Certifiable Robustness

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Today's Agenda

Recap

Obfuscated Gradients

A defense is said to cause gradient masking if it **does not have useful gradients** for generating adversarial examples.

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3. **Exploding & Vanishing Gradients**
   Exploding & Vanishing Gradients are often caused by defenses that consist of **multiple iterations of neural network** evaluation, feeding the output of one computation as the input of the next. This type of computation, when unrolled, can be viewed as an **extremely deep neural network evaluation**, which can cause vanishing/exploding gradients.

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- **Reparameterization** to overcome exploding & vanishing gradients
  - Assume we are given a classifier $f(g(x))$ where $g(.)$ performs some optimization loop to transform the input x to a new input $\hat{x}$.
  - We make a **change-of-variable** $x = h(z)$ for some function $h(.)$ such that $g(h(z)) = h(z)$ for all z, but $h(.)$ is differentiable. This allows us to **compute gradients through** $f(h(z))$ and thereby circumvent the defense.

Case Study: ICLR 2018 Defenses

As a case study for evaluating the prevalence of obfuscated gradients, we study the **ICLR 2018** non-certified defenses that argue robustness in a white-box threat model.

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- To show a defense can be bypassed, it is only necessary to demonstrate one way to do so; in contrast, a defender must show no attack can succeed.
- **Of the 9 accepted papers, 7 rely on obfuscated gradients.**

Defense	Dataset	Distance	Accuracy
Buckman et al. (2018)	CIFAR	0.031 (ℓ_∞)	0%$*$
Ma et al. (2018)	CIFAR	0.031 (ℓ_∞)	5%
Guo et al. (2018)	ImageNet	0.005 (ℓ_2)	0%$*$
Dhillon et al. (2018)	CIFAR	0.031 (ℓ_∞)	0%
Xie et al. (2018)	ImageNet	0.031 (ℓ_∞)	0%$*$
Song et al. (2018)	CIFAR	0.031 (ℓ_∞)	9%$*$
Samangouei et al. (2018)	MNIST	0.005 (ℓ_2)	55%$**$
Madry et al. (2018)	CIFAR	0.031 (ℓ_∞)	47%
Na et al. (2018)	CIFAR	0.015 (ℓ_∞)	15%

Table 1. **Summary of Results:** Seven of nine defense techniques accepted at ICLR 2018 cause obfuscated gradients and are vulnerable to our attacks. Defenses denoted with $*$ propose combining adversarial training; we report here the defense alone, see §5 for full numbers. The fundamental principle behind the defense denoted with $**$ has 0% accuracy; in practice, imperfections cause the theoretically optimal attack to fail, see §5.4.2 for details.

Circumventing Defenses to Adversarial Examples

Adversarial Training

Adversarial training does not cause obfuscated gradients and it **passes all tests** for characteristic behaviors of obfuscated gradients that we list. However it has some limitation

- Adversarial retraining has been shown to be **difficult at ImageNet scale**
- Training exclusively on ℓ_∞ adversarial examples provides only **limited robustness** to adversarial examples under **other distortion metrics**

Thermometer Encoding

Defense Details (Buckman et al. (2018))

- Given an image x, for each pixel color $x_{i,j,c}$, the l-level thermometer encoding $\tau(x_{i,j,c})$ is a l-dimensional vector where $\tau(x_{i,j,c})_k = 1$ if $x_{i,j,c} > \frac{k}{l}$, and 0 otherwise.
  - For a 10-level thermometer encoding, $\tau(0.66) = 1111110000$.

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Discussion

- This defense causes **gradient shattering**
- This can be observed through their **black-box attack evaluation**
  - Adversarial examples generated on a standard adversarially trained model transfer to a thermometer encoded model reducing the accuracy to 67%, well below the 80% robustness to the white-box iterative attack.

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Evaluation

- We use the **BPDA** approach where we let $f(x) = \tau(x)$. Observe that if we define

$$\hat{\tau}(x_{i,j,c})_k = min(max(\frac{x_{i,j,c}}{k/l}, 0), 1)$$

then

$$\tau(x_{i,j,c})_k = floor(\hat{\tau}(x_{i,j,c})_k)$$

So we can let $g(x) = \hat{\tau}(x)$ and replace the backwards pass with the function $g(.)$.

Thermometer Encoding Evaluation

Accuracy of various models on adversarial examples.

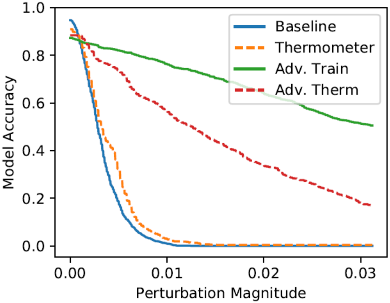

Figure 1. Model accuracy versus distortion (under ℓ_∞). Adversarial training increases robustness to 50% at $\epsilon = 0.031$; thermometer encoding by itself provides limited value, and when coupled with adversarial training performs worse than adversarial training alone.

Input Transformation

Guo et al. (2018) propose five input transformations to counter adversarial examples.

1. Image cropping and rescaling
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4. Total variance minimization
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To overcome transformations

- We circumvent **image cropping and rescaling** with a direct application of **EOT**.
- To circumvent **bit-depth reduction and JPEG compression**, we use BPDA and approximate the **backward pass with the identity function**.
- To circumvent **total variance minimization and image quilting**, which are both non-differentiable and randomized, we apply **EOT** and use **BPDA** to approximate the gradient through the transformation.

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Defenses

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To circumventing both defenses, we estimate the gradients by computing the expectation over instantiations of randomness.

- At each iteration of gradient descent, instead of taking a step in the direction of $\nabla_x f(x)$ we move in the direction of $\sum_{i=1}^{k} \nabla_x f(x)$.

PIXELDEFEND (Exploding and Vanishing Gradient)

Song et al. (2018) propose using a PixelCNN generative model to project a potential adversarial example back onto the data manifold before feeding it into a classifier.

- PixelDefend **purifies** adversarially perturbed images prior to classification by using a greedy decoding procedure to approximate finding the highest probability example within an ϵ-ball of the input image.

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To circumventing PIXELDEFEND

- We sidestep the problem of computing gradients through an unrolled version of PixelDefend by approximating gradients with BPDA (**approximate PixelCNN derivative as the derivative of the identity function**).

Randomized Smoothing

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Certified Adversarial Robustness via Randomized Smoothing

Jeremy Cohen [1] **Elan Rosenfeld** [1] **J. Zico Kolter** [1 2]

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- Classifier $f : \mathbb{R}^d \to [0,1]^K$ is **ϵ-robust** at x, if

$$\forall \ \|\delta\|_p \leq \epsilon, \quad \underset{i \in [K]}{argmax} \ f_i(x+\delta) = \underset{i \in [K]}{argmax} \ f_i(x)$$

where K is the number of classes.

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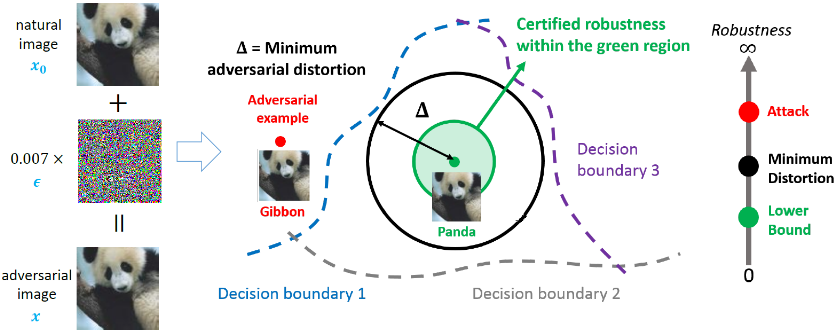

Lipschitz continuity

A function $f : I \rightarrow R$ over some set $I \subseteq \mathbb{R}^d$ is called Lipschitz continuous if there exists a positive real constant L such that, for all $x, y \in I$,

$$|f(y) - f(x)| \leq L\|y - x\|_2$$

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Generally, Let $f = f_1 \circ f_2 \circ .... \circ f_K$ and the Lipschitz constant of f_i be L_i for all $i \in \{1, 2, ..., K\}$, then the Lipschitz constant of f is $L \leq \prod_{k=1}^{K} L_k$.

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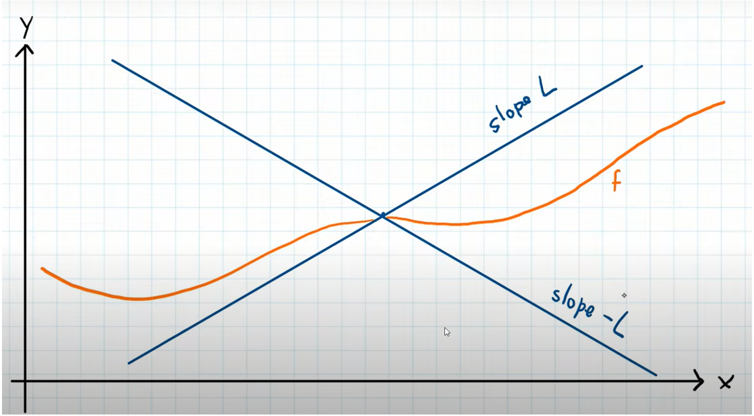

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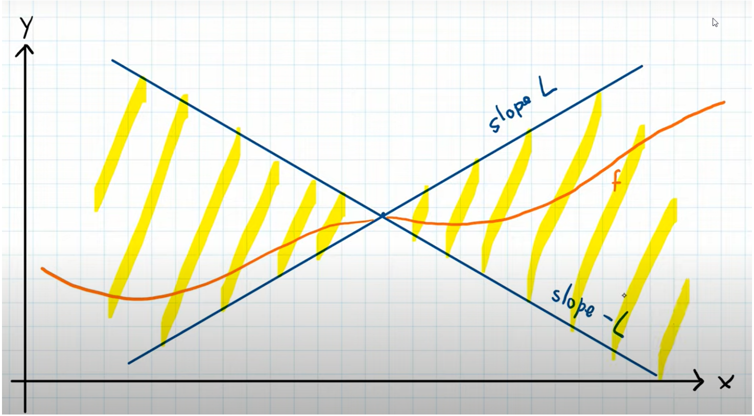

Lipschitz continuity

Let $f : I \rightarrow R$ be a continuous and differntiable function over some set $I \subseteq \mathbb{R}^d$, if we have $\|f'(x)\|_2 \leq m$ for all $x \in I$, then m is the upper Lipschitz constant of f ($L \leq m$).

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Proof sketch:
Mean value theorem: Let $f : I \to R$ be a continuous and differntiable function over some set $I \subseteq \mathbb{R}^d$, For all $a, b \in I$ $(b > a)$, there exists some $c \in (a, b)$ such that:

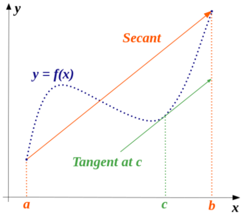
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

For all $a, b \in I$, there exist $c \in (a, b)$, such that:

$$|f(b) - f(a)| = \|f'(c).b - a\|_2 \leq \|f'(c)\|_2 \|b - a\|_2.$$

Since we know that $\|f'(c)\|_2 \leq m$, we have

$$|f(b) - f(a)| \leq m\|b - a\|_2.$$



Certifiable Robustness for L-Lipschitz classifier

Theorem 0: If $f : \mathbb{R}^d \to [0,1]^K$ is L-lipschitz, then f is ϵ-robust at x with $\epsilon = \frac{1}{2L}(P_A - P_B)$, where $P_A = \underset{i}{max}\ f_i(x)$, $P_B = \underset{j \neq i}{max}\ f_j(x)$, and $f_k(x)$ is the k-th element of the probability vector $f(x)$.

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Proof.

Since f is L-lipschitz, we have

$$\forall x, y \in \mathbb{R}^d, \|f(y) - f(x)\|_2 \leq L\|y - x\|_2$$

Denote $x' = x + \delta$ and assume that $\|\delta\| \leq \epsilon$. we get

$$\|f(x') - f(x)\|_2 \leq L\|x' - x\|_2 \rightarrow \|f(x') - f(x)\|_2 \leq L\|\delta\|_2 \leq L\epsilon$$

Hence, P_A can be reduced at most by $L\epsilon$ and P_B can be increased at most by $L\epsilon$. We have ($P_i' = f_i(x')$)

$$P_A' \geq P_A - L\epsilon \quad \text{and} \quad P_B' \leq P_B + L\epsilon$$

Since we want that the label of x' be the same as x, P_A' must be greater than P_B' ($P_A' \geq P_B'$). We have

$$P_A - L\epsilon \geq P_B + L\epsilon \ \rightarrow \ 2L\epsilon \leq P_A - P_B \ \rightarrow \ \epsilon \leq \frac{1}{2L}(P_A - P_B)$$

$\square$

Smoothed Classifier

If we compute the (upper bound of) Lipschitz constant of the classifier, we can determine the radius (ϵ) of the robustness for each sample.

- However, **the lipschitz constant of deep neural networks** is very large and it **grows exponentially** with the number of layers. Hence, the certification is useless ($\epsilon \rightarrow 0$).

Smoothed Classifier

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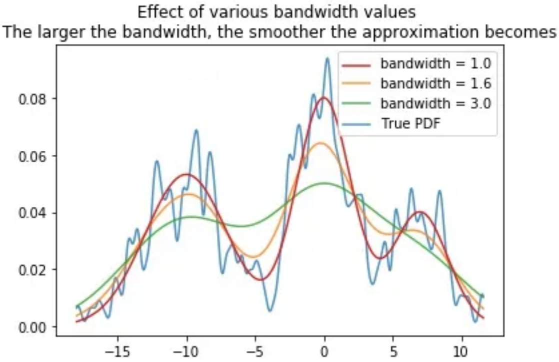

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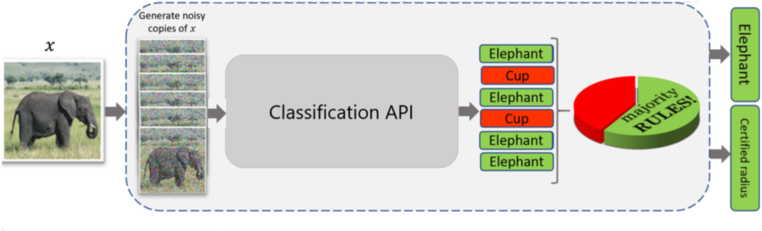
- When queried at x, the smoothed classifier $\hat{f}$ returns whichever class the base classifier f is most likely to return when x is perturbed by isotropic Gaussian noise:

$$\begin{aligned} \hat{f}(x) &= \underset{c \in \mathcal{Y}}{argmax}\ \mathbb{P}(f(x+\epsilon) = c) \\ \text{where} \quad & \epsilon \sim \mathcal{N}(0, \sigma^2 I) \end{aligned} \tag{1}$$

The noise level σ is a hyperparameter of the smoothed classifier $\hat{f}$ which controls a robustness/accuracy tradeoff.



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Notation

Suppose that when the base classifier f classifies $\mathcal{N}(x, \sigma^2 I)$, the most probable class c_A is returned with probability P_A, and the "runner-up" class is returned with probability P_B.

- $\underline{P_A}$ is a lower bound for P_A and $\overline{P_B}$ is a lower bound for P_B.

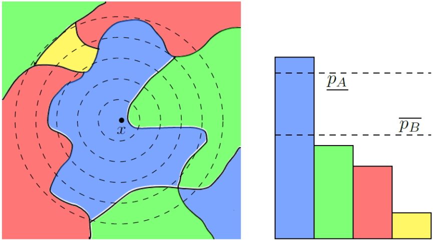

Figure 1. Evaluating the smoothed classifier at an input x. **Left**: the decision regions of the base classifier f are drawn in different colors. The dotted lines are the level sets of the distribution $\mathcal{N}(x, \sigma^2 I)$. **Right**: the distribution $f(\mathcal{N}(x, \sigma^2 I))$. As discussed below, $\underline{p_A}$ is a lower bound on the probability of the top class and $\overline{p_B}$ is an upper bound on the probability of each other class. Here, $g(x)$ is "blue."

Robustness guarantee

Theorem 1. Let $f : \mathbb{R}^d \to \mathcal{Y}$ be any deterministic or random function, and let $\epsilon \sim \mathcal{N}(0, \sigma^2 I)$. Let $\hat{f}$ be defined as in (1). Suppose $C_A \in \mathcal{Y}$ and $\underline{P_A}, \overline{P_B} \in [0, 1]$ satisfy:

$$\mathbb{P}(f(x+\epsilon) = C_A) \geq \underline{P_A} \geq \overline{P_B} \geq \max_{C \neq C_A} \mathbb{P}(f(x+\epsilon) = C)$$

Then $\hat{f}(x+\delta) = C_A$ for all $\|\delta\|_2 \leq R$, where

$$R = \frac{\sigma}{2}(\Phi^{-1}(\underline{P_A}) - \Phi^{-1}(\overline{P_B}))$$

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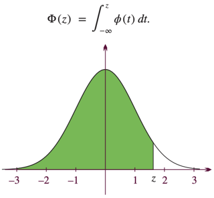
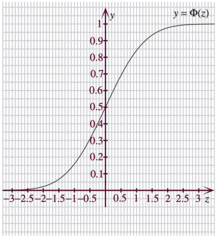
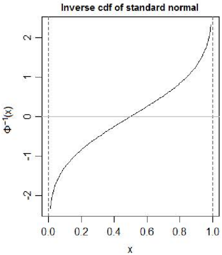


This is the graph of the standard normal probability density function $\phi(z)$.



This is the graph of the standard normal cumulative distribution function $\Phi(z)$.



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- The certified radius R is large when: (1) the noise level σ is high, (2) the probability of the top class C_A is high, and (3) the probability of each other class is low.
- **The certified radius R goes to ∞ as $\underline{P_A} \to 1$ and $\overline{P_B} \to 0$.** This should sound reasonable: the Gaussian distributi is supported on all of $\mathbb{R}^d$, so the only way that $f(x + \epsilon) = C_A$ with probability 1 is if $f = C_A$ almost everywhere.