Interpolation in small lattices

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Abstract. We introduce eight possible properties of an operation on a finite lattice and then prove that some of these properties completely determine the terms of some familiar finite lattices. We then apply our results to the problem of representing some finite lattices as congruence lattices of finite algebras.

1. Introduction

Suppose that $\mathbf{L}$ is the congruence lattice of a finite algebra $\mathbf{A}$ and that $\mathbf{M}$ is a sublattice of $\mathbf{L}$. A natural question to ask is whether or not $\mathbf{M}$ is the congruence lattice of an algebra with the same universe as $\mathbf{A}$. There are operations on $\mathbf{L}$ so that $\mathbf{M}$ is a congruence lattice if and only if $\mathbf{M}$ is closed under these operations (we will make this comment precise later). If each of these operations is a term operation of $\mathbf{L}$ then $\mathbf{M}$ must be closed under the operations and must be a congruence lattice. The operations in question can be shown to have nice properties such as idempotence and monotonicity. In this paper, we address the question: What properties must an operation on a finite lattice satisfy in order to be a term operation of that lattice?

Suppose that $\mathbf{L}$ is a finite lattice and $p : \mathbf{L}^n \to \mathbf{L}$ is an n-ary operation on $\mathbf{L}$. We consider these possible properties of p:

- P1 The operation p is order preserving.
- P2 The operation p is idempotent.
- P3 For any $a, b \in \mathbf{L}$, the sublattice $\{x \in \mathbf{L} : x \leq a \text{ or } b \leq x\}$ is closed under p.
- P4 The operation on $\mathbf{L}^2$ induced by p satisfies P3.
- P5 For any a, b, a', b' in $\mathbf{L}$ with $a < b$ and $a' < b'$, the sets $\{a, b\}$ and $\{a', b'\}$ are closed under p and the algebras $\langle\{a, b\}, p\rangle$ and $\langle\{a', b'\}, p\rangle$ are isomorphic via the map $x \to x'$.
- P6 The operation p is preserved by every automorphism of $\mathbf{L}$.
- P7 The operation p is preserved by every endomorphism of $\mathbf{L}$.
- P8 Every sublattice of $\mathbf{L}$ is closed under p.

If p is equal to a term operation of $\mathbf{L}$, then p must satisfy each of these eight properties. We prove that if $\mathbf{L}$ is a finite distributive lattice or is $\mathbf{N}_5$, then the

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terms of $\mathbf{L}$ are those operations on $\mathbf{L}$ satisfying P1, P2, and P4. If $\mathbf{L}$ is $\mathbf{M}_3$, $\mathbf{D}_1$, or $\mathbf{D}_2$, then the terms of $\mathbf{L}$ are the operations on $\mathbf{L}$ satisfying P1, P2, P4, and P6. We also prove that the terms of a finite lattice in the variety generated by $\mathbf{N}_5$, $\mathbf{M}_3$, $\mathbf{D}_1$, or $\mathbf{D}_2$ are completely determined by properties P1, P2, P4, and P7. Some results concerning congruence heredity are then discussed in light of these results. We then discuss the possible applicability of these notions to a problem of Keith Kearnes.

2. Preliminaries

Suppose that p is an operation on a finite lattice $\mathbf{L}$. Then for each positive n, p induces an operation $p^{\mathbf{L}^n}$ on $\mathbf{L}^n$. We will usually abuse notation and refer to these operations in direct powers also as p. If there is a term operation of $\mathbf{L}$ which is always equal to p, we abuse terminology some and say that p is a term operation. The familiar Galois connection between relations and operations on a finite set declares that p is a term operation of $\mathbf{L}$ if and only if p preserves every subuniverse of every direct power of $\mathbf{L}$. Moreover, since $\mathbf{L}$ has a majority operation, these subuniverses of direct powers are all generated by the subuniverses of $\mathbf{L}^2$ [2]. It follows that:

Lemma 2.1. *Suppose that $\mathbf{L}$ is a finite lattice. An operation p on $\mathbf{L}$ is a term operation of $\mathbf{L}$ if and only if p preserves every subuniverse of $\mathbf{L}^2$.*

Suppose that p is an idempotent, order preserving n-ary operation on a finite lattice $\mathbf{L}$ and that $a < b \in \mathbf{L}$. If $x_1, \ldots, x_n$ are in the interval between a and b, then

$$a = p(a, a, \ldots, a) \leq p(x_1, x_2, \ldots, x_n) \leq p(b, b, \ldots, b) = b$$

This proves that

Lemma 2.2. *Suppose that $\mathbf{L}$ is a finite lattice and that $p : \mathbf{L}^n \to \mathbf{L}$ is an operation satisfying P1 and P2. Every interval in $\mathbf{L}$ is closed under p.*

Hegedűs and Pálfy in [6] give the following characterization of sublattices of the square of a lattice as intersections of special sublattices.

Lemma 2.3. ([6] Lemma 4.7) *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be arbitrary lattices and $\mathbf{L} \subseteq \mathbf{L}_1 \times \mathbf{L}_2$ a sublattice. Let us define*

$$\begin{aligned}
\mathbf{L}_1' &= \{x \in \mathbf{L}_1 : (\exists b \in \mathbf{L}_2)\langle x, b\rangle \in \mathbf{L}\} \\
\mathbf{L}_2' &= \{y \in \mathbf{L}_2 : (\exists a \in \mathbf{L}_1)\langle a, y\rangle \in \mathbf{L}\} \\
\mathbf{L}_1^* &= \{\langle x, y\rangle \in \mathbf{L}_1 \times \mathbf{L}_2 : (\exists \langle a, b\rangle \in \mathbf{L})(x \leq a \textit{ and } b \leq y)\} \\
\mathbf{L}_2^* &= \{\langle x, y\rangle \in \mathbf{L}_1 \times \mathbf{L}_2 : (\exists \langle a, b\rangle \in \mathbf{L})(x \geq a \textit{ and } b \geq y)\}
\end{aligned}$$

Then $\mathbf{L} = (\mathbf{L}_1' \times \mathbf{L}_2) \cap (\mathbf{L}_1 \times \mathbf{L}_2') \cap \mathbf{L}_1^* \cap \mathbf{L}_2^*$. Moreover, if $\mathbf{L}$ is a 0-1 *sublattice, then $\mathbf{L}_1^*$ and $\mathbf{L}_2^*$ are subdirect products in $\mathbf{L}_1 \times \mathbf{L}_2$ with $\langle 0, 1\rangle \in \mathbf{L}_1^*$ and $\langle 1, 0\rangle \in \mathbf{L}_2^*$.*

This lemma makes it easy to prove the following corollary which allows us to only consider very special sublattices of the square when deciding if an operation is a term operation. Theorem 4.8 of [6] states this result for special operations on congruence lattices. The only essential property of these special operations is idempotence. We state the theorem in general here without reference to congruence lattices. This proof is almost identical to that in [6].

Corollary 2.4. *Let* $\mathbf{M}$ *be a finite lattice and let* $p : \mathbf{M}^n \to \mathbf{M}$ *be an idempotent operation on* $\mathbf{M}$*. Then* p *is a term operation of* $\mathbf{M}$ *if and only if* p *preserves every subdirect product in* $\mathbf{M}^2$ *containing* $(\{0\} \times \mathbf{M}) \cup (\mathbf{M} \times \{1\})$*.*

Proof. First, $\{\langle x, y\rangle : x \leq y\}$ is a sublattice of $\mathbf{M}^2$ containing $(\{0\}\times\mathbf{M})\cup(\mathbf{M}\times\{1\})$, so p is order preserving.

According to Lemma 2.1, we need only show that every sublattice of $\mathbf{M}^2$ is closed under p. In order to do so, we will use Lemma 2.3 and will need to know that sublattices of $\mathbf{M}$ are closed under p. Suppose that $\mathbf{L}$ is a sublattice of $\mathbf{M}$. Let $\mathbf{L}'$ be $\mathbf{L} \cup \{0, 1\}$ (it may be that $\mathbf{L} = \mathbf{L}'$). For each $a \in \mathbf{M}$, let a' be the smallest element of $\mathbf{L}'$ above a. Let $\mathbf{L}'' = \{\langle a, b\rangle : a' \leq b\}$. $\mathbf{L}''$ is a sublattice of $\mathbf{M}^2$ containing $(\{0\} \times \mathbf{M}) \cup (\mathbf{M} \times \{1\})$, and so is closed under p. Now, $\mathbf{L}'' \cap \{\langle a, a\rangle : a \in \mathbf{M}\} = \{\langle a, a\rangle : a \in \mathbf{L}'\}$ is closed under p (since applying p to diagonal elements must yield a diagonal element). It follows then that $\mathbf{L}'$ is closed under p. Now, $\mathbf{L}$ is an interval in $\mathbf{L}'$, and p is an idempotent, order preserving operation on $\mathbf{L}'$, so $\mathbf{L}$ is closed under p by 2.2. We have shown that every sublattice of $\mathbf{M}$ is closed under p.

Now let $\mathbf{L}$ be a sublattice of $\mathbf{M}^2$. Let $\mathbf{L}' = \mathbf{L} \cup \{\langle 0, 0\rangle, \langle 1, 1\rangle\}$. Taking $\mathbf{L}_1 = \mathbf{M}$ and $\mathbf{L}_2 = \mathbf{M}$ we can apply Lemma 2.3 to $\mathbf{L}' \subseteq \mathbf{L}_1 \times \mathbf{L}_2$. Let $\mathbf{L}_1'$, $\mathbf{L}_2'$, $\mathbf{L}_1^*$, and $\mathbf{L}_2^*$ be as in Lemma 2.3. The lattices $\mathbf{M}$, $\mathbf{L}_1'$, and $\mathbf{L}_2'$ are closed under p be the previous paragraph. Also, $\mathbf{L}_1^*$ and $\mathbf{L}_2^*$ are closed under p by our assumptions. It follows that $\mathbf{L}' = (\mathbf{L}_1' \times \mathbf{L}_2) \cap (\mathbf{L}_1 \times \mathbf{L}_2') \cap \mathbf{L}_1^* \cap \mathbf{L}_2^*$ is closed under p. Since p is idempotent and order preserving, and since $\mathbf{L}$ is an interval in $\mathbf{L}'$, $\mathbf{L}$ is closed under p.

Since every sublattice of $\mathbf{L}^2$ is closed under p, and since $\mathbf{L}$ has a majority term, p must be a term operation of $\mathbf{L}$ by 2.1. ☐

Suppose that $\mathbf{L}$ is a subdirect product of $\mathbf{M}^2$ which contains $(\{0\} \times \mathbf{M}) \cup (\mathbf{M} \times \{1\})$. For each $x \in \mathbf{M}$, let $e(x)$ be the minimal element of $\{y \in \mathbf{M} : \langle x, y\rangle \in \mathbf{L}\}$. It is easy to show that e must be a join endomorphism of $\mathbf{M}$ which fixes 0 and that $\mathbf{L} = \{\langle x, y\rangle : e(x) \leq y\}$. Moreover, every sublattice of the form $\{\langle x, y\rangle : e(x) \leq y\}$, where e is a join endomorphism of $\mathbf{M}$ which fixes 0, contains $(\{0\}\times\mathbf{M})\cup(\mathbf{M}\times\{1\})$. Therefore, Corollary 2.4 can be restated as:

Corollary 2.5. *Let* $\mathbf{M}$ *be a finite lattice and let* $p : \mathbf{M}^n \to \mathbf{M}$ *be an idempotent operation on* $\mathbf{M}$*. Then* p *is a term operation of* $\mathbf{M}$ *if and only if* p *preserves every subdirect product in* $\mathbf{M}^2$ *of the form* $\{\langle x, y\rangle : e(x) \leq y\}$ *for some join endomorphism* e *of* $\mathbf{M}$ *which fixes* 0*.*

It follows immediately that:

Corollary 2.6. *Let* $\mathbf{M}$ *be a finite lattice and let* $p : \mathbf{M}^n \to \mathbf{M}$ *be an idempotent operation on* $\mathbf{M}$*. Then* p *is a term operation of* $\mathbf{M}$ *if and only if for every join endomorphism* e *of* $\mathbf{M}$ *which fixes* 0 *and for all* $x_1, \ldots, x_n \in \mathbf{M}$*,* $p(e(x_1), \ldots, e(x_n)) \leq e(p(x_1, \ldots, x_n))$*.*

Of course, we could dualize our comments here and focus instead on meet endomorphisms fixing 1. In that case we would end up with:

Corollary 2.7. *Let* $\mathbf{M}$ *be a finite lattice and let* $p : \mathbf{M}^n \to \mathbf{M}$ *be an idempotent operation on* $\mathbf{M}$*. Then* p *is a term operation of* $\mathbf{M}$ *if and only if* p *preserves every subdirect product in* $\mathbf{M}^2$ *of the form* $\{\langle x, y\rangle : x \leq e(y)\}$ *for some meet endomorphism* e *of* $\mathbf{M}$ *which fixes* 1*.*

Automorphisms will prove to be critical when dealing with interpolation in some lattices below. For that reason, we state this automorphism specific lemma relating to 2.5:

Lemma 2.8. *Suppose that* $\mathbf{M}$ *is a finite lattice and let* $p : \mathbf{M}^n \to \mathbf{M}$ *be an order preserving operation on* $\mathbf{M}$*. Let* $e : \mathbf{M} \to \mathbf{M}$ *be an automorphism of* $\mathbf{M}$*. If* e *preserves* p*, then* p *preserves* $\{\langle x, y\rangle : e(x) \leq y\}$*.*

Proof. Let $\mathbf{G}$ (for graph) be the sublattice $\{\langle x, y\rangle : e(x) \leq y\}$ of $\mathbf{M}^2$. Assume that e preserves p. Let $\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle \in \mathbf{G}$. We need to know that

$$\langle p(x_1, \ldots, x_n), p(y_1, \ldots, y_n)\rangle \in \mathbf{G}.$$

We know that $e(x_i) \leq y_i$ for each i, so by the order preserving nature of p, we have $p(e(x_1), \ldots, e(x_n)) \leq p(y_1, \ldots, y_n)$. However, the left hand side of this is $e(p(x_1, \ldots, x_n))$ since e preserves p. This gives $e(p(x_1, \ldots, x_n)) \leq p(y_1, \ldots, y_n)$, which places $\langle p(x_1, \ldots, x_n), p(y_1, \ldots, y_n)\rangle \in \mathbf{G}$ as desired. □

3. The properties P3, P4, and P5

The properties P3, P4, and P5 are intimately related and deserve a few comments. In [15], the first author explored some ways in which new congruence lattices could be constructed from old congruence lattices. This exploration was continued in [16]. In both of those papers, it was necessary to prove that certain sublattices (of congruence lattices) were closed under certain operations. One fact which proved quite useful in this exploration was that these operations satisfied P3. To prove that these operations satisfied P3, the technical property P5 was used. In section 5 we will want to employ property P3 in the square of a lattice. Unfortunately, P3 may not be preserved by products – hence property P4 was introduced. The property P5 is preserved by products. However, P4 is simpler to state. Also, in the presence of idempotence and monotonicity, P4 and P5 are equivalent and imply P3. We prove that here. We also give an example to show that P3 is not equivalent to P4 and P5.

Definition 3.1. Suppose that $\mathbf{L}$ is a finite lattice and that $a, b \in \mathbf{L}$. We will call the sublattice $\{x \in \mathbf{L} : x \leq a \text{ or } b \leq x\}$ a *split interval.*

Lemma 3.2. *Suppose that* $\mathbf{L}$ *is a finite lattice and that* $p : \mathbf{L}^n \to \mathbf{L}$ *is an operation satisfying P1 and P5. Then* p *satisfies P3.*

Proof. Suppose that $r < s \in \mathbf{L}$. Let A be the collection of all subsets B of $\{1, 2, \ldots, n\}$ for which there exists $a_1, a_2, \ldots, a_n \in \{r, s\}$ with $p(a_1, a_2, \ldots, a_n) = r$ and $B = \{i : a_i = r\}$. Then for any $b_1, b_2, \ldots, b_n \in \{r, s\}$, $p(b_1, b_2, \ldots, b_n) = r$ if and only if $\{i : b_i = r\} \in A$. By P5, for any $R < S \in \mathbf{L}$ and for any $b_1, b_2, \ldots, b_n \in \{R, S\}$, $p(b_1, b_2, \ldots, b_n) = R$ if and only if $\{i : b_i = R\} \in A$. Also, notice that if $B \in A$ and if $B \subseteq C$, then $C \in A$. To see this, suppose that $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in \{r, s\}$ so that $\{i : b_i = r\} = B$ and $\{i : c_i = r\} = C$. If $b_i = r$, then $i \in B \subseteq C$, so $c_i = r$. This means that $c_i \leq b_i$ for all i, so $p(c_1, c_2, \ldots, c_n) \leq p(b_1, b_2, \ldots, b_n) = r$ (where the equality follows since $B \in A$). By P5, $p(c_1, c_2, \ldots, c_n)$ must be in $\{r, s\}$, so this implies that $p(c_1, c_2, \ldots, c_n) = r$ and that $C \in A$.

Let $a, b \in \mathbf{L}$ and let $\mathbf{M}$ be the split interval $\{x \in \mathbf{L} : x \leq a \text{ or } b \leq x\}$ and let $y_1, y_2, \ldots, y_n \in \mathbf{M}$. For each i, define

$$x_i = \begin{cases} b & b \leq y_i \\ 0 & else \end{cases} \quad \text{and} \quad z_i = \begin{cases} a & y_i \leq a \\ 1 & else \end{cases}$$

Now, let $u = p(x_1, \ldots, x_n)$, $v = p(y_1, \ldots, y_n)$, and $w = p(z_1, \ldots, z_n)$. Since $x_i \leq y_i \leq z_i$ for each i, we have that $u \leq v \leq w$ by P1. We will argue that either $w = a$ or $u = b$. This will force $v \in \mathbf{M}$. Suppose that $u \neq b$. This means that $u = 0$ by P5. Since $0 = u = p(x_1, \ldots, x_n)$, it has to be that $\{i : x_i = 0\} \in A$ by the arguments above. Suppose that $x_i = 0$. This means that $b \not\leq y_i$. Since $y_i \in \mathbf{M}$, it has to be that $y_i \leq a$. In this case, $z_i = a$. This shows that $\{i : x_i = 0\} \subseteq \{i : z_i = 0\}$. Since $\{i : x_i = 0\} \in A$, we know then that $\{i : z_i = 0\} \in A$. By the arguments above this implies that $a = p(z_1, z_2, \ldots, z_n) = w$. Thus, $v \leq w = a$, so $v \in \mathbf{M}$ as desired. □

Now we prove that for an idempotent, order preserving operation, P4 and P5 are equivalent. To do so, we need to know that P5 is preserved by direct powers. Suppose that p is an idempotent operation on a finite lattice $\mathbf{L}$ satisfying P5. Let $a < b$ and $a' < b'$ in $\mathbf{L}$. If $\{a, b\}$ and $\{a', b'\}$ are closed under p and if p induces isomorphic structures on these sets in an order preserving way, then each of the sets $\{\langle a, a'\rangle, \langle b, b'\rangle\}$, $\{\langle a, a'\rangle, \langle a, b'\rangle\}$, $\{\langle a, a'\rangle, \langle b, a'\rangle\}$ should clearly be closed under $p^{\mathbf{L}^2}$, and the algebraic structures on these sets induced by $p^{\mathbf{L}^2}$ should all be isomorphic in an order preserving way. Since every two element sublattice of $\mathbf{L}^2$ is of one of these forms, we have:

Lemma 3.3. *Suppose that p is an idempotent operation on a finite lattice $\mathbf{L}$ satisfying P5. The operation induced by p on $\mathbf{L}^2$ also satisfies P5.*

We are now ready to show:

Lemma 3.4. *Suppose that p is an operation on a finite lattice $\mathbf{L}$ satisfying P1 and P2. Then p satisfies P4 if and only if p satisfies P5.*

Proof. Suppose first that p satisfies P1, P2, and P5. The coordinate-wise application of p in $\mathbf{L}^2$ satisfies P1 and P2 and (by 3.3) P5. By 3.2, it follows that $p^{\mathbf{L}^2}$ satisfies P3 in $\mathbf{L}^2$. Thus, p satisfies P4.

Now suppose that p satisfies P1, P2, and P4. Let $a < b$ and $a' < b'$ in $\mathbf{L}$. By P4, the split interval

$$\{\langle x, y\rangle \in \mathbf{L}^2 : \langle x, y\rangle \leq \langle a, a'\rangle \text{ or } \langle b, b'\rangle \leq \langle x, y\rangle\}$$

is closed under $p^{\mathbf{L}^2}$, and by 2.2

$$\{\langle x, y\rangle \in \mathbf{L}^2 : \langle a, a'\rangle \leq \langle x, y\rangle \leq \langle b, b'\rangle\}$$

is closed under $p^{\mathbf{L}^2}$. Therefore, their intersection (which is $\{\langle a, a'\rangle, \langle b, b'\rangle\}$) is closed under $p^{\mathbf{L}^2}$ also. This means that $\{a, b\}$ and $\{a', b'\}$ are closed under p. Let $\mathbf{A}$ be the algebra with universe $\{a, b\}$ and one basic operation p. Let $\mathbf{B}$ be the same with $\{a', b'\}$. Then we have that $\{\langle a, a'\rangle, \langle b, b'\rangle\}$ is a subuniverse of $\mathbf{A} \times \mathbf{B}$. Since $\{\langle a, a'\rangle, \langle b, b'\rangle\}$ is the graph of the bijection $x \to x'$, and since this graph is a subuniverse of $\mathbf{A} \times \mathbf{B}$, the map $x \to x'$ is an isomorphism between $\mathbf{A}$ and $\mathbf{B}$. We have shown that p satisfies P3. □

We now know that in the presence of P1 and P2, P4 and P5 are equivalent and imply P3. We give now an example demonstrating that P3 does not imply P5. Let $\mathbf{L}$ be the three element lattice with elements $0 < a < 1$. Define p on $\mathbf{L}$ by

$$p(x, y, z) = \begin{cases} y \vee z & x = 1 \\ y \wedge z & \text{else.} \end{cases}$$

Then p is idempotent and order preserving, and p satisfies P3 (every sublattice of $\mathbf{L}$ is closed under p). However, since $p(1, 1, a) = 1$ and $p(a, a, 0) = 0$, there is not an order preserving isomorphism from $\langle\{a, 1\}, p\rangle$ to $\langle\{0, a\}, p\rangle$ which P5 would require. Thus p does not satisfy P5.

4. δ-convex Sublattices

Before we can move on to actual interpolation results, we need to address certain sublattices related to distributive lattices. This section will give us a tool (Theorem 4.5) which will eliminate many cases in some proofs to follow in later sections.

Definition 4.1. Suppose that $\mathbf{L}$ is any finite lattice. We will use $\mathcal{MP}(\mathbf{L})$ to represent the set of meet prime elements of $\mathbf{L}$. For each $x \in \mathcal{MP}(\mathbf{L})$, let $\bar{x} = \wedge\{l \in \mathbf{L} : l \not\leq x\}$. If $\mathbf{L}$ is distributive, then $\bar{x}$ is join prime and $\mathbf{L}$ is the disjoint union of the intervals $\{l \in \mathbf{L} : \bar{x} \leq l\}$ and $\{l \in \mathbf{L} : l \leq x\}$.

Lemma 4.2. *Suppose that* $\mathbf{D}$ *is a distributive* 0-1 *sublattice of a finite lattice* $\mathbf{L}$*. Then* $\mathbf{D} = \bigcap_{x \in \mathcal{MP}(\mathbf{D})}\{l \in \mathbf{L} : l \leq x \text{ or } \bar{x} \leq l\}$.

Proof. We will establish that $\mathbf{D} = \cap_{u \in \mathcal{MP}(\mathbf{D})}\{x \in \mathbf{L} : x \leq u \text{ or } \bar{u} \leq x\}$. It should be clear that $\mathbf{D} \subseteq \cap_{u \in \mathcal{MP}(\mathbf{D})}\{x \in \mathbf{L} : x \leq u \text{ or } \bar{u} \leq x\}$. To establish the reverse inclusion, suppose that $y \notin \mathbf{D}$. Let $x = \vee\{l \in \mathbf{D} : l \leq y\}$ (note that this is a nonempty set because we assumed that $\mathbf{D}$ contains 0 and 1). Note that since $x \in \mathbf{D}$ and $y \notin \mathbf{D}$, then $x < y$. Since $\mathbf{D}$ is distributive, we know that x is a meet of elements of $\mathcal{MP}(\mathbf{D})$. If every element of $\mathcal{MP}(\mathbf{D})$ which is above x were also above y, then these elements could not meet to x. Thus there is some $u \in \mathcal{MP}(\mathbf{D})$ so that

$x \leq u$ but $y \not\leq u$. The element $\bar{u}$ cannot be below u by the meet primeness of u, so $\bar{u} \not\leq x$. If $\bar{u}$ were below y, then we would have $x < \bar{u} \vee x \leq y$, which contradicts our definition of x since $\bar{u} \vee x \in \mathbf{D}$. Thus we have that $y \not\leq u$ and $\bar{u} \not\leq y$. Then $y \notin \{x \in \mathbf{L} : x \leq u \text{ or } \bar{u} \leq x\}$. This establishes the desired equality. □

This allows us to quickly conclude that:

Lemma 4.3. *Suppose that $\mathbf{L}$ is a finite lattice and that $p : \mathbf{L}^n \to \mathbf{L}$ is an operation satisfying P1, P2, and P3. Every distributive sublattice of $\mathbf{L}$ is closed under p.*

Proof. Suppose that $\mathbf{D}$ is a distributive sublattice of $\mathbf{L}$. Let $\mathbf{D}' = \mathbf{D} \cup \{0, 1\}$. Then $\mathbf{D}'$ is a distributive 0-1 sublattice of $\mathbf{L}$. By 4.2, $\mathbf{D}'$ is an intersection of split intervals and is closed under p by P3. Also, $\mathbf{D}$ is an interval in $\mathbf{D}'$, so $\mathbf{D}$ is also closed under p by Lemma 2.2. □

Definition 4.4. Suppose that $\mathbf{M}$ is a sublattice of a lattice $\mathbf{L}$ and $\alpha \in \mathrm{Con}\mathbf{L}$. $\mathbf{M}$ is *α-convex* if for all $x, y, z \in \mathbf{L}$ if $x, z \in \mathbf{M}$, $x\alpha z$ and $x < y < z$, then $y \in \mathbf{M}$.

Theorem 4.5. *Suppose that $\mathbf{L}$ is a finite lattice and p is an n-ary operation on $\mathbf{L}$ satisfying P1, P2, and P3. If $\delta \in \mathrm{Con}\mathbf{L}$ so that $\mathbf{L}/\delta$ is distributive, then every δ-convex sublattice of $\mathbf{L}$ is closed under p.*

Proof. Suppose that $\mathbf{M}$ is a δ-convex sublattice of $\mathbf{L}$. Assume first that $\mathbf{M}$ is a 0-1 sublattice of $\mathbf{L}$. We will prove that $\mathbf{M}$ is an intersection of split intervals of the form in P3. Let $\mathbf{D}' = \mathbf{L}/\delta$. Let $\mathbf{D} = \mathbf{M}/\delta$. Note that $\mathbf{D}$ is a 0-1 sublattice of $\mathbf{D}'$ by our assumptions. For any $d \in \mathbf{D}$, define

$$d^{\uparrow} = \vee\{m \in \mathbf{M}; m/\delta = d\} \text{ and}$$
$$d^{\downarrow} = \wedge\{m \in \mathbf{M}; m/\delta = d\}.$$

It is not hard to see using δ-convexity that for any $l \in \mathbf{L}$, $l \in \mathbf{M}$ if and only if $l/\delta \in \mathbf{D}$ and $(l/\delta)^{\downarrow} \leq l \leq (l/\delta)^{\uparrow}$. Also, the map $d \to d^{\downarrow}$ is join preserving while $d \to d^{\uparrow}$ is meet preserving. To see this, suppose that $x \leq y \in \mathbf{D}$. Note that $(x^{\downarrow} \wedge y^{\downarrow})/\delta = (x^{\downarrow})/\delta \wedge (y^{\downarrow})/\delta = x \wedge y = x$. This means that $(x^{\downarrow} \wedge y^{\downarrow})\delta x^{\downarrow}$, so $x^{\downarrow} \leq (x^{\downarrow} \wedge y^{\downarrow})$. Since the reverse inequality automatically holds, $(x^{\downarrow} \wedge y^{\downarrow}) = x^{\downarrow}$ and $x^{\downarrow} \leq y^{\downarrow}$. This implies that $d \to d^{\downarrow}$ is order preserving. Next, let $x, y \in \mathbf{D}$ be arbitrary. Note that $(x^{\downarrow} \vee y^{\downarrow})/\delta = (x^{\downarrow})/\delta \vee (y^{\downarrow})/\delta = x \vee y$, so $x^{\downarrow} \vee y^{\downarrow} \geq (x \vee y)^{\downarrow}$ by definition. The reverse inequality holds by monotonicity, so $x^{\downarrow} \vee y^{\downarrow} = (x \vee y)^{\downarrow}$. That $d \to d^{\uparrow}$ is meet preserving is proven similarly.

Let

$$\mathbf{M}' = \bigcap_{x \in \mathcal{MP}(\mathbf{D})} \{l \in \mathbf{L} : l \leq x^{\uparrow} \text{ or } \bar{x}^{\downarrow} \leq l\}.$$

We will prove that $\mathbf{M} = \mathbf{M}'$. First, let $m \in \mathbf{M}$. Suppose that $x \in \mathcal{MP}(\mathbf{D})$. Either $m/\delta \leq x$ or $m/\delta \geq \bar{x}$. Suppose first that $m/\delta \leq x$. Note that

$$(m \vee x^{\uparrow})/\delta = (m/\delta) \vee (x^{\uparrow}/\delta) = (m/\delta) \vee x = x$$

so that $(m \vee x^{\uparrow}) \leq x^{\uparrow}$ (by definition of $x^{\uparrow}$). This implies that $m \leq x^{\uparrow}$ so that $m \in \{l \in \mathbf{L} : l \leq x^{\uparrow} \text{ or } \bar{x}^{\downarrow} \leq l\}$. Now suppose that $m/\delta \geq \bar{x}$. Note that

$$(m \wedge \bar{x}^{\downarrow})/\delta = (m/\delta) \wedge (\bar{x}^{\downarrow}/\delta) = (m/\delta) \wedge \bar{x} = \bar{x}$$

so that $m \wedge \bar{x}^{\downarrow} \geq \bar{x}^{\downarrow}$. This implies $m \geq \bar{x}^{\downarrow}$ so that $m \in \{l \in \mathbf{L} : l \leq x^{\uparrow} \text{ or } \bar{x}^{\downarrow} \leq l\}$. In either case, $m \in \{l \in \mathbf{L} : l \leq x^{\uparrow} \text{ or } \bar{x}^{\downarrow} \leq l\}$. This is true for all $x \in \mathcal{MP}(\mathbf{D})$ so $m \in \mathbf{M}'$ and $\mathbf{M} \subseteq \mathbf{M}'$.

Now suppose that $m \in \mathbf{M}'$. It follows that

$$\begin{array}{rcl} m/\delta & \in & \bigcap_{x \in \mathcal{MP}(\mathbf{D})} \{l \in \mathbf{D}' : l \leq (x^{\uparrow})/\delta \text{ or } (\bar{x}^{\downarrow})/\delta \leq l\} \\ & = & \bigcap_{x \in \mathcal{MP}(\mathbf{D})} \{l \in \mathbf{D}' : l \leq x \text{ or } \bar{x} \leq l\}. \end{array}$$

Using Lemma 4.2, this last intersection is just $\mathbf{D}$. Thus $m/\delta \in \mathbf{D}$. Let $d \in \mathcal{MP}(\mathbf{D})$. We prove that if $m/\delta \leq d$ then $m \leq d^{\uparrow}$ and if $m/\delta \geq \bar{d}$ then $m \geq \bar{d}^{\downarrow}$. Suppose that $m/\delta \leq d$. Since $m \in \mathbf{M}'$, we know that $m \in \{l \in \mathbf{L} : l \leq d^{\uparrow} \text{ or } \bar{d}^{\downarrow} \leq l\}$. Then either $m \geq \bar{d}^{\downarrow}$ (in which case $m/\delta \geq \bar{d}^{\downarrow}/\delta = \bar{d}$) or $m \leq d^{\uparrow}$ (in which case $m/\delta \leq d^{\uparrow}/\delta = d$). Since it cannot be that both $m/\delta \geq \bar{d}$ and $m/\delta \leq d$, it has to be that $m \leq d^{\uparrow}$. Thus if $m/\delta \leq d$ then $m \leq d^{\uparrow}$. That $m/\delta \geq \bar{d}$ implies $m \geq \bar{d}^{\downarrow}$ is proven similarly.

Now, let $e_1, \ldots, e_m, d_1, \ldots, d_n \in \mathcal{MP}(\mathbf{D})$ so that

$$\bar{e}_1 \vee \bar{e}_2 \vee \ldots \vee \bar{e}_m = m/\delta = d_1 \wedge d_2 \wedge \ldots \wedge d_n.$$

Then for each i and j we have $\bar{e}_i \leq m/\delta \leq d_j$. By the previous paragraph, this implies that $\bar{e}_i{}^{\downarrow} \leq m \leq d_j{}^{\uparrow}$ for each i and j. Then

$$\begin{array}{rcl} (m/\delta)^{\downarrow} & = & (\bar{e}_1 \vee \bar{e}_2 \vee \ldots \vee \bar{e}_m)^{\downarrow} \\ & = & \bar{e}_1{}^{\downarrow} \vee \bar{e}_2{}^{\downarrow} \vee \ldots \vee \bar{e}_m{}^{\downarrow} \\ & \leq & m \\ & \leq & d_1{}^{\uparrow} \wedge d_2{}^{\uparrow} \wedge \ldots \wedge d_n{}^{\uparrow} \\ & = & (d_1 \wedge d_2 \wedge \ldots \wedge d_n)^{\uparrow} \\ & = & (m/\delta)^{\uparrow}. \end{array}$$

Since $m/\delta \in \mathbf{D}$ and since $(m/\delta)^{\downarrow} \leq m \leq (m/\delta)^{\uparrow}$, $m \in \mathbf{M}$. We have proven that $\mathbf{M}' \subseteq \mathbf{M}$ and (hence) $\mathbf{M} = \mathbf{M}'$.

Since $\mathbf{M}$ is an intersection of split intervals, $\mathbf{M}$ is closed under the operation p. We have proven that if $\mathbf{M}$ is a δ-convex 0-1 sublattice of $\mathbf{L}$ then $\mathbf{M}$ is closed under p.

Suppose now that $\mathbf{M}$ is any δ-convex sublattice of $\mathbf{L}$. Let $0_{\mathbf{L}}$ and $1_{\mathbf{L}}$ be the smallest and largest elements of $\mathbf{M}$. Let $\mathbf{L}' = \{l \in \mathbf{L} : l \leq 0 \text{ or } 0_{\mathbf{L}} \leq l\} \cap \{l \in \mathbf{L} : l \leq 1_{\mathbf{L}} \text{ or } 1 \leq l\}$. Then $\mathbf{L}'$ is closed under p by P3. Let $\mathbf{M}' = \mathbf{M} \cup \{0, 1\}$. Then $\mathbf{M}'$ is a δ-convex 0-1 sublattice of $\mathbf{L}'$ and so is closed under p. Also, $\mathbf{M}$ is an interval in $\mathbf{M}'$ and is closed under p by 2.2. $\square$

We now want to state a slight variant of 4.5 which will simplify our wording in proofs later.

Definition 4.6. Suppose that $\mathbf{M} \subseteq \mathbf{M}'$ are sublattices of $\mathbf{L}$ and that δ is a congruence relation on $\mathbf{L}$. We will say that $\mathbf{M}$ is relatively δ-convex in $\mathbf{M}'$ if $\mathbf{M}$ is δ'-convex in $\mathbf{M}'$ where δ' is the restriction of δ to $\mathbf{M}'$.

In this situation, if $\mathbf{L}/\delta$ is distributive, then so is $\mathbf{M}'/\delta'$, so we can apply 4.5 to $\mathbf{M}$ within $\mathbf{M}'$. We then get:

Corollary 4.7. *Let p be an operation on a finite lattice $\mathbf{L}$ satisfying P1, P2, and P3. Let δ be a congruence on $\mathbf{L}$ so that $\mathbf{L}/\delta$ is distributive. Let $\mathbf{M}'$ be a sublattice of $\mathbf{L}$ which is closed under p. If $\mathbf{M}$ is a sublattice of $\mathbf{M}'$ which is relatively δ-convex, then $\mathbf{M}$ is closed under p.*

5. Interpolation in small lattices

We now prove in a few cases that some of the properties P1-P8 completely determine the terms of a finite lattice. We begin with distributive lattices (where most of the work has been done for us by 4.3). Many of the results in this section will state that an operation on a finite lattice is a term operation if and only if the operation satisfies some of the properties P1-P8. We note here that every term operation must satisfy these properties, so we will not need to address the "only if" portion of each proof.

Theorem 5.1. *An operation p on a finite distributive lattice is a term operation if and only if p satisfies the properties P1, P2, and P4.*

Proof. Suppose p is an n-ary operation on a distributive lattice $\mathbf{L}$ satisfying P1, P2, and P4. The operation $p^{\mathbf{L}^2}$ on $\mathbf{L}^2$ induced by p also satisfies P1, P2, and P3. Since every sublattice of $\mathbf{L}^2$ is distributive, every sublattice of $\mathbf{L}^2$ is closed under $p^{\mathbf{L}^2}$ by Lemma 4.3. By Lemma 2.1 this means that p is a term operation of $\mathbf{L}$. $\square$

We use $\mathbf{2}$ to represent the two element lattice $\langle\{0,1\},\cdot,+\rangle$. Since $\mathbf{2}$ only has two elements, every operation on $\mathbf{2}$ satisfies P5 vacuously. Then every idempotent and order preserving operation will also satisfy P4 by 3.4. Therefore:

Corollary 5.2. *The terms of $\mathbf{2}$ are precisely the idempotent order preserving operations on the set $\{0,1\}$.*

Theorem 5.3. *An operation p on $\mathbf{N}_5$ is a term operation if and only if p satisfies the properties P1, P2, and P4.*

Proof. Since p satisfies P4, we can use P3 in $\mathbf{N}_5^2$. Let $e : \mathbf{N}_5 \to \mathbf{N}_5$ be a join endomorphism fixing 0 and let $\mathbf{L} = \{\langle x,y\rangle \in \mathbf{N}_5^2 : e(x) \leq y\}$. Denote the elements of $\mathbf{N}_5$ as $\{0,1,a,b,c\}$ with $c \prec b$ the critical cover. If $e(c) = e(b)$, then $\mathbf{L} = X \cap Y$ where

$$X = \{\langle u,v\rangle : \langle u,v\rangle \geq \langle a,e(a)\rangle \text{ or } \langle u,v\rangle \leq \langle b,1\rangle\} \text{ and}$$
$$Y = \{\langle u,v\rangle : \langle u,v\rangle \geq \langle c,e(c)\rangle \text{ or } \langle u,v\rangle \leq \langle a,1\rangle\}.$$

X and Y are closed under p by P4, so $\mathbf{L}$ is also. Suppose now that $e(c) \neq e(b)$ (so $e(c) < e(b)$). If $e(a) \leq e(b)$ then $\mathbf{L} = X \cap Y$ where

$$X = \{\langle u,v\rangle : \langle u,v\rangle \geq \langle c,e(c)\rangle \text{ or } \langle u,v\rangle \leq \langle a,1\rangle\} \text{ and}$$
$$Y = \{\langle u,v\rangle : \langle u,v\rangle \geq \langle 0,e(a)\rangle \text{ or } \langle u,v\rangle \leq \langle c,1\rangle\}.$$

Again, X and Y are closed under p by P4, so $\mathbf{L}$ is also. Now, if $e(a) > e(b)$, then $\mathbf{L} = X \cap Y \cap Z$ where

$$X = \{\langle u,v\rangle : \langle u,v\rangle \geq \langle a,e(a)\rangle \text{ or } \langle u,v\rangle \leq \langle b,1\rangle\},$$
$$Y = \{\langle u,v\rangle : \langle u,v\rangle \geq \langle 0,e(c)\rangle \text{ or } \langle u,v\rangle \leq \langle 0,1\rangle\}, \text{ and}$$
$$Z = \{\langle u,v\rangle : \langle u,v\rangle \geq \langle 0,e(b)\rangle \text{ or } \langle u,v\rangle \leq \langle c,1\rangle\}.$$

Again, it follows that $\mathbf{L}$ is closed under p.

There is one more case to consider. Suppose that $e(c) < e(b)$ and that $e(a)$ and $e(b)$ are not comparable. Since $e(a)$ and $e(b)$ are not comparable, it has to be that $e(1)$ (which equals $e(a) \vee e(b)$) is neither $e(a)$ nor $e(b)$. If $e(c)$ were comparable to $e(a)$, then $e(a) \vee e(c)$ would be either $e(a)$ – in which case $e(a) \vee e(b) = e(a \vee b) = e(a \vee c) = e(a) \vee e(c) = e(a)$, a contradiction – or $e(c)$ – in which case $e(b) < e(a) \vee e(b) = e(a) \vee e(c) = e(c)$, another contradiction. Thus $e(c)$ is also not comparable to $e(a)$. It has to be that $e(x) = x$ for all $x \in \mathbf{N}_5$. Thus $\mathbf{L} = \{\langle x, y\rangle : x \leq y\}$. This is closed under p since p is order preserving.

By Corollary 2.4, we now have that p is a term operation of $\mathbf{N}_5$. □

Let $\mathbf{D}_1$ be the lattice obtained from the four element Boolean lattice by doubling all of the elements but the largest. If we name the atoms of the four element Boolean lattice a and b, then we can label the elements of $\mathbf{D}_1$ as 1, 0_0, 0_1, a_0, a_1, b_0, and b_1, where $x_0 \prec x_1$ for each $x \in \{0, a, b\}$. Let $\mathbf{D}_2$ be the dual of $\mathbf{D}_1$.

Theorem 5.4. *An operation p on $\mathbf{D}_1$ is a term operation if and only if p satisfies the properties P1, P2, P4, and P6.*

Proof. Suppose that $f : \mathbf{D}_1 \to \mathbf{D}_1$ is a join endomorphism and let $\mathbf{M} = \{\langle x, y\rangle \in \mathbf{D}_1^2 : f(x) \leq y\}$. By 2.5, it is enough to show that $\mathbf{M}$ is closed under p. In this proof, we will make repeated use of 4.5 and 4.7. There is a natural homomorphism from $\mathbf{D}_1$ onto the four element Boolean lattice which collapses the doubling. Let α be the kernel of this homomorphism. Let $\delta = \alpha \times \alpha$, a congruence on $\mathbf{D}_1^2$. There are only three classes of δ which contain more than two elements. These are of the form $\{\langle x_0, x_0\rangle, \langle x_0, x_1\rangle, \langle x_1, x_0\rangle, \langle x_1, x_1\rangle\}$ for $x \in \{0, a, b\}$. It is important to observe that

> $\mathbf{M}$ is not δ-convex if and only if $\mathbf{M}$ contains $\langle x_0, x_0\rangle$, $\langle x_0, x_1\rangle$, and $\langle x_1, x_1\rangle$ but not $\langle x_1, x_0\rangle$ for at least one $x \in \{0, a, b\}$.

Suppose that $x \in \{0, a, b\}$ and that $\mathbf{M}$ contains $\{\langle x_0, x_0\rangle, \langle x_0, x_1\rangle, \langle x_1, x_1\rangle\}$ but not $\langle x_1, x_0\rangle$. Now, $f(x_1) \leq x_1$ by the definition of $\mathbf{M}$. If $f(x_1) \neq x_1$, then $f(x_1) \leq x_0$. This would force $\langle x_1, x_0\rangle \in \mathbf{M}$, contrary to our assumption. Therefore it must be that $f(x_1) = x_1$. Also, from the defintion of $\mathbf{M}$, note that $f(x_0) \leq x_0$. It follows that

> $\mathbf{M}$ is not δ-convex if and only if for some $x \in \{0, a, b\}$ $f(x_1) = x_1$ and $f(x_0) \leq x_0$.

We proceed now by cases on $f(0_1)$. In almost every instance, we will be able to refer to 4.5. Suppose first that $f(0_1) = 1$. It follows that $f(a_1) = f(b_1) = f(1) = 1$, so $\mathbf{M}$ is δ-convex and is closed under p. Assume next that $f(0_1) = 0_0$. It follows that $f(a_1) = f(a_0 \vee 0_1) = f(a_0) \vee f(0_1) = f(a_0)$ and that $f(b_1) = f(b_0) \vee f(0_1) = f(b_0)$, so $\mathbf{M}$ is again δ-convex.

Assume now that $f(0_1) = a_0$. Then $f(b_1) \geq a_0$, so $f(b_1) \neq b_1$. If $f(a_1) \neq a_1$, then $\mathbf{M}$ is δ-convex. Assume then that $f(a_1) = a_1$. Since

$$a_1 = f(a_1) = f(a_0) \vee f(0_1) = f(a_0) \vee a_0,$$

it must be that $f(a_0) = a_1$. But then $\mathbf{M}$ is δ-convex. The case where $f(0_1) = b_0$ is similar.

Assume next that $f(0_1) = a_1$. Now, $f(a_1)$ and $f(b_1)$ must be at least as large as $f(0_1) = a_1$, so $f(a_1)$ and $f(b_1)$ each must be 1 or a_1. If $f(a_1) = 1$, then $\mathbf{M}$ is δ-convex by the comments above, so assume that $f(a_1) = a_1$. Since $f(a_0) \leq f(a_1)$, it has to be that $f(a_0)$ is 0_0, 0_1, a_0, or a_1. If $f(a_0) = a_1$ or $f(a_0) = 0_1$, then $\mathbf{M}$ is δ-convex. If $f(a_0) = 0_0$, then

$$f(b_0) = f(b_0) \vee f(a_0) = f(b_0 \vee a_0) = f(b_1 \vee a_0) = f(b_1) \vee f(a_0) = f(b_1).$$

This implies that $f(b_0) \geq f(0_1) = a_1$. In this case, $\mathbf{M}$ is contained completely inside of $\mathbf{M}' = \{\langle x, y\rangle \in \mathbf{D}_1^2 : \langle x, y\rangle \leq \langle a_0, 1\rangle \text{ or } \langle 0_0, a_1\rangle \leq \langle x, y\rangle\}$. Note that $\mathbf{M}'$ is closed under p by P3 (as a consequence of P5). Also, since $\mathbf{M}'$ does not contain $\langle a_1, a_0\rangle$, $\mathbf{M}$ is a relatively δ-convex sublattice of $\mathbf{M}'$ and is closed under p by 4.7. Now suppose that $f(a_0) = a_0$. If $f(b_0) \leq a_0$, then $f(1) = f(a_0) \vee f(b_0) = a_0$. But then $f(a_1) = a_0$ since $f(a_0) \leq f(a_1) \leq f(1)$. This contradicts the assumption that $f(a_1) = a_1$, so it cannot be that $f(b_0) \leq a_0$. If $f(b_0) \geq 0_1$, then $\mathbf{M}$ is a relatively δ-convex sublattice of $\mathbf{M}'' = \{\langle x, y\rangle \in \mathbf{D}_1^2 : \langle x, y\rangle \leq \langle a_0, 1\rangle \text{ or } \langle 0, 0_1\rangle \leq \langle x, y\rangle\}$. If $f(b_0) = b_0$, then $\mathbf{M}$ is contained completely inside of $\{\langle x, y\rangle \in \mathbf{D}_1^2 : x \leq y\}$ – which we assumed to be closed under p. Then $\mathbf{M}$ is a relatively δ-convex sublattice of $\{\langle x, y\rangle \in \mathbf{D}_1^2 : x \leq y\}$ and is closed under p by 4.7. This concludes the case when $f(0_1) = a_1$. The case where $f(0_1) = b_1$ is similar.

Finally, assume that $f(0_1) = 0_1$. If $f(a_0) \geq 0_1$, then $\mathbf{M}$ is relatively δ-convex within the sublattice $\{\langle x, y\rangle \in \mathbf{D}_1^2 : \langle x, y\rangle \leq \langle b_0, 1\rangle \text{ or } \langle 0_0, 0_1\rangle \leq \langle x, y\rangle\}$ which is closed under p. So assume that $f(a_0) \not\geq 0_1$. Similarly, we assume that $f(b_0) \not\geq 0_1$. If $f(a_0) \leq f(b_0)$ then

$$f(b_0) = f(b_0) \vee f(a_0) = f(b_0 \vee a_0) = f(b_1) \geq 0_1$$

contrary to our assumption, so $f(a_0) \not\leq f(b_0)$. Similarly, $f(b_0) \not\leq f(b_0)$. We have then, that $\{f(a_0), f(b_0)\} = \{a_0, b_0\}$. If $f(a_0) = a_0$, then $\mathbf{M} = \{\langle x, y\rangle \in \mathbf{D}_1^2 : x \leq y\}$ – which is closed under p. If $f(a_0) = b_0$, then f is the unique non-identity automorphism of $\mathbf{D}_1$ and $\mathbf{M}$ is closed under p by P6 and 2.8. $\square$

Suppose that $\mathbf{L}$ and $\mathbf{L}^\partial$ are dual lattices with the same universe. Then the terms of $\mathbf{L}$ are identical to the terms of $\mathbf{L}^\partial$. Since $\mathbf{D}_2$ is the dual of $\mathbf{D}_1$, this theorem gives us:

Theorem 5.5. *An operation p on $\mathbf{D}_2$ is a term operation if and only if p satisfies the properties P1, P2, P4, and P6.*

We now address terms on the modular lattices $\mathbf{M}_N$ where N is a positive integer.

Theorem 5.6. *An operation p on $\mathbf{M}_N$ is a term operation if and only if p satisfies the properties P1, P2, P4, P6, and P8.*

Proof. For $N = 1, 2$, this follows from 5.1, so assume that $N \geq 3$. Let $\mathbf{M}_N = \{0, a_1, a_2, \dots, a_N, 1\}$ with 0 and 1 being the minimal and maximal elements. Let $e : \mathbf{M}_N \to \mathbf{M}_N$ be any join endomorphism fixing 0 and let $\mathbf{L} = \{\langle x, y\rangle \in \mathbf{M}_N^2 : e(x) \leq y\}$. Let $\mathcal{C}$ be the clone of all operations on $\mathbf{M}_N$ satisfying P1, P2, P4, P6, and P8 (the reader should check that this is actually a clone). Assume by way of contradiction that $\mathbf{L}$ is not closed under the operations in $\mathcal{C}$. Let $\langle x, y\rangle$ be a minimal element of $\mathbf{M}_N^2$ for which there exists an operation $p : \mathbf{M}_N^n \to \mathbf{M}_N$ in $\mathcal{C}$ and elements $\langle r_1, s_1\rangle, \dots, \langle r_n, s_n\rangle$ in $\mathbf{L}$ so that $\langle x, y\rangle = p(\langle r_1, s_1\rangle, \dots, \langle r_n, s_n\rangle)$ but $\langle x, y\rangle \notin \mathbf{L}$. Since $\{0\} \times \mathbf{M}_N \subseteq \mathbf{L}$, it cannot be that $x = 0$. For each i, $\langle a_i, 1\rangle \in \mathbf{L}$. The operation $q(x_1, \dots, x_{n+1}) = p(x_1, \dots, x_n) \wedge x_{n+1}$ is in $\mathcal{C}$. If $x = 1$, then the element $\langle a_i, y\rangle = \langle x, y\rangle \wedge \langle a_i, 1\rangle = q(\langle r_1, s_1\rangle, \dots, \langle r_n, s_n\rangle, \langle a_i, 1\rangle)$ is strictly less than $\langle x, y\rangle$ and can be obtained by applying an operation in $\mathcal{C}$ to elements of $\mathbf{L}$. Hence, $\langle a_i, y\rangle \in \mathbf{L}$ for $i = 1, 2, \dots, N$ by the minimality of $\langle x, y\rangle$. But then we would have the contradiction that $\langle x, y\rangle = \langle a_1, y\rangle \vee \langle a_2, y\rangle \in \mathbf{L}$. Thus it cannot be that $x = 1$. It has to be that x is an atom of $\mathbf{M}_N$. Without loss of generality, assume that $x = a_1$.

We now proceed by cases on $e(1)$. If $e(1) = 0$, then $\langle a_1, 0\rangle = \langle 1, 0\rangle \wedge \langle a_1, 1\rangle \in \mathbf{L}$. This would imply that $\langle a_1, y\rangle \in \mathbf{L}$, a contradiction, so $e(1) \neq 0$. Suppose next that $e(1) = a_k$ for some $k \in \{1, 2, \dots, N\}$. This means that $e(a_i) \leq e(1) = a_k$ for each i, so each $e(a_i)$ is either a_k or is 0. If $e(a)_i = e(a)_j = 0$ and $i \neq j$, then $e(1) = e(a_i) \vee e(a_j) = 0$, so at most one of the $e(a_i)$'s is 0. The others must equal a_k. Note that $y \neq a_k$ since then we would have $\langle a_1, y\rangle = \langle a_1, 1\rangle \wedge \langle 1, a_k\rangle \in \mathbf{L}$. Note also that $e(a_1) \neq 0$ as this would also imply $\langle a_1, y\rangle \in \mathbf{L}$. Thus y must be incomparable to a_k. If $e(a_j) = 0$ for one of the a_j's, then

$$\mathbf{L} \subseteq \{\langle u, v\rangle : \langle u, v\rangle \geq \langle 0, a_k\rangle \text{ or } \langle u, v\rangle \leq \langle a_j, 1\rangle\}.$$

By P4 the righthand side of this inclusion is closed under $\mathcal{C}$, but this sublattice does not contain $\langle x, y\rangle$. This would make it impossible for $\langle x, y\rangle = p(\langle r_1, s_1\rangle, \dots, \langle r_n, s_n\rangle)$ as we have assumed. If $e(a_j) \neq 0$ for all j, then

$$\mathbf{L} \subseteq \{\langle u, v\rangle : \langle u, v\rangle \geq \langle 0, a_k\rangle \text{ or } \langle u, v\rangle \leq \langle 0, 1\rangle\}.$$

As in the previous case, this would contradict our choice of $\langle x, y\rangle$. Thus if $e(1)$ is an atom we arrive at a contradiction.

We have established that $e(1)$ cannot be 0 and cannot be an atom. Assume now that $e(1) = 1$. Note that $e(a_1) \neq 0$ as this would force $\langle a_1, y\rangle \in \mathbf{L}$. Also, as before, the fact that e is a join endomorphism prevents there being two atoms a_i and a_j with $e(a_i) = e(a_j) = 0$ as this would force $e(1) = 0$. If $e(a_i) = 0$ for some i, then since $1 = e(1) = e(a_j \vee a_i) = e(a_j) \vee (a_i)$ for $j \neq i$, it has to be that $e(a_j) = 1$ for all $j \neq i$. But then $\mathbf{L}$ is contained completely inside the sublattice

$$\{\langle u, v\rangle : \langle u, v\rangle \geq \langle 0, 1\rangle \text{ or } \langle u, v\rangle \leq \langle a_i, 1\rangle\}.$$

As above, this would contradict our choice of $\langle x, y\rangle$. Thus, $e(a_i) \neq 0$ for all i. This means that $\{e(a_1), e(a_2), \dots e(a_N)\} \subseteq \{1, a_1, a_2, \dots, a_N\}$. Furthermore, for $i \neq j$, it cannot be that $e(a_i) = e(a_j)$ unless both of these are 1, since $1 = e(1) = e(a_i) \vee e(a_j)$. Define $\alpha : \mathbf{M}_N \to \mathbf{M}_N$ recursively in the following way. Let $\alpha(0) = 0$ and $\alpha(1) = 1$. If $e(a_1)$ is an atom, then let $\alpha(a_1) = e(a_1)$. Otherwise, let $\alpha(a_1)$ be the first of $a_1, a_2, \dots, a_N$ which is not in $\{e(a_2), e(a_3), \dots, e(a_N)\}$. Now, assuming that $\alpha(a_l)$ has been defined, define $\alpha(a_{l+1})$ to be $e(a_{l+1})$ if this is an atom. Otherwise, let $\alpha(a_{l+1})$ be the first of $a_1, a_2, \dots, a_N$ which is not in $\{e(a_1), \dots, e(a_l), e(a_{l+2}), \dots, e(a_N)\}$. Then α is an automorphism of $\mathbf{M}_N$ with

$\alpha \leq e$ and $\mathbf{L} \subseteq \{\langle u,v\rangle : \alpha(u) \leq v\}$. We will argue that $\langle a_1, y\rangle \notin \{\langle u,v\rangle : \alpha(u) \leq v\}$. This will give a contradiction since p preserves automorphisms. Now

$$\begin{array}{rcl}\langle a_1, y\rangle & = & p(\langle r_1, s_1\rangle, \ldots, \langle r_n, s_n\rangle) \\ & = & \langle p(r_1, \ldots, r_n), p(s_1, \ldots, s_n)\rangle \\ & \geq & \langle p(r_1, \ldots, r_n), p(e(r_1), \ldots, e(r_n))\rangle \\ & = & \langle a_1, p(e(r_1), \ldots, e(r_n))\rangle\end{array}$$

where the inequality follows from the fact that each $\langle r_i, s_i\rangle \in \mathbf{L}$. Denote this last element as $\langle a_1, z\rangle$. Now, $\langle a_1, z\rangle$ is obtained from $\mathbf{L}$ by applying an operation from $\mathcal{C}$. If $z < y$, then by the minimality of $\langle a_1, y\rangle$, $\langle a_1, z\rangle \in \mathbf{L}$, but this would force $y \geq z \geq e(a_1)$, so $\langle a_1, y\rangle \in \mathbf{L}$. This contradiction implies that $z = y$, so $y = p(e(r_1), \ldots, e(r_n))$. Let $\mathbf{M}$ be the sublattice of $\mathbf{M}_N$ generated by $\{e(r_1), \ldots, e(r_n)\}$. Now by our assumption of P8 and by our selection of y, it must be that $y \in \mathbf{M}$. Since $y \neq 1$ (because $\mathbf{M}_N \times \{1\} \subseteq \mathbf{L}$) this implies that either y is 0 or that $y = a_j$ for some $j = 1, 2, \ldots N$ where $a_j = e(a_i)$ for some $i > 1$ (note $e(a_1) \neq y = a_j$). If $y = 0$, then $\langle a_1, y\rangle = \langle a_1, 0\rangle \notin \{\langle u,v\rangle : \alpha(u) \leq v\}$ as desired (since α maps a_1 to an atom). If $y = a_j$ and $a_j = e(a_i)$ for some $i > 1$, then $\alpha(a_i)$ must be a_j and $\alpha(a_1)$ is an atom a_k other than a_j. This means that $\alpha(a_1) \not\leq y$, so $\langle a_1, y\rangle \notin \{\langle u,v\rangle : \alpha(u) \leq v\}$ as desired.

We now have that $\langle x, y\rangle$ is obtained from $\mathbf{L}$ by applying an operation from $\mathcal{C}$ and that $\mathbf{L} \subseteq \{\langle u,v\rangle : \alpha(u) \leq v\}$. However, since the operations in $\mathcal{C}$ satisfy P6, $\{\langle u,v\rangle : \alpha(u) \leq v\}$ is closed under these operations by 2.8. Thus we have a contradiction in the final case where $e(1) = 1$.

We assumed that $\mathbf{L} = \{\langle a,b\rangle : e(a) \leq b\}$ was not closed under the operations of $\mathcal{C}$. We then considered cases on $e(1)$ being 0, an atom, or 1. In each case we arrived at a contradiction. Therefore, the assumption that $\mathbf{L}$ is not closed under the operations in $\mathcal{C}$ must be false. By 2.5, this implies that every operation in $\mathcal{C}$ is a term of $\mathbf{M}_N$ and completes the proof that $\mathrm{Clo}\mathbf{M}_N = \mathcal{C}$. $\square$

If we take $N = 3$ in this theorem, then every proper sublattice of $\mathbf{M}_3$ is distributive, so we get property P8 for free:

Theorem 5.7. *An operation p on $\mathbf{M}_3$ is a term operation if and only if p satisfies the properties P1, P2, P4, and P6.*

We can now use 5.3, 5.4, 5.5, and 5.7 to characterize terms of finite lattices in the varieties generated by $\mathbf{N}_5$, $\mathbf{D}_1$, $\mathbf{D}_2$, and $\mathbf{M}_3$.

Theorem 5.8. *Let $\mathbf{M}$ be $\mathbf{N}_5$, $\mathbf{D}_1$, $\mathbf{D}_2$ or $\mathbf{M}_3$. Suppose that $\mathbf{L}$ is a finite lattice in the variety generated by $\mathbf{M}$. The terms of $\mathbf{L}$ are precisely those operations on $\mathbf{L}$ satisfying P1, P2, P4, and P7.*

Proof. Suppose that $\mathbf{L}$ is a finite lattice in the variety generated by $\mathbf{M}$ and that p is an operation on $\mathbf{L}$ satisfying P1, P2, P4, and P7. If $\mathbf{L}$ is distributive then p is a term operation of $\mathbf{L}$ by 5.1. Suppose then that $\mathbf{L}$ is not distributive. Then $\mathbf{L}$ contains a sublattice isomorphic to one of $\mathbf{N}_5$, $\mathbf{D}_1$, $\mathbf{D}_2$, or $\mathbf{M}_3$. If $\mathbf{L}$ contains a copy of $\mathbf{D}_1$ or $\mathbf{D}_2$ (note that it cannot have both), then let $\mathbf{N}$ be this sublattice.

Otherwise, if $\mathbf{L}$ has a copy of $\mathbf{N}_5$ or $\mathbf{M}_3$ (again, it can only have one), then let $\mathbf{N}$ be this sublattice.

First, note that $\mathbf{L}$ is in the variety generated by $\mathbf{N}$. This is obvious if $\mathbf{N} \cong \mathbf{M}$. The way in which this might not happen is if $\mathbf{M}$ is $\mathbf{D}_i$ for $i = 1$ or 2 and $\mathbf{N} \cong \mathbf{N}_5$. In this case, the only subdirectly irreducibles in the variety generated by $\mathbf{M}$ are $\mathbf{D}_i$, $\mathbf{N}_5$ and $\mathbf{2}$. $\mathbf{D}_i$ is not in the variety generated by $\mathbf{L}$ because every subdirectly irreducible in the variety generated by $\mathbf{L}$ is in $\mathrm{HS}(\mathbf{L})$. No such lattice is isomorphic to $\mathbf{D}_i$ because $\mathbf{D}_i$ is finitely projective and $\mathbf{L}$ contains no sublattice isomorphic to $\mathbf{D}_i$. Then the only subdirectly irreducibles in the variety generated by $\mathbf{L}$ are $\mathbf{N}_5$ and $\mathbf{2}$, and $\mathbf{L}$ is in the variety generated by $\mathbf{N}$.

Since $\mathbf{L}$ is in the variety generated by $\mathbf{N}$, there are homomorphisms $e_1, \ldots, e_n : \mathbf{L} \to \mathbf{N}$ whose product is an injection of $\mathbf{L}$ into $\mathbf{N}^n$. Note that each e_i is an endomorphism of $\mathbf{L}$ and must preserve p. Since $\mathbf{N}$ is subdirectly irreducible and since the homomorphisms $e_1, \ldots, e_n$ separate points in $\mathbf{N}$, it has to be that one of the e_i's is injective on $\mathbf{N}$ – that is, e_i restricts to an automorphism of $\mathbf{N}$. By composing e_i with its inverse on $\mathbf{N}$, we have a homomorphism f from $\mathbf{L}$ to $\mathbf{N}$ which is the identity on $\mathbf{N}$.

Let $x_1, \ldots, x_n \in \mathbf{N}$. Note that

$$p(x_1, \ldots, x_n) = p(f(x_1), \ldots, f(x_n)) = f(p(x_1, \ldots, x_n)) \in \mathbf{N}.$$

Thus, $\mathbf{N}$ is closed under p. The operation p will satisfy P1, P2, and P4 on $\mathbf{N}$ automatically. To address P7, suppose that $g : \mathbf{N} \to \mathbf{N}$ is an endomorphism. Then $g \circ f$ is an endomorphism of $\mathbf{L}$, so p is preserved by $g \circ f$. But the restriction of $g \circ f$ to $\mathbf{N}$ is just g, so g must preserve p on $\mathbf{N}$. Since p satisfies P1, P2, P4, and P7 on $\mathbf{N}$, there is a lattice term T so that $p = T$ in $\mathbf{N}$ by 5.3, 5.4, 5.5, or 5.7.

We now show that $p = T$ in all of $\mathbf{L}$. Suppose that $x_1, \ldots, x_n \in \mathbf{L}$. Then for $i = 1, 2, \ldots, n$,

$$\begin{aligned} e_i(p(x_1, \ldots, x_n)) &= p(e_i(x_1), \ldots, e_i(x_n)) \\ &= T(e_i(x_1), \ldots, e_i(x_n)) \\ &= e_i(T(x_1, \ldots, x_n)). \end{aligned}$$

Since this is true for each e_i, and since $e_1, \ldots, e_n$ separate points in $\mathbf{L}$, it has to be that $p(x_1, \ldots, x_n) = T(x_1, \ldots, x_n)$. $\square$

6. Interpolation modulo certain congruences

In this section, we prove that if p is an operation on a finite lattice satisfying P1, P2, and P5, then p is equal to a term of $\mathbf{L}$ modulo any congruence α on $\mathbf{L}$ for which $\mathbf{L}/\alpha$ is distributive.

Lemma 6.1. *Suppose that $\mathbf{L}$ is a finite lattice and $\alpha \in \mathrm{Con}\mathbf{L}$ so that $\mathbf{L}/\alpha$ is a two element lattice. Every operation on $\mathbf{L}$ satisfying P1, P2, and P5 preserves α.*

Proof. Let $f : \mathbf{L} \to \mathbf{L}$ be an endomorphism of $\mathbf{L}$ so that the range of f is $\{0, 1\}$ and so that $\ker f = \alpha$. Let p be an n-ary operation on $\mathbf{L}$ satisfying P1, P2, and P5. We will prove that f preserves p. There are $a, b \in \mathbf{L}$ so that

$$f(x) = \begin{cases} 0 & x \leq a \\ 1 & x \geq b \end{cases}$$

Let $x_1, \ldots, x_n \in \mathbf{L}$. We will proceed by cases on $p(f(x_1), \ldots, f(x_n)) = 0, 1$. Assume first that $p(f(x_1), \ldots, f(x_n)) = 1$. For each i, let $y_i = b$ if $x_i \geq b$ and let $y_i = 0$ otherwise. By P5, $\langle \{0,1\}, p \rangle \cong \langle \{0, b\}, p \rangle$. Also, $y_i = b$ if and only if $f(x_i) = 1$, so $p(y_1, \ldots, y_n) = b$ by P5. Therefore, $p(x_1, \ldots, x_n) \geq p(y_1, \ldots, y_n) = b$ and $f(p(x_1, \ldots, x_n)) = 1 = p(f(x_1), \ldots, f(x_n))$.

Assume now that $p(f(x_1), \ldots, f(x_n)) = 0$. For each i, let $z_i = 1$ if $x_i \geq b$ and let $z_i = a$ if $x_i \leq a$. Since $p(f(x_1), \ldots, f(x_n)) = 0$, P5 gives us in this case that $p(z_1, \ldots, z_n) = a$. Then $p(x_1, \ldots, x_n) \leq p(z_1, \ldots, z_n) = a$ and $f(p(x_1, \ldots, x_n)) = 0 = p(f(x_1), \ldots, f(x_n))$. In either case, f preserves p, so p preserves $\alpha = \ker f$. $\square$

We can use Lemma 6.1 to step up to homomorphisms onto any distributive lattice.

Corollary 6.2. *Suppose that* $\mathbf{L}$ *is a finite lattice and* $\alpha \in \mathrm{Con}\mathbf{L}$ *so that* $\mathbf{L}/\alpha$ *is distributive. Every operation on* $\mathbf{L}$ *satisfying P1, P2, and P5 preserves* α.

Proof. Let p be an n-ary operation on $\mathbf{L}$ satisfying P1, P2, and P5. Suppose by way of contradiction that p does not preserve α. Then there exist elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ so that $x_i \alpha y_i$ for all i but $p(x_1, \ldots, x_n)$ is not α-related to $p(y_1, \ldots, y_n)$. Suppose that f is a surjective homomorphism from $\mathbf{L}$ to a distributive lattice $\mathbf{D}$ with $\ker f = \alpha$. Let $a = f(p(x_1, \ldots, x_n))$ and let $b = f(p(y_1, \ldots, y_n))$. There is a homomorphism $g : \mathbf{D} \to \mathbf{2}$ so that $g(a) \neq g(b)$. It follows that p does not preserve $\ker(g \circ f)$. This is contrary to Lemma 6.1, so our assumption that p does not preserve α must be false. $\square$

Corollary 6.2 along with 5.1 will help us to establish the interpolation result we desire modulo certain congruences.

Corollary 6.3. *Suppose that* $\mathbf{L}$ *is a finite lattice and* $\alpha \in \mathrm{Con}\mathbf{L}$ *so that* $\mathbf{L}/\alpha$ *is distributive. Suppose that* p *is an* n*-ary operation on* $\mathbf{L}$ *satisfying P1, P2, and P5. There is an* n*-ary lattice term* T *so that* $p(x_1, \ldots, x_n) \alpha T(x_1, \ldots, x_n)$ *for all* $x_1, \ldots, x_n \in \mathbf{L}$.

Proof. Let f be a surjective homomorphism from $\mathbf{L}$ to $\mathbf{D}$ with $\mathbf{D}$ distributive and $\ker f = \alpha$. Suppose that $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{L}$ with $x_i \alpha y_i$ for all i. By the previous corollary, $f(p(x_1, \ldots, x_n)) = f(p(y_1, \ldots, y_n))$. This observation allows us to define an operation $\hat{p}$ on $\mathbf{D}$ in the following way. Suppose $a_1, \ldots, a_n \in \mathbf{D}$. Let $x_1, \ldots, x_n \in \mathbf{L}$ with $f(x_i) = a_i$ for each i. Let $\hat{p}(a_1, \ldots, a_n) = f(p(x_1, \ldots, x_n))$. By the previous observation, the choice of $\hat{p}(a_1, \ldots, a_n)$ is independent of the choice of the x_i's, so this operation is well-defined.

We claim that $\hat{p}$ satisfies P1, P2, and P5. P2 (idempotence) is trivial. To establish P1 and P5 we need to know this fact: for any $a < b$ in $\mathbf{D}$ there exist $x < y$ in $\mathbf{L}$ with $f(x) = a$ and $f(y) = b$. Indeed, suppose that $a < b$ in $\mathbf{D}$. Let x be the least element of $\mathbf{L}$ which f maps to a, and let y be the largest element of $\mathbf{L}$ which

f maps to b. Consider then $f(x \wedge y) = f(x) \wedge f(y) = a \wedge b = a$. Since $f(x \wedge y) = a$, we know that $(x \wedge y)\alpha x$ so $x \leq (x \wedge y)$ by the choice of x. Since $(x \wedge y) \leq x$ also, we have that $x = (x \wedge y)$. This implies that $x < y$.

Now suppose that $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbf{D}$ with $a_i \leq b_i$ for each i. Select $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{L}$ with $x_i \leq y_i$, $f(x_i) = a_i$, and $f(y_i) = b_i$ for all i. Then, since p satisfies P1 (monotonicity), we have

$$\hat{p}(a_1, \ldots, a_n) = f(p(x_1, \ldots, x_n)) \leq f(p(y_1, \ldots, y_n)) = \hat{p}(b_1, \ldots, b_n).$$

Thus $\hat{p}$ satisfies P1.

Suppose now that $a < b$ and $a' < b'$ in $\mathbf{D}$. Find $x < y$ and $x' < y'$ in $\mathbf{L}$ so that $f(x) = a$, $f(y) = b$, $f(x') = a'$ and $f(y') = b'$. Let $d_1, \ldots, d_n \in \{a, b\}$ and let $u_1, \ldots, u_n \in \{x, y\}$ with $f(u_i) = d_i$. Select d_i' and u_i' similarly. Since $\hat{p}(d_1, \ldots, d_n) = f(p(u_1, \ldots, u_n))$, and since $\{x, y\}$ is closed under p, we know that $\hat{p}(d_1, \ldots, d_n) \in \{a, b\}$. Thus $\{a, b\}$ is closed under $\hat{p}$. The same holds for $\{a', b'\}$. Furthermore, f is an isomorphism between $\langle\{x, y\}, p\rangle$ and $\langle\{a, b\}, \hat{p}\rangle$. Therefore

$$\langle\{a, b\}, \hat{p}\rangle \cong \langle\{x, y\}, p\rangle \cong \langle\{x', y'\}, p\rangle \cong \langle\{a', b'\}, \hat{p}\rangle.$$

Since these isomorphisms are all order preserving, $\hat{p}$ satisfies P5.

Now, since $\mathbf{D}$ is distributive and since $\hat{p}$ is an n-ary operation on $\mathbf{D}$ satisfying P1, P2, and P5, then by 5.1 we know that there is an n-ary lattice term T so that $\hat{p} = T$ on $\mathbf{D}$. It follows then that $p\alpha T$ in $\mathbf{L}$. □

7. Our Motivation: Hereditary Congruence Lattices

The notions of congruence heredity and congruence power-heredity were introduced by Hegedűs and Pálfy in [6]. The congruence lattice $\mathbf{L}$ of a finite algebra $\mathbf{A}$ is hereditary if every 0-1 sublattice of $\mathbf{L}$ is the congruence lattice of an algebra with the same universe as $\mathbf{A}$. $\mathbf{L}$ is power-hereditary if every 0-1 sublattice of $\mathbf{L}^n$ is the congruence lattice of an algebra with the same universe as $\mathbf{A}^n$. Note here that the "is" in the definition of congruence (power-)heredity is a literal "is" not "is isomorphic to." We mean that these lattices of equivalence relations are precisely the equivalence relations compatible with an algebra on the underlying set. In [6], Hegedűs and Pálfy characterized which finite, Abelian, prime power ordered groups have (power-)hereditary congruence lattices. In [12], it was proved that every representation of $\mathbf{N}_5$ as the congruence lattice of a finite algebra is power-hereditary. In [10], Pálfy proved that the lattice $\mathbf{M}_3$ does not share this property by giving an example of a finite algebra whose congruence lattice is isomorphic to $\mathbf{M}_3$ but is not power-hereditary. To date, there is no known congruence lattice representation of $\mathbf{M}_4$ which is even hereditary. In [14] it was proved that if a finite algebra $\mathbf{A}$ satisfies a nontrivial idempotent Maltsev condition, and if Con$\mathbf{A}$ contains a copy of an order polynomially complete lattice other than a select few, then Con$\mathbf{A}$ is not hereditary.

Congruence heredity is intimately related to interpolation of certain operations on congruence lattices. To explain this in full, we need to discuss primitive positive definitions. A *primitive positive formula* is a formula of the form $\exists \wedge$ (atomic). If Φ

is a primitive positive formula employing binary relation symbols $r_1, \ldots, r_n$ and if Φ has two free variables, then Φ naturally induces an operation on the set of binary relations of any set. If $\theta_1, \ldots, \theta_n$ are binary relations on a set A, then we will use $\Phi(\theta_1, \ldots, \theta_n)$ to represent the binary relation on A defined by interpreting each r_i in Φ as θ_i. The operation $\langle \theta_1, \ldots, \theta_n \rangle \mapsto \Phi(\theta_1, \ldots, \theta_n)$ is order preserving, and when it is applied to products of relations can be applied coordinate-wise. Closure under primitive positive definitions characterizes those lattices of equivalence relations that are congruence lattices.

Lemma 7.1. ([15] Corollary 2.2) *Suppose* $\mathbf{L}$ *is a* 0-1 *lattice of equivalence relations on a finite set* A. *There is an algebra* $\mathbf{A}$ *on* A *with* $\mathrm{Con}\mathbf{A} = \mathbf{L}$ *if and only if every equivalence relation on* A *which can be defined from* $\mathbf{L}$ *by a primitive positive formula is already in* $\mathbf{L}$.

We will assume from here on that **every primitive positive formula only contains binary relation symbols and has exactly two free variables**. Suppose that Φ is any such primitive positive formula and that $r_1, \ldots, r_n$ are the relations symbols in Φ (or are relations interpreted as the symbols in Φ). Let $x_1, \ldots, x_m$ be the variables in Φ. By the *graph of* $\Phi(r_1, \ldots, r_n)$ we will mean the undirected graph $\mathbf{G}$ with vertices $\{x_1, \ldots, x_m\}$ so that for each occurrence of $r_i(x_j, x_k)$ in Φ, there is an edge in $\mathbf{G}$ labelled by r_i between x_j and x_k. A primitive positive formula Φ will be called *connected* if the corresponding graph is connected. If a primitive positive formula Φ is not connected, then its value is completely determined by the component containing the free variables. If the two free variables are not contained in the same component, then the formula can only define the universal relation. Thus in Lemma 7.1, **it is sufficient to consider connected primitive positive formulas**. In light of this, we will assume that **all primitive positive formulas are connected**. Primitive positive definitions of this sort were considered in [17] where they were called graphical compositions. That paper includes a Theorem (2.6) which is equivalent to 7.1.

The relationship between congruence heredity and power-heredity and interpolation are:

Lemma 7.2. (Lemma 2.3 of [13]) *The congruence lattice of a finite algebra* $\mathbf{A}$ *is hereditary if and only if for every primitive positive formula* $\Phi(x_1, \ldots, x_n)$ *and for all* $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ *if* $\Phi(r_1, \ldots, r_n)$ *is an equivalence relation, then there is a lattice term* $T(x_1, \ldots, x_n)$ *so that* $T(r_1, \ldots, r_n) = \Phi(r_1, \ldots, r_n)$.

Lemma 7.3. (See [6] Lemma 4.5) *The congruence lattice of a finite algebra* $\mathbf{A}$ *is power-hereditary if and only if for every primitive positive formula* $\Phi(x_1, \ldots, x_n)$ *there is a lattice term* $T(x_1, \ldots, x_n)$ *so that if* $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ *and* $\Phi(r_1, \ldots, r_n)$ *is an equivalence relation, then* $T(r_1, \ldots, r_n) = \Phi(r_1, \ldots, r_n)$.

Suppose that $\Phi(r_1, \ldots, r_k)$ is a primitive positive formula. Assume that the free variables in Φ are x_0 and x_1. By $R(x_0, x_1)$ we will mean the statement that the ordered pair $\langle x_0, x_1 \rangle$ is in the relation $\Phi(r_1, \ldots, r_k)$. Note that $R(x_0, x_1)$ is (equivalent to) a primitive positive formula which uses the relation symbols $r_1, \ldots, r_k$.

By $\overline{\Phi}^n$ we will mean the primitive positive formula with relation symbols $r_1, \ldots, r_k$ defined so that $\langle a, b\rangle \in \overline{\Phi}^n(r_1, \ldots, r_k)$ if and only if

$$\exists y_0, \ldots, y_{n+1} \left(\bigwedge_{i=0}^{n} R(y_i, y_{i+1}) \wedge R(y_{i+1}, y_i) \right) \wedge [(y_0 = a) \wedge (y_{n+1} = b)]$$

If $r_1, \ldots, r_k$ are equivalence relations on a set with no more then n elements, then $\overline{\Phi}^n(r_1, \ldots, r_k)$ is the transitive closure of the largest symmetric relation contained in $\Phi(r_1, \ldots, r_k)$. It is easily seen to be reflexive. To see that a particular x is related to itself via this relation, one can take all of the existentially quantified variables to be equal to x (this works since each r_i is reflexive). Thus $\overline{\Phi}^n(r_1, \ldots, r_k)$ is an equivalence relation. Moreover, if $\Phi(r_1, \ldots, r_k)$ is an equivalence relation then the largest symmetric relation contained in $\Phi(r_1, \ldots, r_k)$ is all of $\Phi(r_1, \ldots, r_k)$, so $\overline{\Phi}^n(r_1, \ldots, r_k) = \Phi(r_1, \ldots, r_k)$. We will denote the operation on the equivalence relation lattice of an n element set A which maps $(r_1, \ldots, r_n) \to \overline{\Phi}^n(r_1, \ldots, r_n)$ by P_Φ (technically, the definition of P_Φ is dependent on the size of A, but in our arguments, the set will be understood, so we need not worry about embedding the size of the set in the notation P_Φ). Lemma 7.3 can now be restated as:

Lemma 7.4. *The congruence lattice of a finite algebra* $\mathbf{A}$ *is power-hereditary if and only if for every primitive positive formula* Φ*,* P_Φ *is a term operation of* $\mathrm{Con}\mathbf{A}$*.*

This lemma is the basis of our interest in interpolating certain operations on finite lattices by term operations. A natural question to ask is which of P1-P8 are satisfied by operations of the form P_Φ. Every connected primitive positive definition is trivially idempotent and order preserving, so each P_Φ automatically satisfies P1 and P2. One of the critical lemmas in [15] is following. The lemma in [15] refers to primitive positive formulas of a special form. The proof only requires that the primitive positive formula be connected. We phrase the lemma to refer only to this connectedness condition.

Lemma 7.5. (See [15] Lemma 3.1*) Suppose that* $\alpha < \beta$ *are equivalence relations on a finite set* A*. Let* $\Phi(r_1, \ldots, r_n)$ *be a connected primitive positive definition. Let* $R_1, \ldots, R_n \in \{\alpha, \beta\}$ *and let* $\sigma = \Phi(R_1, \ldots, R_n)$*. Then* $\sigma \in \{\alpha, \beta\}$*. Moreover,* $\sigma = \alpha$ *if and only the free variables in the graph of* σ *are connected by a path of edges labelled by* α*.*

It follows from this lemma that each P_Φ satisfies P5 and (by 3.2 and 3.4) P3 and P4. Thus we have

Lemma 7.6. *If* Φ *is a primitive positive formula and* $\mathbf{A}$ *is a finite algebra, then the operation* P_Φ *satisfies P1, P2, P3, P4, and P5 on* $\mathrm{Con}\mathbf{A}$*.*

Theorem 5.3 along with 7.4 and 7.6 immediately give

Theorem 7.7. (See [12]*) Every representation of* $\mathbf{N}_5$ *as the congruence lattice of a finite algebra is power-hereditary.*

Suppose that $\mathbf{L}$ is the congruence lattice of a finite algebra $\mathbf{A}$. Let $F : \mathbf{L} \to \mathbf{L}$ be an automorphism and $f : \mathbf{A} \to \mathbf{A}$ any function. We will say that F is *carried* by f if $F(\theta) = \{\langle f(x), f(y)\rangle : x\theta y\}$ for all $\theta \in \mathbf{L}$.

Lemma 7.8. *Suppose that* $\mathbf{A}$ *is a finite algebra and that* $F : \mathrm{Con}\mathbf{A} \to \mathrm{Con}\mathbf{A}$ *is an automorphism. If* F *is carried by an automorphism* f *of* $\mathbf{A}$*, then the graph of* F *(as a sublattice of* $(\mathrm{Con}\mathbf{A})^2$*) is closed under all primitive positive definitions yielding equivalence relations.*

Proof. Let $\Phi(x_1, \ldots, x_n)$ be a primitive positive formula. Using the bijectivity of f and the fact that f must map congruences of $\mathbf{A}$ to congruences, it is not hard to show that for any congruence relations $r_1, \ldots, r_n$ on $\mathbf{A}$ $\Phi(f(r_1), \ldots, f(r_n)) = f(\Phi(r_1, \ldots, r_n))$. Let $r_1, \ldots, r_n \in \mathrm{Con}\mathbf{A}$ so that $\Phi(r_1, \ldots, r_n)$ is an equivalence relation. Then

$$\begin{aligned}
\Phi(\langle r_1, F(r_1)\rangle, \ldots, \langle r_n, F(r_n)\rangle &= \langle \Phi(r_1, \ldots, r_n), \Phi(F(r_1), \ldots, F(r_n))\rangle \\
&= \langle \Phi(r_1, \ldots, r_n), \Phi(f(r_1), \ldots, f(r_n))\rangle \\
&= \langle \Phi(r_1, \ldots, r_n), f(\Phi(r_1, \ldots, r_n))\rangle \\
&= \langle \Phi(r_1, \ldots, r_n), F(\Phi(r_1, \ldots, r_n))\rangle.
\end{aligned}$$

Thus, the graph of F is closed under primitive positive definitions. □

Let $\mathbf{2}$ be the two element semilattice. Then $\mathrm{Con}(\mathbf{2}^2)$ is isomorphic to $\mathbf{D}_2$. The lattice $\mathbf{D}_2$ has one non-identity automorphism – one that exchanges the two join irreducible coatoms. This automorphism of $\mathrm{Con}(\mathbf{2}^2)$ is easily seen to be carried by the automorphism of $\mathbf{2}^2$ which exchanges the coatoms. Thus, Theorem 5.5 along with 7.4, 7.6, and 7.8 give:

Theorem 7.9. (E. Martin [9]) *The congruence lattice of the square of the two element semilattice is power-hereditary.*

Our result 4.5 about δ-convex sublattices and 7.4 and 7.6 immediately give

Theorem 7.10. *(See* [13]*) Suppose that* $\mathbf{L}$ *is the congruence lattice of a finite algebra* $\mathbf{A}$ *and that* $\delta \in \mathrm{Con}\mathbf{L}$ *so that* $\mathbf{L}/\delta$ *is distributive. Then every* 0-1 δ*-convex sublattice of* $\mathbf{L}$ *is the congruence lattice of an algebra on the same universe as* $\mathbf{A}$.

8. The Kearnes Problem

In August of 2004, Keith Kearnes posed the following problem to the first author:

Problem 8.1. Is every upper bounded lattice the congruence lattice of a finite algebra which has no Abelian congruence covers?

Every finite upper bounded lattice is representable as the congruence lattice of a finite algebra for the following reasons. Every lower bounded lattice is the congruence lattice of a finite algebra because these lattices are finitely fermentable in the sense of P. Pudlák and J. Tůma [11, 3]. Every upper bounded lattice is the dual of a lower bounded lattice, and it follows from a result of Kurzweil [8] that a finite lattice is representable if and only if its dual is also. Hence our emphasis here is not on *if* these lattices are representable but on *how* they are representable.

If $\mathbf{A}$ is a finite algebra so that no finite algebra in the variety generated by $\mathbf{A}$ has any Abelian congruence covers, then $\mathbf{A}$ generates a congruence meet semidistributive variety by [7]. By [4], every finite algebra in this variety (including $\mathbf{A}$)

has an upper bounded congruence lattice. The condition that a finite algebra $\mathbf{A}$ has no Abelian congruence covers is a somewhat technical condition based on tame congruence theory. It roughly means that $\mathbf{A}$ has (polynomial) operations which in some localities of $\mathbf{A}$ resemble semilattice operations. For our purposes here, it is adequate to know that if $\mathbf{A}$ has a semilattice operation, then $\mathbf{A}$ has no Abelian congruence covers.

Consider the lattice $\mathbf{D}_2$. Let $\mathbf{L}$ be a finite lattice in the variety generated by $\mathbf{D}_2$. Then $\mathbf{L}$ can be embedded as a 0-1 sublattice of a finite direct power of $\mathbf{D}_2$. Since $\mathrm{Con}(\mathbf{2}^2) \cong \mathbf{D}_2$, $\mathbf{L}$ can be embedded as a 0-1 sublattice in a finite direct power of $\mathrm{Con}(\mathbf{2}^2)$. By 7.9, this embedding is the congruence lattice of an algebra whose universe is a direct power of $\mathbf{2}^2$. Now, each congruence in this embedding is a product of congruences of $\mathrm{Con}(\mathbf{2}^2)$ and is therefore compatible with the semilattice operation on $\mathbf{2}^2$. We have proven:

Theorem 8.2. *(E. Martin [9]) Every finite lattice in the variety generated by $\mathbf{D}_2$ is the congruence lattice of a finite algebra with a semilattice operation.*

Early versions of this paper employed a result of K. Adaricheva [1] to prove: If $\mathrm{Con}(\mathbf{2}^n)$ is hereditary for every positive integer n, then every finite upper bounded lattice is the congruence lattice of a finite algebra with a semilattice operation. Based on this, we posed the problem: Is $\mathrm{Con}(\mathbf{2}^n)$ (power-)hereditary for every positive integer n? Recent evidence given by Ralph Freese, Jennifer Hyndman, and J.B. Nation hints that the answer to this question is negative. In light of this, we ask:

Problem 8.3. Which sublattices of $\mathrm{Con}(\mathbf{2}^n)$ are congruence lattices on the universe of $\mathbf{2}^n$? (In particular, is each of these congruence lattices hereditary?)

Similarly, in [6], Hegedűs and Pálfy give an example of a 0-1 sublattice of $\mathrm{Con}(\mathbb{Z}_2^4)$ which is not the congruence lattice of an algebra on the universe of $\mathbb{Z}_2^4$. Hence, we pose the same problem in terms of $\mathbb{Z}_2$:

Problem 8.4. Which sublattices of $\mathrm{Con}(\mathbb{Z}_2^n)$ are congruence lattices on the universe of $\mathbb{Z}_2^n$?

9. Congruence Lattices of Semilattices

One approach to Problem 8.3 would be to begin looking at what properties are satisfied by the operations on $\mathrm{Con}(\mathbf{2}^n)$ induced by primitive positive formulas. As mentioned before, we get P1-P5 for free. In this section, we will prove that every automorphism of $\mathrm{Con}(\mathbf{2}^n)$ is carried by an automorphism of $\mathbf{2}^n$. It will follow that the operations on $\mathrm{Con}(\mathbf{2}^n)$ induced by primitive positive formulas satisfy P6.

We we need first to recall some facts about congruence lattices of semilattices from [4] and [5]. Let $\mathbf{S} = \langle S, \cdot \rangle$ be a finite meet semilattice. Denote the (partial) dual operation of $\cdot$ as $+$. Suppose that $\theta \in \mathrm{Con}\mathbf{S}$. Let J_θ be the set of minimal elements of equivalence classes of θ. Then J_θ contains the minimal element 0 of S and is closed under existing joins. On the other hand, let J be a subuniverse of $\langle S, +, 0\rangle$. Let θ_J be the equivalence relation on S defined so that $x\theta_J y$ if and only if the largest element in J below x is the same as the largest below y. Then $\theta \in \mathrm{Con}\mathbf{S}$. Moreover, the maps $\theta \to J_\theta$ and $J \to \theta_J$ are inverse dual lattice isomorphisms between $\mathrm{Con}\mathbf{S}$ and $\mathrm{Sub}\langle S, +, 0\rangle$. For what follows here, let $2 = \{0, 1\}$. Let $\cdot$ and $+$ be the meet and join operations on 2. Denote by $\mathbf{2}$ the semilattice $\langle 2, \cdot\rangle$. Let $\mathbf{2}_+$ be the semilattice $\langle 2, +\rangle$, and let $\mathbf{2}_0$ be $\langle 2, +, 0\rangle$.

We prove here that every automorphism of $\mathrm{Con}(\mathbf{2}^n)$ is carried by an automorphism of $\mathbf{2}^n$. To do so, we will first show that $\mathrm{Con}(\mathbf{2}^n)$ has no more than $n!$ automorphisms and that $\mathbf{2}^n$ has exactly $n!$ automorphisms. Then we will observe that each automorphism of $\mathbf{2}^n$ induces a unique automorphism of $\mathrm{Con}(\mathbf{2}^n)$. For what remains of this section, fix $n \geq 3$ and let F be an automorphism of $\mathrm{Con}(\mathbf{2}^n)$.

For any $x \in \mathbf{2}^n$, let $c(x)$ be the number of elements of $\mathbf{2}^n$ which are comparable to x and let $h(x)$ be the height of x. The interval $[0, x]$ is a copy of $\mathbf{2}^{h(x)}$ and the interval $[x, 1]$ is a copy of $\mathbf{2}^{n-h(x)}$. The union of these two intervals is the set of all elements of $\mathbf{2}^n$ comparable to x, so we immediately have:

Lemma 9.1. *Let $x \in \mathbf{2}^n$. Then $c(x) = 2^{h(x)} + 2^{n-h(x)} - 1$.*

For any $t \in \mathbf{2}^n$, let θ_t be the equivalence relation on $\mathbf{2}^n$ defined so that $x\theta_t y$ if and only if either $x \geq t$ and $y \geq t$ or $x \not\geq t$ and $y \not\geq t$ (That is, $x \geq t \leftrightarrow y \geq t$). The following lemma is not difficult.

Lemma 9.2. *For each $t \in \mathbf{2}^n$, $\theta_t \in \mathrm{Con}(\mathbf{2}^n)$. Moreover, the coatoms of $\mathrm{Con}(\mathbf{2}^n)$ are precisely the congruences of the form θ_t where $t \neq 0$.*

If $0 \neq a < b \in \mathbf{2}^n$, then $\theta_a \cap \theta_b$ has three classes:

$$\{x : x \geq b\} \text{ and } \{x : x \geq a \text{ and } x \not\geq b\} \text{ and } \{x : x \not\geq a\}.$$

This congruence has depth 2 in $\mathrm{Con}(\mathbf{2}^n)$. On the other hand, if $a, b \in \mathbf{2}^n$ are not comparable, then $\theta_a \cap \theta_b$ has four equivalence classes:

$$\{x : x \geq a \text{ and } x \not\geq b\} \text{ and } \{x : x \geq b \text{ and } x \not\geq a\} \text{ and } \{x : x \geq a + b\} \text{ and}$$
$$\{x : x \not\geq a \text{ and } x \not\geq b\}.$$

Then

$$\theta_a \cap \theta_b < \theta_{a+b} \cap \theta_a < \theta_a < 1$$

so $\theta_a \cap \theta_b$ has depth more than 2. Thus:

Lemma 9.3. *Let $a, b \in \mathbf{2}^n$. Then a is comparable to b if and only if the depth of $\theta_a \cap \theta_b$ is less than or equal to 2.*

Note that if a or b is 0, then the intersection in the lemma has depth 1 or 0. Since F is an automorphism, it must permute the coatoms of $\mathrm{Con}(\mathbf{2}^n)$. For any $a \in \mathbf{2}^n$, let a' be the unique element of $\mathbf{2}^n$ so that $F(\theta_a) = \theta_{a'}$ (Note that this definition also works for 0 as θ_0 is the universal relation and $0' = 0$). The map $a \to a'$ must be bijective.

The automorphism F must preserve the depth of the intersection of coatoms. The map $a \to a'$ must, therefore, preserve comparabilities:

Lemma 9.4. *Let $a, b \in \mathbf{2}^n$. Then a is comparable to b if and only if a' is comparable to b'. Furthermore, $c(a) = c(a')$.*

We now want to show that if a is a coatom of $\mathbf{2}^n$, then so is a'. We will do this partially with 9.4 by counting comparable elements. Suppose that $x, y \in \mathbf{2}^n$ with $c(x) = c(y)$. This means that

$$2^{h(x)} + 2^{n-h(x)} - 1 = 2^{h(y)} + 2^{n-h(y)} - 1.$$

A little high school algebra shows that this implies that either $h(x) = h(y)$ or $h(x) = n - h(y)$. If x is a coatom or an atom, then either y is also a coatom, or y is an atom. Thus 9.4 gives

Lemma 9.5. *If a is a coatom or an atom in $\mathbf{2}^n$, then a' is either a coatom or an atom.*

Now we must eliminate the possibility that a can be a coatom and a' an atom. For each atom a of $\mathbf{2}^n$, let τ_a be the congruence on $\mathbf{2}^n$ which identifies a and 0 but is the identity relation elsewhere. The atoms of $\mathrm{Con}(\mathbf{2}^n)$ are precisely the congruences of the form τ_a for some atom a. If t is a coatom of $\mathbf{2}^n$, then no atom is greater than or equal to t (here we use that $n \geq 3$), so every atom is related to 0 by θ_t. Thus, if a is an atom, $\tau_a \leq \theta_t$. That is, θ_t is exceeds every atom of $\mathrm{Con}(\mathbf{2}^n)$. This is a property that must be preserved by the automorphism F: If t is a coatom of $\mathbf{2}^n$, then $F(\theta_t)$ must exceed every atom of $\mathrm{Con}(\mathbf{2}^n)$.

On the other hand, if t is an atom, then θ_t does not identify t with 0, but τ_t does. Hence θ_t does not exceed the atom τ_t.

Now, let a be a coatom of $\mathbf{2}^n$. The element a' must be either an atom or a coatom by 9.5. Since a is a coatom, θ_a exceeds every atom of $\mathrm{Con}(\mathbf{2}^n)$. Therefore, $F(\theta_a) = \theta_{a'}$ must also exceed every atom of $\mathrm{Con}(\mathbf{2}^n)$. This means that a' cannot be an atom because if a' were an atom then $\theta_{a'}$ would not exceed the atom $\tau_{a'}$. We have shown:

Lemma 9.6. *Suppose that $a, b \in \mathbf{2}^n$ so that $F(\theta_a) = \theta_b$. If a is a coatom, then b is a coatom. If a is an atom, then b is an atom.*

For what remains, let $a_1, a_2, \ldots, a_n$ be the coatoms of $\mathbf{2}^n$. For each i, let b_i be the unique atom of $\mathbf{2}^n$ which is not below a_i. Let $\theta_i = \theta_{a_i}$ and let $\sigma_i = \theta_{b_i}$. We have that F induces a permutation on the congruences $\{\theta_1, \theta_2, \ldots, \theta_n\}$ and that F induces a permutation on $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. We will next show that the action of F on $\mathrm{Con}(\mathbf{2}^n)$ is completely determined by what F does to the θ_i's. This will imply that the number of automorphisms of $\mathrm{Con}(\mathbf{2}^n)$ is no more than $n!$ – the number of permutations of $\{\theta_1, \theta_2, \ldots, \theta_n\}$.

First, note that for each i, since b_i is the unique atom of $\mathbf{2}^n$ which is not below a_i, σ_i is unique among the σ's in having the property that $\theta_i \cap \sigma_i$ has depth more than 2. Therefore, $F(\sigma_i)$ is unique in that $F(\theta_i) \cap F(\sigma_i)$ has depth more than 2. This implies that $a'_i = a_j$ if and only if $b'_i = b_j$ – the permutation on the θ's induced by F is the same as that induced on the σ's.

We observe that the action of F on the coatoms of $\mathrm{Con}(\mathbf{2}^n)$ is determined by the action of F on the θ_i's.

Lemma 9.7. *Let $x \in \mathbf{2}^n$. Then*

$$\theta_x = [\cap\{\theta_i : x \leq a_i\}] \vee [\cap\{\sigma_i : b_i \leq x\}].$$

Proof. Suppose that $u\theta_x v$. Then either $u, v \geq x$, or $u, v \not\geq x$. Assume first that $u, v \geq x$. This means that if $b_i \leq x$, then $b_i \leq u, v$, so $u\sigma_i v$. This places $\langle u, v\rangle \in \cap\{\sigma_i : b_i \leq x\}$, and, hence, in the right hand side of the above inequality. Now assume that $u, v \not\geq x$. If $x \leq a_i$, then it cannot be that u or v exceeds a_i. Therefore, $u\theta_i v$. This places $\langle u, v\rangle \in \cap\{\theta_i : x \leq a_i\}$, and (again) in the right hand side of the equality. We have established that

$$\theta_x \subseteq [\cap\{\theta_i : x \leq a_i\}] \vee [\cap\{\sigma_i : b_i \leq x\}] .$$

For the reverse inclusion, we will show that both $\cap\{\theta_i : x \leq a_i\}$ and $\cap\{\sigma_i : b_i \leq x\}$ are below θ_x. Suppose that $\langle u, v\rangle \in \cap\{\theta_i : x \leq a_i\}$. Suppose by way of contradiction that $\langle u, v\rangle \notin \theta_x$. Then one of u and v is above x and one is not. Without loss of generality, assume $x \leq u$ and $x \not\leq v$. There is some coatom a_j so that $v \leq a_j$ but $x \not\leq a_j$. It cannot be that $u \leq a_j$ (as then $x \leq a_j$). Let a_i be an atom above x. Then $u\theta_i v$ so, $1 = (u+a_j)\theta_i(v+a_j) = a_j$. Since $1 > a_i$, this implies that $a_j \geq a_i \geq x$. This is a contradiction, so the assumption that $\langle u, v\rangle \notin \theta_x$ must be false.

Now suppose that $\langle u, v\rangle \in \cap\{\sigma_i : b_i \leq x\}$. Suppose that $u \geq x$. Let b_i be a coatom with $x \geq b_i$. It must also be that $u \geq b_i$. Since $u\sigma_i v$, it follows that $v \geq b_i$. Thus, v is greater than or equal to every atom which is beneath x. Since x is the join of the atoms below x (using the lattice structure of $\mathbf{2}^n$), this implies $v \geq x$. We have shown that if $u \geq x$, then $v \geq x$. By exchanging u and v, we get that $u \geq x$ if and only if $v \geq x$. This means $u\theta_x v$. We have proven $\cap\{\sigma_i : b_i \leq x\} \leq \theta_x$. □

Lemma 9.8. *Let $\alpha \in \mathrm{Con}(\mathbf{2}^n)$. Then $\alpha = \cap\{\theta_x : x \in J_\alpha\}$.*

Proof. Suppose that $u\alpha v$. Let m be the minimal element of the equivalence class u/α. Let $x \in J_\alpha$. Suppose that $x \leq u$. This means that $m \leq m + x \leq u$. Since the equivalence classes of α must be convex, $m\alpha(m + x)$. But since J_α is closed under joins, $m + x \in J_\alpha$. Since each equivalence class of α has one representative in J_α, this means $m = m + x$, so $x \leq m \leq v$. We have proven that if $x \leq u$, then $x \leq v$. By exchanging u and v, we get that $x \leq u$ if and only if $x \leq v$. Hence, $u\theta_x v$. This places $\langle u, v\rangle \in \cap\{\theta_x : x \in J_\alpha\}$.

For the reverse inclusion, let $\langle u, v\rangle \in \cap\{\theta_x : x \in J_\alpha\}$. Let a be the largest element of J_α below u and let b be the largest below v. By our selection of u and v, we know that $u\theta_a v$. Since $u \geq a$, this means that $v \geq a$ – which places $a \leq b$. A similar argument gives $b \leq a$, so $a = b$. This implies $u\alpha v$. □

By the comments preceeding 9.7, the action of F on $\{\theta_1, \theta_2, \ldots, \theta_n\}$ determines the action of F on $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. This action then determines what F does to the coatoms of $\mathrm{Con}(\mathbf{2}^n)$ by 9.7. The values of F on coatoms then completely determines F by 9.8. Therefore, F is uniquely determined by the permutation it induces on $\{\theta_1, \theta_2, \ldots, \theta_n\}$. Since there are $n!$ such permutations, there are at most $n!$ possibilities for F.

Lemma 9.9. *Let n be a positive integer. The number of automorphisms of $\mathrm{Con}(\mathbf{2}^n)$ is at most $n!$.*

We have established this for $n \geq 3$. For $n = 1$, $\mathbf{2}^1$ is simple, so $\mathrm{Con}(\mathbf{2}^1)$ has only $1 = 1!$ automorphism. For $n = 2$, $\mathrm{Con}(\mathbf{2}^2)$ is the lattice $\mathbf{D}_2$. It has $2 = 2!$ automorphisms – the identity automorphism and the one which exchanges the join irreducible coatoms.

We are finally able to prove:

Theorem 9.10. *Let n be a positive integer. Every automorphism of $\mathrm{Con}(\mathbf{2}^n)$ is carried by an automorphism of $\mathbf{2}^n$.*

Proof. Every automorphism of $\mathbf{2}^n$ induces a permutation of the coatoms of $\mathbf{2}^n$. Moreover, every permutation of these n coatoms induces a unique automorphism of $\mathbf{2}^n$. Hence, $\mathbf{2}^n$ has exactly $n!$ automorphisms.

Let f be an automorphism of $\mathbf{2}^n$. Define $F : \mathrm{Con}(\mathbf{2}^n) \to \mathrm{Con}(\mathbf{2}^n)$ by $F(\alpha) = \{\langle f(x), f(y)\rangle : x\alpha y\}$. Then F is an automorphism of $\mathrm{Con}(\mathbf{2}^n)$ carried by f. Distinct automorphisms of $\mathbf{2}^n$ induce distinct automorphisms of $\mathrm{Con}(\mathbf{2}^n)$ in this way. Thus there are (at least) $n!$ automorphisms of $\mathrm{Con}(\mathbf{2}^n)$ which are carried by automorphisms of $\mathbf{2}^n$. In light of 9.9, these $n!$ automorphisms are all of the automorphisms of $\mathrm{Con}(\mathbf{2}^n)$. □

Combining this with Lemma 7.6 and 7.8, we now have that

Theorem 9.11. *Suppose that Φ is a primitive positive formula. Then P_Φ satisfies P1, P2, P3, P4, P5, and P6 on $\mathrm{Con}(\mathbf{2}^n)$ for all positive integers n.*

10. An Example

What properties characterize the terms of $\mathrm{Con}(\mathbf{2}^n)$ for all n? We give here an example by Ralph McKenzie demonstrating that P1-P6 along with P8 are not enough to determine the terms of every finite lattice in the variety generated by $\mathbf{N}_5$. Compare this with 5.8. Let $\mathbf{L}_0$ and $\mathbf{L}_1$ be two copies of $\mathbf{N}_5$. For $i = 1, 2$, assume that $\mathbf{L}_i = \{0_i, 1_i, a_i, b_i, c_i\}$ with $b_i < c_i$ the critical cover. Let $\mathbf{L}$ be the vertical sum of $\mathbf{L}_0$ and $\mathbf{L}_1$ with 0_1 above 1_0. Define $p(x, y, z)$ on $\mathbf{L}$ to be $z \wedge (x \vee (y \wedge z))$ for $x, y, z \in \mathbf{L}_1$ and to be $(x \wedge z) \vee (y \wedge z)$ for all other x, y, and z. Then p will satisfy P1-P6 and P8. Within the lattice $\mathbf{L} \times \mathbf{L}$, consider the sublattice $\mathbf{M} = \{\langle 0_0, 0_1\rangle, \langle a_0, a_1\rangle, \langle b_0, b_1\rangle, \langle c_0, c_1\rangle, \langle 1_0, 1_1\rangle\}$. Notice that $p(\langle a_0, a_1\rangle, \langle b_0, b_1\rangle, \langle c_0, c_1\rangle) = \langle b_0, c_1\rangle$ – which is not in $\mathbf{M}$. Since $\mathbf{M}$ is not closed under p, then p cannot be equal to a term operation of $\mathbf{L}$.

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