Estimation of Frequency, Amplitude, and Phase from the DFT of a Time Series

Barry G. Quinn

***Abstract*—In a previous paper, a frequency estimator using only three Fourier coefficients was introduced, which has asymptotic variance of order T^{-3}. In this correspondence, a similar technique of Rife and Vincent is shown to have asymptotic variance of larger order. A new estimator is introduced that has asymptotic variance less than 1.65 times the CRLB.**

I. Introduction

As in [1], we consider the estimation of the frequency parameter $\omega \in (0, \pi)$ in the model

$$X_t = \mu + \rho\cos(\omega t + \phi) + \epsilon_t, t = 0, 1, \ldots, T-1 \qquad (1)$$

where $\{\epsilon_t\}$ is some random noise sequence. The least squares estimator of ω and the maximizer of the periodogram are both asymptotically zero mean normally distributed under the conditions of [1, Section III] and have asymptotic variance $48\pi f(\omega)/(T^3\rho^2)$, where $f(\omega)$ is the spectral density of the assumed zero mean stationary sequence $\{\epsilon_t\}$, as shown by Hannan in [2].

II. The Estimator of Rife and Vincent

Put

$$Y_j = \sum_{t=0}^{T-1} X_t \exp(-i2\pi jt/T), U_j = \sum_{t=0}^{T-1} \epsilon_t \exp(-i2\pi jt/T). \qquad (2)$$

Letting $\delta = T\omega/(2\pi) - j$ and considering only the case where $\delta \in [-1/2, 1/2]$, we have, for $k = -1, 0, 1$, as shown in [1]

$$\frac{Y_{j+k}}{Y_j} = \frac{\delta}{\delta - k} + \frac{1}{c}T^{-1}U_{j+k} - \frac{\delta}{c(\delta-k)}T^{-1}U_j + O\big(T^{-1}\log T\big) \qquad (3)$$

almost surely, as $T \to \infty$. As the error term in the above equation is of small order, the term $\delta/(\delta - k)$ may be estimated by the real part of the left side of (3), as in [1]. In [4], however, only the moduli of the left sides are used. The procedure of Rife and Vincent is as follows:

1) Put k_T equal to the maximizer of $\left|Y_j^2\right|, 1 \le j \le [(T-1)/2]$.
2) Put $\widehat{\alpha} = 1$ if $|Y_{k_T+1}|^2 > |Y_{k_T-1}|^2$ and -1 otherwise.
3) Estimate ω by $\widehat{\omega}_T = 2\pi(k_T + \widehat{\delta})/T$, where $\widehat{\delta} = \widehat{\alpha}R_T/(1 + R_T)$, and $R_T = \left|Y_{k_T+\widehat{\alpha}}/Y_{k_T}\right|$.

III. Statistical Properties of the R&V Estimator

Let j_T be the closest integer to $T\omega/(2\pi)$, with either integer being taken in case of ambiguity, and let $\delta_T = T\omega/(2\pi) - j_T$, which is in $[-1/2, 1/2]$. Then, with k_T defined below (3), we have, from [1], the following lemma.

Lemma 1: $\limsup_{T\to\infty} |k_T - j_T| \le 1$, almost surely.

Now, if δ_T is close to $1/2$, k_T may equal $j_T + 1$ with nonzero probability. The estimator $\widehat{\omega}_T$ will then, however, be the same as if k_T were equal to j_T because of the definition of $\widehat{\omega}_T$. The same is true if δ_T is close to $-1/2$. Problems arise, however, if $|\delta_T|$ is small. The following lemma is needed to demonstrate the poor behavior of $\widehat{\omega}_T$.

Lemma 2: Denote by $< \theta >$ the distance between θ and its nearest integer. Then, for $\theta \in (0, 1)$ and irrational, there is an infinite sequence of integers $\{r_n\}$ for which $r_n^{1/4} < r_n\theta >$ converges to 1.

Proof: If θ is irrational, the nth convergent p_n/q_n in the continued fraction of θ satisfies $|\theta - p_n/q_n| < q_n^{-2}$. Put d_n equal to $< h_n q_n \theta >$ for some integer $h_n > 1$, and let $\nu_n = \theta - p_n/q_n$. Then, $d_n =< h_n q_n \nu_n >$, which is equal to $|h_n q_n \nu_n|$ as long as $h_n q_n < |2\nu_n|^{-1}$. Let $r_n = h_n q_n$, where h_n is the integer part of $q_n^{-1}|\nu_n|^{-4/5}$, which will be less than $(2q_n|\nu_n|)^{-1}$ for n large enough as $|\nu_n|$ is less than 1, and which is diverging as $q_n^{-1}|\nu_n|^{-4/5}$ is greater than $q_n^{3/5}$. Then, $r_n^{1/4} < r_n\theta >$ converges to 1, and $\{r_n\}$ is an infinite sequence of integers. ■

Note that the same proof shows that for $\xi > 0$, there is an infinite sequence $\{r_n\}$ for which $r_n^{1/4} < r_n\theta >$ converges to ξ.

The estimator in [1] and the least squares estimator have the property that $T^{3/2-\varepsilon}(\widehat{\omega}_T - \omega)$ converges in probability to zero for any $\varepsilon > 0$. Moreover, the "asymptotic" variances, that is, the variances of the limiting distributions of the estimators, are of order T^{-3}. The remarkable feature of the estimator in [4] is that the correct order is reached if and only if $\omega/(2\pi)$ is rational, as is shown in the following theorem and corollary.

Theorem 1: Let $\widehat{\omega}_T$ be as defined above, with $\omega/(2\pi)$ irrational. Then, $T^{5/4}(\widehat{\omega}_T - \omega)$ does not converge in probability to zero.

Proof: Construct the sequence $\{r_n\}$ as defined above with $\theta = \omega/(2\pi)$. Thus, $\omega = 2\pi(j_T + \delta_T)/T$, where $r_n^{1/4}|\delta_{r_n}|$ converges to 1. For $T \in \{r_n\}$, it is shown in [1] that $\Pr\{k_T = j_T\}$ converges to 1. Thus

$$T(\widehat{\omega}_T - \omega)/(2\pi) \sim \frac{\widehat{\alpha}_T R_T}{1 + R_T} - \delta_T$$

where the T subscripts are now adopted so that there is no confusion. Now, from [1]

$$R_T \sim |\delta_T| + \operatorname{sgn}(\delta_T)\frac{V_{\widehat{\alpha}_T}}{T|c_T|}$$

where $V_\alpha = \Re(U_{j_T+\alpha}\overline{c_T})/|c_T|$. Thus

$$\begin{aligned} &T^{5/4}(\widehat{\omega}_T - \omega)/(2\pi) \\ &\sim T^{1/4}(\widehat{\alpha}_T|\delta_T| - \delta_T) + \operatorname{sgn}(\delta_T)T^{-3/4}V_\alpha/|c_T| \\ &\sim T^{1/4}(\widehat{\alpha}_T|\delta_T| - \delta_T) \end{aligned} \qquad (4)$$

as the distributions of $T^{-1/2}V_1$ and $T^{-1/2}V_{-1}$ converge to normal distributions with zero means and bounded variances. On the subset of the sample space on which $\widehat{\alpha}_T$ does not have the same sign as δ_T, it follows that $T^{5/4}(\widehat{\omega}_T - \omega) \sim -2$. It remains, therefore, to show that the probability of this subspace is nonzero in the limit. Now, again from [1], if $\delta_T < 0$

$$\begin{aligned} \Pr\{\widehat{\alpha}_T = 1\} \sim \Pr\Bigg\{&\frac{\delta_T^2}{(\delta_T - 1)^2} + \frac{2\delta_T}{\delta_T - 1}T^{-1}V_1/|c_T| \\ &> \frac{\delta_T^2}{(\delta_T + 1)^2} + \frac{2\delta_T}{\delta_T + 1}T^{-1}V_{-1}/|c_T|\Bigg\} \\ \sim \Pr\Big\{&T^{-1/2}(V_{-1} + V_1) > T^{1/2}\delta_T^2|c_T|\Big\}. \end{aligned}$$

Manuscript received April 2, 1993; revised September 8, 1994. The associate editor coordinating the review of this paper and approving it for publication was Prof. Kevin M. Buckley.

The author is with the Department of Mathematical and Computing Sciences, Goldsmiths College, University of London, New Cross, London, SE14 6NW U.K. (e-mail: maa01bgq@gold.ac.uk).

Publisher Item Identifier S 1053-587X(97)01876-X.

The same formula holds for $\Pr\{\widehat{\alpha}_T = -1\}$ when $\delta_T > 0$. However, the distribution of $T^{-1/2}(V_{-1}+V_1)$ converges to the normal with mean zero and variance $2\pi f(\omega)$, $T^{1/2}\delta_T^2$ converges to 1, and $|c_T| \sim \rho/2$. Consequently, $\Pr\{\widehat{\alpha}_T \neq \operatorname{sgn}(\delta_T)\}$ converges to $1-\Phi(z)$, where $z = \rho/\left\{2[2\pi f(\omega)]^{1/2}\right\}$, and $\Phi(z)$ is the standard normal distribution function. Hence, $r_n^{5/4}(\widehat{\omega}_{r_n} - \omega)$ does not converge in probability to 0, and $T^{5/4}(\widehat{\omega}_T - \omega)$ can, therefore, not converge in probability to zero. ■

Corollary: If $\omega/(2\pi)$ is rational, $\widehat{\omega}_T$ has "asymptotic variance" of order T^{-3}.

Proof: Let $\omega/(2\pi) = p/q$, where p and q are relatively prime. The fractional parts of $T\omega/(2\pi)$ are thus of the form k/q, where k is a nonnegative integer less than q. Consequently, $T^{\varepsilon}|\delta_T|$ is either zero or diverges to infinity, for any $\varepsilon > 0$. When δ_T is zero, the above analysis shows that

$$T^{3/2}(\widehat{\omega}_T - \omega)/(2\pi) \sim \widehat{\alpha}_T T^{-1/2}\left|U_{j_T+\widehat{\alpha}_T}\right| \tag{5}$$

whereas when δ_T is not zero, it is easily shown that the incorrect sign problem occurs only with probability converging to zero. As the real and imaginary parts of $T^{-1/2}U_{j_T+\widehat{\alpha}_T}$ have distributions converging to the normal with mean zero and variance $\pi f(\omega)$, the "asymptotic variance" is of order T^{-3}. ■

Note that it is straightforward to show that the left side of (5) does not converge in distribution since the limiting distribution for those T for which Tp is a multiple of q is obtained from (5) and involves the exponential distribution, whereas the limiting distribution for those T for which Tp is not divisible by q is normal. Moreover, if q is, for example, a large prime, the incorrect sign can still arise with high probability for moderately large T.

IV. A Modified Estimator and Its Properties

The natural question to ask at this stage is whether the procedure may be easily fixed, for the procedure of [1] requires twice as much storage: the moduli of the (complex) Fourier coefficients are used to select k_T, and the real parts of two successive ratios of these coefficients are then used to construct the estimator. One therefore must keep the full complex Fourier coefficients, instead of their moduli. The answer to this question is that the moduli together with the signs of $\left\{\Re\left(Y_j\bar{Y}_{j+1}\right); 0 \le j \le [(T-3)/2]\right\}$ suffice to construct an estimator which does not have the undesirable feature exhibited in Theorem 1, but which is nevertheless not equivalent to the estimator of [1]:

1) Put k_T equal to the maximizer of $\left|Y_j{}^2\right|, 1 \le j \le [(T-1)/2]$.
2) Let $s_\alpha = \operatorname{sgn}\Re\left(Y_{k_T+\alpha}\bar{Y}_{k_T}\right)$. Let $\widehat{\alpha} = 1$ if $s_{-1} = 1$ and $s_1 = -1$ and let $\widehat{\alpha} = -1$ otherwise.
3) Estimate ω by $\widehat{\omega}_T = 2\pi\left(k_T + \widehat{\delta}\right)/T$, where $\widehat{\delta} = \widehat{\alpha}\frac{R_T}{1+R_T}$ and $R_T = -s_{\widehat{\alpha}}\left|Y_{k_T+\widehat{\alpha}}/Y_{k_T}\right|$.

Note: s_1 is also equal to $\operatorname{sgn}\Re\left(Y_{k_T}\bar{Y}_{k_T+1}\right)$.

From the proofs of theorems 2 and 3 of [1], it follows that

Theorem 2: Under conditions 1-3, $T^{3/2}(\log T)^{-1/2-\nu}(\widehat{\omega}_T - \omega)$ converges almost surely to zero for all $\nu > 0$.

The estimator does not have the same central limit theorem as the estimator in [1], and in fact does not have a normal limit theorem. It may be shown analytically that the estimator behaves in the same way as that of [1] for δ_T fixed as $T \to \infty$, but has a rather odd behavior if $|\delta_T|$ is close to zero, as it will be for some T if ω is fixed. This is borne out by the simulations of Section VII.

V. The Best Estimator and Its Properties

The estimator in [1] was constructed by choosing between the two estimators $\widehat{\delta}_{-1}$ and $\widehat{\delta}_1$ given by

$$\widehat{\delta}_\alpha = \frac{\alpha\widehat{\beta}_\alpha}{\widehat{\beta}_\alpha - 1}, \widehat{\beta}_\alpha = \Re\left(\frac{Y_{k_T+\alpha}}{Y_{k_T}}\right) \tag{6}$$

in such a way that the estimator with smaller "asymptotic variance" was chosen. As the two estimators are jointly asymptotically normal with known covariance matrix, it is possible to find that function of the estimators that is strongly consistent and has the least "variance" possible among such functions. Let $\tilde{\delta}_T = \left(\tilde{\delta}_{-1} + \tilde{\delta}_1\right)/2 + \kappa\left(\tilde{\delta}_1^2\right) - \kappa\left(\tilde{\delta}_{-1}^2\right)$, where

$$\kappa(x) = \frac{1}{4}\log(3x^2+6x+1) - \frac{\sqrt{6}}{24}\log\left(\frac{x+1-\sqrt{\frac{2}{3}}}{x+1+\sqrt{\frac{2}{3}}}\right).$$

$\tilde{\delta}_\alpha$ are constructed from (9) with j_T replacing k_T, and let $\tilde{\omega}_T = 2\pi\left(j_T + \tilde{\delta}_T\right)/T$. Then, we have the following theorem.

Theorem 3: $T^{3/2}(\tilde{\omega}_T - \omega)/s_T$ has a distribution converging to the standard normal, where s_T^2 is given by (8) and is uniformly the smallest possible amongst all consistent estimators constructed from $\tilde{\delta}_{-1}$ and $\tilde{\delta}_1$.

Proof: From (5), $\Re\left(\frac{Y_{j_T+\alpha}}{Y_{j_T}}\right) = \frac{\delta}{\delta-\alpha} + W_\alpha$, where $W_\alpha = T^{-1}\left[\Re\left(c^{-1}U_{j_T+\alpha}\right) - \frac{\delta}{\delta-\alpha}\Re\left(c^{-1}U_{j_T+\alpha}\right)\right]$. Thus

$$\tilde{\delta}_\alpha - \delta \sim \frac{\alpha\left(\frac{\delta}{\delta-\alpha} + W_\alpha\right)}{\frac{\delta}{\delta-\alpha} - 1 + W_\alpha} - \delta \sim -\alpha^{-1}(\delta-\alpha)^2 W_\alpha.$$

It follows from [5] that $T^{1/2}\Omega^{1/2}\left[\tilde{\delta}_{-1} - \delta, \tilde{\delta}_1 - \delta\right]'$ has a distribution converging to the bivariate normal with mean zero and covariance matrix the identity, where Ω is

$$\frac{\pi f(\omega)}{|c|^2}\begin{bmatrix}(\delta+1)^4 + \delta^2(\delta+1)^2 & -\delta^2\left(\delta^2-1\right) \\ -\delta^2\left(\delta^2-1\right) & (\delta-1)^4 + \delta^2(\delta-1)^2\end{bmatrix}.$$

We shall restrict the class of functions of $\tilde{\delta}_{-1}$ and $\tilde{\delta}_1$ to functions of the form $g(\tilde{\delta}_{-1}) + g_1(\tilde{\delta}_1)$ as it turns out there is a member of this class that has asymptotic variance that is the same as the Cramér-Rao lower bound (CRLB) for unbiased estimators of δ, given $\{Y_{j_T+\alpha}; \alpha = -1, 0, 1\}$. Since we must have $\tilde{\delta} - \delta$ converging in probability to zero, g must satisfy $\delta = g(\delta) + g_1(\delta)$. Consequently, we consider only estimators $\tilde{\delta}$ of the form $g(\tilde{\delta}_{-1}) + \tilde{\delta}_1 - g(\tilde{\delta}_1)$. Now, assuming that the derivative h of g is continuous

$$\tilde{\delta} - \delta \sim \tilde{\delta}_1 - \delta + h(\delta)\left(\tilde{\delta}_{-1} - \delta\right) - h(\delta)\left(\tilde{\delta}_1 - \delta\right).$$

The asymptotic variance of $T^{1/2}\left(\tilde{\delta} - \delta\right)$ is thus

$$\frac{\pi f(\omega)}{|c|^2}\left\{h^2\left[(\delta+1)^4 + \delta^2(\delta+1)^2\right] + (1-h)^2\left[(\delta-1)^4 + \delta^2(\delta-1)^2\right] - 2h(1-h)\delta^2(\delta^2-1)\right\}$$

where $h = h(\delta)$. This expression is quadratic in h for fixed δ and has minimum value

$$\frac{\pi f(\omega)}{2|c|^2}\frac{\left(\delta^2-1\right)^2\left(3\delta^4+1\right)}{3\delta^4+6\delta^2+1} \tag{7}$$

when $h = \frac{1}{2} - \frac{3\delta^3+2\delta}{3\delta^4+6\delta^2+1}$. As h is the derivative of g, it follows that

$$g(\delta) = \frac{\delta}{2} - \frac{1}{4}\log(3\delta^4+6\delta^2+1) - \frac{\sqrt{6}}{24}\log\left(\frac{\delta^2+1-\sqrt{\frac{2}{3}}}{\delta^2+1+\sqrt{\frac{2}{3}}}\right)$$

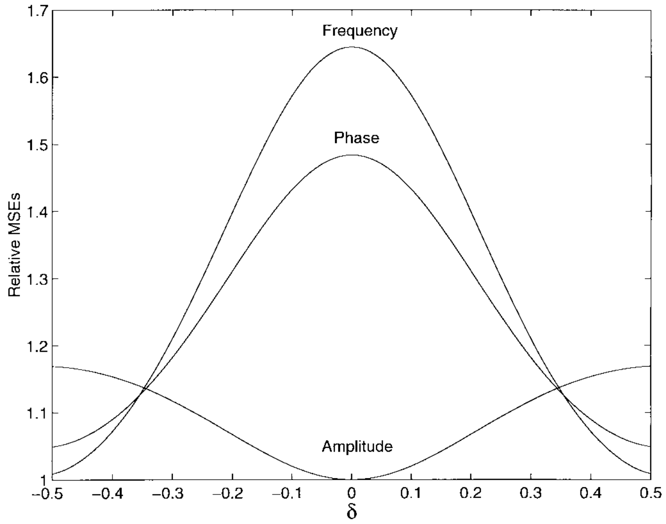

Fig. 1. Plots of MSE's of estimators.

plus an irrelevant arbitrary constant. It is shown in the Appendix that the expression in (7) is T times the CRLB for the variance of unbiased estimators of δ, using only $\{Y_{j_T+\alpha}; \alpha = -1, 0, 1\}$ to construct the likelihood. The theorem therefore follows, with s_T^2 given by $4\pi^2$ times (7) or

$$\frac{8\pi^5 f(\omega)}{\rho^2}\frac{\left(\delta^2-1\right)^2\left(3\delta^4+1\right)}{3\delta^4+6\delta^2+1}\frac{\delta^2}{\sin^2(\pi\delta)} \tag{8}$$

which decreases, as $|\delta|$ increases from 0 to $\frac{1}{2}$, from $8\pi^3 f(\omega)/\rho^2$ to $171\pi^5 f(\omega)/(344\rho^2)$. In other words, the ratio of the asymptotic variance of $\tilde{\omega}_T$ to the asymptotic variance of the least squares estimator decreases from a maximum of $\pi^2/6 \sim 1.6449$ to $171\pi^4/16\,512 \sim 1.0088$ as $|\delta|$ increases from 0 to $\frac{1}{2}$. This should be compared with the corresponding results for the estimator of [1], viz. $\pi^2/3 \sim 3.2899$ to $\pi^4/96 \sim 1.0147$. ■

Corollary: The estimator $\widehat{\omega}_T = \frac{2\pi(k_T+\widehat{\delta}_T)}{T}$, where $\widehat{\delta}_T = \left(\widehat{\delta}_{-1}+\widehat{\delta}_1\right)/2+\kappa(\widehat{\delta}_1^2)-\kappa(\widehat{\delta}_{-1}^2)$ and $\widehat{\delta}_{-1}, \widehat{\delta}_1$ and $\kappa()$ are defined by and below (6), has the same asymptotic properties as $\tilde{\omega}_T$.

Proof: From [1], problems might arise when $|\delta_T|$ is close to $\frac{1}{2}$, for then, it is true with significant probability that $k_T \neq j_T$. A close look at the proof of [1, Theorem 3], however, shows that the difference between $\widehat{\omega}_T$ and $\tilde{\omega}_T$, in that case, is at most $O(T^{-2}\log T)$ almost surely and, therefore, negligible as the root mean square error is $O(T^{-3/2})$. The results of Theorem 3, therefore, extend to $\widehat{\omega}_T$. ■

VI. Best Estimators of Amplitude and Phase

Using [1, (4)], the (asymptotic) log-likelihood given only $\{Y_{j_T+k}; k = -1, 0, 1\}$ is

$$\ell = -3\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{k=-1}^{1}\left|Y_{j_T+k} - T\rho\exp(i\phi)c_k\right|^2$$

plus terms of smaller order, where

$$c_k = [\exp(2\pi i\delta)-1]/[4\pi i(\delta-k)], \sigma^2 = \frac{T}{2}2\pi f(\omega).$$

If δ were known, ℓ would be maximized when

$$\rho = T^{-1}\frac{\left|\sum_{k=-1}^{1} Y_{j_T+k}\bar{c}_k\right|}{\sum_{k=-1}^{1}|c_k|^2}, \phi = \arg\left(\sum_{k=-1}^{1} Y_{j_T+k}\bar{c}_k\right). \tag{9}$$

As is shown in the Appendix, however, the estimator $\widehat{\delta}_T$ derived in Section V asymptotically achieves the CRLB for unbiased estimators

TABLE I
Comparison of the Five Estimators

	i	ii	iii	iv	v
b	-1.08e-4	-1.46e-4	6.83e-5	-3.87e-3	8.64e-4
r	5.74e-3	4.31e-3	3.72e-3	9.95e-3	8.43e-3
t	5.57e-3	4.31e-3	3.38e-3		
b	-1.34e-3	2.04e-4	-4.54e-4	-1.43e-3	-1.35e-3
r	6.51e-3	7.53e-3	3.62e-3	9.35e-3	6.58e-3
t	3.41e-3	3.40e-3	3.38e-3		
b	1.04e-3	2.06e-4	1.17e-5	-1.07e-3	1.22e-3
r	6.49e-3	4.37e-3	3.71e-3	8.98e-3	8.99e-3
t	6.14e-3	4.34e-3	3.38e-3		
b	-5.96e-6	-9.93e-7	2.78e-6	-1.75e-4	4.59e-5
r	2.67e-4	2.06e-4	1.55e-4	5.49e-4	3.27e-4
t	2.46e-4	1.91e-4	1.50e-4		
b	4.01e-6	2.96e-6	3.94e-6	5.58e-6	5.58e-6
r	1.39e-4	1.50e-4	1.38e-4	1.31e-4	1.31e-4
t	1.51e-4	1.50e-4	1.50e-4		
b	4.58e-6	-1.98e-6	-2.02e-6	6.64e-7	-7.28e-7
r	2.72e-4	2.07e-4	1.53e-4	4.26e-4	3.66e-4
t	2.71e-4	1.92e-4	1.50e-4		

of δ. Hence, the estimators of ρ and ϕ obtained by using $\widehat{\delta}_T$ in place of δ in (9) are asymptotically equivalent to the (limited information) maximum likelihood estimators of ρ and ϕ. We therefore have Theorem 4, as follows.

Theorem 4: $T^{1/2}\Sigma^{-1/2}\left[\widehat{\rho}_T - \rho, T(\widehat{\omega}_T-\omega)/(2\pi), \widehat{\phi}_T - \phi\right]'$ is asymptotically normal with mean zero and covariance matrix the identity, where Σ is given by (10).

It follows that the asymptotic variances of the estimators can be obtained from the diagonal elements of Σ. Fig. 1 comprises plots of the mean square errors, as functions of $\delta \in (-\frac{1}{2}, \frac{1}{2})$ of the three estimators relative to those of the least squares estimator (i.e., the Gaussian, not-necessarily-white, CRLB). It is interesting to note that the performance of $\widehat{\phi}_T$ and $\widehat{\omega}_T$ is worst when that of $\widehat{\rho}_T$ is best, and vice versa.

VII. Simulations

Equation (1) was used to generate time series, and five algorithms were used to estimate frequency. The sample sizes T used were 128 and 1024. The added noise was pseudo-Gaussian and white with zero mean and variances σ^2 1. The frequencies ω used were $15\pi/32 + 0.05\pi/T, 15\pi/32 + \pi/T$ and $15\pi/32$ so that the first and third frequencies would cause problems for the Rife and Vincent estimator, whereas the second would not. For each combination of T, σ^2 and ω, 100 replications were carried out. The results of the simulations are contained in Table I. Each horizontal section contains nine rows, in three groups of three, corresponding to each of the three frequencies. The top section contains the results for $T = 128$ and the bottom those for $T = 1024$. The first line in each grouping contains the differences between the average of the 100 estimates and the true frequency and is labeled 'b', for bias, the second, which is labeled 'r,' is the sample root mean square error of the estimates, and the third, which is labeled 't,' is the theoretical asymptotic root mean square error. The five methods used were the following:

i) the method in [1];
ii) the modified method of Section V;
iii) the method of Quinn and Fernandes [6], which is equivalent to least squares;
iv) the method of Rife and Vincent;
v) the modified Rife and Vincent method of Section IV.

None of the results is unexpected. The first three methods behave according to the asymptotic theory, the method of Rife and Vincent yields estimates whose root mean square errors are unacceptably large, as predicted, for the frequency $15\pi/32 + 0.05/T$, and both methods iv) and v) have large root mean square errors for the third frequency $15\pi/32$. The results for other values of σ^2 and T also accorded with the theory but are not reported here for the sake of brevity.

APPENDIX
THE INFORMATION MATRIX

The first derivatives of ℓ are given by

$$\frac{\partial \ell}{\partial \rho} = -\frac{1}{\sigma^2}\left\{\rho T^2 \sum_{k=-1}^{1} |c_k|^2 - T\Re\left[\exp(-i\phi)\sum_{k=-1}^{1} Y_{j_T+k}\bar{c}_k\right]\right\}$$

$$\frac{\partial \ell}{\partial \delta} = -\frac{1}{\sigma^2}\left\{\rho^2 T^2 \sum_{k=-1}^{1} \Re\left(c_k \frac{\partial \bar{c}_k}{\partial \delta}\right)\right.$$

$$\left. -\rho T \Re\left[\exp(-i\phi)\sum_{k=-1}^{1} Y_{j_T+k}\frac{\partial \bar{c}_k}{\partial \delta}\right]\right\}$$

$$\frac{\partial \ell}{\partial \phi} = \frac{\rho T}{\sigma^2}\Re\left[-i\exp(-i\phi)\sum_{k=-1}^{1} Y_{j_T+k}\bar{c}_k\right].$$

After a little algebra, the "information" matrix, that is, minus the expectation of the second derivative of ℓ, may be shown to be

$$T^2\sigma^{-2}\begin{bmatrix} c^*c & -\rho\alpha & 0 \\ -\rho\alpha & \rho^2 d^*d & -\rho^2\beta \\ 0 & -\rho^2\beta & \rho^2 c^*c \end{bmatrix}$$

where $c = [c_{-1}, c_0, c_1]', d = \frac{\partial c}{\partial \delta}, \alpha = \Re(d^*c), \beta = \Im(d^*c)$, and * denotes complex conjugate transpose. The matrix Σ is therefore given by

$$\Sigma = \frac{\pi f(\omega)}{c^*cd^*d - \alpha^2 - \beta^2}\begin{bmatrix} \frac{d^*dc^*c-\beta^2}{c^*c} & \frac{\alpha}{\rho} & \frac{\alpha\beta}{\rho c^*c} \\ \frac{\alpha}{\rho} & \frac{c^*c}{\rho^2} & \frac{\beta}{\rho^2} \\ \frac{\alpha\beta}{\rho c^*c} & \frac{\beta}{\rho^2} & \frac{d^*dc^*c-\alpha^2}{\rho^2 c^*c} \end{bmatrix}. \quad (10)$$

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