Tight Lattices and Congruence Hereditary

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Abstract. We prove that if $\mathbf{A}$ is a finite algebra which satisfies a nontrivial idempotent Maltsev condition, and if Con$\mathbf{A}$ contains a copy of a tight simple lattice other than $\mathbf{2}$, $\mathbf{M}_3$, Con$\mathbb{Z}_2^3$, then Con$\mathbf{A}$ is not hereditary.

1. Introduction

The notions of congruence heredity and power-heredity were introduced in [2] by Hegedűs and Pálfy. A the congruence lattice $\mathbf{L}$ of a finite algebra $\mathbf{A}$ is hereditary if every 0-1 sublattice of $\mathbf{L}$ is the congruence lattice of an algebra with the same universe as $\mathbf{A}$. $\mathbf{L}$ is power hereditary if every 0-1 sublattice of $\mathbf{L}^n$ is the congruence lattice of an algebra with the same universe as $\mathbf{A}^n$. In [2], Hegedűs and Pálfy characterize which finite, Abelian, prime power ordered groups have (power-)hereditary congruence lattices. In [5], this author proves that every representation of $\mathbf{N}_5$ as the congruence lattice of a finite algebra is power-hereditary. In [4], Pálfy proved that the lattice $\mathbf{M}_3$ does not share this property by giving and example of a finite algebra whose congruence lattice is isomorphic to $\mathbf{M}_3$ but is not power-hereditary. To date, there is no known congruence lattice representation of $\mathbf{M}_4$ which is even hereditary. In this manuscript, we prove that if $\mathbf{A}$ is a finite algebra in a variety which satisfies a nontrivial idempotent Maltsev condition and if Con$\mathbf{A}$ contains a copy of $\mathbf{M}_4$, then Con$\mathbf{A}$ cannot be hereditary. This result is used with the Hegedűs and Pálfy characterization to give a characterization of congruence (power-)hereditary vector spaces. We then use the vector space characterization to prove that if $\mathbf{A}$ is a finite algebra which satisfies a nontrivial idempotent Maltsev condition, and if Con$\mathbf{A}$ contains a copy of a tight simple lattice other than $\mathbf{2}$, $\mathbf{M}_3$, Con$\mathbb{Z}_2^3$, then Con$\mathbf{A}$ is not hereditary.

2. Preliminaries

By a *representation* or a *congruence representation* of a finite lattice $\mathbf{L}$ we will mean the congruence lattice Con$\mathbf{A}$ of a finite algebra such that Con$\mathbf{A} \cong \mathbf{L}$. If Con$\mathbf{A}$

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is a representation of $\mathbf{L}$ and Con$\mathbf{A}$ is a (power-)hereditary congruence lattice, then we will say that Con$\mathbf{A}$ is a (power-)hereditary representation.

A *primitive positive formula* is a formula of the form $\exists \wedge$ (atomic). If Φ is a primitive positive formula employing binary relation symbols $r_1, \ldots, r_n$ and if Φ has two free variables, then Φ naturally induces an operation on the set of binary relations of any set. If $\theta_1, \ldots, \theta_n$ are binary relations on a set A, then we will use $\Phi(\theta_1, \ldots, \theta_n)$ to represent the binary relation on A defined by interpreting each r_i in Φ as θ_i. The operation $\langle \theta_1, \ldots, \theta_n \rangle \mapsto \Phi(\theta_1, \ldots, \theta_n)$ is order preserving, and when it is applied to products of relations can be applied coordinate-wise. We need the following lemma from [7].

Lemma 2.1. ([7] Corollary 2.2) *Suppose* $\mathbf{L}$ *is a* 0-1 *lattice of equivalence relations on a finite set* A. *There is an algebra* $\mathbf{A}$ *on* A *with* Con$\mathbf{A} = \mathbf{L}$ *if and only if every equivalence relation on* A *which can be defined from* $\mathbf{L}$ *by a primitive positive formula is already in* $\mathbf{L}$.

We will assume from here on that every primitive positive formula only contains binary relation symbols and has exactly two free variables. Suppose that Φ is any such primitive positive formula and that $r_1, \ldots, r_n$ are the relations symbols in Φ (or are relations interpreted as the symbols in Φ). Let $x_1, \ldots, x_m$ be the variables in Φ. By the *graph of* $\Phi(r_1, \ldots, r_n)$ we will mean the undirected graph $\mathbf{G}$ with vertices $\{x_1, \ldots, x_n\}$ so that for each occurrence of $r_i(x_j, x_k)$ in Φ, there is an edge in $\mathbf{G}$ labelled by r_i extending from x_j to x_k. A primitive positive formula Φ will be called *connected* if the corresponding graph is connected. If a primitive positive formula Φ is not connected, then its value is completely determined by the component containing the free variables. If the two free variables are not contained in the same component, then the formula can only define the universal relation. Thus we actually have:

Corollary 2.2. ([7] Corollary 2.2) *Suppose* $\mathbf{L}$ *is a* 0-1 *lattice of equivalence relations on a finite set* A. *There is an algebra* $\mathbf{A}$ *on* A *with* Con$\mathbf{A} = \mathbf{L}$ *if and only if every equivalence relation on* A *which can be defined from* $\mathbf{L}$ *by a connected primitive positive formula is already in* $\mathbf{L}$.

In light of this corollary, we will assume that **all primitive positive formulas are connected**. Congruence heredity and power-heredity are related to how well primitive positive definitions can be interpolated by lattice terms:

Lemma 2.3. (Lemma 2.3 of [6]) *The congruence lattice of a finite algebra* $\mathbf{A}$ *is hereditary if and only if for every primitive positive formula* $\Phi(x_1, \ldots, x_n)$ *and for all* $r_1, \ldots, r_n \in$ Con$\mathbf{A}$ *if* $\Phi(r_1, \ldots, r_n)$ *is an equivalence relation, there is a lattice term* $T(x_1, \ldots, x_n)$ *so that* $T(r_1, \ldots, r_n) = \Phi(r_1, \ldots, r_n)$.

Lemma 2.4. (See [2] Lemma 4.5) *The congruence lattice of a finite algebra* $\mathbf{A}$ *is power-hereditary if and only if for every primitive positive formula* $\Phi(x_1, \ldots, x_n)$ *there is a lattice term* $T(x_1, \ldots, x_n)$ *so that if* $r_1, \ldots, r_n \in$ Con$\mathbf{A}$ *and* $\Phi(r_1, \ldots, r_n)$ *is an equivalence relation, then* $T(r_1, \ldots, r_n) = \Phi(r_1, \ldots, r_n)$.

Suppose that $\mathbf{A}$ and $\mathbf{B}$ are finite algebras and $f : \mathrm{Con}\mathbf{A} \to \mathrm{Con}\mathbf{B}$ is any function. We will say that f preserves primitive positive definitions if whenever $\Phi(x_1, \ldots, x_n)$ is a primitive positive formula and $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ so that $\Phi(r_1, \ldots, r_n)$ and $\Phi(f(r_1), \ldots, f(r_n))$ are equivalence relations, then

$$f(\Phi(r_1, \ldots, r_n)) = \Phi(f(r_1), \ldots, f(r_n)).$$

Note that any such function must be a lattice homomorphism.

Lemma 2.5. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are finite algebras and that* $f : \mathrm{Con}\mathbf{A} \to \mathrm{Con}\mathbf{B}$ *is a surjection which preserves primitive positive definitions. Then if* $\mathrm{Con}\mathbf{A}$ *is (power-)hereditary then* $\mathrm{Con}\mathbf{B}$ *is also (power-)hereditary.*

Proof. Suppose that $\mathrm{Con}\mathbf{A}$ is hereditary. We will show that $\mathrm{Con}\mathbf{B}$ is hereditary. Let Φ be a primitive positive formula and $s_1, \ldots, s_n \in \mathrm{Con}\mathbf{B}$ with $\Phi(s_1, \ldots, s_n)$ an equivalence relation. Find $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ so that $f(r_i) = s_i$ for all i. Let $\Phi'(x_1, x_2)$ be defined by $\Phi(x_1, x_2) \wedge \Phi(x_2, x_1)$, and define $\Phi''(x_1, x_2)$ by

$$\exists y_0, \ldots, y_{m+1} \left[\left(\bigwedge_{i=0}^{m} \Phi'(y_i, y_{i+1}) \right) \wedge (y_0 = x_1) \wedge (y_{m+1} = x_2) \right]$$

where $m = \max(|A|, |B|)$. Then $\Phi''(r_1, \ldots, r_n)$ is an equivalence relation on A and

$$\Phi''(s_1, \ldots, s_n) = \Phi(s_1, \ldots, s_n).$$

(Φ'' is the equivalence relation closure of Φ on any set with m or fewer elements.) Since $\Phi''(r_1, \ldots, r_n)$ is an equivalence relation and $\mathrm{Con}\mathbf{A}$ is hereditary, then by 2.3 there is a lattice term T so that $T(r_1, \ldots, r_n) = \Phi''(r_1, \ldots, r_n)$. Then

$$\begin{aligned} T(s_1, \ldots, s_n) &= T(f(r_1), \ldots, f(r_n)) \\ &= f(T(r_1, \ldots, r_n)) \\ &= f(\Phi''(r_1, \ldots, r_n)) \\ &= \Phi''(f(r_1), \ldots, f(r_n)) \\ &= \Phi''(s_1, \ldots, s_n) \\ &= \Phi(s_1, \ldots, s_n). \end{aligned}$$

Thus $\mathrm{Con}\mathbf{B}$ is hereditary by 2.3. The case for power-heredity is proven similarly. □

Suppose $\mathcal{V}$ is a variety with a set of basic operation symbols F, and suppose $\mathcal{W}$ is any variety. $\mathcal{W}$ is said to *interpret* $\mathcal{V}$ if for every basic operation t of $\mathcal{V}$ there is a $\mathcal{W}$-term s_t so that for every algebra $\mathbf{A} \in \mathcal{W}$ the algebra $\langle A, \{s_t^A : t \in F\}\rangle$ is a member of $\mathcal{V}$. This relationship is denoted by $\mathcal{V} \leq \mathcal{W}$. A variety $\mathcal{V}$ is *finitely presented* if it has a finite set of basic operation symbols and is defined by a finite set of equations. The variety $\mathcal{V}$ is *idempotent* if every basic operation $t(x_1, \ldots, x_n)$ of $\mathcal{V}$ satisfies the equation $t(x, \ldots, x) \approx x$.

Suppose that $\mathcal{V}$ is a finitely presented variety and $\mathcal{W}$ is any variety. The assertion that $\mathcal{W}$ interprets $\mathcal{V}$ is called a *strong Maltsev condition.* Suppose that $\ldots \mathcal{V}_3 \leq \mathcal{V}_2 \leq \mathcal{V}_1$ are finitely presented varieties. The assertion that $\mathcal{W}$ interprets one of the $\mathcal{V}_i$ is a *Maltsev condition.* An idempotent Maltsev condition is one in which all of the defining varieties are idempotent. A nontrivial Maltsev condition is one which is not satisfied by the variety of sets. When we say that an algebra $\mathbf{A}$ satisfies a

Maltsev condition, we mean that the variety generated by $\mathbf{A}$ satisfies the Maltsev condition.

3. A Minimal Amount of Tame Congruence Theory

We will need a little tame congruence theory to establish our results. We outline the essentials we need here. Any unproven results in this section are established in [3]. The reader who is familiar with tame congruence theory should at least refer to 3.2, 3.4, 3.7, and 3.8.

Suppose that $\mathbf{A}$ is a finite algebra and U is any subset of $\mathbf{A}$. By $\mathbf{A}|_U$ we will denote the *algebra on* U *induced by* $\mathbf{A}$. This algebra has universe U. Its operations are all polynomials of $\mathbf{A}$ under which U is closed.

A unary polynomial e of an algebra $\mathbf{A}$ is *idempotent* if the equality $e(e(x)) = e(x)$ holds for all $x \in \mathbf{A}$. If e is an idempotent unary polynomial of $\mathbf{A}$ and $U = e(A)$ then every operation of $\mathbf{A}|_U$ is of the form $e \circ f$ where f is a polynomial of $\mathbf{A}$. For any $\theta \in \mathrm{Con}\mathbf{A}$, $e(\theta) = \theta \cap (U \times U)$ and the map $\theta \to e(\theta)$ is a surjective lattice homomorphism from $\mathrm{Con}\mathbf{A}$ to $\mathrm{Con}\mathbf{A}|_U$. This is Lemma 2.3 of [3]. We need a slightly stronger version of this lemma:

Lemma 3.1. (See Lemma 2.3 of HobbyMcKenzie1988) *Suppose that* $\mathbf{A}$ *is a finite algebra,* e *is an idempotent unary polynomial of* $\mathbf{A}$*, and* $U = e(\mathbf{A})$*. Then* e *induces a surjective lattice homomorphism from* $\mathrm{Con}\mathbf{A}$ *to* $\mathrm{Con}\mathbf{A}|_U$ *which preserves primitive positive definitions.*

Proof. That e induces a lattice surjection is proven in [3]. We need only prove that this surjection preserves primitive positive definitions. Suppose Φ is the primitive positive formula given by

$$\langle x_1, x_2 \rangle \in \Phi(s_1, \ldots, s_n) \leftrightarrow \exists x_3, \ldots, x_m \bigwedge_{i=1}^{p} s_{j_i}(x_{k_i}, x_{l_i}).$$

Let $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ so that $\Phi(r_1, \ldots, r_n)$ and $\Phi(e(r_1), \ldots, e(r_n))$ are both equivalence relations. That $\Phi(e(r_1), \ldots, e(r_n)) \subseteq e(\Phi(r_1, \ldots, r_n))$ should be clear since $U \subset A$, each $e(r_i) \subseteq r_i$, and e is idempotent.

Next, suppose that $\langle u_1, u_2 \rangle \in e(\Phi(r_1, \ldots, r_n))$. This means that there are $x_1, \ldots, x_m \in A$ so that $e(x_i) = u_i$ for $i = 1, 2$ and $r_{j_i}(x_{k_i}, x_{l_i})$ for $i = 1, \ldots, p$. For $i = 3, \ldots, m$, let $u_i = e(x_i)$. Then $\langle u_{k_i}, u_{l_i} \rangle \in e(r_{j_i})$ holds for $i = 1, \ldots, p$. Thus $\langle u_1, u_2 \rangle \in \Phi(e(r_1), \ldots, e(r_n))$. This establishes the reverse inclusion and equality. □

Combining this with Lemma 2.5 immediately gives us:

Corollary 3.2. *Suppose that* $\mathbf{A}$ *is a finite algebra and* e *is an idempotent unary polynomial of* $\mathbf{A}$ *with* $U = e(A)$*. If* $\mathbf{A}$ *is congruence (power-)hereditary, then so is* $\mathbf{A}|_u$.

We will need similar results for restrictions to equivalence classes. If θ is a congruence on an algebra $\mathbf{A}$ and $U \subseteq A$, then $\theta|_U$ will represent $\theta \cap (U \times U)$. The following lemma comes from Lemma 2.4 of [3].

Lemma 3.3. (See Lemma 2.4 of [3]) *Suppose that* $\mathbf{A}$ *is a finite algebra and* $\beta \in \mathrm{Con}\mathbf{A}$*. Let* B *be any congruence class of* β *and* $\mathbf{B} = \mathbf{A}|_B$*. Then the map* $\theta \to \theta|_B$ *is a lattice homomorphism from the interval* $[0, \beta]$ *in* $\mathrm{Con}\mathbf{A}$ *onto* $\mathrm{Con}\mathbf{B}$ *which preserves primitive positive definitions.*

Proof. Add operations to $\mathbf{A}$ to form an algebra $\mathbf{A}'$ whose congruence lattice is all congruences less than or equal to β along with the universal relation. Let $b \in B$ be arbitrary. Define $e : A \to A$ by

$$e(x) = \begin{cases} x & x \in B \\ \\ b & x \notin B \end{cases}$$

Then e preserves all of the congruences of $\mathbf{A}'$, so we can assume that it was one of the added operations. Also, e is idempotent and $e(A) = B$. The lemma now follows from Lemma 3.1. $\square$

Combining this with Lemma 2.5 immediately gives us:

Corollary 3.4. *Suppose that* $\mathbf{A}$ *is a finite algebra and* $\beta \in \mathrm{Con}\mathbf{A}$*. Let* B *be any congruence class of* β*. If* $\mathrm{Con}\mathbf{A}$ *is (power-)hereditary, then so is* $\mathrm{Con}\mathbf{A}|_B$*.*

Suppose that $\mathbf{A}$ is a finite algebra and $\alpha < \beta$ are congruences of $\mathbf{A}$. Let $U_{\mathbf{A}}(\alpha, \beta)$ be the set of all sets of the form $f(A)$, where f is a unary polynomial of $\mathbf{A}$. Let $M_{\mathbf{A}}(\alpha, \beta)$ be the set of minimal elements of $U_{\mathbf{A}}(\alpha, \beta)$. These will be called the $\langle \alpha, \beta \rangle$-*minimal sets* of $\mathbf{A}$. If $A \in M_{\mathbf{A}}(\alpha, \beta)$, then $\mathbf{A}$ will be called $\langle \alpha, \beta \rangle$-*minimal*. Any $\langle 0, 1 \rangle$-minimal algebra will be called *minimal*.

Suppose that $\mathbf{L}$ is a finite lattice. $\mathbf{L}$ is 0-1 simple if for every nonconstant lattice homomorphism $f : \mathbf{L} \to \mathbf{L}'$ and for $x = 0, 1$ the equality $f^{-1}(f(x)) = \{x\}$ holds. If $\mathbf{L}$ is 0-1 simple and every strictly increasing meet endomorphism of $\mathbf{L}$ is constant, then $\mathbf{L}$ is called *tight*. We need to know that the two element lattice, the lattices $\mathbf{M}_n$, and the congruence lattices of finite vector spaces are tight.

The following lemma extracts the information we will need from Lemmas 2.10, 2.11, and 2.13 of [3].

Lemma 3.5. *Suppose that* $\mathbf{A}$ *is a finite algebra and that* $\alpha < \beta \in \mathrm{Con}\mathbf{A}$ *so that as a lattice the interval* $[\alpha, \beta]$ *is tight. Then*

(1) $\mathbf{A}$ *is* $\langle \alpha, \beta \rangle$*-minimal if and only if for every unary polynomial* f *of* $\mathbf{A}$*, either* f *is a permutation or* $f(\beta) \subseteq f(\alpha)$*.*
(2) If $U \in M_{\mathbf{A}}(\alpha, \beta)$*, then there is an idempotent polynomial* e *of* $\mathbf{A}$ *so that* $U = e(A)$*.*
(3) If $U \in M_{\mathbf{A}}(\alpha, \beta)$*, then* $\mathbf{A}|_U$ *is* $\langle \alpha|_U, \beta|_U \rangle$*-minimal.*

Suppose that $\mathbf{A}$ is a finite algebra and $\alpha < \beta \in \mathrm{Con}\mathbf{A}$. By an $\langle \alpha, \beta \rangle$-*trace* of $\mathbf{A}$, we mean an equivalence class of $\beta|_U$ which is not contained in an α equivalence class for some $U \in M_{\mathbf{A}}(\alpha, \beta)$.

Lemma 3.6. (See Lemma 2.16 of [3]) *Suppose that* $\mathbf{A}$ *is an* $\langle \alpha, \beta \rangle$*-minimal algebra and* N *is an* $\langle \alpha, \beta \rangle$*-trace of* $\mathbf{A}$*. Then* $\mathbf{A}|_N$ *is* $\langle \alpha|_N, \beta|_N \rangle$*-minimal and* $(\mathbf{A}|_N)/(\alpha|_N)$ *is a minimal algebra.*

We will call the minimal algebra $(\mathbf{A}|_N)/(\alpha|_N)$ here the *minimal algebra associated with* the $\langle\alpha,\beta\rangle$-trace N. We will establish a lemma which allows us to pass congruence (power-)heredity all the way down to the minimal algebras associated with traces. To do so, we need the following lemma.

Lemma 3.7. *Suppose that* $\mathbf{N}$ *is a finite algebra and* $\alpha \in \mathrm{Con}\mathbf{N}$*. If* $\mathbf{N}$ *is congruence (power-)hereditary, then so is* $\mathbf{N}/\alpha$*.*

Proof. Add operations to $\mathbf{N}$ to form an algebra $\mathbf{N}'$ whose congruences are those of $\mathbf{N}$ above α along with the identity relation. Let $a_1,\ldots,a_m$ be representatives of the equivalence classes of $\mathbf{N}$. Define $e: N \to N$ by $e(x) = a_i$, where $x\alpha a_i$. Then e preserves all of the congruences of $\mathbf{N}'$, so we can assume that it was one of the added operations. Also, e is idempotent, so we can apply Lemma 3.1 to conclude that restriction of $\mathrm{Con}\mathbf{N}'$ to $U = e(N)$ preserves primitive positive defitions. It follows, then, that $\mathrm{Con}(\mathbf{N}|_U)$ is (power-)hereditary. However, the relational structures $\langle N/\alpha, \mathrm{Con}(\mathbf{N}/\alpha)\rangle$ and $\langle U, \mathrm{Con}(\mathbf{N}|_U)\rangle$ are isomorphic, so $\mathrm{Con}(\mathbf{N}|_U)$ is also (power-)hereditary. □

Combining 3.2, 3.4, and 3.7 we now have:

Lemma 3.8. *Suppose that* $\mathbf{A}$ *is a finite algebra and* $\alpha < \beta$ *are congruences on* $\mathbf{A}$ *so that* $[\alpha,\beta]$ *is tight. If* $\mathrm{Con}\mathbf{A}$ *is (power-)hereditary, then every minimal algebra associated with any* $\langle\alpha,\beta\rangle$*-trace of* $\mathbf{A}$ *is also (power-)hereditary.*

This lemma indicates that the structure of congruence (power-)hereditary minimal algebras might be important in the study of (power-)hereditary congruence lattices. Corollary 4.11 of [3] characterizes minimal algebras up to their polynomials. Two algebras $\mathbf{A}$ and $\mathbf{B}$ are *polynomially equivalent* if and only if they share the same universe and polynomials.

Theorem 3.9. (See chapter 4 of [3]) *A finite algebra* $\mathbf{A}$ *is minimal if and only if* $\mathbf{A}$ *is polynomially equivalent to one of the following.*

(1) A $\mathbf{G}$*-set for some group* $\mathbf{G}$*.*
(2) A vector space.
(3) The two-element Boolean algebra.
(4) The two element lattice.
(5) The two element semilattice.

This list of minimal algebras will allow us to specify what the minimal algebras associated to traces of congruence hereditary algebras must look like.

Finally, we need to know how Maltsev conditions relate to induced algebras and quotient algebras.

Lemma 3.10. (See Lemmas 9.2 and 8.3 of [3]) *Suppose that* $\mathbf{A}$ *is an algebra which satisfies an idempotent Maltsev condition* Ψ*. Let* e *be an idempotent unary polynomial of* $\mathbf{A}$ *and* $U = e(A)$*. Let* α *be a congruence of* $\mathbf{A}$ *and* N *an equivalence class of* α*. Then* $\mathbf{A}|_U$*,* $\mathbf{A}|_N$*, and* $\mathbf{A}/\alpha$ *all satisfy* Ψ*.*

Lemma 3.11. (See Theorem 9.6 of [3]) *The following are equivalent for a finite algebra* $\mathbf{A}$*:*

(1) *No trace on any finite algebra in the variety generated by* $\mathbf{A}$ *is polynomially equivalent to a* $\mathbf{G}$*-set.*
(2) *The variety generated by* $\mathbf{A}$ *satisfies a nontrivial idempotent Maltsev condition.*

4. Representing $\mathbf{M}_4$

In this section, we prove that in the presence of an idempotent Maltsev condition, a copy of $\mathbf{M}_4$ in the congruence lattice of an algebra prevents congruence heredity.

Lemma 4.1. *If a finite vector space* $\mathbf{V}$ *has dimension at least two and does not satisfy* $x + x = 0$*, then* $\mathbf{V}$ *is not congruence hereditary (and hence not power-hereditary).*

Proof. We can assume that for some finite field $\mathbf{F}$, $\mathbf{V}$ is $\mathbf{F}^n$ as a vector space with $n \geq 2$. Define the following four congruences on $\mathbf{V}$:

$$x\eta_0 y \leftrightarrow x_0 = y_0$$

$$x\eta_1 y \leftrightarrow x_1 = y_1$$

$$x\alpha y \leftrightarrow x_0 - x_1 = y_0 - y_1$$

$$x\beta y \leftrightarrow x_0 + x_1 = y_0 + y_1$$

The congruences $0_{\mathbf{V}}$, $1_{\mathbf{V}}$, η_0, η_1, and α form a sublattice $\mathbf{M}$ of Con$\mathbf{V}$ isomorphic to $\mathbf{M}_3$. The primitive positive

$$\Phi(x_0, x_1) \leftrightarrow \exists x_2, x_3\, (x_0\eta_0 x_2 \wedge x_2\eta_1 x_1 \wedge x_0\eta_1 x_3 \wedge x_3\eta_0 x_1 \wedge x_2\alpha x_3)$$

employing only η_0, η_1, and α defines β. But if $\mathcal{V}$ does not satisfy $x + x = 0$, then $\beta \notin \mathbf{M}$. Thus $\mathbf{M}$ is not closed under primitive positive definitions and is not a congruence lattice. Con$\mathbf{V}$ is not hereditary. $\square$

Theorem 4.2. *Suppose that* $\mathbf{A}$ *is a finite algebra which satisfies a nontrivial idempotent Maltsev condition. If* Con$\mathbf{A}$ *contains a copy of* $\mathbf{M}_4$*, then* Con$\mathbf{A}$ *is not hereditary.*

Proof. Suppose that Con$\mathbf{A}$ contains a sublattice $\mathbf{M}$ isomorphic to $\mathbf{M}_4$ and that Con$\mathbf{A}$ is hereditary. Denote the bottom and top elements of $\mathbf{M}$ by 0_M and 1_M. $\mathbf{A}/0_M$ satisfies the idempotent Maltsev condition, has a copy of $\mathbf{M}_4$ in its congruence lattice, and is congruence hereditary by 3.7, so we can assume that $0_M = 0_A$. Suppose that B is an equivalence class of 1_M. Then $\mathbf{A}|_B$ is congruence hereditary by 3.4 and satisfies the same idempotent Maltsev condition by 3.10. Moreover, if $B_1, B_2, \ldots, B_n$ are all of the equivalence classes of 1_M, then the interval $[0_A, 1_M]$ in Con$\mathbf{A}$ can be embedded in $\prod_{i=1}^n \mathrm{Con}\mathbf{A}|_{B_i}$ (as in the proof of Lemma 3.4 in [7]). This implies that for some i the restriction of the equivalence relations in $\mathbf{M}$ to B_i is a lattice injection. Since B_i is an equivalence class of 1_M, this injection must map 1_M to 1_{B_i}. The identity relation 0_M must also map to 0_{B_i}. Thus some $\mathbf{A}|_{B_i}$ has a spanning copy of $\mathbf{M}_4$ in its congruence lattice, is congruence hereditary, and satisfies a nontrivial idempotent Maltsev condition. Replace $\mathbf{A}$ with this $\mathbf{A}|_{B_i}$.

Since Con$\mathbf{A}$ is hereditary, we can add operations to $\mathbf{A}$ so that Con$\mathbf{A} \cong \mathbf{M}_4$ without losing satisfaction of the nontrivial idempotent Maltsev condition.

We have an algebra $\mathbf{A}$ with Con$\mathbf{A}$ hereditary, Con$\mathbf{A} \cong \mathbf{M}_4$, and so that $\mathbf{A}$ satisfies a nontrivial idempotent Maltsev condition. Con$\mathbf{A}$ is tight, so we can apply the tame congruence theory outlined in Section 3. Let U be a $\langle 0_A, 1_A \rangle$-minimal set. We know the following about $\mathbf{A}|_U$:

(1) $\mathbf{A}|_U$ is minimal by 3.5.
(2) Con$\mathbf{A}|_U \cong \mathbf{M}_4$ by 3.5 and 3.1 since Con$\mathbf{A} \cong \mathbf{M}_4$ is simple.
(3) $\mathbf{A}|_U$ satisfies a nontrivial idempotent Maltsev Condition by 3.10.
(4) $\mathbf{A}|_U$ is congruence hereditary by 3.2 and 3.5.

Since $\mathbf{A}|_U$ is minimal, it is (polynomially equivalent to) one of the types of algebras listed in 3.9. Since Con$\mathbf{A}|_U \cong \mathbf{M}_4$, $\mathbf{A}|_U$ must have more than two elements, so $\mathbf{A}|_U$ is either a $\mathbf{G}$-set or a vector space. Since $\mathbf{A}|_U$ satisfies a nontrivial idempotent Maltsev condition, $\mathbf{A}|_U$ cannot be a $\mathbf{G}$-set by 3.11. Thus $\mathbf{A}|_U$ must be a vector space. Since Con$\mathbf{A}|_U$ is hereditary, $\mathbf{A}|_U$ as a vector space must be of characteristic two by 4.1. This implies that Con$\mathbf{A}|_U \cong \mathbf{M}_{2^k+1}$ for some k. But this contradicts that Con$\mathbf{A}|_U \cong \mathbf{M}_4$. Our original assumption that Con$\mathbf{A}$ contains a copy of $\mathbf{M}_4$ and is congruence hereditary must be false. □

5. Congruence lattices of vector spaces

In [2], Hegedűs and Pálfy give the following complete characterization of congruence (power-)hereditary, prime power ordered, Abelian groups:

Theorem 5.1. (Theorem 5.8 of [2]) Suppose that $\mathbf{A}$ is a finite Abelian group of prime power order.

(1) Con$\mathbf{A}$ is hereditary if and only if either $\mathbf{A}$ is cyclic, or $\mathbf{A} = \mathbb{Z}_2 \times \mathbb{Z}_{2^k}$ for some $k \geq 1$, or $\mathbf{A} = \mathbb{Z}_2^3$.
(2) Con$\mathbf{A}$ is power-hereditary if and only if either $\mathbf{A}$ is cyclic, or $\mathbf{A} = \mathbb{Z}_2 \times \mathbb{Z}_{2^k}$ for some $k \geq 1$.

Combining this characterization with the results of the last section, we can easily characterize all congruence (power-)hereditary vector spaces. First, we need this lemma:

Lemma 5.2. *Suppose that* $\mathbf{V}$ *is a congruence hereditary finite vector space. Then* $\mathbf{V}$ *is either simple or is isomorphic to a vector space over the field* $\mathbb{Z}_2$.

Proof. Suppose that $\mathbf{V}$ is not simple. For some finite field $\mathbf{F}$ and for some $k \geq 2$, $\mathbf{V}$ is isomorphic to $\mathbf{F}^k$ as an $\mathbf{F}$-vector space. We prove that $\mathbf{F}$ must be $\mathbb{Z}_2$. By 4.1, we know that $\mathbf{F}$ must be of characteristic two. This implies that $|\mathbf{F}| = 2^n$ for some $n \geq 1$. The congruence lattice of $\mathbf{F}^2$ as an $\mathbf{F}$-vector space is isomorphic to $\mathbf{M}_{2^n+1}$. If $n \geq 2$ then this congruence lattice contains a copy of $\mathbf{M}_4$ and is not hereditary by 4.2. Since $k \geq 2$, we know that $\mathbf{F}^2$ (as a vector space) is a homomorphic image of $\mathbf{F}^k$. If $n \geq 2$, then $\mathbf{F}^2$ is not congruence hereditary, so by 3.7 $\mathbf{F}^k$ is not congruence hereditary. Therefore, we know that $n = 1$. This means that $\mathbf{F} \cong \mathbb{Z}_2$. □

Now, if $\mathbf{V}$ is a vector space over $\mathbb{Z}_2$, then the congruences of $\mathbf{V}$ as a vector space are the same as the congruences of the Abelian group reduct of $\mathbf{V}$. Therefore, we can now apply Theorem 5.1 to get

Theorem 5.3. *Suppose that* $\mathbf{V}$ *is a finite vector space.*

(1) $\mathbf{V}$ *is congruence hereditary if and only if either* $\mathbf{V}$ *is simple or* $\mathbf{V}$ *is* $\mathbb{Z}_2^n$ *for* $n = 2$ *or* $n = 3$*.*
(2) $\mathbf{V}$ *is congruence power-hereditary if and only if either* $\mathbf{V}$ *is simple or* $\mathbf{V}$ *is* $\mathbb{Z}_2^2$*.*

This theorem can now be used to extend Theorem 4.2 to any simple tight lattice. Suppose that $\mathbf{A}$ is a finite algebra satisfying a nontrivial idempotent Maltsev condition so that Con$\mathbf{A}$ contains a copy of a simple tight lattice $\mathbf{L}$. As in the proof of 4.2, we can actually assume that Con$\mathbf{A} \cong \mathbf{L}$. Let U be a $\langle 0_{\mathbf{A}}, 1_{\mathbf{A}} \rangle$-minimal set. Since Con$\mathbf{A}|_U \cong \mathbf{L}$, if $\mathbf{L} \ncong \mathbf{2}$, then the minimal algebra $\mathbf{A}|_U$ must be a vector space. If $\mathbf{L}$ is not isomorphic to $\mathbf{2}$, Con$(\mathbb{Z}_2^2)$ (which is $\mathbf{M}_3$), or Con$(\mathbb{Z}_2^3)$, then by 5.3 $\mathbf{A}$ cannot be congruence hereditary. If $\mathbf{L}$ is not isomorphic to $\mathbf{2}$ or Con$(\mathbb{Z}_2^2)$, then by 5.3 $\mathbf{A}$ cannot be congruence power-hereditary.

Theorem 5.4. *Suppose that* $\mathbf{A}$ *is a finite algebra satisfying a nontrivial idempotent Maltsev condition.*

(1) *If* Con$\mathbf{A}$ *contains a copy of a simple tight lattice other than* $\mathbf{2}$*,* $\mathbf{M}_3$*, or* Con$\mathbb{Z}_2^3$*, then* Con$\mathbf{A}$ *is not hereditary.*
(2) *If* Con$\mathbf{A}$ *contains a copy of a simple tight lattice other than* $\mathbf{2}$ *or* $\mathbf{M}_3$*, then* Con$\mathbf{A}$ *is not power-hereditary.*

6. Affine Complete Algebras

An algebra $\mathbf{A}$ is *affine complete* if every operation on the universe of $\mathbf{A}$ which preserves every congruence of $\mathbf{A}$ is a polynomial operation of $\mathbf{A}$.

Lemma 6.1. *Suppose that* $\mathbf{A}$ *is a finite affine complete algebra and* e *is an idempotent unary polynomial of* $\mathbf{A}$ *with* $U = e(A)$*. Then* $\mathbf{A}|_U$ *is also affine complete.*

Proof. Suppose that f is an n-ary operation on U which preserves the congruences of $\mathbf{A}|_U$. Define g on $\mathbf{A}$ by $g(x_1, \ldots, x_n) = f(e(x_1), \ldots, e(x_n))$. Then g preserves the congruences of $\mathbf{A}$, and in U the equation $f = e \circ g$ holds. This makes f a polynomial operation of $\mathbf{A}|_U$. □

The following corollary implies that information about affine complete $\mathbf{G}$-sets might be useful in addressing the question of which $\mathbf{M}_n$ are congruence lattices of finite algebras.

Corollary 6.2. *Suppose that* n *is a positive integer so that* $\mathbf{M}_n$ *is not the congruence lattice of a finite vector space. If* $\mathbf{M}_n$ *is the congruence lattice of a finite algebra, then for some group* $\mathbf{G}$*, there is an affine complete* $\mathbf{G}$*-set whose congruence lattice is isomorphic to* $\mathbf{M}_n$*.*

Proof. Let $\mathbf{A}$ be a finite algebra with Con$\mathbf{A} \cong \mathbf{M}_n$. By adding the necessary operations to $\mathbf{A}$, we can assume that $\mathbf{A}$ is affine complete. Let U be any $\langle 0,1\rangle$-minimal set of $\mathbf{A}$. The algebra $\mathbf{A}|_U$ is minimal and has a congruence lattice isomorphic to $\mathbf{M}_n$. It must be that $\mathbf{A}|_U$ is polynomially equivalent to a $\mathbf{G}$-set. By the the previous lemma, this $\mathbf{G}$-set is affine complete. □

In [1], it was proven that there exists a finite algebra $\mathbf{A}$ with Con$\mathbf{A}$ distributive but so that there are no operations on the universe of $\mathbf{A}$ compatible with the congruences of $\mathbf{A}$ which satisfy Jónsson's equations for distributivity. Since the lattices $\mathbf{M}_n$ are modular, and since there are n for which $\mathbf{M}_n$ is representable as a congruence lattice but not as the congruence lattice of a vector space (such as $\mathbf{M}_7$), Corollary 6.2 gives:

Corollary 6.3. *There exists a finite algebra* $\mathbf{A}$ *with a modular congruence lattice so that there are not operations on the universe of* $\mathbf{A}$ *compatible with the congruences of* $\mathbf{A}$ *which satisfy Gumm's equations (or Day's equations) for congruence modularity.*

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