SAINT PETER'S UNIVERSITY

HONORS THESIS

Characterizing Cycle Partition in 2-Row Bulgarian Solitaire

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Abstract

This paper studies cyclic partitions under the operation 2-row Bulgarian solitaire. We develop tools such as block notation to make characterizing cyclic partitions easier. Using these blocks, we see that cyclic partition under 2-row Bulgarian solitaire have independently cycling diagonals satisfying one of four conditions. We conclude with an enumeration results that allow us to calculate the number of cyclic partitions for a given integer n.

Acknowledgment

I would like to take this opportunity to thank a number of people without whom this thesis would not have been possible.

I would like to express my sincere gratitude to Dr. Brian Hopkins for his unwavering support and guidance throughout the study and never losing hope. Dr. Hopkins' insight and patience have truly been essential to the completion of this project and I thank him immensely. I would also like to thank the Mathematics Department at Saint Peter's University for giving me the tools to pursue this project.

I am also grateful for the love and reassurance my family gives me everyday. Without their encouragement, I would have never been able to pursue my interests at Saint Peter's University. Even though they reside half a world away, their counsel has remained an important part of my life.

Last, but certainly not least, I would like to thank Dr. Rachel Wifall and the Honors Program at Saint Peter's University for giving me the opportunity to venture out on this research experience, which has certainly put my time here at Saint Peter's to the test.

Contents

1 Introduction

1.1 Finite Dynamical Systems

Given a nonempty finite set X, a finite dynamical system is the pair (X, ϕ) such that $\phi : X \to X$. A repeated composition of the operation ϕ to itself is referred to as an iterations. Given $\alpha \in X$ and $n \in \mathbb{N}$, an n-fold composition of ϕ is denoted by $\phi^n(\alpha) = (\phi \circ \phi \circ \cdots \circ \phi)(\alpha)$.

The set X partitions into two subsets when ϕ is iterated across all elements in X: the cyclic set and the tail set. The cyclic set, $\mathrm{Cyc}(X, \phi)$ of X are a part of a cycle, i.e., $\mathrm{Cyc}(X, \phi) = \{\alpha \in X : \phi^n(\alpha) = \alpha \text{ for some } n \in \mathbb{N}\}$. The tail set, denoted by $\mathrm{Tail}(X, \phi)$, consists of elements that do not form cycles when iterated by ϕ. Given $x \in X$, if m is the least non-negative integer such that $\phi^m(x)$ is cyclic, then $\phi^k(x) \in \mathrm{Tail}(X, \phi)$ for any $0 \leqslant k < m$. The ends of a tail, that is, any $z \in X$ such that there is no $y \in X$ with $\phi(y) = z$, are called a Garden of Eden states (from the language of cellular automata).

1.2 Partitions

Simply stated, a partition of an integer n is a way writing n as the sum of positive integers summands. Since order does not matter, partitions are conventionally written in nonincreasing order of summands. For example, there are 7 partitions of five given below:

$$
\begin{gathered}
5 \\
4+1 \\
3+2 \\
3+1+1 \\
2+2+1 \\
2+1+1+1 \\
1+1+1+1+1
\end{gathered}
$$

A partition 2+1+1+1 can be denoted (2,1,1,1), or using short-hand notation, 2111, or even more succinctly as 21^3.

In general, a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)$ with integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_t > 0$ is a partition of $n = \sum_{i=1}^{t} \lambda_i$ with t parts. We denote the set of all partitions of n by $P(n)$.

We use the Ferrers diagram to represent partitions as an array of dots. For example, the partition 2+1+1+1 has Ferrers diagram

$$\begin{array}{cccc} \circ & & & \\ \circ & \circ & \circ & \circ \\ & & & \\ 2 & 1 & 1 & 1 \end{array} \quad .$$

Each summand λ_i corresponds to a column of λ_i dots. In this paper, we will see that Ferrers diagram are helpful tool for understanding some operations on partitions.

1.3 Bulgarian Solitaire

Bulgarian solitaire combines finite dynamical system with integer partitions. It was first introduced by Brandt in 1982 [3]. Given a partition $\lambda = (\lambda_1, \cdots, \lambda_t)$, Bulgarian solitaire is defined by the operation $\alpha : P(n) \to P(n)$ where

$$\alpha(\lambda) = \{t, \lambda_1 - 1, \lambda_2 - 1, \cdots, \lambda_t - 1\}.$$

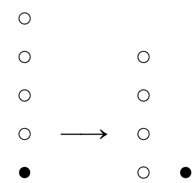
The parts of the image may need to be reordered to be nonincreasing. In terms of the Ferrers diagram, it takes the bottom row of dots and turns it into a column while the rest of the partitions remain unchanged [4].

To understand the operation better, Here are some examples of Bulgarian solitaire operation on partitions of 5. The black dots represent the changes in the Ferrers diagram with each Bulgarian solitaire application.

1. $\alpha((1,1,1,1,1)) = (5)$

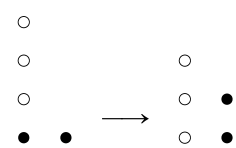
$$\begin{array}{ccccccc} & & & & & & \bullet \\ & & & & & & \bullet \\ & & & & & & \bullet \\ & & & & & \longrightarrow & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & & \bullet \\ & & & & & & \\ 1 & 1 & 1 & 1 & 1 & & 5 \end{array}$$

2. $\alpha((5)) = (4,1)$



5 $\quad\quad$ 4 1

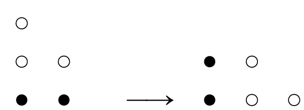
3. $\alpha((4,1)) = (3,2)$



4 1 $\quad\quad$ 3 2

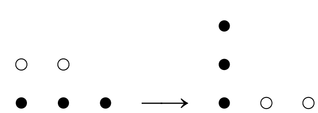
Note: There is a reordering of parts where t is placed after $\lambda_1 - 1$, since $t = 2 < 3 = \lambda_1 - 1$.

4. $\alpha((3,2)) = (2,2,1)$



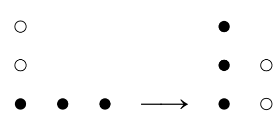
3 2 $\quad\quad$ 2 2 1

5. $\alpha((2,2,1)) = (3,1,1)$



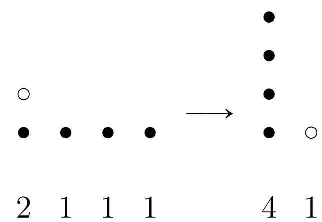
2 2 1 $\quad\quad$ 3 1 1

6. $\alpha((3,1,1)) = (3,2)$



3 1 1 $\quad\quad$ 3 2

7. $\alpha((2,1,1,1)) = (4,1)$



The full map of the Bulgarian solitaire operation on $P(5)$ is

$$\begin{array}{ccccccc} 1^5 & & 21^3 & & & & 311 \\ \downarrow & & \downarrow & & & \swarrow & \uparrow \\ 5 & \rightarrow & 41 & \rightarrow & 32 & \rightarrow & 221 \end{array} \quad .$$

The figure above is an example of a finite dynamic system under the Bulgarian solitaire operation. $\text{Tail}(P(5),\alpha) = \{(5),(4,1),(2,1,1,1),(1,1,1,1,1)\}$. Of these, $(1,1,1,1,1)$ and $(2,1,1,1)$ are Garden of Eden states since they have no pre-images. $\text{Cyc}(P(5),\alpha) = \{(3,2),(2,2,1),(3,1,1)\}$. We observe that all partition of $P(n)$ eventually form cyclic partitions. The cyclic partitions for Bulgarian solitaire have been intensively studied in [3, 4, 5, 6].

Bulgarian solitaire is a variation of one of the basic operations on integer partition, conjugation. Given the partition $\lambda = (\lambda_1, \cdots, \lambda_t)$, the conjugate partition λ' is defined as $\lambda' = (\lambda'_1, \cdots, \lambda'_s)$ where λ' is the number of parts λ_i greater than or equal to i [1].

Take for example the conjugation of partition $(2,2,1)$. According to the definition, when $i = 1$, there are three parts of $(2,2,1)$ that are greater or equal to i, therefore $\lambda_1 = 3$. Note that λ'_1 is equal to t, the number of parts (since every summand is greater than or equal to 1). When $i = 2$, only 2 parts of $(2,2,1)$ are greater or equal to i, thus $\lambda_2 = 2$. There are no parts greater or equal to $i = 3$, therefore, we stop. Thus, $(2,2,1)' = (3,2)$. Similarly, $(3,2)' = (2,2,1)$.

The conjugate of a partition is a reflection of the partition's Ferrers diagram along the block diagonal (or $y = x$ line), swapping the rows and columns. Looking at the Ferrers diagram of the conjugation of $(2, 2, 1)$ and $(3, 2)$, we observe that all the rows of the first partition $(2, 2, 1)$ are simple converted into columns in the second partition, $(3, 2)$.

$$\begin{array}{ccc} & & \\ \circ & \circ & \\ \circ & \circ & \circ \\ & & \\ 2 & 2 & 1 \end{array} \quad \leftrightarrows \quad \begin{array}{cc} \circ & \\ \circ & \circ \\ \circ & \circ \\ & \\ 3 & 2 \end{array}$$

Bulgarian solitaire can be viewed as a first step towards conjugation, in that one row is converted to a column. The next step, converting two rows to columns, is the operation studied in this thesis, defined more formally in the next section. See [4] for some results on this family of operations.

A cycle partition λ for Bulgarian solitaire has the form $(k + \delta_k, k - 1 + \delta_k - 1, ..., 2 + \delta_2, 1 + \delta_1, \delta_0)$ where each δ_i is either 0 or 1 [3].

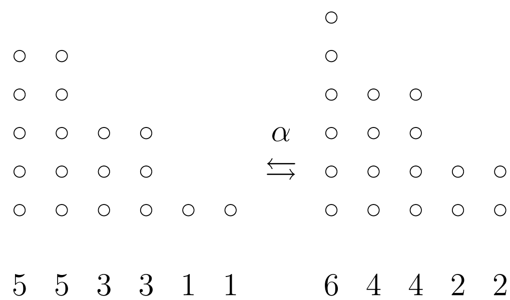
For cases where δ_i is 0 for all $1 \leqslant i \leqslant k$, we see that the partitions is in the form $(k, k-1, \cdots, 2, 1)$ whose sum is $k(k+1)/2$. These types of numbers are called triangular number denoted by T_k. All triangular number T_k form a cyclic partition λ in the form $(k, k-1, \ldots, 3, 2, 1)$ under Bulgarian solitaire [2]. Any partition λ in the form $(k, k-1, \ldots, 2, 1)$ does not change under Bulgarian solitaire. The partition $(3, 2, 1)$ of 6 is shown below.

$$\begin{array}{ccc} \circ & & \\ \circ & \circ & \\ \circ & \circ & \circ \\ & & \\ 3 & 2 & 1 \end{array} \quad \overset{\alpha}{\circlearrowleft}$$

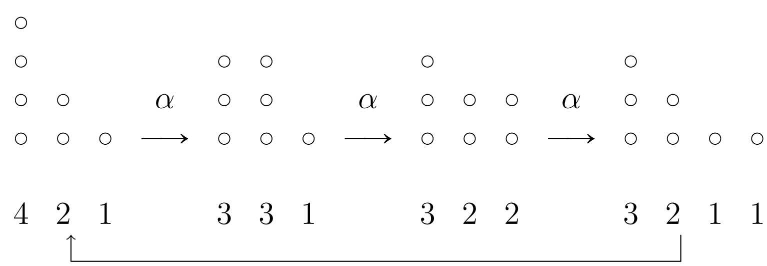
This builds our study to explain characteristics for partitions where δ_i is not 0 for all $1 \leqslant i \leqslant k$. Assuming a nonnegative integer n, write $n = T_k + r$ where k is chosen to be the as large as possible for a nonnegative r. Here, r is equal to $\delta_k + \delta_{k-1} + \cdots + \delta_2 + \delta_1 + \delta_0$. The cyclic partition under Bulgarian solitaire for n will have an unchanging triangular array of T_k dots in the form $(n, n-1, \cdots, 2, 1)$. The variation comes from the remaining r dots which are placed in the $(k+1)$st diagonal [3]. The total number of ways to place r

dots in $k+1$ spaces is simply given by $\binom{k+1}{r}$.

A 2-element cycle between partitions of 18 is shown below.



And a 4-cycle among the partitions of 7.



1.4 2-Row Bulgarian Solitaire Operation

The 2-row Bulgarian solitaire or β operation for a partition $\lambda = (\lambda_1, ..., \lambda_t)$ is defined as

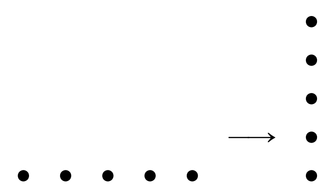
$$\beta(\lambda) = \{\lambda'_1, \lambda'_2, \lambda_1 - 2, ..., \lambda_t - 2\},$$

where λ'_i is the number of parts of $\lambda \geqslant i$. As before, the parts of the image are rearranged in nonincreasing order starting from the left to right.

In terms of the Ferrers diagram, the β operation takes the bottom two rows of the partition and changes them to columns, then rearranges the parts so that the order is non increasing.

Here are several examples of 2-row Bulgarian solitaire.

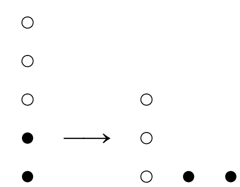
1. $\beta((1,1,1,1,1)) = (5)$



1 1 1 1 1 5

Note: Because there is only one row, the effects of β is the same as α and conjugation.

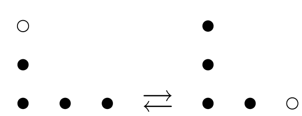
2. $\beta((5)) = (3,1,1)$



5 3 1 1

Note: There is a reordering of parts in this step: since $\lambda'_1 = \lambda'_2 < \lambda_1 - 2$, i.e., $1 < 3$, the image is $(\lambda_1 - 2, \lambda'_1, \lambda'_2)$ or $(3,1,1)$.

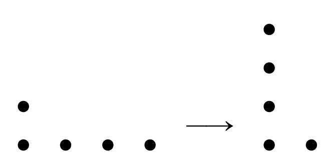
3. $\beta((3,1,1)) = (3,1,1)$



3 1 1 3 1 1

Note: The image produced by β is the same as conjugation.

4. $\beta((2,1,1,1)) = (4,1)$



2 1 1 1 4 1

Note: The image produced by β is the same as α and conjugation.

5. $\beta((4,1)) = (2,2,1)$

$$\begin{array}{cccccc}
\circ & & & & & \\
\circ & & & & & \\
\bullet & & & \bullet & \circ & \\
\bullet & \bullet & \longrightarrow & \bullet & \circ & \bullet \\
 & & & & & \\
4 & 1 & & 2 & 2 & 1
\end{array}$$

Note: The parts have been rearranged to $(\lambda_1', \lambda_1 - 2, \lambda_2')$.

6. $\beta((2,2,1)) = (3,2)$

$$\begin{array}{cccccc}
 & & & & \bullet & \\
\bullet & \bullet & & & \bullet & \bullet \\
\bullet & \bullet & \bullet & \longrightarrow & \bullet & \bullet \\
 & & & & & \\
2 & 2 & 1 & & 3 & 2
\end{array}$$

Note: Image produced by β is the same as conjugation.

7. $\beta((3,2)) = (2,2,1)$

$$\begin{array}{cccccc}
\circ & & & & & \\
\bullet & \bullet & & \bullet & \bullet & \\
\bullet & \bullet & \longrightarrow & \bullet & \bullet & \circ \\
 & & & & & \\
3 & 1 & & 2 & 2 & 1
\end{array}$$

Note: The image produced by β is the same as conjugation.

Like Bulgarian solitaire, the β operation also forms a finite dynamical system on integer partitions. The full map of the 2-row Bulgarian Solitaire operation on $P(5)$ is given below:

$$\begin{array}{ccccccccc}
1^5 & & & & & 21^3 & & 32 \\
\downarrow & & & & & \downarrow & & \uparrow\downarrow \\
5 & \rightarrow & 311 & \circlearrowleft & & 41 & \rightarrow & 221
\end{array}$$

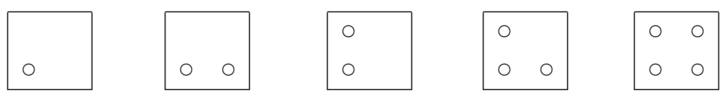
The β map for $P(5)$ has $\mathrm{Tail}(P(5), \beta) = \{(1,1,1,1,1), (5), (2,1,1,1), (4,1)\}$ and $\mathrm{Cyc}(P(5), \beta) = \{(3,1,1), (3,2), (2,2,1)\}$. Out of the Tail elements,

$(1, 1, 1, 1, 1)$ and $(2, 1, 1, 1)$ are Garden of Eden states. The Garden of Eden states for β (and the entire family of operations between Bulgarian solitaire) are characterized in [4]. The focus of this paper is characterizing cyclic partitions of β for all n.

2 Block Notations

To study cyclic partitions under the β operation more effectively, we use a slightly different notation to represent partitions. We divide the Ferrers diagram into 2×2 blocks and represent them via the sum of the dots within the each 2×2 block.

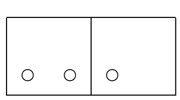
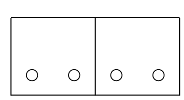
A complication arises when summing the blocks with 2 dots. Since orientation matters, label the vertically aligned two dots as V and the horizontally aligned 2 dots as H; when mentioned together, we call these 2-blocks. Block notation allows only certain combination of block to ensure that the underlying summands are nonincreasing and there are are no gaps within any particular column of the Ferrers diagram which is discussed later in this section. In other words, only certain combinations of blocks correspond to partitions. All possible types of blocks are shown below.



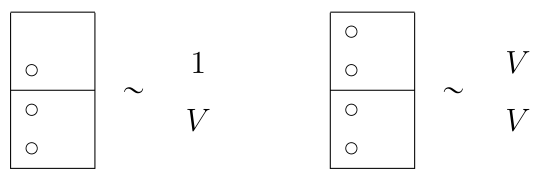
1-block $\quad$ H-block $\quad$ V-block $\quad$ 3-block $\quad$ 4-block

A 1-block has three empty spaces within its block. We cannot have any block to the right of a 1-block to preserve the nonincreasing rule of partitions. In addition, due to the gap in the top half of the 1-block, it cannot support any block above it.

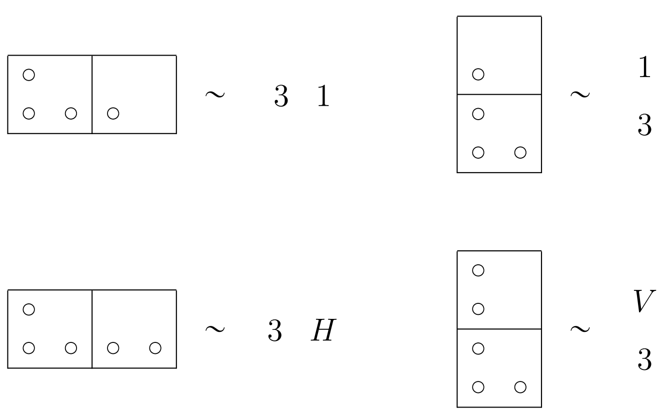
A H-block has two empty spaces in the top half of its block. Thus, to prevent any gaps, the H-block is unable to support block above it. However, the dot in the lower right makes the H-block more flexible than the 1-block: an H-block can support either a 1-block or another H-block to the right. These configurations of H-blocks are given below.

 $\sim \quad H \quad 1$ $\qquad$  $\sim \quad H \quad H$

A V-block also has two empty spaces like the H-block. However, unlike the H-block, these empty spaces are vertically aligned to the right half of the block. Thus, while the V-block is able to support a 1-block or another V-block above it, it cannot support anything to the right. These configurations of V-blocks are given below.



A 3-block only has one empty space to the top right of the block. It is a superposition of the H- and V- block; hence, is able to support any aforementioned configuration of 1 and V-block above itself or 1 and H-block to the right. The 3-block cannot support another 3- or 4-block above or to the right due to an empty space on the top right corner. Some configurations with a 3-block are given below:



A 4-block is the only block that is completely filled. This makes the 4-block gap free, thus provides a gap-less base upon which can support any of the aforementioned configuration of blocks including another 4-block. This also solves the problem of placing blocks in a non increasing order. Since the 4-block has its right column filled, it can also support any block to the right.

This characteristic of 4-blocks plays an important role when talking about cyclic partitions.

Let us consider the partition (6,3,3,3,1).

$$\begin{array}{ccccc}
\circ & & & & \\
\circ & & & & \\
\circ & & & & \\
\circ & \circ & \circ & \circ & \\
\circ & \circ & \circ & \circ & \\
\circ & \circ & \circ & \circ & \circ \\
 & & & & \\
6 & 3 & 3 & 3 & 1
\end{array}$$

Now, divide the number of dots by 2×2 block starting from the bottom left of the Ferrers diagram as shown in the figure below left. The equivalent block diagram is given on the right.

$$\begin{array}{ccccc}
\circ & & & & \\
\circ & & & & \\
\circ & & & & \\
\circ & \circ & \circ & \circ & \\
\circ & \circ & \circ & \circ & \\
\circ & \circ & \circ & \circ & \circ \\
 & & & & \\
6 & 3 & 3 & 3 & 1
\end{array}
\quad \sim \quad
\begin{array}{ccc}
V & & \\
3 & H & \\
4 & 4 & 1
\end{array}$$

The block notation, motivated by β affecting two rows, will be superior to Ferrers diagrams as we analyze cyclic partitions.

We can also define block notation without reference to the Ferrers diagrams. Given a partition $\lambda = (\lambda_1, \cdots, \lambda_t)$, the summands λ_{2i-1} and λ_{2i} determine the blocks for the ith column. If the number of parts t is odd, we add a part $\lambda_{t+1} = 0$ so that parts can be considered in pairs.

If λ_{2i} is even, then the ith column of the block diagram will have $\lambda_{2i}/2$ 4-blocks. If the difference between λ_{2i-1} and λ_{2i} is even then $\lfloor(\lambda_{2i-1} - \lambda_{2i})/2\rfloor$

V-blocks are to added on top of any 4-blocks in the ith column. If the difference is odd, we add $\lfloor(\lambda_{2i-1} - \lambda_{2i})/2\rfloor$ V-blocks and a 1-block above the 4-blocks in the ith column.

For example, partition (5,2) has $\lambda_1 = 5$, $\lambda_2 = 2$, and $i = 1$.

- We have $\lambda_2/2 = 2/2 = 1$ 4-block
- We have $\lfloor(\lambda_1 - \lambda_2)/2\rfloor = \lfloor(5-2)/2\rfloor = \lfloor 3/2\rfloor = 1$ V-block
- Since $(\lambda_1 - \lambda_2) = 3 = 1 \mod 2$, we add a 1-block to the top of the first column.

Thus, partition (5,2) has block notation

$$\begin{array}{c} 1 \\ V \\ 4 \end{array} \quad .$$

If λ_{2i} is odd, then the ith column will have $(\lambda_{2i} - 1)/2$ 4-blocks. If $\lambda_{2i-1} = \lambda_{2i}$, then add one H-block to the ith column. If $\lambda_{2i-1} > \lambda_{2i}$, then add one 3-block and $\lfloor(\lambda_{2i-1} - \lambda_{2i} - 1)/2\rfloor$ V-blocks. Further, if $\lambda_{2i-1} - \lambda_{2i} - 1$ is odd, then add a 1-block at the top of the ith column.

For example, partition $(7,3)$ has $\lambda_1 = 7$, $\lambda_2 = 3$ and $i = 1$.

- We have $(\lambda_2)/2 = (3-1)/2 = 1$ 4-block.
- Since $\lambda_1 > \lambda_2$, i.e., $7 > 3$, we have one 3-block.
- We have $\lfloor(\lambda_1 - \lambda_2 - 1)/2\rfloor = \lfloor(7-3-1)/2\rfloor = \lfloor 3/2\rfloor = 1$ V-block.
- Since $\lambda_1 - \lambda_2 - 1 = 3 = 1 \mod 2$, we also have one 1-block.

Thus, partition (7,3) has block notation

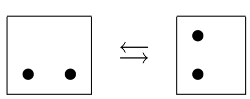
$$\begin{array}{c} 1 \\ V \\ 3 \\ 4 \end{array} \quad .$$

Block notation also gives rise to a concept of column distance which will be very important when dealing with cyclic partitions with multiple incomplete diagonals. Simply stated, the column distance k from a reference block r to a target block t is the number of columns separating r from t. In the block notation of partition $(6, 3, 3, 3, 1)$ above, the 3-block in the first column is two columns away from the 1-block in the third column, thus it has a column distance of 2.

2.1 2-Row Bulgarian Solitaire on Single Blocks

Before we analyze cyclic partition, it is important to understand how the β operations affects a single block. When acting on a single block, the β operation is equivalent to conjugation. In other words, β acting on a single block is the same as reflecting the Ferrers diagram across the line $y = x$.

Due to their symmetry across the block diagonal, the 1-, 3-, and 4-blocks remain unchanged under the β operation. The H-blocks and V-blocks, however, are not symmetric along the block diagonal, so they are not fixed by β. Instead, $\beta(\beta(1, 1)) = \beta((2)) = (1, 1)$. That is, the H-blocks and V-blocks are swapped by β. This is in fact the source of most of the complexity surrounding the characterization problem of this study.



$$H \leftrightarrows V$$

2.2 Cycles in Block Notation

Using block notations, characterizing the cyclic partitions of β operations becomes much easier. It helps us to see the intricate changes that are not as apparent in the Ferrers diagrams.

Let us take a look at the β cycle of (5,4,2,2).

$$\begin{array}{ccccccccccccccc}
\circ & & & & & & & & & & & & & & \\
\circ & \circ & & & & \circ & \circ & & & & \circ & \circ & & & \\
\circ & \circ & & & & \circ & \circ & \circ & & & \circ & \circ & & & \\
\circ & \circ & \circ & \circ & \longrightarrow & \circ & \circ & \circ & \circ & \longrightarrow & \circ & \circ & \circ & \circ & \\
\circ & \circ & \circ & \circ & & \circ & \circ & \circ & \circ & & \circ & \circ & \circ & \circ & \circ \\
 & & & & & & & & & & & & & & \\
5 & 4 & 2 & 2 & & 4 & 4 & 3 & 2 & & 4 & 4 & 2 & 2 & 1
\end{array}$$

Using the block notation, we can see a much clearer pattern in cyclic partitions:

$$\begin{array}{ccccccccc}
1 & & & & & & & & \\
4 & & \longrightarrow & 4 & 1 & \longrightarrow & 4 & & \\
4 & 4 & & 4 & 4 & & 4 & 4 & 1
\end{array}$$

The block notation shows us that the 1-block simply descends along a diagonal over the 4-blocks, as with the cyclic partitions of Bulgarian solitaire. We shall expand this concept of block diagonals in the next section.

In contrast to Bulgarian solitaire, where we considered diagonals of dots, in 2-row Bulgarian solitaire, we need to look at the problem using diagonals of 2×2 dots. The ith block diagonal is composed of i blocks that play an important role to distinguish cyclic partitions from tail partitions.

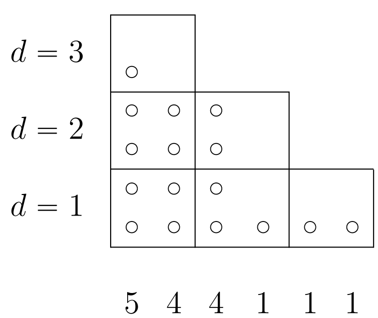
The diagonals in 2-row Bulgarian solitaire can be divided into two categories: complete diagonals and incomplete diagonals.

A complete diagonal consists entirely of 4-blocks, leaving no empty spaces within the diagonal. When dealing with cyclic partitions, we will see that complete diagonals form a triangular array of 4-blocks that remains fixed by β, much like the triangular array of dots in Bulgarian solitaire.

The variation in cyclic partitions for 2-row Bulgarian solitaire β arises from subsequent diagonals, i.e., incomplete diagonals. In contrast to complete block diagonals, any diagonal that is not completely filled with 4-blocks forms an incomplete diagonal. The name is attributed to the empty spaces in the blocks that make the block diagonal.

In the previous section, we looked at the β cycle of $(5, 4, 2, 2)$. We observe that the first and second diagonals are complete, since they are completely filled with 4-blocks while the outermost third diagonal is incomplete, consisting of one 1-block.

Let us look at another example of partition $(5, 4, 4, 1, 1, 1)$ to better understand complete and incomplete diagonals. The d values to the left of the Ferrers diagram and block notation denote the dth diagonal.



$$\sim \quad \begin{array}{r|lll} d=3 & 1 & & \\ d=2 & 4 & V & \\ d=1 & 4 & 3 & H \end{array}$$

We see that the first diagonal, $d = 1$, is the only diagonal complete, since the one 4-block that fills the first diagonal. The second diagonal, $d = 2$, has only one 4-block among its two blocks, thus is incomplete. Similarly, since the third diagonal, $d = 3$, is also not completely filled with three 4-blocks, it is also incomplete.

Cyclic partitions operating under β have the property that blocks stay in the same diagonal. Specifically, the d blocks of the dth diagonal cycle, except that any V- and H-blocks swap when moving from the bottom row to the first column.

In any cyclic partitions, we will show that there may be up to three diagonal cycles operating under β. The length of any particular β cycle can be determined by taking the lowest common multiple of the total amount of steps taken for each diagonal to cycle with a possible factor of 2 when 2-blocks are present.

3 Characterizing Cyclic Partitions

Partitions with only complete block diagonals provide very few cyclic partitions for β. Many more arise with one incomplete diagonal, similar to α. There are some cyclic partitions, though, that have two or three incomplete diagonals. We treat the number of incomplete diagonals in turn and prove that there cannot be four or more in a cyclic partition.

3.1 Cycles with Zero Incomplete Diagonals

Lemma 1. *If a partition λ of n has zero incomplete diagonals then $n = 4T_k$ for some k and $\lambda = (2k, 2k, \ldots, 4, 4, 2, 2)$.*

Proof. Having only complete diagonals, say k of them, means the block diagram of λ consists of k diagonals full of 4-blocks. Thus $n = 4T_k$ and λ must have the desired form. $\square$

Lemma 2. *If a partition λ of n has zero incomplete diagonals, then λ is cyclic and has cycle length one.*

Proof. From Lemma 1, we know $\lambda = (2k, 2k, \ldots, 4, 4, 2, 2)$, with $2k$ parts. Thus, the β image of λ is

$$\begin{aligned}\beta((2k, 2k, \ldots, 4, 4, 2, 2)) &= (2k, 2k, 2k-2, 2k-2, \ldots, 4-2, 4-2, 2-2, 2-2) \\ &= (2k, 2k, 2k-2, 2k-2, \ldots, 2, 2) = \lambda.\end{aligned}$$

In other words, β maps λ to itself. Hence, it has cycle length one. $\square$

3.2 Cycles with One Incomplete Diagonal

Cyclic partitions with one incomplete diagonal have no restrictions as to the content of that diagonal, since every block is supported horizontally and vertically by a 4-block.

3.3 Cycles with Two Incomplete Diagonals

We will see that there are three different types of cyclic partitions with two incomplete diagonals, which we consider in turn. In general, an m-n diagonal means that the lower incomplete diagonal has an m-block as its smallest block the the higher diagonal has an n-block its largest block.

By a 2-1 diagonal we mean that there is at least one type of 2-block on the ith diagonal along with larger blocks and at least one 1-block on the $(i+1)$st diagonal with no larger blocks.

Lemma 3. *Suppose a cyclic partition λ with incomplete diagonals i and $i+1$ satisfies the 2-1 diagonal description. Then i must be odd and there are two conclusions about the content of the incomplete diagonals:*

- *The 1-blocks in the $(i+1)$st diagonal must appear in alternating positions at even distance from one another.*
- *In the ith diagonal, any V-blocks are at an even distance from the 1-blocks while all H-blocks are at an odd distance from the 1-blocks.*

Proof. Consider a partition λ with a triangular array of T_{i-1} 4-blocks, an ith diagonal with one V-block in the first column and $i-1$ 4-blocks filling the rest of the ith diagonal, and an $(i+1)$st diagonal with a single 1-block directly on top of the V-block.

We an represent this as

$$\begin{array}{rcl|llll} d & = & i+1 & 1 & & & \\ d & = & i & V & & & \\ d & = & i-1 & 4 & 4 & & \\ & \vdots & & \vdots & \ddots & \ddots & \\ d & = & 1 & 4 & \cdots & 4 & 4 \end{array}$$
$$\beta^0$$

After i^2 iterations of β, the partition will have arrangement

$$\begin{array}{rcl|llll} d & = & i+1 & 1 & & & \\ d & = & i & 4 & & & \\ d & = & i-1 & 4 & \ddots & & \\ & \vdots & & \vdots & \ddots & 4 & \\ d & = & 1 & 4 & \cdots & 4 & V \end{array} \quad \xrightarrow{\beta} \quad \begin{array}{llll} H & 1 & & \\ 4 & 4 & & \\ \vdots & \ddots & \ddots & \\ 4 & \cdots & 4 & 4 \end{array}$$
$$\beta^{i^2} \qquad\qquad\qquad \beta^{i^2+1}$$

After this arrangement, at the (i^2+i+1)st iteration, the cycle goes back to the original partition.

$$\begin{array}{ccccc|cccc}
d & = & i & + & 1 & 1 & & & \\
d & = & i & & & V & & & \\
d & = & i & - & 1 & 4 & 4 & & \\
 & \vdots & & & & \vdots & \ddots & \ddots & \\
d & = & 1 & & & 4 & \cdots & 4 & 4
\end{array}$$
$$\beta^{i^2+i+1}$$

We observe in this cycle that all the H-blocks occur an odd distances from the 1-block while the V- blocks occur only at even distances from the 1-block. The restrictions are the same when there are aditional 2-blocks on the ith diagonal. The $(i+1)$st diagonal can have 1-blocks in every other position at most. $\square$

A 3-2 diagonal refers to a cyclic partition with at least one 3-block on the ith diagonal along with 4-blocks and at least one 2-block on the $(i+1)$st diagonal along with 1-blocks.

Lemma 4. *Suppose a cyclic partition λ with incomplete diagonal i and $i+1$ satisfies the 3-2 diagonal description. Then i must be even and there are two conclusions about the content of the incomplete diagonals:*

- *The H- and V-block in the $(i+1)$st diagonal must appear at an even distance from like 2-blocks and an odd distance from unlike 2-blocks.*
- *In the ith diagonal, any 3-block is at an even distance from any V-blocks and an odd distance from any H-blocks.*

Proof. Consider a partition λ with a triangular array of T_{i-1} 4-block, an ith diagonal with one 3-block in the first column and 4-blocks filling the rest of the ith diagonal, and an $(i+1)$st diagonal with a V-block directly on top of the 3-block. Thus, λ is

$$
\begin{array}{ccccc|cccc}
d & = & i & + & 1 & V & & & \\
d & = & i & & & 3 & & & \\
d & = & i & - & 1 & 4 & 4 & & \\
 & \vdots & & & & \vdots & \ddots & \ddots & \\
d & = & 1 & & & 4 & \cdots & 4 & 4
\end{array}
$$

$$\beta^0$$

After i^2 iterations of β, the partition will be

$$
\begin{array}{ccccc|cccc}
d & = & i & + & 1 & H & & & \\
d & = & i & & & 4 & & & \\
d & = & i & - & 1 & 4 & \ddots & & \\
 & \vdots & & & & \vdots & \ddots & 4 & \\
d & = & 1 & & & 4 & \cdots & 4 & 3
\end{array}
\quad \xrightarrow{\beta} \quad
\begin{array}{cccc}
3 & H & & \\
4 & 4 & & \\
\vdots & \ddots & \ddots & \\
4 & \cdots & 4 & 4
\end{array}
$$

$$\beta^{i^2} \qquad\qquad \beta^{i^2+1}$$

After this, at $(i^2 + i + 1)$st iteration, the cycle returns to the original partition.

$$
\begin{array}{ccccc|cccc}
d & = & i & + & 1 & V & & & \\
d & = & i & & & 3 & & & \\
d & = & i & - & 1 & 4 & 4 & & \\
 & \vdots & & & & \vdots & \ddots & \ddots & \\
d & = & 1 & & & 4 & \cdots & 4 & 4
\end{array}
$$

$$\beta^{i^2+i+1}$$

We observe in this cycle that all the H-blocks occur an odd distances from the 3-block while the V-blocks occur only at even distances from the 3-block. The restrictions are the same when there are additional 3-blocks on the ith diagonal in at most every other position and additional 2-blocks in the $(i+1)$st diagonal. There are no restrictions for any 1-blocks in the $(i+1)$st diagonal. □

The last type of two incomplete diagonals is the 3-1, refers to a cyclic partitions with at least one 3-block on the ith diagonal along with 4-blocks and at least one 1-block on the $(i+1)$st diagonal with no larger parts. Since a 1-block is supported by a 3-block, and these blocks themselves do not change under β, due to their symmetry, there are no diagonal nor positional restrictions for this type of two incomplete diagonal.

3.4 Cycles with Three Incomplete Diagonals

The three incomplete diagonal case can be reduced to the study of two overlapping two incomplete diagonal cases, namely the 3-2 and the 2-1 diagonal cases.

Lemma 5. *Suppose a cyclic partition λ with incomplete diagonals i, $i+1$, and $i+2$ has at least one 3-block on the ith diagonal along with 4-blocks, 2-blocks on the $(i+1)$st diagonal, and at least one 1-block on the $(i+2)$nd diagonal with no larger blocks. Then i must be even and there are three conclusions about the content of the incomplete diagonals:*

- *The $(i+1)$st diagonal is completely filled with alternating H- and V-blocks.*
- *The 1-blocks in the $(i+2)$nd diagonal must appear in alternating positions at even distances from one another. In addition, all 1-blocks must be an even distance from the V-blocks and an odd distance from the H-blocks.*
- *In the ith diagonal, any 3-block must appear in alternating positions at even distances from one another. In addition, all 3-blocks must be an even distance from the V-blocks and an odd distance from the H-blocks.*

Proof. Let us take a partition λ with a triangular array of T_{i-1} 4-blocks, an ith diagonal with one 3-block in the first column and $i-1$ 4 -blocks, an $(i+1)$st diagonal with alternating H- and V- blocks, and an $(i+2)$nd diagonal with one 1-block in the first column. Thus λ is

$$\begin{array}{ccccc|ccccc}
d & = & i & + & 2 & 1 & & & & \\
d & = & i & + & 1 & V & & & & \\
d & = & i & & & 3 & H & & & \\
d & = & i & - & 1 & 4 & 4 & V & & \\
 & \vdots & & & & \vdots & \ddots & \ddots & \ddots & \\
d & = & 1 & & & 4 & \cdots & 4 & 4 & V
\end{array}$$

$$\beta^0$$

After $i^2/2 - 1$ iterations of β, the partition will be

$$\underset{\beta^{(i^2/2)-1}}{\begin{array}{ccccc|ccccc}
d & = & i & + & 2 & 1 & & & & \\
d & = & i & + & 1 & V & & & & \\
d & = & i & & & 4 & H & & & \\
d & = & i & - & 1 & 4 & \ddots & \ddots & & \\
 & \vdots & & & & \vdots & \ddots & 3 & H & \\
d & = & 1 & & & 4 & \cdots & 4 & 4 & V
\end{array}}
\quad \xrightarrow{\beta^{(i+1)}} \quad
\underset{\beta^{(i^2/2)+i}}{\begin{array}{ccccc|cccccc}
d & = & i & + & 2 & & & & & & \\
d & = & i & + & 1 & H & & & & & \\
d & = & i & & & 4 & V & & & & \\
d & = & i & - & 1 & 4 & \ddots & \ddots & & & \\
 & \vdots & & & & \vdots & \ddots & 4 & V & & \\
d & = & 1 & & & 4 & \cdots & 4 & 3 & H & 1
\end{array}}$$

After this, the $(i^2/2+i+1)$st iteration takes the cycle back to the original partition.

$$\begin{array}{ccccc|ccccc}
d & = & i & + & 2 & 1 & & & & \\
d & = & i & + & 1 & V & & & & \\
d & = & i & & & 3 & H & & & \\
d & = & i & - & 1 & 4 & 4 & V & & \\
 & \vdots & & & & \vdots & \ddots & \ddots & \ddots & \\
d & = & 1 & & & 4 & \cdots & 4 & 4 & V
\end{array}$$

$$\beta^{(i^2/2)+i+1}$$

We observe in this cycle that all the H-blocks occur an odd distances from the 3-block while the V- blocks occur only at even distances from the 3-block. The restrictions are the same when there are additional 3-blocks in at most every other position of the ith diagonal and additional 1-blocks in at most every other position of the $(i+2)$nd diagonal. $\square$

3.5 Impossibility of Four or More Incomplete Diagonals

Lemma 6. *Cyclic partition are limited to no more than three incomplete diagonals.*

Proof. From the previous lemma, we know that three incomplete diagonals is possible only for an $(i+2)$nd even diagonal of alternating 1-blocks completely supported by an $(i+1)$st diagonal of alternating H and V blocks. Similarly, the $(i+1)$st diagonal with alternating H- and V-blocks is supported by an incomplete ith diagonal with at least one 3-block. However, we know that a 3-block can only be supported by a 4 block, so the $(i-1)$st diagonal has to be complete for there to be a cycle. Also, the $(i+2)$nd diagonal with at least one 1-block cannot support anything on the next diagonal. Thus, there can be no more than three incomplete diagonals. $\square$

4 Enumeration of Cyclic Partitions

We complete the thesis with results on counting cyclic partitions under β.

4.1 Counting with Zero or One Incomplete Diagonals

For cyclic partitions of n, write $n = 4T_{i-1} + r$ where i is chosen to be as large as possible for r nonnegative. If $r = 0$, there are no incomplete diagonals, and there is exactly one cyclic partition, described above. Therefore, assume $1 \leqslant r \leqslant 4i - 1$.

We will derive a formula for the number of ways these r dots can be arranged starting in the ith diagonal beginning with just 1-blocks and adding larger blocks until all possibilities are incorporated.

Suppose the r dots are distributed in r 1-blocks on the ith diagonal (this is only sensible for such that $r \leqslant i$). This can be done in $\binom{i}{r}$ ways (using a binomial coefficient or "choose number").

Now, let us assume the r dots can be distributed by 2- and 1-blocks. The number of these combinations is

$$\sum_{j=0}^{\lfloor \frac{r}{2} \rfloor} 2^j \binom{i}{j} \binom{i-j}{r-2j}.$$

In this formula, j is the number of 2-blocks. The 2^j factor accounts for each 2-block being either a V-block or an H-block. Note that the position of H- and V-blocks does not matter here since there is only one incomplete diagonal. The first binomial coefficient counts the ways to place the j 2-blocks among the i positions of the ith diagonal. This accounts for $2j$ of the r dots. The second binomial coefficient chooses which of the remaining $i-j$ positions on the diagonal are filled with the $r-2j$ 1-blocks necessary to place all r dots.

The reasoning is similar when we add the possibility of k 3-blocks:

$$\sum_{k=0}^{\lfloor \frac{r}{3} \rfloor} \binom{i}{k} \sum_{j=0}^{\lfloor \frac{r-3k}{2} \rfloor} 2^j \binom{i-k}{j} \binom{i-j-k}{n-2j-3k}.$$

Finally, we allow ℓ 4-block in the ith block diagonal. We rewrite the resulting formula more concisely using a multinomial coefficient:

$$\sum_{\ell=0}^{\lfloor \frac{r}{4} \rfloor} \sum_{k=0}^{\lfloor \frac{r-4\ell}{3} \rfloor} \sum_{j=0}^{\lfloor \frac{(r-4\ell-3i)}{2} \rfloor} 2^j \binom{i}{\ell, k, j, r-2j-3k-4\ell, i-r+3\ell+2k+j}.$$

Working from the outside in, the limits of summation come from the maximum possible number of 4-, 3-, and 2-blocks, respectively. The multinomial coefficient should be understood as follows: the i positions of the ith diagonal are to be filled with ℓ 4-blocks, k 3-blocks, j various 2-blocks, and $r-2j-3k-4\ell$ 1-blocks, leaving $i-\ell-k-j-(r-2j-3k-4\ell) = i-r+3\ell+2k+j$ empty positions.

4.2 Counting with Two Incomplete Diagonals

For cyclic partition of n with two incomplete diagonals, as mentioned earlier there are three variant. Write $n = 4T_{i-1} + r$ where i is chosen to be as large as possible for a nonnegative r. The types of diagonals depends on whether i is even or odd. Here, r is the number of dots in the outermost diagonal in a one incomplete diagonal arrangement.

For an even i,

- If $\frac{7}{2}i + 2 \leqslant r \leqslant 6i + 1$, then n has a 3-2 diagonal case.
- If $3i + 1 \leqslant r \leqslant 5i$, then n has a 3-1 diagonal case.

For an odd i,

- If $2i + 1 \leqslant r \leqslant \frac{9i-3}{2}$, then n has a 2-1 diagonal case.
- If $3i + 1 \leqslant r \leqslant 5i$, then n has a 3-1 diagonal case.

If r does not meet any of these criteria, this means that there are not enough dots to make two incomplete diagonals. Thus we use dots from the outermost complete diagonal to make the two incomplete diagonals. To do this, we Write $n = 4T_i + m$ where i is chosen to be as large as possible for a nonnegative m. The only difference here is we set $r = n - 4T_{i-1}$. After that like we did previously, depending on whether i is even or odd, we use the previous criteria to see the diagonals n can accommodate depending the value of the new r.

4.2.1 3-1 Diagonals

Let us first look at the case 3-1 diagonal case.

Assume there are k 3-blocks arranged in i positions, while the remaining $i-k$ positions on the diagonal are filled by 4-blocks. The number of ways to arrange this is given by $\binom{i}{k}$, which accounts for the $3k + 4(i-k) = 4i - k$ of the r dots.

The remaining $r - (4i - k) = r + k - 4i$ dots are arranged as 1-blocks in the $(i+1)$st diagonal, leaving $i + 1 - (r + k - 4i) = 5i - r - k + 1$ positions empty, given by $\binom{i+1}{r+k-4i}$.

Thus, we may count all 3-1 diagonal cyclic partitions using the formula

$$\sum_{k=1}^{i} \binom{i}{k} \binom{i+1}{r+k-4i} \delta(r + k - 4i > 0).$$

The Kronecker delta factor $\delta(p)$ is 1 if p is true, 0 if p is false. This is also known as the Iverson bracket. Thus, if there are fewer dots than these blocks can accommodate with at least one 3-block in the ith diagonal and $i+1$ 1-blocks in the $(i+1)$st diagonal, then the delta function reduces the formula to zero since such arrangements are not valid.

4.2.2 2-1 Diagonals

For the 2-1 diagonal case, as mentioned earlier, i must be odd.

As Lemma 3 states, assume the ith diagonal has j 2-blocks with $1 \leqslant j \leqslant i$. The remaining $i - j$ position are filled by k 3-blocks and $i - j - k$ 4-blocks. The number of ways this can be done is given by the product $\binom{i}{j}\binom{i-j}{k}$ and accounts for $2j+3k+4(i-j-k)=4i-2j-k$ dots.

The remaining $r-2j-3k-4(i-j-k) = r+2j+k-4i$ dots are arranged as 1-blocks in $(i+1)/2$ positions in the $(i+1)$st diagonal, leaving $(i+1)-(r+2j+k-4i) = 5i-2j-k-r+1$ empty spaces. The number of these combinations is $\binom{(i+1)/2}{r-4i+2j+k}$.

Thus, the formula

$$\sum_{j=1}^{i}\sum_{k=0}^{i-j} 2\binom{i}{j}\binom{i-j}{k}\binom{(i+1)/2}{r-4i+2j+k}\delta(r-4i+2j+k>0)$$

gives the total number of ways we can arranges r dots in a 2-1 diagonal case. In the formula, the factor two accounts for the two sets of way the H- and V-blocks can be arranged for each configuration. If there are fewer dots than these blocks can accommodate with at least one 2-block in the ith diagonal and $i+1$ 1-blocks in the $(i+1)$st diagonal, then the delta function reduces the formula to zero since such arrangements are not valid.

4.2.3 3-2 Diagonals

For a 3-2 diagonal, i must be even as mention in Lemma 4.

As Lemma 4 states, assume the ith diagonal has k 3-blocks with $1 \leqslant k \leqslant i/2$. The remaining $i-k$ position are filled by 4-blocks. The number of ways this can be done is given by $\binom{i}{k}$. This accounts for $3k+4(i-k) = 4i-k$ dots.

The remaining $r-(4i-k) = r-4i+k$ dots are in the $(i+1)$st diagonal as least one 2-block, counted by j, and $r-4i+k-2j$ 1-blocks. There are $i+1-j-(r-4i+k-2j) = 5i+j-r-k+1$ empty positions in the $(i+1)$st diagonal. The number of ways these arrangements are made can be written as $\binom{i+1-j}{r-4i+k-2j}$.

Thus, the formula

$$\sum_{k=1}^{\frac{i}{2}}\sum_{j=1}^{i+1}\binom{i}{k}2\binom{i+1}{j}\binom{i+1-j}{r-4i+k-2j}$$

gives the total number of ways we can arrange r dots in the 3-2 diagonal case. In the formula, the 2 accounts for the two sets of ways H- and V-blocks can be arranged for each configuration.

4.3 Counting with Three Incomplete Diagonals

For cyclic partitions of n, write $n = 4T_i + m$ where i is chosen to be as large as possible for a nonnegative m. For a 3-2-1 diagonal case, i must be even. The $(i-1)$st diagonal will be the last complete diagonal. Thus, the number dots that make up the three diagonals r is given by $r = n - 4T_{i-1}$, where $3 + \frac{11}{2}i \leqslant r \leqslant 2 + \frac{13}{2}i$

According to Lemma 5, the ith diagonal must have at least one 3-block, the $(i+1)$st diagonal must have alternating H- and V-blocks, and the $(i+2)$nd diagonal must have at least one 1-block.

Let the ith diagonal have k 3-blocks, while the remaining positions are filled with $i - k$ 4-blocks. The number of ways these blocks can be arranged is $\binom{i/2}{k}$. Similarly, the alternating H- and V-blocks in the $(i + 1)$st diagonal have only 2 possible combinations. The ith and $(i + 1)$st diagonal account for $3k + 4(i - k) + 2(i + 1) = 6i - k + 2$ dots. The remaining $r - (6i - k + 2) = r - 6i + k - 2$ dots are arranged as 1-blocks in alternating positions of the $(i+2)$nd diagonal with $(i+2)-(r-6i+k-2) = 7i-r-k+4$ empty spaces; possible configurations are counted by $\binom{(i+2)/2}{r-6i+k+2}$.

Thus the formula

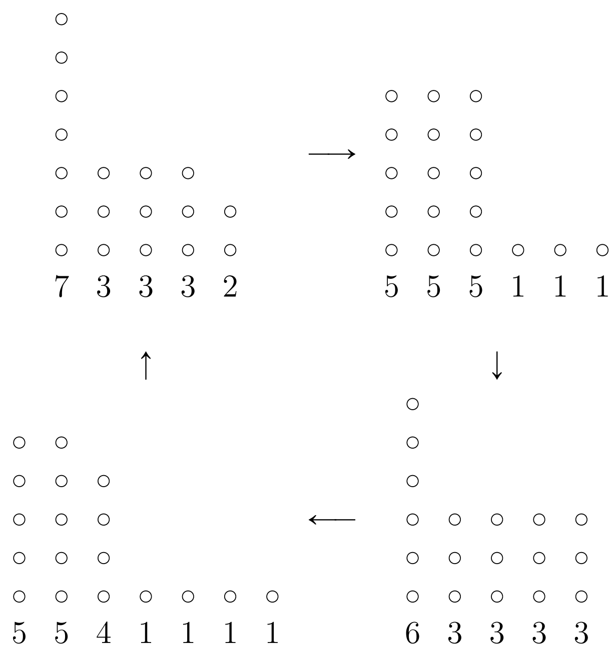
$$\sum_{k=1}^{\frac{i}{2}}\binom{i/2}{k}2\binom{(i+2)/2}{r-6i+k+2}\delta(r-6i+k+2>0)$$

gives the total number of way we can arrange r dots in a 3-2-1 diagonal case. If there are fewer dots than these blocks can accommodate with at least one 3-block in the ith diagonal, alternative H-and V-block for the $(i + 1)$st diagonal and at least one 1-blocks in the $(i+2)$nd diagonal, then the delta function reduces the formula to zero since such arrangements are not valid.

4.4 Examples and data

Here we give some examples of cyclic partitions under β and present a table of numbers of cyclic partitions by type up to $n = 25$, the values we were able to able to verify through other means using Mathematica.

Here is an example of a length 4 cycle among the partitions of 18.



This is clearer when the same partitions are shown in block notation.

$$
\begin{array}{ccc}
\begin{array}{lll} 1 & & \\ V & & \\ 3 & H & \\ 4 & 4 & V \end{array} & \longrightarrow & \begin{array}{lll} H & 1 & \\ 4 & V & \\ 4 & 3 & H \end{array} \\
\uparrow & & \downarrow \\
\begin{array}{llll} H & & & \\ 4 & V & & \\ 4 & 3 & H & 1 \end{array} & \longleftarrow & \begin{array}{lll} V & & \\ 3 & H & 1 \\ 4 & 4 & V \end{array}
\end{array}
$$

These are the smallest partitions with three incomplete diagonals arises in partitions of 18, shown first in standard notation. This leaves $r = 18-4 = 14$ dots to place. Applying the formula derived for three incomplete diagonals in this case corroborates the number we have found:

$$\binom{1}{1} \cdot 2 \cdot \binom{2}{14 - 18 + 1 + 4} = 2 \cdot \binom{2}{1} = 2 \cdot 2 = 4.$$

The following table shows the number of cyclic partitions for $1 \leqslant n \leqslant 25$ by number of incomplete diagonals and, in the case of two incomplete diagonals, the three types we have characterized.

$i \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	.	.	.	1	.	.	.	.	.	.	.	1	.	.	.	.
1	1	2	1	.	2	5	6	8	6	5	2	.	3	9	16	27
2: 2-1	.	.	2	.	.	.	.	.	.	.	.	.	.	.	.	.
2: 3-1	.	.	.	2	1	.	.	.	.	.	3	9	7	2	.	.
2: 3-2	.	.	.	.	.	.	.	.	.	.	.	.	6	12	12	6
3	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
Σ	1	2	3	3	3	5	6	8	6	5	5	10	16	23	28	33

$i \backslash n$	17	18	19	20	21	22	23	24	25
0	.	.	.	.	.	.	.	1	.
1	33	38	33	27	16	9	3	.	4
2: 2-1	.	.	4	14	3	36	24	6	.
2: 3-1	.	.	.	.	.	4	18	34	31
2: 3-2	2	.	.	.	.	.	.	.	.
3	.	4	2	.	.	.	.	.	.
Σ	35	42	39	41	46	49	45	41	35

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