Sensitivity Analysis of the Bipartite Weighted Matching Problem

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Summary: Let G be an undirected bipartite graph where a weight is assigned to every edge. A maximum weighted matching can be computed on G. We define slightly changed problems of the original matching problem. One type of problem is to delete a vertex, a pair of vertices or an edge from G. A second type of problem can be constructed if a vertex or a pair of vertices is doubled by making an identical copy. A third type of problem is the replacement of a vertex by a copy of another vertex. All these problems build up some kind of sensitivity analysis of the original problem and are of special interest for example in man power scheduling. We can show that all these problems can be solved simultaneously by finding the shortest path between all pairs of vertices in a graph G' which can easily be constructed.

1. Introduction

Let $G = (V, E)$ be an undirected bipartite graph with n vertices. The set of vertices is partitioned into the sets X and Y. Therefore it holds that $V = X \cup Y$ and $E \subseteq X \times Y$ describes the possible edges. A weight $w(x, y) \geq 0$ is assigned to every edge $[x, y] \in E$. A maximum weighted matching can be computed on the graph G. The set of matched edges is denoted by M. The overall weight of the maximum weighted matching M of G is $MW(G) = W(M) = \sum_{[x,y]\in M} w(x, y)$ where $MW(G)$ denotes the maximum weight of matching in G and $W(M)$ is the weight of a matching M.

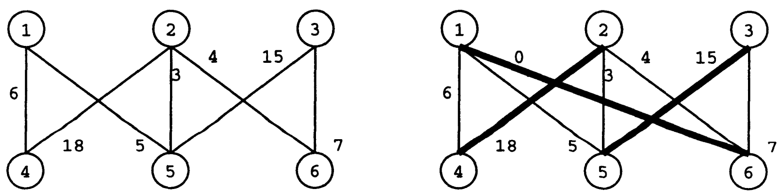
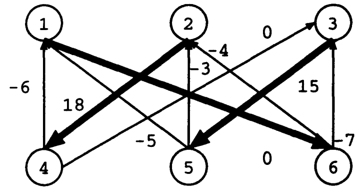


Figure 1: maximum weighted matching

It is known [PS82] that in the weighted matching problem, we can do away with the graph G by adopting the convention that the underlying graph is always complete, and letting the weights of those edges that were missing in G be equal to zero. Because we are dealing with a bipartite graph we may assume that $|X| = |Y|$ – otherwise add new vertices with edges of weight zero.
We may observe that the optimal solutions will always have perfect matchings. The bipartite weighted matching problem is also known as the assignment problem. The objective value of the weighted matching problem in Fig. 1 is 33. Because we only want to look at perfect matchings, there is a matched edge (1,6) with weight zero. Throughout the remaining paper $|X| = |Y|$ and G is a complete bipartite graph.
In this paper, we are interested in the solution of slightly changed problems. We can show that

P. Kall et al. (eds.), *Operations Research Proceedings 1998*



	1	2	3	4	5	6
1	3	-4	-7	14	8	0
2	12	8	5	18	20	12
3	10	6	3	24	15	10
4	-6	-10	-13	8	2	-6
5	-5	-9	-12	9	3	-5
6	3	-4	-7	14	8	3

Figure 2: Directed graph G' and distance matrix

all these solutions can be achieved by solving a shortest path problem on a digraph constructed by means of a known maximum weighted matching. The problems are as follows:

G_u For a vertex u in G let G_u denote the graph formed by deleting u from G. The problem is defined in [KLST97] and is called the all-cavity maximum matching problem which asks for a maximum weighted matching in G_u for all u in G

$G_{x,y}$ For a pair of vertices $x \in X$ and $y \in Y$ let $G_{x,y}$ denote the graph formed by deleting x and y from G. The problem is again defined in [KLST97] and is called the two-vertex all-cavity maximum matching problem which asks for a maximum weighted matching in $G_{x,y}$ for every pair of vertices x and y.

$G_{[x,y]}$ For an edge $[x, y] \in E$ let $G_{[x,y]}$ denote the graph where the weight of $[x, y]$ is set to zero. The problem asks for a maximum weighted matching in $G_{[x,y]}$ for every edge $[x, y] \in E$.

G^u For a vertex u in G let G^u denote the graph formed by copying u and the adjacent edges G. The problem asks for a maximum weighted matching in G^u for all u in G.

$G^{x,y}$ For a pair of vertices $x \in X$ and $y \in Y$ let $G^{x,y}$ denote the graph formed by copying both vertex x and vertex y and all adjacent edges in G. The problem asks for a maximum weighted matching in $G^{x,y}$ for every pair of vertices $x \in X$ and $y \in Y$.

G_u^v For a pair of vertices $u, v \in X$ (or $u, v \in Y$) let G_u^v denote the graph formed by deleting vertex u and copying vertex v and all adjacent edges in G. The problem asks for a maximum weighted matching in G_u^v for every pair of vertices $u, v \in X$ ($u, v \in Y$, respectively).

2. Sensitivity analysis

In order to do the sensitivity analysis we construct a directed graph G'. The set of vertices is the same as before. The arc set of G' is constructed as follows:

- arc (x, y) with weight $w(x, y)$, if $[x, y] \in M$, $x \in X$ and $y \in Y$

- arc (y, x) with weight $-w(x, y)$, if $[x, y] \in E \setminus M$, $x \in X$ and $y \in Y$