MTH 256 Supplement

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Mathematics Faculty
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August 10, 2022

Contents

Chapter 1

A First Look at Differential Equations

1.1 Modeling with Differential Equations

Example 1.1.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 1.1.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$

across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f(\frac{1}{2}x)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f(\frac{1}{2}x)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right

7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x} \qquad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2 \qquad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

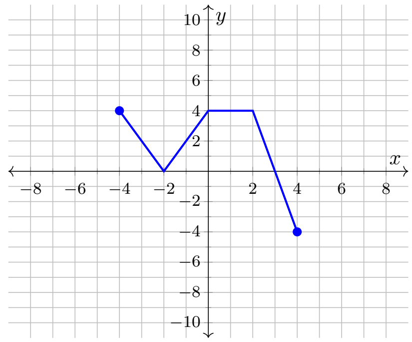
9. $f(x) = \sqrt[3]{x} \qquad g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

Solution.

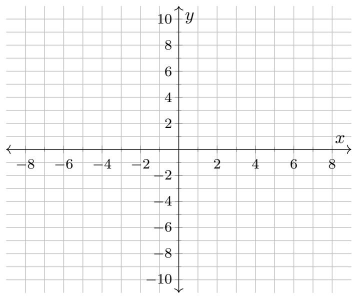
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

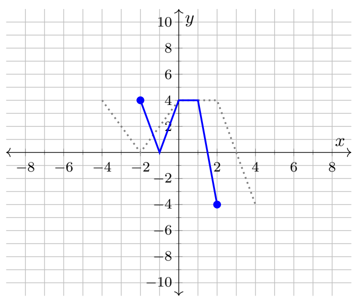
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



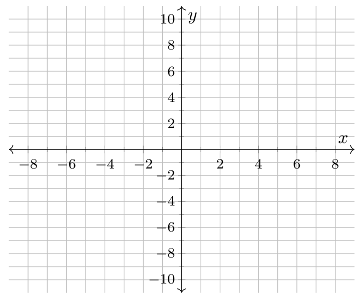
10. $k_1(x) = f(2x)$



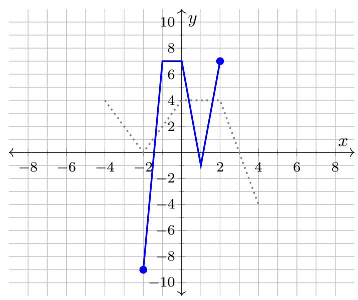
Answer.



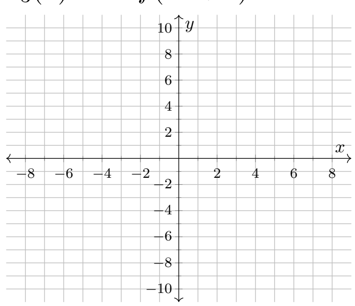
11. $k_2(x) = 2f(-2x) - 1$



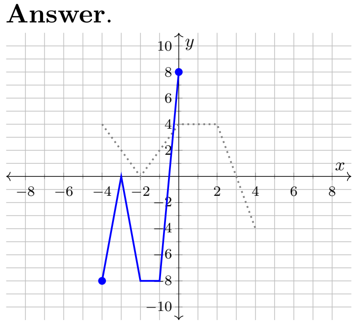
Answer.



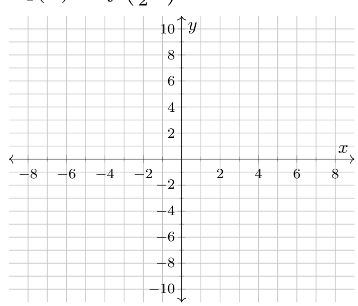
12. $k_3(x) = -2f(2x + 4)$



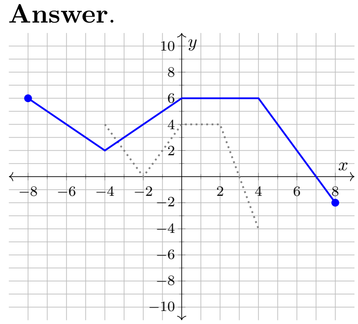
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



1.2 Seperable Differential Equations

Example 1.2.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 1.2.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f(\frac{1}{2}x)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f(\frac{1}{2}x)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\qquad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x}$ $\qquad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2 \qquad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

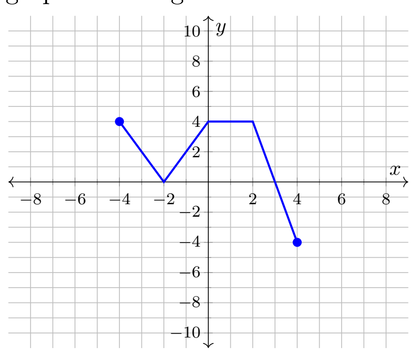
9. $f(x) = \sqrt[3]{x} \qquad g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

Solution.

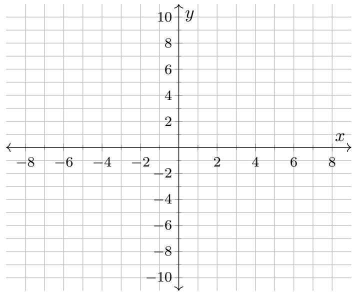
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

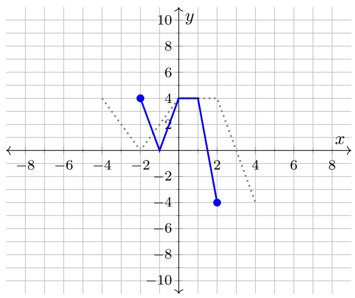
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



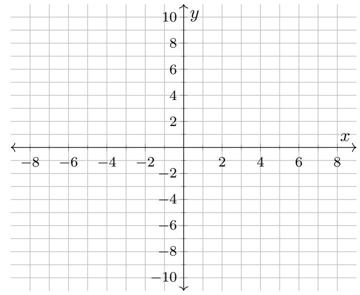
10. $k_1(x) = f(2x)$



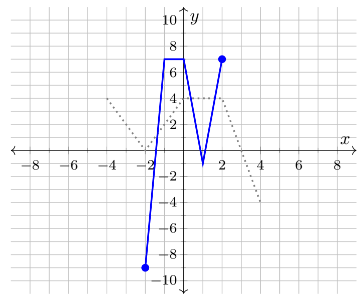
Answer.



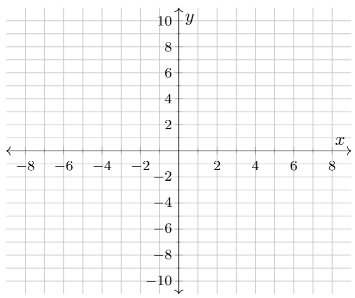
11. $k_2(x) = 2f(-2x) - 1$



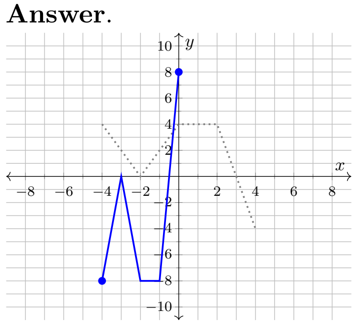
Answer.



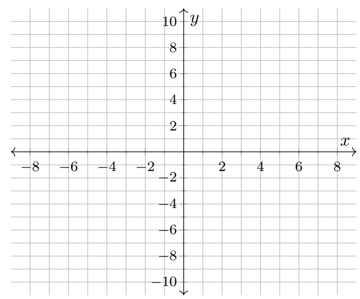
12. $k_3(x) = -2f(2x+4)$



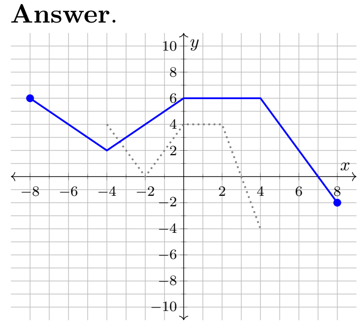
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



1.3 Goemetric and Quantitative Analysis

Example 1.3.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 1.3.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x+5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x+5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f\left(\frac{1}{2}x\right)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x}$ $\quad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\quad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x-5\right)^2+3 \\ &= -4f\left(\frac{1}{2}x-5\right)+3 \\ &= -4f\left(\frac{1}{2}(x-10)\right)+3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

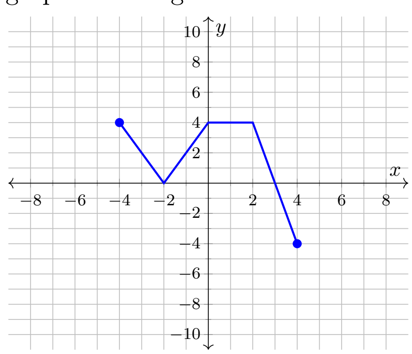
9. $f(x) = \sqrt[3]{x}$ $\qquad g(x) = \frac{1}{2}\sqrt[3]{10x+30} - 6$

Solution.

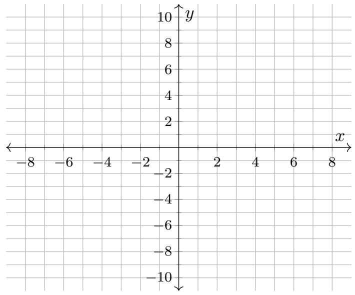
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x+30}-6 \\ &= \frac{1}{2}f(10x+30)-6 \\ &= \frac{1}{2}f(10(x+3))-6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

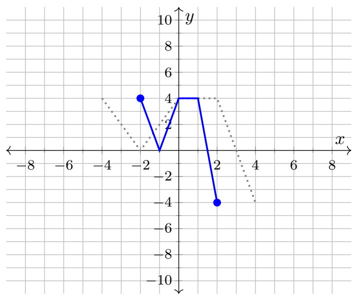
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



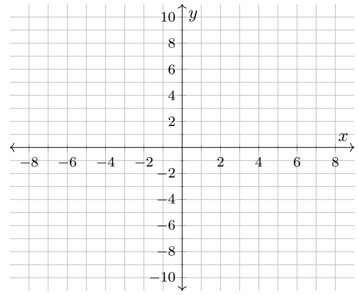
10. $k_1(x) = f(2x)$



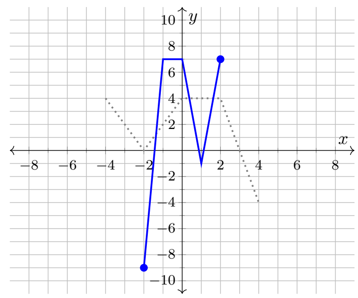
Answer.



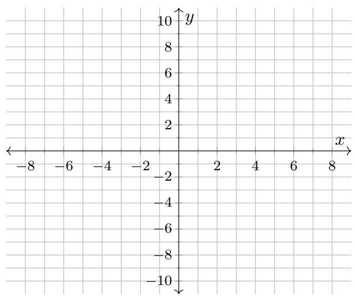
11. $k_2(x) = 2f(-2x) - 1$



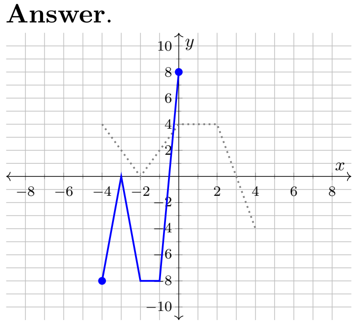
Answer.



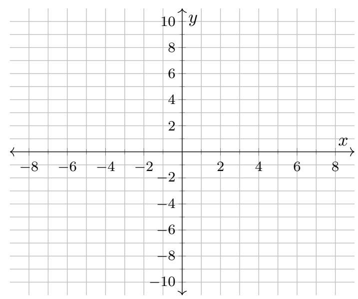
12. $k_3(x) = -2f(2x+4)$



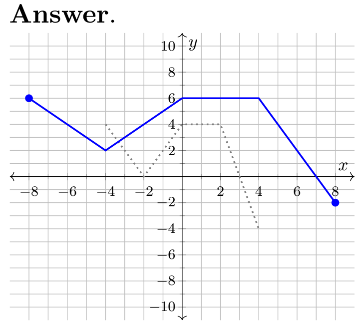
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



1.4 Analyzing Equations Numerically

Example 1.4.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 1.4.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f\left(\frac{1}{2}x\right)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x}$ $\quad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\quad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x-5\right)^2+3 \\ &= -4f\left(\frac{1}{2}x-5\right)+3 \\ &= -4f\left(\frac{1}{2}(x-10)\right)+3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

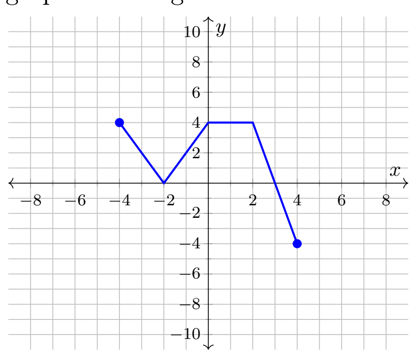
9. $f(x) = \sqrt[3]{x}$ $\qquad g(x) = \frac{1}{2}\sqrt[3]{10x+30} - 6$

Solution.

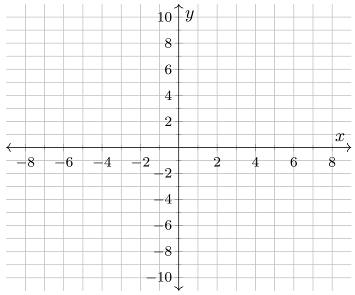
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x+30}-6 \\ &= \frac{1}{2}f(10x+30)-6 \\ &= \frac{1}{2}f(10(x+3))-6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

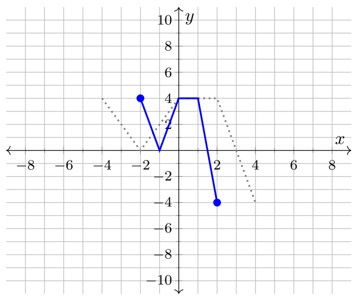
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



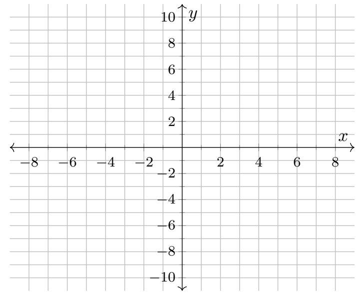
10. $k_1(x) = f(2x)$



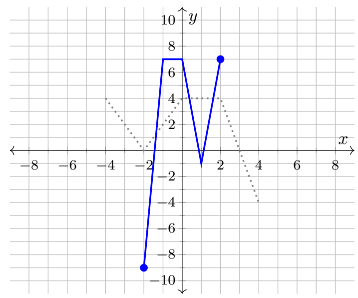
Answer.



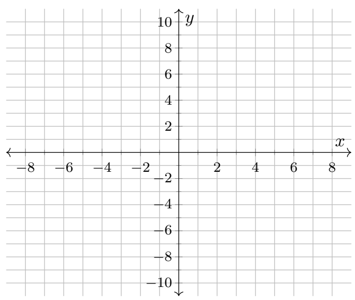
11. $k_2(x) = 2f(-2x) - 1$



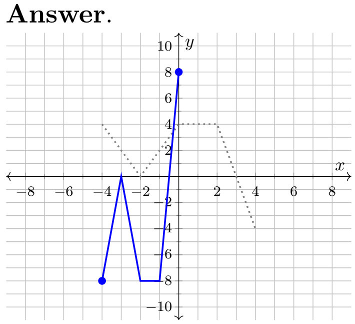
Answer.



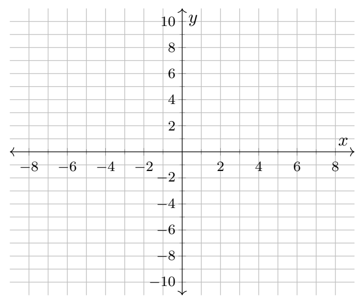
12. $k_3(x) = -2f(2x+4)$



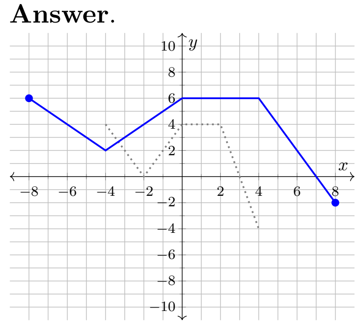
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



1.5 First-Order Linear Equations

Example 1.5.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 1.5.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x+5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x+5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f\left(\frac{1}{2}x\right)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\qquad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x}$ $\qquad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\qquad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

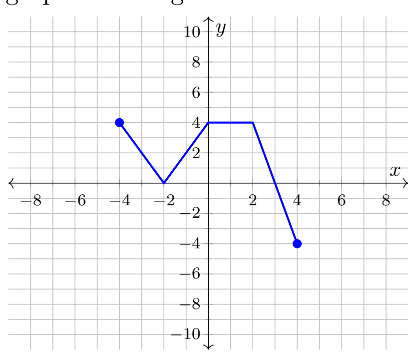
9. $f(x) = \sqrt[3]{x}$ $\quad$ $g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

Solution.

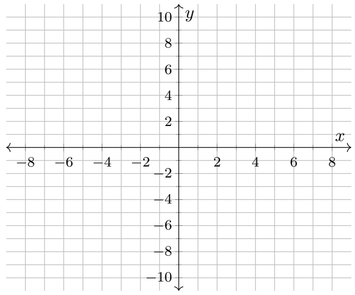
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

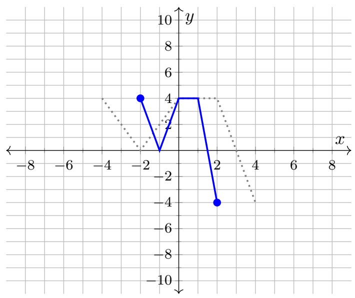
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



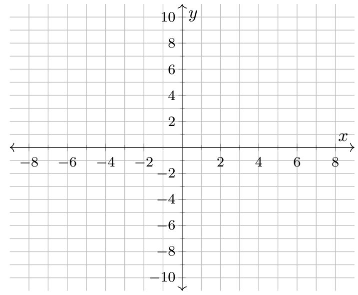
10. $k_1(x) = f(2x)$



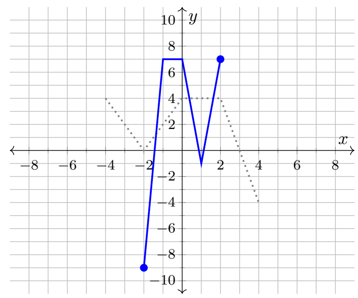
Answer.



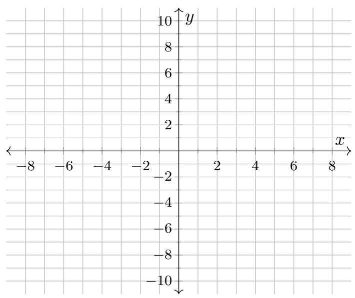
11. $k_2(x) = 2f(-2x) - 1$



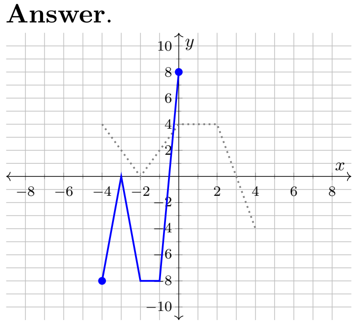
Answer.



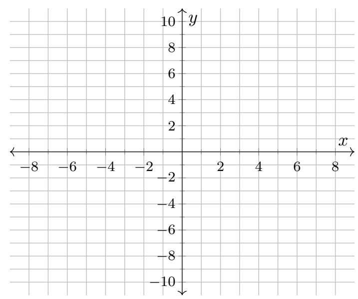
12. $k_3(x) = -2f(2x + 4)$



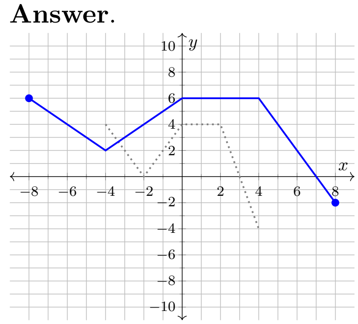
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



1.6 Existence and Uniqueness of Solutions

Example 1.6.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 1.6.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f\left(\frac{1}{2}x\right)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x}$ $\quad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\quad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

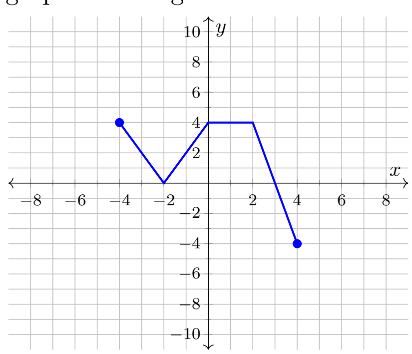
9. $f(x) = \sqrt[3]{x}$ $\quad g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

Solution.

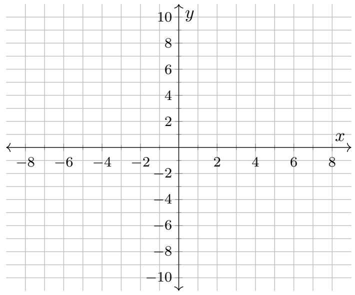
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

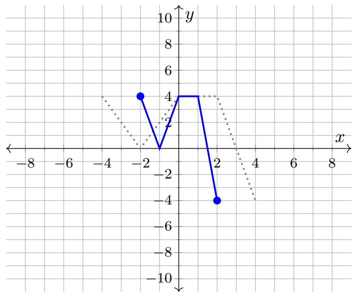
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



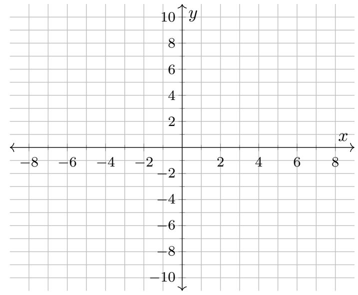
10. $k_1(x) = f(2x)$



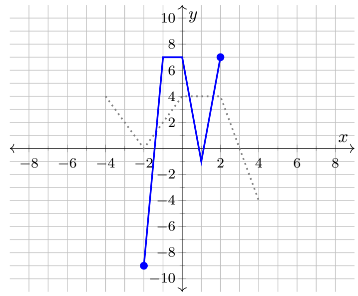
Answer.



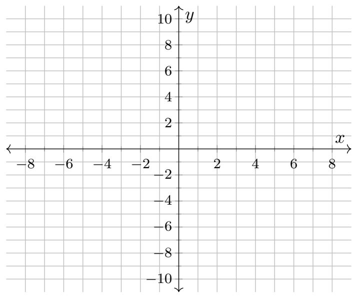
11. $k_2(x) = 2f(-2x) - 1$



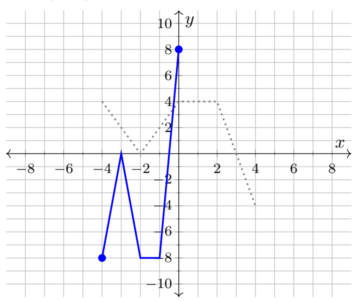
Answer.



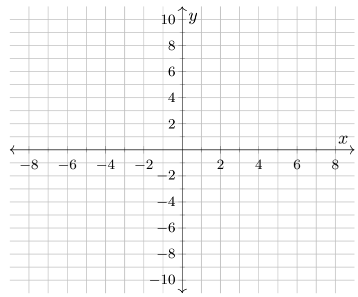
12. $k_3(x) = -2f(2x + 4)$



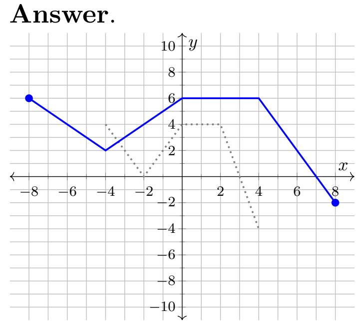
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



1.7 Bifurcations

Example 1.7.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 1.7.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f\left(\frac{1}{2}x\right)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

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$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x}$ $\quad g(x) = \frac{2}{x} + 3$

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$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\quad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x-5\right)^2+3 \\ &= -4f\left(\frac{1}{2}x-5\right)+3 \\ &= -4f\left(\frac{1}{2}(x-10)\right)+3 \end{aligned}$$

So we can transform $y=f(x)$ into $y=g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

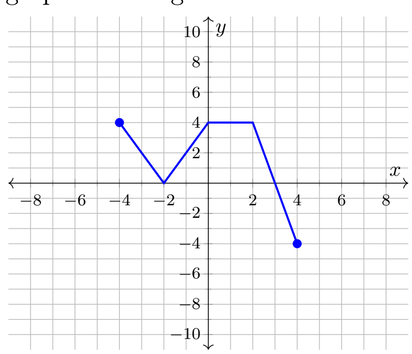
9. $f(x)=\sqrt[3]{x}$ $\qquad g(x)=\frac{1}{2}\sqrt[3]{10x+30}-6$

Solution.

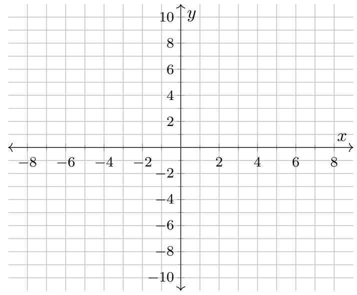
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x+30}-6 \\ &= \frac{1}{2}f(10x+30)-6 \\ &= \frac{1}{2}f(10(x+3))-6 \end{aligned}$$

So we can transform $y=f(x)$ into $y=g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

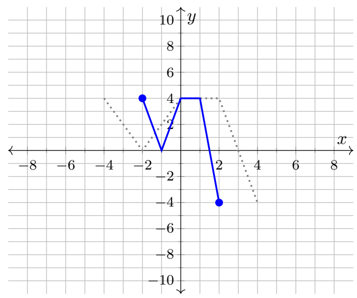
Sketch Transformations. In Exercises 10–13, use the provided graph of $y=f(x)$ to sketch a graph of each given function.



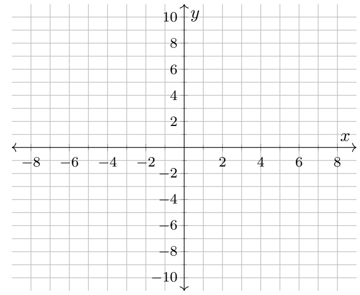
10. $k_1(x) = f(2x)$



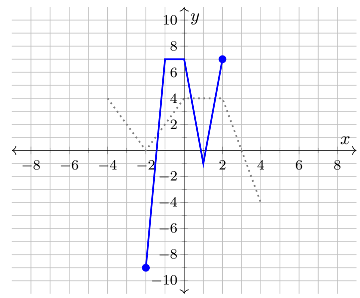
Answer.



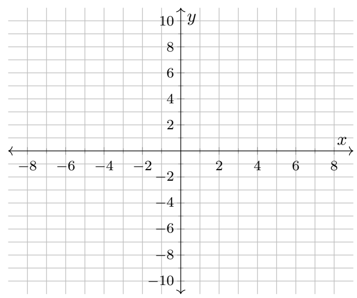
11. $k_2(x) = 2f(-2x) - 1$



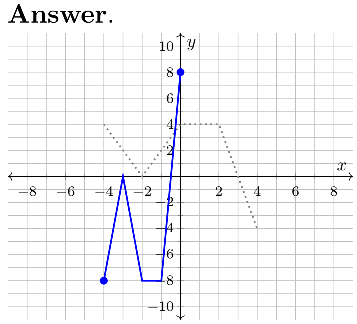
Answer.



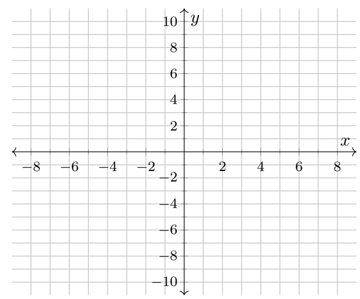
12. $k_3(x) = -2f(2x + 4)$



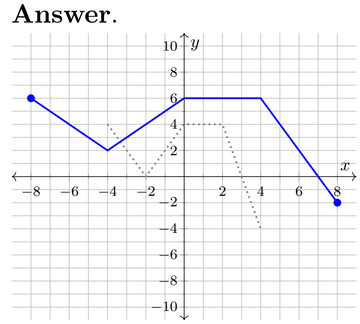
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



Chapter 2

Systems of Differential Equations

2.1 Modeling with Systems

Example 2.1.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 2.1.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x+5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x+5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$

across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

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$f(x)+5$									
$f(x+2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
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So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right

7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x}$ $\qquad g(x) = \frac{2}{x} + 3$

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So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

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Solution.

$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

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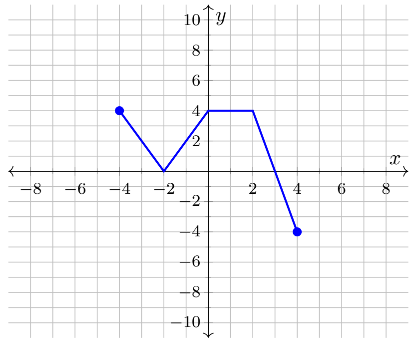

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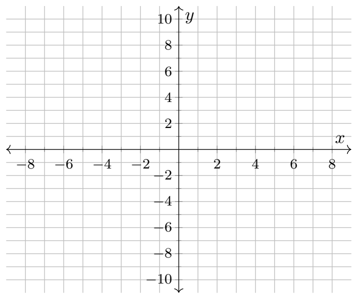
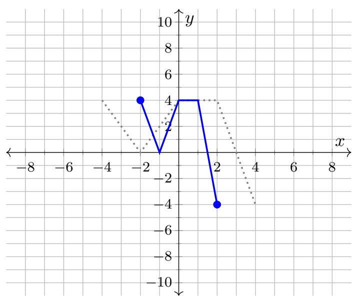


Answer.



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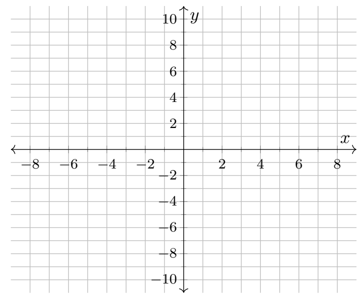
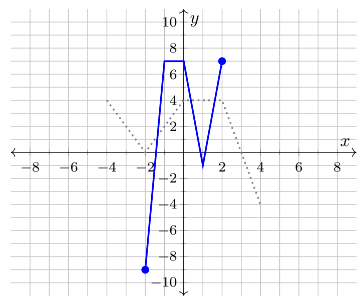


Answer.



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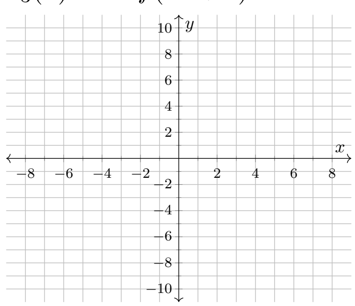
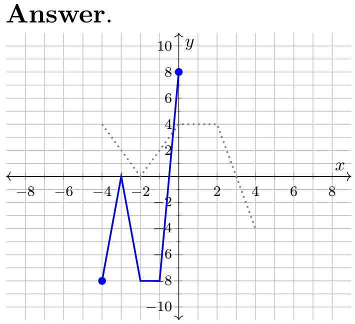


Answer.



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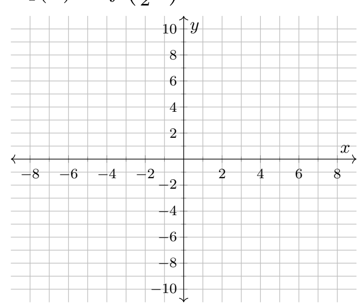
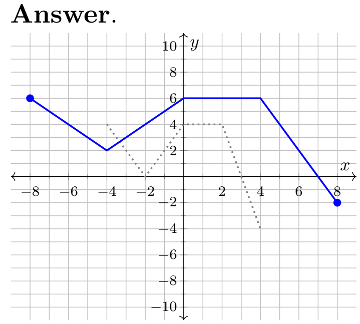


Answer.



2.2 The Geometry of Systems

Example 2.2.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 2.2.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x+5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x+5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f(\frac{1}{2}x)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f(\frac{1}{2}x)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x}$ $\quad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\quad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

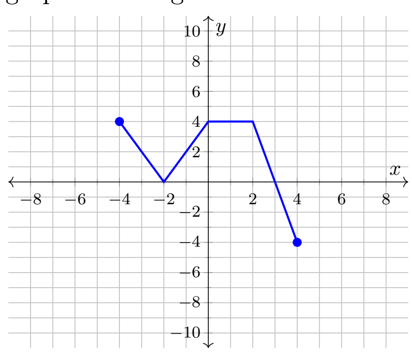
9. $f(x) = \sqrt[3]{x}$ $\quad g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

Solution.

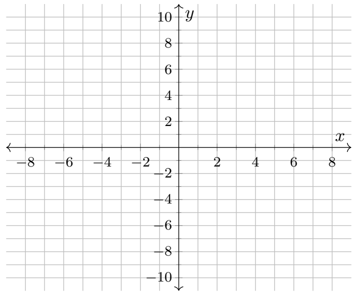
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

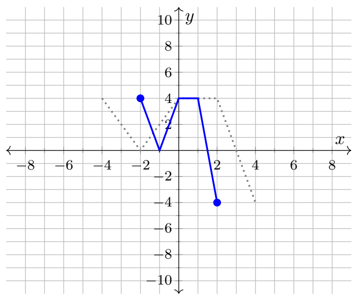
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



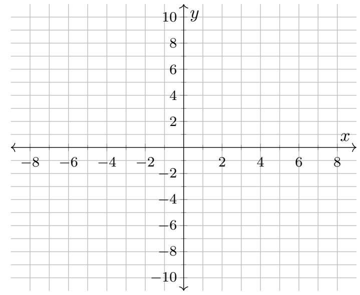
10. $k_1(x) = f(2x)$



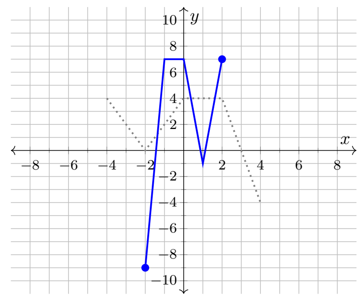
Answer.



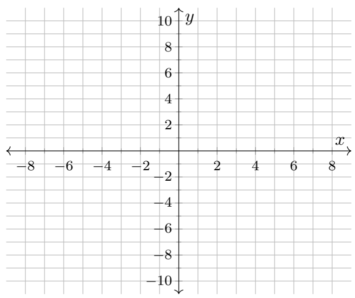
11. $k_2(x) = 2f(-2x) - 1$



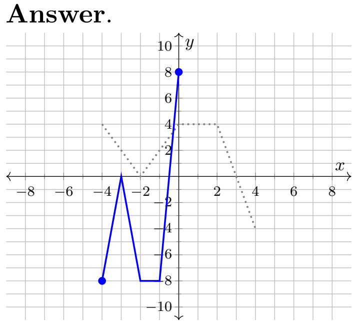
Answer.



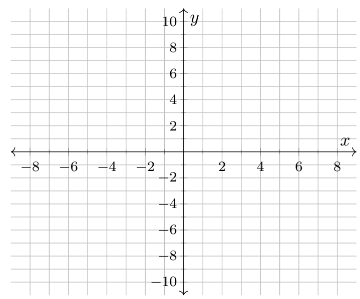
12. $k_3(x) = -2f(2x + 4)$



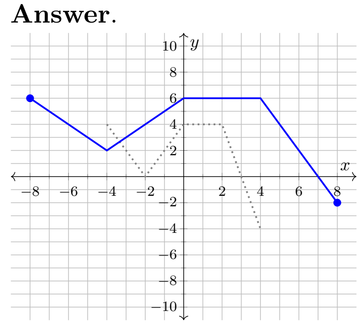
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



2.3 Not Covered

We are not covering this section in our course.

2.4 Solving Systems Anaytically

Example 2.4.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 2.4.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f\left(\frac{1}{2}x\right)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y=f(x)$ into the graph of $y=g(x)$.

6. $f(x)=\sqrt{x}$ $\qquad g(x)=\frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y=f(x)$ into $y=g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x)=\frac{1}{x}$ $\qquad g(x)=\frac{2}{x}+3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x}+3 \\ &= 2\cdot\frac{1}{x}+3 \\ &= 2f(x)+3 \end{aligned}$$

So we can transform $y=f(x)$ into $y=g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x)=x^2$ $\qquad g(x)=-4\left(\frac{1}{2}x-5\right)^2+3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x-5\right)^2+3 \\ &= -4f\left(\frac{1}{2}x-5\right)+3 \\ &= -4f\left(\frac{1}{2}(x-10)\right)+3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

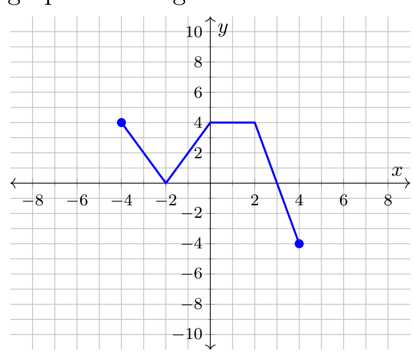
9. $f(x) = \sqrt[3]{x}$ $\qquad g(x) = \frac{1}{2}\sqrt[3]{10x+30} - 6$

Solution.

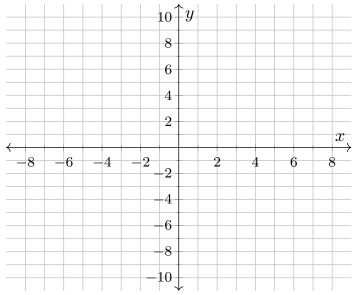
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x+30}-6 \\ &= \frac{1}{2}f(10x+30)-6 \\ &= \frac{1}{2}f(10(x+3))-6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

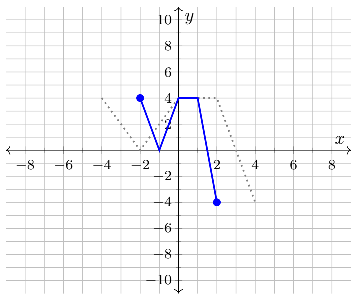
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



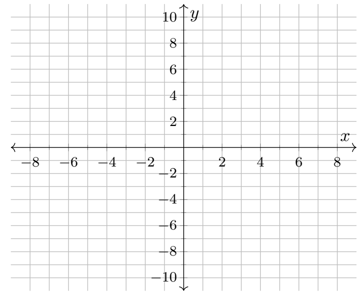
10. $k_1(x) = f(2x)$



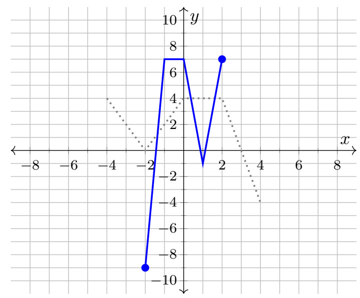
Answer.



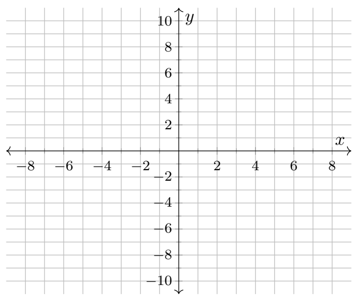
11. $k_2(x) = 2f(-2x) - 1$



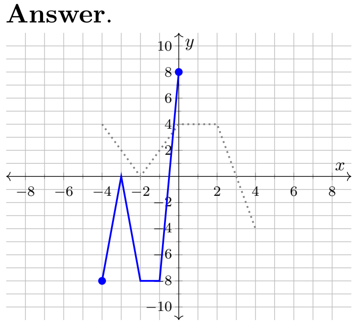
Answer.



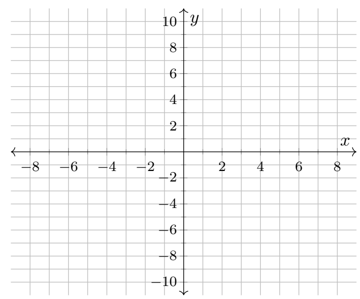
12. $k_3(x) = -2f(2x + 4)$



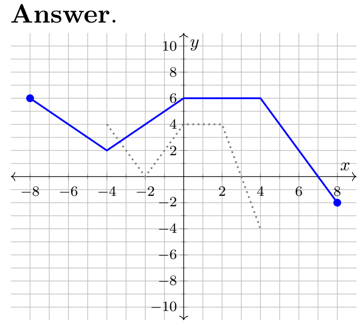
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



Chapter 3

Linear Systems

3.1 Graph Transformations

Example 3.1.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 3.1.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f(\frac{1}{2}x)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f(\frac{1}{2}x)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are

other correct answers.)

7. $f(x) = \frac{1}{x}$ $\qquad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\qquad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

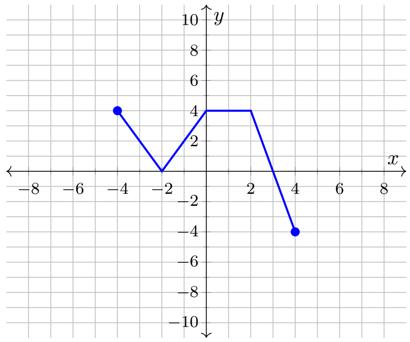
9. $f(x) = \sqrt[3]{x}$ $\qquad g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

Solution.

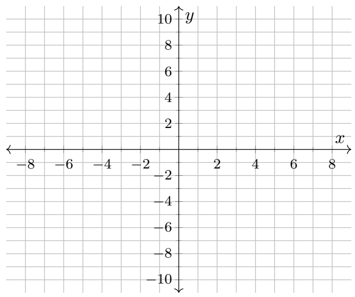
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

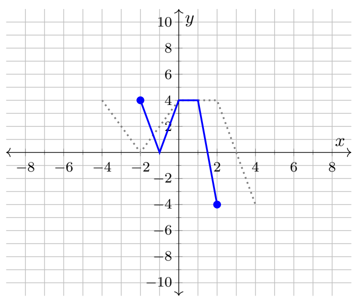
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



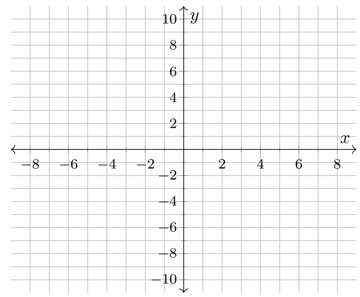
10. $k_1(x) = f(2x)$



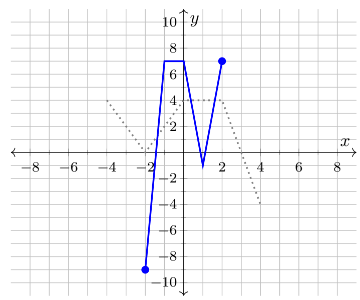
Answer.



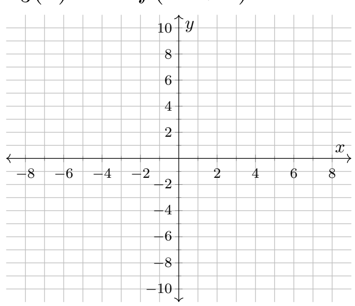
11. $k_2(x) = 2f(-2x) - 1$



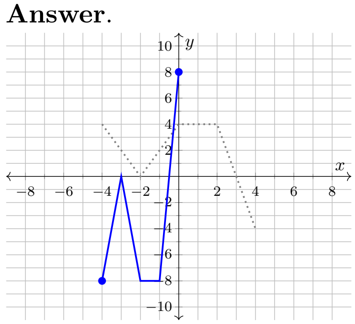
Answer.



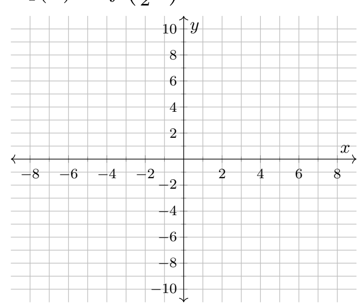
12. $k_3(x) = -2f(2x + 4)$



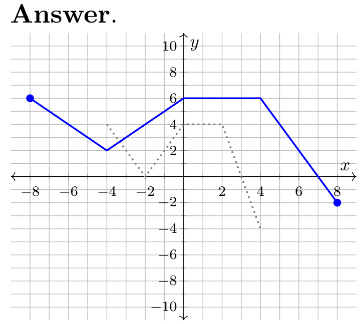
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



3.2 Inverse Functions

These exercises examine the invertibility of a function defined using a table.

Exercises

1. The table below defines the function m. Is m an invertible function? Why or why not? If your answer is yes, construct a table-of-values for m^{-1}.

x	1	2	3	4	5
$m(x)$	0	5	10	15	20

Solution. m is an invertible function since it is one-to-one, i.e., each output corresponds to exactly one input. Here is a table-of-values for m^{-1}.

x	0	5	10	15	20
$m^{-1}(x)$	1	2	3	4	5

2. The table below defines the function p. Is p an invertible function? Why or why not? If your answer is yes, construct a table-of-values for p^{-1}.

x	1	2	3	4	5
$p(x)$	4	0	−2	0	2

Solution. p isn't an invertible function since it isn't one-to-one. Notice how the output 0 corresponds to two distinct output values.

3.3 Exponential Functions

These exercises find the formula for an exponential funciton given a pair of input-output coordinates.

Exercises

Find the Formula. In Exercises 1–6, find an algebraic rule for an exponential function f that passes through the given two points.

1. $(0, 50)$ and $(3, 400)$

Answer. $f(x) = 50 \cdot 2^x$

2. $(0, 4)$ and $\left(4, \frac{1}{4}\right)$

Answer. $f(x) = 4 \cdot \left(\frac{1}{2}\right)^x$

3. $\left(-1, \frac{2}{3}\right)$ and $(2, 18)$

Answer. $f(x) = 2 \cdot 3^x$

4. $\left(-2, \frac{125}{8}\right)$ and $(1, 8)$

Answer. $f(x) = 10 \cdot \left(\frac{4}{5}\right)^x$

5. $(-2, 125)$ and $\left(3, \frac{1}{25}\right)$

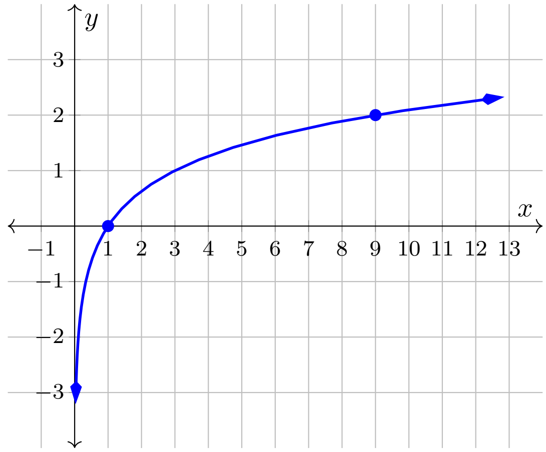
Answer. $f(x) = 5 \cdot \left(\frac{1}{5}\right)^x$

6. $\left(-3, \frac{27}{16}\right)$ and $\left(3, \frac{4}{27}\right)$

Answer. $f(x) = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^x$

3.4 Logarithmic Functions

Example 3.4.1 The graph of $f(x) = \log_a(x)$ is given in the graph below. Find the value of a. Note, the points $(1, 0)$ and $(9, 2)$ are on the graph of f.



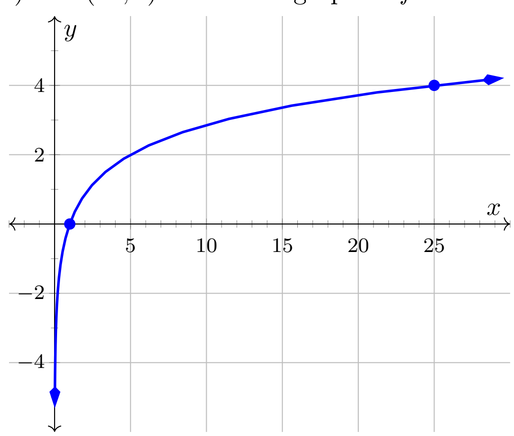
Solution. Since the function has the form $f(x) = \log_a(x)$ and $(9, 2)$ is on the graph, we know that $f(9) = 2$. Thus,

$$\begin{aligned} f(9) = 2 &\implies \log_a(9) = 2 && (\text{since } f(9) = \log_a(9)) \\ &\implies a^2 = 9 && (\text{translate to an exponential statement}) \\ &\implies a = 3 && (\text{positive square root because bases are positive}) \end{aligned}$$

Notice that we didn't attempt to use $(1, 0)$, the other obvious point on the graph of $f(x) = \log_a(x)$, to find the value of a. Why not? The point $(1, 0)$ is on the graph of all functions of the form $f(x) = \log_a(x)$, so it doesn't provide information that will help us find the paerticular function graphed here. □

Exercises

1. The graph of $f(x) = \log_a(x)$ is given below. Find the value of a. Note, the points $(1, 0)$ and $(25, 4)$ are on the graph of f.



Answer. $a = \sqrt{5}$

Find the Base. In Exercises 2–3, each table represents a table-of-values for a function $f(x) = \log_a(x)$. Find the value of a.

2.

x	0.000125	0.05	1	$2\sqrt{5}$	400
$f(x)$	-3	-1	0	0.5	2

Answer. $a = 20$

3.

x	$\frac{1}{9}$	1	3	81	243
$f(x)$	-4	0	2	8	10

Answer. $a = \sqrt{3}$

Chapter 4

Second-Order Linear Equations

4.1 Graph Transformations

Example 4.1.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 4.1.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$

across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f\left(\frac{1}{2}x\right)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\qquad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right

7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are other correct answers.)

7. $f(x) = \frac{1}{x}$ $\qquad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\qquad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

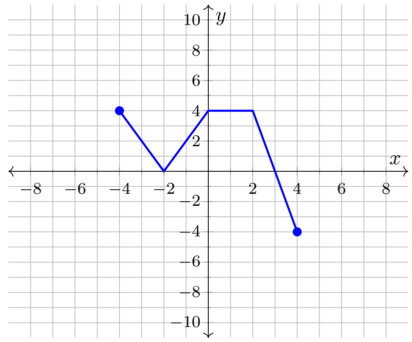
9. $f(x) = \sqrt[3]{x}$ $\qquad g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

Solution.

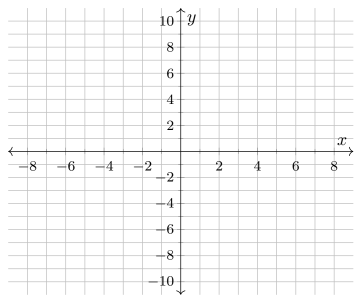
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

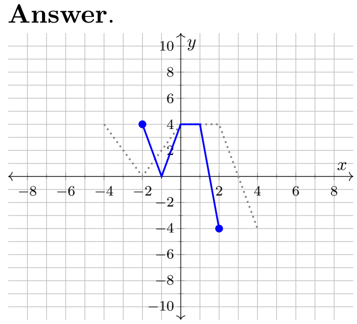
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



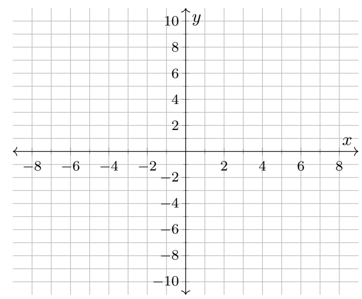
10. $k_1(x) = f(2x)$



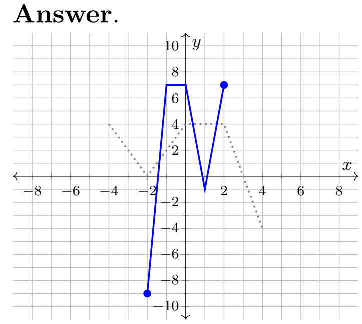
Answer.



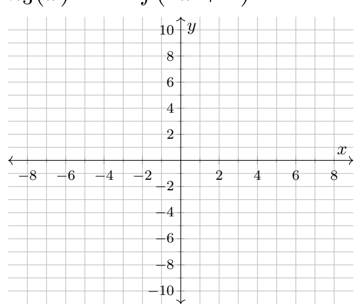
11. $k_2(x) = 2f(-2x) - 1$



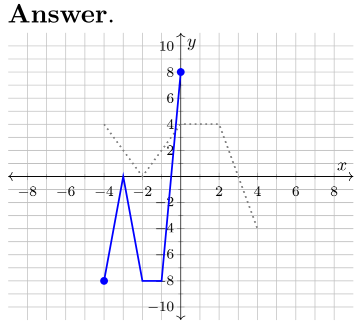
Answer.



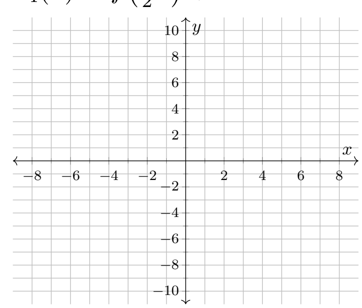
12. $k_3(x) = -2f(2x + 4)$



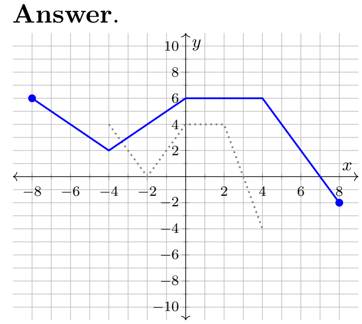
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



4.2 Inverse Functions

These exercises examine the invertibility of a function defined using a table.

Exercises

1. The table below defines the function m. Is m an invertible function? Why or why not? If your answer is yes, construct a table-of-values for m^{-1}.

x	1	2	3	4	5
$m(x)$	0	5	10	15	20

Solution. m is an invertible function since it is one-to-one, i.e., each output corresponds to exactly one input. Here is a table-of-values for m^{-1}.

x	0	5	10	15	20
$m^{-1}(x)$	1	2	3	4	5

2. The table below defines the function p. Is p an invertible function? Why or why not? If your answer is yes, construct a table-of-values for p^{-1}.

x	1	2	3	4	5
$p(x)$	4	0	-2	0	2

Solution. p isn't an invertible function since it isn't one-to-one. Notice how the output 0 corresponds to two distinct output values.

4.3 Exponential Functions

These exercises find the formula for an exponential funciton given a pair of input-output coordinates.

Exercises

Find the Formula. In Exercises 1–6, find an algebraic rule for an exponential function f that passes through the given two points.

1. $(0, 50)$ and $(3, 400)$

Answer. $f(x) = 50 \cdot 2^x$

2. $(0, 4)$ and $\left(4, \frac{1}{4}\right)$

Answer. $f(x) = 4 \cdot \left(\frac{1}{2}\right)^x$

3. $\left(-1, \frac{2}{3}\right)$ and $(2, 18)$

Answer. $f(x) = 2 \cdot 3^x$

4. $\left(-2, \frac{125}{8}\right)$ and $(1, 8)$

Answer. $f(x) = 10 \cdot \left(\frac{4}{5}\right)^x$

5. $(-2, 125)$ and $\left(3, \frac{1}{25}\right)$

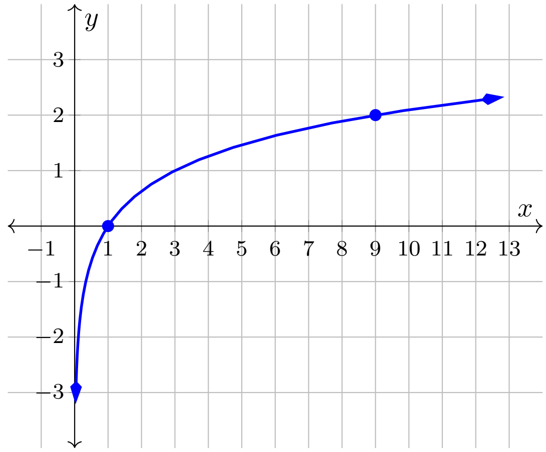
Answer. $f(x) = 5 \cdot \left(\frac{1}{5}\right)^x$

6. $\left(-3, \frac{27}{16}\right)$ and $\left(3, \frac{4}{27}\right)$

Answer. $f(x) = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^x$

4.4 Logarithmic Functions

Example 4.4.1 The graph of $f(x) = \log_a(x)$ is given in the graph below. Find the value of a. Note, the points $(1, 0)$ and $(9, 2)$ are on the graph of f.



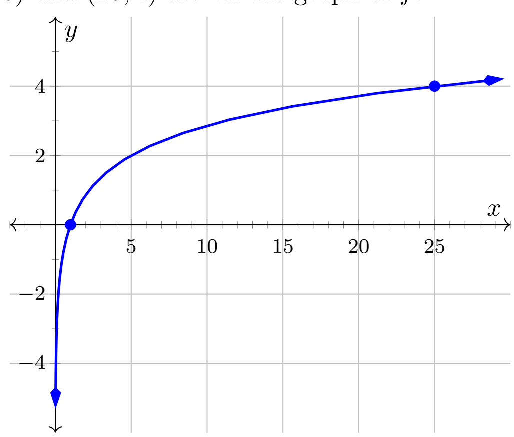
Solution. Since the function has the form $f(x) = \log_a(x)$ and $(9, 2)$ is on the graph, we know that $f(9) = 2$. Thus,

$$\begin{aligned} f(9) = 2 &\implies \log_a(9) = 2 && \text{(since } f(9) = \log_a(9)\text{)} \\ &\implies a^2 = 9 && \text{(translate to an exponential statement)} \\ &\implies a = 3 && \text{(positive square root because bases are positive)} \end{aligned}$$

Notice that we didn't attempt to use $(1, 0)$, the other obvious point on the graph of $f(x) = \log_a(x)$, to find the value of a. Why not? The point $(1, 0)$ is on the graph of all functions of the form $f(x) = \log_a(x)$, so it doesn't provide information that will help us find the paerticular function graphed here. □

Exercises

1. The graph of $f(x) = \log_a(x)$ is given below. Find the value of a. Note, the points $(1, 0)$ and $(25, 4)$ are on the graph of f.



Answer. $a = \sqrt{5}$

Find the Base. In Exercises 2–3, each table represents a table-of-values for a function $f(x) = \log_a(x)$. Find the value of a.

2.

x	0.000125	0.05	1	$2\sqrt{5}$	400
$f(x)$	-3	-1	0	0.5	2

Answer. $a = 20$

3.

x	$\frac{1}{9}$	1	3	81	243
$f(x)$	-4	0	2	8	10

Answer. $a = \sqrt{3}$

Chapter 5

Not Covered

We are not covering this chapter in our course.

Chapter 6

The Laplace Transform

6.1 Graph Transformations

Example 6.1.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 6.1.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f(\frac{1}{2}x)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f(\frac{1}{2}x)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are

other correct answers.)

7. $f(x) = \frac{1}{x}$ $\qquad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\qquad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

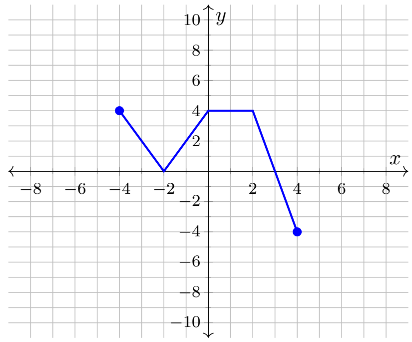
9. $f(x) = \sqrt[3]{x}$ $\qquad g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

Solution.

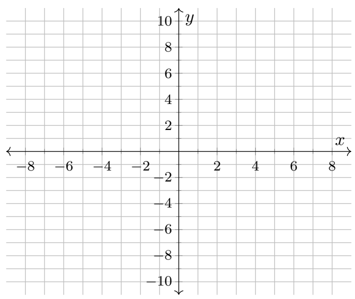
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

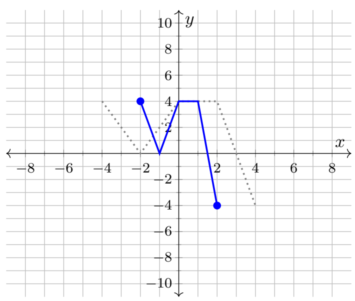
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



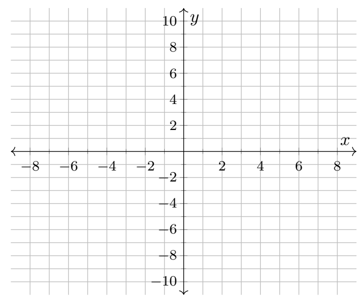
10. $k_1(x) = f(2x)$



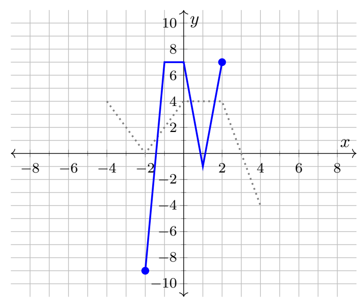
Answer.



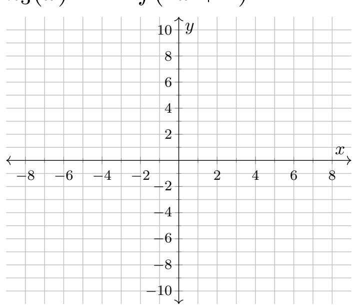
11. $k_2(x) = 2f(-2x) - 1$



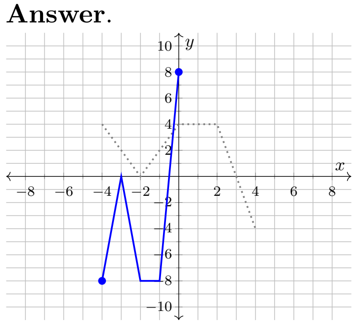
Answer.



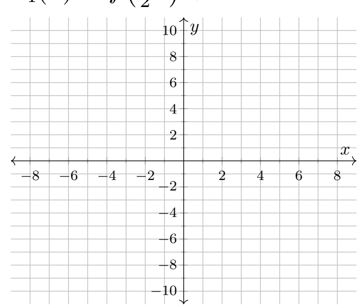
12. $k_3(x) = -2f(2x + 4)$



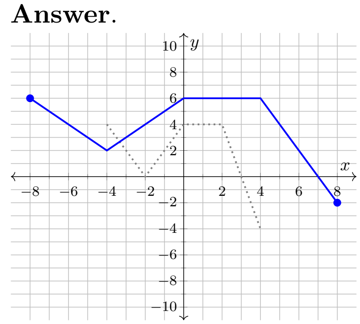
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



6.2 Inverse Functions

These exercises examine the invertibility of a function defined using a table.

Exercises

1. The table below defines the function m. Is m an invertible function? Why or why not? If your answer is yes, construct a table-of-values for m^{-1}.

x	1	2	3	4	5
$m(x)$	0	5	10	15	20

Solution. m is an invertible function since it is one-to-one, i.e., each output corresponds to exactly one input. Here is a table-of-values for m^{-1}.

x	0	5	10	15	20
$m^{-1}(x)$	1	2	3	4	5

2. The table below defines the function p. Is p an invertible function? Why or why not? If your answer is yes, construct a table-of-values for p^{-1}.

x	1	2	3	4	5
$p(x)$	4	0	-2	0	2

Solution. p isn't an invertible function since it isn't one-to-one. Notice how the output 0 corresponds to two distinct output values.

6.3 Exponential Functions

These exercises find the formula for an exponential funciton given a pair of input-output coordinates.

Exercises

Find the Formula. In Exercises 1–6, find an algebraic rule for an exponential function f that passes through the given two points.

1. $(0, 50)$ and $(3, 400)$

Answer. $f(x) = 50 \cdot 2^x$

2. $(0, 4)$ and $\left(4, \frac{1}{4}\right)$

Answer. $f(x) = 4 \cdot \left(\frac{1}{2}\right)^x$

3. $\left(-1, \frac{2}{3}\right)$ and $(2, 18)$

Answer. $f(x) = 2 \cdot 3^x$

4. $\left(-2, \frac{125}{8}\right)$ and $(1, 8)$

Answer. $f(x) = 10 \cdot \left(\frac{4}{5}\right)^x$

5. $(-2, 125)$ and $\left(3, \frac{1}{25}\right)$

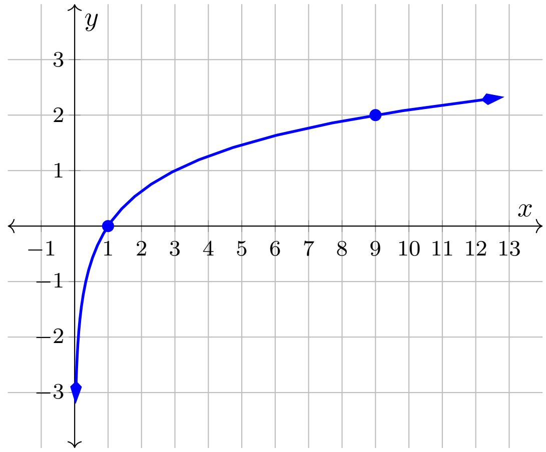
Answer. $f(x) = 5 \cdot \left(\frac{1}{5}\right)^x$

6. $\left(-3, \frac{27}{16}\right)$ and $\left(3, \frac{4}{27}\right)$

Answer. $f(x) = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^x$

6.4 Logarithmic Functions

Example 6.4.1 The graph of $f(x) = \log_a(x)$ is given in the graph below. Find the value of a. Note, the points $(1, 0)$ and $(9, 2)$ are on the graph of f.



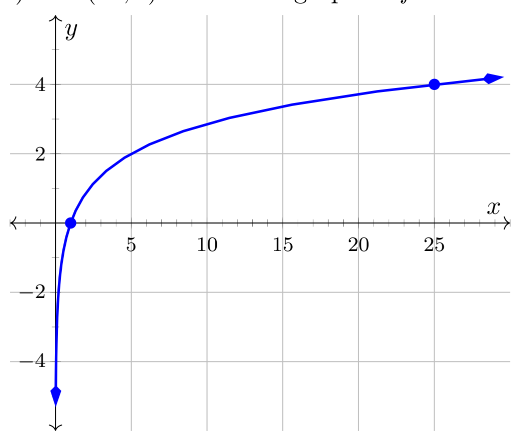
Solution. Since the function has the form $f(x) = \log_a(x)$ and $(9, 2)$ is on the graph, we know that $f(9) = 2$. Thus,

$$\begin{aligned} f(9) = 2 &\implies \log_a(9) = 2 && (\text{since } f(9) = \log_a(9)) \\ &\implies a^2 = 9 && (\text{translate to an exponential statement}) \\ &\implies a = 3 && (\text{positive square root because bases are positive}) \end{aligned}$$

Notice that we didn't attempt to use $(1, 0)$, the other obvious point on the graph of $f(x) = \log_a(x)$, to find the value of a. Why not? The point $(1, 0)$ is on the graph of all functions of the form $f(x) = \log_a(x)$, so it doesn't provide information that will help us find the paerticular function graphed here. $\square$

Exercises

1. The graph of $f(x) = \log_a(x)$ is given below. Find the value of a. Note, the points $(1, 0)$ and $(25, 4)$ are on the graph of f.



Answer. $a = \sqrt{5}$

Find the Base. In Exercises 2–3, each table represents a table-of-values for a function $f(x) = \log_a(x)$. Find the value of a.

2.

x	0.000125	0.05	1	$2\sqrt{5}$	400
$f(x)$	-3	-1	0	0.5	2

Answer. $a = 20$

3.

x	$\frac{1}{9}$	1	3	81	243
$f(x)$	-4	0	2	8	10

Answer. $a = \sqrt{3}$

Chapter 7

MTH 111 Supplement

7.1 Graph Transformations

Example 7.1.1 The table below defines the functions f, g, and h. Express $g(x)$ and $h(x)$ in terms of f.

x	-3	-2	-1	0	1	2	3
$f(x)$	8	6	4	2	0	-1	-2
$g(x)$	-8	-6	-4	-2	0	1	2
$h(x)$	5	3	1	1	-3	-4	-5

Answer. $g(x) = -f(x)$ and $h(x) = f(x) - 3$. □

Example 7.1.2

(a) If $f(x) = x^2$ and $g(x) = 2x^2 + 5$, express $g(x)$ in terms of f.

Answer. $g(x) = 2f(x) + 5$

(b) If $f(x) = x^2$ and $h(x) = (x + 5)^2 - 3$, express $h(x)$ in terms of f.

Answer. $h(x) = f(x + 5) - 3$

□

Exercises

One Function in Terms of Another. In Exercises 1–4, the table below defines the functions f, g, h, k, and l.

x	-2	-1	0	1	2
$f(x)$	0	1	2	3	4
$g(x)$	4	3	2	1	0
$h(x)$	0	-1	-2	-3	-4
$k(x)$	6	7	8	9	10
$l(x)$	0	3	6	9	12

1. Express $g(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = g(x)$.

Answer. $g(x) = f(-x)$. So, we can reflect the graph of $y = f(x)$ across the y-axis to obtain $y = g(x)$.

2. Express $h(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = h(x)$.

Answer. $h(x) = -f(x)$. So, we can reflect the graph of $y = f(x)$ across the x-axis to obtain $y = h(x)$.

3. Express $k(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = k(x)$.

Answer. $k(x) = f(x) + 6$. So, we can shift the graph of $y = f(x)$ up 6 units to obtain $y = k(x)$.

4. Express $l(x)$ in terms of f and describe how the graph of $y = f(x)$ can be transformed into the graph of $y = l(x)$.

Answer. $l(x) = 3f(x)$. So, we can stretch the graph of $y = f(x)$ vertically by a factor of 3 to obtain $y = l(x)$.

5. The second row in the table below gives values for the function f. Complete the rest of the table. If you don't have sufficient information to fill in some of the cells, leave those cells blank.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$									
$-2f(x)$									
$f(x)+5$									
$f(x+2)$									
$f\left(\frac{1}{2}x\right)$									
$f(2x)$									
$f(x-3)$									

Answer.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	-2	-1	0	1	2	3	4	5	6
$\frac{1}{2}x$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-2f(x)$	4	2	0	-2	-4	-6	-8	-10	-12
$f(x)+5$	3	4	5	6	7	8	9	10	11
$f(x+2)$	0	1	2	3	4	5	6		
$f\left(\frac{1}{2}x\right)$	0		1		2		3		4
$f(2x)$			-2	0	2	4	6		
$f(x-3)$				-2	-1	0	1	2	3

Find the Transformations. In Exercises 6–9, first write $g(x)$ in terms of f. Then compose a sequence of transformations that will transform the graph of $y = f(x)$ into the graph of $y = g(x)$.

6. $f(x) = \sqrt{x}$ $\quad g(x) = \frac{\sqrt{x-7}}{4}$

Solution.

$$\begin{aligned} g(x) &= \frac{\sqrt{x-7}}{4} \\ &= \frac{1}{4}\sqrt{x-7} \\ &= \frac{1}{4}f(x-7) \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first shifting right 7 units and then compressing vertically by a factor of $\frac{1}{4}$. (There are

other correct answers.)

7. $f(x) = \frac{1}{x}$ $\qquad g(x) = \frac{2}{x} + 3$

Solution.

$$\begin{aligned} g(x) &= \frac{2}{x} + 3 \\ &= 2 \cdot \frac{1}{x} + 3 \\ &= 2f(x) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching vertically by a factor of 2 and then shifting up 3 units. (There are other correct answers.)

8. $f(x) = x^2$ $\qquad g(x) = -4\left(\frac{1}{2}x - 5\right)^2 + 3$

Solution.

$$\begin{aligned} g(x) &= -4\left(\frac{1}{2}x - 5\right)^2 + 3 \\ &= -4f\left(\frac{1}{2}x - 5\right) + 3 \\ &= -4f\left(\frac{1}{2}(x - 10)\right) + 3 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first stretching horizontally by a factor of 2 and then shifting right 10 units. Then, stretching vertically by a factor of 4 and reflecting across the x-axis, and finally shifting up 3 units. (There are other correct answers.)

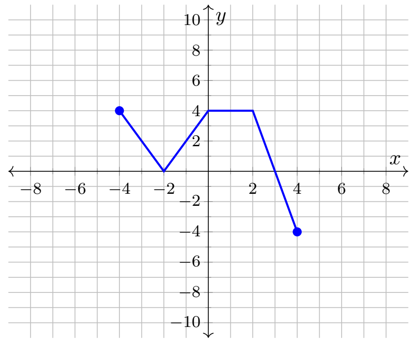
9. $f(x) = \sqrt[3]{x}$ $\qquad g(x) = \frac{1}{2}\sqrt[3]{10x + 30} - 6$

Solution.

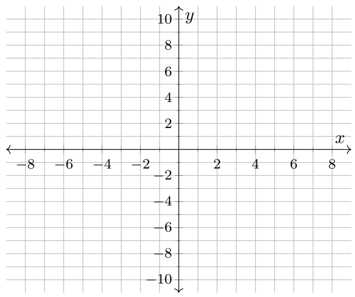
$$\begin{aligned} g(x) &= \frac{1}{2}\sqrt[3]{10x + 30} - 6 \\ &= \frac{1}{2}f(10x + 30) - 6 \\ &= \frac{1}{2}f(10(x + 3)) - 6 \end{aligned}$$

So we can transform $y = f(x)$ into $y = g(x)$ by first compressing horizontally by a factor of $\frac{1}{10}$ and then shifting left 3 units. Then, compressing vertically by a factor of $\frac{1}{2}$ and finally shifting down 6 units. (There are other correct answers.)

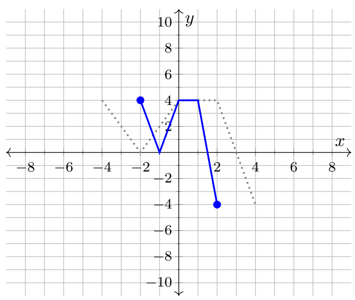
Sketch Transformations. In Exercises 10–13, use the provided graph of $y = f(x)$ to sketch a graph of each given function.



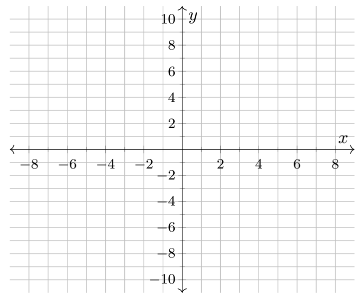
10. $k_1(x) = f(2x)$



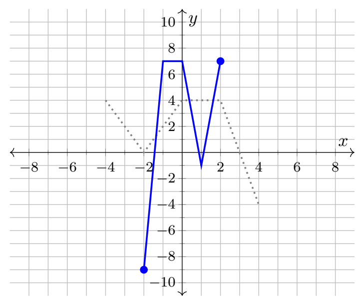
Answer.



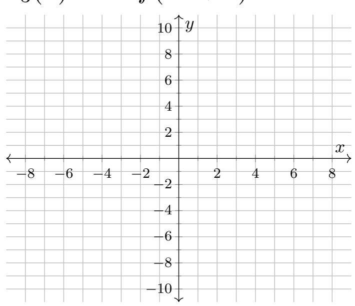
11. $k_2(x) = 2f(-2x) - 1$



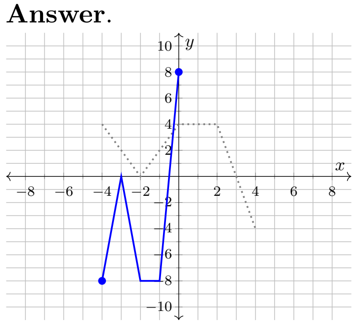
Answer.



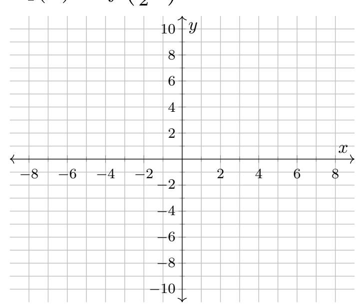
12. $k_3(x) = -2f(2x+4)$



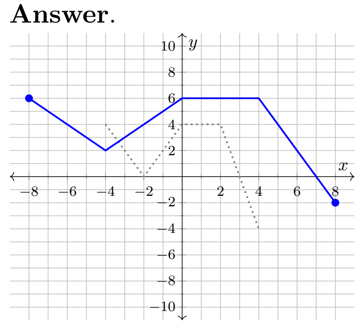
Answer.



13. $k_4(x) = f\left(\frac{1}{2}x\right) + 2$



Answer.



7.2 Inverse Functions

These exercises examine the invertibility of a function defined using a table.

Exercises

1. The table below defines the function m. Is m an invertible function? Why or why not? If your answer is yes, construct a table-of-values for m^{-1}.

x	1	2	3	4	5
$m(x)$	0	5	10	15	20

Solution. m is an invertible function since it is one-to-one, i.e., each output corresponds to exactly one input. Here is a table-of-values for m^{-1}.

x	0	5	10	15	20
$m^{-1}(x)$	1	2	3	4	5

2. The table below defines the function p. Is p an invertible function? Why or why not? If your answer is yes, construct a table-of-values for p^{-1}.

x	1	2	3	4	5
$p(x)$	4	0	-2	0	2

Solution. p isn't an invertible function since it isn't one-to-one. Notice how the output 0 corresponds to two distinct output values.

7.3 Exponential Functions

These exercises find the formula for an exponential funciton given a pair of input-output coordinates.

Exercises

Find the Formula. In Exercises 1–6, find an algebraic rule for an exponential function f that passes through the given two points.

1. $(0, 50)$ and $(3, 400)$

Answer. $f(x) = 50 \cdot 2^x$

2. $(0, 4)$ and $\left(4, \frac{1}{4}\right)$

Answer. $f(x) = 4 \cdot \left(\frac{1}{2}\right)^x$

3. $\left(-1, \frac{2}{3}\right)$ and $(2, 18)$

Answer. $f(x) = 2 \cdot 3^x$

4. $\left(-2, \frac{125}{8}\right)$ and $(1, 8)$

Answer. $f(x) = 10 \cdot \left(\frac{4}{5}\right)^x$

5. $(-2, 125)$ and $\left(3, \frac{1}{25}\right)$

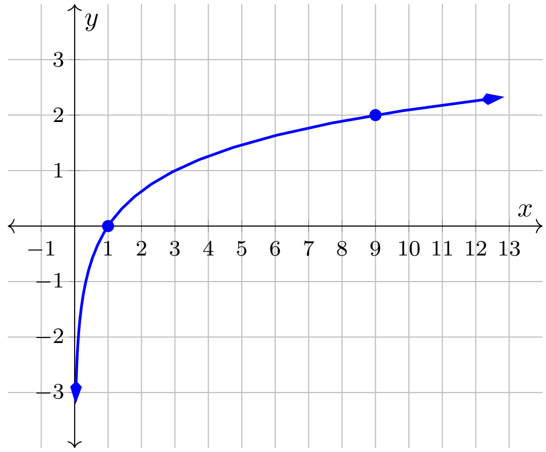
Answer. $f(x) = 5 \cdot \left(\frac{1}{5}\right)^x$

6. $\left(-3, \frac{27}{16}\right)$ and $\left(3, \frac{4}{27}\right)$

Answer. $f(x) = \frac{1}{2} \cdot \left(\frac{2}{3}\right)^x$

7.4 Logarithmic Functions

Example 7.4.1 The graph of $f(x) = \log_a(x)$ is given in the graph below. Find the value of a. Note, the points $(1, 0)$ and $(9, 2)$ are on the graph of f.



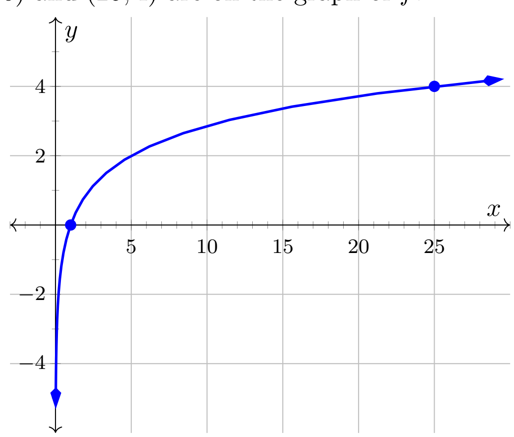
Solution. Since the function has the form $f(x) = \log_a(x)$ and $(9, 2)$ is on the graph, we know that $f(9) = 2$. Thus,

$$\begin{aligned} f(9) = 2 &\implies \log_a(9) = 2 && \text{(since } f(9) = \log_a(9)\text{)} \\ &\implies a^2 = 9 && \text{(translate to an exponential statement)} \\ &\implies a = 3 && \text{(positive square root because bases are positive)} \end{aligned}$$

Notice that we didn't attempt to use $(1, 0)$, the other obvious point on the graph of $f(x) = \log_a(x)$, to find the value of a. Why not? The point $(1, 0)$ is on the graph of all functions of the form $f(x) = \log_a(x)$, so it doesn't provide information that will help us find the paerticular function graphed here. □

Exercises

1. The graph of $f(x) = \log_a(x)$ is given below. Find the value of a. Note, the points $(1, 0)$ and $(25, 4)$ are on the graph of f.



Answer. $a = \sqrt{5}$

Find the Base. In Exercises 2–3, each table represents a table-of-values for a function $f(x) = \log_a(x)$. Find the value of a.

2.

x	0.000125	0.05	1	$2\sqrt{5}$	400
$f(x)$	-3	-1	0	0.5	2

Answer. $a = 20$

3.

x	$\frac{1}{9}$	1	3	81	243
$f(x)$	-4	0	2	8	10

Answer. $a = \sqrt{3}$

Chapter 8

MTH 112 Supplement

8.1 Angles

8.1.1 Coterminal Angles

Definition 8.1.1 Two angles are **coterminal** if they have the same terminal side when in standard position. ◊

Since $360°$ represents a complete revolution, if we add integer-multiples of $360°$ to an angle measured in degrees, we'll obtain a coterminal angle. Similarly, since 2π represents a complete revolution in radians, if we add integer-multiples of 2π to an angle measured in radians, we'll obtain a coterminal angle. We can summarize this information as follows

If θ is measured in degrees, θ and $\theta + 360° \cdot k$, where $k \in \mathbb{Z}$, are coterminal.

If θ is measured in radians, θ and $\theta + 2\pi \cdot k$, where $k \in \mathbb{Z}$, are coterminal.

Example 8.1.2 The angles $45°$, $405°$, and $-315°$ are coterminal as illustrated in Figure 8.1.3.

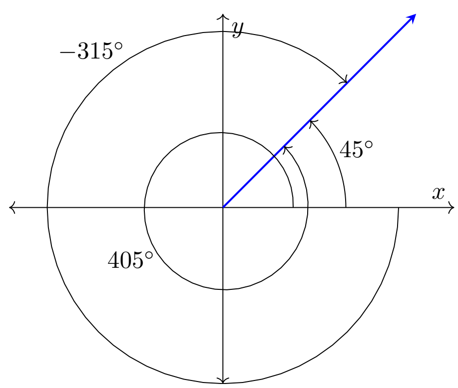


Figure 8.1.3 Coterminal angles

□

8.1.2 Reference Angles

Definition 8.1.4 The **reference angle** for an angle in standard position is the positive acute angle formed by the x-axis and the terminal side of the angle. ◊

Depending on the location of the angle's terminal side, we'll have to use a different calculation to determine the angle's reference angle.

Example 8.1.5 The angles $\frac{\pi}{3}$ and $30°$ are their own reference angles since they are acute angles; seen in Figure 8.1.6 and Figure 8.1.7.

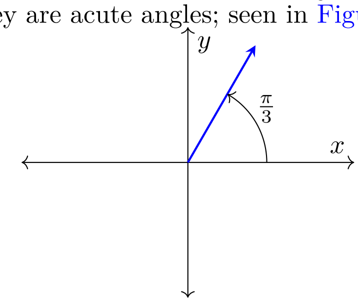


Figure 8.1.6

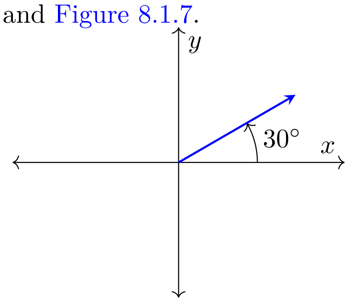


Figure 8.1.7

□

Example 8.1.8 The reference angle for $\frac{2\pi}{3}$ is $\pi - \frac{2\pi}{3} = \frac{\pi}{3}$ (see Figure 8.1.9), while the reference angle for $150°$ is $180° - 150° = 30°$ (see Figure 8.1.10).

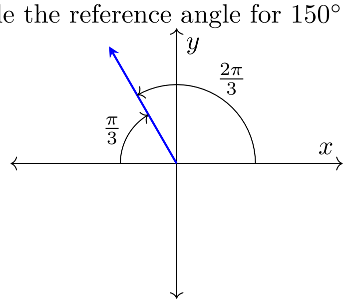


Figure 8.1.9

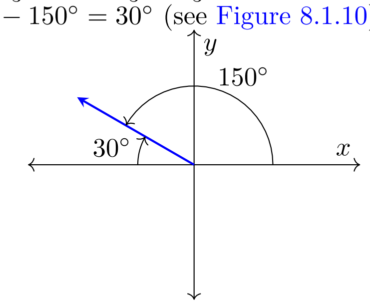


Figure 8.1.10

□

Example 8.1.11 The reference angle for $\frac{4\pi}{3}$ is $\frac{4\pi}{3} - \pi = \frac{\pi}{3}$ (see Figure 8.1.12), while the reference angle for $210°$ is $210° - 180° = 30°$ (see Figure 8.1.13).

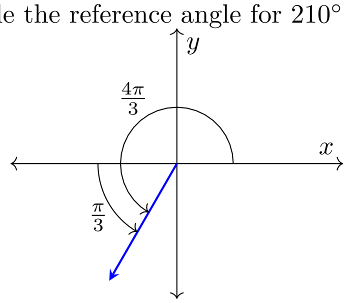


Figure 8.1.12

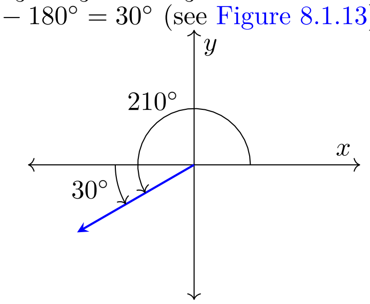


Figure 8.1.13

□

Example 8.1.14 The reference angle for $\frac{5\pi}{3}$ is $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$ (see Figure 8.1.15), while the reference angle for $330°$ is $360° - 330° = 30°$ (see Figure 8.1.16).

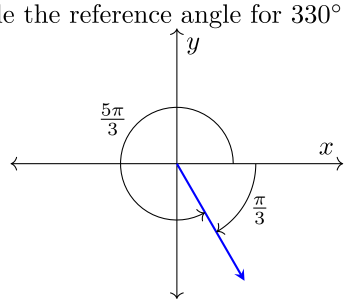


Figure 8.1.15

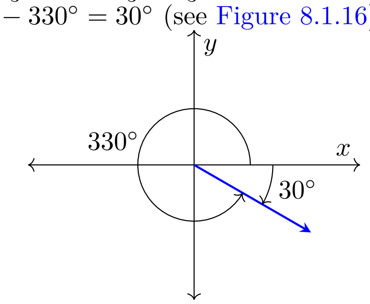


Figure 8.1.16

□

Example 8.1.17 The reference angle for 7.5 radians is $7.5 - 2\pi \approx 1.2$ radians (see Figure 8.1.18), and the reference angle for $-137°$ is $180° + (-137°) = 43°$ (see Figure 8.1.19).

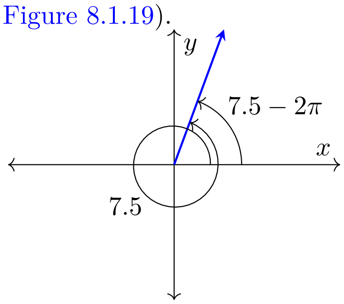


Figure 8.1.18

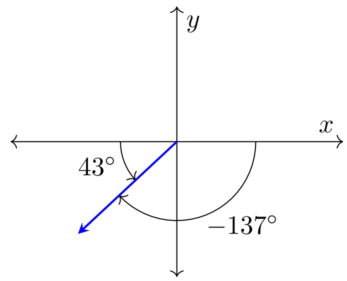


Figure 8.1.19

□

8.1.3 Exercises

Coterminal Angles. In Exercises 1–3, find both a positive and negative angle that is coterminal angle with the following angles.

1. $63°$

Answer. $423°$ and $-297°$ are coterminal with $63°$.

2. $\frac{\pi}{9}$

Answer. $\frac{19\pi}{9}$ and $-\frac{17\pi}{9}$ are coterminal with $\frac{\pi}{9}$.

3. $\frac{13\pi}{8}$

Answer. $\frac{29\pi}{8}$ and $-\frac{3\pi}{8}$ are coterminal with $\frac{13\pi}{8}$.

Reference Angles. In Exercises 4–12, find the reference angle for the following angles.

4. $120°$

Answer. $60°$

5. $\frac{5\pi}{4}$

Answer. $\frac{\pi}{4}$

6. $400°$

Answer. $40°$

7. $\frac{13\pi}{8}$

Answer. $\frac{3\pi}{8}$

8. 2

Answer. $\pi - 2 \approx 1.14$

9. $\frac{10\pi}{11}$

Answer. $\frac{\pi}{11}$

10. $2000°$

Answer. $20°$

11. $-\frac{9\pi}{5}$

Answer. $\frac{\pi}{5}$

12. $-100°$

Answer. $80°$

8.2 Graphing Sinusoidal Functions: Phase Shift vs. Horizontal Shift

Let's consider the function $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$. Using what we study in MTH 111 about graph transformations, it should be apparent that the graph of $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ can be obtained by transforming the graph of $g(x) = \sin(x)$. (To confirm this, notice that $g(x)$ can be expressed in terms of $f(x) = \sin(x)$, as $g(x) = f\left(2x - \frac{2\pi}{3}\right)$.) Since the constants "2" and "$\frac{2\pi}{3}$" are multiplied by and subtracted from the input variable, x, what we study in MTH 111 tells us that these constants represent a horizontal stretch/compression and a horizontal shift, respectively.

It is often recommended in MTH 111 that we factor-out the horizontal stretching/compressing factor before transforming the graph, i.e., it's often recommended that we first re-write $g(x) = \sin\left(2x - \frac{2\pi}{3}\right)$ as $g(x) = \sin\left(2\left(x - \frac{\pi}{3}\right)\right)$.