Quantum Field Theory and the Electroweak Standard Model

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High Energy Theory

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Outline

1. Particles, fields and symmetries
   - ▷ Basics: Poincaré symmetry
   - ▷ Particle physics with quantum fields
   - ▷ Global and gauge symmetries
     - Internal symmetries and the gauge principle
     - Quantization of gauge theories
     - Spontaneous Symmetry Breaking
2. The Standard Model
   - ▷ Gauge group and field representations
   - ▷ Electroweak interactions
     - One generation of quarks *or* leptons
     - Electroweak SSB: Higgs sector, gauge boson and fermion masses
     - Additional generations: fermion mixings (quarks *vs* leptons)
3. Electroweak phenomenology

⋆ *Appendices*: kinematics, loop calculations, . . .

Exercises [ugr.es/∼jillana/qft-ewsm-ex.pdf]

Bibliography

[1] **M. Maggiore**,
A Modern Introduction to Quantum Field Theory,
Oxford University Press, 2005.

[2] **M. D. Schwartz**,
Quantum Field Theory and the Standard Model,
Cambridge University Press, 2014.

[3] **M. E. Peskin and D. V. Schroeder**,
An Introduction to Quantum Field Theory,
Addison-Wesley, 1995.

[4] **A. Lahiri, P. B. Pal**,
A first book of Quantum Field Theory,
Narosa Publishing House, 2nd edition, 2005.

[5] **S. Pokorski**,
Gauge Field Theories,
Cambridge University Press, 2nd edition, 2000.

[6] **Particle Data Group**,
Review of Particle Properties,
pdg.web.cern.ch

[•] *QFT and SM at Corfu 2021* (lecture notes),
ugr.es/∼jillana/qft-sm-notes.pdf

[•] *QFT at UGR* (slides),
ugr.es/∼jillana/qft.pdf

1. Particles, fields and symmetries

Why Quantum Field Theory to describe Particle Physics?

- QFT is the (only) way to reconcile Quantum Mechanics and Special Relativity

[−] Wave equations (relativistic or not) cannot account for changing # of particles. And the relativistic versions suffer pathologies:

$\ast$ negative probability densities
$\ast$ negative-energy solutions
$\ast$ violation of causality

[+] Quantum fields:

$\ast$ provide a natural framework (*Fock space* of multiparticle states)
$\ast$ make sense of negative-energy solutions (*antiparticles*)
$\ast$ solve causality problem (*Feynman propagator*)
$\ast$ explains spin-statistics connection (*theorem*)
$\ast$ arguably, solve the wave-particle duality puzzle (no particles, *only fields*)

Basics: Poincaré symmetry

Guided by symmetry

- Relativistic fields are *irreps* of Poincaré group (rotations, boosts, translations)

$$\text{scalar } \phi(x), \text{ vector } V_\mu(x), \text{ tensor } h_{\mu\nu}(x), \ldots$$
$$\text{Weyl } \psi_L(x), \psi_R(x); \text{ Dirac } \psi(x), \ldots$$

- Lagrangian densities: **local** $\mathcal{L}(x) = \mathcal{L}(\phi, \partial_\mu \phi)$ (maybe several "ϕ_i", ψ, V_μ, ...)

  invariant under Poincaré transformations

– e.g. for a free Dirac field $\psi(x)$:

$$\boxed{\mathcal{L}_0 = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi} \quad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu \,, \quad \overline{\psi} \equiv \psi^\dagger \gamma^0$$

⋆ Field dynamics

⋆ Noether's theorem: (continuous) symmetry implies conservation laws
(spacetime symmetries ⇒ *energy, momentum, angular momentum*)

Lagrangians **Dynamics** (classical)

- Principle of least action: $\delta S = 0$ where $S = \int \mathrm{d}^4x\, \mathcal{L}(x)$

  $\Rightarrow$ Field EoM (E-L equations)

$$\delta S = \int \mathrm{d}^4x \sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta(\partial_\mu \phi_i) \right)$$

$$= \int \mathrm{d}^4x \sum_i \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \delta\phi_i = 0 \quad , \quad \forall \phi_i$$

(integrating by parts and assuming fields vanish at boundary)

  – e.g. EoM of a free Dirac field is the Dirac equation

$$\boxed{(\mathrm{i}\partial\!\!\!/ - m)\psi(x) = 0}$$

$$\rightsquigarrow \quad \psi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(a_{\vec{p},s} u^{(s)}(\vec{p}) \mathrm{e}^{-\mathrm{i}px} + b^*_{\vec{p},s} v^{(s)}(\vec{p}) \mathrm{e}^{\mathrm{i}px} \right)$$

with $p^2 = E^2_{\vec{p}} - |\vec{p}|^2 = m^2, \quad (p\!\!\!/ - m)u(\vec{p}) = 0, \quad (p\!\!\!/ + m)v(\vec{p}) = 0.$

Lagrangians **Quantization** («particle» concept emerges)

- Impose canonical quantization rules:

commutation/anticommutation of fields with conjugate momenta $\Pi_i(x) = \dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi_i)}$

$$[\phi(t,\vec{x}),\Pi_\phi(t,\vec{y})] = \mathrm{i}\delta^3(\vec{x}-\vec{y}), \quad \{\psi(t,\vec{x}),\Pi_\psi(t,\vec{y})\} = \mathrm{i}\delta^3(\vec{x}-\vec{y})$$

so that the Hamiltonian ($\mathcal{H} = \sum_i \Pi_i\dot{\Phi}_i - \mathcal{L}$) is bounded from below.

– e.g for a free fermion field, *anticommutation* is enforced! implying

$$\{a_{\vec{p},r}, a^\dagger_{\vec{k},s}\} = \{b_{\vec{p},r}, b^\dagger_{\vec{k},s}\} = (2\pi)^3\delta^3(\vec{p}-\vec{k})\delta_{rs}\ , \quad \{a_{\vec{p},r}, a_{\vec{k},s}\} = \cdots = 0$$

– After *normal ordering* :: (all creation to left of annihilation opts) to subtract zero-point energy,

$$H = \int \mathrm{d}^3x : \mathcal{H}(x) : = \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\vec{p}} \sum_{s=1,2} (a^\dagger_{\vec{p},s}a_{\vec{p},s} + b^\dagger_{\vec{p},s}b_{\vec{p},s})$$

⇒ Fields become operators that annihilate/create particles/antiparticles

$$|0\rangle \ \text{(vacuum)}\ , \quad a^\dagger_{\vec{p},s}|0\rangle \ \text{(1 particle)}\ , \quad b^\dagger_{\vec{p},s}|0\rangle \ \text{(1 antiparticle)}\ , \quad \ldots$$

⇒ Multiparticle states symmetric/antisymmetric under exchange (spin-statistics!)

One-particle representations

- One-particle states are unitary irreps of the Poincaré group, so that

$$\langle\psi_1|\psi_2\rangle = \langle\psi_1|\,\mathcal{P}^\dagger\mathcal{P}\,|\psi_2\rangle \qquad \text{(invariant matrix elements)}$$

  $\mathcal{P}$ are represented by unitary operators in this space, and the generators $\underbrace{J^i \text{ (rotations)}, K^i \text{ (boosts)}}_{J_{\mu\nu}=-J_{\nu\mu}}$, P^μ (translations) by Hermitian operators.

$$(J^i = \tfrac{1}{2}\epsilon^{ijk}J^{jk}, \quad K^i = J^{0i})$$

- Rotations form a compact subgroup (its finite dimensional irreps are unitary). But Lorentz group and Poincaré group are non-compact. Therefore:

  The *unitary* representations of the Poincaré group are *infinite-dimensional* (differential operators acting on fields).

- Poincaré group has two Casimir operators (commute with all generators)

$$\boxed{m^2 = P_\mu P^\mu, \quad W_\mu W^\mu} \qquad (W_\mu = \text{Pauli-Lubanski vector})$$

  whose eigenvalues label the irreps. Lorentz invariant (choose convenient frame).

- Two cases, characterized by $\boxed{\text{mass } m \text{ and spin } j}$

  – $m \neq 0$: choose $P^\mu = (m, 0, 0, 0) \Rightarrow W_\mu W^\mu = -m^2 j(j+1)$
  $\Rightarrow$ **massive particles of spin *j*** have $2j+1$ dof ($j_3 = -j, -j+1, \ldots, j$)
  because $SU(2)$ is the *little group* (transformations leaving P^μ invariant)

  – $m = 0$: choose $P^\mu = (\omega, 0, 0, \omega) \Rightarrow W_\mu W^\mu = -\omega^2[(J^1 + K^2)^2 + (J^2 - K^1)^2]$
  $\Rightarrow$ **massless particles of spin *j*** have 2 dof (helicity $h = \pm j$)
  because now $SO(2)$ is the *little group* (rotations in plane $\perp \vec{P}$)

- Embedding unitary reps of Poincaré group (particles) in a field theory not trivial.
  Note: To construct a **unitary field theory with** $\boxed{V_\mu}$ (contains both spin 0 and 1)
  one has to choose carefully the Lagrangian so that the *physical theory* **never excites**:

  – the spin-0 component

  – neither the longitudinal spin-1 component (if massless) $\Leftrightarrow$ gauge invariance

Particle physics

- Observables (cross sections, decays widths) expressed in terms of S-matrix elements ($m \to n$ processes)

$$
{}_{\text{out}}\left\langle \vec{p}_1 \vec{p}_2 \cdots \vec{p}_n \middle| \vec{k}_1 \vec{k}_2 \cdots \vec{k}_m \right\rangle_{\text{in}}
$$

  (scalar fields/particles to simplify)

- Only free fields are related to particles/antiparticles ($a^\dagger_{\vec{p}}$, $b^\dagger_{\vec{p}}$).

  We expect

$$
\phi(x) \xrightarrow[t\to-\infty]{} Z_\phi^{1/2} \phi_{\text{in}}(x) \,, \quad \phi(x) \xrightarrow[t\to+\infty]{} Z_\phi^{1/2} \phi_{\text{out}}(x) \,,
$$

$$
\begin{aligned}
\phi(x)&: \quad \text{interacting fields} \\
\phi_{\text{in}}(x),\, \phi_{\text{out}}(x)&: \quad \text{free fields (before, after interaction)} \\
Z_\phi&: \quad \textit{wave function} \text{ renormalization}
\end{aligned}
$$

- LSZ reduction formula relates S-matrix elements with the (Fourier transform of) vacuum expectation values of *time-ordered* field products (correlators):

$$\left(\prod_{i=1}^{m}\frac{\mathrm{i}\sqrt{Z_\phi}}{k_i^2-m^2}\right)\left(\prod_{j=1}^{n}\frac{\mathrm{i}\sqrt{Z_\phi}}{p_j^2-m^2}\right)\,{}_{\text{out}}\left\langle \vec{p}_1\vec{p}_2\cdots\vec{p}_n\middle|\vec{k}_1\vec{k}_2\cdots\vec{k}_m\right\rangle_{\text{in}}$$

$$=\int\left(\prod_{i=1}^{m}\mathrm{d}^4x_i\;\mathrm{e}^{-\mathrm{i}k_ix_i}\right)\int\left(\prod_{j=1}^{n}\mathrm{d}^4y_j\;\mathrm{e}^{+\mathrm{i}p_jy_j}\right)\langle 0|\,T\{\underbrace{\phi(x_1)\cdots\phi(x_m)\phi(y_1)\cdots\phi(y_n)}_{\text{interacting fields}}\}\,|0\rangle$$

The correlator = the Green's function of $m+n$ points $G(\vec{p}_1\cdots\vec{p}_n;\vec{k}_1\cdots\vec{k}_m)$

▷ Physical particles (asymptotic states) are on-shell ($p^2-m^2=0$).

For on-shell incoming and outgoing particles, the rhs of LSZ formula (correlator) will have **poles** that cancel those in the prefactor of the lhs, yielding a regular S-matrix element [*residues* of the correlator].

- The correlators can be expressed in terms of free fields "ϕ_I" (interaction picture):

$$\langle 0|\, T\{\phi_(x_1)\cdots\phi(x_n)\}\,|0\rangle = \frac{\langle 0|\, T\left\{\phi_I(x_1)\cdots\phi_I(x_n)\exp\left[\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\text{int}}[\phi_I(x)]\right]\right\}|0\rangle}{\langle 0|\, T\left\{\exp\left[\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\text{int}}[\phi_I(x)]\right]\right\}|0\rangle}$$

- In perturbation theory one expands the exponential and computes every correlator using *Wick's theorem* (all possible "contractions")

$$\textit{contraction} \equiv \overbracket{\phi_I(x)\phi_I}(y) = D_F(x-y) = \langle 0|\, T\{\phi_I(x)\phi_I(y)\}\,|0\rangle$$

Feynman propagator

- Feynman diagrams/rules provide a systematic procedure to organize/compute the perturbative series in terms of *propagators* (and vertices)

- **Note:** functional quantization (path integral) provides an *alternative* method

$$\langle 0|\, T\{\hat\phi(x_1)\cdots\hat\phi(x_n)\}\,|0\rangle = \frac{\int \mathcal{D}\phi\ \phi(x_1)\cdots\phi(x_n)\ \mathrm{e}^{\mathrm{i}S}}{\int \mathcal{D}\phi\ \mathrm{e}^{\mathrm{i}S}}$$

perturbatively *or* not! (*lattice*)

- **Causality** requires $[\phi(x), \phi^\dagger(y)] = 0$ if $(x-y)^2 < 0$ (*spacelike* interval)

▷ Recall that a (free) field is a combination of positive and negative energy waves:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{E_{\vec{p}}}} \left(a_{\vec{p}} \mathrm{e}^{-\mathrm{i}px} + b^\dagger_{\vec{p}} \mathrm{e}^{\mathrm{i}px} \right)$$

▷ From the commutation relations of creation and annihilation operators:

$$[\phi(x), \phi^\dagger(y)] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 \sqrt{E_{\vec{p}}}} \int \frac{\mathrm{d}^3 q}{(2\pi)^3 \sqrt{E_{\vec{q}}}} \left(\mathrm{e}^{-\mathrm{i}(px-qy)} [a_{\vec{p}}, a^\dagger_{\vec{q}}] + \mathrm{e}^{\mathrm{i}(px-qy)} [b^\dagger_{\vec{p}}, b_{\vec{q}}] \right)$$

$$= \Delta(x-y) - \Delta(y-x)$$

where the first (second) contribution comes from particles (antiparticles) and

$$\Delta(x-y) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 E_{\vec{p}}} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

▷ If $(x-y)^2 < 0$ choose frame where $x - y \equiv (0, \vec{r})$. Then

$$\Delta(x-y) = \Delta(y-x) \propto \frac{m}{r} \mathrm{e}^{-mr} \neq 0, \quad \text{for } mr \gg 1$$

Therefore:

$$\text{If only particles:} \quad [\phi(x), \phi^\dagger(y)] = \Delta(x-y) \neq 0 \text{ (!!)}$$

$$\text{If } \textit{both} \text{ particles and antiparticles:} \quad [\phi(x), \phi^\dagger(y)] = \Delta(x-y) - \Delta(y-x) = 0 \ (\checkmark)$$

- In fact the **Feynman propagator** contains *both* contributions:

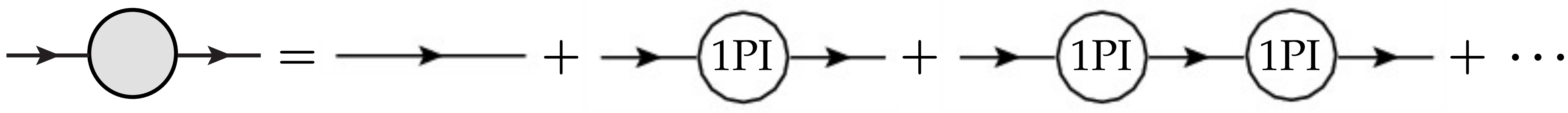
$$D_F(x-y) = \langle 0 | \, T\{\phi(x)\phi^\dagger(y)\} \, |0\rangle = \theta(x^0 - y^0)\Delta(x-y) + \theta(y^0 - x^0)\Delta(y-x)$$

  – Probability amplitude that particle created in y propagates to x, if $x^0 > y^0$

  – Probability amplitude that antiparticle created in x propagates to y, if $y^0 > x^0$

$$D_F(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\varepsilon} \mathrm{e}^{-\mathrm{i}p\cdot(x-y)} \quad \text{where} \quad \varepsilon \to 0^+ \text{(usually omitted)}$$

- Quantum corrections to external legs (external propagators) can be resummed:

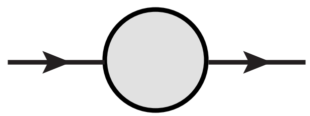


$$= \frac{\mathrm{i}}{p^2 - m_0^2} + \frac{\mathrm{i}}{p^2 - m_0^2}[-\mathrm{i}M^2(p^2)]\frac{\mathrm{i}}{p^2 - m_0^2} + \ldots$$

$$= \frac{\mathrm{i}}{p^2 - m_0^2 - M^2(p^2)} \qquad (m_0 = \text{mass in } \mathcal{L})$$

and Taylor expanding about $p^2 = m^2$ (*physical* mass):

$$p^2 - m_0^2 - M^2(p^2) = (p^2 - m^2)\left(1 - \left.\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\right|_{p^2=m^2}\right)$$

$\Rightarrow$  $= \dfrac{\mathrm{i}Z_\phi}{p^2 - m^2} +$ regular near $p^2 = m^2$

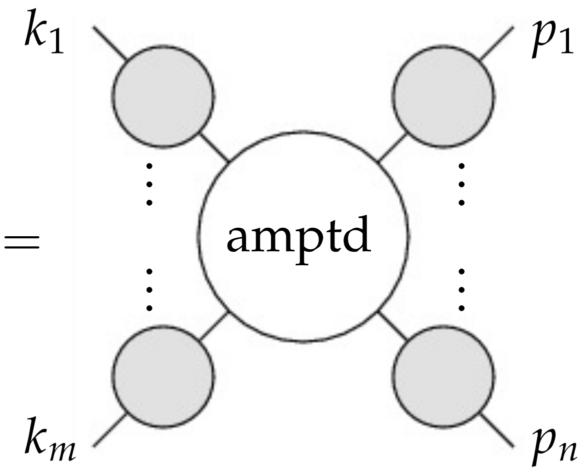
$$\text{with} \quad m^2 = m_0^2 + M^2(m^2)\,, \quad Z_\phi = \left(1 - \left.\frac{\mathrm{d}M^2}{\mathrm{d}p^2}\right|_{p^2=m^2}\right)^{-1}$$

(mass and wave function renormalization)

Particle physics | Perturbation theory | LSZ (rewritten)

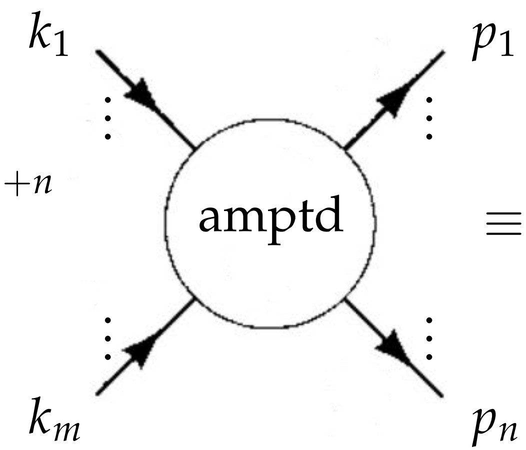
▷ Then we may factor out external legs from *amputated* diagrams:

$$G(\vec{p}_1 \cdots \vec{p}_n; \vec{k}_1 \cdots \vec{k}_m) =$$



and express the LSZ formula in a simpler form:

$${}_{\text{out}}\left\langle \vec{p}_1 \vec{p}_2 \cdots \vec{p}_n \middle| \vec{k}_1 \vec{k}_2 \cdots \vec{k}_m \right\rangle_{\text{in}} = \left(\sqrt{Z}\right)^{m+n}$$



$$\equiv (2\pi)^4 \delta^4\left(\sum_i p_i - \sum_j k_j\right) \mathrm{i}\mathcal{M}$$

- Feynman rules require integration over loop momenta resulting *sometimes* in divergent expressions.

$$\mathcal{M} = \mathcal{M}^{(0)} + \underbrace{\mathcal{M}^{(1)}}_{\text{divergent?}} + \dots$$

(the loop expansion is also an expansion in powers of $\hbar$: *quantum* corrections)

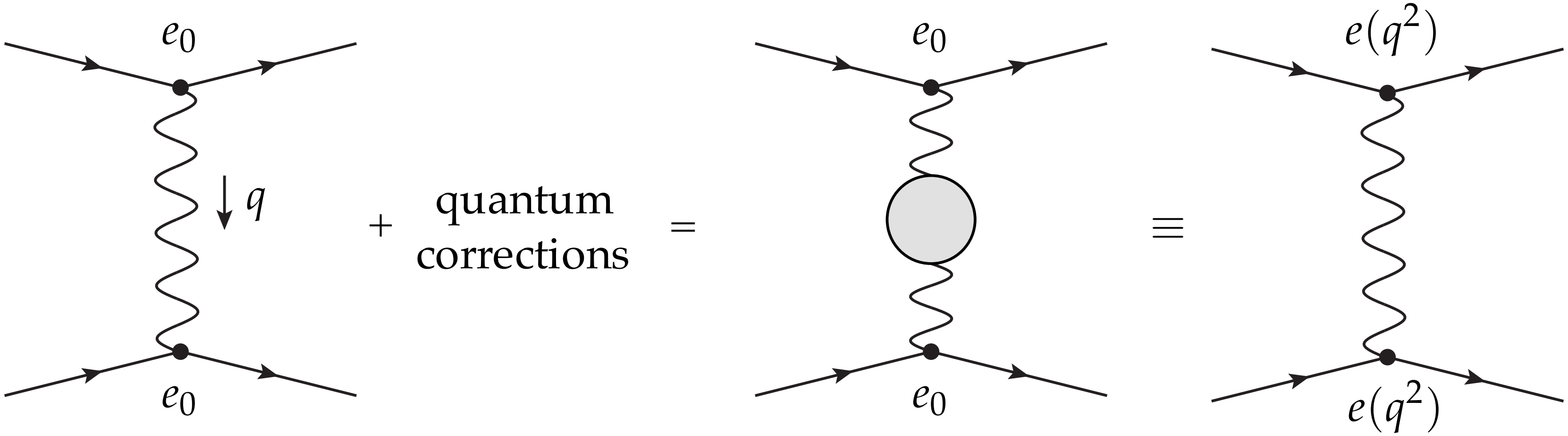
- **Regularization** and **renormalization** needed to make sense of these divergences.

▷ One assumes that fields and parameters in the Lagrangian (*bare*) must be redefined order by order in terms of new ones (*renormalized*) so that physical predictions are finite

$$\mathcal{M} = \mathcal{M}^{(0)} + \underbrace{\widehat{\mathcal{M}}^{(1)}}_{\text{finite}} + \dots$$

▷ As a consequence, *coupling constants run* (depend on a scale)

e.g.



$$\frac{{e_0}^2}{1-\Pi(q^2)} = \frac{e_R^2}{1-[\Pi(q^2)-\Pi(0)]} \equiv e^2(q^2)$$

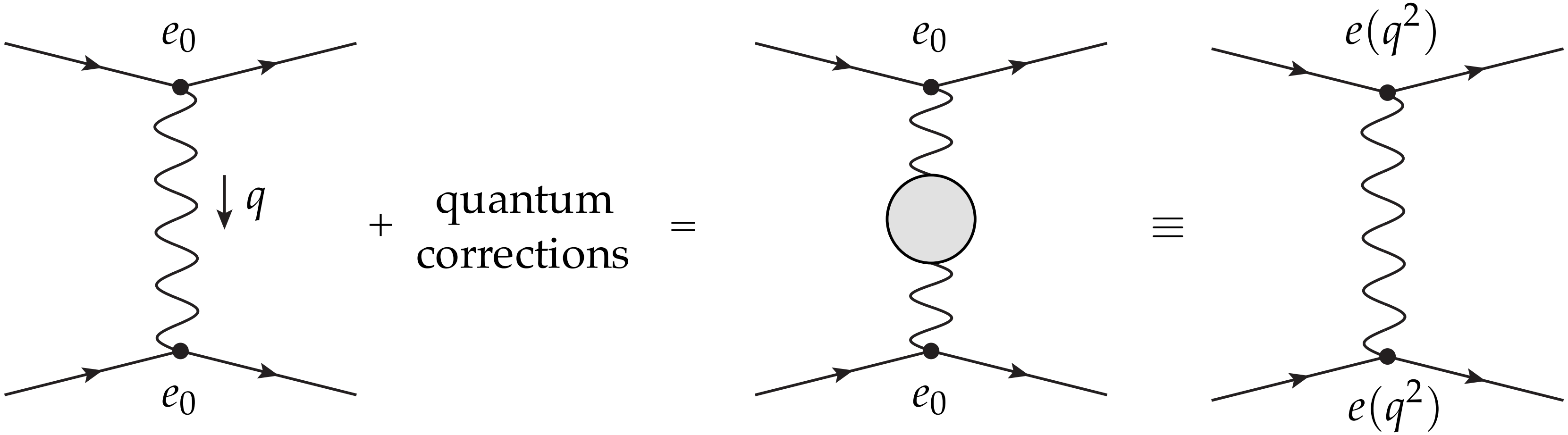
introducing renormalization constants $Z_e = Z_A^{-1/2}$ [Ward-Takahashi identity]:

$$e_0 \equiv Z_e e_R = Z_A^{-1/2} e_R \quad \text{and} \quad Z_A = [1-\Pi(0)]^{-1} \quad (\textit{universal})$$

Note: both e_0 and $\Pi(q^2)$ are infinite but $e(q^2)$ is finite.

▷ As a consequence, *coupling constants run* (depend on a scale)

e.g.



$$
\begin{aligned}
e_0 &= \textit{bare}\text{ coupling} \\
e(q^2) &= \textit{running}\text{ coupling} \\
e_R &= \textit{renormalized}\text{ coupling } [= e(0)] \\
q^2 &= \textit{renormalization scale}\text{ (at which } e \text{ is ``measured'')}
\end{aligned}
$$

Note: e is not an observable

Global symmetries and gauge invariance

Internal symmetries | free Lagrangian

- In addition to **spacetime** (Poincaré) symmetries, the free Lagrangian

(Dirac) $$\boxed{\mathcal{L}_0 = \overline{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi} \quad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu \,, \quad \overline{\psi} \equiv \psi^\dagger \gamma^0$$

$\Rightarrow$ Invariant under **internal** global U(1) phase transformations:

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta}\psi(x) \,, \quad Q,\ \theta \text{ (constants)} \in \mathbb{R}$$

$\Rightarrow$ By Noether's theorem, divergentless current:

$$\mathcal{J}^\mu = Q\,\overline{\psi}\gamma^\mu\psi \,, \quad \partial_\mu \mathcal{J}^\mu = 0$$

and a conserved «charge»

$$\mathcal{Q} = \int \mathrm{d}^3x\ \mathcal{J}^0, \quad \partial_t \mathcal{Q} = 0$$

Internal symmetries | free Lagrangian

- For a free fermion $\boxed{\text{quantum}}$ field:

  $\Rightarrow$ The Noether charge is an operator:*

$$\mathcal{Q} = Q \int \mathrm{d}^3x \; :\overline{\psi}\gamma^0\psi: = Q \int \frac{\mathrm{d}^3p}{(2\pi)^3} \sum_{s=1,2} \left(a^\dagger_{\vec{p},s} a_{\vec{p},s} - b^\dagger_{\vec{p},s} b_{\vec{p},s} \right)$$

$$\mathcal{Q}\; a^\dagger_{\vec{k},s} |0\rangle = +Q\; a^\dagger_{\vec{k},s} |0\rangle \;\; (\text{particle}) \,, \quad \mathcal{Q}\; b^\dagger_{\vec{k},s} |0\rangle = -Q\; b^\dagger_{\vec{k},s} |0\rangle \;\; (\text{antiparticle})$$

* normal ordering prescription for fermionic operators

$$: a_{\vec{p},r} a^\dagger_{\vec{q},s} : \equiv -a^\dagger_{\vec{q},s} a_{\vec{p},r} \,, \quad : b_{\vec{p},r} b^\dagger_{\vec{q},s} : \equiv -b^\dagger_{\vec{q},s} b_{\vec{p},r}$$

The gauge principle | gauge symmetry dictates interactions

- To make $\mathcal{L}_0$ invariant under local ≡ gauge transformations of U(1):

$$\psi(x) \mapsto \psi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta(x)}\psi(x)\,, \quad \theta = \theta(x) \in \mathbb{R}$$

perform the minimal substitution:

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}eQA_\mu \quad \text{(covariant derivative)}$$

where a gauge field $A_\mu(x)$ is introduced transforming as:

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x) \quad \Leftarrow \quad \boxed{D_\mu\psi \mapsto \mathrm{e}^{-\mathrm{i}Q\theta(x)}D_\mu\psi} \quad \overline{\psi}\not{D}\psi \text{ inv.}$$

1

⇒ The new Lagrangian contains **interactions** between ψ and A_μ:

$$\boxed{\mathcal{L}_{\text{int}} = -e\, Q\, \overline{\psi}\gamma^\mu\psi A_\mu} \quad \propto \begin{cases} \text{coupling} & e \\ \text{charge} & Q \end{cases}$$

$$(= -e\, \mathcal{J}^\mu A_\mu)$$

The gauge principle — gauge invariance dictates interactions

- Dynamics for the gauge field ⇒ add gauge invariant kinetic term:

$$\text{(Maxwell)} \qquad \boxed{\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \qquad \Leftarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \mapsto F_{\mu\nu}$$

– The full U(1) gauge invariant Lagrangian for a fermion field $\psi(x)$ reads:

$$\mathcal{L}_{\text{sym}} = \overline{\psi}(\mathrm{i}\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (= \mathcal{L}_0 + \mathcal{L}_{\text{int}} + \mathcal{L}_1) \quad \text{(QED)}$$

– The same applies to a complex scalar field $\phi(x)$:

$$\mathcal{L}_{\text{sym}} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad \text{(sQED)}$$

- A general gauge symmetry group G is a *compact* N-dimensional Lie group

$$\mathrm{g} \in G\,, \quad \mathrm{g}(\vec{\theta}) = \mathrm{e}^{-\mathrm{i}T_a\theta^a}\,, \quad a = 1, \ldots, N$$

$$\theta^a = \theta^a(x) \in \mathbb{R}\,, \quad T_a = \text{Hermitian generators}\,, \quad [T_a, T_b] = \mathrm{i}f_{abc}T_c \quad \text{(Lie algebra)}$$

structure constants: $f_{abc} = 0$ Abelian

$f_{abc} \neq 0$ non-Abelian

$\Rightarrow$ *Unitary* finite-dimensional irreducible representations:

$g(\vec{\theta})$ represented by $U(\vec{\theta})$

$d \times d$ matrices : $U(\vec{\theta})$ [given by $\{T_a\}$ algebra representation]

$$d\text{-multiplet}: \quad \Psi(x) \mapsto \Psi'(x) = U(\vec{\theta})\Psi(x)\,, \quad \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_d \end{pmatrix}$$

- Examples:

G	N	Abelian	
U(1)	1	Yes	
SU(n)	n^2-1	No	($n \times n$ unitary matrices with det $= 1$)

  - U(1): 1 generator (q), one-dimensional irreps only

  - SU(2): 3 generators

    $f_{abc} = \epsilon_{abc}$ (Levi-Civita symbol)
    * Fundamental irrep ($d = 2$): $T_a = \frac{1}{2}\sigma_a$ (3 Pauli matrices)
    * Adjoint irrep ($d = N = 3$): $(T_a^{\text{adj}})_{bc} = -\mathrm{i} f_{abc}$

  - SU(3): 8 generators

    $f^{123} = 1$, $f^{458} = f^{678} = \frac{\sqrt{3}}{2}$, $f^{147} = f^{156} = f^{246} = f^{247} = f^{345} = -f^{367} = \frac{1}{2}$
    * Fundamental irrep ($d = 3$): $T_a = \frac{1}{2}\lambda_a$ (8 Gell-Mann matrices)
    * Adjoint irrep ($d = N = 8$): $(T_a^{\text{adj}})_{bc} = -\mathrm{i} f_{abc}$

(for SU(n): f_{abc} totally antisymmetric)

The gauge principle — non-Abelian gauge theories

- To make $\mathcal{L}_0$ invariant under local $\equiv$ gauge transformations of G:

$$\mathcal{L}_0 = \overline{\Psi}(\mathrm{i}\partial\!\!\!/ - m)\Psi\,, \qquad \Psi(x) \mapsto \Psi'(x) = U(\vec{\theta})\Psi(x)\,, \quad \vec{\theta} = \vec{\theta}(x) \in \mathbb{R}$$

substitute the covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}g\widetilde{W}_\mu\,, \quad \widetilde{W}_\mu \equiv T_a W^a_\mu$$

where a gauge field $W^a_\mu(x)$ per generator is introduced, transforming as:

$$\widetilde{W}_\mu(x) \mapsto \widetilde{W}'_\mu(x) = \underbrace{U\widetilde{W}_\mu(x)U^\dagger}_{\text{adjoint irrep}} - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \quad \Leftarrow \quad \boxed{D_\mu\Psi \mapsto UD_\mu\Psi} \quad \overline{\Psi}D\!\!\!\!/\,\Psi \text{ inv.}$$ 1

$\Rightarrow$ The new Lagrangian contains interactions between Ψ and W^a_μ:

$$\boxed{\mathcal{L}_{\text{int}} = g\,\overline{\Psi}\gamma^\mu T_a \Psi W^a_\mu} \quad \propto \begin{cases} \text{coupling} & g \\ \text{charge} & T_a \end{cases}$$

$$(= g\,\mathcal{J}^\mu_a W^a_\mu)$$

- Dynamics for the gauge fields $\Rightarrow$ add gauge invariant kinetic terms:

$$\text{(Yang-Mills)} \quad \boxed{\mathcal{L}_{\rm YM} = -\frac{1}{2}\mathrm{Tr}\left\{\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\right\} = -\frac{1}{4}W^a_{\mu\nu}W^{a,\mu\nu}}$$

$$\widetilde{W}_{\mu\nu} \equiv T_a W^a_{\mu\nu} \equiv D_\mu \widetilde{W}_\nu - D_\nu \widetilde{W}_\mu = \partial_\mu \widetilde{W}_\nu - \partial_\nu \widetilde{W}_\mu - ig[\widetilde{W}_\mu, \widetilde{W}_\nu] \;\Leftrightarrow\; \widetilde{W}_{\mu\nu} \mapsto U\widetilde{W}_{\mu\nu}U^\dagger$$

$$\Rightarrow \quad W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu$$

2

$\Rightarrow \mathcal{L}_{\rm YM}$ contains cubic and quartic $\boxed{\text{self-interactions}}$ of the gauge fields W^a_μ:

$$\mathcal{L}_{\rm kin} = -\frac{1}{4}(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu)(\partial^\mu W^{a,\nu} - \partial^\nu W^{a,\mu})$$

$$\mathcal{L}_{\rm cubic} = -\frac{1}{2} g f_{abc}\,(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu) W^{b,\mu} W^{c,\nu}$$

$$\mathcal{L}_{\rm quartic} = -\frac{1}{4} g^2 f_{abe} f_{cde}\, W^a_\mu W^b_\nu W^{c,\mu} W^{d,\nu}$$

- The (Feynman) propagator of a scalar field:

$$D_F(x-y) = \langle 0|\, T\{\phi(x)\phi^\dagger(y)\}\,|0\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

(Feynman *prescription* $\varepsilon \to 0^+$)

is a Green's function of the Klein-Gordon operator:

$$(\Box_x + m^2)D_F(x-y) = -\mathrm{i}\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{D}_F(p) = \frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}$$

- The propagator of a fermion field:

$$S_F(x-y) = \langle 0|\, T\{\psi(x)\overline{\psi}(y)\}\,|0\rangle = (\mathrm{i}\partial\!\!\!/_x + m)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{i}}{p^2-m^2+\mathrm{i}\varepsilon}\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}$$

is a Green's function of the Dirac operator:

$$(\mathrm{i}\partial\!\!\!/_x - m)S_F(x-y) = \mathrm{i}\delta^4(x-y) \quad \Leftrightarrow \quad \widetilde{S}_F(p) = \frac{\mathrm{i}}{p\!\!\!/ - m + \mathrm{i}\varepsilon}$$

Quantization of gauge theories | propagators

- HOWEVER a gauge field propagator cannot be defined unless $\mathcal{L}$ is modified:

(e.g. modified Maxwell) $\qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2$

Euler-Lagrange: $\quad \frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad \Rightarrow \quad \left[g^{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial^\nu\right] A_\mu = 0$

– In momentum space the propagator is the inverse of:

$$-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right)k^\mu k^\nu \quad \Rightarrow \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$

$\Rightarrow$ Note that $(-k^2 g^{\mu\nu} + k^\mu k^\nu)$ is singular!

$\Rightarrow$ One may argue that $\mathcal{L}$ above will not lead to Maxwell equations …
unless we fix a (Lorenz) gauge where: (remove redundancy)

$$\partial^\mu A_\mu = 0 \quad \Leftarrow \quad A_\mu \mapsto A'_\mu = A_\mu + \partial_\mu\Lambda \;\text{ with }\; \partial^\mu\partial_\mu\Lambda \equiv -\partial^\mu A_\mu$$

Quantization of gauge theories — gauge fixing (Abelian case)

- The extra term is called Gauge Fixing:

$$\boxed{\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}(\partial^\mu A_\mu)^2}$$

$\Rightarrow$ modified $\mathcal{L}$ equivalent to Maxwell Lagrangian just in the gauge $\partial^\mu A_\mu = 0$

$\Rightarrow$ the ξ-dependence always cancels out in physical amplitudes

- Several choices for the gauge fixing term (simplify calculations): R_ξ gauges

$$(\text{'t Hooft-Feynman gauge}) \quad \xi = 1: \quad \widetilde{D}_{\mu\nu}(k) = -\frac{\mathrm{i}g_{\mu\nu}}{k^2 + \mathrm{i}\varepsilon}$$

$$(\text{Landau gauge}) \quad \xi = 0: \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right]$$

Quantization of gauge theories | gauge fixing (non-Abelian case)

- For a non-Abelian gauge theory, the gauge fixing terms:

$$\boxed{\mathcal{L}_{\rm GF} = -\sum_a \frac{1}{2\xi_a}(\partial^\mu W^a_\mu)^2}$$

allow to define the propagators:

$$\widetilde{D}^{ab}_{\mu\nu}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2+\mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi_a)\frac{k_\mu k_\nu}{k^2}\right]$$

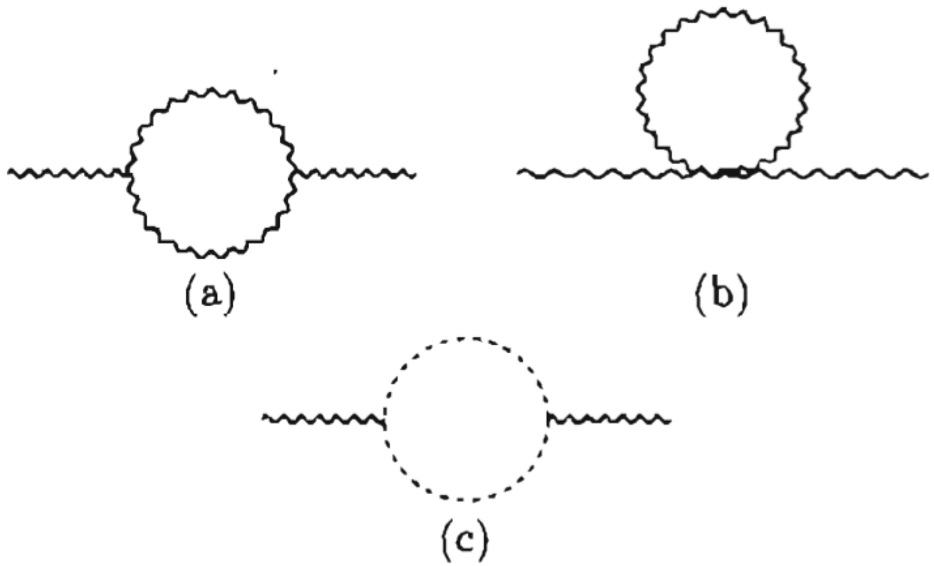
HOWEVER, unlike the Abelian case, this is not the end of the story ...

- Add Faddeev-Popov *ghost* fields $c_a(x)$, $a = 1, \ldots, N$: ('t Hooft-Feynman gauge)

$$\boxed{\mathcal{L}_{\text{FP}} = (\partial^\mu \overline{c}_a)(D_\mu^{\text{adj}})_{ab} c_b = (\partial^\mu \overline{c}_a)(\partial_\mu c_a - g f_{abc} c_b W_\mu^c)} \quad \Leftarrow \quad D_\mu^{\text{adj}} = \partial_\mu - \mathrm{i} g T_c^{\text{adj}} W_\mu^c$$

Computational trick: *anticommuting* scalar fields, just in loops as virtual particles

$\Rightarrow$ Faddeev-Popov ghosts needed to preserve gauge symmetry: 3



Self Energy

$$= \Pi_{\mu\nu} = \mathrm{i}(k^2 g_{\mu\nu} - k_\mu k_\nu)\Pi(k^2)$$ 4

Ward identity: $k^\mu \Pi_{\mu\nu} = 0$

with

$$\widetilde{D}_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 + \mathrm{i}\varepsilon}$$

[(−1) sign for closed loops! (like fermions)]

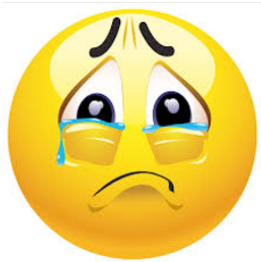
- Then the full quantum Lagrangian is

$$\boxed{\mathcal{L}_{\text{sym}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}}$$

$\Rightarrow$ Note that in the case of a massive vector field

$$\text{(Proca)} \qquad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu$$

it is not gauge invariant!!!



What about the gauge principle???

– The propagator is:

5

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\varepsilon}\left(-g_{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right)$$

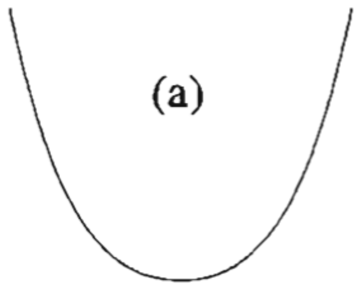
Spontaneous Symmetry Breaking | discrete symmetry

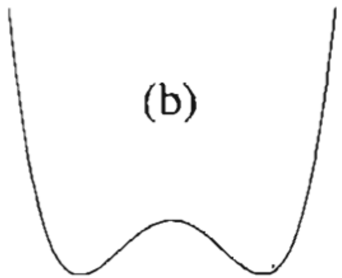
- Consider a real scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4 \quad \text{invariant under} \quad \phi \mapsto -\phi$$

$$\Rightarrow \quad \mathcal{H} = \frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2) + V(\phi)$$

$$V = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$$





$\mu^2, \lambda \in \mathbb{R}$ (Real/Hermitian Hamiltonian) and $\lambda > 0$ (existence of a ground state)

(a) $\mu^2 > 0$: min of $V(\phi)$ at $\phi = 0$

(b) $\mu^2 < 0$: min of $V(\phi)$ at $\phi = v \equiv \pm\sqrt{\dfrac{-\mu^2}{\lambda}}$, in QFT $\langle 0|\,\phi\,|0\rangle = v \neq 0$ (VEV)

– A quantum field must have $v = 0$

$$a\,|0\rangle = 0 \qquad \Rightarrow \qquad \phi(x) \equiv v + \eta(x)\,, \quad \langle 0|\,\eta\,|0\rangle = 0$$

Spontaneous Symmetry Breaking | discrete symmetry

- At the quantum level, the same system is described by $\eta(x)$ with Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{4}\lambda v^4 \quad \text{not invariant under} \quad \eta \mapsto -\eta$$

$$(m_\eta = \sqrt{2\lambda}\, v)$$

⇒ Lesson:

$\mathcal{L}(\phi)$ has the symmetry but the parameters can be such that the ground state of the Hamiltonian is not symmetric (Spontaneous Symmetry Breaking)

⇒ Note:

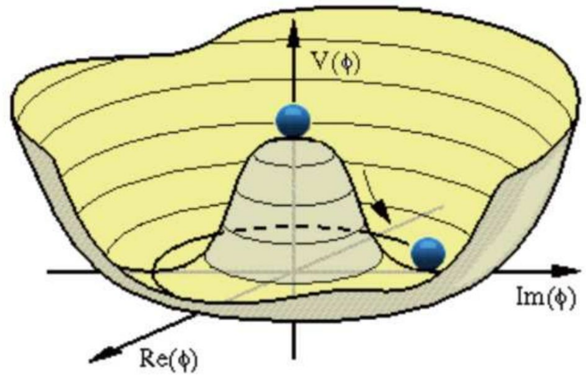
One may argue that $\mathcal{L}(\eta)$ exhibits an explicit breaking of the symmetry. However this is not the case since the coefficients of terms η^2, η^3 and η^4 are determined by just two parameters, λ and v (remnant of the original symmetry)

Spontaneous Symmetry Breaking | continuous symmetry

- Consider a complex scalar field $\phi(x)$ with Lagrangian:

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \quad \text{invariant under U(1):} \quad \phi \mapsto \mathrm{e}^{-\mathrm{i}Q\theta}\phi$$

$$\lambda > 0,\ \mu^2 < 0 : \quad \langle 0|\,\phi\,|0\rangle \equiv \frac{v}{\sqrt{2}}, \quad |v| = \sqrt{\frac{-\mu^2}{\lambda}}$$



Take $v \in \mathbb{R}^+$. In terms of quantum fields:

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + \mathrm{i}\chi(x)], \quad \langle 0|\,\eta\,|0\rangle = \langle 0|\,\chi\,|0\rangle = 0$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \frac{1}{2}(\partial_\mu \chi)(\partial^\mu \chi) - \lambda v^2 \eta^2 - \lambda v \eta(\eta^2 + \chi^2) - \frac{\lambda}{4}(\eta^2 + \chi^2)^2 + \frac{1}{4}\lambda v^4$$

Note: if $v\mathrm{e}^{\mathrm{i}\alpha}$ (complex) replace η by $(\eta \cos\alpha - \chi \sin\alpha)$ and χ by $(\eta \sin\alpha + \chi \cos\alpha)$

$\Rightarrow$ The actual quantum Lagrangian $\mathcal{L}(\eta, \chi)$ is not invariant under U(1)

U(1) broken $\Rightarrow$ one scalar field remains massless: $m_\chi = 0$, $m_\eta = \sqrt{2\lambda}\, v$

Spontaneous Symmetry Breaking | continuous symmetry

- Another example: consider a real scalar SU(2) triplet $\Phi(x)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi^\mathsf{T})(\partial^\mu \Phi) - \frac{1}{2}\mu^2 \Phi^\mathsf{T}\Phi - \frac{\lambda}{4}(\Phi^\mathsf{T}\Phi)^2 \quad \text{inv. under SU(2):} \quad \Phi \mapsto \mathrm{e}^{-\mathrm{i}T_a\theta^a}\Phi$$

that for $\lambda > 0$, $\mu^2 < 0$ acquires a VEV $\langle 0|\,\Phi^\mathsf{T}\Phi\,|0\rangle = v^2 \qquad (\mu^2 = -\lambda v^2)$

Assume $\Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \\ v + \varphi_3(x) \end{pmatrix}$ and define $\varphi \equiv \frac{1}{\sqrt{2}}(\varphi_1 + \mathrm{i}\varphi_2)$

$$\mathcal{L} = (\partial_\mu \varphi^\dagger)(\partial^\mu \varphi) + \frac{1}{2}(\partial_\mu \varphi_3)(\partial^\mu \varphi_3) - \lambda v^2 \varphi_3^2 - \lambda v(2\varphi^\dagger\varphi + \varphi_3^2)\varphi_3 - \frac{\lambda}{4}(2\varphi^\dagger\varphi + \varphi_3^2)^2 + \frac{1}{4}\lambda v^4$$

$\Rightarrow$ Not symmetric under SU(2) but invariant under U(1):

$$\varphi \mapsto \mathrm{e}^{-\mathrm{i}Q\theta}\varphi \quad (Q = \text{arbitrary}) \qquad\qquad \varphi_3 \mapsto \varphi_3 \quad (Q = 0)$$

SU(2) broken to U(1) $\Rightarrow 3 - 1 = 2$ broken generators

$\Rightarrow$ 2 (real) scalar fields (= 1 complex) remain massless: $m_\varphi = 0$, $m_{\varphi_3} = \sqrt{2\lambda}\, v$

Spontaneous Symmetry Breaking | continuous symmetry

⇒ Goldstone's theorem: [Nambu '60; Goldstone '61]

The number of massless particles (Nambu-Goldstone bosons) is equal to the number of spontaneously broken generators of the symmetry

$$\text{Hamiltonian symmetric under group } G \quad \Rightarrow \quad [T_a, H] = 0\,, \quad a = 1, \ldots, N$$

$$\text{By definition: } H|0\rangle = 0 \quad \Rightarrow \quad H(T_a|0\rangle) = T_a H|0\rangle = 0$$

– If $|0\rangle$ is such that $T_a|0\rangle = 0$ for all generators

⇒ non-degenerate minimum: *the* vacuum

– If $|0\rangle$ is such that $T_{a'}|0\rangle \neq 0$ for some (broken) generators a'

⇒ degenerate minimum: chose one (*true* vacuum) and $\mathrm{e}^{-\mathrm{i}T_{a'}\theta^{a'}}|0\rangle \neq |0\rangle$

⇒ excitations (particles) from $|0\rangle$ to $\mathrm{e}^{-\mathrm{i}T_{a'}\theta^{a'}}|0\rangle$ cost no energy: massless!

Spontaneous Symmetry Breaking | gauge symmetry

- Consider a U(1) gauge invariant Lagrangian for a complex scalar field $\phi(x)$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \,, \quad D_\mu = \partial_\mu + \mathrm{i}eQA_\mu$$

inv. under $\quad \phi(x) \mapsto \phi'(x) = \mathrm{e}^{-\mathrm{i}Q\theta(x)}\phi(x) \,, \quad A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x)$

If $\lambda > 0$, $\mu^2 < 0$, the $\mathcal{L}$ in terms of quantum fields η and χ with null VEVs:

$$\phi(x) \equiv \frac{1}{\sqrt{2}}[v + \eta(x) + \mathrm{i}\chi(x)] \,, \quad \mu^2 = -\lambda v^2$$

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) \\ &\boxed{-\lambda v^2\eta^2} - \lambda v\eta(\eta^2+\chi^2) - \frac{\lambda}{4}(\eta^2+\chi^2)^2 + \frac{1}{4}\lambda v^4 \\ &\boxed{+\, eQvA_\mu\partial^\mu\chi} + eQA_\mu(\eta\partial^\mu\chi - \chi\partial^\mu\eta) \\ &\boxed{+\, \frac{1}{2}(eQv)^2A_\mu A^\mu} + \frac{1}{2}(eQ)^2A_\mu A^\mu(\eta^2 + 2v\eta + \chi^2)\end{aligned}$$

Comments:

(i) $m_\eta = \sqrt{2\lambda}\,v$
$m_\chi = 0$

(ii) $M_A = |eQv|$ (!)

(iii) Term $A_\mu\partial^\mu\chi$ (?)

(iv) Add $\mathcal{L}_{\mathrm{GF}}$

Spontaneous Symmetry Breaking | gauge symmetry

- Removing the cross term and the (new) gauge fixing Lagrangian:

$$\mathcal{L}_{\rm GF} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M_A \chi)^2$$

$$\Rightarrow \quad \mathcal{L} + \mathcal{L}_{\rm GF} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \overbrace{M_A\partial_\mu(A^\mu\chi)}^{\substack{M_A[\partial_\mu A^\mu \chi + A_\mu \partial^\mu \chi] \\ \text{total deriv.}}}$$
$$+ \frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \frac{1}{2}\xi M_A^2 \chi^2 + \ldots$$

and the propagators of A_μ and χ are: 6

$$\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right]$$
$$\widetilde{D}(k) = \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\varepsilon}$$

$\Rightarrow$ χ has a gauge-dependent mass: actually it is not a physical field!

Spontaneous Symmetry Breaking | gauge symmetry

- A more transparent parameterization of the quantum field ϕ is

$$\phi(x) \equiv \mathrm{e}^{\mathrm{i}Q\zeta(x)/v}\frac{1}{\sqrt{2}}[v+\eta(x)]\,, \quad \langle 0|\,\eta\,|0\rangle = \langle 0|\,\zeta\,|0\rangle = 0$$

$$\phi(x) \mapsto \mathrm{e}^{-\mathrm{i}Q\zeta(x)/v}\phi(x) = \frac{1}{\sqrt{2}}[v+\eta(x)] \quad \Rightarrow \quad \zeta \text{ gauged away!}$$

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta)\\ &-\lambda v^2\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4 + \frac{1}{4}\lambda v^4\\ &+\frac{1}{2}(eQv)^2A_\mu A^\mu + \frac{1}{2}(eQ)^2A_\mu A^\mu(2v\eta+\eta^2)\end{aligned}$$

Comments:

(i) $m_\eta = \sqrt{2\lambda}\,v$

(ii) $M_A = |eQv|$

(iii) No need for $\mathcal{L}_{\mathrm{GF}}$

$\Rightarrow$ This is the unitary gauge ($\xi \to \infty$): just physical fields

$$\widetilde{D}_{\mu\nu}(k) \to \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_A^2}\right] \quad \text{and} \quad \widetilde{D}(k) \to 0$$

Spontaneous Symmetry Breaking | gauge symmetry

⇒ Brout-Englert-Higgs mechanism:

[Anderson '62]

[Higgs '64; Englert, Brout '64; Guralnik, Hagen, Kibble '64]

The gauge bosons associated with the spontaneously broken generators become massive, the corresponding would-be Goldstone bosons are unphysical and can be absorbed, the remaining massive scalars (Higgs bosons) are physical (the smoking gun!)

- The would-be Goldstone bosons are 'eaten up' by the gauge bosons ('get fat') and disappear (gauge away) in the unitary gauge ($\xi \to \infty$)
  ⇒ Degrees of freedom are preserved

  Before SSB: 2 (massless gauge boson) + 1 (Goldstone boson)

  After SSB: 3 (massive gauge boson) + 0 (absorbed would-be Goldstone)

- For loops calculations, 't Hooft-Feynman gauge ($\xi = 1$) is more convenient:
  ⇒ Gauge boson propagators are simpler, but
  ⇒ Goldstone bosons must be included in internal lines

- Comments:

  – After SSB the FP ghost fields (unphysical) acquire a gauge-dependent mass, due to interactions with the scalar field(s):

$$\widetilde{D}_{ab}(k) = \frac{\mathrm{i}\delta_{ab}}{k^2 - \xi_a M_{W^a}^2 + \mathrm{i}\varepsilon}$$

  – Gauge theories with SSB are renormalizable ['t Hooft, Veltman '72]

    UV divergences appearing at loop level can be removed by renormalization of parameters and fields of the classical Lagrangian ⇒ predictive!