**SBW RDA MODEL**

## 1. Model set-up

We consider the spatial version of Ludwig-Aronson-Weinberger model [1] describing dynamics of the spruce budworm (SBW) population. The habitat of SBW is represented by a finite interval $\left[-\frac{L}{2}, \frac{L}{2}\right]$ on the real line. The linear density of SBW population at location $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$ at time $t$ is given by the function $u(x,t)$. Following [1], the boundary conditions are of the hostile type:

$$u\left(-\frac{L}{2}, t\right) = u\left(\frac{L}{2}, t\right) = 0.$$

We assume that individuals move randomly and are subject to a unidirectional movement bias given in the form of wind transporting the larval population, represented by an advection term $q\frac{\partial u}{\partial x}$ (where $q \geq 0$). In addition, we assume that the population experiences the logistic growth and is subject to predation, via Holling type II function.

Therefore, combining the non-dimensionalized version of a reaction-diffusion equation presented in [1] (denoting $\alpha = \frac{1}{Q}$ and $\beta = \frac{1}{R}$) with the advection term, we get:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - q\frac{\partial u}{\partial x} + u - \alpha u^2 - \beta\frac{u^2}{1+u^2}. \tag{1.1}$$

We denote the reaction term of (1.1) as follows: $g(u) = -u + \alpha u^2 + \beta\frac{u^2}{1+u^2}$.

We start our analysis by looking for steady state solutions $u = u(x)$ of the boundary value problem

$$\begin{cases} u'' - qu' - g(u) = 0 \\ u\left(-\frac{L}{2}\right) = u\left(\frac{L}{2}\right) = 0. \end{cases} \tag{1.2}$$

By introducing $v(x) = u'(x)$, we rewrite the above boundary value problem as:

$$\begin{cases} \frac{du}{dx} = v \\ \frac{dv}{dx} = qv + g(u) \\ u\left(-\frac{L}{2}\right) = u\left(\frac{L}{2}\right) = 0. \end{cases} \tag{1.3}$$

Equilibria of (1.3) are given by $(u(x), v(x)) = (u^*, 0)$ where $g(u^*) = 0$. Note that the number of equilibria varies between 2 and 4. Indeed, they are the roots of the equation

$$u(1+u^2) = \alpha u^2(1+u^2) + \beta u^2,$$

or

$$u(\alpha u^3 - u^2 + (\alpha+\beta)u - 1) = 0.$$

We will denote the zero solution of the above equation by $u_0$. Next, we focus on the case when the above equation has three distinct positive roots.

Let $f(u) = \alpha u^3 - u^2 + (\alpha + \beta)u - 1$. Then

$$f'(u) = 3\alpha u^2 - 2u + (\alpha + \beta).$$

The necessary condition for having three positive equilibria of (1.3) is the existence of two positive roots of $f'(u) = 0$ (note that any root of $f'(u) = 0$ would be positive since the vertex of $v = f'(u)$ has positive $u$-coordinate). Thus, $f'(u) = 0$ has two positive roots exactly when $4 - 12\alpha(\alpha + \beta) > 0$, or

$$\alpha(\alpha + \beta) < \frac{1}{3}.$$

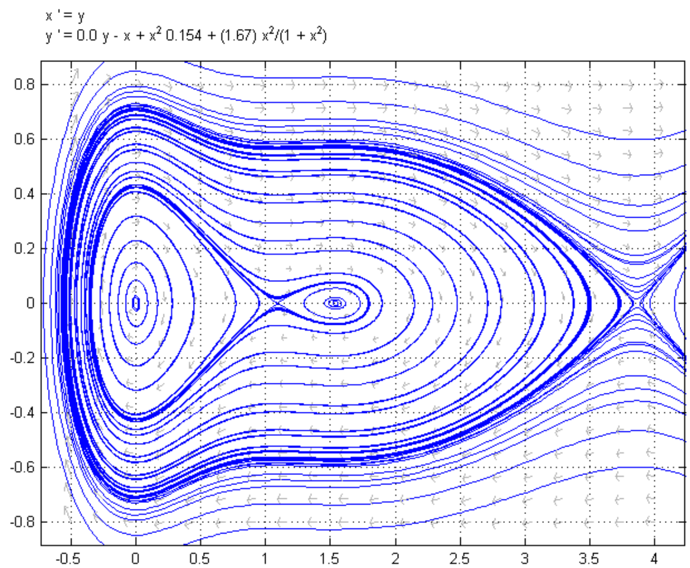


FIGURE 1.1. Phase plane portrait for system (1.3) when $\alpha = 0.154, \beta = 1.67$ and $q = 0$. The four equilibria of (1.3) are easily identified as two centers and two saddle points. The portions of the closed orbits in the first and fourth quadrants form endemic and outbreak solutions of (1.3) corresponding to different habitat lengths.

From now on we assume that $g(u)$ has three distinct positive real roots $u_1, u_2, u_3$, such that $u_1 < u_2 < u_3$. Thus, the system (1.3) has four distinct equilibria:

$$(u_0, 0) = (0, 0), (u_1, 0), (u_2, 0), (u_3, 0).$$

In this case, $g'(u_0), g'(u_2) < 0$ and $g'(u_1), g'(u_3) > 0$ (see the graph in Figure 1.2).

Consider the Jacobian matrix of system (1.3):

$$J(u, v) = \begin{bmatrix} 0 & 1 \\ g'(u) & q \end{bmatrix}.$$

Its trace and determinant are given by $\text{tr}(J(u, v)) = q \geq 0$ and $\det(J(u, v)) = -g'(u)$. We observe the following regarding each equilibrium point of (1.3):

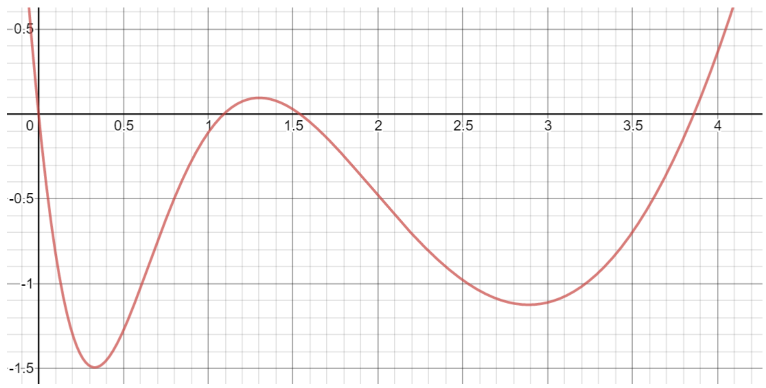


FIGURE 1.2. Graph of $g(u) = -u+\alpha u^2+\beta\frac{u^2}{1+u^2}$ for $\alpha = 0.154, \beta = 1.67$ with a vertical scaling by factor of 10.

- $(u_0, 0) = (0, 0)$: Note that $g'(0) = -1$. When $q = 0$, this equilibrium point is a center. When $0 < q < 2$, we have $4\det(J(0, 0)) - \mathrm{tr}^2(J(0, 0)) = 4-q^2 < 0$ while $\mathrm{tr}(J(0, 0)) = q > 0$, so the equilibrium is an unstable spiral. When $q \geq 2$, we have an unstable node.
- $(u_1, 0)$: Since $\det(J(u_1, 0)) = -g'(u_1) < 0$, this equilibrium point is a saddle for any value of $q$.
- $(u_2, 0)$: Note that $g'(u_2) < 0$. When $q = 0$, the equilibrium point is a center. When $0 < q < 2\sqrt{-g'(u_2)}$, we have $4\det(J(u_2, 0)) - \mathrm{tr}^2(J(u_2, 0)) = 4(g'(u_2))^2 - q^2 < 0$ while $\mathrm{tr}(J(u_2, 0)) = q > 0$, so the equilibrium is an unstable spiral. When $q \geq 2\sqrt{-g'(u_2)}$, we have an unstable node.
- $(u_3, 0)$: As in the case of $(u_1, 0)$, we have $\det(J(u_3, 0)) = -g'(u_3) < 0$, and therefore this equilibrium point is a saddle point for any value of $q$.

When $q = 0$ (see Fig. 1.1), we observe five different separatrix curves. By $S_1^{left}$ and $S_1^{right}$ we denote the two separatrix curves having $(u_1, 0)$ as their $\alpha$ and $\omega$-limits, and enclosing the points $(0, 0)$ and $(u_2, 0)$, respectively. The curve $S_3$ has its $\alpha$ and $\omega$-limit at $(u_3, 0)$ and encircles the other three equilibria. The curve $S_3^+$ has $(u_3, 0)$ as its $\alpha$-limit and stays in the first quadrant, whereas the surve $S_3^-$ has $(u_3, 0)$ as its $\omega$-limit and lies in the fourth quadrant.

As we increase $q > 0$ (see Fig. 1.3), the phase plane portrait changes as follows. The two centers, $(0, 0)$ and $(u_2, 0)$ become instable spirals. We observe appearance of the following separatrix curves:

- $S_{01}$ connects $(0, 0)$ to $(u_1, 0)$ by spiraling out of $(0, 0)$ ($\alpha$-limit) and approaching $(u_1, 0)$ ($\omega$-limit) from the first quadrant;
- $S_{03}$ connects $(0, 0)$ to $(u_3, 0)$ by spiraling out of $(0, 0)$ ($\alpha$-limit) and approaching $(u_3, 0)$ ($\omega$-limit) from the first quadrant (while staying above $S_{01}$ after completing the rotation);
- $S_1^-$ starts at $(u_1, 0)$ ($\alpha$-limit), follows the vector field through the quadrants IV, III, II, and then stays in quadrant I;
- $S_1^+$ starts at $(u_1, 0)$ ($\alpha$-limit), follows the vector field through the quadrants I. IV, III, II, and then stays in quadrant I;

- $S_2$ spirals out of $(u_2, 0)$ ($\alpha$-limit) and follows the vector field through quadrants I, IV, III, II, and then stays in quadrant I;
- $S_3$ starts at $(u_3, 0)$ ($\alpha$-limit) and follows the vector field through quadrants IV, III, II, and then stays in quadrant I;
- $S_3^+$ has $(u_3, 0)$ as its $\alpha$-limit and stays in the first quadrant, whereas the surve $S_3^-$ has $(u_3, 0)$ as its $\omega$-limit and lies in the fourth quadrant.

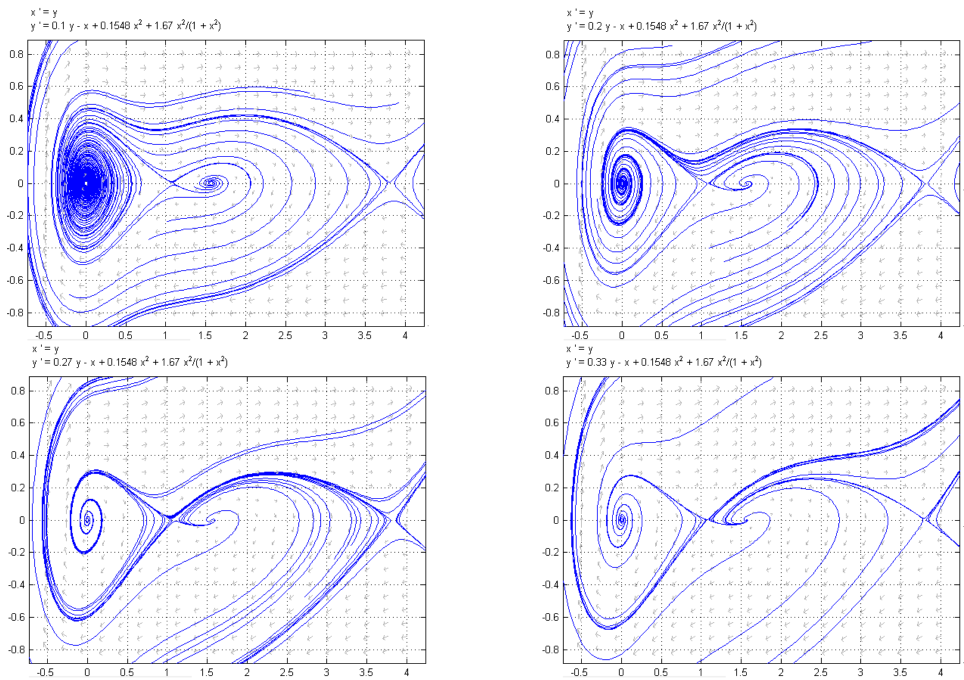


FIGURE 1.3. Phase plane portraits for system (1.3) when $\alpha = 0.154, \beta = 1.67$, for $q = 0.1, 0.2, 0.27, 0.33$. The nature of saddle points remains the same, while both centers become unstable spirals. For $q = 0.1$ (top left) and $q = 0.2$ (top right), both endemic and outbreak solutions still exist, but for $q = 0.2$ the point $(u^*, 0)$ where the curve $S_1^+$ intersects the $u$-axis is shifted to the right. The value $q = 0.27$ (bottom left) is very close to $q_{cr}$. As the result, the "outer" spiral-to-saddle connection $S_{03}$ seems to be lost while a new saddle-to-saddle connection $S_{13}$ seems to appear. As a result, there no more outbreak solutions. For $q = 0.33$ (bottom right), there is a curve ($S_{23}$) connecting the second spiral to the second saddle. There are no outbreak solutions, but there are still endemic solutions.

Let $(u^*, 0)$ be the intersection point of curve $S_1^+$ with the positive $u$-axis. As we increase $q$ (see Fig. 1.3), we observe that $(u^*, 0)$ is shifting to the right and approaching $(u_3, 0)$. This shift can be explained by the fact that while the $u$-component of the vector field given by system (1.3) does not depend on $q$, and thus is not changing, the $v$-component is increasing with $q$. At a certain critical value of $q$, which we will denote $q_{cr}$, the separatrix curve $S_{03}$ connecting $(0, 0)$ to

$(u_3, 0)$ disappears and is replaced by a separatrix $S_{13}$ connecting $(u_1, 0)$ and $(u_3, 0)$ in the first quadrant. For $q > q_{cr}$, $S_{13}$ disappears and is replaced by the curve $S_{23}$ connecting $(u_2, 0)$ to $(u_3, 0)$ . Note that for $q < q_{cr}$ we have "outbreak solution" curves that pass between $S_1^+$ and $S_{03}$. However, for $q \geq q_{cr}$ outbreak solutions disappear along with the disappearance of $S_{03}$ and appearance of $S_{13}$ and then $S_{23}$. The dynamics as we increase $q$ is illustrated in Fig. 1.4.

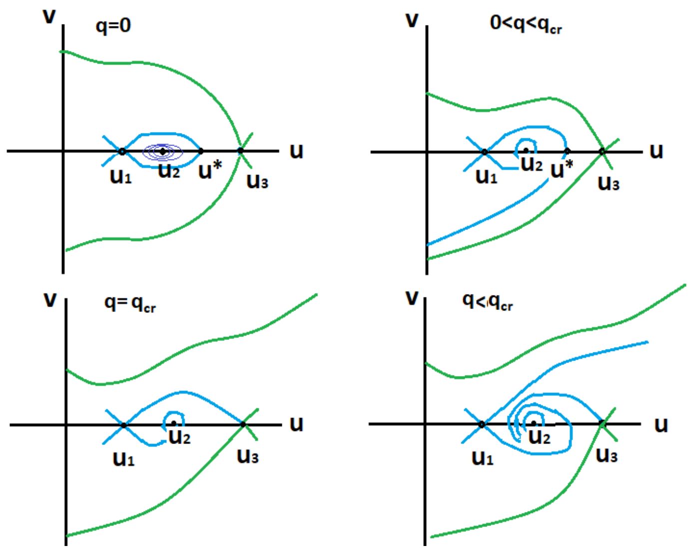


FIGURE 1.4. Changes in the phase plane portrait of system (1.3) as $q$ increases. For $q = 0$, $S_1^{right}$ is present. Outbreak solution curves pass between $S_1^{right}$ and the equilibrium point $(u_3, 0)$. For positive subcritical values of $q$, $S_1^{right}$ is replaced with $S_1^+$, but outbreak solution curves still exist. At $q = q_{cr}$, $S_1^{right}$ is replaced with $S_{13}$ thus preventing outbreak solutions. For $q > q_{cr}$, $S_{13}$ is replaced with $S_{23}$ (and there are no outbreak solutions as well).

## 2. ESTIMATING THE CRITICAL ADVECTION FOR OUTBREAK SOLUTIONS

In this section we will determine conditions for the existence of outbreak solutions in terms of the advection speed $q$. As discussed in the previous section, the outbreak solutions disappear when $q$ reaches a critical value $q_{cr}$ and the "outer" spiral-saddle connection $S_{03}$ is replaced with a saddle-to-saddle connection $S_{13}$. In particular, we will find a lower bound for $q_{cr}$.

**Remark 2.1.** *Since every solution $(u(x), v(x))$ of system (1.3) satisfies $u'(x) = v(x)$, the portions of the solution curves of (1.3) located either in the upper half of the uv-plane $\{(u, v) : v > 0\}$ or in the lower half $\{(u, v) : v < 0\}$ satisfy the vertical line test. Given such a curve located in either half of the uv-plane, we can view it as the graph of a function $v = v(u)$ defined on a certain interval. Such a function will be a solution of the first order differential equation*

$$\frac{dv}{du} = q + \frac{g(u)}{v}. \tag{2.1}$$

Let $0 \le q \le q_{cr}$. As before, let $S_1^+$ be solution curve having $(u_1, 0)$ as its $\alpha$-limit that follows the vector field into the first quadrant, and let $(u^*, 0)$ be the point where $S_1^+$ intersects the $u$-axis. Note that $u_2 < u^* \le u_3$.

Thus, the portion of $S_1^+$ located in the first quadrant is the graph of a function $v = v(u)$ satisfying (2.1) and defined for $u_1 \le u \le u^*$, such that $v(u_1) = v(u^*) = 0$ and $v(u) > 0$ for $u_1 < u < u^*$. Multiplying both sides of (2.1) by $v(u)$ and integrating from $u_1$ to $u^*$ gives:

$$\int_{u_1}^{u^*} (qv(u) + g(u))du = \int_{u_1}^{u^*} v(u)\frac{dv}{du}du =$$
$$\int_{v(u_1)}^{v(u^*)} vdv = \frac{(v(u^*))^2}{2} - \frac{(v(u_1))^2}{2} = 0 - 0 = 0.$$

Therfore, we obtain the following:

$$-q\int_{u_1}^{u^*} v(u)du = \int_{u_1}^{u^*} g(u)du.$$

Since, $\int_{u_1}^{u^*} v(u)du > 0$, we get

$$q = -\frac{\int_{u_1}^{u^*} g(u)du}{\int_{u_1}^{u^*} v(u)du}. \tag{2.2}$$

Note that $g(u) > 0$ for $u_1 < u < u_2$ and $g(u) < 0$ for $u_2 < u < u_3$.

Let $A_1 = \int_{u_1}^{u_2} g(u)du$, $A^* = -\int_{u_2}^{u^*} g(u)du$ and $A_2 = -\int_{u_2}^{u_3} g(u)du$. Then $A_1, A^*, A_2 > 0$ and $A^* \le A_2$.

Notice that $\int_{u_1}^{u^*} g(u)du = A_1 - A^*$ and $\int_{u_1}^{u_3} g(u)du = A_1 - A_2$.

**Remark 2.2.** *Observe that boundary value problem (1.3) possesses an outbreak solution for some $L > 0$ and $q = 0$ exactly when $u^* < u_3$. This is also equivalent to existence of an outbreak solution (for some $L > 0$) for some $q \ge 0$, or for all sufficiently small $q \ge 0$.*

Next, we obtain the following necessary and sufficient condition for existence of an outbreak solution of (1.3) for sufficiently small $q \ge 0$.

**Proposition 2.3.** *Suppose $g(u)$ has three positive zeros. Then boundary value problem (1.3) has an outbreak solution for sufficiently small $q \ge 0$ if and only if $A_2 > A_1$.*

*Proof.* Suppose that for some $q \ge 0$, an outbreak solution exists. Then the orbit $S_1^+$ intersects the $u$-axis at a point $(u^*, 0)$ where $u_2 < u^* < u_3$, and by (2.2)

$$0 \le q = \frac{A^* - A_1}{\int_{u_1}^{u^*} v(u)du},$$

and thus, $A^* \ge A_1$. Note that since $u^* < u_3$, we have $A_2 > A^*$. Therefore $A_2 > A_1$.

Suppose that for $q = 0$, an outbreak solution does not exist, but $A_2 > A_1$. Then there exists an orbit (a solution curve satisfying $(u'(x), v'(x)) = (v, g(u))$) lying in the first quadrant, having $(u_1, 0)$ as its $\alpha$-limit and either having $(u_3, 0)$ as its $\omega$-limit, or satisfying $u(x) \to \infty$ and $v(x) \to \infty$ as $x \to \infty$ (see Fig. **??**). Note that the portion of this curve between $u = u_1$ and $u = u_3$ is the graph of a function $v(u)$ satisfying the first order differential equation

$$v\frac{dv}{du} = g(u)$$

for $u_1 \leq u \leq u_3$, such that $v(u_1) = 0$.

Then we have:

$$0 > A_1 - A_2 = \int_{u_1}^{u_3} g(u)du = \int_{u_1}^{u_3} v\frac{dv}{du}du = \frac{(v(u_3))^2}{2} - \frac{(v(u_1))^2}{2} = \frac{(v(u_3))^2}{2} - 0 = \frac{(v(u_3))^2}{2},$$

a contradiction.

Thus, an outbreak solution exists.

□

Figure 2.1 shows the phase plane of system (1.3) for $\alpha = 0.154$ and $q = 0$ and three different values of $\beta$:
(a) $\beta = 1.7, A_1 < A_2$, outbreak solutions exist;
(b) $\beta = 1.743, A_1 = A_2$, no outbreak solution is possible;
(c) $\beta = 1.75, A_1 > A_2$, no outbreak solution is possible.

By Proposition 2.3, when $A_1 < A_2$, for small enough $q$, the orbit $S_1^+$ intersects the $u$-axis at $(u^*, 0)$ where $u_2 < u^* < u_3$, and there are outbreak solutions (for sufficiently large $L$), i.e. orbits connecting the positive $v$-semiaxis to negative $v$-semiaxis, intersecting the $u$-axis between $u^*$ and $u_3$. In order to emphasize the dependence of $u^*$ on $q$ (for sufficiently small values of $q$), we will denote $u^* = u^*(q)$.

Note that $u^*(q)$ is an increasing function of $q$. Let $q_{cr}$ be the supremum of all $q > 0$ for which outbreak solutions exist. Then for $q = q_{cr}$ there exists an orbit connecting $(u_1, 0)$ and $(u_3, 0)$, and

$$\lim_{q \to q_{cr}^-} u^*(q) = u_3.$$

Passing to the limit as $q \to q_{cr}^-$ in (2.2), we get

$$q_{cr} = -\frac{\int_{u_1}^{u_3} g(u)du}{\int_{u_1}^{u_3} v(u, q_{cr})du}. \tag{2.3}$$

In the formula above, we use the notation $v = v(u, q)$ instead of $v = v(u)$, due to the dependence of $v$ on the value of the parameter $q$ as well. Note that the denominator of the fraction in (2.3) represents the area of the region in the $uv$-plane bounded above by the curve $v = v(u)$ (the orbit $S_{13}$), where $u_1 \leq u \leq u_3$, and the $u$-axis. Our next objective is to find an upper bound for this area. Note that the curve $v = v(u)$ passes through $(u_1, 0)$ and $(u_3, 0)$ and is concave down. Since $v(u_1) = v(u_3) = 0$ it follows that $v'(u_1) > 0$ and $v'(u_3) < 0$.

Thus, we consider the triangular region formed by the segment of the $u$-axis between $u_1$ and $u_3$, the tangents to the graph of $v = v(u)$ at these points (see Fig. 2.2). Since $v(u)$ is concave down,

$$\int_{u_1}^{u_3} v(u, q_{cr})du < A,$$

where $A$ is the area of the triangle.

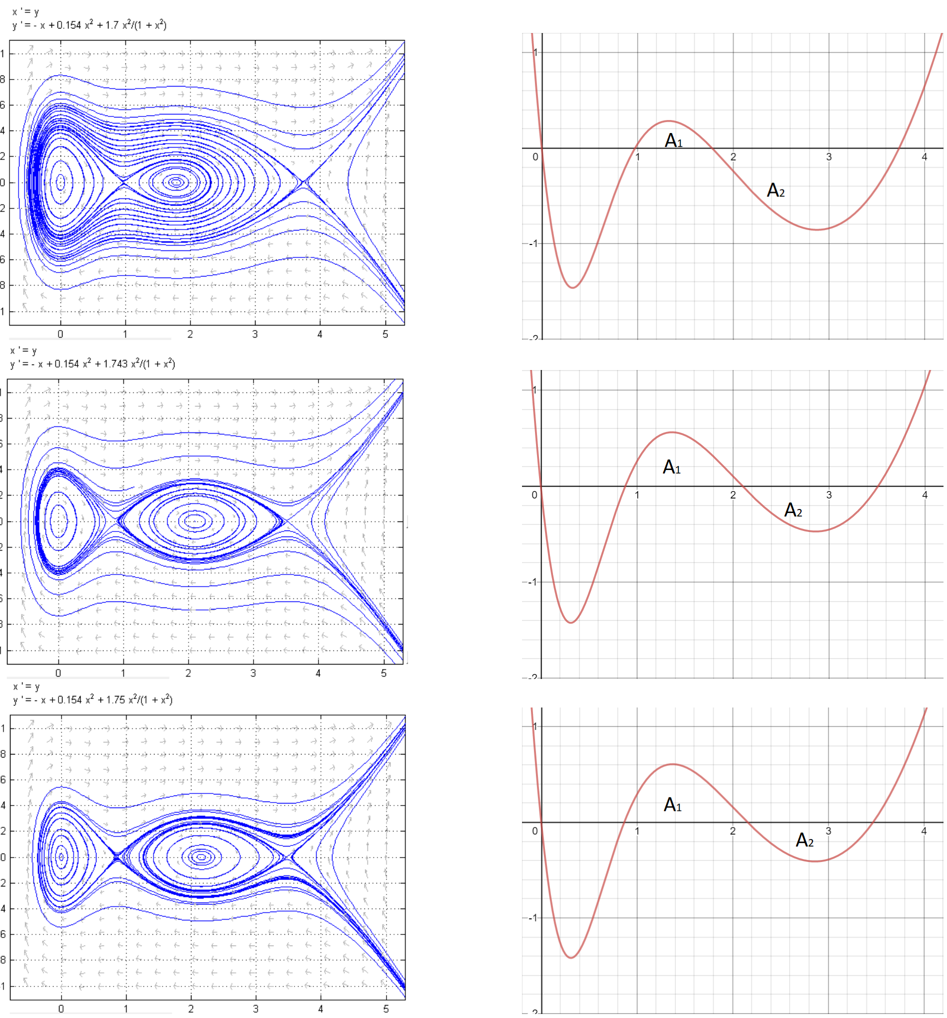


FIGURE 2.1. The phase plane of system (1.3) for $\alpha = 0.154, q = 0$ and $\beta = 1.7, 1.743, 1.75$. Graphs of $g(u)$ (scaled by factor of 10) with the areas $A_1$ and $A_2$ indicated are shown on the right.

In order to compute $A$ we will need to find the slopes of the curve $v = v(u)$ at $u = u_1$ and $u = u_3$. Noticing that $g(u_1) = v(u_1) = g(u_3) = v(u_3) = 0$ and using L'Hospital's rule, we get:

$$v'(u_1) = \lim_{u \to u_1^+} v'(u) = q_{cr} + \lim_{u \to u_1^+} \frac{g(u)}{v(u)} = q + \lim_{u \to u_1^+} \frac{g'(u)}{v'(u)} = q_{cr} + \frac{g'(u_1)}{v'(u_1)}.$$

The above equation can be written as

$$(v'(u_1))^2 - q_{cr} v'(u_1) - g'(u_1) = 0.$$

Similarly, letting $u \to u_3^-$ gives

$$(v'(u_3))^2 - q_{cr} v'(u_3) - g'(u_3) = 0.$$

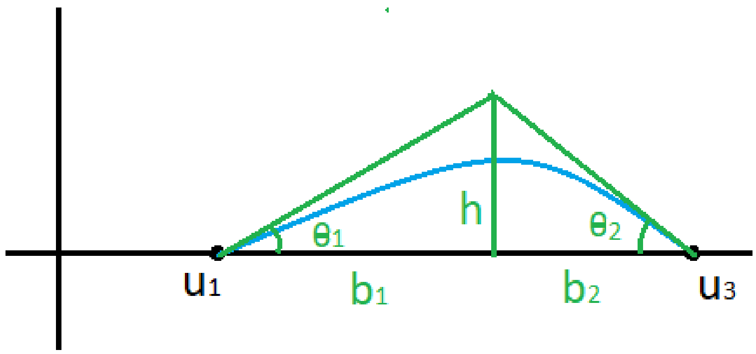


FIGURE 2.2. Estimating $\int_{u_1}^{u_3} v(u)du$ using a triangle formed by tangent lines.

Note that $v'(u_1) > 0$, $v'(u_3) < 0$, $g'(u_1) > 0$ and $g'(u_3) > 0$.
Thus, we have

$$v'(u_1) = \frac{q_{cr} + \sqrt{q_{cr}^2 + 4g'(u_1)}}{2}$$

and

$$v'(u_3) = \frac{q_{cr} - \sqrt{q_{cr}^2 + 4g'(u_3)}}{2}.$$

Let $\theta_1$ be the acute angle formed by the tangent to $v = v(u)$ at $u = u_1$ and the $u$-axis, and let $\theta_2$ be the acute angle formed by the tangent to $v = v(u)$ at $u = u_3$ and the $u$-axis. Then we have

$$\tan\theta_1 = \frac{q_{cr} + \sqrt{q_{cr}^2 + 4g'(u_1)}}{2}$$

and

$$\tan\theta_2 = \frac{-q_{cr} + \sqrt{q_{cr}^2 + 4g'(u_3)}}{2}.$$

Thus, the area of the triangular region, is given by

$$A = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(u_3 - u_1)h.$$

To find $h$, we note that $h = b_1 \tan\theta_1 = b_2 \tan\theta_2 = (u_3 - u_1 - b_1)\tan\theta_2$. Solving for $b_1$, we get

$$b_1 = \frac{(u_3 - u_1)\tan\theta_2}{\tan\theta_1 + \tan\theta_2},$$

and, therefore,

$$h = \frac{(u_3 - u_1)\tan\theta_1 \tan\theta_2}{\tan\theta_1 + \tan\theta_2}.$$

Hence,

$$A = \frac{1}{2} \cdot \frac{(u_3 - u_1)^2 \tan\theta_1 \tan\theta_2}{\tan\theta_1 + \tan\theta_2} = A(q_{cr}),$$

where

$$A(q) = \frac{(\sqrt{q^2 + 4g'(u_1)} + q)(\sqrt{q^2 + 4g'(u_3)} - q)(u_3 - u_1)^2}{4(\sqrt{q^2 + 4g'(u_1)} + \sqrt{q^2 + 4g'(u_3)})}$$

is viewed as a function of $q \geq 0$.

Next, we recall that $\int_{u_1}^{u_3} v(u, q_{cr})du < A(q_{cr})$ and $\int_{u_1}^{u_3} g(u)du < 0$, and thus,

$$q_{cr} = \frac{-\int_{u_1}^{u_3} g(u)du}{\int_{u_1}^{u_3} v(u, q_{cr})du} > \frac{-\int_{u_1}^{u_3} g(u)du}{A(q_{cr})}.$$

Note that $A(q)$ is well-defined, continuous and positive for all $q \geq 0$. For $q \geq 0$, let

$$F(q) = q + \frac{\int_{u_1}^{u_3} g(u)du}{A(q)},$$

which is a continuous function for $q \geq 0$. Then

$$F(0) = \frac{\int_{u_1}^{u_3} g(u)du}{A(0)} < 0,$$

and $F(q_{cr}) > 0$. Let $\tilde{q} > 0$ be the smallest positive zero of $F(q)$. Then $0 < \tilde{q} < q_{cr}$, that is, $\tilde{q}$ serves as a lower bound for $q_{cr}$.

## 3. A lower bound for critical domain size for outbreak solutions

In this section, our goal is to estimate the critical domain size for the purpose of outbreak solution, i.e. the minimal habitat length capable of supporting an outbreak, provided the avection is less than $q_{cr}$ (as defined in the previous section).

Suppose $q < q_{cr}$. Then for large enough $L > 0$ there exist solutions of (1.3) whose trajectories cross the $u$-axis between $u^*$ and $u_3$.

Let $(u(x), v(x))$, $-\frac{L}{2} \leq x \leq \frac{L}{2}$, be such a solution. Let $-\frac{L}{2} \leq x^* \leq \frac{L}{2}$ be such that $v(x^*) = 0$. Thus, $u(x^*)$ is the maximal density of the outbreak population reached on the given habitat of length $L$.

Applying the Mean Value Theorem to the function $u(x)$, we get:

$$u(x^*) - u\left(-\frac{L}{2}\right) = u'(\xi_1)\left(x^* + \frac{L}{2}\right) = v(\xi_1)\left(x^* + \frac{L}{2}\right), \quad -\frac{L}{2} < \xi_1 < x^*,$$

$$u\left(\frac{L}{2}\right) - u(x^*) = u'(\xi_2)\left(\frac{L}{2} - x^*\right) = v(\xi_2)\left(\frac{L}{2} - x^*\right), \quad x^* < \xi_2 < \frac{L}{2}.$$

Noticing that $u\left(-\frac{L}{2}\right) = u\left(\frac{L}{2}\right) = 0$ and $v(\xi_1), v(\xi_2) \neq 0$, we obtain:

$$x^* + \frac{L}{2} = \frac{u(x^*)}{v(\xi_1)}$$

and

$$\frac{L}{2} - x^* = -\frac{u(x^*)}{v(\xi_2)}.$$

Adding the above equations, we get

$$L = \frac{u(x^*)}{v(\xi_1)} - \frac{u(x^*)}{v(\xi_2)} = \frac{u(x^*)}{|v(\xi_1)|} + \frac{u(x^*)}{|v(\xi_2)|}.$$

To get a lower bound on $L$, we will need upper bounds $M_1$ and $M_2$ on $|v(x)|$ for $-\frac{L}{2} < x < x^*$ (upper portion of the solution orbit) and for $x^* < x < \frac{L}{2}$ (lower portion of the solution orbit), respectively, as well as a lower bound on $u(x^*)$, which can be chosen to be just $u_2$.

Thus, if we denote by $L_c$ the smallest domain that can support an outbreak population, we get

$$L_c \geq \frac{u_2}{M_1} + \frac{u_2}{M_2}.$$

Next, we will obtain the estimates $M_1$ and $M_2$.

In order to obtain $M_1$, it suffices to ensure that for all $u \geq 0$ and $v \geq M_1$, we will have $qv + g(u) > 0$ which means that the $v$-component of the vector field given by the system (1.3) will be positive in the region $[0, \infty) \times [M_1, \infty)$. Geometrically, the inequality means that whenever an orbit satisfying $(u', v') = (v, qv + g(u))$ crosses into that region, it will stay in it and will never intersect the $u$-axis. Thus, every solution orbit of (1.3) will stay below the line $v = M_1$.

Therefore, we need $M_1$ such that $v > -\frac{g(u)}{q}$ for all $v \geq M_1$ and $u \geq 0$. Since $g(u) > 0$ for $u > u_3$, we can take $M_1$ to be the maximum of the function $-\frac{g(u)}{q}$ on the interval $[0, u_3]$. Thus,

$$M_1 = \frac{1}{q} \max_{0 \leq u \leq u_3} \left(-g(u)\right).$$

Next, we will obtain the estimate $M_2$ such that $|v(x)| \leq M_2$ for $x^* \leq x \leq \frac{L}{2}$. The approach is to find a straight line in the $uv$-plane, passing through $(u_3, 0)$, with a positive slope $m$, such that the vector field $(v, qv + g(u))$ when restricted to the portion of the line in the fourth quadrant, will have a non-obtuse angle with the normal vector (to the line) pointing towards the origin. Geometrically, this guarantees that the portion of the solution orbit $(u(x), v(x))$ for $x^* \leq x \leq \frac{L}{2}$ lies above the line.

The equation of the line is given by $v = m(u - u_3)$, and its normal vector pointing towards the origin can be chosen to be $(-m, 1)$. Thus, the condition on $m$ is $(-m, 1) \cdot (v, qv + g(u)) \geq 0$ where $v = m(u - u_3)$ and $0 < u < u_3$. Hence, we obtain:

$$m(u - u_3)(q - m) + g(u) \geq 0.$$

Simplifying, we get:

$$m^2 - mq \geq \frac{g(u)}{u - u_3}.$$

This inequality has to hold for all $u \in (0, u_3)$. To find a value of $m$ satisfying the above inequality, we will first find an upper bound on its right hand side.

Define a function $h : [0, u_3] \to \mathbb{R}$ as follows

$$h(u) = \begin{cases} \frac{g(u)}{u - u_3}, & u < u_3 \\ g'(u_3), & u = u_3. \end{cases}$$

Clearly, $h$ is continuous on $[0, u_3]$, and by the Extreme Value Theorem, it attains its maximal value, $h_{max} (> 0)$ in $[0, u_3]$. Then it suffices to let $m$ be the positive root of the equation

$$m^2 - mq - h_{max} = 0,$$

i.e.

$$m = \frac{q + \sqrt{q^2 + 4h_{max}}}{2}.$$

Let $M_2 = mu_3$.

Then for all $x^* \le x \le \frac{L}{2}$, we have $-M_2 \le v(x) \le 0$, i.e. $|v(x)| \le M_2$.

To summarize, we have obtained the following lower bound on $L_c$:

$$L_c \ge \frac{u_2}{M_1} + \frac{u_2}{M_2}, \tag{3.1}$$

where $M_1 = \frac{1}{q}\max_{0\le u\le u_3}(-g(u))$, $M_2 = u_3 m = u_3 \frac{q+\sqrt{q^2+4h_{max}}}{2}$ and $h_{max}$ is the maximal value of $h[0,u_3] \to \mathbb{R}$ as defined above.

Next, we will improve the lower bound on $L_c$ by changing the way we estimate the "parametric length" of the lower portion of a solution orbit.

Recall that the line $v = m(u - u_3)$ as defined above stays below the curve $(u(x), v(x))$ for $x^* \le x \le \frac{L}{2}$. That is, for all $x^* \le x \le \frac{L}{2}$ we have $v(x) \ge m(u(x) - u_3)$.

Let $\varepsilon > 0$ be sufficiently small so that $x^* + \varepsilon < \frac{L}{2}$. Note that $u'(x) = v(x) < 0$ for all $x^* + \varepsilon < x < \frac{L}{2}$. Note that the curve $(u(x), v(x))$, $x^* \le x \le \frac{L}{2}$, satisfies the vertical line test and thus can be viewed as the graph of a function $v = v(u)$ where $0 \le u \le u(x^*)$. Then

$$\frac{L}{2} - x^* - \varepsilon = \int_{x^*+\varepsilon}^{\frac{L}{2}} dx = \int_{x^*+\varepsilon}^{\frac{L}{2}} \frac{u'(x)dx}{u'(x)} = \int_{x^*+\varepsilon}^{\frac{L}{2}} \frac{u'(x)dx}{v(u(x))} = \int_{u(x^*+\varepsilon)}^{u\left(\frac{L}{2}\right)} \frac{du}{v(u)} = \int_{u(x^*+\varepsilon)}^{0} \frac{du}{v(u)}.$$

Taking the limit as $\varepsilon \to 0^+$, we observe the convergence of the improper intergal:

$$\frac{L}{2} - x^* = \int_{u(x^*)}^{0} \frac{du}{v(u)} = -\int_{0}^{u(x^*)} \frac{du}{v(u)} = \int_{0}^{u(x^*)} \frac{du}{-v(u)} \ge \int_{0}^{u(x^*)} \frac{du}{m(u_3 - u)}.$$

Noticing that $u_3 - u > 0$ for all $0 \le u \le u(x^*)$, we get:

$$\frac{L}{2} - x^* \ge \frac{1}{m} \ln(u_3 - u)\,\Big|_{u=0}^{u=u(x^*)} = \frac{1}{m} \ln \frac{u_3}{u_3 - u(x^*)}.$$

Since $u(x^*) > u_2$, we obtain

$$\frac{L}{2} - x^* > \frac{1}{m} \ln \frac{u_3}{u_3 - u_2}.$$

As before, using the inequality $x^* + \frac{L}{2} \ge \frac{u_2}{M_1}$, we get an estimate on $L_c$ alternative to (3.1):

$$L_c > \frac{u_2}{M_1} + \frac{1}{m} \ln \frac{u_3}{u_3 - u_2}. \tag{3.2}$$

Note that (3.2) is a better estimate on $L_c$ than (3.1) since

$$\frac{1}{m} \ln \frac{u_3}{u_3 - u_2} > \frac{u_2}{M_2} = \frac{u_2}{m u_3}.$$

Indeed, we have $0 < \frac{u_2}{u_3} < 1$ and $\ln \frac{1}{1-x} > x$ for all $0 < x < 1$.

## References

[1] D. Ludwig, D.G. Aronson, H.F. Weinberger, Spatial patterning of the bruce budworm, J. Math. Biol., 8(3), 217-258, 1979.