Differential Privacy

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Today's Agenda

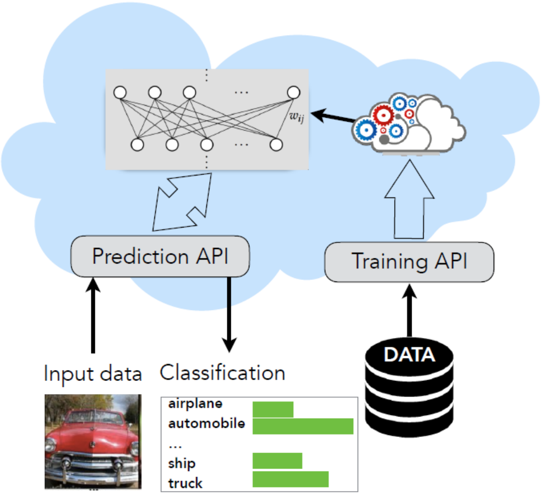
Recap

Machine Learning as a Service

Machine Learning as a Service

Google Cloud Platform

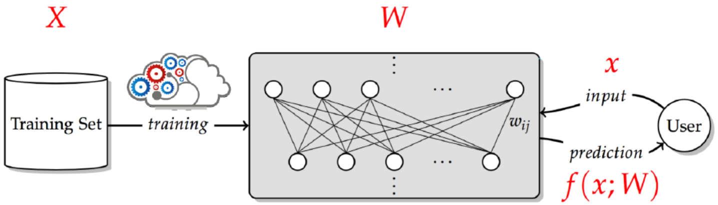
amazon web services™



(Shokri, 2020)

Privacy Risks in Machine Learning

- What is training data leakage? Inferring information about members of X, beyond what can be learned about its underlying distribution.



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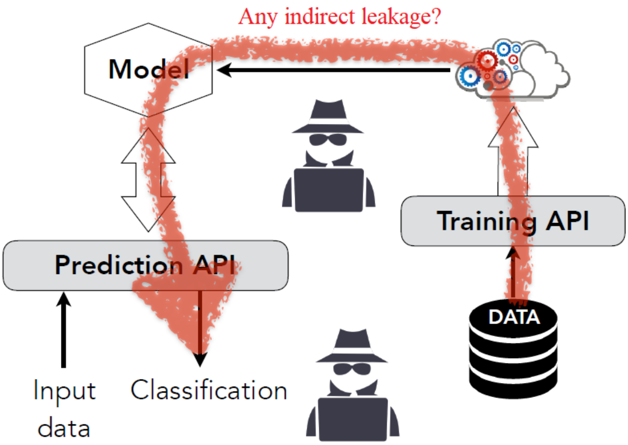
Membership Inference Attack

- Given a model, can an adversary infer whether data point x is part of its training set?



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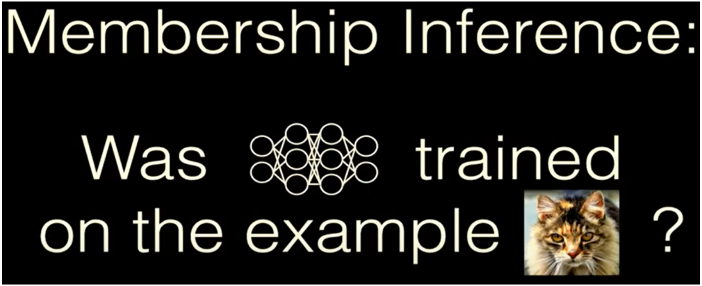

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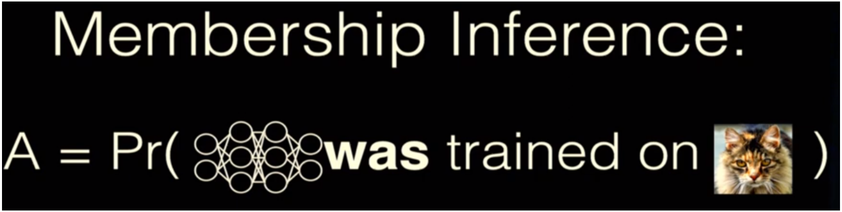

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Differential Privacy

Differential Privacy

source: http://www.recode.net/2016/6/15/11940908/mossberg-apple-is-still-a-world-of-its-own

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Differential privacy describes a promise, made by a data holder, or curator, to a data subject:

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Differential privacy addresses the paradox of **learning nothing about an individual** while **learning useful information about a population**.

Privacy

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- Perhaps
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Differential privacy: the impact on the smoker is the same **independent** of whether or not he was in the study.

- Differential privacy promises that the probability of harm/benefit **was not significantly increased** by their choice to participate.

Defining Private Data Analysis

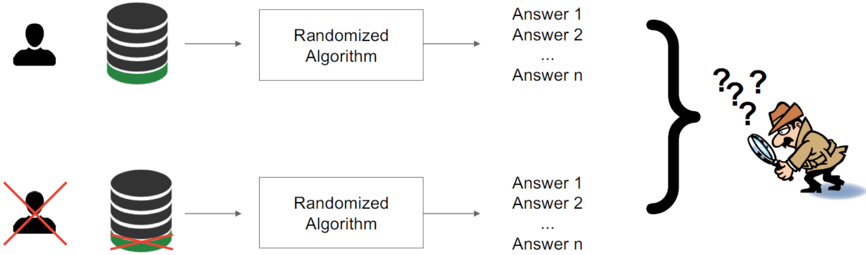
The analyst **knows no more about any individual** in the data set **after the analysis** is completed than she knew before the analysis was begun.

- The adversary's **prior and posterior views about an individual** (i.e., before and after having access to the database) **shouldn't be too different**"
  - Access to the database shouldn't change the adversary's views about any individual **too much**.

Differential Privacy

Differential Privacy ensures that any sequence of outputs (responses to queries) is **essentially equally likely** to occur, **independent** of the presence or absence of **any individual**.

- If **nothing is learned about an individual** then the individual **cannot be harmed by the analysis**.



(Papernot , 2019)

Randomized Response

A technique developed in the social sciences to collect **statistical information about embarrassing or illegal behavior**

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Let p is the true fraction of participants having property P the expected number of "Yes" answers is

$$\mathbb{E}(\text{"}yes\text{"}|p) = \frac{1}{4}(1-p) + \frac{3}{4}p = \frac{1}{4} + \frac{p}{2} \rightarrow p = 2\mathbb{E}(\text{"}yes\text{"}|p) - \frac{1}{2}$$

Randomization Is Essential

Suppose, for the sake of contradiction, that we have a non-trivial deterministic algorithm.

- Non-triviality says that there exists a query and two databases that yield different outputs under this query.

Changing one row at a time we see there exists a **pair of databases differing only in the value** of a single row, on which the **same query yields different outputs**.

- An adversary knowing that the database is one of these two almost identical databases learns the value of the data in the unknown row.

name	DOB	sex	weight	smoker	lung cancer
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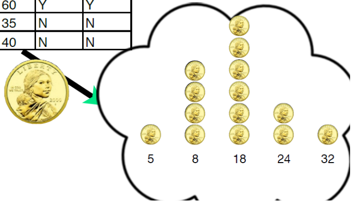

(Katrina Ligett, 2017)

A Randomized Algorithm

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Definition 2.1 (Probability Simplex)**.** Given a discrete set B, the *probability simplex* over B, denoted $\Delta(B)$ is defined to be:

$$\Delta(B) = \left\{ x \in \mathbb{R}^{|B|} : x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{|B|} x_i = 1 \right\}$$

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Definition 2.2 (Randomized Algorithm). A randomized algorithm $\mathcal{M}$ with domain A and discrete range B is associated with a mapping $M : A \to \Delta(B)$. On input $a \in A$, the algorithm $\mathcal{M}$ outputs $\mathcal{M}(a) = b$ with probability $(M(a))_b$ for each $b \in B$. The probability space is over the coin flips of the algorithm $\mathcal{M}$.

Database

We will think of databases x as being collections of records from a universe $\mathcal{X}$. It will often be convenient to represent databases by their histograms: $x \in \mathbb{N}^{|\mathcal{X}|}$, in which each entry x_i represents the number of elements in the database x of type $i \in \mathcal{X}$.

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Distance Between Databases

Definition 2.3 (Distance Between Databases)**.** The ℓ_1 norm of a database x is denoted $\|x\|_1$ and is defined to be:

$$\|x\|_1 = \sum_{i=1}^{|\mathcal{X}|} |x_i| \, .$$

The ℓ_1 distance between two databases x and y is $\|x - y\|_1$

Note that $\|x\|_1$ is a measure of the *size* of a database x (i.e., the number of records it contains), and $\|x - y\|_1$ is a measure of how many records *differ* between x and y.

Differential Privacy

Differential Privacy: A randomized algorithm $\mathcal{M}$ with domain $\mathbb{N}^{|\mathcal{X}|}$ is (ϵ,δ)-differentially private if for all $\mathcal{S} \subseteq \text{Range}(\mathcal{M})$ and for all $x, y \in \mathbb{N}^{|\mathcal{X}|}$ such that $\|x - y\|_1 \leq 1$:

$$\Pr[\mathcal{M}(x) \in \mathcal{S}] \approx_\epsilon \Pr[\mathcal{M}(y) \in \mathcal{S}]$$

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- $\epsilon \approx 1$ and $\delta \ll \frac{1}{N}$ (generally speaking, one digit ϵ is good) where N is the size of dataset.

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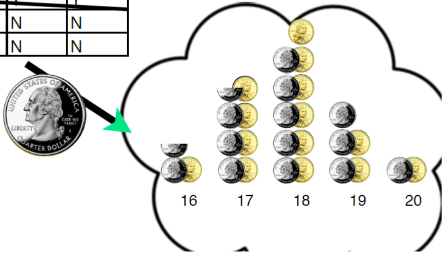

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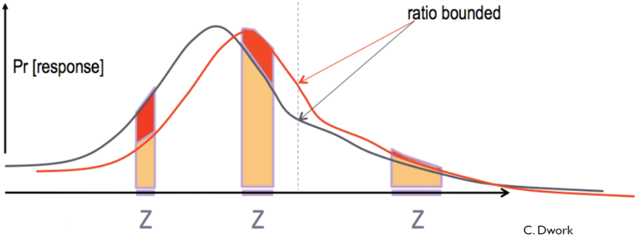
Differential Privacy

ϵ-differential privacy

$$\Pr[M(x_1) \in S] \leq e^{\varepsilon} \Pr[M(x_2) \in S]$$



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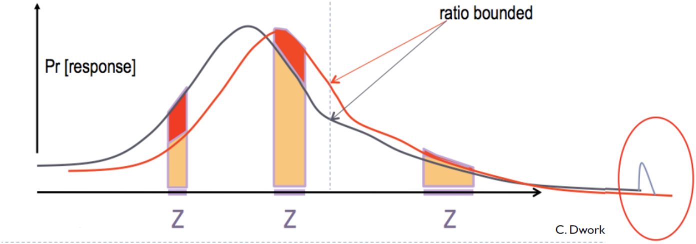

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Privacy Loss

(ϵ, δ)-differential privacy ensures that for all adjacent x, y, the **absolute value of the privacy loss will be bounded by** ϵ with probability at least $1 - \delta$.

The quantity

$$\mathcal{L}^{(\xi)}_{\mathcal{M}(x)\|\mathcal{M}(y)} = \ln\left(\frac{\Pr[\mathcal{M}(x) = \xi]}{\Pr[\mathcal{M}(y) = \xi]}\right)$$

is important to us; we refer to it as the *privacy loss* incurred by observing ξ. This loss might be positive (when an event is more likely under x than under y) or it might be negative (when an event is more likely under y than under x).

What is the privacy loss of Randomized response mechanism?

Claim 3.5. The version of randomized response described above is $(\ln 3, 0)$-differentially private.

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Claim 3.5. The version of randomized response described above is $(\ln 3, 0)$-differentially private.

Proof. Fix a respondent. A case analysis shows that $\Pr[\text{Response} = \text{Yes}|\text{Truth} = \text{Yes}] = 3/4$. Specifically, when the truth is "Yes" the outcome will be "Yes" if the first coin comes up tails (probability $1/2$) or the first and second come up heads (probability $1/4$)), while $\Pr[\text{Response} = \text{Yes}|\text{Truth} = \text{No}] = 1/4$ (first comes up heads and second comes up tails; probability $1/4$). Applying similar reasoning to the case of a "No" answer, we obtain:

$$\frac{\Pr[\text{Response} = \text{Yes}|\text{Truth} = \text{Yes}]}{\Pr[\text{Response} = \text{Yes}|\text{Truth} = \text{No}]}$$

$$= \frac{3/4}{1/4} = \frac{\Pr[\text{Response} = \text{No}|\text{Truth} = \text{No}]}{\Pr[\text{Response} = \text{No}|\text{Truth} = \text{Yes}]} = 3.$$

$\square$

Laplace Mechanism

Sensitivity

The ℓ_1 sensitivity of a function f captures the magnitude by which a single individual's data can change the function f in the worst case

- Intuitively, the uncertainty in the response that we must introduce in order to hide the participation of a single individual.

The sensitivity of a function gives an upper bound on how much we must perturb its output to preserve privacy.

Definition 3.1 (ℓ_1-sensitivity). The ℓ_1-sensitivity of a function $f : \mathbb{N}^{|\mathcal{X}|} \to \mathbb{R}^k$ is:

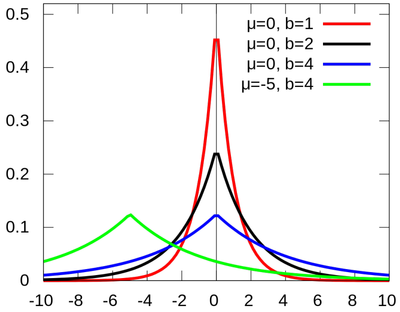
$$\Delta f = \max_{\substack{x,y \in \mathbb{N}^{|\mathcal{X}|} \\ \|x-y\|_1 = 1}} \|f(x) - f(y)\|_1.$$

The Laplace Distribution

Definition 3.2 (The Laplace Distribution). The Laplace Distribution (centered at 0) with scale b is the distribution with probability density function:

$$\text{Lap}(x|b) = \frac{1}{2b} \exp\left(-\frac{|x|}{b}\right).$$

The variance of this distribution is $\sigma^2 = 2b^2$. We will sometimes write $\text{Lap}(b)$ to denote the Laplace distribution with scale b.



The Laplace Mechanism

Definition 3.3 (The Laplace Mechanism). Given any function $f : \mathbb{N}^{|\mathcal{X}|} \to \mathbb{R}^k$, the Laplace mechanism is defined as:

$$\mathcal{M}_L(x, f(\cdot), \varepsilon) = f(x) + (Y_1, \ldots, Y_k)$$

where Y_i are i.i.d. random variables drawn from $\mathrm{Lap}(\Delta f/\varepsilon)$.

The Laplace Mechanism

Theorem 3.6. The Laplace mechanism preserves $(\varepsilon, 0)$-differential privacy.

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Proof. Let $x \in \mathbb{N}^{|\mathcal{X}|}$ and $y \in \mathbb{N}^{|\mathcal{X}|}$ be such that $\|x - y\|_1 \leq 1$, and let $f(\cdot)$ be some function $f : \mathbb{N}^{|\mathcal{X}|} \to \mathbb{R}^k$. Let p_x denote the probability density function of $\mathcal{M}_L(x, f, \varepsilon)$, and let p_y denote the probability density function of $\mathcal{M}_L(y, f, \varepsilon)$. We compare the two at some arbitrary point $z \in \mathbb{R}^k$

$$\frac{p_x(z)}{p_y(z)} = \prod_{i=1}^{k} \left(\frac{\exp(-\frac{\varepsilon|f(x)_i - z_i|}{\Delta f})}{\exp(-\frac{\varepsilon|f(y)_i - z_i|}{\Delta f})} \right) = \prod_{i=1}^{k} \exp\left(\frac{\varepsilon(|f(y)_i - z_i| - |f(x)_i - z_i|)}{\Delta f} \right)$$

$$\leq \prod_{i=1}^{k} \exp\left(\frac{\varepsilon|f(x)_i - f(y)_i|}{\Delta f} \right) = \exp\left(\frac{\varepsilon \cdot \|f(x) - f(y)\|_1}{\Delta f} \right) \leq \exp(\varepsilon),$$

where the first inequality follows from the triangle inequality, and the last follows from the definition of sensitivity and the fact that $\|x - y\|_1 \leq 1$. That $\frac{p_x(z)}{p_y(z)} \geq \exp(-\varepsilon)$ follows by symmetry. $\square$

Example: Counting Queries

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A fixed but arbitrary list of m **counting queries** can be viewed as a vector-valued query.

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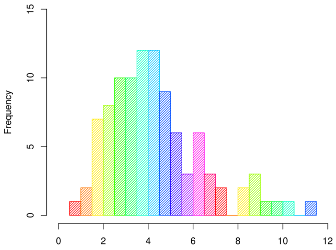
A fixed but arbitrary list of m **counting queries** can be viewed as a vector-valued query.

- The sensitivity is m
- $(\epsilon, 0)$-differential privacy can be achieved by adding noise scaled to m/ϵ to the true answer to each query.

Histogram Queries

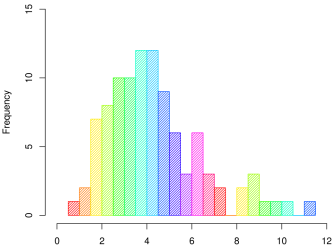
In this type of query the universe $\mathbb{N}^{|\mathcal{X}|}$ is **partitioned into cells**, and the query asks how many database elements lie in each of the cells.



Histogram Queries

In this type of query the universe $\mathbb{N}^{|\mathcal{X}|}$ is **partitioned into cells**, and the query asks how many database elements lie in each of the cells.

- Because the cells are disjoint, the addition or removal of a single database element can affect the count in exactly one cell. Hence the **sensitivity is 1**.
- $(\epsilon, 0)$-differential privacy can be achieved by adding noise scaled to $1/\epsilon$, that is, by adding noise drawn from $\mathrm{Lap}(1/\epsilon)$ to the true count in each cell.



Report Noisy Max

Report Noisy Max. Consider the following simple algorithm to determine which of m counting queries has the highest value: Add independently generated Laplace noise $\text{Lap}(1/\varepsilon)$ to each count and return the index of the largest noisy count (we ignore the possibility of a tie). Call this algorithm Report Noisy Max.

Claim 3.9. The Report Noisy Max algorithm is $(\varepsilon, 0)$-differentially private.

The Accuracy of the Laplace Mechanism

Fact 3.7. If $Y \sim \text{Lap}(b)$, then:

$$\Pr[|Y| \geq t \cdot b] = \exp(-t).$$

Theorem 3.8. Let $f : \mathbb{N}^{|\mathcal{X}|} \to \mathbb{R}^k$, and let $y = \mathcal{M}_L(x, f(\cdot), \varepsilon)$. Then $\forall \delta \in (0, 1]$:

$$\Pr\left[\|f(x) - y\|_\infty \geq \ln\left(\frac{k}{\delta}\right) \cdot \left(\frac{\Delta f}{\varepsilon}\right)\right] \leq \delta$$

Proof. We have:

$$\begin{aligned}
\Pr\left[\|f(x) - y\|_\infty \geq \ln\left(\frac{k}{\delta}\right) \cdot \left(\frac{\Delta f}{\varepsilon}\right)\right] &= \Pr\left[\max_{i \in [k]} |Y_i| \geq \ln\left(\frac{k}{\delta}\right) \cdot \left(\frac{\Delta f}{\varepsilon}\right)\right] \\
&\leq k \cdot \Pr\left[|Y_i| \geq \ln\left(\frac{k}{\delta}\right) \cdot \left(\frac{\Delta f}{\varepsilon}\right)\right] \\
&= k \cdot \left(\frac{\delta}{k}\right) \\
&= \delta
\end{aligned}$$

where the second to last inequality follows from the fact that each $Y_i \sim \text{Lap}(\Delta f / \varepsilon)$ and Fact 3.7. $\square$

Example: First Names

Suppose we wish to calculate which first names, from a list of 10,000 potential names, were the most common among participants of the 2010 census.

- This question can be represented as a query $f : \mathbb{N}^{|\mathcal{X}|} \rightarrow \mathbb{R}^{10000}$

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We can simultaneously calculate the frequency of all 10000 names with $(\epsilon = 1, \delta = 0)$-differential privacy $(y = \mathcal{M}(\mathbb{N}^{|\mathcal{X}|}, f))$, and with probability 95% $(\delta = 0.05)$.

$$Pr[\|f(x) - y\|_\infty \geq \ln(\frac{k}{\delta}).(\frac{\Delta f}{\epsilon})] \leq \delta$$

$$Pr[\|f(x) - y\|_\infty \geq \ln(\frac{10000}{0.05}).(\frac{1}{1})] \leq 0.05$$

$$Pr[\|f(x) - y\|_\infty \geq 12.2] \leq 0.05$$

Hence, no estimate will be off by more than an additive error of $\ln(10000/.05) \approx 12.2$, with probability 95%.

- That's pretty low error for a nation of more than 300,000,000 people!

The Gaussian Mechanism

Let $f : \mathbb{N}^{|\mathcal{X}|} \to \mathbb{R}^d$ be an arbitrary d-dimensional function, and define its ℓ_2 sensitivity to be $\Delta_2 f = \max_{\text{adjacent}x,y} \|f(x) - f(y)\|_2$. The *Gaussian Mechanism with parameter* σ adds noise scaled to $\mathcal{N}(0, \sigma^2)$ to each of the d components of the output.

Theorem A.1. Let $\varepsilon \in (0, 1)$ be arbitrary. For $c^2 > 2\ln(1.25/\delta)$, the Gaussian Mechanism with parameter $\sigma \geq c\Delta_2 f/\varepsilon$ is (ε, δ)-differentially private.

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