C.M

HW #6 CHAPT 5

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Prob #'s 2, 6, 8, 18, 22, 30

#2

5.2 ★ The potential energy of two atoms in a molecule can sometimes be approximated by the Morse function,

$$U(r) = A\left[\left(e^{(R-r)/S} - 1\right)^2 - 1\right]$$

where r is the distance between the two atoms and A, R, and S are positive constants with $S \ll R$. Sketch this function for $0 < r < \infty$. Find the equilibrium separation r_o, at which $U(r)$ is minimum. Now write $r = r_o + x$ so that x is the displacement from equilibrium, and show that, for small displacements, U has the approximate form $U = \text{const} + \frac{1}{2}kx^2$. That is, Hooke's law applies. What is the force constant k?

MORSE FUNCTION:

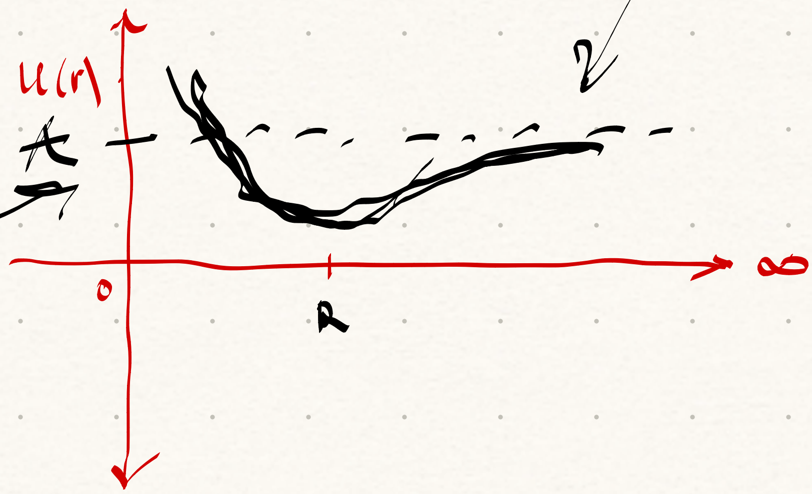
$$U(r) = A\left[\left(e^{(R-r)/S} - 1\right)^2 - 1\right]$$

r: distance 2 atoms

A, R, S: positive constants $(S \ll R)$

as $r \to \infty$, $U \to A$

as $r \to 0$, $U \to \infty$



r_{eq} at $\frac{dU}{dr} = 0$, $\frac{d}{dr} A\left[\left(e^{(R-r)/s} - 1\right)^2 - 1\right]$

$$= A \cdot 2\left(e^{(R-r)/s} - 1\right) \cdot \frac{d}{dr}\left(e^{(R-r)/s} - 1\right)$$

$$= A \cdot 2\left(e^{(R-r)/s} - 1\right) \cdot \frac{-1}{s}\left(e^{(R-r)/s}\right)$$

$$= \underbrace{-\frac{2A}{s} e^{(R-r)/s}}_{\text{These aren't zero}} \left(e^{(R-r)/s} - 1\right) = 0$$

$$e^{(R-r)/s} = 1 \rightarrow R - r/s = 0 \rightarrow \underline{\underline{R = r}}$$

$$U(r_0 + x) = A\left[\left(e^{(R-(r_0+x))/s} - 1\right)^2 - 1\right]$$

$R - r_0 - x$
$= -x$ $\rightsquigarrow A\left[\left(e^{-x/s} - 1\right)^2 - 1\right]$

expand $e^{-x/s} \approx 1 - \frac{x}{s} + \frac{x^2}{2s^2}$

sub
$$U(r_0+x) = A\left[\left(1 - x/s + \frac{x^2}{2s^2} - 1\right)^2 - 1\right]$$

$$= A\left[\left(x/s + \frac{1}{2}\left(\frac{x}{s}\right)^2\right)^2 - 1\right]$$

$$A\left(\left(\frac{x}{s}\right)^2 - \left(\frac{x}{s}\right)^3 + \frac{1}{4}\left(\frac{x}{s}\right)^4 - 1\right)$$

$$U(r_0+x) \approx A\left[\left(\frac{x}{s}\right)^2 - 1\right] \simeq A\frac{x^2}{s^2} - A$$

$$\approx \underbrace{\frac{A}{s^2}}_{\text{const}} x^2 \underbrace{-A}_{+\text{const}} \Rightarrow U(r_0+x) \approx \text{const} + \frac{1}{2}kx^2$$

if $\frac{1}{2}k = \frac{A}{s^2}$

5.6 ⋆ A mass on the end of a spring is oscillating with angular frequency ω. At $t = 0$, its position is $x_o > 0$ and I give it a kick so that it moves back toward the origin and executes simple harmonic motion with amplitude $2x_o$. Find its position as a function of time in the form (III) of Problem 5.5.

Gen Sol : $A\cos(\omega t + \phi)$

initial kick $\rightarrow x(0) = 2x_0 \cos(\phi) = \cos(\phi) = \frac{1}{2}$

$\Rightarrow \phi = \frac{\pi}{3}, -\frac{\pi}{3}$

Put back $\Rightarrow A\cos(\omega t + \phi) = 2x_0 \cos(\omega t + \pi/3)$

Probs: 8, 18, 22, 30 ...

#8.

5.8 ⋆ **(a)** If a mass $m = 0.2$ kg is tied to one end of a spring whose force constant $k = 80$ N/m and whose other end is held fixed, what are the angular frequency ω, the frequency f, and the period τ of its oscillations? **(b)** If the initial position and velocity are $x_o = 0$ and $v_o = 40$ m/s, what are the constants A and δ in the expression $x(t) = A\cos(\omega t - \delta)$?

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$m = 0.2\,\text{kg}$ $\quad \omega = \frac{2\pi}{T} \quad f = \frac{1}{T} \quad \tau = \frac{2\pi}{\omega} \quad \omega = \sqrt{k/m}$

$k = 80\,\text{N/m}$ $\quad \omega = \sqrt{80/0.2} = \sqrt{400} = 20 \quad \cancel{f}\ \tau = \frac{2\pi}{20} = \frac{\pi}{10}$

$\omega = 20\,\text{rad/s}$

$\tau = \pi/10 \approx 0.3\,\text{s}$ $\quad f = \frac{1}{\tau} = \frac{1}{\pi/10} = \frac{10}{\pi} \approx 3$

$f \approx 3\,\text{cycles/s}$

$\approx 3\,\text{Hz}$

$x(0) = A\cos(\omega t + \delta) = 0$

$A \neq 0$, so $\cos(\omega t + \delta) = 0$

$\cos(\pi/2) = 0$

$A = -2\,\text{m}$

$\delta = \pi/2\,\text{rad}$

$\Rightarrow \pi/2 = \omega t + \delta$

$t = 0$, so $\delta = \pi/2$

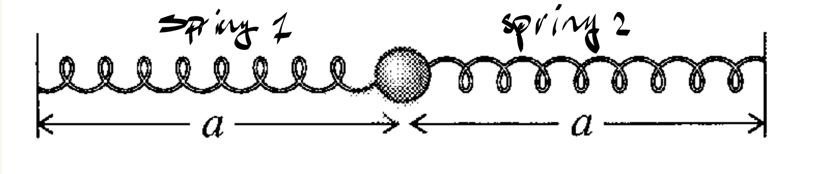
$v(0) = 40\,\text{m/s} \rightsquigarrow \frac{d}{dt}x(t) = -A\omega\sin(\omega t + \delta)$

$40\,\text{m/s} = -A\omega\sin(\omega(0) + \pi/2)$

$40\,\text{m/s} = -A(20\,\text{rad/s})\sin(\pi/2)$

$\frac{40\,\text{m/s}}{20\,\text{rad/s}} = 2\,\text{m} = -A \Rightarrow A = -2\,\text{m}$

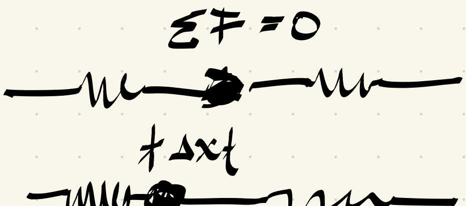
5.18 ★★★ The mass shown from above in Figure 5.27 is resting on a frictionless horizontal table. Each of the two identical springs has force constant k and unstretched length l_o. At equilibrium the mass rests at the origin, and the distances a are not necessarily equal to l_o. (That is, the springs may already be stretched or compressed.) Show that when the mass moves to a position (x, y), with x and y small, the potential energy has the form (5.104) (Problem 5.14) for an anisotropic oscillator. Show that if $a < l_o$ the equilibrium at the origin is unstable and explain why.



$\delta = 0$ My idea here is that, since both springs are identical, we can model this as a 1-D, simple harmonic one-spring system where the one-spring has k-value twice that of the 2-spring system.

$\bullet \longrightarrow F_s = 2kx$

2-spring system:



$\Sigma F = 2k\Delta x$

$\vec{F}_{s1} = kx$ $\vec{F}_{spring2} = kx$

The potential E $U(x)$ is the net force integrated over displacement (Work). $\int_{x_{initial}}^{x} \Sigma F \, dx = \int k \Delta x \, dx$

$= k \int x \, dx = \frac{1}{2} k x^2$. Now, I'm not sure why we require x, y coordinates here but it's simple to show.

If displacement $= |\vec{r}| = \sqrt{x^2+y^2}$ $U(r) = \frac{1}{2} k r^2$, then

$\longrightarrow$

$|\vec{r}| = \sqrt{x^2+y^2} \Rightarrow u(r) = \frac{1}{2}k(\sqrt{x^2+y^2})^2$

$= \frac{1}{2}k(x^2+y^2)$ and if the k's

were not identical, $\frac{1}{2}(k_x x^2 + k_y y^2)$.

#22

5.22 ★ **(a)** Consider a cart on a spring which is critically damped. At time $t = 0$, it is sitting at its equilibrium position and is kicked in the positive direction with velocity v_o. Find its position $x(t)$ for all subsequent times and sketch your answer. **(b)** Do the same for the case that it is released from rest at position $x = x_o$. In this latter case, how far is the cart from equilibrium after a time equal to $\tau_o = 2\pi/\omega_o$, the period in the absence of any damping?

A critically damped system doesn't oscilate. It looks something like this:  where the arrow shows the system reaches equilibrium in quickest time. Starting from an equation of the form $a\ddot{x} + b\dot{x} + cx = 0$ or $m\ddot{x} - \gamma\dot{x} - kx = 0$, using characteristic equation and finding roots to get solution (provided upon request), you get eq. of motion: $x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t}$.

Apply initial conditions $\Longrightarrow$

$$x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t}$$

$x(0) = c_1 = 0 \rightarrow$ first constant found, now for second (c_2), find $v(t)$.

$$v(t) = x'(t) = -\gamma c_1 e^{-\gamma t} + \frac{d}{dt} c_2 t e^{-\gamma t}$$

prod. rule: $+ c_2 t(-\gamma) e^{-\gamma t} + c_2 e^{-\gamma t}$

$$v(t) = -\gamma c_1 e^{-\gamma t} + c_2 t(-\gamma) e^{-\gamma t} + c_2 e^{-\gamma t}$$

$$c_1 = 0, \Rightarrow v(t) = e^{-\gamma t} c_2 (-\gamma t + 1)$$

$$v_0 = v(0) = e^{-\gamma(0)} c_2 (0 + 1) = c_2$$

$$\boxed{\begin{aligned} x(t) &= v_0 t e^{-\gamma t} \\ v(t) &= v_0 t(-\gamma) e^{-\gamma t} + v_0 e^{-\gamma t} = v_0 e^{-\gamma t}(-\gamma t + 1) \end{aligned}}$$

For $x(0)=x_0$, $x(t) = c_1 e^{-\gamma t} + c_2 t e^{-\gamma t}$

$$x(0) = c_1 = x_0$$

$$x(t) = x_0 e^{-\gamma t} + c_2 t e^{-\gamma t}$$

$$v(t) = -\gamma x_0 e^{-\gamma t} + c_2 e^{-\gamma t}(-\gamma t + 1)$$

$$v(0) = -\gamma x_0 + c_2 = 0$$

$$c_2 = \gamma x_0$$

$$\Rightarrow x(t) = x_0 e^{-\gamma t} + \gamma x_0 t e^{-\gamma t}$$

$$\boxed{= x_0 e^{-\gamma t}(1+\gamma t)}$$

b) How far from equilibrium after a time $\tau_0 = 2\pi/\omega_0$ w/out damping?

$\gamma = 0 \Rightarrow$ no damping, so $x(t) = v_0 t$

$$\boxed{x(2\pi/\omega_0) = v_0 (2\pi/\omega_0) \text{ meters}}$$

$v_0 = m/s$, $\omega_0 = s$, $\cancel{m/s}\left(\frac{\text{const}}{\cancel{s}}\right) = m$ ✓

5.30 ★★ The position $x(t)$ of an overdamped oscillator is given by (5.40). **(a)** Find the constants C_1 and C_2 in terms of the initial position x_o and velocity v_o. **(b)** Sketch the behavior of $x(t)$ for the two cases that $v_o = 0$ and that $x_o = 0$. **(c)** To illustrate again how mathematics is sometimes cleverer than we (and check your answer), show that if you let $\beta \to 0$, your solution for $x(t)$ in part (a) approaches the correct solution for undamped motion.

$$x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_o^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_o^2}\right)t}. \qquad (5.40)$$

$x(0) = x_0 = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$

$v(t)$, let's call $\left.\begin{array}{l}(\beta - \sqrt{\beta^2 - \omega_0^2}) = r_1 \\ (\beta + \sqrt{\beta^2 - \omega_0^2}) = r_2\end{array}\right\}$ const.

$$\frac{d}{dt} x(t) = \left(C_1 e^{-r_1 t} + C_2 e^{-r_2 t}\right)' = -\left(C_1 r_1 e^{-r_1 t} + C_2 r_2 e^{-r_2 t}\right)$$

for $v(0) = v_0 = -(C_1 r_1 + C_2 r_2)$

So... $\left.\begin{array}{l} x_0 = C_1 + C_2 \\ v_0 = -(C_1 r_1 + C_2 r_2) \end{array}\right\}$ solve

$C_1 = C_2 - x_0$

$v_0 = -\left[(C_2 - x_0) r_1 + C_2 r_2\right]$

$$V_0 = -\left[(C_2 - x_0) r_1 + C_2 r_2\right]$$

$$= -\left[C_2 r_1 - x_0 r_1 + C_2 r_2\right]$$

$$= -C_2(r_1 + r_2) - x_0 r_1$$

$$x_0 r_1 = C_2(r_1 + r_2) \Rightarrow C_2 = \frac{x_0 r_1}{r_1 + r_2}$$

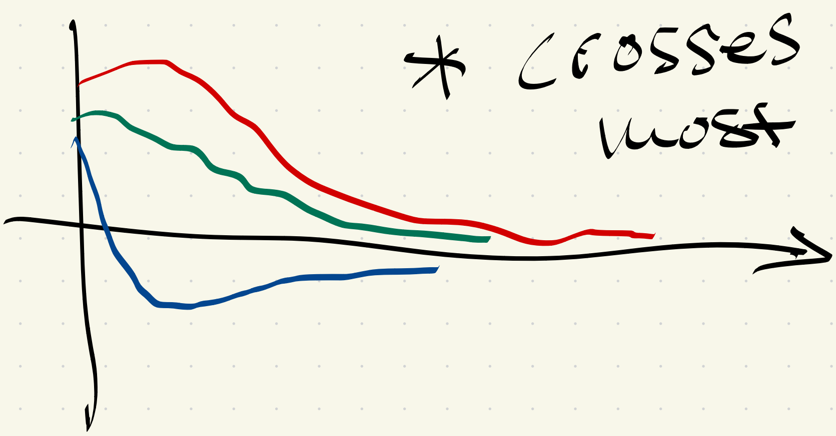
$$x(t) = C_1 e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t}. \qquad (5.40)$$

$$x_0 = C_1 + \quad \Rightarrow C_1 = x_0\left(1 - \frac{r_1}{r_1 + r_2}\right)$$

$$\boxed{x(t) = x_0\left(1 - \frac{r_1}{r_1 + r_2}\right) e^{-r_1 t} + x_0\left(\frac{r_1}{r_1 + r_2}\right) e^{-r_2 t}}$$

where $r_1 = \beta - \sqrt{\beta^2 - \omega_0^2}$ and $r_2 = \beta + \sqrt{\beta^2 - \omega_0^2}$

b)



* crosses axis at most once

c) Let $\beta \to 0$

$$x(t) = x_0\left(1 - \frac{r_1}{r_1+r_2}\right)e^{-r_1 t} + x_0\left(\frac{r_1}{r_1+r_2}\right)e^{-r_2 t}$$

$$r_1 = \beta - \sqrt{\beta^2 - \omega_0^2} \to 0 - \sqrt{-\omega_0^2} \to -i\omega_0$$

$$r_2 = \beta + \sqrt{\beta^2 - \omega_0} \to 0 + \sqrt{-\omega_0} \to i\omega_0$$

$$x(t) = x_0\left[1 - \frac{(-i\omega_0^2)^{\to 0}}{\cancel{i\omega_0^2 - i\omega_0^2}}\right]e^{i\omega_0}$$

$$= x_0 e^{i\omega_0}$$

$$e^{i\theta} \begin{cases} \to \cos(\theta) + i\sin(\theta) \\ \to \cos(\omega_0) + i\sin(\omega_0) \end{cases} \quad \theta = \omega_0$$

$$= x_0\cos(\omega_0) + \cancel{i\sin(\omega_0)}$$

no physical effect

$$x(t) = x_0\cos(\omega_0 t)$$

↳ undamped motion