Modular Arithmetic and Cryptography

Recall

- Steganography - hide messages

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- Frequency analysis

A time question

If it is 7:00 now, what time will it be in 8 hours?

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- $7 + 8 = 15$

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- 15 is "too big" subtract 12
- In 8 hours it will be 3:00.

A time question

If it is 7:00 now, what time will it be in 19 hours?

A time question

If it is 7:00 now, what time will it be in 19 hours?

- $7 + 19 = 26$

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- $7 + 19 = 26$
- 26 is too big

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If it is 7:00 now, what time will it be in 19 hours?

- $7 + 19 = 26$
- 26 is too big
- $26 - 12 = 14$

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- $26 - 12 = 14$ still too big
- $14 - 12 = 2$

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- 26 is too big
- $26 - 12 = 14$ still too big
- $14 - 12 = 2$
- In 19 hours, it will be 2:00.

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If it is 7:00 now, what time will it be in 100 hours?

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- We could subtract repeatedly...
- Repeated subtraction is (more or less) division.
- What is left over after repeatedly subtracting 12 is the remainder when dividing by 12
- $107 \div 12$ is 9 with a remainder of 11
- In 100 hours, it will be 11:00.

Mod

- There is nothing special about 12.
- If n is any positive integer and if m is any integer, then

$$m \pmod{n} \text{ is the remainder when } m \text{ is divided by } n.$$

- Note: The remainder should be one of $0, 1, 2, \ldots, (n-1)$.

Mods

$$17 \pmod 5 =$$

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17 \pmod{5} &= 2 \\
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Mods

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\end{aligned}
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Using a calculator

Find $12345 \pmod{678}$

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- Subtract 18 to get: $0.20796460176991150442477876106195$

Using a calculator

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- My calculator says $12345 \div 678 = 18.207964601769911504424778761062$
- Subtract 18 to get: $0.207964601769911504424778761061 95$
- Multiply by 678 to get the remainder: 141

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- Subtract 18 to get: 0.2079646017699115044247787610 6195
- Multiply by 678 to get the remainder: 141
- Note, you may have to round.

Using a calculator

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- We are looking for $4 + 365 \pmod 7 = 5$

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Today is Thursday, April 12, 2018. What day of the week will April 12, 2018 be?

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- Friday.

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- This gives a small number system - with only n numbers.
- We can do much of our usual arithmetic and algebra in $\mathbb{Z}_n$

Addition and Multiplication in $\mathbb{Z}_5$

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

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- Subtract 1 to get $2x = 3$
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- This is the same as $1x = 4$ or $x = 4$

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- Try brute force: $2 \times 0 = 0$, $2 \times 1 = 2$, $2 \times 2 = 4$, $2 \times 3 = 0$, $2 \times 4 = 2$, $2 \times 5 = 4$
- There is no solution to this equation in $\mathbb{Z}_6$.

Weird

In $\mathbb{Z}_6$, $2 \times 3 = 0$

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2 and 3 are divisors of 0

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- $x = 3$ or $x = 4$

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- Factor to get $(x + 2)(x + 1) = 0$
- But in $\mathbb{Z}_6$, the products of some non-zero numbers is 0!
- This does not help!

An equation in $\mathbb{Z}_6$

- Solve $x^2 + 3x + 2 = 0$ in $\mathbb{Z}_6$ by BRUTE FORCE

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- $3^2 + 3 \times 3 + 2 = 2$ so 3 is not a solution

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- $4^2 + 3 \times 4 + 2 = 0$ so 4 is a solution

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- $3^2 + 3 \times 3 + 2 = 2$ so 3 is not a solution
- $4^2 + 3 \times 4 + 2 = 0$ so 4 is a solution
- $5^2 + 3 \times 5 + 2 = 0$ so 5 is a solution

Another oddity

In $\mathbb{Z}_{17}$, there is a solution to $x^2 + 1 = 0$.

Another oddity

In $\mathbb{Z}_{17}$, there is a solution to $x^2 + 1 = 0$.

This does not happen in the real numbers.

Back to cryptography

Letters and numbers

We can identify letters and numbers:

A=0	N = 13
B=1	O =14
C=2	P =15
D=3	Q =16
E=4	R =17
F=5	S =18
G=6	T =19
H=7	U =20
I=8	V =21
J=9	W =22
K=10	X =23
L=11	Y =24
M=12	Z =25

Letters and numbers

Identify the alphabet with $\mathbb{Z}_{26}$ and can do arithmetic with letters.

Letter calculations

Calculate:

- P+W=

Letter calculations

Calculate:

- $P+W=L$
- $H \times C=$

Letter calculations

Calculate:

- $P+W=L$
- $H\times C=O$
- $F\times(E+K)=$

Letter calculations

Calculate:

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Letter calculations

Calculate:

- $P+W=L$
- $H \times C=O$
- $F \times (E+K)=F \times O=S$

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A Caesar cipher with key k replaces any plaintext letter p with $p + k$.

- Encrypt "cat" with a Caesar Cipher and key D

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- Encrypt "cat" with a Caesar Cipher and key D
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$$\begin{array}{cccc} & \text{c} & \text{a} & \text{t} \\ + & \text{D} & \text{D} & \text{D} \\ \hline & \text{F} & \text{D} & \text{W} \end{array}$$

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- Ciphertext: FDW
- To decypt, subtract.

$$\begin{array}{cccc} & \text{F} & \text{D} & \text{W} \\ - & \text{D} & \text{D} & \text{D} \\ \hline & \text{c} & \text{a} & \text{t} \end{array}$$

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Atbash encrypts any plaintext letter p as $Z - p$.

- Encrypt "holy"

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$$\begin{array}{ccccc} & Z & Z & Z & Z \\ - & h & o & l & y \\ \hline & S & L & O & B \end{array}$$

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Atbash encrypts any plaintext letter p as $Z - p$.

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- Ciphertext: SLOB
- Decrypt also by subtracting from ZZZZZZ

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- Substitution for pairs of letters rather than single letters
- Homophonic substitution - more than one symbol can be used to represent a plaintext letter

Homophonic substitution cipher

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
09	48	13	10	14	10	06	23	32	15	04	26	22	18	00	38	94	29	11	17	08	34	60	28	21	02
12	81	41	03	16	31	25	39	70			37	27	58	05	95		35	19	20	61		89		52	
33		62	45	24			50	73			51		59	07			40	36	30	63					
47			79	44			56	83			84		66	54			42	76	43						
53				46			65	88					71	72			77	86	49						
67				55			68	93					91	90			80	96	69						
78				57										99					75						
92				64															85						
				74															97						
				82																					
				87																					
				98																					

t	u	r	n	t	o	t	h	e	e	a	s	t
17	08	29	18	20	00	30	23	14	16	09	11	43

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- Poly-alphabetic substitution:
  - A plaintext letter may be encrypted multiple ways.
  - A ciphertext symbol may represent different plaintext letters.

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- The Viginère cipher was not widely accepted because it was somewhat tedious (even though machines like Alberti's wheel made it easier).
- A mod-2 version of the Viginère cipher is present in many modern ciphers.

Caesar Cipher

Recall...

- To encrypt "secret message" with a key of D
- Add D to each letter of the plaintext:

	s	e	c	r	e	t	m	e	s	s	a	g	e
+	D	D	D	D	D	D	D	D	D	D	D	D	D
	V	H	F	U	H	W	P	H	V	V	D	J	H

- Every e is encrypted as H.
- Every H represents an e.
- Decrypt by subtracting D.

Viginère

Viginère suggested that instead of adding a SINGLE LETTER to the plaintext, we could add a WORD.

Viginère

- To encrypt "secret message" with a key of MATH

Viginère

- To encrypt "secret message" with a key of MATH
- Copy MATH repeatedly under the plaintext and add

$$
\begin{array}{cccccccccccccc}
 & s & e & c & r & e & t & m & e & s & s & a & g & e \\
+ & M & A & T & H & M & A & T & H & M & A & T & H & M \\
\hline
 & E & E & V & Y & Q & T & F & L & E & S & T & N & Q
\end{array}
$$

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- The e's are encrypted as E, Q, L
- E represents both s and e
- Decrypt by subtracting the keyword.
- Frequency analysis is hard (but not impossible)

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- This is absolutely secure.

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  - Every cipher we have talked about has this problem.
- Some have used the One Time Pad with their favorite novel as a key pad.

Breaking Viginère

- If the key for Viginère is shorter than the plaintext and repeated, then Viginère can be broken.
- If the key has length, say, 5, then every fifth letter is encrypted with a Caesar Cipher.
- A frequence analysis on every fifth letter would easily give the key.
- All that is needed is the key-length.
- The key-length can be found with some “statistical-voodoo”
- Methods developed independently by Charles Babbage and Wilhelm Kasiski around 1850

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- Best cryptography methods in homophonic substitution and Viginère cipher.
- Homophonic substitutions can be broken with complex frequency analysis.
- Babbage and Kasiski had discovered how to break Viginère around 1850.
- Radio brought the need for fast, mechanical encryption.

Column Scytale – Transposition Cipher

- To encrypt the plaintext "attack the south gate at dawn" with 5 columns

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- Write the text in a grid of 5 columns:

a	t	t	a	c
k	t	h	e	s
o	u	t	h	g
a	t	e	a	t
d	a	w	n	x

  filling in extra space with gibberish.

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  filling in extra space with gibberish.
- Copy the ciphertext down the columns:

  AKOADTTUTATHTEWAEHANCSGTX

Tabular Substitution

	A	D	F	G	X
A	K	E	Y	W	O
D	R	D	A	B	C
F	F	G	H	I/J	L
G	M	N	P	Q	S
X	T	U	V	X	Z

- A letter is encrypted as a pair of symbols - the row and column containing the letter.

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- A letter is encrypted as a pair of symbols - the row and column containing the letter.
- Example: G is encrypted as FD

Combining

The most successful means of cryptography combine methods.

$$\begin{array}{ccccc} \text{Tabular} & & \text{Column} & & \text{Collapse} \\ \text{Substitution} & \rightarrow & \text{Scytale} & \rightarrow & \text{Tabular} \\ & & & & \text{Substitution} \end{array}$$

Example

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- Encrypt the result of the previous step with a column scytale with 5 columns
- Collapse the result of the last step with a tabular substitution with keyword ZIPF

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- (no collapse).
- Breaking the ADFGVX cipher helped to prevent Germany from taking Paris.

Most modern ciphers include these ingredients:

- Break apart the letters of the plaintext into parts. (To diffuse information)

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- Rearrange and substitute the parts.
- Glue the parts back together.

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- Ease of Communication vs. Ease of Interception.
- Partially drove the movement toward mechanized cryptography.

Alberti and the Cipher Disk

Almost five hundred years before WWI, Alberti made a "cipher disk" to help perform substitution ciphers (Caesar) more quickly.

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- You could rotate the inner disk after each letter to accomplish a poly-alphabetic substition (like Viginère).
- Predecessor of the most formidable cipher machine of WWII.

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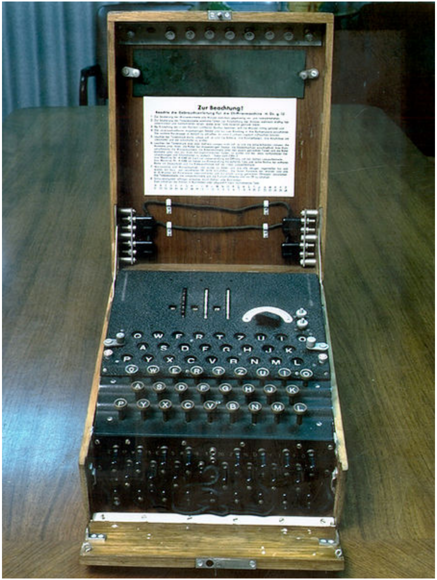
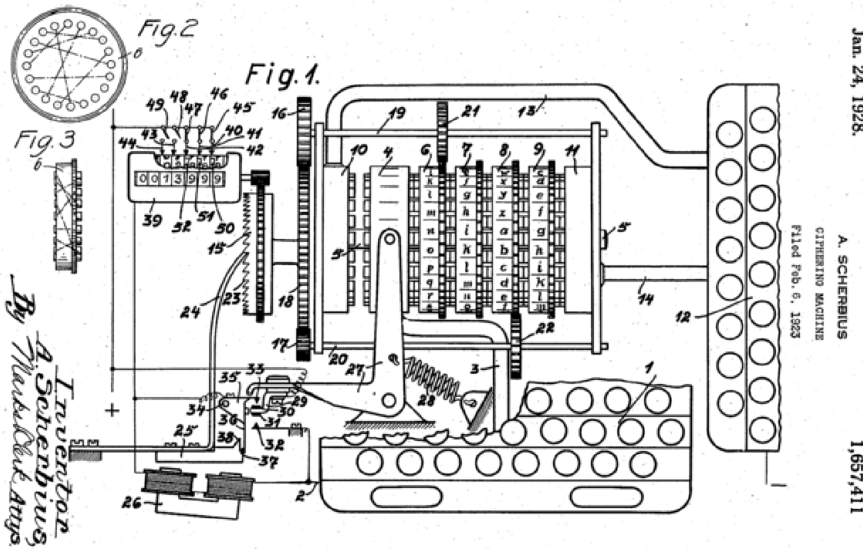
- The only interested customer was the German government.
- Others devised similar machines, but their businesses all failed.

Enigma

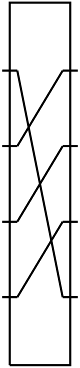


Enigma


Jan. 24, 1928.
A. SCHERBIUS
CIPHERING MACHINE
Filed Feb. 6, 1923
1,657,411
Fig.1.
Fig.2
Fig.3
Inventor
A. Scherbius

Enigma

- Enigma was (in a way) a glorified electronic cipher disk.
- Imagine a rubber disk with 26 electric contacts on each side.
- Wires within the disk connected the contacts on one side with the ones on the other side.
- The wires were “scrambled” inside the disk.



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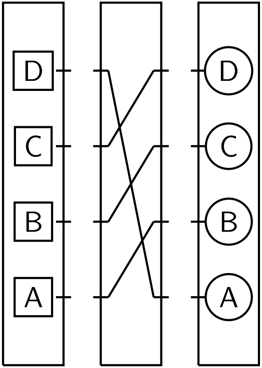
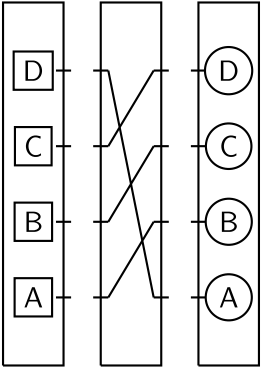


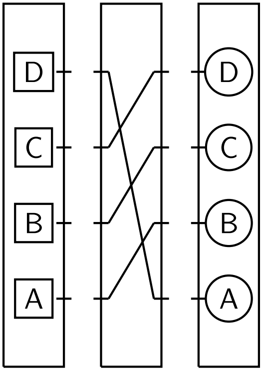
A keyboard was connected to the contacts on one side of the disk, and lights were connected to the other side.

Enigma



When a key was pressed, elecrticity entered the disk-scrambler at one letter and exited the other side to illuminate the light for a different letter.

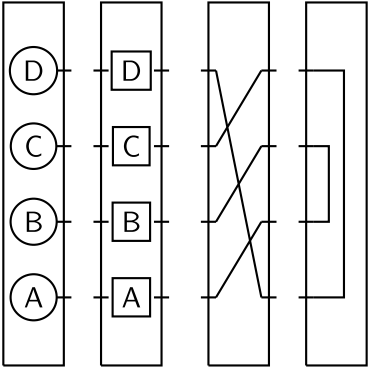
Enigma



- Each time a key was pressed, the disk rotated one step.
- An Alberti Cipher Disk with a keyboard and lights.

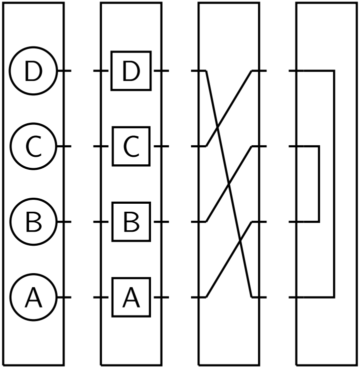
Enigma

- A second second scrambler was added next to the first with contacts on only one side.
- The keyboard and lights were on the same side of the first scrambler.
- The effect of this *reflector* was that encryption and decryption worked the same way - just type and see what lights light up.



Enigma

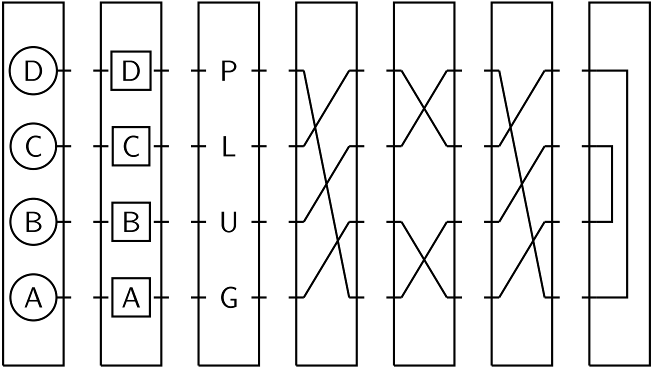
- At this point, the Enigma is an electronic cipher disk (the reflector makes it easy to use).
- The cipher depends only on the initial position of the scrambler-disk...which has 26 positions.
- For more configurations, two more scramblers were added between the initial disk and the reflector.



Enigma

- After each keypress, the first disk turned.
- After the first disk made a complete revolution, the second disk turned.
- After the second disk made a complete revolution, the third one turned.
- Like an odometer.
- For even more configurations, the three scramblers could be rearranged.

To provide the Enigma with more configurations, a plugboard was added that allowed six pairs of letters to be exchanged.



Enigma Configurations

The total number of Enigma Configurations

- The number of ways to re-arrange the disks: $3! = 6$
- The number of starting positions for three disks:

$$26 \times 26 \times 26 = 17,576$$

- The number of ways to exchange 6 pairs of letters from 26:

$$100,391,791,500$$

- Total configurations (multiply): $10,586,916,764,424,000$

Enigma

- The plugboard is the largest factor in the number of keys.
- The scramblers make for a poly-alphabetic substitution to prevent frequency analysis.
- Other changes were eventually made to provide even more security.

Bletchley Park

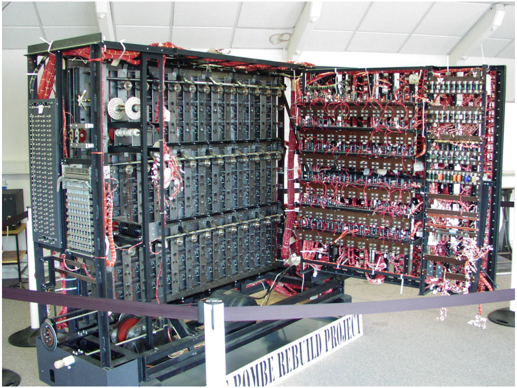
Enigma and other German codes were (eventually) broken by British Intelligence working at Bletchley Park.

Crossword enthusiasts, scientists, mathematicians, experts in literature and language.

Alan Turing

Enigma required the efforts of legendary mathematician Alan Turing.

Enigma also required the construction of computing machines to search for keys.



The design of these machines was similar to that of Difference Engine No. 2