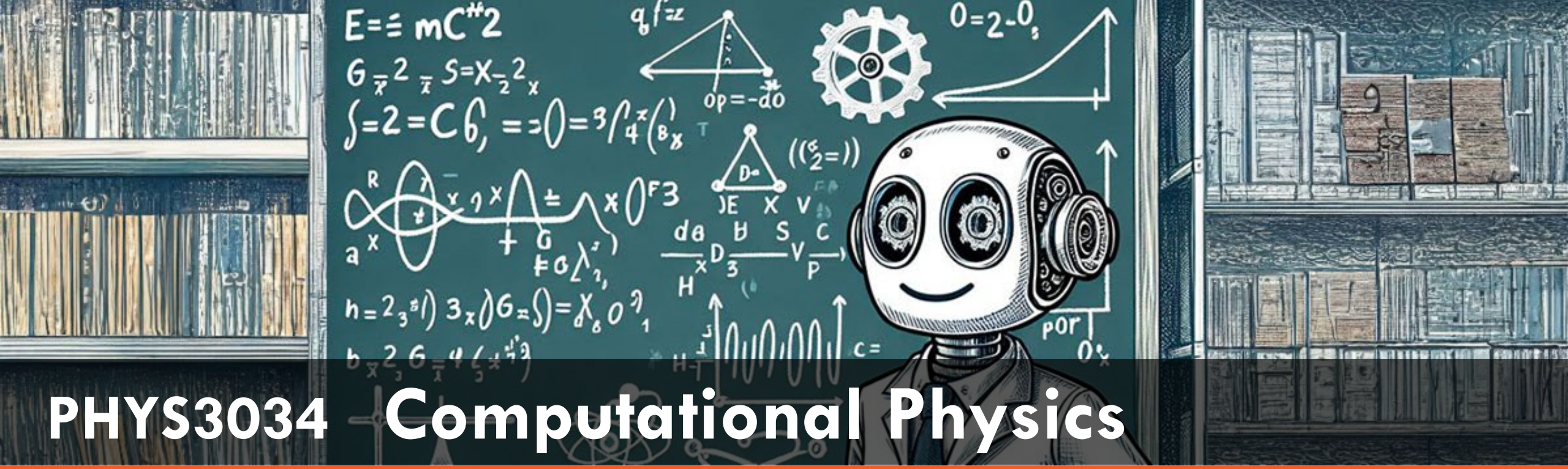

PHYS3034 Computational Physics

A/Professor Sveta Postnova

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We recognise and pay respect to the Elders and communities – past, present, and emerging – of the lands that the University of Sydney's campuses stand on. For thousands of years they have shared and exchanged knowledges across innumerable generations for the benefit of all.

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Celebrating 175 years

Module Structure

Five Lectures: Tuesday 9-10 AM, weeks 1-5

All materials on Canvas

Assessment

Five Computer Labs (10%)

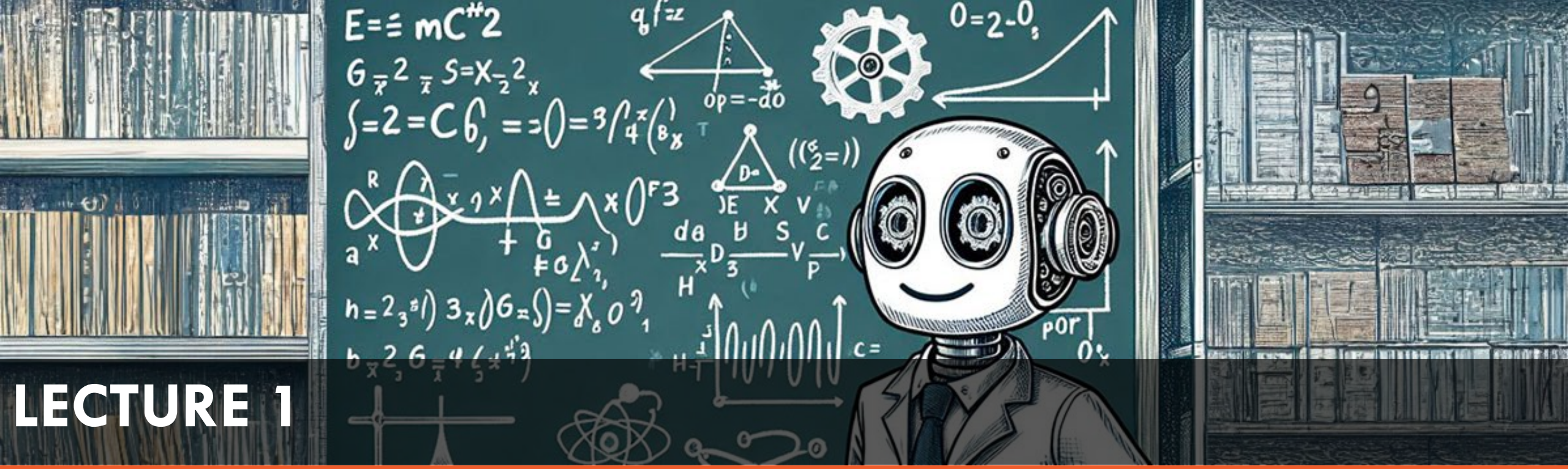
- 2-h labs, Thursday 10 AM – 12 PM, weeks 1-5
- Tutors: Christian Canete, Will Edibam, Akshaya Rajesh, Denis Furtel
- Assessed via Canvas Quiz (2% each) due 2 weeks after the associated lecture

Assignment (10%, no exam)

- Released in week 4, due in week 6

Communication and feedback

- Post questions to Ed Discussion → Computational Physics
- Only email me if you have a question that concerns only you
- All lectures and codes for this module are on GitHub (see Canvas for the link):
  - We will use Python
  - Matlab codes available on Canvas if needed



LECTURE 1

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Computation in Physics

In this course, we learn the theory of how to solve physics problems on computers.

- how to develop numerical algorithms to solve physical equations.
- and how to analyse them for stability and accuracy.

We don't teach coding; we focus on the algorithms and how they work.

Why do we need computation?

Physics has theory, experiment, and *computation*.

Undergrad physics is training in the simplest cases (where analytic techniques can often work). In the real world, physical equations often need to be solved numerically.

- E.g.,: molecular dynamics, N-body problems, nonlinear systems, etc.

Modern physics research relies heavily on computational methods.

- Solving for electric and magnetic fields in stars (Mike Wheatland).
- Dynamics of online social networks (Tristram Alexander).
- Biological rhythms (Svetlana Postnova), and brain dynamics (Pulin Gong, Ben Fulcher), …

Types of systems we will work with

Ordinary Differential Equations (ODEs):

- Numerical error, Euler method – *projectile motion*
- Verlet methods – *planetary motion*
- Runge-Kutta – *motion of a pendulum*

Partial Differential Equations (PDEs):

- FTCS – *heat diffusion*
- Lax method – *wave propagation*

Lecture 1: content

Types of numerical error:

- Floating point error (representation of numbers).
- Truncation error (numerical method).

Dynamics problems: the motion of a particle.

- Simple projectile motion: motion under gravity.

Non-dimensionalisation

- A standard procedure prior to numerical solution.
- Applied to projectile motion.

Euler's method

- Applied to simple projectile motion.

Global truncation error

- Estimating the error of a numerical solution.

Types of numerical error

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Number representation on computer (recap)

Computers don't know numbers; they operate in bits: 0s and 1s

Instead of base-10 (0–9), computers use **base-2**.

Each bit represents a power of 2

Bit position	7	6	5	4	3	2	1	0
Bit value	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

Example: $\mathbf{13_{10} = 00001101_2}$

$\rightarrow 2^3 + 2^2 + 2^0$

$= 8 + 4 + 1$

Unsigned integers:

With **N bits**, you can represent:

$0 \ to \ 2^N - 1$

- 8-bit: 0 → 255
- 16-bit: 0 → 65,535
- 32-bit: 0 → 4,294,967,295

Signed integers:

With **N bits**, you can represent:

$-2^{N-1} \ to \ 2^{N-1} - 1$

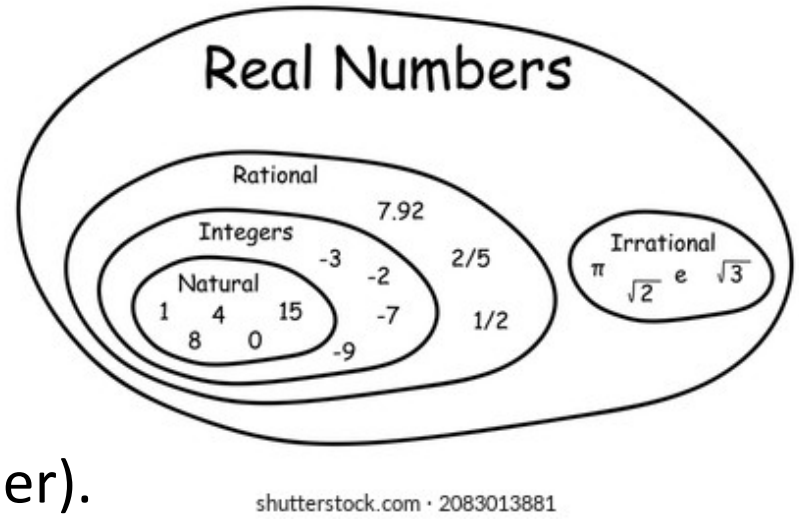
- Negative numbers are stored by inverting bits and adding 1
- The **leftmost bit** is the sign bit
- E.g., $-13_{10} = 11110011_2$

How to represent real numbers?

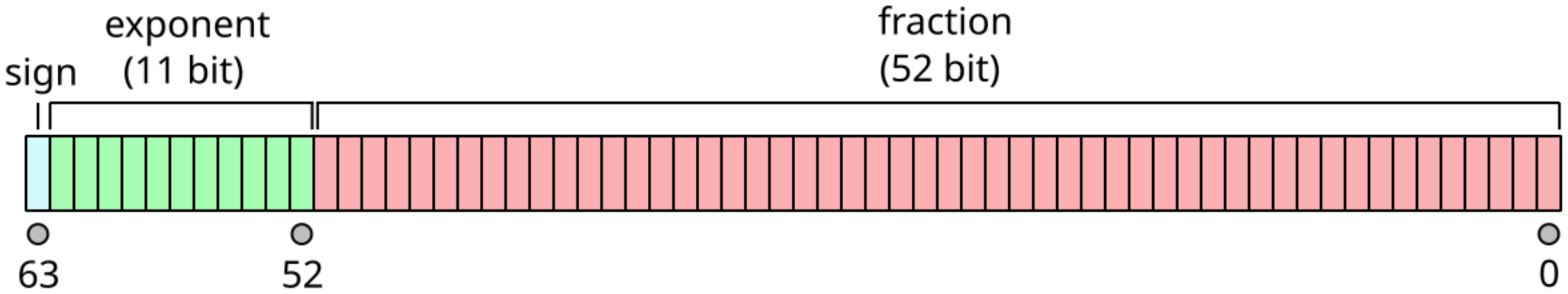
- **Floating point** is an approximate representation of real numbers.
- IEEE double precision (**binary64**) is the standard in modern coding languages.
- Uses base 2 with 64 bits (0s or 1s) per number.
- A binary64 number, $\boldsymbol{x}$, is represented as

$$\boldsymbol{x = (-1)^s \times M \times 2^E}$$

  - s is the sign bit.
  - M is a 52-bit significand (the fractional part of the number).
  - E is the exponent (represented with 11 bits).



How to represent real numbers?



- **Sign bit (1 bit)** → Determines positive (0) or negative (1).
- **Exponent (11 bits)** → 2048 → Stored in **biased** form (E = exponent + 1023).
- **Mantissa (52 bits)** → Stores the significant digits (without the leading 1).

$$x = (-1)^s \times M \times 2^E$$

There are infinitely many reals, but only a finite number of floating-point numbers...

-Yields **range error** and **rounding error**.

Overflow and Underflow errors

Overflow: Number is *too large* for binary64.

Underflow: Number is *too small* for binary64

Precision limit, machine epsilon (eps, ϵ): the fractional spacing between numbers

Quantity	Meaning	Python Value (approx.)	Python command
Real maximum	Largest finite normal number	1.7976931348623157e+308	np.finfo(np.float64).max
Real minimum (normal)	Smallest positive normal number	2.2250738585072014e-308	np.finfo(np.float64).tiny
Smallest subnormal	Smallest positive subnormal number	5e-324	np.finfo(np.float64).smallest_subnormal
Precision limit (ε)	Machine epsilon (spacing at 1)	2.220446049250313e-16	np.finfo(np.float64).eps

Rounding errors

- ***Rounding error*: Every real number is 'rounded' to the nearest floating-point number**
  - Spaced, multiplicatively, by relative factors, eps.

- Let's predict the results of:

❓ >> 1 + eps - 1?

❓ >> 1 + 0.5*eps - 1?

❓ >> 10 + eps - 10?

A real number is stored **exactly** in binary64 **if and only if** it can be written as:

$\frac{k}{2^n}$ *with integers* k, n within the representable exponent range.

That's it. Everything else is rounded.

Rounding errors: test

Calculate: 0.3 + 0.1 - 0.3 - 0.1 = ?

You: ans = 0
Computer: ans = 2.7756e-17

❓ How could this happen?

Are you still worried about computers taking over the world?

Non-Numbers (NaN, nan)

Non-numbers: Me is *too wild* for binary64:

- NaN (or nan) stands for 'not a number’.
- Example: 0/0 doesn't have a numeric answer.
- NaNs 'propagate': NaN + x gives NaN for all x.

Summary: Float-point errors

- *Range error*: due to there being the largest and smallest floats.
  - *Underflow*: $|x| \lesssim 5 \times 10^{-324}$replaced by 0.
  - *Overflow*: $|x| \gtrsim 2 \times 10^{308}$ replaced by Inf.

- *Rounding error* is due to the discrete spaces between floats.
  - Results are rounded to the nearest float.
  - The fractional error is $\leq \frac{1}{2}\epsilon$
  - $\epsilon = 2^{-52} \approx 2.2 \times 10^{-16}$is the *machine epsilon/precision*

Truncation Errors

- A *truncation error* is the error you introduce when you replace an exact mathematical process with a finite approximation.

- When we write down algorithms to *solve equations*, they make numerical approximations of quantities like derivatives. This leads to truncation error.

  - Example: *estimating the derivative* $\boldsymbol{f'(x)}$ numerically, where we can evaluate $\boldsymbol{f(x)}$ for any input, $\boldsymbol{x}$.

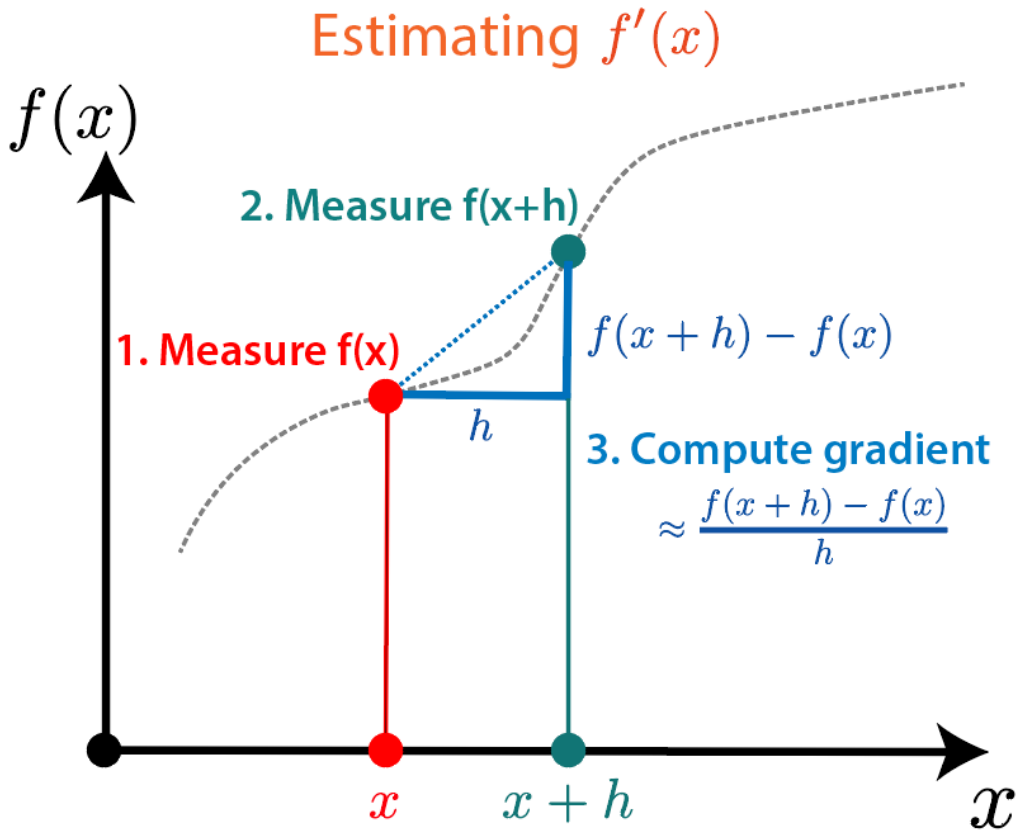
❓ What would be your strategy ('algorithm', 'numerical method')?

Estimating Derivatives: the forward difference approximation

- Formally: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

- We approximate: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$
  - Measure $f(x)$.
  - Measure $f(x+h)$ for some (small) increment, h .
  - Compute the gradient.

If h is small enough such that the function does not vary too much over h, this should be a good numerical approximation



Truncation Error due to forward difference approximation

- Function $f(x+h)$ is described **exactly** with Taylor expansion:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \cdots$$

- Rearranging for $f'(x)$ gives an exact expression for the derivative

$$\underbrace{f'(x) = \frac{f(x+h) - f(x)}{h}}_{\text{approximation}} \underbrace{- \frac{h}{2}f''(x) - \frac{h^2}{6}f'''(x) + \cdots}_{O(h),\ \text{truncation error}}$$

- We say that the local truncation error is of order h
- We *truncate* the full series when terms become proportional to (small) h
- The *truncation error* limits accuracy (how close you are to the correct answer)
  - And typically *dominates floating point errors*

See a detailed video on Canvas

Dynamics Problems

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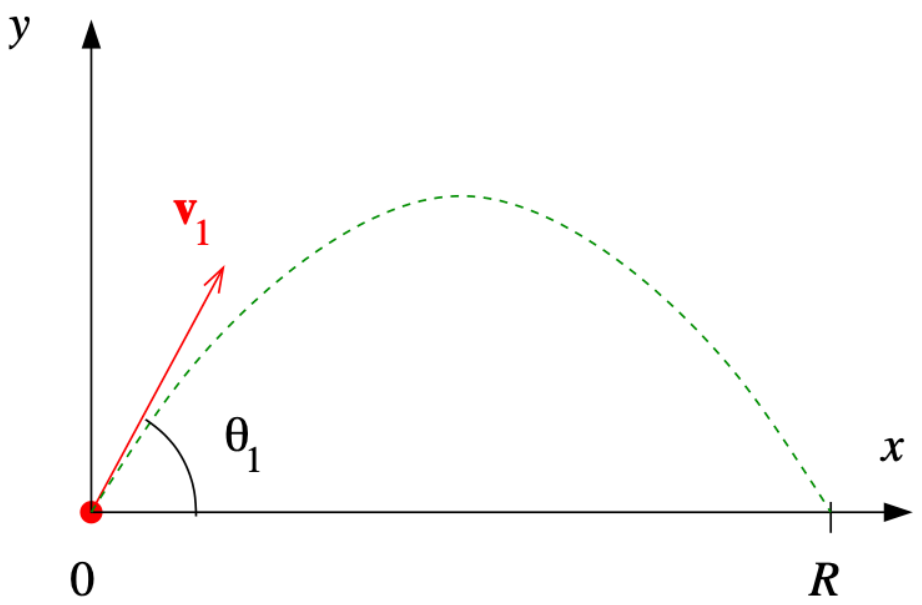
Dynamics problems: solving the motion of a particle

- $\frac{d\mathbf{r}}{dt} = \mathbf{v}$, and $\frac{d\mathbf{v}}{dt} = \mathbf{a}(\mathbf{r}, \mathbf{v}, t)$
  - *position*: $\mathbf{r} = (x(t), y(t), z(t)) = \mathbf{r}(t)$,
  - *velocity*: $\mathbf{v} = \mathbf{v}(t) = (v_x(t), v_y(t), v_z(t))$
  - *acceleration*: $\mathbf{a}(\mathbf{r}, \mathbf{v}, t)$
- Six coupled Ordinary Differential Equations (ODEs).
- The problem is defined by specifying:
  - *(i)* acceleration, $\mathbf{a}(\mathbf{r}, \mathbf{v}, t)$, and *(ii)* initial condition: $\mathbf{v}(t = 0)$, $\mathbf{r}(t = 0)$.
- *Our goal*: take equations we want to solve, and construct a method for us to walk forward in time to piece together a trajectory 🏃,
  - *Our worry*: we make a truncation error at each step…

Example: Simple projectile motion

- Constant gravity: $\mathbf{a} = -g\hat{\mathbf{y}}$ (a constant vector), where
  - $g \approx 9.81\,\text{ms}^{-2}$: acceleration due to gravity close to the Earth.
  - $\hat{\mathbf{y}}$: vertical unit vector.
- Because $\mathbf{a}$ is a constant, we can solve analytically (using integration) for initial condition $\mathbf{r}(0) = \mathbf{r}_1$ and $\mathbf{v}(0) = \mathbf{v}_1$:
  - $\mathbf{r} = \mathbf{r}_1 + \mathbf{v}_1 t - \frac{1}{2} g t^2 \hat{\mathbf{y}}$
  - $\mathbf{v} = \mathbf{v}_1 - g t \hat{\mathbf{y}}$



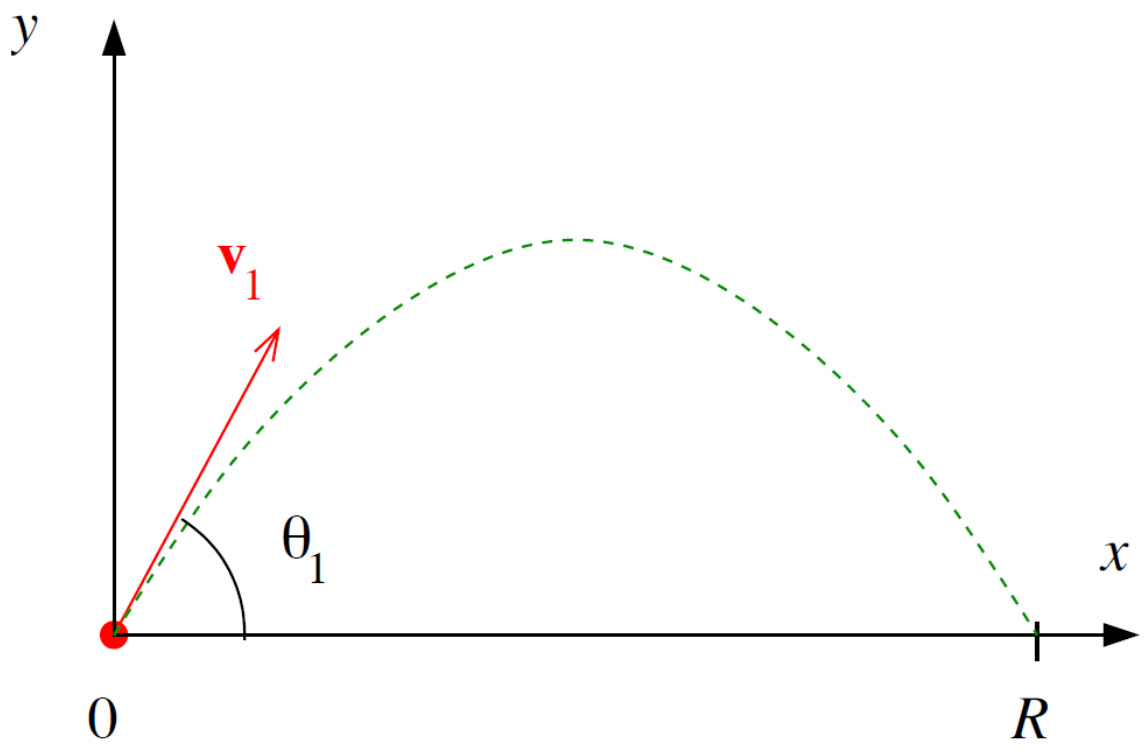
This week's test problem: projectile motion

- Fire an object at some initial velocity, $\mathbf{v}_1$...
- How far will it go? The range, R.
- *Analytic solution*: parabolic trajectory,

$$y = -\frac{g}{2v_1^2 \cos^2 \theta_1} x(x - R).$$

  - The range, $R = \frac{1}{g} v_1^2 \sin(2\theta_1)$.

- We can compare against this to *test the accuracy* of a numerical method.



Non-dimensionalisation

Non-dimensionalising equations

- Physical equations use *dimensional* variables:
  - Time (s), position (m), velocity (ms^{-1}) have physical units.
  - Constants can also be dimensional: e.g., $g = 9.8\ ms^{-2}$.

- In numerical work, it's simpler to *non-dimensionalise*
  - *The goal*: variables and constants in our equations have *no units*.

- *How?*: Rescale dimensional variables to *non-dimensional* (ND) ones:
  - We divide a distance by a distance, a time by a time, etc.
  - When the dust settles, *dimensional constants* can be eliminated and replaced by ND constants.

Non-dimensionalising: projectile motion equations

- *Projectile motion*: $\frac{d\mathbf{r}}{dt} = \mathbf{v}$ and $\frac{d\mathbf{v}}{dt} = -g\hat{\mathbf{y}}$.
- We introduce new *dependent*, $\mathbf{r}$ and $\mathbf{v}$, and *independent*, t, variables:
  - $\bar{\mathbf{r}} = \frac{\mathbf{r}}{L_s}, \quad \bar{t} = \frac{t}{t_s}, \quad \bar{\mathbf{v}} = \frac{\mathbf{v}}{L_s/t_s}.$
- L_s and t_s are a *chosen scale of length and time*.
  - *We can choose* L_s and t_s to be whatever we want, with units length and time.
- The new variables, $\bar{\mathbf{r}}$, $\bar{t}$, and $\bar{\mathbf{v}}$ are *dimensionless* (have no units).
- Careful: *We rescale variables, not constants/parameters*.
  - E.g., we do not rescale g.

Non-dimensionalising: projectile motion equations

- *Velocity*: $\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad \Rightarrow \quad \frac{L_s}{t_s}\frac{d\bar{\mathbf{r}}}{d\bar{t}} = \frac{L_s}{t_s}\bar{\mathbf{v}} \quad \Rightarrow \quad \frac{d\bar{\mathbf{r}}}{d\bar{t}} = \bar{\mathbf{v}}$
- *Acceleration*: $\frac{d\mathbf{v}}{dt} = -g\hat{\mathbf{y}} \quad \Rightarrow \quad \frac{L_s}{t_s^2}\frac{d\bar{\mathbf{v}}}{d\bar{t}} = -g\hat{\mathbf{y}} \quad \Rightarrow \quad \frac{d\bar{\mathbf{v}}}{d\bar{t}} = -\frac{gt_s^2}{L_s}\hat{\mathbf{y}}$
- Our constants occur in a big 📦 as $\frac{gt_s^2}{L_s}$.
  - ❓ How might we set the timescale, t_s, to remove the other constants, g and L_s?

Non-dimensionalising: projectile motion equations

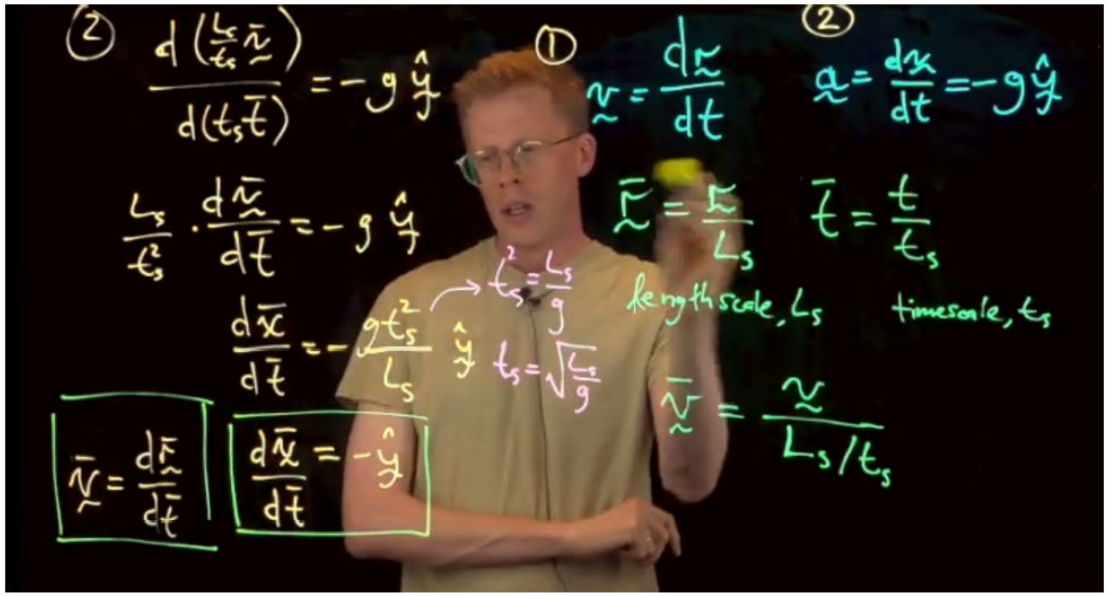
- *Velocity*: $\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad \Rightarrow \quad \frac{L_s}{t_s}\frac{d\bar{\mathbf{r}}}{d\bar{t}} = \frac{L_s}{t_s}\bar{\mathbf{v}} \quad \Rightarrow \quad \frac{d\bar{\mathbf{r}}}{d\bar{t}} = \bar{\mathbf{v}}$
- *Acceleration*: $\frac{d\mathbf{v}}{dt} = -g\hat{\mathbf{y}} \quad \Rightarrow \quad \frac{L_s}{t_s^2}\frac{d\bar{\mathbf{v}}}{d\bar{t}} = -g\hat{\mathbf{y}} \quad \Rightarrow \quad \frac{d\bar{\mathbf{v}}}{d\bar{t}} = -\frac{gt_s^2}{L_s}\hat{\mathbf{y}}$
- Our constants occur in a big 📦 as $\frac{gt_s^2}{L_s}$.
  - ❓ How might we set the timescale, t_s, to remove the other constants, g and L_s?
- Choosing $t_s = \sqrt{\frac{L_s}{g}}$ yields non-dimensional projectile-motion equations: $\boxed{\frac{d\bar{\mathbf{r}}}{d\bar{t}} = \bar{\mathbf{v}}\,, \quad \frac{d\bar{\mathbf{v}}}{d\bar{t}} = -\hat{\mathbf{y}}\,.}$
  - No more annoying constants! 💖
- We perform calculations using the ND equations!
  - If we ever need dimensional values, we can always rescale back!

Non-dimensional tips

- This is a skill you need to have 💪
  - It will come up again and again throughout this course.
- *Note*: We often omit the bars for convenience, and work with non-dimensional quantities.
- When you non-dimensionalize, you should be careful to clearly write down what variables are being rescaled.
- E.g., for dynamics problems we have $\bar{t} = t/t_s$.
  - So whenever we see a t, we need to convert it to $t = \bar{t}\, t_s$.
- For a second derviative, we are rescaling $t^2 = (t_s \bar{t})^2 = t_s^2 \bar{t}^2$, noting that linear rescalings are constants that also rescale derivatives (can be 'pulled out').
  - Yielding $\dfrac{d^2}{dt^2} = \dfrac{d^2}{d(\bar{t}\, t_s)^2} = \dfrac{1}{t_s^2}\dfrac{d^2}{d\bar{t}^2}$

Non-dimensionalisation strips a model down to its essential physics: it reveals the true control parameters, reduces complexity, improves numerical behaviour, and makes results universal rather than unit-dependent.

👨‍🏫 There's a video on Canvas if you want to work through this process a little more slowly.



Euler’s method

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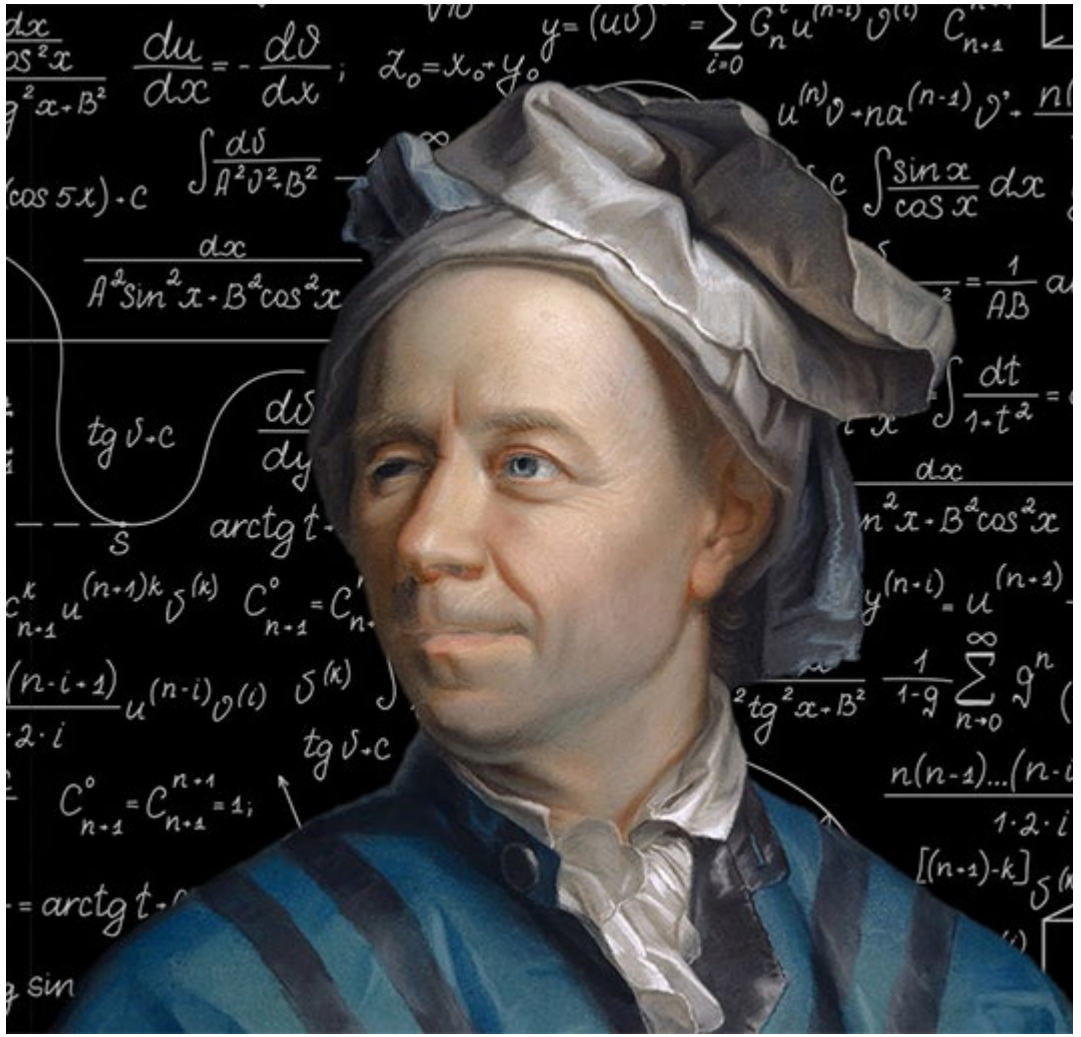
Euler's Method

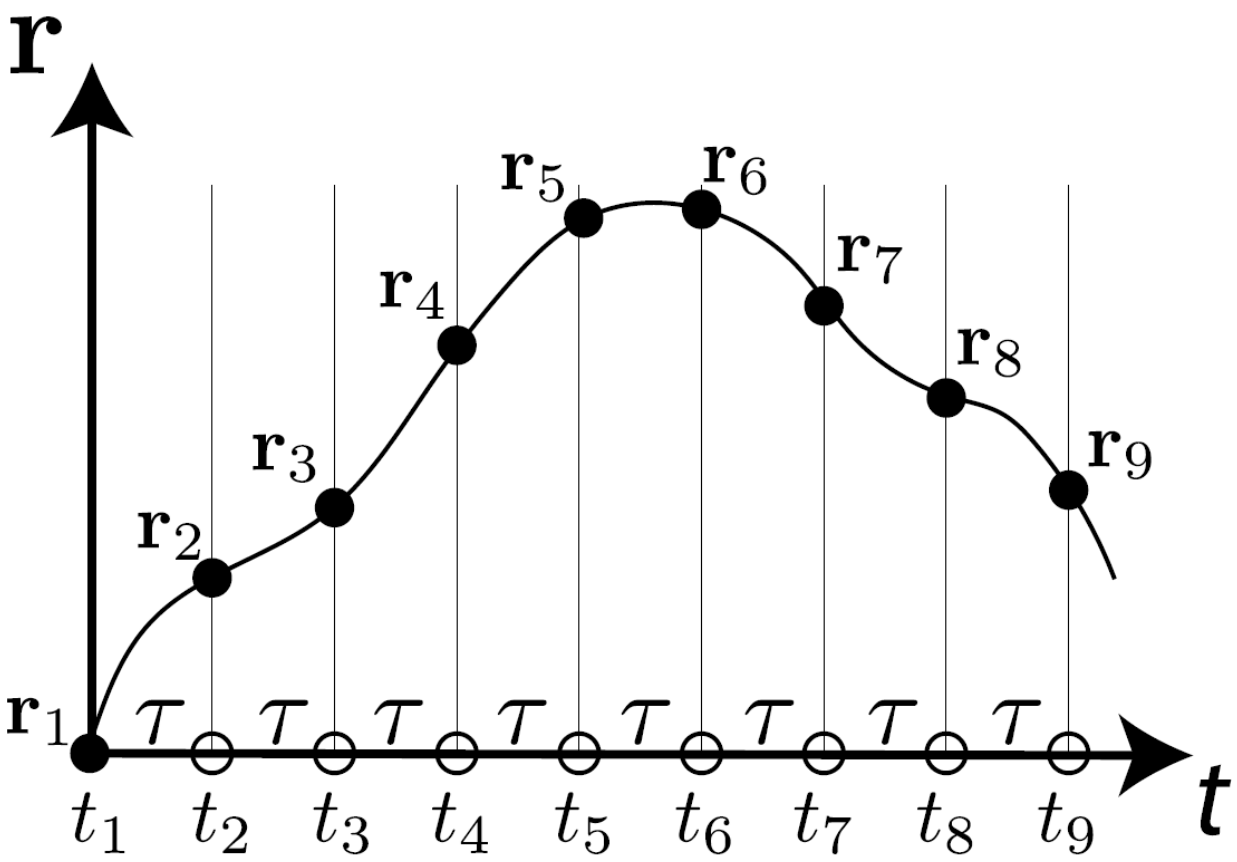
- Our very first numerical method 😄
- From Euler's book, *Institutionum calculi integralis* (1768!)
- Uses the *forward-difference approximation* to evaluate the time derivatives.



Leonhard Euler (1707–1783)

Discretising time

- Numerical methods typically treat time across a *discrete grid*.
- We evaluate at a set of times, $t_n = (n-1)\tau$ $(n = 1, 2, 3 \ldots)$
  - τ is the *time step*.
- The projectile motion equations, $\frac{d\bar{\mathbf{r}}}{d\bar{t}} = \bar{\mathbf{v}}$, $\frac{d\bar{\mathbf{v}}}{d\bar{t}} = -\hat{\mathbf{y}}$, tell us how position and velocity update over time.
  - But to solve them on a computer, *we need to compute these derivatives*.
- We want a set of rules (an *algorithm*) to march forward through time, piecing together the trajectory one τ increment at a time 🏃



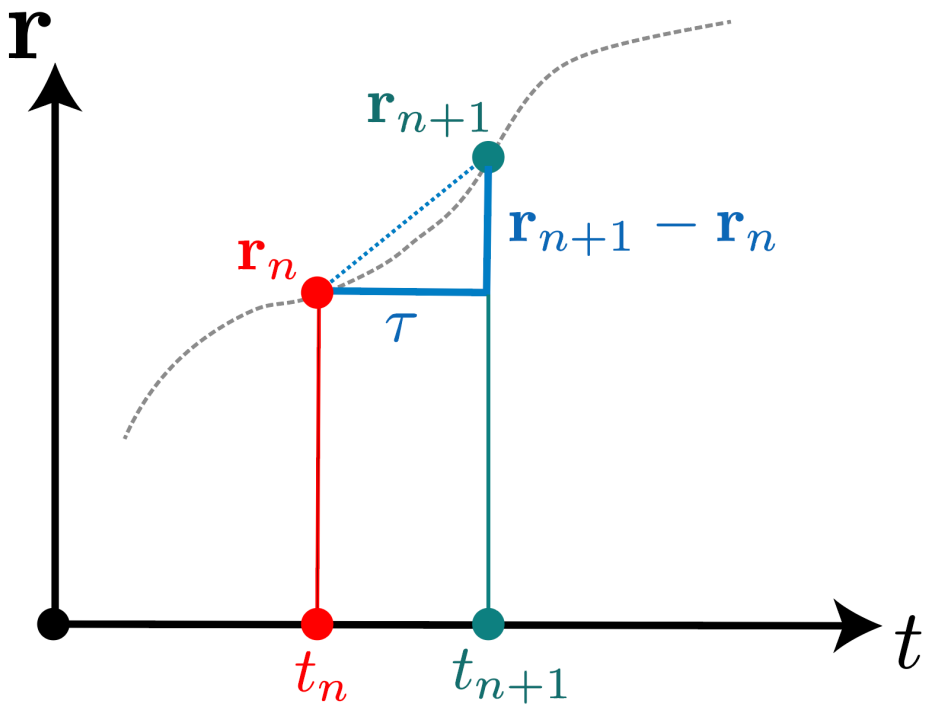
Time derivative: the forward-difference approximation

Estimate time derivative as before: $f'(t) = \frac{f(t+\tau)-f(t)}{\tau} + O(\tau)$

Simple approximation gives us a relationship between two successive time points!

$$\left.\frac{d\mathbf{r}}{dt}\right|_{t_n} = \frac{\mathbf{r}_{n+1} - \mathbf{r}_n}{\tau} + O(\tau)$$
$$\approx \frac{\mathbf{r}_{n+1} - \mathbf{r}_n}{\tau}.$$

Estimating $\left.\frac{d\mathbf{r}}{dt}\right|_{t_n}$



Euler's method for dynamics

- Applies the forward-difference approximation to the derivatives in $\mathbf{r}$ and $\mathbf{v}$:
  - $$\left.\frac{d\mathbf{r}}{dt}\right|_{t_n} = \frac{\mathbf{r}(t_{n+1}) - \mathbf{r}(t_n)}{\tau} + O(\tau)$$
    - We write as $$\left.\frac{d\mathbf{r}}{dt}\right|_{t_n} = \frac{\mathbf{r}_{n+1} - \mathbf{r}_n}{\tau} + O(\tau).$$
  - And similarly for $$\left.\frac{d\mathbf{v}}{dt}\right|_{t_n} = \frac{\mathbf{v}(t_{n+1}) - \mathbf{v}(t_n)}{\tau} + O(\tau) = \frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{\tau} + O(\tau).$$

Applying the forward-difference approximation to ND dynamics equations

$$\frac{d\bar{\mathbf{r}}}{d\bar{t}} = \bar{\mathbf{v}}\,, \qquad \frac{d\bar{\mathbf{v}}}{d\bar{t}} = \bar{\mathbf{a}}\,.$$

$$\frac{1}{\tau}(\mathbf{r}_{n+1} - \mathbf{r}_n) + O(\tau) = \mathbf{v}_n\,, \qquad \frac{1}{\tau}(\mathbf{v}_{n+1} - \mathbf{v}_n) + O(\tau) = \mathbf{a}_n\,.$$

or:

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \tau\mathbf{v}_n + O(\tau^2)\,, \qquad \mathbf{v}_{n+1} = \mathbf{v}_n + \tau\mathbf{a}_n + O(\tau^2)\,.$$

Euler's Method for Dynamics

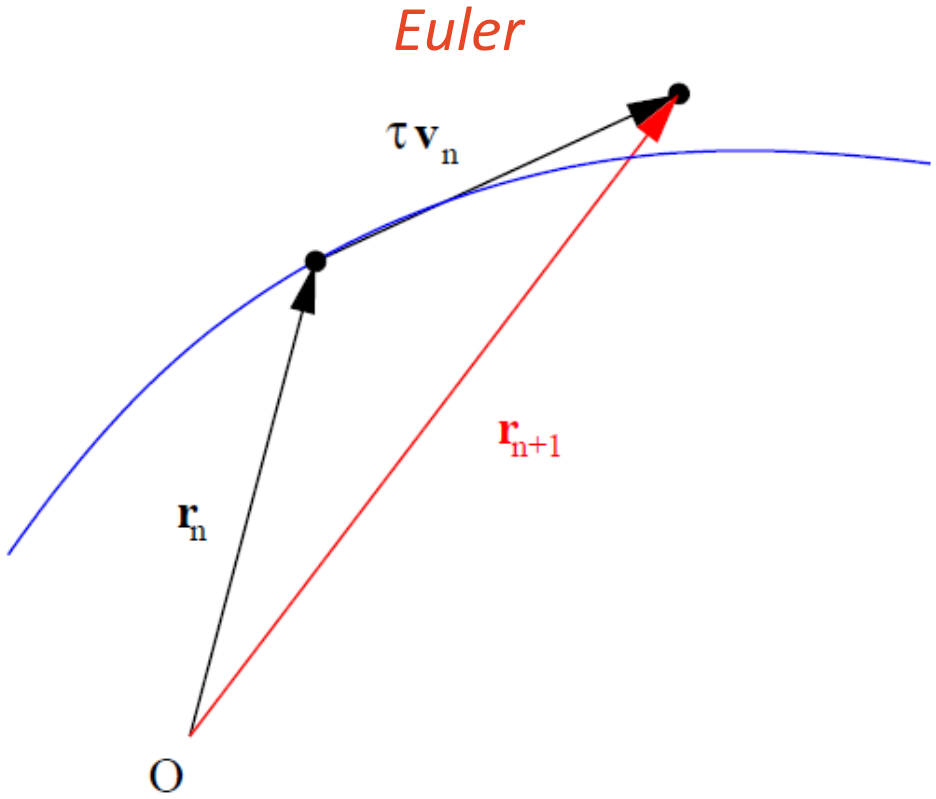
- Excludes terms of order τ^2:
  - $$\boxed{\begin{aligned}\mathbf{r}_{n+1} &= \mathbf{r}_n + \tau \mathbf{v}_n\,, \\ \mathbf{v}_{n+1} &= \mathbf{v}_n + \tau \mathbf{a}_n\,.\end{aligned}}$$
- Yields a simple rule for stepping forward in time 🏃:
  - $(\mathbf{r}_1, \mathbf{v}_1) \xrightarrow{\tau} (\mathbf{r}_2, \mathbf{v}_2) \xrightarrow{\tau} \cdots$
- In general, we make a *local truncation error* in $\mathbf{r}$ and $\mathbf{v}$ at *every time step*.
  - This error is $O(\tau^2)$ per step in general
    - (but could be less in special situations: e.g., uniform motion).

Stepping forward in time

- $$\mathbf{r}_{n+1} = \mathbf{r}_n + \tau \mathbf{v}_n \,,$$
  $$\mathbf{v}_{n+1} = \mathbf{v}_n + \tau \mathbf{a}_n \,.$$
- We use the current velocity estimate, $\mathbf{v}_n$, to take a step:
  - $(\mathbf{r}_n, \mathbf{v}_n) \xrightarrow{\tau} (\mathbf{r}_{n+1}, \mathbf{v}_{n+1})$.

❓ How much worse is it to take a 'big step' (high τ) than a 'small step' (low τ)?

We make an error at each time step (of order τ^2) and they can *accumulate*.

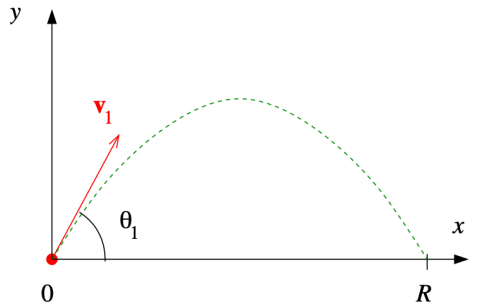


Euler's method for simple projectile motion

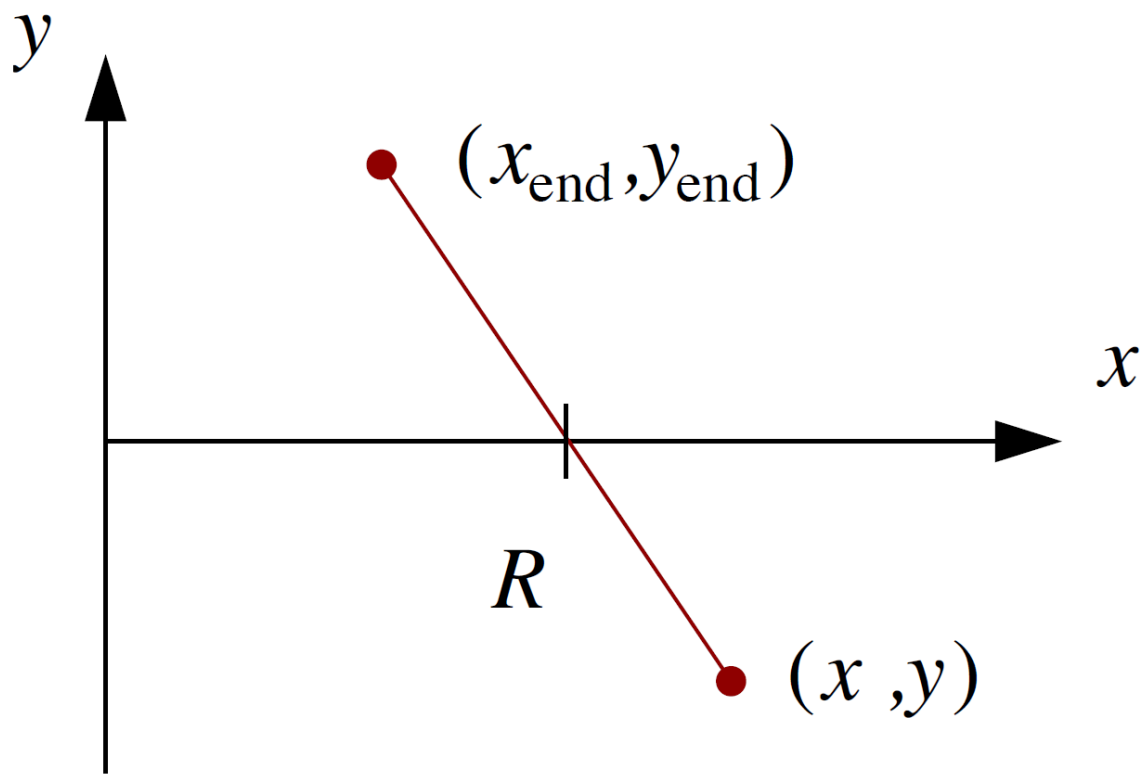
`proj_euler.ipynb` and `Euler_ODE_ipynb`

- Simple *projectile motion* has $\mathbf{a} = -\hat{\mathbf{y}}$, so our Euler algorithm for solving it is:
  - $$\boxed{\begin{aligned}\mathbf{r}_{n+1} &= \mathbf{r}_n + \tau \mathbf{v}_n\,, \\ \mathbf{v}_{n+1} &= \mathbf{v}_n - \tau \hat{\mathbf{y}}\,.\end{aligned}}$$
- The code `proj_euler` solves these *non-dimensional* equations using Euler's method.
  - The user supplies values of v_1 (in m/s) and θ_1 (degrees).
  - Converts to non-dimensional values (L_s and τ are specified within the code).
  - Steps forward ('integrates') using a `while` loop.
  - The loop exits when $y_n \leq 0$ (projectile 'goes underground').
  - Uses *linear interpolation* to estimate the range, R (see next slide).

Estimating R



- `proj_euler` uses linear interpolation
- (*Careful to note*: this approximation is another source of numerical error).
- $$\frac{0 - y_{\text{end}}}{R - x_{\text{end}}} = \frac{y - 0}{x - R}$$
- Solving for R: $R = x - y\frac{x - x_{\text{end}}}{y - y_{\text{end}}}$.



you're linearly interpolating velocity to update position, and **linearly interpolating acceleration to update velocity.**

Global Truncation Error

THE UNIVERSITY OF SYDNEY

Global truncation error

- We know that we make *local truncation errors* at each step, say $O(\tau^k)$ (in general).
  - $O(\tau^2)$, so $k = 2$, for Euler.
- Then we can (naively) estimate the *global error*, E_g, across a total integration time T:
  - $E_g \approx (\text{number of steps}) \times (\text{error in each step})$
  - $E_g \approx \frac{T}{\tau} \times O(\tau^k)$
  - $E_g \approx O(\tau^{k-1})$
- ⭐ If local error is *order* k, then the global error is *order* $k-1$.
  - (A lower order means we're truncating early in the series: *less accurate*).
- This simple argument and calculation assumes that errors in successive steps are independent
  - (*Be careful*: errors can be non-independent).

Truncation errors: Solving dynamics using Euler's method

Consider the full Taylor series for a position update:

$$\begin{aligned}\mathbf{r}_{n+1} &= \mathbf{r}(t_{n+1}) = \mathbf{r}(t_n + \tau)\,, \\ &= \mathbf{r}(t_n) + \tau \left.\frac{d\mathbf{r}}{dt}\right|_{t_n} + \tfrac{1}{2}\tau^2 \left.\frac{d^2\mathbf{r}}{dt^2}\right|_{t_n} + \dots\,, \\ &= \mathbf{r}_n + \tau\mathbf{v}_n + \tfrac{1}{2}\tau^2\mathbf{a}_n + \dots\,, \\ &= \mathbf{r}_n + \tau\mathbf{v}_n - \tfrac{1}{2}\tau^2 g\hat{\mathbf{y}}\,.\end{aligned}$$

- The last step uses the (constant) projectile motion acceleration: $\mathbf{a} = -\frac{1}{2}\tau^2 g\hat{\mathbf{y}}$
  - Higher-order derivatives are zero in this case.
- Comparing to the Euler method update: $\mathbf{r}_{n+1} = \mathbf{r}_n + \tau\mathbf{v}_n$ 😉
  - So we find an *exact expression* for the local error we're making: $\frac{1}{2}\tau^2 g$ (in the y-direction).

Global Error, Δy

- The local error in $\mathbf{r}$ is $\frac{1}{2}\tau^2 g$ (in the y-direction).
- Using our simple assumption of independent errors to estimate the global error, Δy, after a time T:
  - T/τ steps
  - $\frac{1}{2}\tau^2 g$ error per step
  - So $\Delta y \approx \frac{1}{2} T \tau g$.
- For the figure example, $g \approx 10\,\mathrm{m\,s^{-2}}$, $\tau \approx 0.03\,\mathrm{s}$, $T \approx 2\,\mathrm{s}$:
  - global error $\approx \frac{1}{2} \times 2 \times 0.03 \times 10\,\mathrm{m} \approx 30\,\mathrm{cm}$
- This is consistent with the observed error in R.

The Computer Lab

Computational Physics Lab 1

Not available until Feb 23 at 0:00 Due Mar 9 at 10:00 2 pts

- You will simulate projectile motion using the Euler method.
- You have 2 weeks and only 1 attempt to complete the Canvas Quizz
- Lab attendance is optional, but this is your only chance to get help
- *Extra*: How are these ideas used in gaming? Check this YouTube video

1 1 point

Run the code **proj_euler.m** with initial conditions $v_1 = 50$ m/s and $\theta_1 = 40$ deg. Determine the range for the Euler's method solution with the non-dimensional time step $\tau = 0.1$.

Write your answer as a number below, in units of m.

Type your answer...

2 1 point

What is the percentage error in the Euler's method result for the range *R* compared with the (exact) analytic range given by $R = v_1^2 \sin(2\theta_1)/g$ where v_1 is the initial speed and θ_1 is the initial angle to the horizontal?

Write your answer as a number below.

Type your answer...

3 1 point

What is the error in the Euler's method result due to?

Give a brief answer in the text window below.

12pt Paragraph

supplementary

Common Binary Representations of Numeric Values

Unsigned Integers

n bit Positional Binary

value $= d_{n-1} \cdot 2^{n-1} + \cdots + d_1 \cdot 2^1 + d_0 \cdot 2^0$ (n digits)

Example: $n = 8$

00000000	$= 0$	
00000001	$= 2^0$	$= 1$
00000010	$= 2^1$	$= 2$
00000011	$= 2^1 + 2^0$	$= 3$
00000100	$= 2^2$	$= 4$
⋮	⋮	⋮
10000000	$= 2^{n-1}$	$= 128$
⋮	⋮	⋮
11111111	$= 2^n - 1$	$= 255$

Signed Integers

n bit Two's Complement

$(i + \text{2sComplement}(i) = 2^n)$

Example: $n = 8$

01111111	$= 2^{n-1} - 1$	$= +127$
⋮		⋮
00000010	Negation:	$= +2$
00000001	1. invert	$= +1$
00000000	2. add 1	$= \pm 0$
11111111		$= -1$
11111110		$= -2$
⋮		⋮
10000000	$= -2^{n-1}$	$= -128$

Floating-Point Numbers

IEEE 754 Format and derivatives

sign bit | exponent (x bits) | mantissa (m bits)

S | E | M

$n = (1 + x + m)$ bits

Example: half precision ($n = 16,\ x = 5,\ m = 10$)

subnormal	S 00000 M	$= (-1)^S \cdot 2^{-2^{x-1}+2} \cdot \left(0 + \frac{M}{2^m}\right)$		
	0 00000 0000000000	$= -1^0 \cdot 2^{-14} \cdot \left(0 + \frac{0}{1024}\right) = +0$		negative zero
	1 00000 0000000000	$= -1^1 \cdot 2^{-14} \cdot \left(0 + \frac{0}{1024}\right) = -0$		
	0 00000 0000000001	$= -1^0 \cdot 2^{-14} \cdot \left(0 + \frac{1}{1024}\right) \approx +0.0000000596$		smallest positive subnormal
	0 00000 1111111111	$= -1^0 \cdot 2^{-14} \cdot \left(0 + \frac{1023}{1024}\right) \approx +0.0000609756$		largest subnormal
normal	S 0⋮01 M S 1⋮10 M	$= (-1)^S \cdot 2^{(E - (2^{x-1}-1))} \cdot \left(1 + \frac{M}{2^m}\right)$ exponent − bias		exponent bias: $2^{x-1} - 1 = 15$
	0 00001 0000000000	$= -1^0 \cdot 2^{-14} \cdot \left(1 + \frac{0}{1024}\right) \approx +0.000061$		smallest positive normal
	0 01110 0000000000	$= -1^0 \cdot 2^{-1} \cdot \left(1 + \frac{0}{1024}\right) = +0.5$		
	0 01110 1111111111	$= -1^0 \cdot 2^{-1} \cdot \left(1 + \frac{1023}{1024}\right) \approx +0.999512$		largest less than one
	0 01111 0000000000	$= -1^0 \cdot 2^0 \cdot \left(1 + \frac{0}{1024}\right) = +1$		
	1 01111 0000000000	$= -1^1 \cdot 2^0 \cdot \left(1 + \frac{0}{1024}\right) = -1$		
	0 01111 0000000001	$= -1^0 \cdot 2^0 \cdot \left(1 + \frac{1}{1024}\right) \approx +1.000977$		smallest larger than one
	0 11110 0000000000	$= -1^0 \cdot 2^{15} \cdot \left(1 + \frac{0}{1024}\right) = +32768$		
	0 11110 1111111111	$= -1^0 \cdot 2^{15} \cdot \left(1 + \frac{1023}{1024}\right) = +65504$		largest normal
	1 11110 1111111111	$= -1^1 \cdot 2^{15} \cdot \left(1 + \frac{1023}{1024}\right) = -65504$		smallest normal
special	0 11111 0000000000	$= +\infty$		infinity
	0 11111 0000000001	= signaling NaN	on x86	"not a number"
	0 11111 1000000001	= quiet NaN	on x86	"not a number"

	n	x	m
half precision	16	5	10
bfloat16	16	8	7
TensorFloat	19	8	10
fp24	24	7	16
PXR24	24	8	15
single precision / "float"	32	8	23
double precision	64	11	52
x86 extended prec.	80	15	64
quadruple precision	128	15	112
octuple precision	256	19	236