Continuity

Definition 3.1.1. Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is any function. The number L is a *limit* of f at z if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ so that for all $x \in D$, if $0 < |x - z| < \delta$ then $|f(x) - L| < \epsilon$.

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Theorem 3.2.1. *Suppose that z is an accumulation point of $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is any function. The number L is the limit of f at z if and only if for every sequence $\langle x_n \rangle$ in $D - \{z\}$ converging to z the sequence $\langle f(x_n) \rangle$ converges to L.*

Definition 3.3.1. Suppose that $a \in D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is *continuous* at a if for every real number $\epsilon > 0$ there is a real number $\delta > 0$ so that for all $x \in D$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. If f is continuous at all $x \in D$, then f is *continuous on* D.

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Theorem 3.4.1. *Suppose that $z \in D \subseteq \mathbb{R}$ is an accumulation point of D. A function $f : D \to \mathbb{R}$ is continuous at z if and only if*

$$\lim_{x \to z} f(x) = f(z).$$

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Theorem 3.4.3. *If $z \in D \subseteq \mathbb{R}$ is not an accumulation point of D, then any function $f : D \to \mathbb{R}$ is continuous at z.*

Theorem 3.4.5. *Suppose that* $z \in D \subseteq \mathbb{R}$*. A function* $f : D \to \mathbb{R}$ *is continuous at* z *if and only if for every sequence* $\langle s_n \rangle$ *in* D *converging to* z*,* $\lim f(s_n) = f(z)$*.*

Theorem 3.4.7. *Suppose that $f : D \to \mathbb{R}$ is continuous at $z \in D$. If $f(z) > 0$, then there is an open interval (a, b) containing z so that $f(x) > 0$ for all $x \in (a, b) \cap D$.*

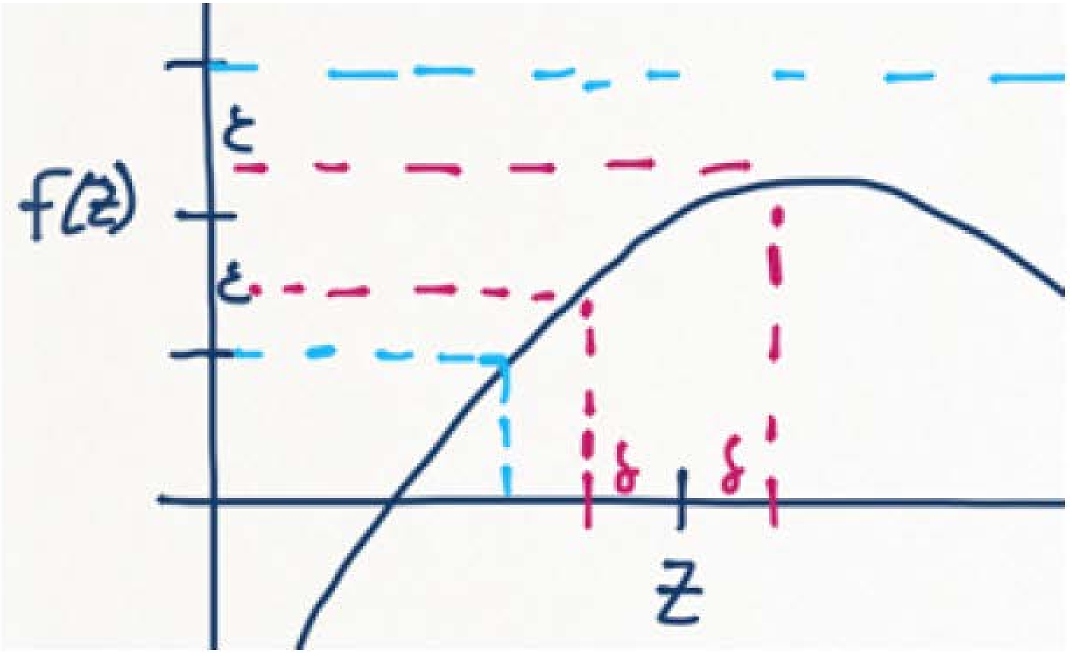

Figure 3.4: Suppose that $f(z) > 0$. Let $\epsilon = f(z)/2 > 0$. There is a $\delta > 0$ so that if x is within δ of z then $f(x)$ is within ϵ of $f(z)$. But this means that $f(x) > f(z) - \epsilon = f(z)/2 > 0$.

Theorem 3.4.8. (Algebraic Properties of Continuity) *Suppose that $z \in D \subseteq \mathbb{R}$ and that $f, g : D \to \mathbb{R}$ are functions which are continuous at z. Suppose also that $k \in \mathbb{R}$ and that p is a polynomial. These functions are continuous at z:*

1. *The constant function k*
2. kf.
3. $f + g$.
4. $f - g$.
5. p.
6. fg.
7. $|f|$.
8. $\sqrt{f}$ (*if* $f(x) \geq 0$ *for* $x \in D$).
9. f/g (*if* $g(z) \neq 0$).

$\square$

Theorem 3.4.9. *Suppose that* $z \in D \subseteq \mathbb{R}$*, that* $f : D \to \mathbb{R}$ *is continuous at* z*, that* $f(D) \subseteq E \subseteq \mathbb{R}$ *and that* $g : E \to \mathbb{R}$ *is continuous at* $f(z)$*. Then* $g \circ f$ *is continuous at* z*.*