Circles

Some Definitions

- A **circle** is a set consisting of all points whose distance from a given point is equal to a certain given positive number.
- The given point is **the center** of the circle.
- The given distance is **the radius** of the circle.
- Twice the radius is **the diameter** of the circle.
- We denote the circle with center O and radius r as $C(O, r)$.

More Definitions

For a circle $C = C(O, r)$:

- A segment with O as one endpoint and with the other endpoint on C is **a radius** of C.
- A segment with endpoints on C is a **chord** of C.
- A chord of C which contains O is **a diameter** of C.
- Two circles are concentric if they have the same center.
- A point P is an **interior point** of C if $OP < r$.
- A point P is an **exterior point** of C if $OP > r$.
- A line ℓ is a **tangent line of** C if $\ell \cap C$ contains exactly one point.
- A line ℓ is a **secant line of** C if $\ell \cap C$ contains exactly two points.
- Two circles are **tangent to each other** if they intersect in exactly one point.

Two Easy Lemmas

Lemma 14.1: If C is a circle of radius r, then every diameter of C has length $2r$, and the center of C is the midpoint of each diameter.

Lemma 14.2: Two concentric circles with a point in common are equal.

Circles and Lines

Theorem 14.3: No circle contains three distinct collinear points.

Theorem 14.4 (Properties of Secants): Suppose that C is a circle and ℓ is a secant line that intersects C at A and B. Then every interior point of the chord $\overline{AB}$ is in the interior of C, and every point of ℓ that is not in $\overline{AB}$ is in the exterior of C.

Theorem 14.5 (Properties of Chords): Suppose C is a circle an $\overline{AB}$ is a chord of C.

1. The perpendicular bisector of $\overline{AB}$ passes through the center of C.
2. If $\overline{AB}$ is not a diameter, a radius of C is perpendicular to $\overline{AB}$ if and only if it bisects $\overline{AB}$.

Circles and Lines

Theorem 14.6 (Line-Circle Theorem): Suppose C is a circle and ℓ is a line that contains a point in the interior of C. Then ℓ is a secant line of C, and there are exactly two points where ℓ intersects C.

Theorem 14.7 (Tangent Line Theorem): Suppose C is a circle and ℓ is a line that intersects C at a point P. Then ℓ is tangent to C if and only if ℓ is perpendicular to the radius through P.

Corollary 14.8 (Existence of Tangents): If C is a circle and $P \in C$, then there is a unique line tangent to C at P.

Theorem 14.9 (Properties of Tangents): If C is a circle and ℓ is a line that is tangent to C at P, then every point of ℓ except P lies in the exterior of C, and every point of C except P lies on the same side of ℓ as the center of C.

Circles and Circles

Theorem 14.10 (Two Circles Theorem): Suppose that C and D are two circles with radii r and s, respectively, and let d be the distance between their centers. If either of these two conditions is satisfied, the there exist exactly two points where the circles intersect, one on either side of the line containing their centers.

1. $d < r + s$, $r < d + s$, and $s < d + r$
2. D contains a point in the interior of C and a point in the exterior of C.

Theorem 14.11 (Tangent Circles Theorem): Suppose that $C(O, r)$ and $C(Q, s)$ are tangent to teach other at a point A. Then O, Q, and A are distinct and collinear, and $C(O, r)$ and $C(Q, s)$ have a common tangent line at A.

Arcs and Angles

Suppose that $C = C(O, r)$ is a circle and A and B are distinct points on C.

- If $\angle AOB$ is a proper angle, then the **minor arc bounded by** A **and** B is the set consisting of A, B, and all of the points on C in the *interior* of $\angle AOB$.
- If $\angle AOB$ is a proper angle, then the **major arc bounded by** A **and** B is the set consisting of A, B, and all of the points on C in the *exterior* of $\angle AOB$.
- If $\angle AOB$ is a straight angle, then each set composed of A, B, and all of the points of C on one side of $\overleftrightarrow{AB}$ is a **semicircle**.
- An **arc** is a major arc, a minor arc, or a semicircle.
- Two arcs whose union is all of C and which intersect in exactly two points are **conjugate arcs**.
- If X is an interior point of an arc with endpoints A and B, we denote the arc by $\widehat{AXB}$.
- We sometimes use $\widehat{AB}$ if it is understood which arc we are talking about.

Lemma 14.12: Suppose that C is a circle and $\widehat{AB}$ is a minor arc of C. If X is any point on C, then X lies on $\widehat{AB}$ if and only if the radius OX intersects the chord $\overline{AB}$.

Theorem 14.13: Suppose that C is a circle with center O and that A and B are two distinct points on C. The two arcs bounded by A and B are the intersections of C with the closed half-planes determined by $\overleftrightarrow{AB}$.

Central Angles

Suppose that C is a circle with center O and that $\widehat{AB}$ is an arc on C.

- The angle $\angle AOB$ is the **central angle associated with** $\widehat{AB}$.
- If $\widehat{AB}$ is a minor arc, then the measure of $\widehat{AB}$ is $m\widehat{AB} = m\angle AOB$.
- If $\widehat{AB}$ is a major arc, then the measure of $\widehat{AB}$ is $m\widehat{AB} = rm\angle AOB$.
- If $\widehat{AB}$ is a semicircle, then the measure of $\widehat{AB}$ is $m\widehat{AB} = 180$.

Inscribed Angles

- An angle $\angle ADB$ is **inscribed** in a circle C if its vertex is on C and the interiors of both of its rays intersect C.
- Suppose that $\angle ADB$ is inscribed in a circle C, and suppose that A and B are on C. The arc $\widehat{ADB}$ is the **inscribed arc** for $\angle ADB$.
- The arc $\widehat{AB}$ is the **intercepted arc** for $\angle ADB$.
- An angle **inscribed in a semicircle** is an angle whose vertex is on a circle and whose sides intersect the circle at the endpoints of a diameter.

Lemma 14.15: Any two conjugate arcs have measures adding to 360.

Theorem 14.16 (Thale's Theorem): Any angle inscribed in a semicircle is a right angle.

Theorem 14.17 (Converse to Thale's Theorem): The hypotenuse of a right triangle is a diameter of a circle that contains all three vertices.

Theorem 14.18 (Existence of Tangent Lines through an Exterior Point): Let C be a circle, and let A be a point in the exterior of C. There are exactly two distinct tangent lines to C containing A. The two points of tangency X and Y are equidistant from A, and the center of C lies on the bisector of $\angle YAX$.

Theorem 14.19 (Inscribed Angle Theorem): The measure of a proper angle inscribed in a circle is one-half the measure of its intercepted arc.

Corollary 14.20 (Arc Addition Theorem): Suppose A, B, and C are three distinct points on a circle D and that $\widehat{AB}$ and $\widehat{BC}$ are arcs that intersect only at B. Then $m\widehat{ABC} = m\widehat{AB} + m\widehat{BC}$.

Corollary 14.21 (Intersecting Chords Theorem): Suppose that $\overline{AB}$ and $\overline{CD}$ are two distinct chords of a circle S that intersect at a point P in the interior of S. Then $(PA)(PB) = (PC)(PD)$.

Corollary 14.22 (Intersecting Secants Theorem): Suppose that $\overline{AB}$ and $\overline{CD}$ are two distinct chords of a circle S and that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point P in the exterior of S. Then $(PA)(PB) = (PC)(PD)$.

Inscribed Polygons

Suppose that P is a polygon and C is a circle. If all of the vertices of P are on C then:

- P is inscribed in C.
- C is circumscribed about P.
- P is a cyclic polygon.
- C is the circumcircle of P.
- The center of C is the circumcenter for P.

Theorem 14.23: Every polygon inscribed in a circle is convex.

14.24 (Circumcircle Theorem): A polygon is cyclic if and only if the perpendicular bisectors of all of its edges are concurrent. If this is the case, the point O where these perpendicular bisectors intersect is the unique circumcenter for P, and the circle with center O and radius equal to the distance from O to any vertex is the unique circumcenter.

Theorem 14.25 (Cyclic Triangle Theorem): Every triangle is cyclic.

Corollary 14.26 (Perpendicular Bisector Concurrence Theorem): In any triangle, the perpendicular bisectors of all three sides are concurrent.

Corollary 14.27: Given any three noncollinear points, there is a unique circle that contains all of them.

Theorem 14.28 (Cyclic Quadrilateral Theorem): A quadrilateral is cyclic if and only if it is convex and both pairs of opposite angles are supplementary.

Circumscribed Polygons

Suppose that C is a circle and P is a polygon.

- A segment $\overline{AB}$ is **tangent** to C if $\overleftrightarrow{AB}$ is tangent to C and the point of tangency is in $\overline{AB}$.
- P is **circumscribed** around C or C is **inscribed** in P if every edge of P is tangent to C.
- If C is inscribed in P, then C is an **incircle** of P.
- If C is inscribed in P, then the center of C is an **incenter** of P.
- A polygon with an incircle is a **tangential polygon**.

Circumscribed Polygons

Lemma 14.29: If C is an incircle for a polygon P, then the point of tangency for each edge is an interior point of the edge.

Theorem 14.31: Every tangential polygon is convex.

Corollary 14.32: If P is a tangential polygon, then the interior of its incircle is contained in the interior of P.

Circumscribed Polygons

Theorem 14.33 (Incircle Theorem): A polygon P is tangential if and only if it is convex and the bisectors of all of its angles are concurrent. If this is the case, the point O where these bisectors intersect is the unique incenter of P, and the circle with center P and radius equal to the distance from O to any edge is the unique incircle.

Theorem 14.34 (Tangential Triangle Theorem): Every triangle is tangential.

Corollary 14.35 (Angle Bisector Concurrence): In any triangle, the bisectors of the three angles are concurrent.

Theorem 14.36 (Altitude Concurrence): In any triangle, the lines containing the three altitudes are concurrent.

Triangles and Concurrence

Theorem 12.13 (Median Concurrence Theorem – Centroid): The medians of a triangle are concurrent, and the distance from the point of intersection to each vertex is twice the distance to the midpoint of the opposite side.

Corollary 14.26 (Perpendicular Bisector Concurrence Theorem – Circumcenter): In any triangle, the perpendicular bisectors of all three sides are concurrent.

Corollary 14.35 (Angle Bisector Concurrence – Incenter): In any triangle, the bisectors of the three angles are concurrent.

Theorem 14.36 (Altitude Concurrence – Orthocenter): In any triangle, the lines containing the three altitudes are concurrent.

Regular Polygons and Circles

Theorem 14.39: Every regular polygon is both cyclic and tangential, and its incenter is equal to its circumcenter.

Theorem 14.40 (Central Angles of a Regular Polygon): Every central angle of a regular n-gon has measure $360/n$.

Lemma 14.41 Every equilateral polygon inscribed in a circle is regular.