Composition series and Maltsev algebras

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Abstract. We extend the notion of composition series to general algebras in such a way that the Jordan-Hölder Theorem holds for any finite algebra with a Maltsev operation and a one element algebra. We then prove that for semigroups, satisfaction of this variant of the Jordan-Hölder theorem across a variety is equivalent to having a Maltsev term. We also establish a connection between this Jordan-Hölder variant and n-permutability for arbitrary idempotent varieties.

1. Preliminaries

For algebras $\mathbf{B}$ and $\mathbf{A}$, we will use the notation $\mathbf{B} \leq \mathbf{A}$ to indicate that $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. If $\mathbf{B} \leq \mathbf{A}$ and if there is a congruence θ on $\mathbf{A}$ so that $\mathbf{B}$ is an equivalence class of θ then we will say that $\mathbf{B}$ is a **normal subalgebra** of $\mathbf{A}$. We will denote this as $\mathbf{B} \trianglelefteq \mathbf{A}$. If $\mathbf{B} \trianglelefteq \mathbf{A}$, then $\mathbf{A}/\mathbf{B}$ will denote the quotient algebra $\mathbf{A}/\theta$ where θ is the *greatest* congruence on $\mathbf{A}$ for which $\mathbf{B}$ is an equivalence class.

Suppose that $\mathbf{A}$ is an algebra with a one element subuniverse $\{a\}$. In this case, we will abuse notation and say that a is a subalgebra of $\mathbf{A}$ and write $a \leq \mathbf{A}$. If $\theta \in \mathrm{Con}\mathbf{A}$, then the equivalence class of a modulo θ is a subuniverse of $\mathbf{A}$ (because $a \leq \mathbf{A}$). As a subalgebra of $\mathbf{A}$, we will denote this equivalence class as $\theta(a)$.

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are similar algebras with one element subalgebras a and b respectively. Let η be the kernel of the projection homomorphism of $\mathbf{A} \times \mathbf{B}$ onto $\mathbf{B}$. The map $x \mapsto \langle x, b\rangle$ is clearly an isomorphism of $\mathbf{A}$ and $\eta(\langle a, b\rangle)$. Thus we have:

Lemma 1.1. *Suppose that $\mathbf{A}$ and $\mathbf{B}$ are similar algebras with one element subalgebras a and b respectively. Let η be the kernel of the projection homomorphism of $\mathbf{A} \times \mathbf{B}$ onto $\mathbf{B}$. Then $\mathbf{A} \cong \eta(\langle a, b\rangle)$.*

If $\mathbf{B} \leq \mathbf{A}$ and $\theta \in \mathrm{Con}\mathbf{A}$, then $\theta \cap (\mathbf{B} \times \mathbf{B})$ is a congruence on $\mathbf{B}$. We will denote this congruence as $\theta|_{\mathbf{B}}$.

The following generalization of the Second Isomorphism Theorem from group theory will accomplish most of our work later.

Lemma 1.2. *(Diamond Lemma) Suppose that α and β are permuting congruences on an algebra $\mathbf{A}$ with a one element subalgebra a. Let $\gamma = \alpha \vee \beta$ and $\delta = \alpha \cap \beta$. Then*

$$\gamma(a)/\alpha|_{\gamma(a)} \cong \beta(a)/\delta|_{\beta(a)}.$$

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Proof. First, note that $\beta(a) = \beta|_{\gamma(a)}(a)$ and $\delta|_{\beta(a)} = (\delta|_{\gamma(a)})|_{\beta(a)}$. Also note that $\alpha|_{\gamma(a)}$ and $\beta|_{\gamma(a)}$ commute and join to $1_{\gamma(a)}$. Therefore, we can replace $\mathbf{A}$ by $\gamma(a)$, α by $\alpha|_{\gamma(a)}$, and β by $\beta|_{\gamma(a)}$.

We have an algebra $\mathbf{A}$ with a one element subalgebra a and permuting congruences α and β which join to 1_A. The congruence δ is still defined to be $\alpha \cap \beta$. To establish the Lemma, we need to show that $\mathbf{A}/\alpha \cong \beta(a)/\delta|_{\beta(a)}$.

Since $\alpha, \beta \supseteq \delta$, $\mathbf{A}/\alpha \cong (\mathbf{A}/\delta)/(\alpha/\delta)$ and $\beta(a)/\delta|_{\beta(a)} \cong (\beta/\delta)(a/\delta)$ (The element a/δ is a one element subuniverse of $\mathbf{A}/\delta$). Also, α/δ and β/δ permute since α and β are above δ and permute. Therefore, we can replace $\mathbf{A}$ by $\mathbf{A}/\delta$ and assume that $\delta = 0_A$.

We now have an algebra $\mathbf{A}$ with a one element subalgebra a and permuting congruences α and β which join to 1_A and meet to 0_A. We need to show that $\mathbf{A}/\alpha \cong \beta(a)$. With this arrangement, $\mathbf{A} \cong (\mathbf{A}/\alpha) \times (\mathbf{A}/\beta)$ and the elements a/α and a/β are one element subalgebras of $\mathbf{A}/\alpha$ and $\mathbf{A}/\beta$ respectively. The isomorhism $\mathbf{A}/\alpha \cong \beta(a)$ follows from Lemma 1.1. $\square$

Notice that Lemma 1.2 does not in general hold for non-permuting congruences. For example, let $\mathbf{A}$ be the three element chain considered as a lattice with least element 0. Then 0 is a one element subalgebra. Con$\mathbf{A}$ contains four elements with two atoms. In one atom (call it α), 0 is related to the element directly above it. In the other (call it β), 0 forms a singleton equivalence class. The congruences α and β do not permute but do join to 1_A and meet to 0_A. With this arrangement, $\mathbf{A}/\alpha$ has two elements, but $\beta(0)$ is a singleton.

2. Composition Series

Suppose that $\mathbf{A}$ is an algebra with a one element subalgebra a. A **composition series** for $\mathbf{A}$ over a is a sequence of subalgebras of $\mathbf{A}$

$$a = \mathbf{B}_0 \leq \mathbf{B}_1 \leq \cdots \leq \mathbf{B}_n = \mathbf{A}$$

so that $\mathbf{B}_i \trianglelefteq \mathbf{B}_{i+1}$ and $\mathbf{B}_{i+1}/\mathbf{B}_i$ is simple for each i.

We establish the familiar Jordan-Hölder Theorem for composition series in this environment through the following two theorems. Their proofs mimic the traditional proofs for group theory.

Theorem 2.1. *(Generalized Jordan-Hölder Part 1) If $\mathbf{A}$ is a finite algebra with a one element subalgebra a, then $\mathbf{A}$ has a composition series over a.*

Proof. The proof is by induction on the size of $\mathbf{A}$. If $|\mathbf{A}| = 2$, then $\mathbf{A}$ is simple and $a \leq \mathbf{A}$ is a composition series for $\mathbf{A}$ over a. Assume then that $|\mathbf{A}| > 2$ and that every finite algebra smaller than $\mathbf{A}$ which has a one element subalgebra has a composition series.

Let θ be a coatom in Con$\mathbf{A}$. If a/θ is a singleton equivalence class, then $a \leq \mathbf{A}$ is a composition series for $\mathbf{A}$ over a. Assume then that a/θ contains more than one element. $\theta(a)$ is an algebra smaller than $\mathbf{A}$ and has a one element subalgebra a.

By induction, $\theta(a)$ has a composition series

$$a = \mathbf{B}_0 \leq \mathbf{B}_1 \leq \cdots \leq \mathbf{B}_n = \theta(a).$$

Since θ is maximal in Con$\mathbf{A}$, θ must be the greatest congruence on $\mathbf{A}$ which has $\mathbf{B}_n = \theta(a)$ as an equivalence class. It follows then that $\mathbf{A}/\mathbf{B}_n$ is simple, so

$$a = \mathbf{B}_0 \leq \mathbf{B}_1 \leq \cdots \leq \mathbf{B}_n = \theta(a) \leq \mathbf{B}_{n+1} = \mathbf{A}$$

is a composition series for $\mathbf{A}$. □

Two composition series

$$a = \mathbf{B}_0 \leq \mathbf{B}_1 \leq \cdots \leq \mathbf{B}_n = \mathbf{A}$$

and

$$a = \mathbf{C}_0 \leq \mathbf{C}_1 \leq \cdots \leq \mathbf{C}_m = \mathbf{A}$$

for an algebra $\mathbf{A}$ over a one element subalgebra a are **equivalent** if $m = n$ and if the simple quotients $\mathbf{B}_{i+1}/\mathbf{B}_i$ $(i = 1, 2, \ldots, n)$ and $\mathbf{C}_{i+1}/\mathbf{C}_i$ $(i = 1, 2, \ldots, n)$ are the same up to a permutation.

In the next theorem, the presence of a Maltsev term is assumed to force congruences on all subalgebras to permute. Since we actually only need certain congruences on certain subalgebras to permute, a more relaxed assumption might work.

Theorem 2.2. *(Generalized Jordan-Hölder Part 2) Suppose that* $\mathbf{A}$ *is a finite algebra with a one element subalgebra* a*. If* $\mathbf{A}$ *has a Maltsev term, then any two composition series for* $\mathbf{A}$ *over* a *are equivalent.*

Proof. Note that since $\mathbf{A}$ has a Maltsev operation every subalgebra of $\mathbf{A}$ has permuting congruences.

We prove this by induction on the size of $\mathbf{A}$. If $|\mathbf{A}| = 2$, then $\mathbf{A}$ is simple and $a \leq \mathbf{A}$ is trivially the only composition series for $\mathbf{A}$ over a. Assume then that $|\mathbf{A}| > 2$ and that the theorem holds for algebras smaller than $\mathbf{A}$.

Suppose that

$$a = \mathbf{B}_0 \leq \mathbf{B}_1 \leq \cdots \leq \mathbf{B}_{n+1} = \mathbf{A}$$

and

$$a = \mathbf{C}_0 \leq \mathbf{C}_1 \leq \cdots \leq \mathbf{C}_{m+1} = \mathbf{A}$$

are composition series for $\mathbf{A}$ over a. Suppose first that $n = 0$. Let α be the largest congruence on $\mathbf{A}$ with $\mathbf{B}_0 = \{a\}$ as an equivalence class. Since $\mathbf{A}/\alpha = \mathbf{B}_1/\mathbf{B}_0$ is simple, we know that α is a coatom in Con$\mathbf{A}$. Let β be the largest congruence on $\mathbf{A}$ with $\mathbf{C}_m$ as an equivalence class. Note that β must also be a coatom. It must be the case that $\alpha = \beta$. If this were not the case then $\alpha \circ \beta = 1_A$. This means that for all $y \in \mathbf{A}$ there is an $x \in \mathbf{A}$ with $a\alpha x\beta y$. Since a/α is a singleton, this means that $a\beta y$. Thus a/β is all of $\mathbf{A}$. This contradicts the fact that $\mathbf{A}/\beta = \mathbf{C}_{m+1}/\mathbf{C}_m$ is simple. It has to be that $\beta = \alpha$, so $\mathbf{C}_m = \mathbf{C}_0 = \mathbf{B}_0 = \{a\}$. Then $m = 0$ and these two composition series are identical – and hence equivalent. A similar argument establishes equivalence if $m = 0$, so we can assume that m and n are greater than 0.

If $\mathbf{B}_n = \mathbf{C}_m$, then the induction hypothesis applied to $\mathbf{B}_n$ establishes that

$$a = \mathbf{B}_0 \leq \mathbf{B}_1 \leq \cdots \leq \mathbf{B}_n$$

and

$$a = \mathbf{C}_0 \leq \mathbf{C}_1 \leq \cdots \leq \mathbf{C}_m$$

are equivalent composition series of $\mathbf{B}_n$ over a. It follows that our two composition series are equivalent.

Suppose now that $\mathbf{B}_n \neq \mathbf{C}_m$ and let $\mathbf{D} = \mathbf{B}_n \cap \mathbf{C}_m$. It must be that $\mathbf{D} \neq \mathbf{B}_n$ and $\mathbf{D} \neq \mathbf{C}_m$. Otherwise, $\mathbf{B}_n$ and $\mathbf{C}_m$ would be comparable. This cannot be because $\mathbf{A}/\mathbf{B}_n$ and $\mathbf{A}/\mathbf{C}_m$ are simple. $\mathbf{D}$ has a composition series over a:

$$a = \mathbf{D}_0 \leq \mathbf{D}_1 \leq \cdots \leq \mathbf{D}_k = \mathbf{D}.$$

Let α and β be the largest congruences with $\mathbf{B}_n$ and $\mathbf{C}_m$ as equivalence classes respectively. Notice that since $\mathbf{A}/\mathbf{B}_n$ and $\mathbf{A}/\mathbf{C}_m$ are simple, it has to be that α and β are coatoms which join to 1_A. Let $\delta = \alpha \cap \beta$. Then $\mathbf{D}$ is an equivalence class of δ, $\mathbf{D} \trianglelefteq \mathbf{B}_n$ and $\mathbf{D} \trianglelefteq \mathbf{C}_m$. By Lemma 1.2 we know that

$$\mathbf{C}_{m+1}/\mathbf{C}_m = \mathbf{A}/\beta \cong \alpha(a)/\delta|_{\alpha(a)} = \mathbf{B}_n/\delta|_{\mathbf{B}_n}.$$

in particular, $\mathbf{B}_n/\delta|_{\mathbf{B}_n}$ is simple. This means that $\delta|_{\mathbf{B}_n}$ must be the largest congruence on $\mathbf{B}_n$ with $\mathbf{D}$ as an equivalence class. Thus, $\mathbf{B}_n/\mathbf{D} = \mathbf{B}_n/\delta|_{\mathbf{B}_n}$ is simple. We now have that

$$a = \mathbf{B}_0 \leq \mathbf{B}_1 \leq \cdots \leq \mathbf{B}_n$$

and

$$a = \mathbf{D}_0 \leq \mathbf{D}_1 \leq \cdots \leq \mathbf{D}_k = \mathbf{D} \leq \mathbf{D}_{k+1} = \mathbf{B}_n$$

are composition series for $\mathbf{B}_n$ over a. By induction, these series are equivalent. That is, $n = k + 1$ and – up to a rearrangement – the simple quotients $\mathbf{B}_{i+1}/\mathbf{B}_i$ and $\mathbf{D}_{i+1}/\mathbf{D}_i$ are the same for $i = 0, 1, \ldots, n-1$. A similar argument shows the same is true for the series of $\mathbf{C}_i$'s. We have then that $n = m$ and that for $i = 0, 1, \ldots, n-1$ all but one of the factors $\mathbf{B}_{i+1}/\mathbf{B}_i$ can be matched injectively with the factors $\mathbf{C}_{i+1}/\mathbf{C}_i$ and vice-versa. The ones which are omitted correspond to those factors isomorphic to $\mathbf{B}_n/\mathbf{D}$ and $\mathbf{C}_n/\mathbf{D}$. However, these missing factors can be matched with $\mathbf{C}_{n+1}/\mathbf{C}_n \cong \mathbf{B}_n/\mathbf{D}$ and $\mathbf{B}_{n+1}/\mathbf{B}_n \cong \mathbf{C}_n/\mathbf{D}$ by Lemma 1.2 so that these two composition series are equivalent. □

Example 2.3. We give here an example of an algebra (actually, six) without a Maltsev operation in which this variant of the Jorday-Hölder Theorem fails. Let $\mathbf{L}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{S}$, $\mathbf{D}$, and $\mathbf{T}$ be two element algebras on the set $\{0, 1\}$ so that

- $\mathbf{L}$ is a left zero semigroup;
- $\mathbf{R}$ is a right zero semigroup;
- $\mathbf{C}$ is a semigroup in which $xy = 0$ for all x and y;
- $\mathbf{S}$ is a meet semilattice with least element 0;
- $\mathbf{D}$ is a lattice; and
- $\mathbf{T}$ is a set (with no operations).

Let A be the four element subset of $\{0,1\}^4$ containing these elements:

$$\begin{aligned} a &= \langle 0,0,0,0\rangle \\ b &= \langle 0,0,0,1\rangle \\ c &= \langle 0,0,1,1\rangle \\ d &= \langle 0,1,1,1\rangle. \end{aligned}$$

Then A is a subuniverse of the algebras induced on $\{0,1\}^4$ induced by each of $\mathbf{L}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{S}$, $\mathbf{D}$, and $\mathbf{T}$. Let $\mathbf{A}$ be the algebra on A induced by one of these algebras. Define these subalgebras of $\mathbf{A}$: $\mathbf{A}_0 = \{a\} = \mathbf{B}_0$, $\mathbf{A}_1 = \{a,b\} = \mathbf{B}_1$, $\mathbf{A}_2 = \{a,b,c\}$, and $\mathbf{A}_3 = \{a,b,c,d\} = \mathbf{B}_2$. Then $\mathbf{A}_0 \trianglelefteq \mathbf{A}_1 \trianglelefteq \mathbf{A}_2 \trianglelefteq \mathbf{A}_3$ and $\mathbf{B}_0 \trianglelefteq \mathbf{B}_1 \trianglelefteq \mathbf{B}_n$. Since each successive $\mathbf{A}_i$ is only one element larger than the previous, the $\mathbf{A}_{i+1}/\mathbf{A}_i$ has exactly two elements for each i and, so, is simple. Thus the $\mathbf{A}_i$'s form a composition series for $\mathbf{A}$ over a. $\mathbf{B}_1/\mathbf{B}_0$ is obviously simple. The largest congruence on $\mathbf{A} = \mathbf{B}_2$ for which $\mathbf{B}_1 = \{a,b\}$ is an equivalence class is the congruence α so that $a/\alpha = \{a,b\} = \mathbf{B}_1$ and $c/\alpha = \{c,d\}$. Therefore, $\mathbf{B}_2/\mathbf{B}_1$ has two elements and is simple. Thus, the $\mathbf{B}_i$'s also give a composition series for $\mathbf{A}$ over a. Since these composition series are different lengths, they cannot be equivalent.

3. Semigroup Varieties

Definition 3.1. If $\mathcal{V}$ is any variety, then we will say that $\mathcal{V}$ satisfies $\mathcal{JH}$ if for any finite algebra $\mathbf{A}$ in $\mathcal{V}$ and for any one element subalgebra a of $\mathbf{A}$, all composition series for $\mathbf{A}$ over a are equivalent.

We will need the following result of T. Evans:

Lemma 3.2. (T. Evans [1]) *If a nontrivial variety of semigroups does not contain* $\mathbf{L}$*,* $\mathbf{R}$*,* $\mathbf{C}$*, or* $\mathbf{S}$*, then the variety is a subvariety of a Burside variety of groups.*

Theorem 3.3. *The following are equivalent for any variety* $\mathcal{V}$ *of semigroups:*

(1) Every semigroup in $\mathcal{V}$ *is a group.*
(2) $\mathcal{V}$ *has a Maltsev term.*
(3) $\mathcal{V}$ *satisfies* $\mathcal{JH}$*.*

Proof. Suppose (1) holds. Since $\mathbf{L}$, $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{S}$ are not groups, then by the Lemma of Evans, $\mathcal{V}$ must be a subvariety of a variety of Burnside groups. In this variety, the group inverse is a semigroup term, so the term $xy^{-1}z$ is a Maltsev term of $\mathcal{V}$. Thus (1)→(2). That (2)→(3) follows from 2.2. For (3)→(1), suppose that (1) does not hold. By 3.2, $\mathcal{V}$ must contain $\mathbf{L}$, $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{S}$. In any case, that (3) does not hold follows from 2.3. Thus we have that (3)→(1). □

4. Special Varieties

Theorem 4.1. *Suppose that* $\mathcal{V}$ *is an idempotent variety satisfying* $\mathcal{JH}$*. Then for every locally finite variety* $\mathcal{W}$ *which interprets* $\mathcal{V}$*, there is a natural number* n *so that* $\mathcal{W}$ *is* n*-permutable.*

Proof. Suppose that $\mathcal{W}$ is locally finite, that that $\mathcal{V}$ is an idempotent variety satisfying $\mathcal{V} \leq \mathcal{W}$, and that $\mathcal{W}$ is not n-permutable for any n. By Theorem 9.14 of [2] typ$\{\mathcal{V}\}$ contains **1**, **4**, or **5**, and by arguments in Theorems 9.6, 9.8, and 9.14 of [2] $\mathcal{V}$ must be interpretable in one of the varieties of sets, semilattices, or distributive lattices. This means that one of the algebas **T**, **S**, or **D** from Example 2.3 has terms which model the equations of $\mathcal{V}$. This implies that for one of these cases the algebra **A** of Example 2.3 has terms modelling the equations of $\mathcal{V}$, so this algebra **A** has a reduct $\mathbf{A}'$ in $\mathcal{V}$. The two non-equivalent composition series from the example are non-equivalent composition series for $\mathbf{A}'$. Thus, $\mathcal{V}$ cannot satisfy $\mathcal{JH}$. □

References

[1] T. Evans. The lattice of semigroup varieties. *Semigroup Forum*, 2:1–43, 1971.
[2] D. Hobby and R. McKenzie. *The Structure of Finite Algebras (tame congruence theory)*. Contemporary Mathematics. American Mathematical Society, Providence, RI, 1988.