THE ONSET OF CHAOS

PURPOSE

To study the onset of chaos in a nonlinear RLC circuit and to explore with MATLAB the approach to chaos demonstrated by the Logistic Equation.

REFERENCES

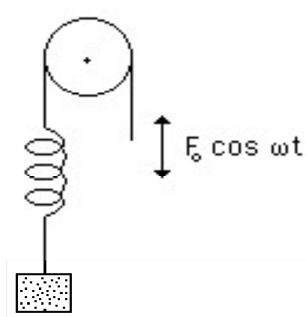
"Period Doubling and Chaotic Behavior in a Driven Anharmonic Oscillator", Paul Lindsay, Physical Review Letters 47 1349 (1981)

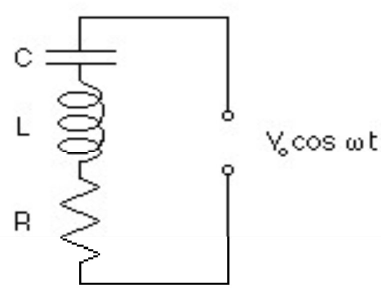
"Mathematical methods for Scientists and Engineers", Peter Kahn, Chapter 16. Q158.5.K34 1990

Useful websites:
https://en.wikipedia.org/wiki/Logistic_map
https://geoffboeing.com/2015/03/chaos-theory-logistic-map/
https://www.nathaniel.ai/logistic-map/

BACKGROUND

A few weeks ago you did an experiment on the forced harmonic motion of a resonant spring-mass system, which demonstrated several important features: (1) the mass undergoes a transient initial motion determined by the initial conditions (such as initial displacement and velocity), (2) the transient motion damps out leaving a steady state motion independent of the initial conditions, (3) the steady state motion is a linear function of the amplitude of the driving force (i.e., if the force is doubled, the amplitude of oscillation doubles). This last feature is a characteristic of a linear system. You also may have done or studied an analogous experiment on a resonant series RLC circuit, where L would play the role of the mass, $1/C$ that of the spring constant, and the charge Q stored on the capacitor the role of the amplitude of oscillation, which showed exactly the same behavior.





In this experiment you will study a series resonant RLC circuit, but with one crucial change. In place of using a capacitor whose capacitance is constant, you will use one whose capacitance depends on the magnitude of the voltage across it. This capacitor voltage will then no longer be a linear function of the driving voltage and we have a nonlinear or anharmonic system.

You will view the voltage V_R across the resistance in the circuit shown above as you increase V_0. At first you will observe a voltage oscillating at the frequency ω, with each successive peak of the oscillation having the same magnitude. But when V_0 becomes large enough, the peaks will alternate in amplitude, that is, V_R will no longer have period $T = 2\pi/\omega$, but rather twice the period, $T = 4\pi/\omega$. There has been a period doubling of the response of the circuit. As V_0 is further increased, the pattern of peaks again changes so that the same amplitude only occurs every fourth cycle; there has been another period doubling and the period is now $T = 8\pi/\omega$. As V_0 is still further increased there is a succession of period doublings which require smaller and smaller increases in V_0. You will rapidly find that no matter how good your apparatus, you cannot control V_0 accurately enough to discern successive period doublings and the voltage across the resistor appears to vary randomly with time. But we know from what we've seen for smaller V_0 that the response of the circuit varies in a perfectly well determined manner. This deterministic yet apparently random behavior is called chaos to distinguish it from truly random behavior.

It has been only since the advent of high-speed computers that scientists have realized the extent to which nonlinear systems may exhibit chaotic behavior and that the approach to chaos through period doubling is a universal behavior that does not depend on the details of the system studied. In this experiment we will first observe the chaotic behavior of the nonlinear RLC circuit, and then use MATLAB to study the approach to chaos for the Logistic Equation, an extremely simple nonlinear difference equation that exhibits chaos. The Logistic Equation,

$$x(n+1) = ax(n)[1 - x(n)] \tag{1}$$

has been used to model natural populations such as the yearly population swings of lemmings. If the population $x(n)$ during the n-th interval is known, then Eq. (1) gives the population $x(n+1)$ in the next interval (i.e., the next year, for lemmings). a is called the growth parameter and is essentially the difference between the birth and death rates. We will allow $x(n)$ to vary between 0 and 1, which limits a to being between 0 and 4.

Just as for the oscillating mass or the RLC circuit, the Logistic Equation shows an initial behavior which depends on the starting condition [$x(0)$] and damps out leaving a steady state response, i.e., a constant value for $x(n)$ for large n. It is important to notice that there is no driving term so that the "normal" steady state behavior will be a constant value rather than the steady state oscillation seen for the RLC circuit. Despite this difference the approach to chaos will be the same as that of the RLC circuit.

The behavior of the steady state response of the Logistic Equation depends strongly on a as follows:

$\mathbf{0 \leq a < 1:}$ The growth parameter is too small to sustain any population. Regardless of $x(0)$ the population crashes to zero after a few intervals. $x(n) = 0$ is called a fixed point of the system since $x(n) = x(n+1)$. It is a stable fixed point because after any small disruption the population returns to zero. Stable fixed points are also called attractive fixed points or attractors.

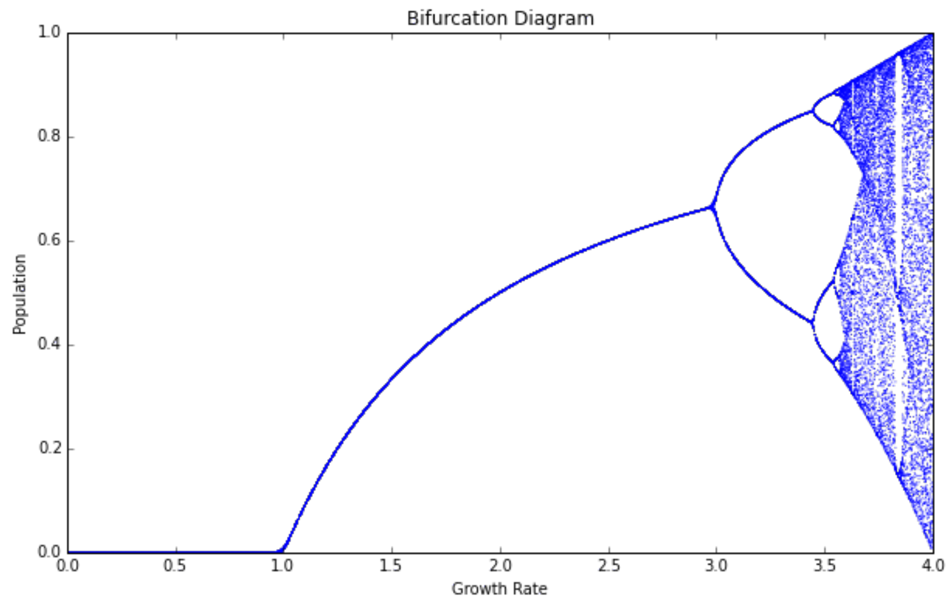
$\mathbf{1 \leq a < 3:}$ For $a = 1$, the fixed point at zero becomes unstable and is called a repellent fixed point or a repeller. At the same time, a new attractor appears, $x(n) = 1 - 1/a$. The growth parameter is now large enough to sustain a nonzero steady state population. The transient solution monotonically approaches $x(n) = 1 - 1/a$ for a close to 1, but as a gets larger the solution oscillates about the attractor before eventually being damped out.

$\mathbf{3 \leq a < 3.4473:}$ At $a = 3$, the attractor $x(n) = 1 - 1/a$. become unstable and the steady state solution oscillates about $x(n) = 1 - 1/a$ with alternate $x(n)$ having the same value. For $a < 3$, the interval between like values for $x(n)$ in the steady state solution was $\Delta n = 1$, while for $a \geq 3$ the interval is $\Delta n = 2$; <u>the period has doubled</u>. Do not confuse the appearance of this oscillation with the steady state behavior of the RLC circuit before frequency doubling has set in.

$\mathbf{3.4473 \leq a < 3.5433:}$ At $a = 3.4473$ the period doubles again so that the interval between equal values of $x(n)$ is now $\Delta n = 4$.

$\mathbf{3.5433 \leq a < 3.6:}$ The steady state solution goes through a series of period doublings as shown in the graph on the next page until at $a \approx 3.6$ chaos sets in. The graph on the next page show this period doubling and approach to chaos.

In your MATLAB calculations you will see that in the chaotic region, the steady state is so sensitive to the value of a that it is impossible for us to predict the solution. The slightest error in our determination of a will completely throw off our predicted behavior.



We also noted in a linear system or in the nonchaotic region for the Logistic Equation that the steady state solution is independent of the starting conditions. Calculations with the Logistic Equation will also show that the steady state solution is no longer independent of

the starting condition and is in fact extremely sensitive to the exact value of $x(0)$. We must conclude then that when chaos appears we cannot theoretically predict the response of a system even though it is in principle completely deterministic.

The sequence of period doublings that you will observe in this experiment for both the RLC circuit and the Logistic Equation has a universal character that is independent of the particular system. Most of the details of this universal behavior are beyond the scope of this laboratory. However, one aspect is accessible; Feigenbaum (see references in Phys. Rev. Letter listed above) has demonstrated that if a_1 is the value of a where the first period doubling sets in, a_2 the value for the second doubling etc., then for large k

$$a_\infty = a_k + S\delta^{-k} \tag{2}$$

, where a_∞ is the value of a where chaos sets in. S is a constant characteristic of the particular system, but $\delta = 4.6692\ldots$ is a universal constant called the Feigenbaum number. Since δ is large, this series converges rapidly and Eq. (2) will allow us to estimate δ and predict a_∞ with only a little data. In this experiment you will measure the first few a_k and use this data to estimate δ from Eq. (2). To do this, use data from three successive period doublings to eliminate S and a_∞ from Eq. (2) and obtain

$$\delta_k = \frac{a_k - a_{k-1}}{a_{k+1} - a_k} \tag{3}$$

If k is large enough, the value of δ_k calculated this way should converge to Feigenbaum's theoretical value. To get a_∞ from your data, solve Eq. (2) for S,

$$S = \frac{(a_k - a_{k-1})\delta^k}{\delta - 1} \tag{4}$$

, and then use Eq. (2) to get a_∞. Keep in mind that Eqs. (2) and (4) are correctly only when the value of k is large enough.

EQUIPMENT

The RLC circuit is pre-wired for you. For the nonlinear capacitor we use a silicon diode. The diode consists of a layer of p-type silicon in contact with a layer of n-type silicon. When the diode is reverse biased the positive charges in the p-type silicon and the negative charges in the n-type silicon are moved away from the junction by the electric field produced by the applied voltage. When the junction is forward biased they are moved toward the junction. Thus the distance of separation of the charge in the diode and hence, its capacitance depends on the voltage. It turns out that this dependence is very nonlinear. The voltage applied across the diode will not be large enough to turn the diode on and make it stop acting as a capacitor.

The equipment needed:

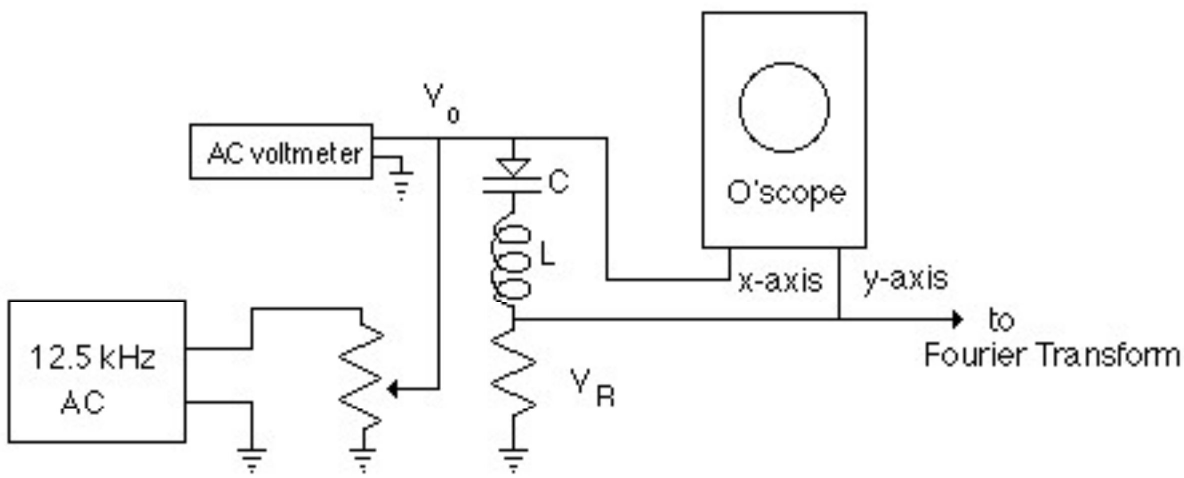
RLC circuit
Oscilloscope
Sine wave generator (12.5 kHZ)
SR760 FFT
4 digit AC voltmeter

PROCEDURE

Lab Activity for Week 1

A. Nonlinear RLC circuit

1. Set up the nonlinear RLC circuit as shown below:



2. Set the sine wave generator to a frequency of about 12 kHz and maximum 10 V amplitude (i.e., **20 V peak-to-peak**). Use an oscilloscope to see both V_0 and V_R signals; Set V_R as the triggering source. When you use your oscilloscope, you will use two different modes to display V_0 and V_R in a useful manner; you can either look at V_0 and V_R versus time using the time sweep for the x-axis, or you can look at V_R versus V_0, which is known as a Lissajous figure. You will find the Lissajous figure most useful for detecting frequency doubling. Practice using both modes of operation.

3. Adjust V_R to a small value by turning the knob on the right (i.e., NOT the one labeled Drive Voltage) **counter-clockwise until it stops then turn back about one-half turn**. Adjust the drive level until the voltage on the voltmeter reads about 0.1 V. Look at V_R on the oscilloscope. Adjust the frequency until the amplitude of V_R is near maximum (i.e., near the resonance frequency) without distorting the sinusoidal waveform.

4. Look at the spectrum on the FFT in dB (i.e., log scale). Adjust the span to 12.5 kHz. You should see a peak around 9-12 kHz. Any other peaks should be smaller than the main peak by at least 40 dB. You may see double peaks at about half the driving frequency. If so, adjust slightly the driving frequency until the two peaks merge. The peaks at the half the driving frequency may be more apparent when AUTORANGE is ON in

SR760. You can manually specify the input range via INPUT menu of SR760 and setting it to 0 dBV (default value) may make these undesirable peaks less apparent.

5. **Observation:** Set the oscilloscope to look at the Lissajous figure. Adjust the Volts/Div scales to give an ellipse about two cm in diameter. Starting with V_0 small, slowly increase its value and look for evidence for period doubling using the Lissajous figure. What happens to the FFT spectrum when this happens? Observe the amplitude and frequencies of peaks in the FFT spectrum. Also change the oscilloscope view mode to see V_0 and V_R signals in time. Adjust the trigger level properly to see the V_R signal stably. Can you confirm the period doubling in the waveform signal too? As you increase V_0 still further, observe additional doubling and the onset of chaos. You may notice that once you have reached the chaotic region, there are still values of V_0 for which you get simple deterministic behavior (period tripling and quintupling).

6. **Measurement**: Now that you have a good qualitative understanding about the system, you need to make measurements to study how chaos is set on in more detail.

<u>Task 6-1: Bifurcation diagram for nonlinear RLC circuit</u>
Use AC voltmeter to measure the amplitude (or RMS value of V_0) and let's term this value as $\bar{V}_0$.

Use oscilloscope to record V_0 vs time; and V_R vs time. Set the Time/Div knob on the oscilloscope such that <u>~100 periods of V_0 waveform is recorded.</u> **Make sure that V_R signal is triggered to be shown stably.** Use RUN/STOP button to capture the data array first and then save it to USB drive, especially if you encounter troubles with triggering. Export a sample waveform data showing twice the period doubling (i.e., 4 peaks on SR760 and 4-loop Lissajous curve on oscilloscope) into MATLAB and plot V_0 vs V_R (i.e., Lissajous curve) to first check whether you get a curve that is similar to the Lissajous curve displayed on the oscilloscope before further data acquisition. If the curve looks too noisy, try AVERAGE function in the oscilloscope setup (can be accessed via ACQUIRE button). Try different number of averages (4, 16, 64), save the waveforms, import the data into MATLAB, and plot the Lissajous curves. Do averaged waveforms data show less noisy Lissajous curves? If so, pick a reasonable number of averages and use the setting for all your data acquisition.

Save the corresponding Fourier spectrum using SR760 [*Using AVERAGE menu to first capture a time-averaged spectrum and acquire it on PC with LabView program will make the task easy.]

Use these data and your own MATLAB code to obtain the values of V_R every one-period of V_0 signal (e.g., whenever V_0 reaches the peaks, or crosses zero from negative to positive, or crosses zero from positive to negative, etc.). Let's term these ~100 values as $\bar{V}_R$. The values of $\bar{V}_R$ for any given $\bar{V}_0$ value will well represent the periodicity of $\bar{V}_R$. For example, the values $\bar{V}_R$ will be almost a constant before the first period doubling event; and will be split into two clusters after period doubling. **Try different schemes of determining $\bar{V}_R$ and choose the one that best separates the clusters after period doubling events.**

Make a plot of $\bar{V}_R$ vs $\bar{V}_0$ to generate a bifurcation diagram similar to the earlier graph for period doubling for the Logistic Equation on page 4 of the lab manual. Carefully determine the values of $\bar{V}_0$ where period doubling occurs (easier with the spectrum analyzer) to identify the period doubling threshold values a_n as precisely as possible [*Once the frequency is close to a doubling point, average the spectrum over many samples (Use AVERAGE menu of SR760) to better determine the actual point. For the 3rd or higher order period-doubling events, you should try setting the frequency span to half as large to better isolate the point.] Try to detect and record as many period-doubling events as possible (i.e., see if you can detect n larger than 4 or 5). Try to include data for at least 5 different values of $\bar{V}_0$ (evenly spaced more or less) within each range between a_{n+1} and a_n for $n = 1, 2, 3, \ldots$ Also include at least 5 data points before the first period double event. Try to include data showing periodic signals that intermittently appear in the middle of the chaotic regime. In your lab report, you should show a representative set of these plots: 1) V_R vs time; 2) V_0 vs V_R (i.e., Lissajous curve); and 3) FFT spectrum from SR760 for each range between a_{n+1} and a_n in the bifurcation diagram.

<u>Task 6-2: Feigenbaum constant</u>

Make a table showing a_k, $a_{k+1} - a_k$, and $\delta_k \left(\equiv \frac{a_{k+1} - a_k}{a_{k+2} - a_{k+1}}\right)$. Does δ_k converges to Feigenbaum constant? Use Eqs. (2)-(4) to estimate a_∞, S, and δ, and report the values in your report. [Note that $\bar{V}_0$ plays the role of the growth parameter a.]

Lab Activity for Week 2

Helpful sample MATLAB code for this lab activity:
https://www.mathworks.com/matlabcentral/fileexchange/135932-logistic-map-and-bifurcation-diagram

(*You are allowed to adapt parts of the sample code for your own m-file if you need help, but simply copying and pasting the code is prohibited; you should understand every single line of the code you write. Students are encouraged to write their own code from scratch and use the sample code only as a reference.)

B. The Logistic Equation

1. Set up a MATLAB m-file to calculate $x(n)$ for various values of $x(0)$ and a.

[General Tip/Instruction: In order to examine the transient response of the equation, graph $x(n)$ for the first 20 values of n, and to examine the steady state response, graph $x(n)$ for about 40 cycles when n is large. (To get the steady state solution near $a = 3$, you may have to go up to $n \approx 1000$ to avoid the transient.) You can use subplot() command to put several plots on the same page in order to see the effect of changing a or $x(0)$.**]**

2. Examine the behavior of the solutions for $a < 3$ for various values of $x(0)$ in order to verify that the equation behaves as described in the BACKGROUND section. In particular verify that the steady state solution is independent of the initial condition.

[To do]: Show $x(n)$ plots for $a = [0.5, 1, 1.5, 2.9]$ in your report. Each figure for a given value of a should include three $x(n)$ curves for three initial value $x(0)$ randomly chosen by using rand() function. Describe and discuss about the main characteristics of the results.

3. Examine the behavior of the solutions for the four cases $a = 2.99$ and 3.01 with $x(0) = 0.667$ and 0.5, where period doubling first sets in.

[To do]: Does the steady state value depend on the initial value for any of these cases? Show the four plots in your report to support your claim. Describe and discuss about the main characteristics of the results.

4. Determine to four decimal places (3.xxxx) the values of a where period doubling sets in at least for the first 4 period doublings, i.e., $a_1, a_2, a_3, a_4, \ldots$ in Eq.(2) **(Make sure you understand the definition of these values accurately)**. To do this, examine the steady state solution values of $x(n)$ for $n, n+1, n+2, \ldots$ to see how many cycles you have to go through before the pattern repeats. [Using MATLAB plot() to visualize $x(n), x(n+1), x(n+2), \ldots$ only with markers (without lines) will be useful.] Then vary a until you find the value where the period doubles. Use a value of n near 1000 and increase the accuracy of $x(n)$ as much as necessary to see a change in pattern.

[To do]: For each value of a_k you determined, show two x vs n graphs demonstrating the period doubling event. For example, if you determined a_3 =3.4567 to four decimal places as required then x vs n should show period four for $a = 3.4566$ but period eight for $a = 3.4567$. Make a table showing a_k, $a_{k+1} - a_k$, and $\delta_k \left(\equiv \frac{a_{k+1} - a_k}{a_{k+2} - a_{k+1}}\right)$. Does δ_k converges to Feigenbaum constant? Use Eqs. (2)-(4) to estimate a_∞, S, and δ, and report the values in your report.

5. For a in the chaotic regime, $a > 3.6$, examine the steady state solution to see the dependence on extremely small changes in a and in $x(0)$. The effect is easiest to see on a small graph if you use values near 3.9. For example, use a starting point of 0.6000 and get x-values up to the 200^{th} iteration; and label it as $x_1(n)$. Then repeat the process by very slightly changing the starting point (e.g., 0.6001); and label it as $x_2(n)$. Plotting $x_2(n) - x_1(n)$ vs n will visualize how the two trajectories diverge as n increases.

[To do]: Make $x_2(n) - x_1(n)$ vs n for two values of a in the chaotic regime. Make 5 similar plots for a values representing period one (e.g., for $a = 2.0$), period two, period four, period 8, and period 16 (*You can easily determine these a values based on your work in Part 4 above). Make a similar plot for period three due to unique period tripling event in the middle of the Chaotic regime neat $a = 3.8$ (See the bifurcation diagram on page 4 of this lab manual). Discuss about the results.

6. Using and summarizing all these results, reproduce a bifurcation diagram similar to the plot on page 4 of this lab manual. Refer to the sample code linked above if you need help.

[To do]: Show your bifurcation diagram in the lab report.

Lab Report Submission Instruction

Submit a single lab report that includes all the results for A-B by the due date set for the week-2 lab. You can include multiple m-files if it is more desirable to separate your work for Part A and B, but you should briefly explain what each m-file does in your lab report. **All the plots you use in your lab report should be generated by running your MATLAB codes.**

The total lab grade will be 60 points.

20 points: Introduction / Experimental Methods / Part A

40 points: Part B / Conclusions