University of Central Florida

School of Computer Science

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Prof. Rene Peralta

Homework 1 Solutions

2. Page 27, Problem 0.10. Please be precise and concise.

Find the error in the following proof that $2 = 1$. Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, $(a + b)(a - b) = b(a - b)$, and divide each side by $(a - b)$, to get $a + b = b$. Finally, let a and b equal 1, which shows that $2 = 1$.

Solution The division by $(a-b)$ is illegal, because the assumption was made that $a = b$.

3. Page 27, Consider the proof of problem 0.11. Find the <u>first</u> incorrect statement (i.e., a statement that is not an assumption and does not follow from previous statements). Give a counterexample that shows the statement is not always true.

CLAIM: In any set of h horses, all horses are the same color.

PROOF: By induction on h.

Basis: For $h = 1$. In any set containing just one horse, all horses clearly are the same color.

Induction step: For $k \geq 1$ assume that the claim is true for $h = k$ and prove that it is true for $h = k+1$. Take any set H of $k+1$ horses. We show that all the horses in this set are the same color. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are the same color. Now replace the removed horse and remove a different one to obtain the set H_2. By the same argument, all the horses in H_2 are the same color. Therefore all the horses in H must be the same color, and the proof is complete.

Solution The last statement, "Therefore all the horses in H must be the same color, and the proof is complete," is incorrect. If $h = 2$ and the two horses are of different colors, the proof works except for this last statement.

4. Page 27, Problem 0.13.

Use Theorem 0.15 to derive a formula for calculating the size of the monthly payment for a mortgage in terms of the principal P, interest rate I, and the number of payments t. Assume that, after t payments have been made, the loan amount is reduced to 0. Use the formula to calculate the dollar amount of each monthly payment for a 30-year mortgage with 360 monthly payments on an initial loan amount of \$100,000 with an 8% annual interest rate.

Solution Theorem 0.15 states that

$$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$$

where P_t is the amount of the loan outstanding after the tth month, $M = 1 + I/12$ is the monthly multiplier, and Y is the amount of the monthly payment. We are given that $P_t = 0$ and we wish to solve for Y:

$$\begin{aligned} 0 &= PM^t - Y\left(\frac{M^t - 1}{M - 1}\right) \\ Y\left(\frac{M^t - 1}{M - 1}\right) &= PM^t \\ Y &= PM^t\left(\frac{M - 1}{M^t - 1}\right) \\ &= P(1 + I/12)^t\left(\frac{I/12}{(1 + I/12)^t - 1}\right). \end{aligned}$$

Using this formula with $t = 360$, $P = \$100,000$, and $I = 0.08$, we obtain

$$Y = \$100000(1 + 0.08/12)^{360}\left(\frac{0.08/12}{(1 + 0.08/12)^{360} - 1}\right) = \$733.76$$

when rounded to the nearest cent.

5. It is actually not true that an annual interest rate I is the same as a monthly rate of $I/12$. If it were true you could do the following (assume an annual rate of 12 percent):

- borrow \$100 from a bank and promise to pay back \$112 in 12 months.
- deposit the \$100 in a savings account at 1 percent a month and leave the money there for 12 months.
- withdraw the money from the savings account and pay back \$112 to the bank.

(a) Do you have any money left?

Solution A savings account at 1 percent a month multiplies the principal by 1.01 each month. After 12 months, the \$100 has grown to

$$\$100 \cdot (1.01)^{12} = \$112.68$$

when rounded to the nearest cent. After you pay back \$112 to the bank, you are left with \$0.68.

(b) Does it get better if you divide the year into days rather than months (i.e., you deposit the \$100 at 12/365 percent per day)? If so, how much better?

Solution Now we have a daily multiplier is $1 + 0.12/365 \approx 1.0003288$. After 365 days, the \$100 has grown to

$$\$100 \cdot (1.0003288)^{365} = \$112.75$$

when rounded to the nearest cent. Hence you are left with about 7 cents more than before.

6. Prove by induction that a non-negative integer $x = b_n \cdots b_0$ written in binary notation is divisible by 3 if and only if $\sum_{i=0}^{n}(-1)^i b_i$ is divisible by 3. (Hint: using the notation of example 0.7 of the text, we have that x is divisible by three iff $x \equiv_3 0$. Therefore the claim is true if $x \equiv_3 \sum_{i=0}^{n}(-1)^i b_i$.)

Solution Two methods will be given; Method 1 is shorter and more elegant than Method 2, but both are acceptable.

<u>**Method 1**</u> Proof by induction on n.

Base case ($n = 0$):

$$\text{LHS} = b_0$$

$$\begin{aligned} \text{RHS} &= \sum_{i=0}^{0}(-1)^i b_i \\ &= (-1)^0 b_0 \\ &= b_0 \end{aligned}$$

Notice that $b_0 - b_0 = 0 = 3(0)$ is a multiple of 3. Therefore LHS $\equiv_3$ RHS, and the base case is proved.

Induction hypothesis ($n = k \geq 0$): Assume that

$$b_k \cdots b_0 \equiv_3 \sum_{i=0}^{k}(-1)^i b_i.$$

Induction step ($n = k+1$): We must prove that

$$b_{k+1} \cdots b_0 \equiv_3 \sum_{i=0}^{k+1} (-1)^i b_i.$$

We will remove the last digit b_0 of the binary number on the LHS and perform a change of variable in order to apply the induction hypothesis.

$$\begin{aligned}
\text{LHS} &= b_{k+1} \cdots b_0 \\
&= b_{k+1} \cdot 2^{k+1} + b_k \cdot 2^k + \cdots + b_1 \cdot 2^1 + b_0 \cdot 2^0 \text{ by definition of a binary number} \\
&= (c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \cdots + c_0 \cdot 2^1) + b_0 \text{ by change of variable } c_{i-1} = b_i \\
&= 2(c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_0 \cdot 2^0) + b_0 \text{ by factoring out a 2} \\
&= 2(c_k \cdots c_0) + b_0 \text{ by definition of a binary number} \\
&\equiv_3 2\left[\sum_{i=0}^{k} (-1)^i c_i\right] + b_0 \text{ by applying the induction hypothesis to } (c_k \cdots c_0) \\
&\equiv_3 (-1)\left[\sum_{i=0}^{k} (-1)^i c_i\right] + b_0 \text{ because } 2 \equiv_3 -1 \\
&= \left[\sum_{i=1}^{k+1} (-1)^i c_{i-1}\right] + b_0\text{, multiplying through by } (-1) \\
&= \left[\sum_{i=1}^{k+1} (-1)^i b_i\right] + b_0 \text{ because } c_{i-1} = b_i \\
&= \sum_{i=0}^{k+1} (-1)^i b_i. \text{ QED.}
\end{aligned}$$

Therefore the induction step is proved, and the induction is complete.

Method 2 Proof by induction on n.

Base case $(n = 0)$:

$$\text{LHS} = b_0$$

$$\begin{aligned} \text{RHS} &= \sum_{i=0}^{0} (-1)^i b_i \\ &= (-1)^0 b_0 \\ &= b_0 \end{aligned}$$

Notice that $b_0 - b_0 = 0 = 3(0)$ is a multiple of 3. Therefore LHS $\equiv_3$ RHS, and the base case is proved.

Induction hypothesis $(n = k \geq 0)$: Assume that

$$b_k \cdots b_0 \equiv_3 \sum_{i=0}^{k} (-1)^i b_i.$$

Using the definition of the equivalence relation $\equiv_3$, this is equivalent to the following statement:

$$(b_k \cdots b_0) - \sum_{i=0}^{k} (-1)^i b_i = 3u$$

for some $u \in \mathbb{Z}$.

Induction step ($n = k{+}1$): We must prove that

$$b_{k+1} \cdots b_0 \equiv_3 \sum_{i=0}^{k+1} (-1)^i b_i.$$

Again, using the definition of the equivalence relation $\equiv_3$, this is equivalent to the following statement:

$$(b_{k+1} \cdots b_0) - \sum_{i=0}^{k+1} (-1)^i b_i = 3v$$

for some $v \in \mathbb{Z}$. We will prove this equivalent statement. Begin with the LHS of the statement to be proved:

$$\begin{aligned}
\text{LHS} &= (b_{k+1} \cdots b_0) - \sum_{i=0}^{k+1} (-1)^i b_i \\
&= b_{k+1} \cdot 2^{k+1} + (b_k \cdots b_0) - \sum_{i=0}^{k+1} (-1)^i b_i \text{ by definition of a binary number} \\
&= b_{k+1} \cdot 2^{k+1} + (b_k \cdots b_0) - \left(\sum_{i=0}^{k} (-1)^i b_i \right) - (-1)^{k+1} b_{k+1} \text{ by removing} \\
&\quad \text{the last term of the summation} \\
&= b_{k+1} \cdot 2^{k+1} - (-1)^{k+1} b_{k+1} + \left[(b_k \cdots b_0) - \sum_{i=0}^{k} (-1)^i b_i \right] \\
&= b_{k+1} \cdot 2^{k+1} - (-1)^{k+1} b_{k+1} + [3u] \text{ by the Induction Hypothesis}
\end{aligned}$$

Because b_{k+1} is a binary digit, it can take the values 0 or 1. We will consider these two cases separately.

Induction step, case 1: $b_{k+1} = 0$. In this case,

$$\begin{aligned} \text{LHS} &= 0 \cdot 2^{k+1} - (-1)^{k+1} \cdot 0 + 3u \\ &= 3u \end{aligned}$$

where $u \in \mathbb{Z}$. Thus, by setting $v = u$,

$$(b_{k+1} \cdots b_0) - \sum_{i=0}^{k+1} (-1)^i b_i = 3u = 3v$$

for $v = u \in \mathbb{Z}$. It follows that

$$b_{k+1} \cdots b_0 \equiv_3 \sum_{i=0}^{k+1} (-1)^i b_i.$$

Therefore the induction step is proved, and the induction is complete.

Induction step, case 2: $b_{k+1} = 1$. In this case,

$$\begin{aligned} \text{LHS} &= 1 \cdot 2^{k+1} - (-1)^{k+1} \cdot 1 + 3u \\ &= 2^{k+1} - (-1)^{k+1} + 3u \end{aligned}$$

The $3u$ term is clearly divisible by 3. It remains to be proved that $2^{k+1} - (-1)^{k+1}$ is divisible by 3 for $k \geq 0$. We will prove this by induction on k.

Second base case ($k = 0$): $2^{0+1} - (-1)^{0+1} = 2 - (-1) = 3 = 3(1)$ is divisible by 3. Hence the base case is proved.

Second induction hypothesis ($k = h \geq 0$): Assume that $2^{h+1} - (-1)^{h+1}$ is divisible by 3; in other words, $2^{h+1} - (-1)^{h+1} = 3m$ for some $m \in \mathbb{Z}$.

Second induction step ($k = h + 1$): We must prove that $2^{h+2} - (-1)^{h+2}$ is divisible by 3. This expression is a little tricky to work with, so we will split this proof into two cases: either $h + 2$ is even or it is odd.

Second induction step, case 1: $h+2$ is even. It follows that $h+1$ is odd, so the second induction hypothesis becomes

$$\begin{aligned} 2^{h+1}-(-1)^{h+1} &= 3m \\ 2^{h+1}-(-1) &= 3m \\ 2^{h+1} &= 3m-1 \end{aligned}$$

We have

$$\begin{aligned} 2^{h+2}-(-1)^{h+2} &= 2^{h+2}-1 \text{ because } h+2 \text{ is even} \\ &= 2(2^{h+1})-1 \\ &= 2(3m-1)-1 \text{ by the second induction hypothesis} \\ &= 6m-2-1 \\ &= 6m-3 \\ &= 3(2m-1) \end{aligned}$$

Thus $2^{h+2}-(-1)^{h+2}$ is divisible by 3, and the second induction step is proved.

Second induction step, case 2: $h+2$ is odd. It follows that $h+1$ is even, so the second induction hypothesis becomes

$$\begin{aligned} 2^{h+1}-(-1)^{h+1} &= 3m \\ 2^{h+1}-1 &= 3m \\ 2^{h+1} &= 3m+1 \end{aligned}$$

We have

$$\begin{aligned} 2^{h+2}-(-1)^{h+2} &= 2^{h+2}+1 \text{ because } h+2 \text{ is odd} \\ &= 2(2^{h+1})+1 \\ &= 2(3m+1)+1 \text{ by the second induction hypothesis} \\ &= 6m+2+1 \\ &= 6m+3 \\ &= 3(2m+1) \end{aligned}$$

Thus $2^{h+2}-(-1)^{h+2}$ is divisible by 3, and the second induction step is proved.

This completes the second induction, proving that $2^{k+1} - (-1)^{k+1}$ is divisible by 3 for $k \geq 0$. Thus we can write $2^{k+1} - (-1)^{k+1} = 3w$ for some $w \in \mathbb{Z}$. Returning to the first induction step, case 2, we have

$$\begin{aligned} \text{LHS} &= 2^{k+1} - (-1)^{k+1} + 3u \\ &= 3w + 3u \\ &= 3(w+u) \end{aligned}$$

where $w + u \in \mathbb{Z}$, because the integers are closed under addition. Thus, by setting $v = w + u$,

$$(b_{k+1} \cdots b_0) - \sum_{i=0}^{k+1} (-1)^i b_i = 3(w+u) = 3v$$

for $v = w + u \in \mathbb{Z}$. It follows that

$$b_{k+1} \cdots b_0 \equiv_3 \sum_{i=0}^{k+1} (-1)^i b_i.$$

Therefore the induction step is proved, and the induction is complete.