Modular Arithmetic and Cryptography

Recall

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- Frequency analysis

A time question

If it is 7:00 now, what time will it be in 8 hours?

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- $7 + 8 = 15$
- 15 is "too big" subtract 12
- In 8 hours it will be 3:00.

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If it is 7:00 now, what time will it be in 19 hours?

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If it is 7:00 now, what time will it be in 19 hours?

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- $7 + 19 = 26$
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- 26 is too big
- $26 - 12 = 14$

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- 26 is too big
- $26 - 12 = 14$ still too big
- $14 - 12 = 2$

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- $7 + 19 = 26$
- 26 is too big
- $26 - 12 = 14$ still too big
- $14 - 12 = 2$
- In 19 hours, it will be 2:00.

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If it is 7:00 now, what time will it be in 100 hours?

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- $107 \div 12$ is 9 with a remainder of 11

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- Repeated subtraction is (more or less) division.
- What is left over after repeatedly subtracting 12 is the remainder when dividing by 12
- $107 \div 12$ is 9 with a remainder of 11
- In 100 hours, it will be 11:00.

Mod

- There is nothing special about 12.
- If n is any positive integer and if m is any integer, then

$$m \pmod{n} \text{ is the remainder when } m \text{ is divided by } n.$$

- Note: The remainder should be one of $0, 1, 2, \ldots, (n-1)$.

Mods

$$17 \pmod{5} \quad =$$

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\end{aligned}
$$

Mods

$$\begin{aligned}
17 \pmod 5 &= 2 \\
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\end{aligned}$$

Using a calculator

Find $12345 \pmod{678}$

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- My calculator says $12345 \div 678 = 18.20796460176991150442477876106\,2$

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- My calculator says
  $12345 \div 678 = 18.2079646017699115044247787610 62$
- Subtract 18 to get: 0.207964601769911504424778761061 95

Using a calculator

Find 12345 $\pmod{678}$

- My calculator says
  $12345 \div 678 = 18.2079646017699115044247787610 62$
- Subtract 18 to get: 0.20796460176991150442477876106195
- Multiply by 678 to get the remainder: 141

Using a calculator

Find $12345 \pmod{678}$

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  $12345 \div 678 = 18.207964601769911504424778761062$
- Subtract 18 to get: $0.20796460176991150442477876106195$
- Multiply by 678 to get the remainder: 141
- Note, you may have to round.

Using a calculator

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$$
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The ancient Chinese invented modular arithmetic to answer questions like:
Today is Thursday, April 12, 2018. What day of the week will April 12, 2018 be?

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- Friday.

- If n is a positive integer, $\mathbb{Z}_n = \{0, 1, 2, \ldots, (n-1)\}$

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- This gives a small number system - with only n numbers.
- We can do much of our usual arithmetic and algebra in $\mathbb{Z}_n$

Addition and Multiplication in $\mathbb{Z}_5$

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

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- This is the same as $1x = 4$ or $x = 4$

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- Try brute force: $2 \times 0 = 0$, $2 \times 1 = 2$, $2 \times 2 = 4$, $2 \times 3 = 0$, $2 \times 4 = 2$, $2 \times 5 = 4$
- There is no solution to this equation in $\mathbb{Z}_6$.

Weird

In $\mathbb{Z}_6$, $2 \times 3 = 0$

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2 and 3 are divisors of 0

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- $x = 3$ or $x = 4$

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- But in $\mathbb{Z}_6$, the products of some non-zero numbers is 0!
- This does not help!

An equation in $\mathbb{Z}_6$

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- $4^2 + 3 \times 4 + 2 = 0$ so 4 is a solution

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- $3^2 + 3 \times 3 + 2 = 2$ so 3 is not a solution
- $4^2 + 3 \times 4 + 2 = 0$ so 4 is a solution
- $5^2 + 3 \times 5 + 2 = 0$ so 5 is a solution

Another oddity

In $\mathbb{Z}_{17}$, there is a solution to $x^2 + 1 = 0$.

Another oddity

In $\mathbb{Z}_{17}$, there is a solution to $x^2 + 1 = 0$.

This does not happen in the real numbers.

Back to cryptography

Letters and numbers

We can identify letters and numbers:

A=0	N = 13
B=1	O =14
C=2	P =15
D=3	Q =16
E=4	R =17
F=5	S =18
G=6	T =19
H=7	U =20
I=8	V =21
J=9	W =22
K=10	X =23
L=11	Y =24
M=12	Z =25

Letters and numbers

Identify the alphabet with $\mathbb{Z}_{26}$ and can do arithmetic with letters.

Letter calculations

Calculate:

- P+W=

Letter calculations

Calculate:

- $P+W=L$
- $H \times C=$

Letter calculations

Calculate:

- $P+W=L$
- $H\times C=O$
- $F\times(E+K)=$

Letter calculations

Calculate:

- $P+W=L$
- $H\times C=O$
- $F\times(E+K)=F\times O=$

Letter calculations

Calculate:

- P+W=L
- H×C=O
- F×(E+K)=F×O=S

Caesar Cipher

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- Encrypt "cat" with a Caesar Cipher and key D

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$$\begin{array}{rccc} & \text{c} & \text{a} & \text{t} \\ + & \text{D} & \text{D} & \text{D} \\ \hline & \text{F} & \text{D} & \text{W} \end{array}$$

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- Ciphertext: FDW
- To decypt, subtract.

$$\begin{array}{cccc} & \mathsf{F} & \mathsf{D} & \mathsf{W} \\ - & \mathsf{D} & \mathsf{D} & \mathsf{D} \\ \hline & \mathsf{c} & \mathsf{a} & \mathsf{t} \end{array}$$

Atbash

Atbash encrypts any plaintext letter p as $Z - p$.

- Encrypt "holy"

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- Subtract

	Z	Z	Z	Z
-	h	o	l	y
	S	L	O	B

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$$\begin{array}{ccccc} & Z & Z & Z & Z \\ - & h & o & l & y \\ \hline & S & L & O & B \end{array}$$

- Ciphertext: SLOB

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$$\begin{array}{ccccc} & Z & Z & Z & Z \\ - & h & o & l & y \\ \hline & S & L & O & B \end{array}$$

- Ciphertext: SLOB
- Decrypt also by subtracting from ZZZZZZ

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- Use some symbols to represent entire words.
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- Homophonic substitution - more than one symbol can be used to represent a plaintext letter

Homophonic substitution cipher

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
09	48	13	10	14	10	06	23	32	15	04	26	22	18	00	38	94	29	11	17	08	34	60	28	21	02
12	81	41	03	16	31	25	39	70			37	27	58	05	95		35	19	20	61		89		52	
33		62	45	24			50	73			51		59	07			40	36	30	63					
47			79	44			56	83			84		66	54			42	76	43						
53				46			65	88					71	72			77	86	49						
67				55			68	93					91	90			80	96	69						
78				57										99					75						
92				64															85						
				74															97						
				82																					
				87																					
				98																					

t	u	r	n		t	o		t	h	e		e	a	s	t
17	08	29	18		20	00		30	23	14		16	09	11	43

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  - Every occurrence of a symbol in the ciphertext represents the same plaintext letter.
- Poly-alphabetic substitution:
  - A plaintext letter may be encrypted multiple ways.
  - A ciphertext symbol may represent different plaintext letters.

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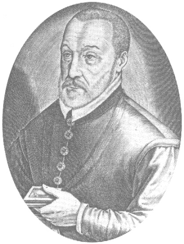
- The Viginère cipher was not widely accepted because it was somewhat tedious (even though machines like Alberti's wheel made it easier).
- A mod-2 version of the Viginère cipher is present in many modern ciphers.



Caesar Cipher

Recall...

- To encrypt "secret message" with a key of D
- Add D to each letter of the plaintext:

	s	e	c	r	e	t	m	e	s	s	a	g	e
+	D	D	D	D	D	D	D	D	D	D	D	D	D
	V	H	F	U	H	W	P	H	V	V	D	J	H

- Every e is encrypted as H.
- Every H represents an e.
- Decrypt by subtracting D.

Viginère suggested that instead of adding a SINGLE LETTER to the plaintext, we could add a WORD.

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- Decrypt by subtracting the keyword.
- Frequency analysis is hard (but not impossible)

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- Some have used the One Time Pad with their favorite novel as a key pad.

Breaking Viginère

- If the key for Viginère is shorter than the plaintext and repeated, then Viginère can be broken.
- If the key has length, say, 5, then every fifth letter is encrypted with a Caesar Cipher.
- A frequence analysis on every fifth letter would easily give the key.
- All that is needed is the key-length.
- The key-length can be found with some “statistical-voodoo”
- Methods developed independently by Charles Babbage and Wilhelm Kasiski around 1850

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- Radio brought the need for fast, mechanical encryption.

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a	t	t	a	c
k	t	h	e	s
o	u	t	h	g
a	t	e	a	t
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  filling in extra space with gibberish.

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  filling in extra space with gibberish.
- Copy the ciphertext down the columns:

  AKOADTTUTATHTEWAEHANCSGTX

Tabular Substitution

	A	D	F	G	X
A	K	E	Y	W	O
D	R	D	A	B	C
F	F	G	H	I/J	L
G	M	N	P	Q	S
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- Example: G is encrypted as FD

Combining

The most successful means of cryptography combine methods.

Tabular		Column		Collapse
Substitution	$\rightarrow$	Scytale	$\rightarrow$	Tabular
				Substitution

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- Encrypt the result of the previous step with a column scytale with 5 columns
- Collapse the result of the last step with a tabular substitution with keyword ZIPF

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- Breaking the ADFGVX cipher helped to prevent Germany from taking Paris.

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- Glue the parts back together.

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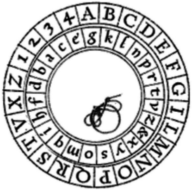
- Ease of Communication vs. Ease of Interception.
- Partially drove the movement toward mechanized cryptography.

Alberti and the Cipher Disk

Almost five hundred years before WWI, Alberti made a "cipher disk" to help perform substitution ciphers (Caesar) more quickly.



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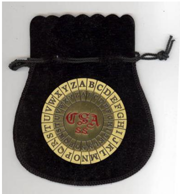

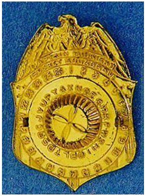

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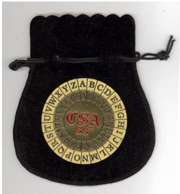

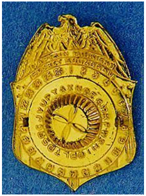

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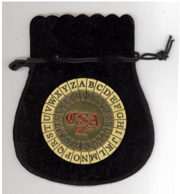

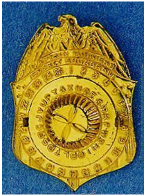

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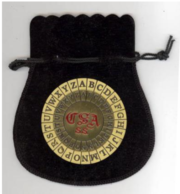

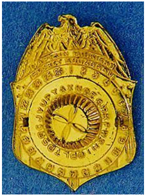

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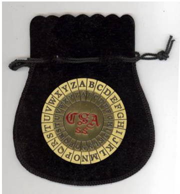

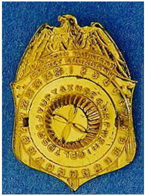

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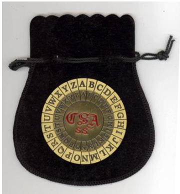
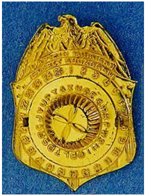
- Predecessor of the most formidable cipher machine of WWII.





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