluation metric is **certified test set accuracy** at radius r defined as

- The fraction of the test set which $\hat{f}$ classifies correctly with a prediction that is certifiably robust within an ℓ_2 ball of radius r.
- Thus, to compute **certified accuracy**, we pick a target radius and count the number of points in the test set whose certified radius $r \geq T$ and where the predicted c_A matches the test set label. Standard accuracy is instantiated with $T = 0$.

CIFAR10 model

- ResNet-110
- $n = 100000$ (use n samples from $f(x + \epsilon)$ to obtain some $\underline{P_A}$ and $\overline{P_B}$.)
- Certifying each example took 15 seconds on an NVIDIA RTX 2080 Ti.

ImageNet model

- ResNet-50
- $n = 100000$
- certifying each example took 110 seconds.

Experiments

	$r = 0.0$	$r = 0.5$	$r = 1.0$	$r = 1.5$	$r = 2.0$	$r = 2.5$	$r = 3.0$
$\sigma = 0.25$	**0.67**	**0.49**	0.00	0.00	0.00	0.00	0.00
$\sigma = 0.50$	0.57	0.46	**0.37**	**0.29**	0.00	0.00	0.00
$\sigma = 1.00$	0.44	0.38	0.33	0.26	**0.19**	**0.15**	**0.12**

Table 2. Approximate certified test accuracy on ImageNet. Each row is a setting of the hyperparameter σ, each column is an ℓ_2 radius. The entry of the best σ for each radius is bolded. For comparison, random guessing would attain 0.001 accuracy.

Experiments

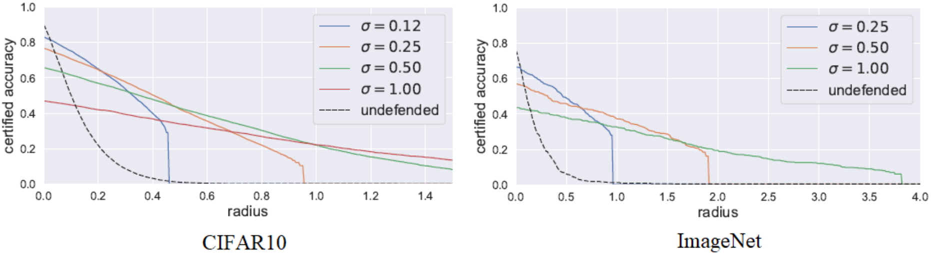


Figure 6. Approximate certified accuracy attained by randomized smoothing on CIFAR-10 (left) and ImageNet (right). The hyper-parameter σ controls a robustness/accuracy tradeoff. The dashed black line is an upper bound on the empirical robust accuracy of an undefended classifier with the base classifier's architecture.

Gaussian Smoothing for L_P Attacks

If we use Gaussian smoothing against L_P attacks, for $p \geq 2$ we get

$$r_p = \frac{\sigma}{2d^{\frac{1}{2}-\frac{1}{p}}}\left(\Phi^{-1}(p_1(x)) - \Phi^{-1}(p_2(x))\right)$$

Curse of dimensionality: For L_P attacks where $p > 2$, the smoothing-based certificate upper bound decreases as d increases.

(See: Curse of Dimensionality on Randomized Smoothing for Certifiable Robustness, Kumar et al., 2020)