# University of Central Florida
# School of Computer Science
# COT 4210      Fall 2004

## Prof. Rene Peralta
## Homework 2 Solutions (by TA Robert Lee)

1. A correspondence between $\mathbb{N}$ and $\mathbb{Z}$ is defined by ordering the latter thus

$$0, 1, -1, 2, -2, 3, -3, ...$$

Give a mathematical formula for this correspondence. (i.e., find $f : \mathbb{N} \to \mathbb{Z}$ which is one-to-one and onto).

Hint: $(-1)^i$ can be used to alternate positive and negative signs. You might also want to use the “floor” and/or “ceiling” integer functions.

**Solution** The mapping

$$f(n) = (-1)^n \left\lfloor \frac{n}{2} \right\rfloor$$

is one-to-one and onto and produces the given ordering of $\mathbb{Z}$.

2. (from Test 1) Describe a way to list the set of finite subsets of the natural numbers. Hint: recall the technique I used in lecture 4 to show that the set of languages over the binary alphabet is not countable.

**Solution** We can represent the finite subsets of the natural numbers using finite binary strings. Create a table with the natural numbers listed across in increasing order, and subsets of the natural numbers listed down:

| $\mathbb{N}:$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $S_1$ | 0 | 1 | 0 | 1 | 0 | 1 | 0 | $\cdots$ |
| $S_2$ | 1 | 0 | 1 | 0 | 0 | 0 | 0 | $\cdots$ |

Each finite subset of the natural numbers is represented by a unique "presence vector", a binary string indicating which natural numbers are in the subset. If the binary string for a subset has a 1 in the $i$-th position, then the natural number $i$ is in the subset; if there is a 0 in the $i$-th position, then $i$ is not in the subset. For example, the binary string for set $S_1$ has a 1 in the 2nd, 4th, and 6th positions, so set $S_1$ contains the natural numbers 2, 4, and 6.

Because a subset must be finite, we can make the corresponding binary string finite by ending it at the last 1. (There must be a last 1 because a finite subset must have a largest element.) The only exception is the string 0, which represents the empty set.

Now the problem of listing the finite subsets of the natural numbers is reduced to listing the finite binary strings. Here is one listing:

| $\mathbb{N}:$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|
| $(0)_{10}$ | 0 | | | | | | | |
| $(1)_{10}$ | 1 | | | | | | | |
| $(2)_{10}$ | 0 | 1 | | | | | | |
| $(3)_{10}$ | 1 | 1 | | | | | | |
| $(4)_{10}$ | 0 | 0 | 1 | | | | | |
| $(5)_{10}$ | 1 | 0 | 1 | | | | | |
| $\vdots$ | | | | | | | | |

(This corresponds to the listing of finite subsets $\{\}, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \cdots$.) By reading the finite binary strings "backwards", that is, from right to left, we can interpret them as integers written in binary. The decimal values of these binary numbers are given in the left-most column. Thus by listing the integers in binary in increasing order, starting from 0, we obtain a listing of the finite subsets of the natural numbers.

3. Denote by $\neg u$ the negation of the Boolean variable $u$. Here is a proof that

$$\neg(x \vee y) \Leftrightarrow (\neg x \wedge \neg y)$$

**proof:** If $x = T$ then

$$\begin{aligned} \neg(x \vee y) &\Leftrightarrow \neg(T \vee y) \\ &\Leftrightarrow \neg T \\ &\Leftrightarrow F \end{aligned}$$

and

$$\begin{aligned} (\neg x \wedge \neg y) &\Leftrightarrow (\neg T \wedge \neg y) \\ &\Leftrightarrow (F \wedge \neg y) \\ &\Leftrightarrow F. \end{aligned}$$

Since the formula is symmetric in $x, y$, the claim is also true if $y = T$. The remaining case is $X = Y = F$. In this case

$$\begin{aligned} \neg(x \vee y) &\Leftrightarrow \neg(F \vee F) \\ &\Leftrightarrow \neg F \\ &\Leftrightarrow T \end{aligned}$$

and

$$\begin{aligned} (\neg x \wedge \neg y) &\Leftrightarrow (\neg F \wedge \neg F) \\ &\Leftrightarrow (T \wedge T) \\ &\Leftrightarrow T. \end{aligned}$$

Give an analogous proof that

$$\neg(x \wedge y) \Leftrightarrow (\neg x \vee \neg y).$$

**Solution** If $x = F$ then

$$\begin{aligned}
\neg(x \wedge y) &\Leftrightarrow \neg(F \wedge y) \\
&\Leftrightarrow \neg F \\
&\Leftrightarrow T
\end{aligned}$$

and

$$\begin{aligned}
(\neg x \vee \neg y) &\Leftrightarrow (\neg F \vee \neg y) \\
&\Leftrightarrow (T \vee \neg y) \\
&\Leftrightarrow T.
\end{aligned}$$

Since the formula is symmetric in $x, y$, the claim is also true if $y = F$. The remaining case is $X = Y = T$. In this case

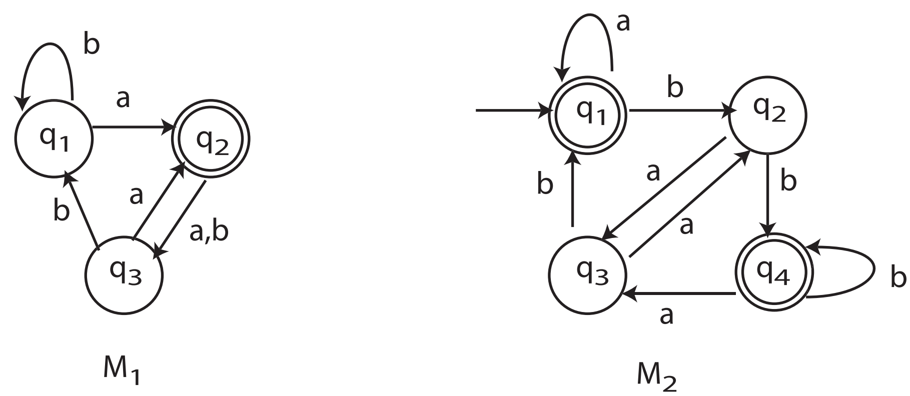
$$\begin{aligned}
\neg(x \wedge y) &\Leftrightarrow \neg(T \wedge T) \\
&\Leftrightarrow \neg T \\
&\Leftrightarrow F
\end{aligned}$$

and

$$\begin{aligned}
(\neg x \vee \neg y) &\Leftrightarrow (\neg T \vee \neg T) \\
&\Leftrightarrow (F \vee F) \\
&\Leftrightarrow F.
\end{aligned}$$

4. Exercise 1.2, page 83 of text.

Give the formal description of the machines $M_1$ and $M_2$:




**Solution** The formal description of a DFA is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$.

$$M_1 = (\{q_1, q_2, q_3\}, \{a, b\}, \delta_1, q_1, \{q_2\}),$$

where $\delta_1$ is given by the following table:

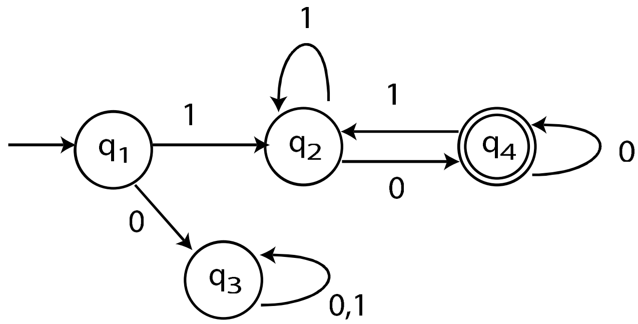
| $\delta_1$ | $a$ | $b$ |
|---|---|---|
| $q_1$ | $q_2$ | $q_1$ |
| $q_2$ | $q_3$ | $q_3$ |
| $q_3$ | $q_2$ | $q_1$ |

$$M_2 = (\{q_1, q_2, q_3, q_4\}, \{a, b\}, \delta_2, q_1, \{q_1, q_4\}),$$

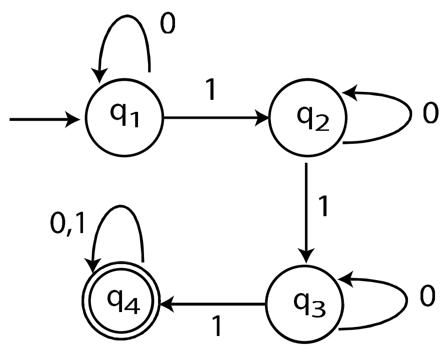
where $\delta_2$ is given by the following table:

| $\delta_2$ | $a$ | $b$ |
|---|---|---|
| $q_1$ | $q_1$ | $q_2$ |
| $q_2$ | $q_3$ | $q_4$ |
| $q_3$ | $q_2$ | $q_1$ |
| $q_4$ | $q_3$ | $q_4$ |

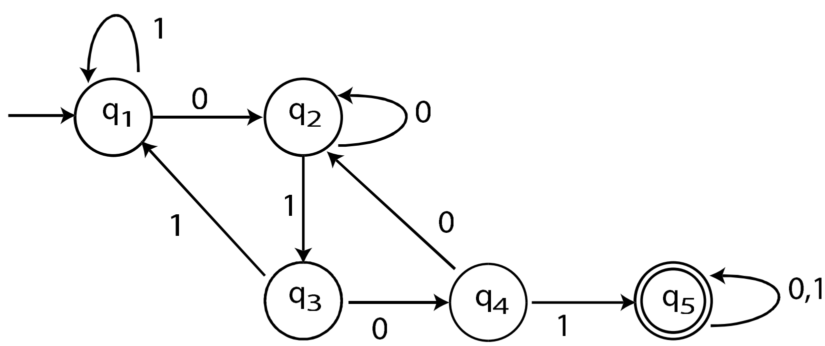
5a. {w | w begins with a 1 and ends with a 0}



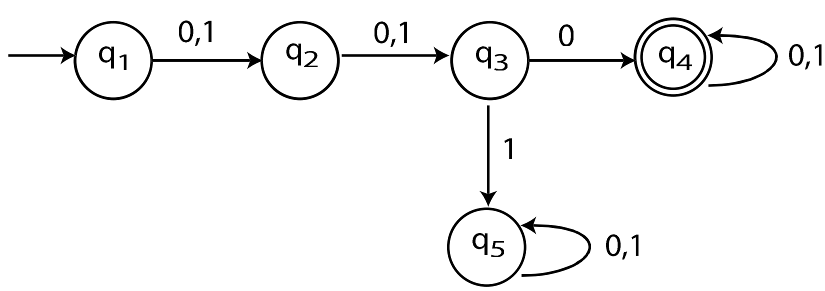

5b. {w | w contains at least three 1s}



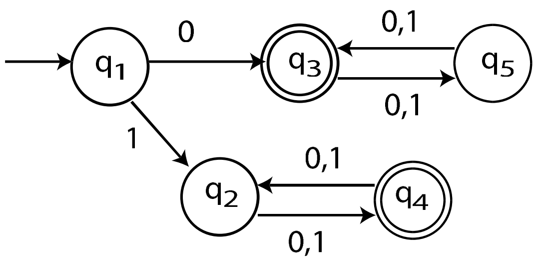

5c. {w | w contains the substring 0101}



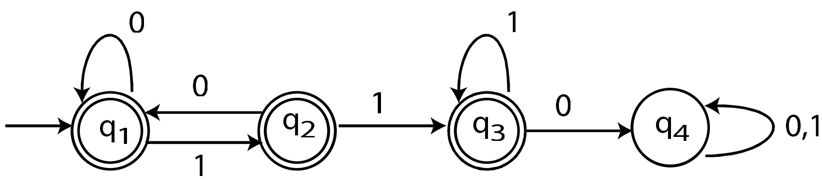

5d. {w | w has length at least 3 and its third symbol is a 0}



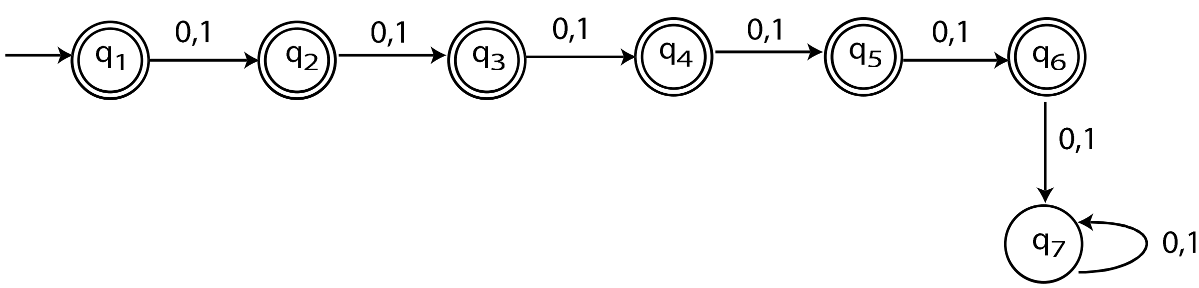
5e. {w | w starts with 0 and has odd length, or starts with 1 and has even length}



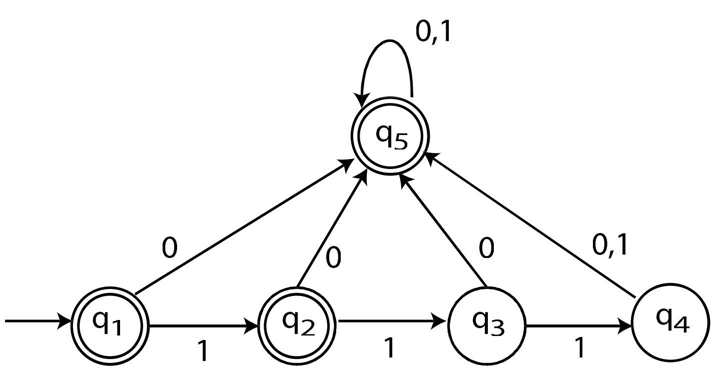

5f. {w | w doesn't contain the substring 110}



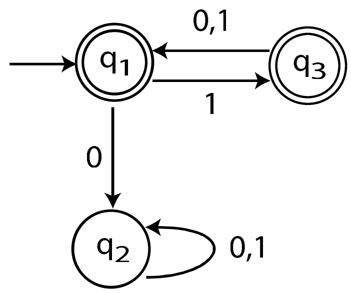

5g. {w | the length of w is at most 5}



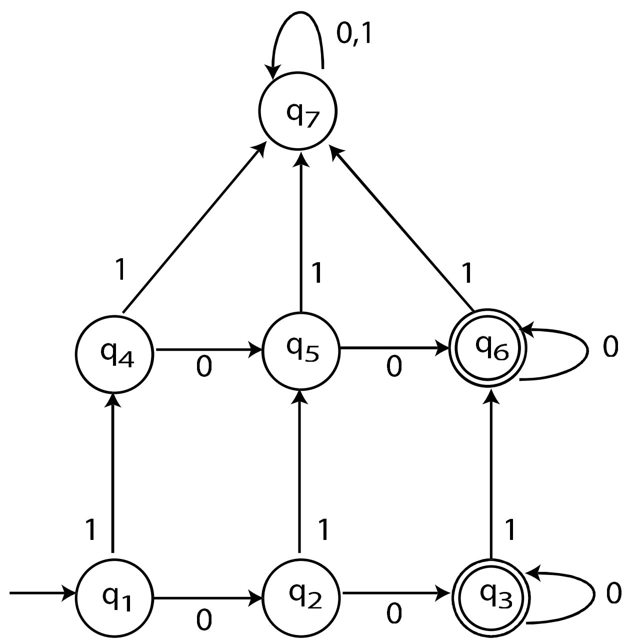

5h. {w | w is any string except 11 and 111}



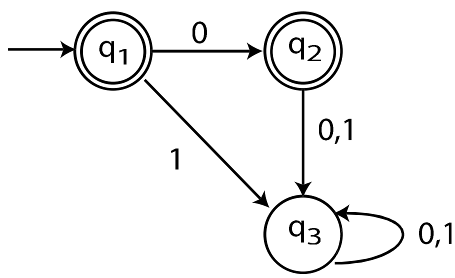
5i. {w | every odd position of w is a 1}



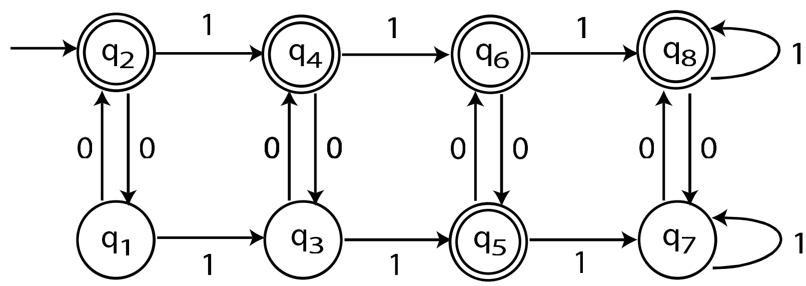

5j. {w | w contains at least two 0s and at most one 1}



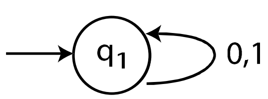

5k. {ε, 0}



5l. {w | w contains an even number of 0s, or exactly two 1s}




5m. The empty set.




5n. All strings except the empty string.