

FOURIER ANALYSIS AND SYNTHESIS

BACKGROUND

The French mathematician J. B. Fourier showed in 1807 that any piecewise continuous periodic function with a frequency ν can be expressed as the sum of an infinite series of sines and cosines with frequencies of integer multiples of ν ; this relation is called the Fourier series. It was later shown that any function, including non-periodic ones, could be expressed as a sum of sines and cosines of all frequencies from 0 to infinity; this relation is called the Fourier transform. These remarkable relations have widespread applications in mathematics, physics, and engineering. In this lab we will investigate some of the properties of the Fourier series and Fourier transform and their applications.

First, let's discuss the Fourier series for a periodic function. Let $f(t)$ be a periodic function with period T , that is let $f(t) = f(t + T)$. Fourier's theorem says that $f(t)$ can be expressed as a sum of harmonics (that is sine and cosine waves whose frequencies are multiples of the fundamental frequency $1/T$). Mathematically

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \{A_n \cos(2\pi n \nu_0 t) + B_n \sin(2\pi n \nu_0 t)\} \quad (1)$$

where $\nu_0 = 1/T$,

$$A_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt, \quad (2)$$

$$A_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(2\pi n \nu_0 t) dt, \quad B_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(2\pi n \nu_0 t) dt$$

Notice that A_n and B_n have the same units as $f(t)$. Also, $A_0/2$ is just the time average of $f(t)$. If this average is zero, as is frequently the case, then A_0 is zero. Although the integrals shown have limits of $-T/2$ and $T/2$, any limits covering one period can be used (e.g., 0 to T).

This theorem can be simplified if $f(t)$ is a symmetric (even) function, that is if $f(t) = f(-t)$, since the cosine is a symmetric (even) function while the sine is anti-symmetric (odd), i.e., $f(t) = -f(-t)$. Then B_n must be zero for all n . Likewise, if $f(t)$ is anti-symmetric (odd), A_n must be zero for all n . Many functions can be expressed as either

symmetric or anti-symmetric by careful choice of the starting time. Others are inherently asymmetric and must be expressed using both sines and cosines.

The Fourier series can be expressed more elegantly by replacing sines and cosines with $e^{i2\pi n\nu_0 t}$ and $e^{-i2\pi n\nu_0 t}$:

$$f(t) = \sum_{-\infty}^{\infty} \hat{f}_n \exp(i2\pi n\nu_0 t), \quad \text{where} \quad \hat{f}_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \exp(-i2\pi n\nu_0 t) dt \quad (3)$$

The Fourier transform is the extension of the Fourier series to a non-periodic function. The Fourier transform of a function of time in terms of frequency ν is given by

$$\Phi(\nu) = \int_{-\infty}^{\infty} F(t) \exp(-i2\pi \nu t) dt$$

Interestingly, the original function is related to its Fourier transform by the function

$$F(t) = \int_{-\infty}^{\infty} \Phi(\nu) \exp(i2\pi \nu t) d\nu$$

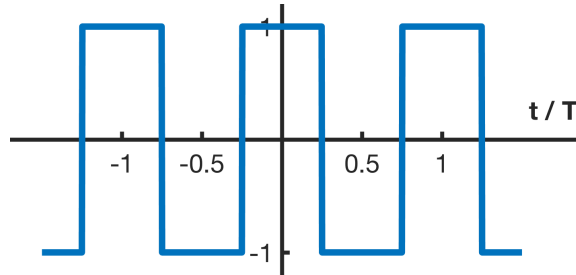
Thus, we have two mathematically identical ways to express some function - as either amplitudes as a function of time or amplitudes as a function of frequency. Variables related by a Fourier transform (such as time and frequency) are called conjugate variables. Notice that $\Phi(\nu)$ is a complex function. For voltages, the power per unit frequency is proportional to $\Phi(\nu)\Phi^*(\nu) = |\Phi(\nu)|^2$. This is called the Power Spectrum or Spectral Power Density (SPD) of $F(t)$.

The Fourier transform of a function can be obtained numerically. A method to perform the transform quickly, called the Fast Fourier Transform (FFT) was discovered by Cooley and Tukey in 1965. We will use a device called a Spectrum Analyzer, based on the FFT method, to study the Fourier transform of various functions.

In this experiment you will also use Fourier's theorem to synthesize periodic functions by adding up the harmonics. The table below gives the coefficients for three commonly encountered periodic functions. Note that the origin is chosen so that the functions are symmetric, although for the square and triangular wave it could equally well have been chosen so that the functions would have been anti-symmetric.

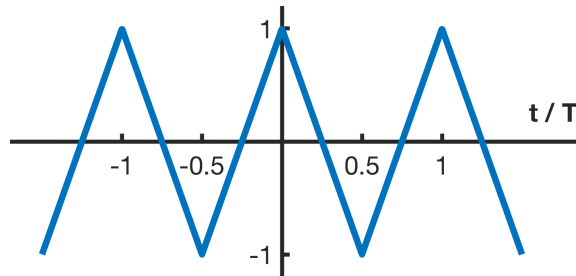
Questions:

1. Square Wave: $A_n = -\frac{4}{n\pi}(-1)^{\frac{n+1}{2}}$ for n odd; zero otherwise



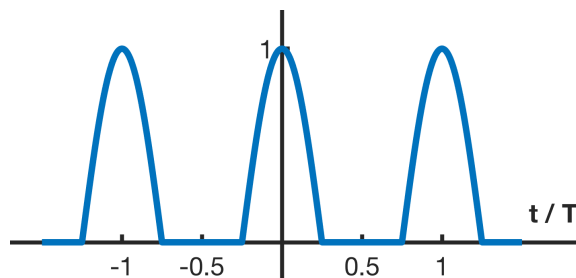
Calculate the Fourier coefficients \hat{f}_n for this square wave using Eq. (3). Show your derivation in the lab report.

2. Triangular Wave: $A_n = \frac{8}{\pi^2 n^2}$ for n odd; zero otherwise



Calculate the Fourier coefficients \hat{f}_n for this triangular wave using Eq. (3). Show your derivation in the lab report.

3. Half-wave Rectified Cosine Wave: $A_0 = \frac{1}{\pi}$; $A_1 = \frac{1}{2}$; $A_n = -\frac{2(-1)^{\frac{n}{2}}}{\pi(n^2-1)}$ for n even; zero otherwise



Calculate the Fourier coefficients \hat{f}_n for this rectified cosine wave using Eq. (3). Show your derivation in the lab report.

EQUIPMENT

Computer,
 Function Generator (User manual is available),
 Oscilloscope (User manual is available),
 Stanford Research Systems (SRS) SR760 FFT Spectrum Analyzer (User manual is available),
 Tuning Forks,
 Microphone,

PROCEDURE

Lab Activity for Week 1 (Fourier Analysis of Waveforms from a Function Generator)

A. WAVE SYNTHESIS

Evaluate the Fourier coefficients \hat{f}_n for a square wave that is symmetric in time (i.e., $f(-t) = f(t)$, as shown in the previous page) using Eq. (3). Give this derivation in your lab report.

Use MatLab to synthesize the Fourier components up to $n = 9$ for the square wave given in the introduction (**Make sure you synthesize a square wave that is symmetric in time! The MatLab function may generate an antisymmetric wave in default and you may have to make some modifications to generate a symmetric wave**). Prepare a graph showing the sums of harmonics 1+3, 1+3+5, 1+3+5+7, and 1+3+5+7+9 all superimposed on the graph (**Here '1' harmonics means both plus and minus frequency Fourier coefficients, i.e., \hat{f}_1 and \hat{f}_{-1} ; make sure you include all of them**). [In preparing the graph take care that the superposition of the four graphs is not hopelessly confusing. One way to do this is to plot it as a line plot. Make the time steps small enough that the plots are smooth curves.] Use an arbitrary time interval (say, $T = 1s$) and plot for the time interval $-T/2$ to $T/2$. What do you observe near the discontinuities for the best approximation (i.e., the 1+3+5+7+9 curve)?

B. SAWTOOTH WAVE

Evaluate the Fourier coefficients for a sawtooth wave using Eq. (3) and the function form below:

$$f(t) = 2t \quad \text{for } -T/2 < t < T/2$$

Give this derivation in your lab report. Use MatLab to add up these Fourier components out to $n = 5$ and prepare a graph comparing this sum and the exact value for $f(t)$ given above similarly to Part A.

C. SPECTRUM ANALYZER

The SR760 FFT Spectrum Analyzer will be used to record waveforms and do the Fourier transform.

The SR760 takes an input voltage and performs a Fourier transform to find the frequency spectrum of the signal. It has many features to provide a display which presents the information in the most useful fashion. We will use only a small subset of these features. The discussion and exercises below will help you get acquainted with the basic features of the SR760; more detailed information can be found in its manual that is available the course website. Follow the steps outlined below to become familiar with the controls and functions of SR760 FFT. The instrument is delicate and expensive. **Do not apply voltages greater than 1-volt amplitude. It can damage the instrument!!!**

C.1 Getting Started:

First check no cable is connected to the input terminals of SR760 analyzer. Press the power switch (located at the left corner) to it turn on.

If you want to return all settings to the default values, turn on the analyzer while holding down the ←key (on the bottom row of the keypad) until all internal tests are completed. This is a useful thing to remember if you find you've changed so many settings that nothing works anymore!

Turn on the oscilloscope. Set the probe attenuation factor to 1X (default is set to 10X). Set the input coupling to dc and sensitivity to 0.1 V/cm (you can later adjust the sensitivity according to your signal strength). With no signal connected to channel A, adjust the vertical position knob such that the trace is displayed along the middle horizontal line. Refer to the user manual for more information on how to change these settings and use the oscilloscope properly.

Turn on the function generator. Select sine wave output. Adjust the dc level control to zero and amplitude control to zero. Connect the output of the function generator to the oscilloscope channel A. Set the amplitude control to produce a sine wave with a frequency of 10kHz and amplitude of about 0.25 V (which is equivalent to 0.5 V peak-to-peak voltage). Check whether the expected waveform is generated using the oscilloscope. Use MEAS key on the oscilloscope to measure the pk-to-pk voltage and frequency accurately. If the measured voltage or frequency are very different from what you expected, check the probe attenuation factor and/or the settings of function generator to fix the issue.

Now connect the signal to the input A of SR760 analyzer using a BNC T adapter. The initial display will be set to show the frequency spectrum from 0 to 100 kHz (the maximum allowed) in bins of 250 Hz. You should now see a spectrum with a large peak at 10 kHz and smaller peaks at integral multiples of 10 kHz (i.e., 20 kHz, 30 kHz, etc.). If not, press the key AUTO-RANGE (to the right of the number keypad). This will cause the analyzer to adjust the amplification to fit the signal size. You may also notice that from time to time the SR760 will stop to recalibrate the signal. Just be patient, it will be done in 15 or 20 seconds. Next press AUTO-SCALE which will expand the display to fill the screen.

C.II Changing the Display

Now that the initial excitement has worn off, you'll want to learn how to more effectively use the analyzer. Take a look at the front panel. To the right of the keypad you will see keys ("menu keys") labeled MEAS, FREQ, DISPLAY, INPUT, and AVERAGE, along with a few others we won't be using for now. Just to the right of the screen you will see some other keys (called "soft keys" in the SR760 manual). After you push one of the menu keys, a number of options will appear on the screen. These can be selected by pushing the nearest soft key. Try pushing few buttons to find out their functions. As you read through each section, try out the appropriate keys and see what happens to the display.

1. MEAS

This key will cause a menu to be displayed with five other options: measure, display, units, window, and calculator. We will use only the first three. The window option changes the filtering of the input. The default is BMH and we will always use that. The calculator allows manipulation of the trace, but we won't need to do that either. Some of the MEAS options are described below.

DISPLAY

The display option presents another menu with five more options. We will use only two of these options: log mag. and lin. mag. These change the display to either a logarithmic or linear scale. The other options allow one to view the real or imaginary parts of the transform (or the magnitude and the phase of the transform). For our purposes, only the magnitude is important.

MEASURE

This option gives four choices: Spectrum, PSD, Time Record, and Octave. We will be interested only in the Spectrum (default) and Time Record. The Spectrum shows the magnitude of the FFT, hence the magnitude of each frequency. The Time Record shows what the input looks like (after some filtering).

UNITS

This changes the units of the display. We will use either Volt Pk, the peak voltage amplitude of each frequency, or dBV, the voltage in decibels. A decibel is $20 \log_{10}(V_i/V_0)$, where V_0 is taken as one. Decibels are a convenient way of expressing voltages which differ by several orders of magnitude. Each 20 dB is a factor of 10 smaller.

2. FREQ

The FREQ key will give a menu with several options. We will use only Span, Full Span, and Center Frequency. The Span is the frequency range to be analyzed. The default is 100 kHz.

When the Span soft key is pushed, the span range will be highlighted on the screen. Its value can be changed by rotating the knob next to the screen (the usual way) or by entering a value on the keypad (which will invoke another set of soft keys as soon as you start pushing numbers). The Span can be changed to increase the frequency resolution. For example, if the signal has two frequencies close together, say 10,000 Hz and 10,025 Hz, the default setting will put them in one bin. By changing the Span, the frequency resolution is increased. The resolution is given on the screen under the heading "line width", so you don't even have to calculate anything. The Center Frequency sets the center value of the display and can be set with the knob or by entering the value. If you want to carefully examine the region from 9.0 to 11.0 kHz, you'll need to reduce the Span to 3.125 kHz (the Span is adjusted in powers of 2 below 100 kHz - 50 kHz, 25 kHz, etc.) then set the center frequency to 10 kHz. The Full Span soft key sets the span to the maximum 100 kHz, along with other defaults—useful for getting back home when you've lost your way among the keys.

3. DISPLAY

The DISPLAY menu gives a few options. We will use only the Marker Width and Marker Seeks options. Marker Width changes the range of the marker (two vertical dashed lines). We will use only Norm or Spot (one channel width). The Marker Seeks option determines where the marker is displayed: the minimum of the range, the maximum of the range, or the mean of the range. To read out the spectrum channel by channel, set Marker Width to Spot.

4. INPUT

The INPUT key will offer a wide menu of choices. We will use only one option, Trigger. The trigger option will give another menu. The first option is again called Trigger and has four options: Cont, Int, Ext., and Ext. TTI. Each press of the soft key nearest this option will change the current trigger mode. Cont. means continuous

mode and is the default. The analyzer takes a new reading as soon as it is ready. We will also use the Int. (internal) mode in which a read of the signal is triggered by the signal. In Int. mode, the analyzer will make a new reading as soon as the input voltage passes the value given in the option Trigger Level. The Trigger Slope sets whether the trigger is on the rising or falling edge of the pulse. If the level is set to 0.5 V and slope to positive, the analyzer will trigger when the signal goes from less than 0.5 V to greater than 0.5V. If the slope is set to negative, it will trigger when the signal goes from greater than 0.5 V to smaller than 0.5 V. The Trigger Delay sets the time when the analyzer begins to record the signal to a time after the trigger. After selection of Trigger Delay, the value can be set with the knob or by entering a value from the keypad. To see the effect of the trigger level and delay, set the DISPLAY option of the MEAS menu to Time Record and then change the trigger level and delay.

5. AVERAGE

The AVERAGE menu key allows us to average several signals. This is particularly useful if the variation in the signal is large from pulse to pulse, or if signal is small compared to the background noise. The AVERAGE key will bring up another menu. Averaging can be turned on or off and the number of pulses to average can be set (again, with the knob or keypad). The maximum number of pulses to average is 32,767, but the most you will need in this lab is about 500. We will always use RMS average type and Linear mode. If you are in averaging mode, you will need to press the START key (top of the keypad) to start averaging. The averaged display will be held until START is pressed again, which will clear the data and begin a new average.

C.III Other Keys

You have already used two keys, AUTORANGE and AUTOSCALE. These adjust the input range to match the signal size and the display to fill as most of the screen as possible. The key MARKER, on the top, will exit any soft key options and allow you to move the marker with the knob. The ALT key changes the key definitions to those written by the side of the keys. Alphabetical characters can then be entered (when saving a spectrum to disk, for example).

C.IV Saving a Waveform Signal and Spectrum

1. Saving a waveform from oscilloscope

Use 'Save All to Files' function of the oscilloscope to save the waveform signal (.csv), screenshot (.bmp), and the setting (.set) to a USB thumb drive. Refer to the oscilloscope user manual for the detail. The waveform signal (.csv) can then be imported into MatLab using either readtable or importdata.

2. Saving a Fourier spectrum from SR760

The lab desktop PC can communicate with SR760 via GPIB (General Purpose Interface Bus) and LabView. Run a LabView program SR760_spectrum.vi (LabView program has extension '.vi' meaning 'virtual instrument') at Desktop > Documents > SR760. Pressing on 'right arrow' icon will start acquisition of the FFT data from SR760 and ask user to specify a file name at the end. The file can then be imported into MatLab using either readtable or importdata.

D. ANALYSIS OF WAVEFORMS

Now that you are familiar with the function generator, oscilloscope, and SR760, examine the analysis with a few different waveforms.

D.I Square Waves

Set up a square wave in the function generator with the linear display in the SR760. Record the waveform (using oscilloscope) and the Fourier spectrum (using SR760 and LabView). Record the waveform for three different SEC/DIV settings such that ~50, ~10, ~5 pulse trains are acquired within each signal, respectively. Save them in separate folders (Save All to File will automatically add numberings to a new folder and file names). Write a MatLab m-file that imports the data and analyzes them. Make a plot where you compare the three followings: 1) the Fourier coefficients amplitude obtained from the waveform data and MatLab function FFT (***be careful about the additional normalization to convert the FFT result to Fourier coefficients!**); 2) theoretically calculated Fourier coefficients; 3) the Fourier spectrum obtained from SR760. How well do they agree? If any of the three does not with others, describe about it and provide possible explanations you can think of. Compare the results among the three different SEC/DIV settings and describe/explain their differences.

Question: Now look at the SR760 output on a log scale. You will notice smaller peaks (about 30 dB smaller than the main peaks) with constant magnitude as a function of frequency. Did you expect these peaks according to the theoretical calculation of Fourier coefficients for the square wave? If not, explain what do you think these peaks correspond to and why they appear? **To support your explanation, you should provide analytic calculation or numerical simulation (using MatLab) that can reproduce the SR760 output for square waves.**

D.II Triangular Waves

Repeat **D.I** for a triangular wave except the 'log scale' part later.

Lab Activity for Week 2 (Fourier Analysis of Real World Signals)**E. TUNING FORK**

A tuning fork is supposed to provide a tone of a given frequency. Connect a microphone to the BNC input of the SR760. Analyze the frequency components of three tuning forks similarly to Part D (you don't have to include theoretical values because we don't know about it but compare your own FFT result of waveform signal and the corresponding SR760 spectrum). How pure are the tones? This is best examined using the log scale. You have to make a plot similar to the one for Part D for each fork.

F. VOICE

Repeat using your own voice. Examine the frequency components as you attempt to sing a single tone. How does the frequency distribution differ for spoken sounds? Repeat the analysis for at least three different tones. Make some qualitative comments on the distribution for vowels compared to consonants. You have to make a plot similar to the one for Part D for each tone.

Lab Report Submission Instruction

Submit a single lab report that includes all the results for A-F by the due date set for the week-2 lab.

The total lab grade will be 40 points.

5 points: Three questions about calculating Fourier coefficients for the square wave, triangular wave, and the rectified cosine wave.

20 points: Introduction / Experimental Methods / Parts A-D

15 points: Parts E-F / Conclusions