Continuity

Definition 3.1.1. Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is any function. The number L is a *limit* of f at z if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ so that for all $x \in D$, if $0 < |x - z| < \delta$ then $|f(x) - L| < \epsilon$.

Theorem 3.2.1. *Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is any function. The number L is the limit of f at z if and only if for every sequence $\langle x_n \rangle$ in $D - \{z\}$ converging to z the sequence $\langle f(x_n) \rangle$ converges to L.*

Definition 3.3.1. Suppose that $a \in D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is *continuous* at a if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ so that for all $x \in D$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. If f is continuous at all $x \in D$, then f is *continuous on* D.

Theorem 3.4.1. *Suppose that $z \in D \subseteq \mathbb{R}$ is an accumulation point of D. A function $f : D \to \mathbb{R}$ is continuous at z if and only if*

$$\lim_{x \to z} f(x) = f(z).$$

Theorem 3.4.3. *If $z \in D \subseteq \mathbb{R}$ is not an accumulation point of D, then any function $f : D \to \mathbb{R}$ is continuous at z.*

Theorem 3.4.5. *Suppose that $z \in D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is continuous at z if and only if for every sequence $\langle s_n \rangle$ in D converging to z, $\lim f(s_n) = f(z)$.*

Theorem 3.4.8. (Algebraic Properties of Continuity) *Suppose that $z \in D \subseteq \mathbb{R}$ and that $f, g : D \to \mathbb{R}$ are functions which are continuous at z. Suppose also that $k \in \mathbb{R}$ and that p is a polynomial. These functions are continuous at z:*

1. *The constant function k*
2. kf.
3. $f + g$.
4. $f - g$.
5. p.
6. fg.
7. $|f|$.
8. $\sqrt{f}$ *(if $f(x) \geq 0$ for $x \in D$).*
9. f/g *(if $g(z) \neq 0$).*

$\square$

Theorem 3.4.7. *Suppose that $f : D \to \mathbb{R}$ is continuous at $z \in D$. If $f(z) > 0$, then there is an open interval (a, b) containing z so that $f(x) > 0$ for all $x \in (a, b) \cap D$.*

Theorem 3.4.9. *Suppose that $z \in D \subseteq \mathbb{R}$, that $f : D \to \mathbb{R}$ is continuous at z, that $f(D) \subseteq E \subseteq \mathbb{R}$ and that $g : E \to \mathbb{R}$ is continuous at $f(z)$. Then $g \circ f$ is continuous at z.*

3.3.5 A function $f : D \to \mathbb{R}$ satisfies the Lipschitz condition on D if there is an $M \in \mathbb{R}$ so that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in D$. Prove that if f satisfies the Lipschitz condition on D and if $z \in D$ then f is continuous at z.

Definition 3.5.1. Suppose that $E \subseteq D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is *uniformly continuous* on E if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ so that for all $x, y \in E$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

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Theorem 3.5.6. *If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*

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Theorem 3.5.6. *If f is continuous on a closed interval $[a, b]$, then f is uniformly continuous on $[a, b]$.*
Lemma 3.5.7. *If $f : D \to \mathbb{R}$ is uniformly continuous and $\langle x_n \rangle$ is a convergent sequence in D then $\langle f(x_n) \rangle$ is a convergent sequence.*

Theorem 3.5.8. *A function $f : (a, b) \to \mathbb{R}$ is uniformly continuous if and only if f has limits at a and at b.*