Notes and Exercises from Sergei Winitzki's Book.

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Contents

1 "Linear Algebra Without Coordinates"

We are going to need the definition of a *field*. This is what Wikipedia says:

> Formally, a field is a set together with two operations called *addition* and *multiplication*. An operation is a mapping that associates an element of the set to *every* pair of its elements. The result of the addition of a and b is called the sum of a and b and denoted $a + b$. Similarly, the result of the multiplication of a and b is called the product of a and b, and denoted ab or $a \cdot b$. These operations are required to satisfy the following properties, referred to as field axioms. In the sequel, a, b and c are arbitrary elements of F.
>
> - Associativity of addition and multiplication: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
> - Commutativity of addition and multiplication: $a+b = b+a$ and $a \cdot b = b \cdot a$.
> - Additive and multiplicative identity: there exist two different elements 0 and 1 in F such that $a+0 = a$ and $a \cdot 1 = a$.
> - Additive inverses: for every a in F, there exists an element in F, denoted $-a$, called additive inverse of a, such that $a + (-a) = 0$.
> - Multiplicative inverses: for every $a \neq 0$ in F, there exists an element in F, denoted by a^{-1}, $1/a$, or $\frac{1}{a}$, called multiplicative inverse of a such that $a \cdot a^{-1} = 1$.
> - Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Page 14 (a) Q: Is this a number field: $\{x + iy\sqrt{2} \mid x, y \in \mathbb{Q}\}$?
A: I am going to refer to objects like this as (x, y). Then plainly the additive identity is $(0, 0)$ and the multiplicative identity is $(1, 0)$.

- Closure under addition? $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ so yes, the set is closed under addition.

- Closure under multiplication? $(a_1, a_2) \cdot (b_1, b_2) = (a_1 + ia_2\sqrt{2})(b_1 + ib_2\sqrt{(2)}) = (a_1b_1 - 2a_2b_2 + i\sqrt{2}[a_1b_2 + a_2b_1]) = (c_1, c_2)$ with $c_1 = a_1b_1 - 2a_2b_2$ and $c_2 = a_1b_2 + a_2b_1$. So yes, the set is closed under multiplication.

- Commutativity of addition and multiplication? Obviously, yes.

- Additive inverse? Yes, obviously $-(x, y) = (-x, -y)$.

- Additive commutativity? Obviously yes,

- Multiplicative commutatativity? I omit the calculation, but yes.

- Multiplicative inverse? $\frac{1}{x+i\sqrt{2}y} = \frac{x-i\sqrt{2}}{(x+i\sqrt{2}y)(x-i\sqrt{2})} = \frac{x-i\sqrt{2}y}{x^2+2y^2} = \left(\frac{x}{x^2+2y^2}, \frac{-y}{x^2+2y^2}\right)$. So yes.

All the requirements for the set to be a field are met, so the set *is* a field.

Page 14 (b) Q: Is this a number field: $\{x + y\sqrt{2} \mid x, y \in \mathbb{Z}\}$?
A: No, because $(0, 1) = \sqrt{2}$ does not have a multiplicative inverse in which both numbers are integers.

Examples on Pages 16–17.
I am going to need the axioms of a vector space, as supplied by [Mac10]: a vector space $\boldsymbol{V}$ is a set of objects called *vectors*. There are two operations defined on $\boldsymbol{V}$: scalar multiplication $a\boldsymbol{v}$ and vector addition $\boldsymbol{v} + \boldsymbol{w}$. There is a zero vector $\mathbf{0}$. The following axioms must be satisfied for all vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$, and all scalars a and b.

$V0$ $a\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{v} + \boldsymbol{w} \in \boldsymbol{V}$.

$V1$ $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$.

$V2$ $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$.

$V3$ $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$.

$V4$ $0\boldsymbol{v} = \mathbf{0}$.

$V5$ $1\boldsymbol{v} = \boldsymbol{v}$.

$V6$ $a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$.

$V7$ $a(\boldsymbol{v} + \boldsymbol{w}) = a\boldsymbol{v} + a\boldsymbol{w}$.

$V8$ $(a+b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$.

I will also need the test for a subspace, also found in [Mac10]: Let $\boldsymbol{U}$ be a sut of vectors in a vector space $\boldsymbol{V}$, with $\mathbf{0} \in \boldsymbol{U}$. Let $\boldsymbol{U}$ inherit scalar multiplication and vector addition from $\boldsymbol{V}$. Then

$$\boldsymbol{U} \text{ is a subspace of } \boldsymbol{V} \tag{1.1}$$
$$\Longleftrightarrow \tag{1.2}$$
$$\boldsymbol{U} \text{ is closed under scalar multiplication and vector addition.} \tag{1.3}$$

References

[Mac10] Alan Macdonald. *Linear and Geometric Algebra.* Createspace Independent Publishing Platform, 2010. ISBN: 978-1-4538-5493-8.

[Win10] Sergei Winitzki. *Linear Algebra Via Exterior Products.* lulu.com, 2010. ISBN: 978-1-4092-9496-2. Version: 1.2.