# Integration

**Definition 5.5.1.** A *partition* of a closed interval $[a, b]$ is a finite subset $P$ of $[a, b]$ which includes $a$ and $b$. We will usually number the elements of $P$ in an increasing manner so that $P = \{x_0, x_1, \ldots, x_n\}$ where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

The *mesh* of this partition $P$ is

$$|P| = \max\{(x_i - x_{i-1}) : i = 1, 2, \ldots, n\}.$$

If $f : [a, b] \to \mathbb{R}$ is a bounded function, then the *upper sum* of $f$ for partition $P$ is

$$U(f, P) = \sum_{i=1}^{n} \sup\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}).$$

The *lower sum* of $f$ for partition $P$ is

$$L(f, P) = \sum_{i=1}^{n} \inf\{f(x) : x \in [x_{i-1}, x_i]\}(x_i - x_{i-1}).$$

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}.$$

Next, we let $\Delta x_i = x_i - x_{i-1}$. Finally, for any $i = 1, 2, \ldots, n$, let

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} \text{ and } M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}.$$

We will refer to this notation as the *standard notation* for partitions. With the standard notation,

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \text{ and } U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i.$$

**Lemma 5.5.7.** *Suppose that* $f : [a, b] \to \mathbb{R}$ *is a bounded function and* $P$ *is a partition of* $[a, b]$*. Further suppose that* $M \in \mathbb{R}$ *so that* $|f(x)| \leq M$ *for all* $x \in [a, b]$*. Then*

$$-M(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a).$$

**Lemma 5.5.7.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and $P$ is a partition of $[a, b]$. Further suppose that $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $x \in [a, b]$. Then*

$$-M(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a).$$

**Lemma 5.5.8. (Partition Refinement Lemma)** *If $P \subseteq Q$ are partitions of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is bounded then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

**Lemma 5.5.7.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function and $P$ is a partition of $[a, b]$. Further suppose that $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $x \in [a, b]$. Then*

$$-M(b - a) \leq L(f, P) \leq U(f, P) \leq M(b - a).$$

**Lemma 5.5.8. (Partition Refinement Lemma)** *If $P \subseteq Q$ are partitions of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is bounded then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

**Theorem 5.5.9.** *Suppose that $P$ and $Q$ are partitions of $[a, b]$ and that $f : [a, b] \to \mathbb{R}$ is bounded. Then $L(f, P) \leq U(f, Q)$.*

**Definition 5.6.1.** Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. The *upper integral* of $f$ over $[a, b]$ is

$$\overline{\int_a^b} f = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}.$$

The *lower integral* of $f$ over $[a, b]$ is

$$\underline{\int_a^b} f = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$

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The *lower integral* of $f$ over $[a, b]$ is

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**Theorem 5.6.2.** *Suppose that $P$ and $Q$ are partitions of $[a, b]$ and that $f : [a, b] \to \mathbb{R}$ is bounded. Let $M \in \mathbb{R}$ so that $|f(x)| \leq M$ for all $x \in [a, b]$. Then*

$$-M(b-a) \leq L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, Q) \leq M(b-a).$$

**Definition 5.6.3.** Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. If

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

then $f$ is *integrable* on $[a, b]$. The *integral* of $f$ over $[a, b]$ is

$$\int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f.$$

**Theorem 5.7.1. (The $\epsilon$-Partition Integrability Condition)** *A bounded function $f : [a, b] \to \mathbb{R}$ is integrable if and only if for every $\epsilon > 0$ there is a partition $P$ of $[a, b]$ so that $U(f, P) - L(f, P) < \epsilon$.*

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**Theorem 5.7.3.** *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and that $R \in \mathbb{R}$ so that $L(f, P) \leq R \leq U(f, P)$ for all partitions $P$ of $[a, b]$. Then $\displaystyle\int_a^b f = R$.*

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**Theorem 5.7.4.** *Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and that $R \in \mathbb{R}$. If for every $\epsilon > 0$ there is a partition $P$ of $[a, b]$ so that $U(f, P) - L(f, P) < \epsilon$ and $L(f, P) \leq R \leq U(f, P)$, then $f$ is integrable on $[a, b]$ and $\displaystyle\int_a^b f = R$.* $\square$

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**Theorem 5.7.6.** *If $f : [a, b] \to \mathbb{R}$ is bounded and monotonic, then $f$ is integrable.*

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**Theorem 5.7.6.** *If $f : [a, b] \to \mathbb{R}$ is bounded and monotonic, then $f$ is integrable.*

**Theorem 5.7.7.** *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ is integrable.*

**Theorem 5.7.8.** *Suppose that* $f : [a, b] \to \mathbb{R}$ *is integrable and*

$$a \leq u < v \leq b.$$

*Then* $f$ *is integrable on* $[u, v]$.

**Theorem 5.7.8.** *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and*

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*Then $f$ is integrable on $[u, v]$.*

**Theorem 5.7.9.** *Suppose that $f : [a, c] \to \mathbb{R}$ is bounded and $a < b < c$. If $f$ is integrable on $[a, b]$ and on $[b, c]$, then $f$ is integrable on $[a, c]$. Moreover, $\displaystyle\int_a^c f = \int_a^b f + \int_b^c f$.*

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**Theorem 5.7.10.** *Suppose that $f, g : [a,b] \to \mathbb{R}$ are bounded. If $f(x) = g(x)$ for all $x \in [a,b)$ and if $f$ is integrable on $[a,b]$, then $g$ is integrable on $[a,b]$ and $\displaystyle\int_a^b f = \int_a^b g$.*

**Theorem 5.7.8.** *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and*

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**Theorem 5.7.11.** *If $f : [a, b] \to \mathbb{R}$ is piecewise integrable on $[a, b]$, then $f$ is integrable $[a, b]$.* $\square$

**Theorem 5.8.1.** *If $f : [a,b] \to \mathbb{R}$ is integrable and $k \in \mathbb{R}$, then $kf$ is integrable on $[a,b]$ and* $\displaystyle\int_a^b (kf) = k \int_a^b f$.

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**Theorem 5.8.2.** *Suppose that $f, g : [a,b] \to \mathbb{R}$ are integrable. Then $f+g$ and $f-g$ are integrable on $[a,b]$ and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \text{ and } \int_a^b (f-g) = \int_a^b f - \int_a^b g.$$

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**Lemma 5.8.3.** *If $f : [a,b] \to \mathbb{R}$ is integrable and $k \in \mathbb{R}$, so that $k \leq f(x)$ for all $a \in [a,b]$, then $k(b-a) \leq \int_a^b f$.*

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**Theorem 5.8.4.** *Suppose that $f, g : [a,b] \to \mathbb{R}$ are integrable and that $f(x) \leq g(x)$ for all $x \in [a,b]$. Then $\int_a^b f \leq \int_a^b g$.*

**Theorem 5.8.5.** *If $f : [a,b] \to \mathbb{R}$ is integrable then $|f|$ is integrable on $[a,b]$ and $\left| \int_a^b f \right| \leq \int_a^b |f|$.*

**Definition 5.9.1.** If $f$ is a function defined at a real number $a$, then

$$\int_a^a f = 0.$$

Suppose that $a < b$ in $\mathbb{R}$ and that $f : [a, b] \to \mathbb{R}$ is integrable. Then

$$\int_b^a f = -\int_a^b f.$$

**Theorem 5.9.2.** *Suppose that $f$ is integrable on an interval containing $a$, $b$, and $c$. Then* $\displaystyle\int_a^c f = \int_a^b f + \int_b^c f.$ $\square$

**Theorem 5.9.4. (Fundamental Theorem of Calculus)** *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. If $f'$ is integrable on $[a, b]$, then*

$$\int_a^b f' = f(b) - f(a).$$

**Theorem 5.9.6. (Fundamental Theorem of Calculus)** *Suppose that $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ and that $F : [a,b] \to \mathbb{R}$ is defined by*

$$F(x) = \int_a^x f.$$

*Then $F$ is continuous on $[a,b]$. If $f$ is continuous at $z \in (a,b)$, then $F$ is differentiable at $z$ and $F'(z) = f(z)$.*