Characterizing Cyclic Partitions in 2-Row Bulgarian Solitaire

Sabin K. Pradhan

August 4, 2016

Saint Peter's University

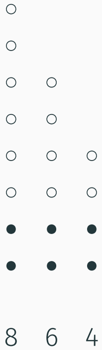
Short History on Bulgarian Solitaire

- Bulgarian Solitaire is neither Bulgarian nor a form of Solitaire.
- 1980, Konstantin Oskolkov.
- Moscow → Sofia → Stockholm → San Diego.
- Henrik Eriksson coined the term “Bulgarian Solitaire.”
- 1983, Martin Gardner.

2-Row Bulgarian Solitaire β

How Does β Work?

The 2-row Bulgarian Solitaire operation β takes the bottom two rows of dots and turns them into columns, rearranging the piles as necessary to be in nonincreasing order.



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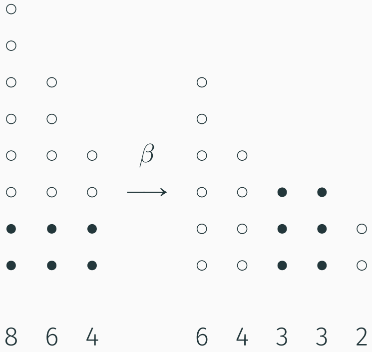
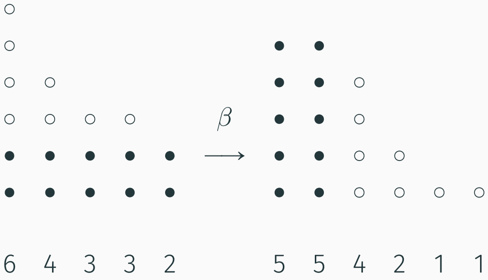


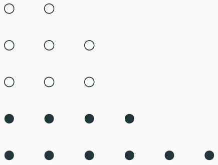
β Operation on $(6, 4, 3, 3, 2)$

$$\begin{array}{ccccc}
\circ & & & & \\
\circ & & & & \\
\circ & \circ & & & \\
\circ & \circ & \circ & \circ & \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & \\
6 & 4 & 3 & 3 & 2
\end{array}$$

β Operation on $(6, 4, 3, 3, 2)$

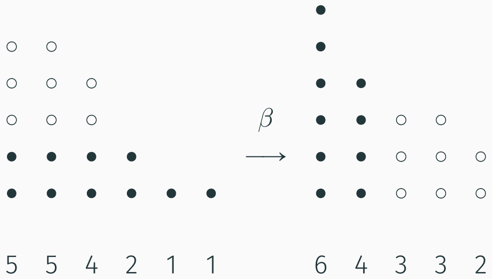


β Operation on $(5, 5, 4, 2, 1, 1)$

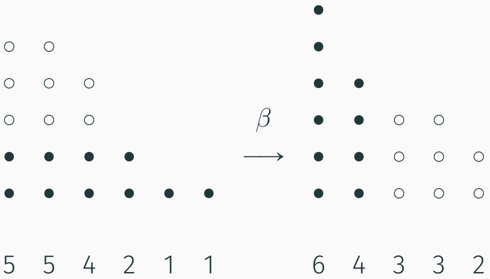


5 5 4 2 1 1

β Operation on $(5, 5, 4, 2, 1, 1)$



β Operation on $(5, 5, 4, 2, 1, 1)$



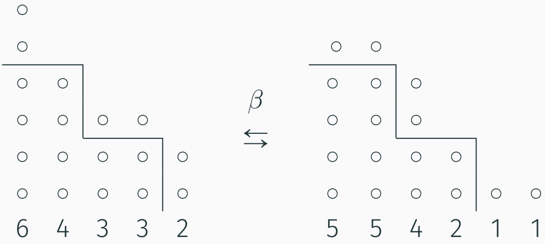
The partition $(5, 5, 4, 2, 1, 1)$ goes back to $(6, 4, 3, 3, 2)$, forming a cycle.

β Map of (8,6,4)

$$(8,6,4) \quad \longrightarrow \quad (6,4,3,3,2) \quad \leftrightarrows \quad (5,5,4,2,1,1)$$

Focus of the research: Characterizing and counting cyclic partitions under β.

Taking a closer look at the β cycle

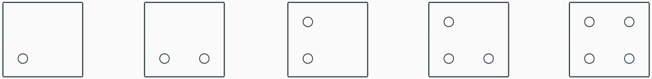


- The lower left 2×2 block acts as a pivot to form the cycle.
- Notice the triangular nature of the 2×2 dots; this plays an important determining number of cyclic partitions.

Block Notation

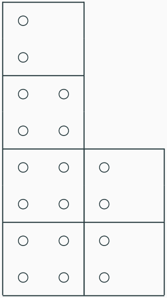
How to convert Ferrers Diagram into Block Notation

- Divide the Ferrers diagram into blocks 2×2 dots.
- Represent each block by summing the number of dots inside the block.



1-block *H*-block *V*-block 3-block 4-block

$(8, 6, 4)$ in Block Notation

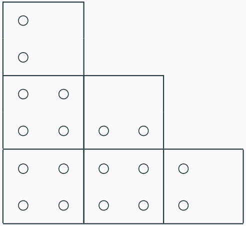


$\sim$

V	
4	
4	V
4	V

8 6 4

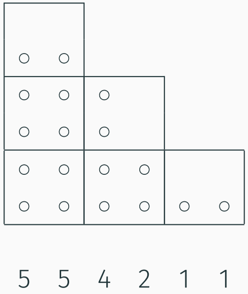
$(6, 4, 3, 3, 2)$ in Block Notation



6 4 3 3 2

$$\sim \begin{array}{ccc} V & & \\ 4 & H & \\ 4 & 4 & V \end{array}$$

$(5, 5, 4, 2, 1, 1)$ in Block Notation



$$\sim \begin{array}{ccc} H & & \\ 4 & V & \\ 4 & 4 & H \end{array}$$

β Map of $(8,6,4)$

The block map of the β iteration on $(8,6,4)$ is

$$\begin{array}{ccccc}
\begin{array}{lll} V & & \\ 4 & & \\ 4 & V & \\ 4 & V & \end{array} & \longrightarrow & \begin{array}{lll} V & & \\ 4 & H & \\ 4 & 4 & V \end{array} & \leftrightarrows & \begin{array}{lll} H & & \\ 4 & V & \\ 4 & 4 & H \end{array} \\
(8,6,4) & & (6,4,3,3,2) & & (5,5,4,2,1,1)
\end{array}$$

Discussing Cycle Partitions

Complete vs. Incomplete Diagonals

$$\begin{matrix} H & & \\ 4 & V & \\ 4 & 4 & H \end{matrix}$$

- Complete diagonals consist entirely of 4 blocks.
- Complete diagonals remain the same under β.
- Variation in cycle partitions stems from incomplete diagonals (i.e., ones with empty spaces or any block other than a 4).

Zero Incomplete Diagonals

Some examples of zero incomplete diagonals:

$$
\begin{array}{l} 4 \circlearrowleft \end{array}
\qquad
\begin{array}{ll} 4 & \circlearrowleft \\ 4 & 4 \end{array}
\qquad
\begin{array}{lll} 4 & & \\ 4 & 4 & \circlearrowleft \\ 4 & 4 & 4 \end{array}
\qquad
\begin{array}{llll} 4 & & & \\ 4 & 4 & & \\ 4 & 4 & 4 & \circlearrowleft \\ 4 & 4 & 4 & 4 \end{array}
$$

- Has to be in form $4T_k = 4(1 + 2 + \cdots + k)$.
- Always points to itself ,i.e., cycle length 1.

One Incomplete Diagonal

$$\begin{array}{ccc} V & & \\ \cline{1-1} 4 & \multicolumn{1}{|c}{H} & \\ \cline{2-2} 4 & 4 & \multicolumn{1}{|c}{V} \end{array} \quad \overset{\beta}{\leftrightarrows} \quad \begin{array}{ccc} H & & \\ \cline{1-1} 4 & \multicolumn{1}{|c}{V} & \\ \cline{2-2} 4 & 4 & \multicolumn{1}{|c}{H} \end{array}$$

- Foundation of unchanging $4T_2 = 4(1+2) = 4 * 3 = 12$ dots.
- $18 - 12 = 6$ dots alternate in the third diagonal.

One Incomplete Diagonal

Other one incomplete diagonal cycles for 18 are

$$\begin{tikzcd}
\begin{matrix}3&&\\4&3&\\4&4&\end{matrix} \arrow[rr] & & \begin{matrix}&&\\4&3&\\4&4&3\end{matrix} \arrow[dl] \\
& \begin{matrix}3&&\\4&&\\4&4&3\end{matrix} \arrow[ul] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{matrix}1&&\\4&1&\\4&4&4\end{matrix} \arrow[rr] & & \begin{matrix}4&&\\4&1&\\4&4&1\end{matrix} \arrow[dl] \\
& \begin{matrix}1&&\\4&4&\\4&4&1\end{matrix} \arrow[ul] &
\end{tikzcd}$$

Enumerating One Incomplete Diagonal

How many one incomplete diagonal cyclic partitions can there exist for 18?

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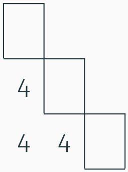

The 6 dots over the 12 in the triangular foundation need to fill the 3 boxes using a combination of $1, H, V, 3, 4$ blocks.

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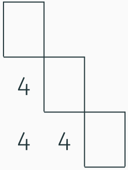

The 6 dots over the 12 in the triangular foundation need to fill the 3 boxes using a combination of $1, H, V, 3, 4$ blocks. We can do this by using the formula:

$$\sum_{h=0}^{\lfloor \frac{r}{4} \rfloor} \sum_{i=0}^{\lfloor \frac{r-4h}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{(r-4h-3i)}{2} \rfloor} 2^j \binom{k}{h, i, j, r-2j-3i-4h, k+3h+2i+j-r}$$

and plugging in $r = 6$ and $k = 3$ into which gives us 38 cyclic partitions of 18 with one incomplete diagonal.

However, Hopkins computed that out of the 385 partition of 18, there are 42 cyclic partitions for 18, so where are the other 4?

Three Incomplete Diagonals in Ferrers Diagram

$$
\begin{array}{ccc}
\begin{array}{ccccc}
\circ & & & & \\
\circ & & & & \\
\circ & & & & \\
\circ & & & & \\
\circ & \circ & \circ & \circ & \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
7 & 3 & 3 & 3 & 2
\end{array}
& \longrightarrow &
\begin{array}{cccccc}
\circ & \circ & \circ & & & \\
\circ & \circ & \circ & & & \\
\circ & \circ & \circ & & & \\
\circ & \circ & \circ & & & \\
\circ & \circ & \circ & \circ & \circ & \circ \\
5 & 5 & 5 & 1 & 1 & 1
\end{array}
\\
\uparrow & & \downarrow \\
\begin{array}{ccccccc}
\circ & \circ & & & & & \\
\circ & \circ & \circ & & & & \\
\circ & \circ & \circ & & & & \\
\circ & \circ & \circ & & & & \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
5 & 5 & 4 & 1 & 1 & 1 & 1
\end{array}
& \longleftarrow &
\begin{array}{ccccc}
\circ & & & & \\
\circ & & & & \\
\circ & & & & \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
6 & 3 & 3 & 3 & 3
\end{array}
\end{array}
$$

Three Incomplete Diagonals in Block Notation

$$
\begin{array}{ccc}
\begin{matrix} 1 & & \\ V & & \\ 3 & H & \\ 4 & 4 & V \end{matrix} & \longrightarrow & \begin{matrix} H & 1 & \\ 4 & V & \\ 4 & 3 & H \end{matrix} \\
\uparrow & & \downarrow \\
\begin{matrix} H & & & \\ 4 & V & & \\ 4 & 3 & H & 1 \end{matrix} & \longleftarrow & \begin{matrix} V & & \\ 3 & H & 1 \\ 4 & 4 & V \end{matrix}
\end{array}
$$

Enumerating Three Incomplete diagonal Cyclic Partitions

The following formula counts three incomplete diagonal for any $P(n)$

$$\sum_{i=1}^{\frac{k-1}{2}} 2\binom{\frac{k-1}{2}}{i}\binom{\frac{k+1}{2}}{r-6k+i+4} \cdot \delta(r-6k+i+4>0)$$

All three diagonal cycle partitions for $P(18)$ can be counted by plugging $k=3$ and $r=14$ into the formula above which yields

$$2\binom{1}{1}\binom{2}{14-18+1+4} = 2\binom{2}{1} = 2\cdot 2 = 4.$$

Therefore, there are four three incomplete diagonal cyclic partitions of 18.

How many other forms are there?

Proved that there are only 4 Types of cyclic partitions under β:

- Partitions with zero, one and three incomplete diagonal.
- Partitions with 2 incomplete diagonals: root of all the variation in the problem.

Cyclic partitions with more than 3 incomplete diagonals is not possible.

Two Incomplete Diagonals

- Three incomplete diagonals problem is in fact an overlapping two incomplete diagonal problem.

$$\begin{array}{ccc} 1 & & \\ V & & \\ 3 & H & \\ 4 & 4 & V \end{array}$$

- We showed that in a diagonal above 3-blocks or below 1-blocks, any H or V-blocks must appear in alternating positions.
- Main source of problems regarding characterizing variations.

Enumerating Two Incomplete diagonal Cyclic Partitions

2-1 block for any diagonals k and $k+1$, where k is odd

$$\sum_{j=1}^{k}\sum_{i=0}^{k-j} 2\binom{k}{j}\binom{k-j}{i}\binom{\frac{k+1}{2}}{r-4k+2j+i} \cdot \delta(r-4k+2j+i>0)$$

3-2 block for any diagonals $k-1$ and k, where k is odd

$$\sum_{i=1}^{\frac{k-1}{2}}\sum_{j=1}^{k} 2\binom{\frac{k-1}{2}}{i}\binom{k}{j}\binom{k-j}{r-2j+i-4k+4} \cdot \delta(r-2j+i-4k+4 \geqslant 0)$$

3-1 block for any diagonal k and $k+1$, where k is an integer

$$\sum_{i=1}^{k}\sum_{j=1}^{k+1} \binom{k}{i}\binom{k+1}{r+i-4k} \cdot \delta(r+i-4k>0)$$

Conclusion

Summary

The Task:

Find a way to characterize and count cyclic partitions in 2-row Bulgarian solitaire for any positive integer.

Summary

The Task:

Find a way to characterize and count cyclic partitions in 2-row Bulgarian solitaire for any positive integer.

Accomplishments:

1. Developed Tools: Block Notation, Complete and Incomplete diagonals
2. Proved that there are four variant of cycle partitions with zero, one, two or three incomplete diagonals, no more.
3. Came up with formulas to count all the cyclic partition for any positive integer n under β.

References

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