University of Central Florida
School of Computer Science
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Prof. Rene Peralta
Test 1 Solutions (by TA Robert Lee)

1. Prove the following by induction

$$2\sum_{i=1}^{n} i = n(n+1)$$

for all $n \geq 1$.

Solution

Base case ($n = 1$):

$$\text{LHS} = 2\sum_{i=1}^{1} i = 2(1) = 2$$

$$\text{RHS} = 1(1+1) = 1(2) = 2$$

Therefore LHS = RHS and the base case is proved.

Induction hypothesis ($n = k$): For some $k \geq 1$, assume

$$2\sum_{i=1}^{k} i = k(k+1).$$

Induction step ($n = k+1$): We will show that

$$2\sum_{i=1}^{k+1} i = (k+1)(k+2).$$

$$\begin{aligned}
\text{LHS} &= 2\sum_{i=1}^{k+1} i \\
&= \left[2\sum_{i=1}^{k} i\right] + 2(k+1) \\
&= k(k+1) + 2(k+1) \text{ by IH} \\
&= (k+1)(k+2) \text{ by factoring.}
\end{aligned}$$

Therefore LHS = RHS and the induction step is proved. This completes the induction.

2. For each part, give a relation on the set $A = \{1, 2, 3\}$ that satisfies the condition.

   (a) Reflexive and symmetric but not transitive

   Solution

   $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$. This relation is not transitive because $(1, 2)$ and $(2, 3)$ are in R but $(1, 3)$ is not.

   (b) Reflexive and transitive but not symmetric

   Solution

   $R = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$. This relation is not symmetric because $(1, 2)$ is in R but $(2, 1)$ is not.

   (c) Transitive and symmetric but not reflexive

   Solution

   $R = \{(1, 1), (2, 2)\}$. This relation is not reflexive because $(3, 3)$ is missing. Also note that the "empty relation" $R = \{\}$ is a correct answer.

3. Answer true or false and briefly justify your answer.

   (a) The set of finite subsets of the natural numbers is countably infinite.

   Solution True. The justification will be assigned as a problem in Homework 2.

   (b) Every graph with five nodes must have either a complete subgraph of size three or an independent set [1] of size three.

   Solution False. The graph that is a pentagon has neither a complete subgraph of size three nor an independent set of size three.

   (c) All bipartite graphs must have an even number of vertices.

   Solution False. If set A has one vertex and set B has two vertices, if there is no edge between the two vertices in set B (but any other edges are permitted), these three vertices form a bipartite graph with an odd set of vertices.

   (d) $10^{93} \equiv_{11} 10$.

   Solution True. Notice that $10^2 \equiv_{11} 1$. Multiply LHS with itself 46 times and RHS with itself 46 times to get $(10^2)^{46} \equiv_{11} 1^{46}$; in other words, $10^{92} \equiv_{11} 1$. Now since $10 \equiv_{11} 10$, multiply the LHS and RHS of these two equations to get $10^{93} \equiv_{11} 10$.

   (e) Suppose there exists an island where if you pick any set of two horses, they have the same color. Then in this island all horses have the same color.

   Solution True. This is exactly the case where the "in any set of h horses, all horses are the same color" proof in Homework 1 failed; if it is true, then the proof works for all cases.

   (f) If $x \in \{1, 2, 3, 4\}$ then there exists y such that $xy \equiv_5 1$.

   Solution True. If $x = 1$, let $y = 1$; if $x = 2$, let $y = 3$; if $x = 3$, let $y = 2$; if $x = 4$, let $y = 4$. To prove an implication, you need only be concerned with the cases where the premise is true; here the premise $x \in \{1, 2, 3, 4\}$ is true in four different cases. You need to prove the conclusion ("there exists y such that $xy \equiv_5 1$") for each of the cases in which the premise is true.

   (g) If $x \in \{1, 2, 3, 4, 5\}$ then there exists y such that $xy \equiv_6 1$.

   Solution False. If $x = 2$, then xy must be an even number; thus $xy - 1$ is an odd number. Since a multiple of 6 is always an even number, $xy - 1$ cannot be divisible by 6, hence $xy \not\equiv_6 1$. To disprove an implication, you need only a single counterexample in which the premise is true but the conclusion is false.

[1] An independent set is a set of vertices with no edges between them.