Derivation of the Geodesic Equation and Riemann Tensor from the free particle Lagrangian in curved spacetime

Geodesic Equation

For a metric which varies with spacetime, or at least, the spacetime coordinates, $g_{uv} = g_{uv}(x)$, $\partial r \to \partial_\alpha$ for each of the coordinates in spacetime, which means 4 independent variables in the Lagrangian $(t; x, \dot{x}) \to (t, x_\alpha, \dot{x_\alpha})$. Then, $\frac{\partial}{\partial \dot{x}^v} L = \frac{m}{2} g_{\alpha\beta} \frac{\partial(\dot{x}^\alpha \dot{x}^\beta)}{\partial \dot{x}^v}$. Since, ∂x^α are orthogonal basis vectors, $\frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta$ $\to \frac{\partial}{\partial \dot{x}^v} L = \frac{m}{2} g_{\alpha\beta}(\delta^\alpha{}_v \dot{x}^\beta + \delta^\beta{}_v \dot{x}^\alpha)$.

Summing over repeated indices, either the α or the β in either order since they are merely dummy indices, $\frac{\partial}{\partial \dot{x}^v} L = \frac{m}{2}(g_{v\beta}\dot{x}^\beta + g_{\alpha v}\dot{x}^\alpha) = \frac{m}{2}(\dot{x}_v + \dot{x}_v) = m\dot{x}_v$.

Also, $\frac{\partial L}{\partial x^v} = \frac{m}{2} \partial_v(g_{\alpha\beta})\dot{x}^\alpha \dot{x}^\beta$. Then, $\frac{d}{dt}(\dot{x}_v) = \frac{dx^\alpha}{dt} \partial_\alpha(\dot{x}_v)$. Then, $\frac{m}{2} \partial_v(g_{\alpha\beta})\dot{x}^\alpha \dot{x}^\beta = m\dot{x}^\alpha \partial_\alpha \dot{x}_v$.

Noticing that one side has a lowered velocity index and the other side has a raised velocity index, to make it more readable,
$\dot{x}_v = g_{v\gamma}\dot{x}^\gamma \Rightarrow \frac{m}{2} \partial_v(g_{\alpha\beta})\dot{x}^\alpha \dot{x}^\beta = m\dot{x}^\alpha \partial_\alpha(g_{v\gamma}\dot{x}^\gamma) \Rightarrow \frac{1}{2} \partial_v(g_{\alpha\beta})\dot{x}^\alpha \dot{x}^\beta = \partial_\alpha(g_{v\gamma})\dot{x}^\gamma \dot{x}^\alpha + g_{v\gamma}\dot{x}^\alpha$

Then, using the symmetry of the lower two indices of g,
$\partial_\alpha(g_{v\gamma})\dot{x}^\gamma \dot{x}^\alpha = \frac{1}{2}(\partial_\gamma(g_{v\alpha})\dot{x}^\alpha \dot{x}^\gamma + \partial_\alpha(g_{v\gamma})\dot{x}^\gamma \dot{x}^\alpha) = \frac{1}{2}(\partial_\gamma(g_{v\alpha}) + \partial_\alpha(g_{v\gamma}))\dot{x}^\gamma \dot{x}^\alpha$.

This implies that $\frac{1}{2}(\partial_v(g_{\alpha\gamma}) - (\partial_\gamma(g_{v\alpha}) + \partial_\alpha(g_{v\gamma})))\dot{x}^\gamma \dot{x}^\alpha = g_{v\gamma}\ddot{x}^\gamma$.

Multiplying both sides by $g^{\sigma v}$ which recall is the same as multiplying both sides on an equation by a matrix, which is completely valid,
$g^{\sigma v}\frac{1}{2}(\partial_v(g_{\alpha\gamma}) - \partial_\gamma(g_{v\alpha}) - \partial_\alpha(g_{v\gamma}))\dot{x}^\gamma \dot{x}^\alpha = \delta^\sigma{}_\gamma \ddot{x}^\gamma \Rightarrow \frac{1}{2} g^{\sigma v}(\partial_v(g_{\alpha\gamma}) - \partial_\gamma(g_{v\alpha}) - \partial_\alpha(g_{v\gamma}))\dot{x}^\gamma$

With the definition $-\Gamma^\sigma_{\gamma\alpha} = \frac{1}{2} g^{\sigma v}(\partial_v(g_{\alpha\gamma}) - \partial_\gamma(g_{v\alpha}) - \partial_\alpha(g_{v\gamma}))$ then, $\ddot{x}^\sigma + \Gamma^\sigma_{\gamma\alpha}\dot{x}^\gamma \dot{x}^\alpha = 0$.

This is the classic equation for geodesic motion, or the motion of a particle along a geodesic, in GR. Usually derived in other, slightly less intuitive ways, this derivation of the geodesic equation came simply from the free particle Lagrangian in a curved spacetime. The only difference between the usual constant velocity motion in flat spacetime and this more general derivation is that the metric g is allowed to change as an equation of x on the spacetime. No external force but rather the curvature itself is what gives this notion of a force acting on the particle. However, there is no force, but the particle is rather simply following the shortest path in this spacetime, which is called a geodesic.

Christoffel Symbols

The symbols Γ defined in the geodesic equation are called the Christoffel symbols. As will be later explained, these are not the components of an invariant tensor on the spacetime like the metric tensor is for example, but are rather coordinate dependent.

Also, the dimensionality of the Christoffel symbols seems to be $Dim(V)^3$ where $Dim(V)$ is the dimension of the underlying vector space the spacetime is defined on, which is 4 in this case. However, $\Gamma^{\sigma}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$ from where this definition arose implies that the commutativity of $\dot{x}$ implies that Γ is invariant under exchange of the lower two indices, which is easy to confirm. Then, there are actually $4 * Dim(Sym(M^{4\times 4})) = 4 * 10 = 40$ independent components. Note there are 16 components in just the lower two indices, and the symmetry upon exchange, means there are 10 independent components. This is easier to see when treating it as a symmetric 4×4 matrix, which is what the above notation signifies.

Now, in the case of flat spacetime, the shortest path between any two points is a line as is well known. The geodesic equation should reduce to $\ddot{x} = 0$ in the case of the metric being constant as in this case. Obviously, if the coordinate system used was Cartesian, $g_{00} = 1, g_{ij} = \delta_{ij}$, all the components of the metric being constant should reduce this zero as can easily be seen. However, this should be true in any coordinate system one uses on this spacetime. Consider then spherical coordinates, the length of an element is $(d\tau)^2 = (dr)^2 + r^2(d\theta)^2 + (r\sin\theta)^2(d\phi)^2$. The metric elements are $g_{rr} = 1, g_{\theta\theta} = r^2, g_{\phi\phi} = (r\sin\theta)^2$. Note the diagonality of this metric ensures that $g^{uv} = \frac{1}{g_{uv}}$The only non-zero components of the Christoffel symbols are $\Gamma^{r}_{\theta\theta} = -r, \Gamma^{r}_{\phi\phi} = -r\sin^2\theta, \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r}, \Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} = \frac{1}{r}, \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} =$. There are 9 non-zero components, but they should all reduce to 0 in some way to ensure that the description of the spacetime is coordinate independent. Because the Christoffel symbols are different depending on the coordinate system being used, they are not invariant under coordinate transformations and therefore do not form the part of an coordinate invariant tensor.

Geodesic Deviation

Consider now the problem of ensuring a coordiante invariant description of spacetime. This is where the concept of geodesic deviation comes in. A particle will move exactly the same regardless of what coordinate system it is being described in. For the case of free particle motion, it will move along geodesics. Consider a geodesic on the spacetime being described a single parameter r. Obviously, there is an infinite number of possible geodesics on the spacetime, just like how there is any number of straight lines in regular 3d space. Describe this family of geodesics by another parameter λ. These are both uncountabily infinite parameters where the function $x^{\alpha}(r, \lambda)$ descibes any point on a given geodesic. Consider the partial derivatives $\frac{\partial x^{\alpha}}{\partial r} = t^{\alpha}$, $\frac{\partial x^{\alpha}}{\partial \lambda} = V^{\alpha}$, $\frac{\partial^2 x^{\alpha}}{\partial r \partial \lambda} = \frac{\partial^2 x^{\alpha}}{\partial \lambda \partial r} \Rightarrow \frac{\partial V^{\alpha}}{\partial r} = \frac{\partial t^{\alpha}}{\partial \lambda}$. V is the displacement vector from one geodesic to a neighboring geodesic, which in the case of a flat spacetime, would be constant and normal always. However, if V does change as the particle moves along a geodesic, then the spacetime is not flat, but rather curved. How could this be expressed? Well, the motion of a particle that is free was described by the geodesic equation. A particle that moves according to the geodesic equation means that it is not accelerating, but rather moving at a constant 'velocity', which means its velocity vector which is tangent to its motion, $\dot{x}$, follows the geodesic equation. Then for there to be no geodesic deviation, $\ddot{V}^{\sigma} + \Gamma^{\sigma}_{\gamma\alpha}\dot{V}^{\gamma}\dot{V}^{\alpha} = 0$ is the condition V must satisfy. The tangent vector t is always parallel to $x(r)$ by definition. In flat space, the condition that a particle has no acceleration is $\ddot{x} = 0 \Rightarrow \dot{t} = \frac{\partial t}{\partial r} = 0$. Here the 'dot' is to indicate derivative with respect to the arc length parameter r. In curved space, what is the analog? The geodesic equation which described constant motion of a particle in curved space $\frac{\partial \dot{x}^{\alpha}}{\partial \tilde{t}} + \Gamma^{\alpha}_{uv}\dot{x}^{u}\dot{x}^{v} = 0$. The parameter $\tilde{t}$ is just time in this case, but note that since time is an axis in spacetime, it can very well be considered a parameter just like r. In fact, the parameter used to describe a path a particle takes in spacetime, or a line, does not determine the line itself, obviously. Only the rate at which the line is 'traveresed' is changed. If $x(t)$ is the path a particle takes, $t \to r = r(t) \Rightarrow dr = \frac{\partial r}{\partial t}dt \Rightarrow x(t) \to x(r) = x(t) = x$. Then, $\frac{\partial x}{\partial \tilde{t}} \to \frac{\partial x}{\partial r}$. This makes the geodesic equation $\frac{\partial t^{\alpha}}{\partial r} + \Gamma^{\alpha}_{uv}t^{u}t^{v} = 0$. Define the operator $\nabla_r t^{\alpha} = \frac{D}{\partial r}t^{\alpha} = \frac{\partial t^{\alpha}}{\partial r} + \Gamma^{\alpha}_{uv}\frac{\partial t^{u}}{\partial r}\frac{\partial t^{v}}{\partial r}$ for a vector t and a parameter r. In flat spacetime, the interpretation of acceleration is how much the velocity vector of a particle changes. Implictly, the direction is also included in this definition. The acceleration is considered to be 0 if the velocity vector does not change in any given direction. Circular motion is the most common example of where the velocity is not changing in the direction it is pointing, meaning in the direction of a vector dr which is tangent to it, but rather in the direction of a vector $d\tilde{r}, d\tilde{r} \cdot dr = 0$ completely perpendicular to it. With this intuition, define $\frac{D}{\partial r}t$ as the amount the vector t changes in the direction of dr. This is the definition of the covariant derivative on spacetime. Since a vector can change in an infinite number of directions, the direction of the derivative must be defined. Then, a particle traveling along a geodesic then, or without any external acceleration, satisfies the condition $\nabla_t t = 0$ where t is the tangent vector of that particle at any given point.

Going back to the family of geodesics $x(r, \lambda)$, note that λ is a parameter too but along the direction of traversal between all the geodesics, meaning $d\lambda$ points in the direction of an neighboring geodesic. Then, the equation for geodesic deviation seems to be $\frac{D}{D\lambda}t$ which describes how much the tangent vector changes in the direction of the 'normal vector' meaning how much a particle along a geodesic is getting closer to another geodesic or not. However, this is not an equation one can set to 0 to describe geodesic deviation since a constant change is possible and would be indicative of the coordinate system being used. For example, two straight lines in flat spacetime being described by polar coordinates would seem to be moving away from each other, but that is because of the nature of the coordinates being used. Then, to accurately describe geodesic deviation, one really needs $\frac{D}{Dr}\frac{D}{D\lambda}t = 0$. This means the acceleration of nearby geodesics is actually 0. As the particle moves along a geodesic with a tangent vector t, the rate of change of its motion in the direction of all nearby geodesics does not change. This is the equivalent in flat spacetime of parallel lines never meeting and staying parallel forever. This equation for geodesic deviation ensures that a path satisfying it is equivalent to that path being a geodesic.

Riemann Tensor

To look at this more quantitatively,

$$\frac{D}{Dr}\frac{D}{D\lambda}t^\alpha = \frac{D}{Dr}\left(\frac{\partial t^\alpha}{\partial\lambda} + \Gamma^\alpha_{uv}V^u t^v\right) = \frac{\partial^2 t^\alpha}{\partial r\partial\lambda} + \Gamma^\alpha_{\eta\sigma}t^\eta\frac{\partial t^\sigma}{\partial\lambda} + \frac{\partial}{\partial r}(\Gamma^\alpha_{uv}V^u t^v) + \Gamma^\alpha_{\eta\sigma}t^\eta\Gamma^\sigma_{uv}V^u t^v = \frac{\partial^2 t^\alpha}{\partial r\partial\lambda} + \Gamma^\alpha_{\eta\sigma}t^\eta\frac{\partial t^\sigma}{\partial\lambda} + \partial r(\Gamma^\alpha_{uv})V^u t^v + \Gamma^\alpha_{uv}\frac{\partial V^u}{\partial r}t^v + \Gamma^\alpha_{uv}V^u\frac{\partial t^v}{\partial r} + \Gamma^\alpha_{\eta\sigma}\Gamma^\sigma_{uv}t^\eta V^u t^v.$$

Since $\partial_r = \partial_r x^\alpha\partial_\alpha = \frac{\partial x^\alpha}{\partial r}\partial_\alpha = t^\alpha\partial_\alpha$,

then

$$\frac{D^2}{DrD\lambda}t^\alpha = \frac{\partial^2 t^\alpha}{\partial r\partial\lambda} + \Gamma^\alpha_{\eta\sigma}t^\eta\frac{\partial t^\sigma}{\partial\lambda} + \partial_\eta(\Gamma^\alpha_{uv})t^\eta V^u t^v + \Gamma^\alpha_{uv}\frac{\partial V^u}{\partial r}t^v + \Gamma^\alpha_{uv}V^u\frac{\partial t^v}{\partial r} + \Gamma^\alpha_{\eta\sigma}\Gamma^\sigma_{uv}t^\eta V^u t^v.$$

Notice from the geodesic equation which t satisfies $\frac{\partial t^v}{\partial r} = -\Gamma^v_{\eta\sigma}t^\eta t^\sigma$, and from before, $\frac{\partial V^\alpha}{\partial r} = \frac{\partial t^\alpha}{\partial\lambda}$, the equation for geodesic deviation becomes

$$\frac{\partial}{\partial\lambda}(-\Gamma^\alpha_{\eta\sigma}t^\eta t^\sigma) + \Gamma^\alpha_{\eta\sigma}t^\eta\frac{\partial t^\sigma}{\partial\lambda} + \partial_\eta(\Gamma^\alpha_{uv})t^\eta V^u t^v + \Gamma^\alpha_{uv}\frac{\partial t^u}{\partial\lambda}t^v + \Gamma^\alpha_{uv}V^u(-\Gamma^v_{\eta\sigma}t^\eta t^\sigma) + \Gamma^\alpha_{\eta\sigma}\Gamma^\sigma_{uv}t^\eta V^u t$$
$$\frac{\partial}{\partial\lambda}(-\Gamma^\alpha_{\eta\sigma})t^\eta t^\sigma - \Gamma^\alpha_{\eta\sigma}\frac{\partial}{\partial\lambda}(t^\eta t^\sigma) + \Gamma^\alpha_{\eta\sigma}t^\eta\frac{\partial t^\sigma}{\partial\lambda} + \Gamma^\alpha_{\eta\sigma}\frac{\partial t^\eta}{\partial\lambda}t^\sigma + \partial_\eta(\Gamma^\alpha_{uv})t^\eta V^u t^v + \Gamma^\alpha_{uv}V^u(-\Gamma^v_{\eta\sigma}t^\eta t^\sigma)$$
$$\frac{\partial x^u}{\partial\lambda}\partial_u(-\Gamma^\alpha_{\eta\sigma})t^\eta t^\sigma + \partial_\eta(\Gamma^\alpha_{uv})t^\eta V^u t^v + \Gamma^\alpha_{u\sigma}V^u(-\Gamma^\sigma_{\eta v}t^\eta t^v) + \Gamma^\alpha_{\eta\sigma}\Gamma^\sigma_{uv}t^\eta V^u t^v$$

With the definition of $\frac{\partial x^u}{\partial\lambda}$
$$\Rightarrow \frac{D^2}{DrD\lambda}t^\alpha = (-\partial_u\Gamma^\alpha_{\eta v} + \partial_\eta\Gamma^\alpha_{uv} - \Gamma^\alpha_{u\sigma}\Gamma^\sigma_{\eta v} + \Gamma^\alpha_{\eta\sigma}\Gamma^\sigma_{uv})t^\eta V^u t^v = 0.$$

The quantity $-\partial_u\Gamma^\alpha_{\eta v} + \partial_\eta\Gamma^\alpha_{uv} - \Gamma^\alpha_{u\sigma}\Gamma^\sigma_{\eta v} + \Gamma^\alpha_{\eta\sigma}\Gamma^\sigma_{uv}$ is defined as the Riemann tensor

$R^\alpha_{v\eta u} = -\partial_u\Gamma^\alpha_{\eta v} + \partial_\eta\Gamma^\alpha_{uv} - \Gamma^\alpha_{u\sigma}\Gamma^\sigma_{\eta v} + \Gamma^\alpha_{\eta\sigma}\Gamma^\sigma_{uv}$ meaning that if $R^\alpha_{v\eta u}t^\eta V^u t^v = 0$,

then there is no geodesic deviation and the particle is traveling along a geodesic.

This derivation of the Riemann tensor is different from how it is usually derived in most approaches. One usually takes the argument of a particle moving around an infinitesimal square and the deviation of the velocity vector of this particle from before moving to after moving around the square is captured by the Riemann tensor. This parallel transport of a vector around a closed loop would result in no deviation if there was no curvature on the surface, as is the case in flat spacetime, but since there is, it is a non-zero quantity in curved spacetime, which is true in GR.

Also note that the derivation of the Riemann tensor was done based only on the assumption of a non-constant metric g and the principle of least action. This shows the correspondence between curvature and how it arises naturally as a non-zero quantity if there is a non-constant metric. Even if a particle is traveling along a geodesic, the non-constant metric creates an acceleration which is independent from any external force on the particle, but rather is because of the surface it is traveling 'on'.

As an example, consider the case of spherical coordinates with the already calculated Christoffel symbols.
$\Gamma^r_{\theta\theta} = -r, \Gamma^r_{\phi\phi} = -r\sin^2\theta, \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}, \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} =$
. There being no curvature means the Riemann tensor should have all its components be 0.

This analysis will be made much simpler when one looks at the symmetries of this tensor. Looking at the Riemann tensor plainly, it seems there should be 4^4 different components in 4d spacetime. However, as shall be shown, there are actually a lot of slightly hidden symmetries in it.