Forced Harmonic Motion

(Lecture 07)

Phys 326

Fall 2024

Rutgers

Sang-Hyuk Lee

⊙ 1-D Damped Harmonic Oscillator Under Harmonic Ext. Force

$$m\frac{d^2x(t)}{dt^2} = \underbrace{-kx(t)}_{\text{Spring Force}} \underbrace{-R\frac{dx(t)}{dt}}_{\text{Friction Force}} + \underbrace{F_0\cos(\omega t+\phi_0)}_{\text{External Harmonic Force}} \quad (1)$$

where $\begin{cases} F_0 : \text{External Harmonic Force Amplitude} \\ \omega : \text{" " Angular Frequency} \\ \phi_0 \text{ " " Initial Phase.} \end{cases}$

↳ Regardless of the initial motion (i.e., $x(0)$ and $v(0)$) the motion reaches the steady state in a while that oscillates with the same frequency as the external harmonic force!

$\Rightarrow \quad x(t) \propto \cos\omega t \propto \mathrm{Re}\{e^{i\omega t}\}$

$\Rightarrow$ Dealing with $e^{i\omega t}$ is mathematically easier than $\cos\omega t$. So we use complex position variable

(2) → $x(t) = x_0 e^{i(\omega t+\phi_0)}$ to solve for the steady state solution and take Real part at the end.

Here, x_0 is a complex variable as well.

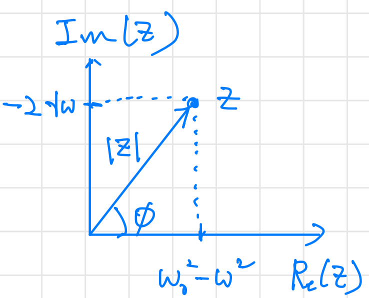
Plugging Eq.(2) into Eq.(1) :

$$m\frac{d^2x(t)}{dt^2} + R\frac{dx(t)}{dt} + kx(t) = F_0\, e^{i(\omega t+\phi_0)}$$

$$\left[m(i\omega)^2 + R(i\omega) + k\right] x_0 \cancel{e^{i(\omega t+\phi_0)}} = F_0\, \cancel{e^{i(\omega t+\phi_0)}}$$

$$x_0 = \frac{F_0/m}{\left(\underbrace{\cancel{\tfrac{k}{m}}}_{\omega_0^2} - \omega^2\right) + i\underbrace{\cancel{\tfrac{R}{m}}}_{2\gamma}\omega} = \frac{F_0/m}{(\omega_0^2-\omega^2)+i2\gamma\omega}$$

$$= \frac{F_0}{m}\,\frac{\overbrace{(\omega_0^2-\omega^2) - i2\gamma\omega}^{\equiv Z}}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2} = A\, e^{i\phi}$$: Euler Form of a complex variable



$$|Z| = \sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}$$

A (>0) : Amplitude

ϕ : Phase difference between $x(t)$ and $F(t)$

$(\because)\ x(t) = A e^{i(\omega t+\phi_0+\phi)}$

$F(t) = F_0 e^{i(\omega t+\phi_0)}$

Therefore

$$\boxed{A = \frac{F_0}{m}\,\frac{|Z|}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2} = \frac{F_0}{m}\,\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}}$$

$$\boxed{\tan\phi = \frac{-2\gamma\omega}{\omega_0^2-\omega^2} = \frac{2\gamma\omega}{\omega^2-\omega_0^2}}$$

*Note :

$$A = \frac{F_0}{2m\gamma\omega}\,|\sin\phi| \;!$$

Normalization of ω and γ by ω_0 makes the physics behind clearer :

$$\bar{\omega} = \omega/\omega_0 \quad , \quad \bar{\gamma} = \gamma/\omega_0$$

$$A = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}} = \underbrace{\left(\frac{F_0}{m\omega_0^2}\right)}_{\equiv A_0 = \frac{F_0}{m(k/m)} = \frac{F_0}{k}}\frac{1}{\sqrt{\underbrace{(\bar{\omega}^2-1)^2+4\bar{\gamma}^2\bar{\omega}^2}_{\equiv\,\eta(\bar{\omega})}}}$$

$\frac{F_0}{k}$: Equilibrium displacement by static force F_0

$$\eta(\bar{\omega}) = \bar{\omega}^4 - 2(1-2\bar{\gamma}^2)\bar{\omega}^2 + 1$$

$$= \left[\bar{\omega}^2 - \underbrace{(1-2\bar{\gamma}^2)}_{}\right]^2 + 1 - (1-2\bar{\gamma}^2)^2$$

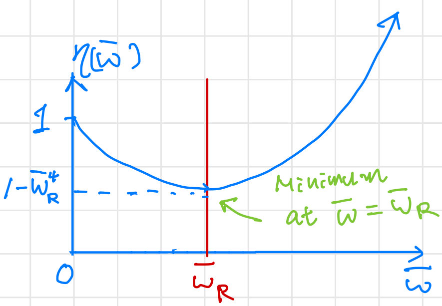
$> \bar{\omega}_R^2 \equiv 1-2\bar{\gamma}^2 \le 1$

$$= (\bar{\omega}^2 - \bar{\omega}_R^2)^2 + (1-\bar{\omega}_R^4) \Rightarrow \begin{cases} \eta(\bar{\omega}=0) = 1 \\ \eta(\bar{\omega}\to\infty) \to \infty \end{cases}$$

Case 1 :

$\bar{\omega}_R^2 = 1-2\bar{\gamma}^2 \ge 0$

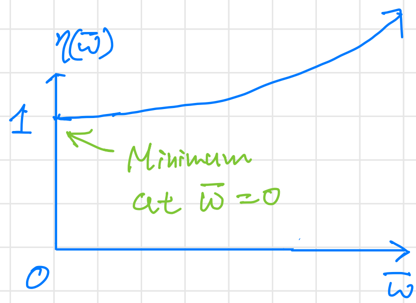
(i.e. $\bar{\gamma}^2 \le \frac{1}{2}$)



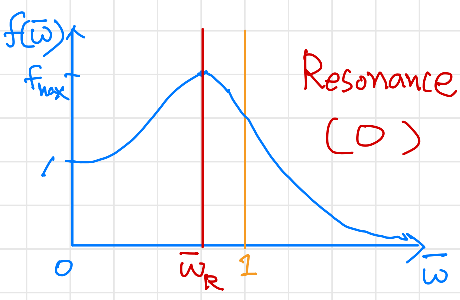
Case 2 :

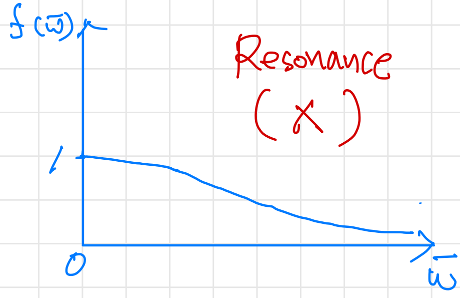
$\bar{\omega}_R^2 = 1-2\bar{\gamma}^2 < 0$

(i.e, $\bar{\gamma}^2 > \frac{1}{2}$)



$$A(\bar{\omega}) = \frac{A_0}{\sqrt{\eta(\bar{\omega})}} = \frac{A_0}{\sqrt{(\bar{\omega}^2-1)^2+4\bar{\gamma}^2\bar{\omega}^2}} = \frac{A_0}{\sqrt{(\bar{\omega}^2-\bar{\omega}_R^2)^2+1-\bar{\omega}_R^4}} \Rightarrow \underline{A_0 f(\bar{\omega})}\,_{\text{//}}$$





In case of resonance (i.e. $\bar{\omega}_R^2 = 1 - 2\gamma^2 > 0$)

$$f_{max} = \frac{1}{\sqrt{1-\bar{\omega}_R^4}} = \frac{1}{\sqrt{1-(1-2\gamma^2)^2}}$$

For mildly damped oscillation $\Rightarrow$ $\gamma \ll 1$.

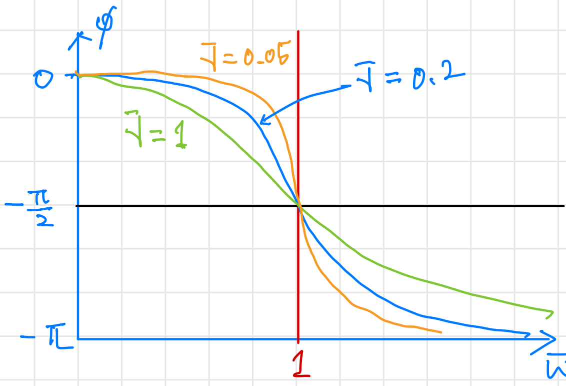
Then

$$1-(1-2\gamma^2)^2 \simeq 1-(1-4\gamma^2) = 4\gamma^2$$

$$(\therefore) \quad f_{max} \simeq \frac{1}{2\gamma} \quad \text{for } \gamma \ll 1$$

Now consider phase shift ϕ.

$$\tan\phi = \frac{2\gamma\omega}{\omega^2-\omega_0^2} = \frac{2\gamma\bar{\omega}}{\bar{\omega}^2-1} \quad \left(\begin{array}{l}\omega \text{ and } \gamma \text{ are} \\ \text{normalized by } \omega_0\end{array}\right)$$



* Remember

$$F(t) \propto e^{i(\omega t + \phi_0)}$$

$$x(t) \propto e^{i(\omega t + \phi_0 + \phi)}$$

$\hookrightarrow$ $\phi(\bar{\omega}) \leq 0 \Rightarrow x(t)$ lags $F(t)$ by $|\phi|$

At $\bar{\omega} \ll 1 \Rightarrow$ " " " ~ 0 (i.e., in phase)

At $\bar{\omega} = 1 \Rightarrow$ " " " $\pi/2$

At $\bar{\omega} \gg 1 \Rightarrow$ " " " $\sim \pi$ (i.e., out of phase)

⓪ Velocity $v(t)$

(Phase shift of $v(t)$ with respect to $F(t)$) $\rightarrow \theta$

$$F(t) = F_0 e^{i(\omega t + \phi_0)}$$

$$x(t) = x_0 e^{i(\omega t + \phi_0)} = A e^{i(\omega t + \phi_0 + \phi)}$$

$$\Rightarrow \quad v(t) = \frac{dx(t)}{dt} = i\omega A e^{i(\omega t + \phi_0 + \phi)} = \underbrace{\omega A}_{\equiv V} e^{i(\omega t + \phi_0 + \underbrace{\phi + \frac{\pi}{2}}_{\theta})}$$

(Amplitude of velocity oscillation)

Then,

$$V(\bar{\omega}) = \omega A = \omega \frac{A_0}{\sqrt{(\quad)}} = \frac{\omega_0 A_0 \bar{\omega}}{\sqrt{(\quad)}}$$

$$= \frac{\omega_0 A_0 \bar{\omega}}{\sqrt{(\bar{\omega}^2 - 1)^2 + 4\gamma^2 \bar{\omega}^2}} = \underbrace{\frac{\omega_0 A_0}{2\gamma}}_{\equiv V_R} \frac{2\gamma \bar{\omega}}{\sqrt{(\bar{\omega}^2 - 1)^2 + 4\gamma^2 \bar{\omega}^2}}$$

$$= V_R \frac{2\gamma}{\sqrt{(\bar{\omega} - \frac{1}{\bar{\omega}})^2 + 4\gamma^2}}$$

$$= V_R |\sin\phi|$$

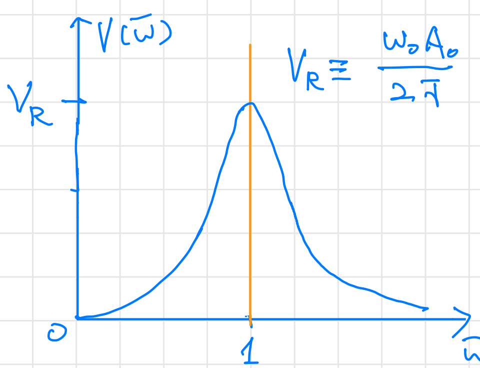
$$\Rightarrow V_R\, g(\bar{\omega})$$

(Maximum condition: $\frac{dg}{d\bar{\omega}} = 0$

$$0 = 2\left(\bar{\omega} - \frac{1}{\bar{\omega}}\right)\left(1 - \frac{1}{\bar{\omega}^2}\right)$$

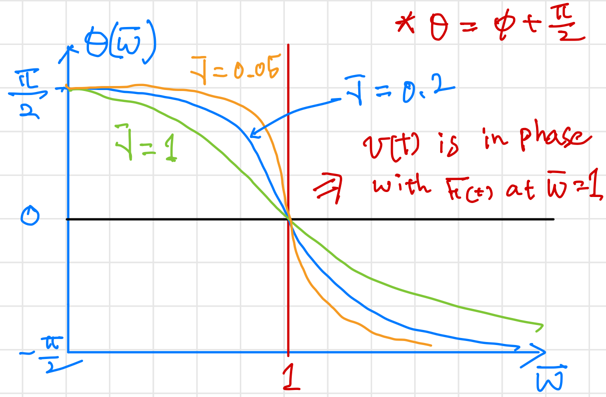
$\therefore$ Max at $\bar{\omega} = 1$ with $g(\bar{\omega} = 1) = 1$.)

Velocity Amplitude $V(\bar{\omega})$:



Velocity Phase $\theta(\bar{\omega})$:

$* \ \theta = \phi + \frac{\pi}{2}$



⓪ Power 'P' supplied by external force

$$P(t) = \frac{(\text{Work done by } F)}{(\text{Unit time})} = \frac{F(t)\,\Delta x}{\Delta t} = F(t)\,v(t)$$

where
$$F(t) = \mathrm{Re}\left\{ F_0\, e^{i(\omega t + \phi_0)} \right\}$$
$$v(t) = \mathrm{Re}\left\{ V\, e^{i(\omega t + \phi_0 + \theta)} \right\}$$

Then
$$P(t) = F_0 V \cos(\overbrace{\omega t + \phi_0}^{\alpha})\cos(\omega t + \phi_0 + \theta)$$
$$= F_0 V \cos(\alpha)\cos(\alpha + \theta) = \frac{F_0 V}{2}\left[\cos(2\alpha + \theta) + \cos\theta\right]$$

Consider average over one oscillation cycle $\tau = \frac{2\pi}{\omega}$.

$$\langle P \rangle = \frac{1}{\tau}\int_0^{\tau} P(t)\,dt = \frac{F_0 V}{2}\int_0^{\tau}\left[\underbrace{\cos(2\omega t + 2\phi + \theta)}_{\text{Two full oscillations for } t\in[0,\tau]} + \cos\theta\right]dt$$

$$= \frac{F_0 V}{2}\cos\theta = \frac{F_0}{2} V_R |\sin\phi| \cos\left(\phi + \frac{\pi}{2}\right)$$

$$= \frac{F_0 V_R}{2}\sin^2\phi = \frac{F_0 V_R}{2} g^2(\bar{\omega}) = \langle P \rangle_{max}\, g^2(\bar{\omega})$$

$$= \underline{\langle P \rangle_{max}\, g^2(\bar{\omega})}$$

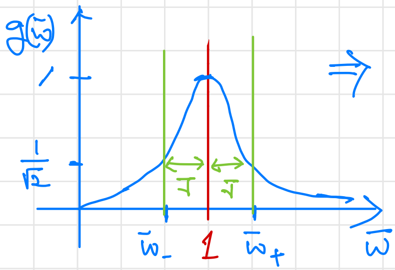
where

$$\langle P \rangle_{max} = \frac{F_0 V_R}{2} = \frac{F_0}{2}\frac{\omega_0 A_0}{2\gamma} = \frac{F_0}{4\gamma}\overset{k/m}{\cancel{\omega_0^2}}\frac{F_0}{k}$$

$$= \frac{F_0^2}{4\gamma m}$$

⊙ Quality Factor Q

$$\langle P \rangle = \langle P_{max} \rangle\, g(\bar{\omega})^2 \quad , \quad g(\bar{\omega}) = \frac{2\bar{\gamma}}{\sqrt{\left(\bar{\omega} - \frac{1}{\bar{\omega}}\right)^2 + 4\bar{\gamma}^2}}$$



$\Rightarrow$

* Find $\bar{\omega} = \bar{\omega}_\pm$ where $g(\bar{\omega}_\pm) = \frac{1}{\sqrt{2}}$

$$(\bar{\omega}_\pm^2 - 1)^2 = 4\bar{\gamma}^2 \bar{\omega}_\pm^2$$

$$\bar{\omega}_\pm^4 - 2(1 + 2\bar{\gamma}^2)\bar{\omega}_\pm^2 + 1 = 0$$

$$\bar{\omega}_\pm^2 = (1 + 2\bar{\gamma}^2) \pm \sqrt{(1 + 2\bar{\gamma}^2)^2 - 1}$$

$$\approx (1 + 2\bar{\gamma}^2) \pm 2\bar{\gamma} \approx 1 \pm 2\bar{\gamma}$$

if $\bar{\gamma} \ll 1$ (mild damping)

$$\Rightarrow \bar{\omega}_\pm = \sqrt{1 \pm 2\bar{\gamma}} \simeq 1 \pm \bar{\gamma}$$

Therefore at $\bar{\omega} = \bar{\omega}_\pm$

$$\Rightarrow g(\bar{\omega}_\pm) = \frac{1}{\sqrt{2}}$$

$$V(\bar{\omega}_\pm) = \frac{V_R}{\sqrt{2}}$$

$$\langle P \rangle (\bar{\omega}_\pm) = \frac{1}{2} \langle P \rangle_{max}.$$

Quality factor Q is defined as

$$Q = \frac{\omega_0}{2\Delta\omega} = \frac{\omega_0}{\omega_+ - \omega_-}$$

$$\Rightarrow Q = \frac{\omega_0}{2\gamma}$$

//

⓪ Physical Meaning of Q ?

$$Q = \frac{\text{maximum energy stored at resonance}}{\text{energy dissipated at resonance in one radian}}$$

$$= \frac{\text{maximum energy stored at resonance}}{(\text{energy dissipated at resonance in one cycle})/2\pi}$$

$$(\text{Numerator}) = \text{max kinetic energy}$$

$$= \frac{1}{2} m V_R^2 = \frac{1}{2} m \left(\frac{F_0}{2\gamma m}\right)^2$$

$$(\text{Denominator}) = \langle P \rangle_{max} \tau / 2\pi = \langle P \rangle_{max} / \omega_0$$

$$= \frac{F_0 V_R}{2} \frac{1}{\omega_0} = \frac{F_0^2}{4\gamma m} \frac{1}{\omega_0}$$

$$\Rightarrow \quad Q = \frac{\frac{1}{2} \frac{F_0^2}{4\gamma^2 m}}{\frac{F_0^2}{4\gamma m} \frac{1}{\omega_0}} = \frac{\omega_0}{2\gamma}.$$

Same result as the previous result !