# 1-Geometry of Gauge Theories (An Overview)

varunkannan2000

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## 1 Introduction

The only explicitly made fundamental geometric theory is that of gravity in general relativity. In quantum field theory, usually the particles are represented as just operators simply on Minkowski space. Of course, this is very accurate for the electromagnetic, strong, and weak interaction forces. In fact, electrodynamics (the quantum field thy of the electromagnetic force), and quantum chromodynamics, (the quantum field thy of the strong force) are some of the best tested, most complete theories ever. However, it is possible to make a theory of the electromagnetic and, especially, the strong force through geometric means, like what is done for gravity in general relativity. This may provide further insights and give some kind of interpretation where one can reconcile the quantum field theories of these forces with more geometrical theories, such as the ones provided here. (Note: this will not be mathematically precise or formal but more focused on physical interpretations and the meaning of the mathematical objects). (Note: An overview, not a full detailed account of these topics) (Note: Background in GR and QFT assumed to at least a basic level).

## 2 Background

To explicitly formulate the theory in terms of geometric operations, the appropriate background material is needed mathematically and in the already existing physical,geometric theory of general relativity. Then, define the 4 manifold of spacetime, $M$, intrinsically with local coordinates at any given point $x$ on the spacetime manifold by $x = (t, x, y, z)$ . (Will assume topological triviality for now then with each point locally seeming like Euclidean space, at least differentially, meaning it is locally diffeomorphic to Euclidean space) . There will be defined a vector space, say $V$ at each point on the manifold with basis vectors $e_u$ , and so can then define a vector field say $V(x) = V(t, x, y, z)$ at each point of the manifold. Of course,with each vector space, one can define a corresponding dual space of 1-forms with basis $w^u$ with the relation between them given by $e_u\omega^v = \delta_u^v$ . These should be thought of as functions with vectors taking one forms to scalars $V = V^u e_u = V^u e_u(B_v\omega^v) = V(B)$ , and one forms can be thought of as just taking vectors to scalars by$B = B_v\omega^v = B(V)$ . Of course, $B(V) = V(B)$. With this viewpoint, one can define the tensor product as $e_u \otimes \omega^v = e_u\omega^v$ . By convention, the one-forms are put on the left and the basis vectors on the right. One can also form arbitrary tensor spaces of rank $pr$ where p is the number of covariant indices(number of one forms) and r is the number of contravariant indies(number of basis vectors) with this tensor product. Each tensor can be viewed as multi linear function on the spacetime. For example, one usually defines an metric with basis one forms $\omega^u$ as that of $g = g_{uv}\omega^u\omega^v$ , where $g_{uv}$ is the components of the metric and the basis of this metric is defined as $\omega^u\omega^v = \omega^u \otimes \omega^v$ , making the metic just a 2-form. This should be viewed as an multi linear operation taking two vectors as input and outputting a scalar like $g_{uv}dx^u dx^v = g_{uv}dx^u(V)dx^v(W)$ where V and W are vectors. Then, there is a natural one to one correspondence such that for each vector $V$ one gets a one-form $\omega$ given by $w = g_{uv}dx^u dx^v(V) = g_{uv}dx^u dx^v(V^\alpha(d_\alpha)) = g_{uv}V^\alpha dx^u(dx^v(d_\alpha)) = g_{uv}V^\alpha dx^u\sigma^v_\alpha = g_{uv}V^v dx^u = V_u dx^u$. Then, of course, on can define also any given tensor field of arbitrary dimension at each point in spacetime with the standard definition or extension from that of vector fields with all its mathematical properties. Finally, to compare vectors at nearby vector spaces, one needs a way of moving a vector through the spacetime, which can be done with the covariant derivative, $\nabla$ and so the connection coefficients $\Gamma = \Gamma(x)$ defined with respect to basis vectors $\nabla_v e_u = d_v e_u + \Gamma^\sigma_{uv}e_\sigma$ . The most common metric is the Levi-Cita connection in general relativity, but any connection can be defined depending on the physical restraints of the system as long as the usual properties of differential operators are held. Then, the given metric along with differential structures on the manifold defined by the connection $\nabla$ for GR can be used to define the curvature, $R^n_{uvp} = ([\nabla_u, \nabla_v] - \nabla_{[uv]})(e_p)(w^n)$, torsion, $T_{uv} = \nabla_u e_v - \nabla_v e_u - [e_u, e_v]$, and other "isometrical" (quantities defined with a metric) structures on the manifold. However, let us drop that for now by saying that we will not define the metric yet. So, we will only use structures up to the differential level of complexity for now.

## 3 Electromagnetism in terms of Differential forms in Minkowski Space

Then, in electromagnetism, the basic equations are just Maxwell's equations defined in 3-D space with time as an independent variable. However, they can be expressed covariantly with the electric potential and the electromagnetic tensor, $F_{uv} = d_u A_v - d_v A_u$ where $A = A_u\omega_u = A(x)$ is the potential, a scalar field all over spacetime, which is just Minkowski spacetime in this case. A defines B and E by $E = \nabla A_t, B = \nabla \times A$ . Going through this derivation is constructive. Maxwell's equations are, in traditional vector calculus form, with the usual electric and magnetic fields in 3 dimensions, $E, B$ , $\nabla \cdot E = \rho$ , $\nabla \cdot B = 0, \nabla \times B = J + \frac{dE}{dt}, \nabla \times E = -\frac{dB}{dt}$ . Then, Maxwell's equations are just $d_u F_{u,v} = \mu_0 J_v$and then $d_{[u}F_{v\sigma]} = 0$ However, another way of expressing it is in terms of exterior calculus. This is where we can define the wedge product $\omega^u \wedge \omega^v = \omega^u\omega^v - \omega^v\omega^u$ , the antisymmetric tensor product of two basis one forms., and the Hodge star of a tensor $*(A_{ij}) = \epsilon_{ijkl}A^{ij} = A_{kl}$, where the tensor in this case is of rank 2. Generally it takes a tensor of rank n and takes it to a rank of $4-n$, or $d-n$ in the general d-dimensional case. Then,$*dx^u = dx^v \wedge dx^\alpha \wedge dx^l, u \neq, v, \alpha, l$ , for example then with similar results for the other cases. In general, in 4 dimensional spacetime, this will just be the contraction of any given tensor with the 4 dimensional Levi-Cita tensor. One can also define the gradient, or exterior derivative, as an operation $d$ which takes a tensor, of say n covariant entries, and creates one with n+1 covariant entries. For example, for a scalar $f = f(x)$ , it result in just the gradient $df = f_{,\alpha}dx^\alpha, f_{,\alpha} = \partial_\alpha(f)$ (just a convenience notation) For a 1 one form $\rho = \rho_u dx^u, d\rho = (\rho_u)_{,\sigma}dx^u dx^\sigma$, a 2-form. Note that for any given p form, given $R, dR = (R_\alpha)_{,u}dx^\alpha dx^u$ .Then , $d^2(R) = ddR = ((R_\alpha)_{,u})_{,\sigma}dx^\alpha \wedge dx^\sigma$ . By commutativity of any partial derivatives, the component is symmetric but the wedge product is anti-symmetric, so it vanishes meaning $((R_\alpha)_{,u})_{,\sigma}dx^\alpha \wedge dx^\sigma = (-1)((R_\alpha)_{,\sigma})_{,u}dx^\alpha \wedge dx^\sigma = 0$ .Now any tensor can be expressed in terms of a symmetric part and some anti-symmetric part $T = T_s + T_a = AB = \frac{1}{2}([A,B] + \{A,B\})$ , with the standard definitions of the commutator and the anti-commutator. However, looking at the curl, one can write $\nabla \times A = \varepsilon_{ijl}\partial_i(A_j)\omega^l$ with the time component being 0, so it is expressed in 3 spatial dimensions. This operation is antisymmetric in i and j, so only the anti symmetric part is needed of the corresponding two form expressed, $\rho_{ji}dx^l = ((A_j)_{,i} - (A_i)_{,i})dx^l + [cp](A, ijl)$ , where $[cp](A, ijl)$ cyclic permutations of $A$ with respect to labels $ijl$. Then $\rho_{ij} = (A_i)_{,j} - (A_j)_{,i}$ . Then, the magnetic field is actually a 2-form with the corresponding components, and can be represented with $B(V,W) = B_{ij}dx^i dx^j = \frac{1}{2}B_{ij}dx^i \wedge dx^j$ , since it is completely anti- symmetric. The electric field is a one form, as it is the exterior derivative, or gradient in this case, of a scalar,. Then, $E = dA = \partial_u(A)dx^u$ . For convenience, For convenience, $\{dt, dx, dy, dz\} = dx^i$.Now, wanting to represent both E and B in spacetime and both together, we can define the two form or rank 2 tensor $F = E\wedge dt + B = A_\alpha dx^\alpha \wedge dt + B = \frac{1}{2}(A_{,\alpha}dx^\alpha \wedge dt + B_{i\alpha}dx^i \wedge dx^\alpha)$. Then, also defining the current and charge as an one-form instead of a vector , $\rho dt + J_i dx^i = J_u dx^u$ . Maxwell's two equations can be put in the form of differential, exterior calculus , $d(E \wedge dt) = d(dA \wedge dt) = 0$, $dB = 0 \rightarrow dF = 0$ . Now, one can get rest of the Maxwell equations, or more specifically the second pair, by taking the Hodge star of F using $*F = *E \wedge (dt) + *B = (E_\sigma dx^\sigma \wedge dt + \frac{1}{2}B_{ij}dx^i dx^j) = (E_k\varepsilon_{j0} + B_{ij})dx^k \wedge dx^\sigma, i \neq j \neq k \neq \sigma$. Then, $*(E \wedge dt) = *(E_x dx^1 + E_y dx^2 + E_z dx^3) \wedge dt = (E_x dx^2 \wedge dx^3 + E_y dx^3 \wedge dx^1 + E_z dx^1 \wedge dx^2) \wedge dx^\alpha \rightarrow *B = (B_z dx^1 \wedge dx^2 + B_y dx^3 \wedge dx^1 + B_x dx^2 \wedge dx^(3)) = (B_z dx^3 \wedge dt + Bxdt \wedge dx^2 + Bydx^0 t \wedge dx^1) wedgedx^\alpha$ Then, $d*F = d*E\wedge dt + d*B = Ex_{,\alpha}dx^2 \wedge dx^3 + Ey_{,\alpha}dx^3 \wedge dx^1 + Ez_{,\alpha}dx^1 \wedge dx^2 + (Bx)_{,\alpha}dx^3 \wedge dt + (By)_{,\alpha}d(x^3) \wedge dt + (B_z), \alpha dt \wedge dx$ $*J = \rho dx^1 \wedge dx^2 \wedge dx^3 + j_x dx^2 \wedge dx^3 \wedge dt + j_y dx^3 \wedge dt \wedge dx^1 + jzdt \wedge dx^1 \wedge dx_2$ Of course, each of the wedge products is a basis element, so then setting separate basis elements with their components equal to each other, one get the remaining Maxwell's equations in the form $d * F = *J$ These are Maxwell's equations in the form of differential, exterior calculus.

## 4 Yang-Mills fields in Minkowski space

A given Yang-Mills field are just the generalization of the electromagnetic tensor and field: in fact, a Yang Mills field reduces to the electromagnetic tensor in the case that the generators of the field commute. Generalizing, considering the electromagnetic tensor which was expressed in terms of geometric quantities, or as a tensor field defined over the spacetime manifold. Another viewpoint of looking at the same thing is the case where one can consider the Lagrangian density, which is defined over the spacetime as then $L = -\frac{1}{4}(F_{uv}F^{uv} + JA)$ with the usual current and vector potential definitions. The equations of motion, Maxwells equations in whatever form, come from the stationary points of this action, $S = \int d^4xL(x)$. Varying it is similar to varying it individually at each point in spacetime as it is true for any volume that is integrated over. Now the point of this is to look at the nature of $F$ , more specifically A,and consider the case of non-Abelian fields $A_u$ ,such that $[A_u, A_v] \neq 0$ , like an quantum mechanical operator field, Then, $F$ ,generalizing, is then in component form with respect to some local basis is then$F_{uv} = d_uA_v - d_vA_u + [A_u, A_v]$ . Of course, for the A to not commute there has to be multiple of them, which are called the generators of the corresponding transformation on spacetime that leaves the Lagrangian invariant. The relationship can be defined as $[A_i, A_j] = f_{ijl}A^l$ where the $f_{ijl}$ are called the fine-structure constants. Denoting each of these generators by some index, one can write $F^l_{up} = d_uA^l_p - -d_p(A)^l_u + f^l_{ijn}A^n$ where $l, f$ is a internal index which has nothing to do with the spacetime,but rather the number of generators of the particular corresponding transformation. For the electromagnetic transformation, of course, the last term trivially disappears, as it has only one generator (A). Now, these have to do with symmetries of the Lagrangian, meaning that they are algebraic symmetries rather than local symmetries defined on the spacetime itself.

Now, if one defines an abstract vector space and then a vector field where the transformation can take place, it would be viewed as a geometric symmetry like that in general relativity of diffeomorphsim invariance of the metric being unchanged under local Lorentz transformations. This leads to the concepts of fiber bundles and principle bundles, and how to generalize what is going on geometrically. A fiber bundle is defined by the following structure: $(E, M, \sigma)$ where E is a manifold of some dimension, M is another manifold, and $\sigma = \sigma(E) = M$ is a map from the manifold E to the manifold M. Then, the fiber part of this bundle is the space $E_p = \sigma(E(r) = p))$, where p is a point on M and r is a point on E. It is the projection of $E(r)$ onto M at a given point p and so is called the fiber over p. So this usually would correspond to say the tangent space at a given point in spacetime if E is the set of all tangent spaces and M is the spacetime manifold in general relativity. The set of all such $E_p$ for all points p on E is called a bundle of fibers, meaning just E itself. The point of this is it allows one to define multiple bundles or sets of different vector spaces on the spacetime. This allows for the definition of several possible transformations on the spacetime, not just the Lorentz transformations. Then, exactly the way it is done in general relativity, one can associate a connection and therefore separate curvature with each fiber bundle. Then, for example, the standard fiber bundle the tangent bundle $TM$, with the corresponding connection, the Levi-Cita connection which acts on a locally defined basis vector at a point on the manifold by $\nabla_u(e_v) = \Gamma^{\beta}_{vu} e_{\beta}$ , with properties that $T_{uv} = \nabla_u(e_v) - \nabla_v(e_u) - [e_u, e_v] = 0, \nabla_{\epsilon}(g_{uv}) = 0$. It is the unique connection that is torsion free and leaves the metric invariant. Now, of course one can define other connections that do not have these properties, but this connection is the one which yields the best results in general relativity. Note that bundles associate with forces are called principal bundles and those associated with the matter fields on the spacetime that travel on the manifold are called vector bundles (obvious definition)

Then, let's describe the other possible fibers that have corresponding connections and curvatures. One can look at the electromagnetic tensor $F_{uv} = d_u A_v - d_v A_u$ and see that it has the structure that is similar to the Riemann tensor and its expression written in terms of connection coefficients $R^n_{puv} = d_u \Gamma^n_{vp} - d_v \Gamma up^n + \Gamma^n_{uq} \Gamma^q_{vp} - \Gamma^n_{vq} \Gamma^q_{up}$. The last two terms vanish in this case, meaning a coordinate basis is being used to describe the local space. Let's say then to associate the given e electromagnetic field with a fiber bundle, $PM$ such that one can associate the transformation associated with the given electromagnetic field as acting in this space of the bundle. There are corresponding $v = v^u e_u$ vectors and the rest of the structure, as there is for the tangent bundle, on this bundle. Now in quantum field theory, recall that if a particle moves according to the standard derivative with no external forces acting on it, then it moves according to the action of the covariant derivative $D_u = d_u - i\theta A$ , where A is the associated boson field, interacting with it. Then, realizing that this covariant derivative has the same structure as that of the covariant derivative of vectors being transported in curved spacetime, one can say that this is the covariant derivative associated with the fiber bundle of the space related to the electromagnetic field, and the A, the 4-vector potential in this case, is the corresponding connection coefficient. Then, one can define the curvature tensor, similar to the Riemann tensor , as then $X_{uv} = D_u A_v - D_v A_u$ , exactly the same or is in this case as the electromagnetic tensor since $X_{uv} = (d_u - i\theta A)(A_v) - (d_v - i\theta A)A_u = F_{uv}$. The fiber bundle in this case is called the principle bundle associated with the corresponding transformation generated by the corresponding gauge boson. Now, one can associate principle bundles corresponding to all the possible gauge boson forces or interactions with corresponding matter fields like that is done in quantum field theory. For a Yang Mills theory, then the associated principle bundle would have the covariant derivative $D_u = d_u - i\theta_u A^u$ where the A correspond to the possible generators of the transformation associated with the gauge boson. The associated curvature tensor is then $X_{uv} = D_u A_v - D_v A_u - [A_u, A_v]$ . It is then possible to associate multiple gauge boson fields with corresponding fiber bundles on the spacetime simultaneously.

Now, the interesting part is the dimensionality of these manifolds and the corresponding geometric interpretations. There is a connection $D_u$ , and therefore curvature $F^{\alpha}_{uvj}$ associated with each boson and its transformation, but only tangent bundles seem to have geometrical properties that are physically manifest. The other bosons have geometrical properties which are associated with the internal symmetry space of the corresponding fiber bundle. In fact, on e think of these bundles as implying, or mathematically inducing, a corresponding manifold exactly similar to that of spacetime in general relativity. These are like "images" or copies of the spacetime manifold say with the corresponding curvatures and connections defined or interpreted exactly analogous to that the geometrical interpretations in general relativity. However, let's look at this from a different perspective since these image manifolds are not physical apparently. The usual approach is to leave the base manifold, say M, as it is, defined to be the spacetime of GR. The overall dimension of the structure say with the usual tangent bundle of GR and connection, with the metic that describes gravity, and say one other principle bundle that describes a given interaction of the matter field with the corresponding gauge boson. The total dimension of the structure is then $D_T = D_M + D_G = 4 + d$ where $d$ is the dimension of the gauge group. Then, one can either add some of the dimensions of the interactions of the space described by the fiber bundle (principle bundle) to that described by the tangent bundle. The total dimensionality of the structure remains , or should be to maintain the structure of theory, the same, but it is just a question of what should be included physically or not. There are of course different approaches to this which result in different models or theories of quantum gravity. I will go over all this in another time, but would just like to point out one possible approach that is novel seemingly or at least not mainstream in therms of a possible idea.

# 5 Possible idea of a theory of uniting the interpretations of both Quantum fields and Gravity

However, I would like to suggest an alternate theory that may make more physical sense or follow the logic of quantum field theory while also including the symmetries of general relativity. Take the manifold in GR that exists say with a global matter or fermion field on it by description of its corresponding vector bundle. Suppose there is no say internal vector space for each fiber bundle of the two gauge forces and the 1 matter field, but rather there is only one vector space $L$ that takes different interpretations or acts in different ways depending on what part of the manifold it is on. Now, imagine the manifold itself can be described not just by each connection distinctly, bu rather a linear superposition of possible connections $\Sigma = x_u \eta^u$ . This means the connection can take on different forms depending on the part of the manifold it is on, or more properly, maybe what part of the matter or fermion field it interacts with. This comes form realizing the , we know that from quantum field theory and the standard model that bosons are force carriers and fermions are the matter fields which interact with these force carries and get "acted on", meaning some of their proprieties change in some way. However it is know that fundamentally there are only or generally speaking two types of fermions, leptons and quarks. The only difference between them is that the strong force acts o n the quarks, while it does not for the rest of the fermions meaning the leptons. Now, it is experimentally known that the number of quarks in the universe, roughly identified as the Baryon number, is much more than the number of leptons in the universe. These are conserved in any given interaction, at least those described by the Standard Model. However, lets take this viewpoint that the only observed difference between leptons and quark comes in the fact that quarks are observed to have a quantity called color charge associated with them. Leptons do not. This means we will leave the quantity difference of a type of property alone for now, such as the fractional difference in electric charge between quarks and leptons. However, this property comes only because of their interaction wit the strong nuclear force, alternative described by the gluon field, with 8 corresponding gauge bosons called gluons represented by the generators of the SU(3) transformation- this is an algebraic symmetry. However, one needs to realize that participles or the properties that define them only come out of the invariance of these participles in their equations of motion to certain gauge transformations. Hypothetically, if these gauge interaction did not exist, then threw would be no way to tell particles apart from one another,meaning they will be indistinguishable. Looked at in this way, all the properties that participles have , such as mass, charge, color, and most of the rest come from these transformations or the interaction of the fermions with these different sets of bosons. Spin is the only seemingly intrinsic or definite quantity. However, lets take a view at the picture that spacetime is made of up a superposition of connections, meaning the curvature many not be describable with just a single connection, but rather a quantum m superposition of possible ones. Actually, let s just not even define it as quantum and just call it a linear superposition. Say this field of fermions is interacting with this field of gauge bosons. If the fermion field is interacting at a point where the connection say corresponding to the strong fore is more, or has a larger component in the overall state vector of the system, then there will be the definition of the fermion associated with that execration at that point in spacetime will have primarily the property of he strong force, meaning a quark. If there is no such excitation there but rather at another point, it will have the excitation as that of a lepton. Also ,one can say that a strong interaction is just the electromagnetic interaction looked at from a non - coordinate basis. This initially seems like a contradictory statement as not even the dimension o of the transformations are the soma, but say that there is such a transformation, though it will be unphsyical, meaning the associated reference "observer" is moving faster than that of light such that a point on the lightcone inside is transformed to a point outside the light cone as usually defined, Then however, there will be a so called "critical point" or more accurately a hypersurface where the interaction of the fermion with the boson field becomes quantified. This means that the gravity part of the connection no longer dominates compared to the rest of the parts of the connection. However,then, this can also have to do with the length scale or that of the mass related or proportional to the "size of the excitation of the fermion field", so this will also address the scale question. Looking at the overall structure, all there is a base manifold with two fiber bundles defined on it, one principle bundle and one vector bundle. Since these describe or can take the form of different gauge forces and different matter fields, it means that these are the only two actually fully separate bundles. How does or can that be described simultaneously? Well, spacetime or many manifold with reasonable properties can be intrinsically defined can be fully by the first and second fundamental from. These are of course curvature and torsion, and since curative is already associated with the boson field and so one can then naturally associate the fermion field with the torsion. This can also make sense physically, as the deviation by how much a particle or more accurately a vector field, or even more accurately, space itself, twists as it moves in circle results in an so called excitation that is related to he fermion field and can be described the torsion. This effect is not really seen in GR because of the fact that spin does not really play a role because of the scale of the particle involved, and so a torsion free theory is accurate, as it is for GR. This is of course no the only possibility, and the aspect of spin for bosons and fermions needs to be included explicitly. Perhaps a vector than can be like two fold or have a dual aspect, such as that of a spinor describing two states simultaneously, which has the properties of describing both the fermions, but then acts as an curvature measure too then for the gauges. Of course, there a lot of similar theories and quantum gravity models already out there, such as twistor theory, loop quantum gravity, etc which incorporate some of these aspects in some ways. I will go into more detail in a later time.