Ch:1

1 Quick Note

The General Theory of Relativity, GR, is a fascinating subject which touches upon and has contributed to various aspects of modern theoretical physics. However, it is unique in the sense that is is distinctly disconnected from all other areas in modern physics because of its decidedly non quantum nature. Since its description is built on the concept of continuity at its core, as defined in the traditional math sense (the limit exists for a function $f(x)$ when approached from all possible directions), there is no at least obvious discrete aspect to this theory. This is why GR is treated differently from all other fundamental branches of modern theoretical physics and the question of quantum gravity, the formulation of a theory which combines GR and quantum field theory, is the biggest open question currently in all of theoretical physics.

2 Quick Review of Special Relativity

This will be an extremely fast review of the core ideas in special relativity.

2.1 There are two main postulates in the formation of this theory as it is usually described.

- The first is that all inertial frames have the same laws of physics.
- The second is that the speed of light is constant for all observers.

In full accuracy for theory formulation and to keep the minimum number of postulates, it's easy to show the second actually follows from the first.

Consider Maxwell's equations, taken in a vacuum. Let E, B be the electric and magnetic fields respectively. $\nabla \cdot\ E = \mathbf{0}$ $\mu_0\,\epsilon_0\,\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B}$ $\nabla \cdot\ B = \mathbf{0}$ $\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$

These equations must hold in all reference frames if the first postulate is to hold. But then, the solution to these equations is, using the vector calculus identity $\nabla \cdot (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot A) - \nabla^2 A$, where A is any 3 dimensional vector.

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0\,\epsilon_0\,\frac{\partial \mathbf{E}}{\partial t} \\ \Rightarrow \nabla \cdot (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot B) - \nabla^2 B &= - \\ \nabla^2 B &= \nabla \cdot \mu_0\,\epsilon_0\,\frac{\partial \mathbf{E}}{\partial t} \\ \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ \Rightarrow -\nabla^2 B &= -\mu_0\,\epsilon_0\,\frac{\partial^2 \mathbf{B}}{\partial t} \\ \Rightarrow \nabla^2 B &= \mu_0\,\epsilon_0\,\frac{\partial^2 \mathbf{B}}{\partial t} \end{aligned}$$

This is the wave equation in 3D space, where for any vector field $\mathbf{A(x)}$ describing a 'particle' moving as a wave with velocity of magnitude v

$$\Rightarrow \nabla^2 A = \frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t}\text{with}v = \frac{1}{(\mu_0\,\epsilon_0)^{1/2}} = c$$

Note however, that if Maxwell's equations where to be true in any reference frame as the first postulate states, they must always move at speed c, no matter the reference frame an observer is in, since c is equal to merely a constant. Then, this implies that the speed of light is always c in any reference frame, showing that the second postulate follows almost directly from the first.

2.2 The main results of the special theory of relativity are

Consider the setup of an observer O in a given reference frame with coordinates **x** and an observer O' in a reference frame moving at velocity v with respect to O with coordinates **x** Take the direction of the velocity to be in the positive i-th direction, which can always be done by the respective choice of coordinate axis.

They would both expect the observation of a light 'wave' to be the same, which is of magnitude c. However, by classical intuition, if O observes the light wave to have velocity v, O' would expect it to have v' = c - v. This is not true however if the laws of physics are invariant in all reference frames. What is the solution to this paradox?

Well, consider $v = \frac{dx}{dt}$ in O, $v' = \frac{dx'}{dt'}$ in O'. If they are to be the same, $\frac{dx}{dt} = \frac{dx'}{dt'}$. This means that each observer has a different time as well, not just a position in space.

Note from before $(\nabla^2 - \frac{\partial^2}{v^2 \partial^2 t})B(x) = 0$ must have the same form in all inertial frames. This implies that $(\nabla'^2 - \frac{\partial^2}{v^2 \partial^2 t'})B(x') = 0$. B(x) is just the magnitude of the field and since there exist solutions, such as a constant magnetic field, which satisfy Maxwell's equations, it follows that B(x) $=$ B(x)'. This is the definition of a Lorentz scalar, which will be looked at in more detail at a later topic. This implies that defining

$$\partial^2 = \nabla^2 - \frac{\partial^2}{v^2 \partial^2 t} \Rightarrow \partial^2 = (\partial')^2$$

Inverting the differential operator, which can be done for most purposes including in the given context,

$$\Rightarrow (c\,dt)^2 - ((dx)^2 + (dy)^2 - (dz)^2) = (dt')^2 - ((dx')^2 + (dy')^2 - (dz')^2)$$

Defining $(d\tilde{x})^2 = (c\,dt)^2 - ((dx)^2 + (dy)^2 - (dz)^2)$, one can see that $(d\tilde{x})^2$ is conserved in all reference frames. This is defined as a metric which must be preserved by its corresponding transformations

Then, given the coordinates O, what are the coordinates relative to those in O'?$((x,t) \to (x',t'))$. Consider an object at rest in O' $x' = 0$ and x is where the object is which is the 'same' as where the frame is. $T(\tilde{x}) = \tilde{x'} \Rightarrow (d\tilde{x'})^2 = (d(\tilde{x}))^2$ where T is the transformation. Let's take T to be linear. Looking at just one spatial dimension for simplicity,

$$\Rightarrow (c\,t, x) \to (c\,t', x') = (a\,c\,t + b\,x, a_1\,t + b_1\,x)$$
$$(\tilde{dx})^2 = (c\,dt)^2 - (dx)^2$$
$$\Rightarrow ((a^2)(ct) + (b^2)(x))^2 - ((a_1^2)(ct) + (b_1^2)(x))^2 = ((a^2) - (a^1)^2)(ct)^2 +$$
$$((b^2) - (b_1)^2)x^2 + 2ctx((a^2)(b^2) - (a_1)^2(b_1)^2) = (ct)^2 - x^2$$
$$: a^2 - (a^1)^2 = 1, ((b^2) - (b_1)^2 = 1, (a^2)(b^2) - (a_1)^2(b_1)^2$$
$$\Rightarrow ct' = ct\cosh\phi + x\sinh\phi, x' = ct\sinh\phi + x\cosh\phi \Rightarrow$$
$$(\tilde{dx})^2 = (c\,t(\cosh\phi)^2 - (\sinh\phi)^2 = 1$$

Solutions of the form $(c\,t, x)$: $(c\,t', x') = (ct\cosh\phi + x\sinh\phi, -ct\sinh\phi + x\cosh\phi)$ naturally satisfy the given constraint, with ϕ being some constant parameter. To get the explicit form of ϕ note for $x' = 0$ the origin of O'

$$x\cosh\phi = ct\sinh\phi \Rightarrow \frac{\sinh\phi}{\cosh\phi} = \tanh\phi = \frac{x}{ct} \Rightarrow \tanh\phi = \frac{v}{c}$$

Then, note that the identity $\tanh x^2 = 1 - \operatorname{sech}^2 x$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\cosh^2\phi} \Rightarrow \frac{1}{\cosh^2\phi} = 1 - \frac{v^2}{c^2} \Rightarrow \cosh^2 x = \frac{1}{1 - \frac{v^2}{c^2}} \Rightarrow \cosh\phi = (\frac{1}{1 - \frac{v^2}{c^2}})^{1/2}$$

This factor is defined as γ and is called the Lorentz factor. Note that the relation $\sinh\phi = \cosh\phi\tan\phi = v\gamma$

Time Dilation: A time Δt passes in O. How much time $\Delta t'$ has passed in O'? Well, $(ct')^2 = -x^2 + (ct)^2 = -(vt)^2 + (ct)^2 = ((ct)^2)(1 - \frac{v^2}{c^2}) \Rightarrow t' = (1 - \frac{v^2}{c^2})^{1/2} t$ Then, $t' = \frac{1}{\gamma} t$ This is the famous time dilation formula, where in the rest frame of an object, time is always 'slower' than in any other frame. Length contraction follows from this naturally. Consider an object at rest in O' l'. What is the length of the object in O?

To make it more precise what measurement means in this context, consider that both frames start at the same point. In O, the object passes by them in the time t at v. In O' the object starts at the same point as O but he measures it as having length l'. For further clarity, both frames starting at the same point mark the origin as one end of the object. A measurement is made when the object has fully passed that benchmark, which is the origin of O, but O' has moved from. In O the object fully passes its origin at time t but in O' it passes that point at time t'. $\gamma t' = t \Rightarrow l = vt = vt'\gamma = l'\gamma$

Now the Lorentz transformations can be expressed more compactly as a matrix since it is a linear transformation.

$$x' = \begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\phi & \sinh\phi \\ -\sinh\phi & \cosh\phi \end{bmatrix} x = L_x \begin{bmatrix} ct \\ x \end{bmatrix}$$

where L_x is defined as the Lorentz transformation in the x direction, meaning the frames are moving at velocity v_x in only $\hat{x}$

Now let's expand this to the y and z direction. In this case,

$$L(x) = \begin{bmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$, L(y) = \begin{bmatrix} \cosh\phi & 0 & \sinh\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh\phi & 0 & \cosh\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$, L(z) = \begin{bmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{bmatrix}$$

These are linearly independent of each other, meaning a boost in an arbitrary direction can be represented as a linear combination of these 'basis' matrices $L(r) = a_x L(x) + a_y L(y) + a_z L(z) = a_i L(x^i)$ for constants a_i and an implied sum being used over the index i in the final expression. This is common notation for convenience, Einstein's summation convention.