AMDiS – Adaptive Multi-Dimensional Simulations

Eine Einführung in das FEM-Framework

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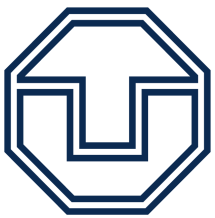
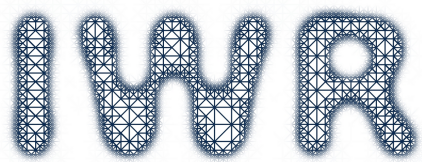


TECHNISCHE UNIVERSITÄT DRESDEN



About this Course

- Goals: Introductory course
  - You know how to use the software AMDiS
  - You can solve a scalar equation, or a system of elliptic equations
  - You can handle instationary problems, nonlinearities and complex boundary conditions
  - (You can run your simulation in parallel)
- References:
  - Some theoretical background and basic design ideas of AMDiS software: ALBERTA-FEM especially the ALBERT 1.0 documentation.

    `http://goo.gl/ZMI2kA`
  - The AMDiS-Wiki

    `https://goo.gl/Jy3u1u`
  - Some (old) PDF Documentation on Fusionforge

    `https://goo.gl/5ngfYd`
- Workshop material can be found on GitHub

  `https://goo.gl/3IxKYx`

Outline

1. Scalar linear second order PDEs
2. Handling data on unstructured grids
3. Adaptivity and systems of equations
4. Time-dependent and nonlinear problems

AMDiS **A**daptive **M**ulti-**Di**mensional **S**imulations

AMDiS developed around 2005. Basis: C-library **ALBERTA**. Now: an object-oriented C++-Framework. Basic concepts

- **High abstraction level**: (Physical) Problems can be formulated with little knowledge about numerical details
- **Generality**: Solve a broad class of PDE problems. Linear and nonlinear problems, stationary and instationary. Multiple dimensions and coupling of different dimensions
- **Extensibility**: Interface to extend AMDiS in several aspects, e.g. own error estimators, linear solvers, preconditioners, time-stepping schemes
- **Efficiency**: Several tools for highly efficient simulations, e.g. adaptive meshes, parallelization, multi-mesh, fast linear solver libraries

AMDiS **A**daptive **M**ulti-**Di**mensional **S**imulations

Short history:

- 2002: Beginning of development (based on C-library ALBERT(A))
- 2005: First release
- 2007: PhD Thesis of Simon Vey: "*Adaptive Finite Elements for Systems of PDEs*"
- 2007: Development in the IWR at TU-Dresden
- 2008: First parallel version
- 2011: Release of stable version 0.9
- 2013: PhD Thesis of Thomas Witkowski: "*Software concepts and algorithms for an efficient and scalable parallel finite element method*"
- 2014: Generic expression terms introduced
- 2016(?): AMDiS 2.0

Developers

Axel Voigt, Simon Vey, Christina Stöcker, Thomas Wittkowski, Andreas Naumann, Simon Praetorius, ...

AMDiS **A**daptive **M**ulti-**Di**mensional **S**imulations

Some features

- Solve (sequence of) Systems of **stationary linear PDEs** of **2nd order**
- **Time integrators**: e.g. Rosenbrock method
- **Nonlinear solvers**: e.g. Newton method
- **Adaptivity** in space and time
- Lagrange **basis functions** (deg. 1–4), and center-bubble function
- **Mixed finite-elements**, e.g. P^2/P^1, Mini-Element
- **Multi-Mesh** method (i.e. different components with different mesh)
- Sequential and **parallel** (tested with up to 16K cores, and 10^9 DOFs)
- **Multi-Grid**: geometric (in AMDiS), algebraic (external library)
- Interface to **linear solvers**: (P)MTL4, PETSc, Hypre
- **FETI-DP** / Schur-complement solvers in parallel

Session 1

Second order PDE

General problem formulation:

$$[\partial_t u] + cu - \nabla \cdot (\underline{b} u) - \nabla \cdot (\mathbb{A} \nabla u) = f, \qquad \text{in } \Omega \times (0, T]$$

with $u = u(x)$ the unknown function and coefficients $c, \underline{b}, \mathbb{A}$ that can depend on space, time, and other quantities v living in Ω:

$$\begin{aligned} c &= c(t, x, v, \nabla v), \\ \underline{b} &= \underline{b}(t, x, v, \nabla v), \\ \mathbb{A} &= \mathbb{A}(t, x, v, \nabla v), \\ f &= f(t, x, v, \nabla v), \end{aligned}$$

equipped with (initial- and) boundary conditions on $\partial\Omega$.

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System of second order PDEs

General problem formulation:

$$\sum_j \left[M^{i,j} \partial_t u^j \right] + c^{i,j} u^j + \underline{b}^{i,j} \cdot \nabla u^j - \nabla \cdot (\mathbb{A}^{i,j} \nabla u^j) = f^i, \qquad \text{in } \Omega \times (0, T]$$

with $u^j = u^j(x)$ the unknown functions and coefficients $c, \underline{b}, \mathbb{A}$ that can depend on space, time, and other quantities v living in Ω:

$$\begin{aligned} c^{i,j} &= c^{i,j}(t, x, \{v_k\}, \{\nabla v_k\}), \\ \underline{b}^{i,j} &= \underline{b}^{i,j}(t, x, \{v_k\}, \{\nabla v_k\}), \\ \mathbb{A}^{i,j} &= \mathbb{A}^{i,j}(t, x, \{v_k\}, \{\nabla v_k\}), \\ f^i &= f^i(t, x, \{v_k\}, \{\nabla v_k\}), \end{aligned}$$

equipped with (initial- and) boundary conditions on $\partial\Omega$.

Second order PDEs

AMDiS is an FEM framework and FEM is based on the **weak formulation** of the equations, thus the basic formulation for problems is:
Find $u \in L_2(0, T; V^{(1)})$, s.t.

$$[\langle \partial_t u, \theta \rangle_\Omega] + \langle cu, \theta \rangle_\Omega + \langle \underline{b}u, \nabla\theta \rangle_\Omega + \langle \mathbb{A}\nabla u, \nabla\theta \rangle_\Omega = \langle f, \theta \rangle_\Omega, \quad \forall \theta \in V^{(0)}$$

with $\langle a, b \rangle_\Omega := \int_\Omega a \cdot b \, \mathrm{d}x$ and $V^{(0)}, V^{(1)}$ compatible spaces, e.g.

- $V^{(0)} = V^{(1)} = H^1(\Omega)$, or
- $V^{(1)} = V_G := \{u \in H^1(\Omega) : u|_{\partial\Omega} = G\}$, $V^{(0)} = V_0$.

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Basic ingredients for an AMDiS program

What data/information do you need to formulate your problem?

1. Description of your **domain** Ω + a **triangulation** $\mathcal{T}_h$ of the domain $\rightarrow$ `Mesh`
2. Function-space V, or its **basis-functions** respectively.
   - `P1` := Lagrange elements with polynomial degree $p = 1$:

$$V = \{v \in H^1(\Omega) : v|_T \in \mathbb{P}_p(T), \forall T \in \mathcal{T}_h(\Omega)\}$$

   - `P1+bubble` := `P1` + center bubble-function

   $V^{(0)}$ is called `RowFeSpace` and $V^{(1)}$ is called `ColumnFeSpace`.
3. **Solution** vector u, based on a numbering of all DOFs $\rightarrow$ `DOFVector`

$$u(x) = \sum_i u_i \phi_i(x),$$

with $\{\phi_i\}$ a basis of $V^{(1)}$.

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Basic ingredients for an AMDiS program

4. **Coefficient functions** $c, \underline{b}, \mathbb{A} \rightarrow$ `OperatorTerm`
   Categorized by degree of derivative:

$$c u - \nabla \cdot (\underline{b} u) - \nabla \cdot (\mathbb{A} \nabla u) = f$$

   - c, f: `ZeroOrderTerm` (ZOT)
   - $\underline{b}$: `FirstOrderTerm` (FOT)
     - ★ Derivative is on trial function: `GRD_PHI`: $\langle \underline{b} \cdot \nabla \phi, \psi \rangle$
     - ★ Derivative is on test function: `GRD_PSI`: $\langle \underline{b} \phi, \nabla \psi \rangle$
   - $\mathbb{A}$: `SecondOrderTerm` (SOT)

5. **Boundary terms**, i.e. coefficient functions α, g and properties of the pair of function spaces $V^{(0)}, V^{(1)}$.

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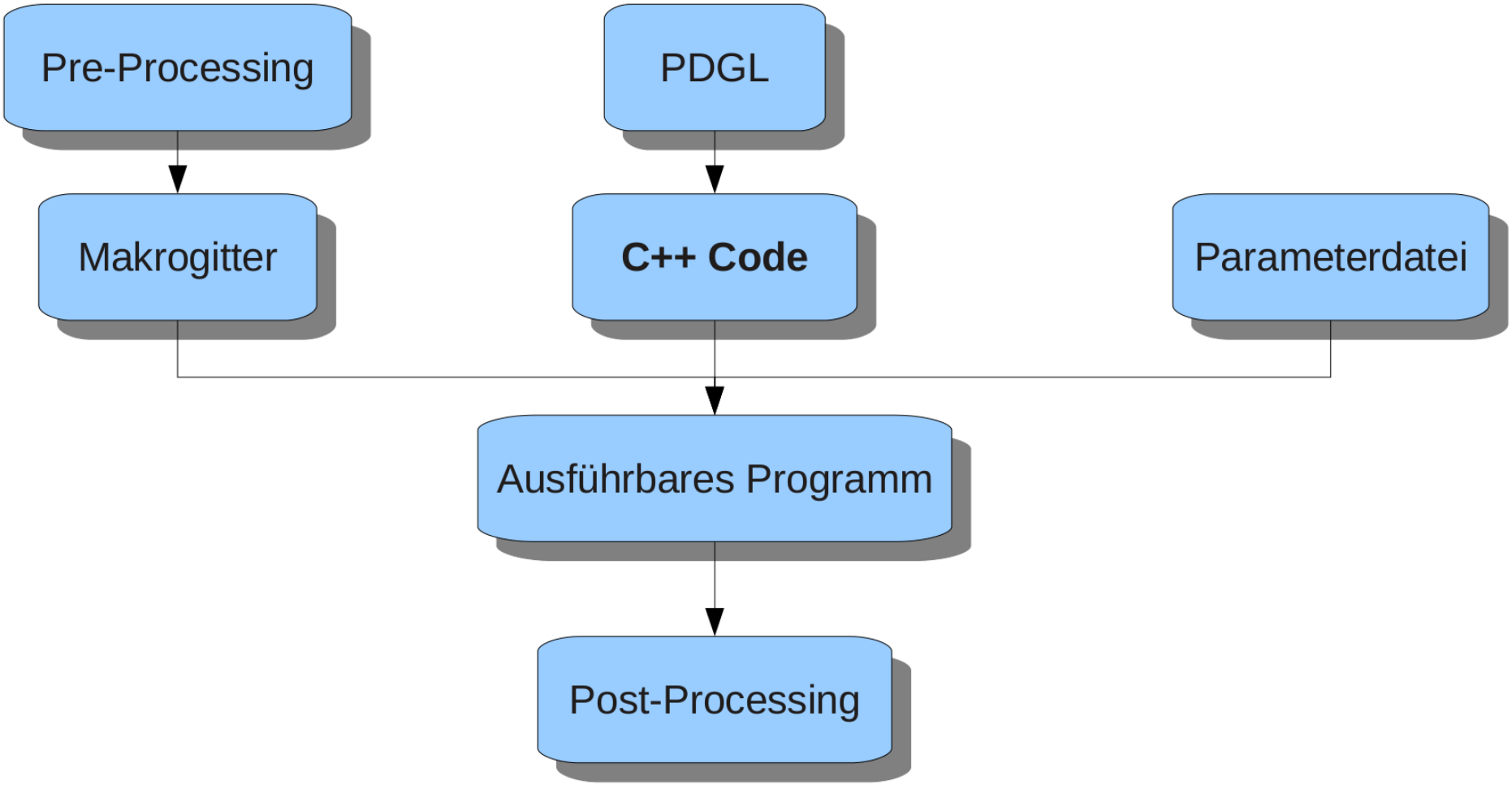
General procedure



Basic structure of AMDiS program (C++ code)

- Header file:

```
#include "AMDiS.h"
```

- AMDiS initialization:

```
using namespace AMDiS;
int main(int argc, char** argv) {
   AMDiS::init(argc, argv);
   ...
   AMDiS::finalize();
}
```

init(...):

- initializes *PMTL4*, *PETSc*, *MPI*, *Zoltan*
- parses command line arguments (run program with `--help` to see available options)
- reads a parameter-file

finalize():

- finalizes *PMTL4*, *PETSc*, *MPI*

Stationary equation

We start with the most simple equation: Poisson equation

$$-\Delta u = f(x) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,\ f(x) \equiv 1$$

$\Rightarrow$ coefficient functions are: $c = 0$, $\underline{b} = 0$, $\mathbb{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv 1$.

Weak formulation: Find $u \in V_0$, s.t.

$$\langle \nabla u, \nabla \theta \rangle_\Omega = \langle f, \theta \rangle_\Omega,\ \forall \theta \in V_0.$$

- Problem definition

```
ProblemStat prob("poisson");
prob.initialize(INIT_ALL); // init FeSpace, Mesh, ...
```

- Operator definition

```
Operator opL( prob.getFeSpace(), prob.getFeSpace() );
addSOT( opL, 1.0 ); // ⟨∇u, ∇θ⟩_Ω, 𝔸 = 1.0
Operator opF( prob.getFeSpace() );
addZOT( opF, 1.0 ); // ⟨f, θ⟩_Ω, f = 1.0
```

- Add Operators to Problem

```
prob.addMatrixOperator( opL, 0, 0 );
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```

- Add Operators to Problem

```
prob.addMatrixOperator( opL, 0, 0 );
prob.addVectorOperator( opF, 0 );
```

- Define boundary conditions

```
prob.addDirichletBC( nr, 0, 0, new Constant(0.0) );
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where `nr` is a boundary number and `Constant` a predefined functor, that returns always 0.

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Assemble and solve equation

In order to solve the PDE system: assemble matrix, solve linear system:

- Store Informations about solution process:

```
AdaptInfo adaptInfo("adapt");
```

- Assemble and solve system:

```
prob.assemble(&adaptInfo);
prob.solve(&adaptInfo);
```

- Write solution to file:

```
prob.writeFiles(&adaptInfo, true);
```

```
// create and init the scalar problem
ProblemStat prob("poisson");
prob.initialize(INIT_ALL);

// define operators
Operator opLaplace(prob.getFeSpace(), prob.getFeSpace());
  addSOT(opLaplace, 1.0); // <grad(u), grad(theta)>
Operator opF(prob.getFeSpace());
  addZOT(opF, 1.0); // <f(x), theta>

// add operators to problem
prob.addMatrixOperator(opLaplace, 0, 0);
prob.addVectorOperator(opF, 0);

// add boundary conditions
BoundaryType nr = 1;
prob.addDirichletBC(nr, 0, 0, new Constant(0.0));

AdaptInfo adaptInfo("adapt");

// assemble and solve linear system
prob.assemble(&adaptInfo);
prob.solve(&adaptInfo);

prob.writeFiles(&adaptInfo, true);
```

Compile and run AMDiS programs

- CMake configuration:

```
project("workshop")
find_package(AMDIS REQUIRED)
if (AMDIS_FOUND)
   include($AMDIS_USE_FILE)
endif (AMDIS_FOUND)

add_executable("poisson1" src/poisson1.cc)
target_link_libraries("poisson1" $AMDIS_LIBRARIES)
```

  - CMake requires variable `AMDIS_DIR` to point to directory, that contains the file `AMDISConfig.cmake`
  - See `http://goo.gl/kVe0Z2` for documentation of cmake commands.

- Run an AMDiS program by

```
./executable INIT-FILE
```

The Parameter file

A parameter file (init-file) controls various parameters of the solution process, defines the FiniteElemSpace and sets the mesh and is mandatory.

- Init-files have the suffix `.dat.Xd`, where `X` is in $\{1, 2, 3\}$.
- Parameter definition:

```
parameter_name:  parameter_value  % a comment
```

  - The ':' sign is the delimiter between parameter name and value
  - The '%' sign indicates a starting comment

- Read a parameter from init-file:

```
value_type value = value0;
Parameters::get("parameter_name", value);
```

  - value0 is default, if parameter not found
  - value_type can be number, vector, string
  - arithmetic expressions are parsed, interpreted and cast to value_type

The Initfile manual: `https://goo.gl/Dhm9Bx`

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The Parameter file

```
dimension of world:  2

mesh->macro file name:  ./macro/macro1.2d
mesh->global refinements:  10

poisson->mesh:  mesh
poisson->dim:  2
poisson->components:  1
poisson->feSpace[0]:  P1
poisson->name[0]:  u

poisson->solver:  cg
poisson->solver->max iteration:  1000
poisson->solver->tolerance:  1.e-8
poisson->solver->left precon:  diag

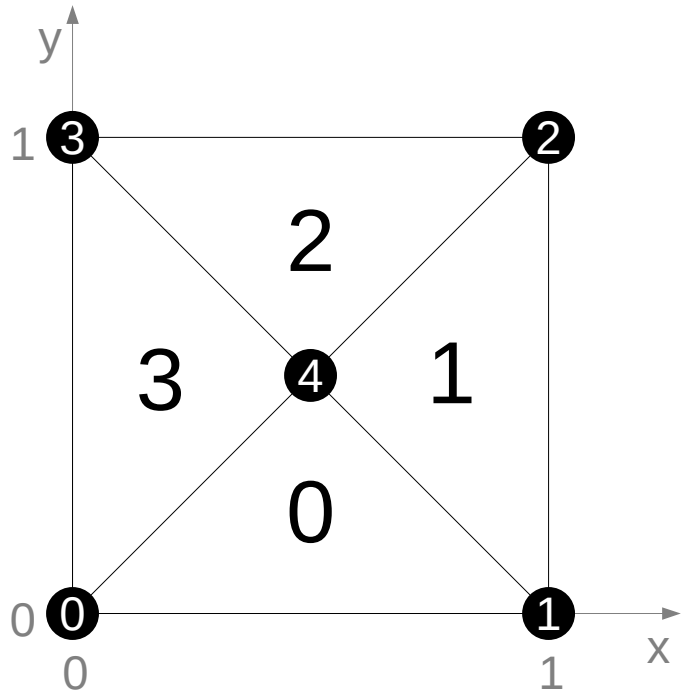
poisson->output->filename:  ./output/poisson1.2d
poisson->output->ParaView format:  1
```

The Macro-Mesh

In AMDiS the geometry and coarse triangulation is called `Macro-Mesh`

Header

```
DIM: 2
DIM_OF_WORLD: 2
number of elements:  4
number of vertices:  5
```



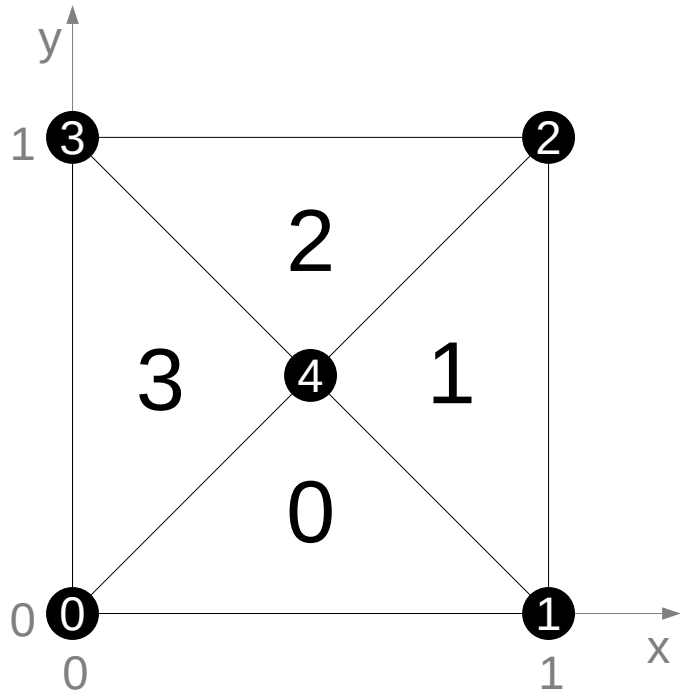
ASCII-File, that describes dimension of the elements (`DIM`), dimension of the world (`DIM_OF_WORLD`), the vertex coordinates, element connectivity, element boundary numbers and a neighborship relation.

The Macro-Mesh

In AMDiS the geometry and coarse triangulation is called `Macro-Mesh`

Mesh description

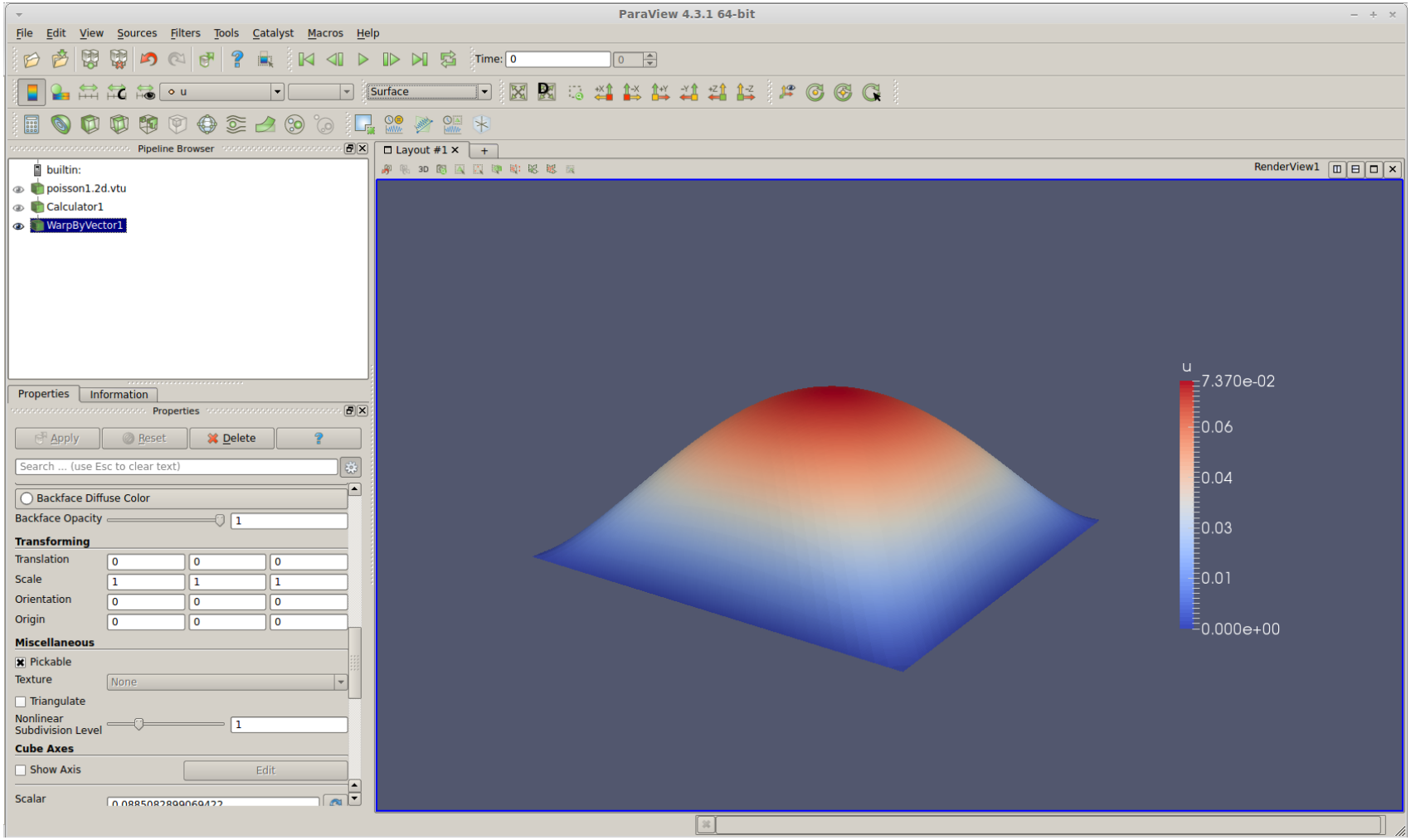
```
element vertices:
0 1 4
1 2 4...
element boundaries:
0 0 1
0 0 1...
vertex coordinates:
0.0 0.0
1.0 0.0...
```



ASCII-File, that describes dimension of the elements (`DIM`), dimension of the world (`DIM_OF_WORLD`), the vertex coordinates, element connectivity, element boundary numbers and a neighborship relation.

Visualization

Plot Solution (.vtu file), using ParaView



Exercise1: Poisson equation

We want to solve the Poisson equation

$$-\Delta u = f(x) \quad \text{in } \Omega,$$

and (in)homogeneous Dirichlet boundary conditions on $\partial\Omega$.

1. Setup the CMake configuration:

```
> mkdir build
> cd build
> cmake ..
```

2. Compile and run the example code `exercise1.cc`

```
> make exercise1
> cd ..
> build/exercise1 init/exercise1.dat.2d
```

3. Visualize the file `exercise1.2d.vtu` created in directory `output` with the program `ParaView`.
4. What is the concrete expression for f and dirichlet value g?

Advanced Exercise1: modify the Poisson equation

1. Change the expression for f and g.
2. Modify the parameters in the init-file `exercise1.dat.2d`, e.g.
   - Change the nr. of global refinements
   - Set a different Finite-Element space
   - Use another linear solver

   See also AMDiS-Wiki/Initfile: `https://goo.gl/Dhm9Bx`
   What parameters can be changed, and where do you get an error, or a crash of the program?
3. Solve the same equation in 1d or 3d, by changing the mesh, dimension of world and problem dimension in the init-file.
4. Modify the macro-mesh, e.g.
   - change the boundary nr for the right edge
   - change the coordinates of the vertices

Some hints

1. Used functions/classes:

```
addSOT(Operator, EXPRESSION);
addZOT(Operator, EXPRESSION);
// Functor class with return-type double and
// argument-type WorldVector<double>:
AbstractFunction<double, WorldVector<double>>;
```

2. Parameters to modify:

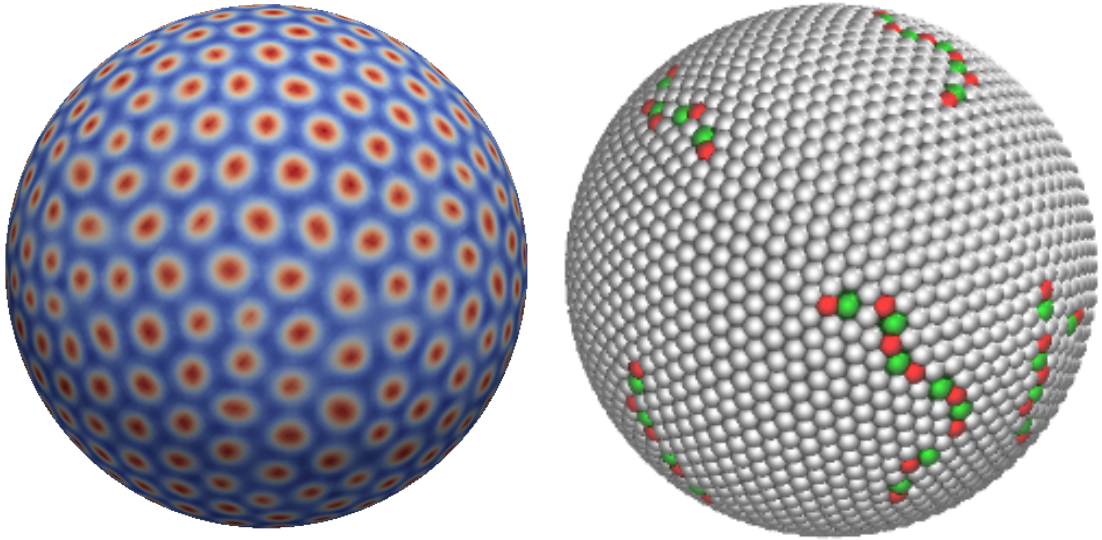
```
mesh->global refinements:  INTEGER
poisson->feSpace[0]:  {P1, P2, P3, P4, P1+bubble}
poisson->solver:  {cg, gmres, direct, ...}
```

3. Sources for help:
   - AMDiS-Wiki: `https://goo.gl/Jy3u1u`
   - Init-file manual: `https://goo.gl/Dhm9Bx`
   - Expressions manual: `https://goo.gl/JK8EUI`
   - List of init-file parameters: `https://goo.gl/LWJzq9`

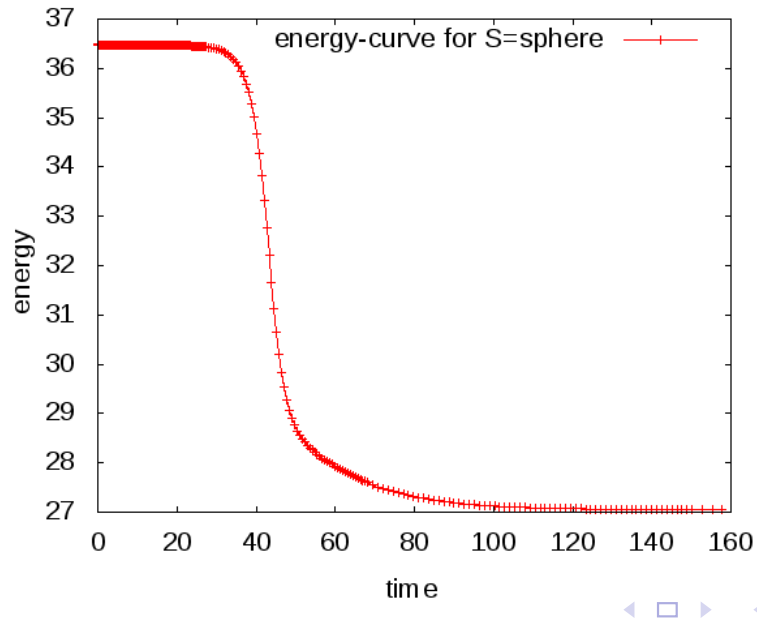
Session 2

Processing the data

Extract maxima of solution



Calculate integrals



Working with the discrete solution

Finite-Element function u_h expressed as linear combination of basis-functions ϕ_i:

$$u_h(x) = \sum_i u_i \phi_i(x),$$

with $u_i :=$ Degrees of Freedom, stored in `DOFVector<value_type> U`:

- Evaluate at DOF index: `U[idx]`
- Iterate over all DOFs:

```
DOFIterator<value_type> it(&U [, USED_DOFS]);
for (it.reset(); !it.end(); ++it)
   *it = value; // idx = it.getDOFIndex()
```

- Evaluate DOFVector at coordinate x:

```
WorldVector<double> x;
x[0] = 0.2; x[1] = 0.3;
value_type value = U(x);
```

Working with the discrete solution

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```

- Evaluate DOFVector at coordinate x:

```
WorldVector<double> x;
x[0] = 0.2; x[1] = 0.3;
value_type value = U(x);
```

Fill a DOFVector

```
U << EXPRESSION;
```

where `EXPRESSION` can contain e.g.

- The coordinates: `X()`, `X(i)`
- Scalar values: `1.0`, `constant(1.0)`
- Matrix and vector expressions: `two_norm(...)`, `vec * vec`
- Other DOFVectors: `valueOf(V)`, `gradientOf(V)`

Expressions manual: `https://goo.gl/JK8EUI`

Example: Let C and U be of type DOFVector<double>:

$$U := \frac{1}{2}\left(C^2(1-C)^2 + \frac{1}{\epsilon}\|\nabla C\|^2\right)$$

Assign expression on rhs to DOFVector U:

```
U << 0.5*( pow<2>(valueOf(C) * (1.0 - valueOf(C))
        + (1.0/eps) * unary_dot(gradientOf(C)) );
```

Fill a DOFVector

```
U << EXPRESSION;
```

where `EXPRESSION` can contain e.g.

- The coordinates: `X()`, `X(i)`
- Scalar values: `1.0`, `constant(1.0)`
- Matrix and vector expressions: `two_norm(...)`, `vec * vec`
- Other DOFVectors: `valueOf(V)`, `gradientOf(V)`

Expressions manual: `https://goo.gl/JK8EUI`

Example: Let C and U be of type DOFVector<double>:

$$U(x) := \frac{1}{2}\left(C(x)^2\,(1 - C(x))^2 + \frac{1}{\epsilon}\|\nabla C(x)\|^2\right)$$

Assign expression on rhs to DOFVector U:

```
U << 0.5*( pow<2>(valueOf(C) * (1.0 - valueOf(C))
        + (1.0/eps) * unary_dot(gradientOf(C)) );
```

Working with the discrete solution

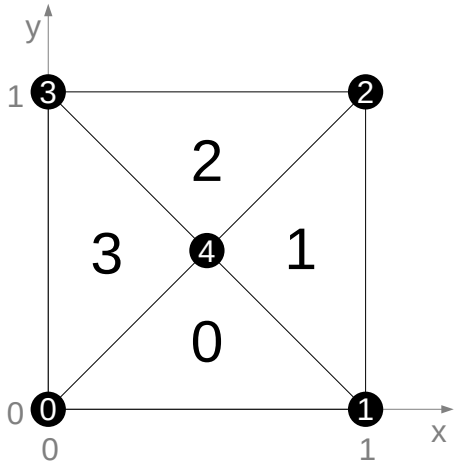
- Reduce the DOFVector, i.e. calc integrals, norms:

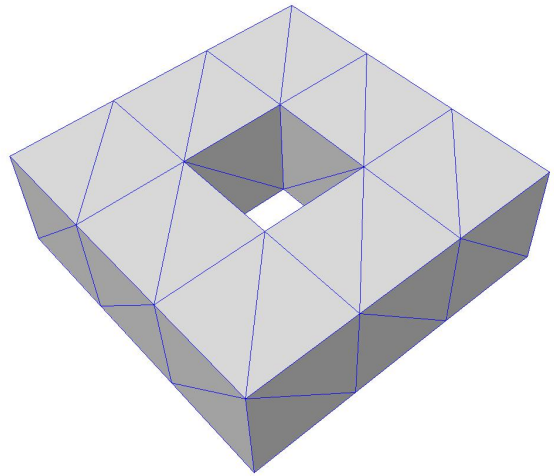
```
integrate( EXPRESSION [, Mesh*] )
```

```
integrate( valueOf(U) ); ⇒ ∫_Ω u dx
integrate( two_norm(gradientOf(U)) ); ⇒ ∫_Ω ‖∇u‖_2 dx
integrate(pow<2>(valueOf(U)) + unary_dot(gradientOf(U)));
⇒ ‖u‖²_{H¹(Ω)}
```

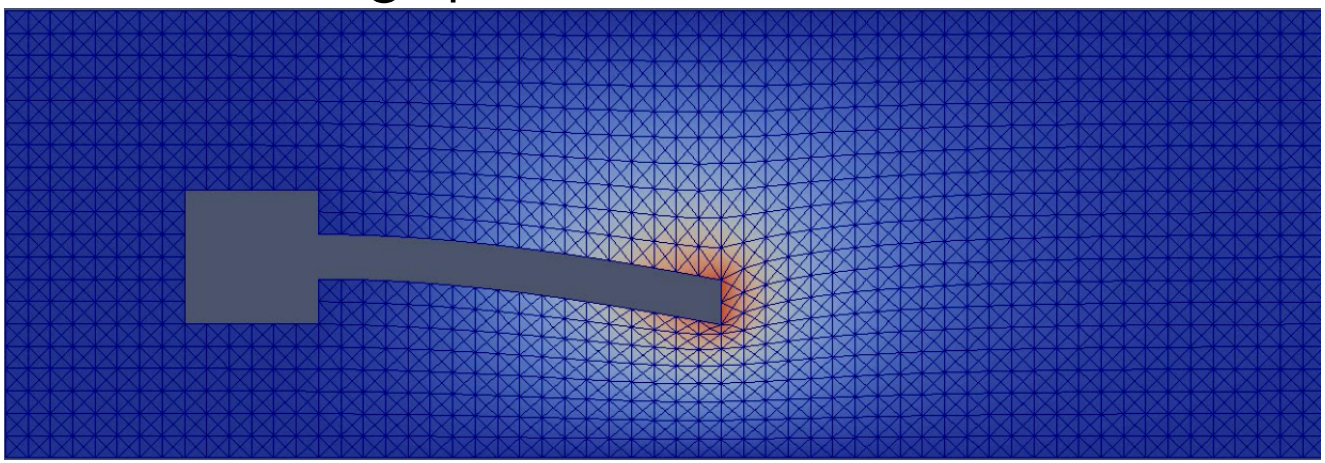
- Other reduction operations (over DOFs):
  - `max( EXPRESSION )` $\Rightarrow \max\{expr(x_i) : x_i \in \Omega\}$
  - `min( EXPRESSION )` $\Rightarrow \min\{expr(x_i) : x_i \in \Omega\}$
  - `abs_max( EXPRESSION )` $\Rightarrow \max\{|expr(x_i)| : x_i \in \Omega\}$
  - `abs_min( EXPRESSION )` $\Rightarrow \min\{|expr(x_i)| : x_i \in \Omega\}$

Working with the mesh





Change position of mesh vertices:



Working with the mesh

A Mesh holds all information about a triangulation. Get the mesh from `ProblemStat`:

```
Mesh& mesh = *prob.getMesh();
```

- Number of vertices: `mesh.getNumberOfVertices()`
- Number of triangles (elements): `mesh.getNumberOfElements()`
- World coordinates are stored in `WorldVector<double>`.
- Get the coordinates of all DOFs:

```
DOFVector<WorldVector<double> >
    Coords(prob.getFeSpace(), "coords");
mesh.getDofCoords(Coords);
```

- Change coordinates of the mesh:

```
ParametricSimple parametric(Coords);
mesh.setParametric(&parametric);
```

Working with the mesh:

- (Advanced) Traverse the mesh element-wise:

```
Flag traverseFlag = Mesh::CALL_LEAF_EL | (FILL_FLAGS);
TraverseStack stack;
ElInfo *elInfo = stack.traverseFirst(&mesh, -1, traverseFlag);
while (elInfo) {
  // do something with elInfo
  elInfo = stack.traverseNext(elInfo);
}
```

- (Advanced) Global Refinement of the mesh: Use `RefinementManager`

```
RefinementManager* refManager = prob.getRefinementManager();

Flag f = refManager->globalRefine(&mesh, 5);
if (f == MESH_REFINED) {
  MSG("Mesh globally refined by 5 levels!\n");
}
```

Example: Print solution with coordinates

Writing out the solution to the screen:

```
DOFVector<WorldVector<double>> C(U.getFeSpace(), "c");
mesh.getDofCoords(C);

DOFIterator<double> it(U);
DOFIterator<WorldVector<double>> c_it(C);
for (it.reset(),c_it.reset();  !it.end();  ++it,++c_it)
  std::cout << "U(" << *c_it << ") = " << *it << "\n";
```

This will print out:

```
U(0 0) = 0
U(1 0) = 0
U(1 1) = 0
U(0 1) = 0
U(0.5 0.5) = 0.0833333
```

Input/Output

Write DOFVectors to file, for visualization, serialization, ...

- File writer with automatic file-type detection:

```
io::writeFile(DOFVECTOR, FILENAME);
```

  Detection by filename extension: ".vtu", ".arh", ".dat", ".2d", ".gnu",...
- Use **ParaView** for visualization
- Tool **MeshConv** can convert beween mesh formats
- File reader with automatic file-type detection:

```
io::readFile(FILENAME, DOFVECTOR);
```

- Use ARH format to exchange data (can be converted to VTU by MeshConv.)

Exercise2: Poisson equation

We want to solve the Poisson equation

$$-\Delta u = f(x) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g$$

in a rectangular domain Ω, with

$$f(x) = 40(1 - 10\|x\|^2)e^{-10.0\|x\|^2}$$
$$g(x) := e^{-10.0\|x\|^2}$$

and with exact solution $u^* = g$.

1. Assemble and solve the equation.
2. Interpolate error $ERR := |u_h - g|$ to DOFVector and write it to a file.
3. Calculate the error-norms $\|u_h - g\|_{L_2(\Omega)}$ and $\|u_h - g\|_{H_1(\Omega)}$
4. Evaluate the error at the grid-point $x = (0.5, 0.5)$.
5. Refine the mesh globally and compare error-norms to old error-norms.

Advanced Exercise2: Mesh adaption

The datastructure `AdaptInfo` provides parameters for tolerance and nr. of iterations for an adaption process:

```
AdaptInfo adaptInfo("adapt");
adaptInfo.getMaxSpaceIteration(); // => adapt->max iteration
adaptInfo.getSpaceTolerance(0);   // => adapt[0]->tolerance
```

1. Use AdaptInfo to write an adaption loop that refines the mesh globally to reduce the error until a tolerance is reached.
2. Write the error DOFVector in every adaption iteration to a file.
3. Calculate the exponent in the error estimate $\|u_h - I_h u^*\|_{\#} \leq Ch^k$
4. (optional) Transform the mesh, by $x \mapsto 2x$ and solve again.

Some hints

1. Used functions/classes:

```
DOFVector << EXPRESSION;
integrate( EXPRESSION );

// EXPRESSION: {valueOf(U), gradientOf(U), X(), X(i), +,-,*,/,
//              unary_dot(EXPR.), pow<2>(EXPR.), absolute(EXPR.)}

RefinementManager* refManager = prob.getRefinementManager();
refManager->globalRefine(prob.getMesh(), NR_OF_REFINEMENTS);

adaptInfo.getSpaceTolerance(0);
adaptInfo.getMaxSpaceIteration();

io::writeFile( DOFVECTOR, FILENAME );
```

2. Parameters to modify:

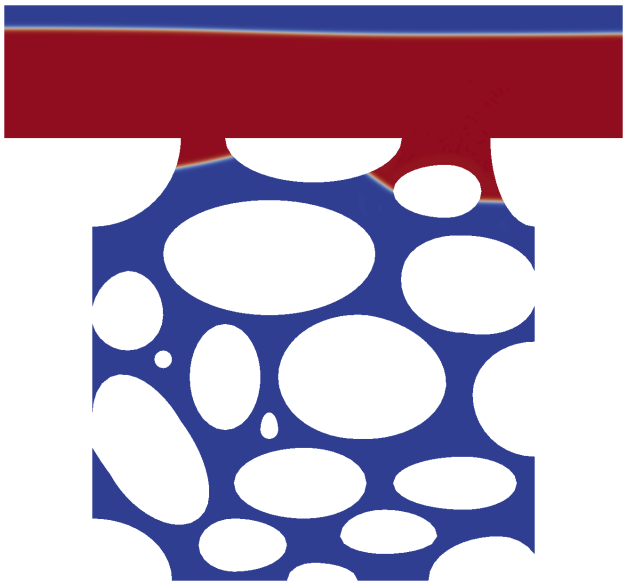
```
adapt[0]->tolerance:  DOUBLE
adapt->max iteration:  INTEGER
```

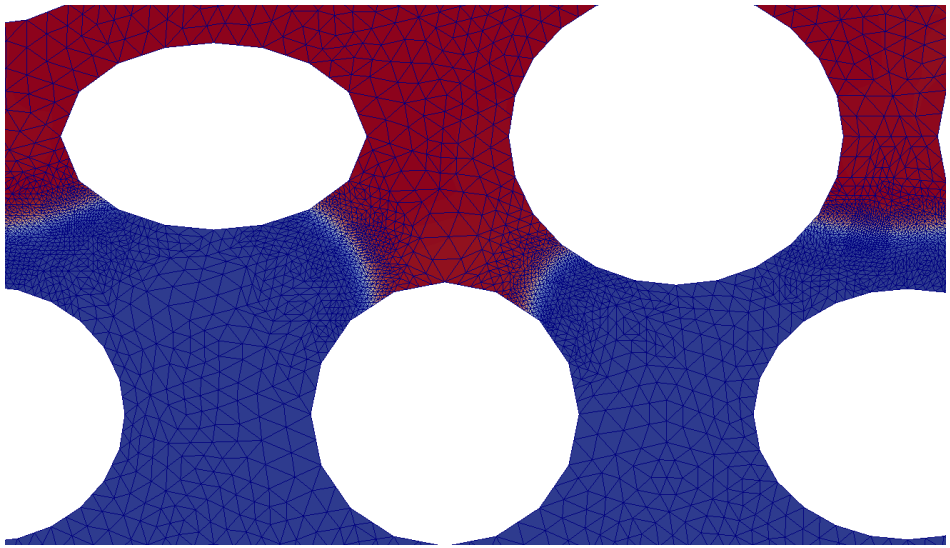
Session 3

Mesh adaptivity

Motivation:

- Two-phase flow problem
- Steep transition between two phases

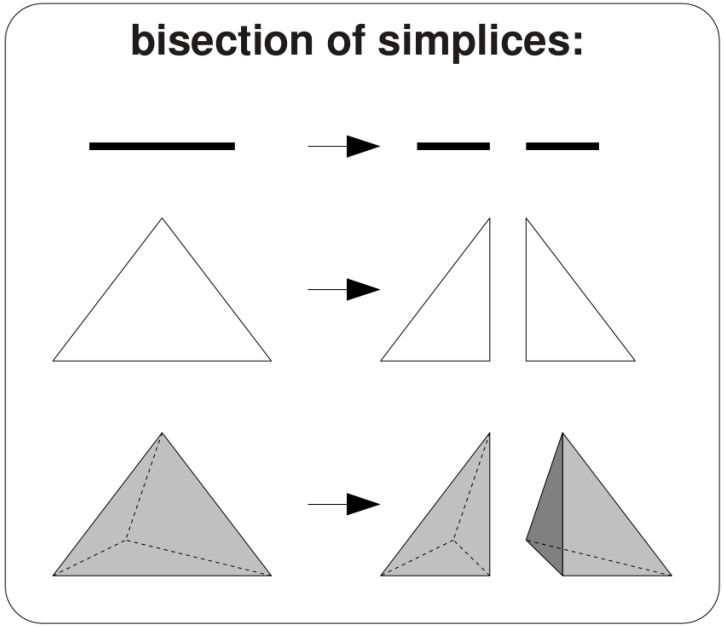


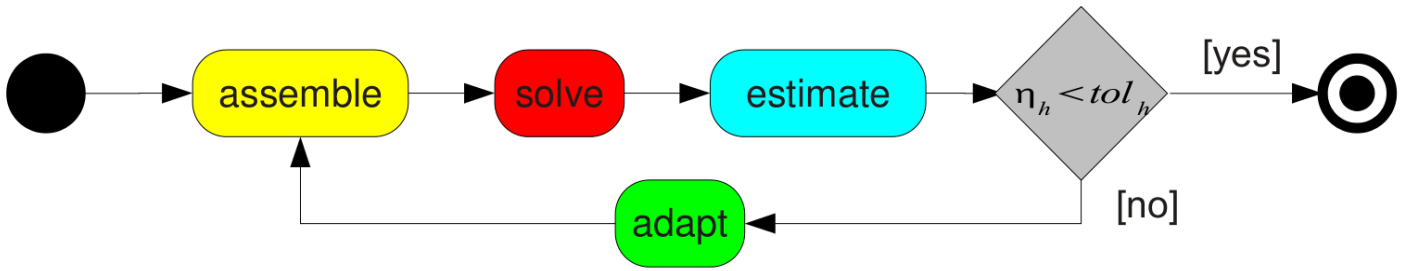


Adaptivity and error estimators

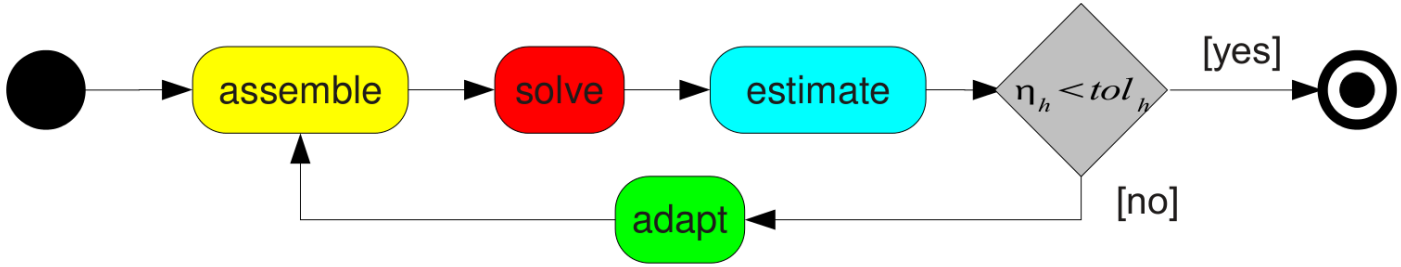
Adative solution strategy:

- Refine elements to reduce global error
- Local refinement instead of global refinement
- Repeated solution with adapted mesh:





Adaptivity and error estimators

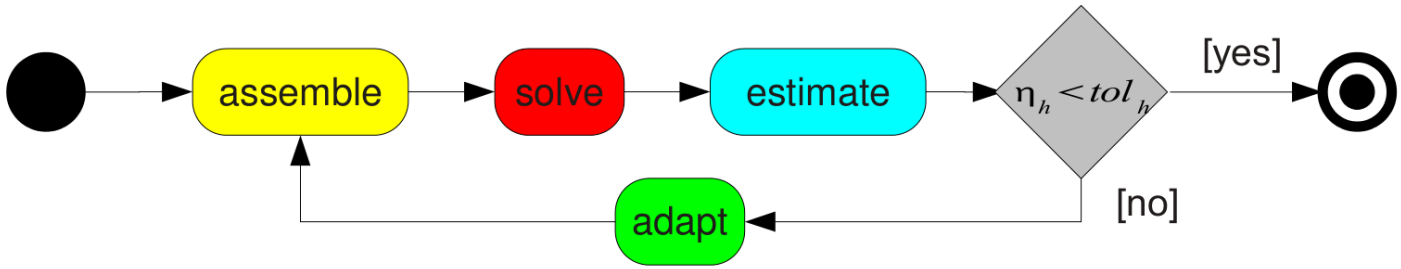



```
prob.assemble(&adaptInfo);
prob.solve(&adaptInfo);
prob.estimate(&adaptInfo); // local error indicator

Flag adapted;
for (int i = 0; i < MAX_ITER; ++i)
{
   if (adaptInfo.getEstSum(0) < TOL)
       break;
   Flag markFlag = prob.markElements(&adaptInfo); // mesh adaption
   adapted = prob.refineMesh(&adaptInfo);
   adapted|= prob.coarsenMesh(&adaptInfo);

   prob.buildAfterCoarsen(&adaptInfo, markFlag); // assemble
   prob.solve(&adaptInfo);
   prob.estimate(&adaptInfo);
}
```

Adaptivity and error estimators




```
prob.assemble(&adaptInfo);
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prob.estimate(&adaptInfo); // local error indicator

Flag adapted;
for (int i = 0; i < MAX_ITER; ++i)
{
   if (adaptInfo.getEstSum(0) < TOL)
       break;
   Flag markFlag = prob.markElements(&adaptInfo); // mesh adaption
   adapted = prob.refineMesh(&adaptInfo);
   adapted|= prob.coarsenMesh(&adaptInfo);

   prob.buildAfterCoarsen(&adaptInfo, markFlag); // assemble
   prob.solve(&adaptInfo);
   prob.estimate(&adaptInfo);
}
```
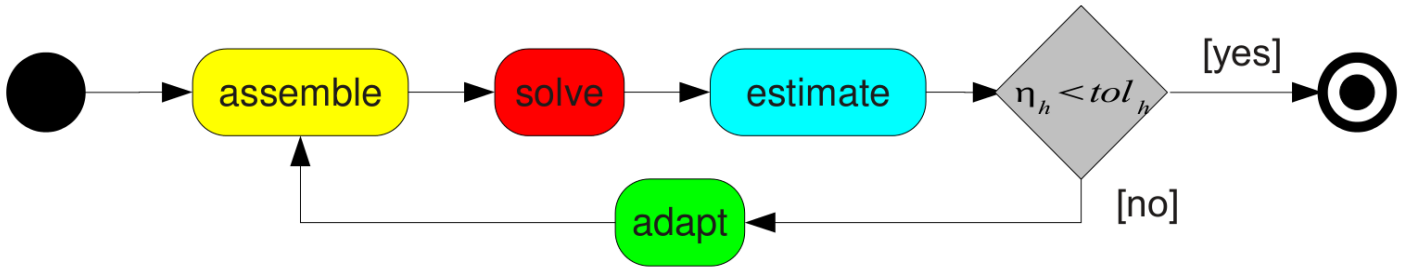

Better: AdaptStationary

Adaptivity and error estimators



Encapsulation into class:

```
AdaptStationary adaptStat("adapt", prob, adaptInfo);
adaptStat.adapt(); // start adaptive solution process
```

Adaptivity and error estimators

Solution strategy: assemble $\rightarrow$ solve $\rightarrow$ estimate $\rightarrow$ mark $\rightarrow$ refine/coarsen

- Estimator type can be set in init-file
- e.g. Residual-Estimator: Element error-indicator η_T:

$$\eta_T := C_0 h_T^2 \| f + \nabla \cdot \mathbb{A} \nabla u_h \|_{L_2(T)} + C_1 \left(\sum_{e \subset T} h_e \left[[\mathbb{A} \nabla u_h \cdot \underline{\nu_e}] \right]^2 \right)^{1/2}$$

- e.g. Recovery Estimator:

$$\eta_T := \| \nabla u_h^* - \nabla u_h \|_{L_2(T)}$$

- Various **marking strategies**: global refinement, maximum strategy, equidistribution strategy, guaranteed error reduction strategy

Adaptivity and error estimators

Solution strategy: assemble $\rightarrow$ solve $\rightarrow$ estimate $\rightarrow$ mark $\rightarrow$ refine/coarsen

- Estimator type can be set in init-file
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- e.g. Recovery Estimator:

$$\eta_T := \|\nabla u_h^* - \nabla u_h\|_{L_2(T)}$$

- Various **marking strategies**: global refinement, maximum strategy, equidistribution strategy, guaranteed error reduction strategy

Adaptivity and error estimators

Solution strategy: assemble $\rightarrow$ solve $\rightarrow$ estimate $\rightarrow$ mark $\rightarrow$ refine/coarsen

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$$\eta_T := C_0 h_T^2 \|f + \nabla \cdot \mathbb{A} \nabla u_h\|_{L_2(T)} + C_1 \left(\sum_{e \subset T} h_e \left[\left[\mathbb{A} \nabla u_h \cdot \underline{\nu_e} \right] \right]^2 \right)^{1/2}$$

- e.g. Recovery Estimator:

$$\eta_T := \|\nabla u_h^* - \nabla u_h\|_{L_2(T)}$$

- Various **marking strategies**: global refinement, maximum strategy, equidistribution strategy, guaranteed error reduction strategy

Stationary equation with adaptivity

Source-code: see "poisson4.cc"

```
ProblemStat prob("poisson5");
prob.initialize(INIT_ALL);

Operator opL( prob.getFeSpace(), prob.getFeSpace() );
  addSOT( opL, A ); // e.g. A = 1.0
Operator opF( prob.getFeSpace() );
  addZOT( opF, f ); // e.g. f = X()*X() + 1

prob.addMatrixOperator( opL, 0, 0 );
prob.addVectorOperator( opF, 0 );

prob.addDirichletBC( nr, 0, 0, new G );

AdaptInfo adaptInfo("adapt");
AdaptStationary("adapt", prob, adaptInfo).adapt();

prob.writeFiles(&adaptInfo, true);
```

Stationary equation with adaptivity

Init-file: see "poisson4.dat.2d"

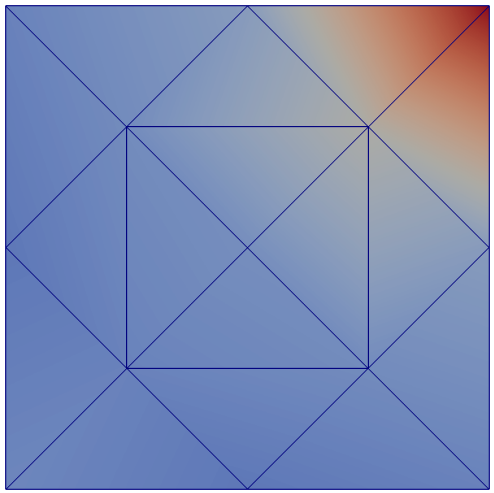
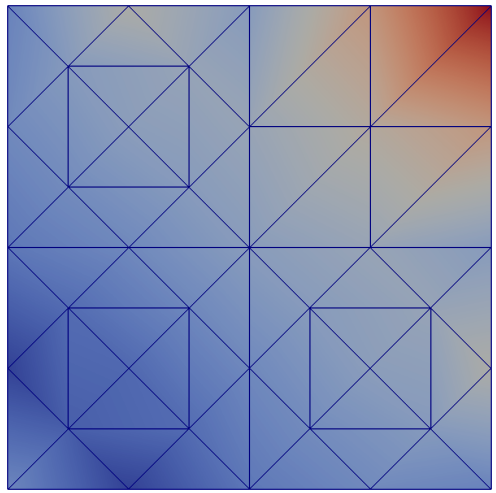
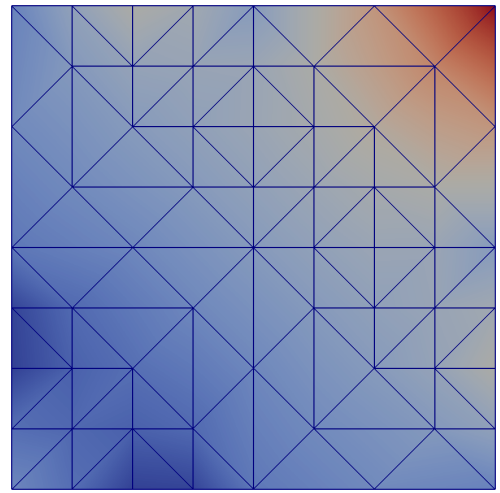
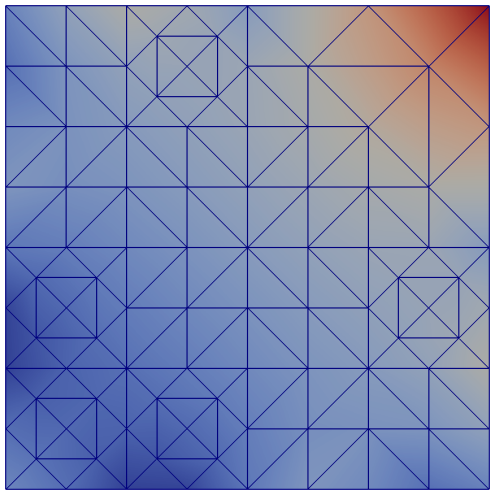
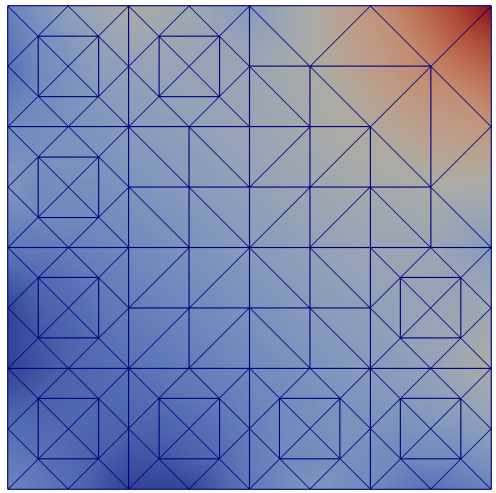
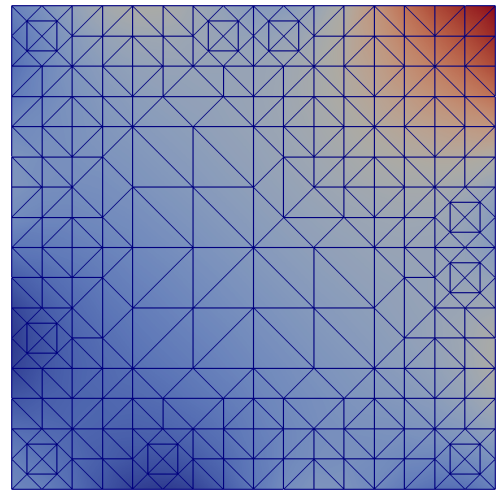
```
poisson->estimator[0]: residual % or recovery
poisson->estimator[0]->C0: 0.1 % weight of element residual
poisson->estimator[0]->C1: 0.1 % weight of jump residual

poisson->marker[0]->strategy: 2
% 1... global refinement
% 2... maximum strategy
% 3... equidistribution strategy
% 4... guaranteed error reduction strategy

adapt[0]->tolerance: 1e-4
adapt->max iteration: 5
```

Stationary equation with adaptivity

Visualization of the locally refined grid:

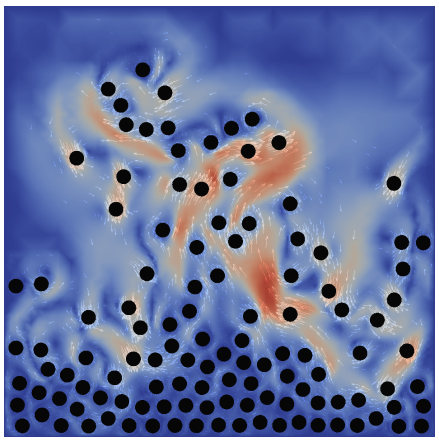







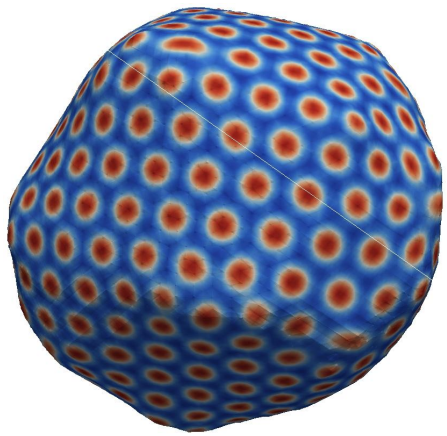
Lesson 3b

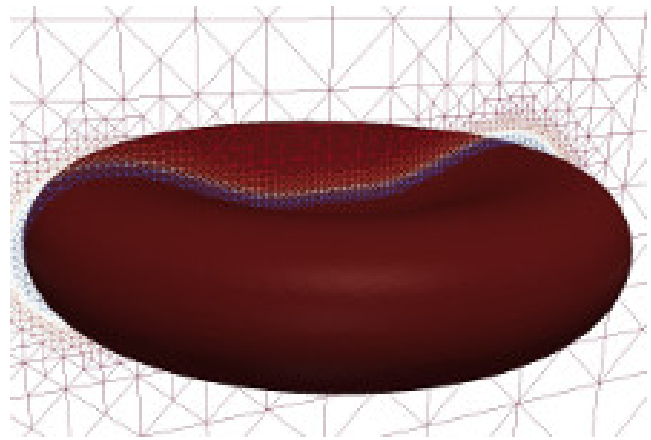
System of equations

Motivation:

- Navier-Stokes equations
- Phase-Field (Crystal) equation
- Willmore flow







Example: Biharmonic equation

We consider the stream-function formulation for the Stokes equation (2D):

$$-\nu\Delta^2\psi = f, \qquad \text{in } \Omega, \qquad \left(\mathbf{v} = -\nabla \times \mathbf{e}_z\psi\right)$$

with $\psi|_{\partial\Omega} = \Delta\psi|_{\partial\Omega} = 0$ and $f = (\nabla \times \mathbf{f})_z$.

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$$\begin{aligned} -\nu\Delta\phi &= f, \quad \text{in } \Omega \\ -\Delta\psi + \phi &= 0, \end{aligned}$$

or in weak form:

$$\begin{aligned} \langle \nu\nabla\phi, \nabla\theta\rangle - \langle \nu\partial_n\phi, \theta\rangle_{\partial\Omega} &= \langle f, \theta\rangle, \quad \forall\theta \in V \\ \langle \nabla\psi, \nabla\theta'\rangle - \langle \partial_n\psi, \theta'\rangle_{\partial\Omega} + \langle \phi, \theta'\rangle &= 0, \qquad \forall\theta' \in V' \end{aligned}$$

with solution components ($\psi \in V, \phi \in V'$).

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or in weak form:

$$\begin{aligned} \langle \nu\nabla\phi, \nabla\theta\rangle - \cancel{\langle \nu\partial_n\phi, \theta\rangle_{\partial\Omega}} &= \langle f, \theta\rangle, \quad \forall\theta \in V_0 \\ \langle \nabla\psi, \nabla\theta'\rangle - \cancel{\langle \partial_n\psi, \theta'\rangle_{\partial\Omega}} + \langle \phi, \theta'\rangle &= 0, \quad \forall\theta' \in V_0' \end{aligned}$$

with solution components ($\psi \in V_0, \phi \in V_0'$).

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or in weak form:

$$\begin{aligned} \overbrace{\langle \nu\nabla\phi, \nabla\theta \rangle}^{\text{SOT}} &= \overbrace{\langle f, \theta \rangle}^{\text{ZOT}}, && \forall\theta \in V_0 \\ \underbrace{\langle \nabla\psi, \nabla\theta' \rangle}_{\text{SOT}} + \underbrace{\langle \phi, \theta' \rangle}_{\text{ZOT}} &= 0, && \forall\theta' \in V_0' \end{aligned}$$

with solution components ($\psi \in V_0, \phi \in V_0'$).

Example: Biharmonic equation

We consider the stream-function formulation for the Stokes equation (2D):

$$-\nu\Delta^2\psi = f, \quad \text{in } \Omega, \qquad \left(\mathbf{v} = -\nabla \times \mathbf{e}_z\psi\right)$$

with $\psi|_{\partial\Omega} = \Delta\psi|_{\partial\Omega} = 0$ and $f = (\nabla \times \mathbf{f})_z$. Reformulated as system of second order equation, we obtain:

$$\begin{aligned} -\nu\Delta\phi &= f, \quad \text{in } \Omega \\ -\Delta\psi + \phi &= 0, \quad \text{with } \psi|_{\partial\Omega} = 0 \text{ and } \phi|_{\partial\Omega} = 0 \end{aligned}$$

or in weak form:

$$\begin{aligned} \overbrace{\langle \nu\nabla\phi, \nabla\theta \rangle}^{\text{SOT(0,1)}} &= \overbrace{\langle f, \theta \rangle}^{\text{ZOT(0)}}, \quad \forall\theta \in V_0 \\ \underbrace{\langle \nabla\psi, \nabla\theta' \rangle}_{\text{SOT(1,0)}} + \underbrace{\langle \phi, \theta' \rangle}_{\text{ZOT(1,1)}} &= 0, \quad \forall\theta' \in V_0' \end{aligned}$$

with solution components ($\psi \in V_0, \phi \in V_0'$).

The biharmonic equation

Source-code: see "biharmonic1.cc"

$$\overbrace{\langle \nu \nabla \phi, \nabla \theta \rangle}^{\texttt{opL\_phi}} = \overbrace{\langle f, \theta \rangle}^{\texttt{opF}},$$

$$\underbrace{\langle \nabla \psi, \nabla \theta' \rangle}_{\texttt{opL\_psi}} + \underbrace{\langle \phi, \theta' \rangle}_{\texttt{opM\_phi}} = 0$$

```
Operator opL_phi(prob.getFeSpace(0), prob.getFeSpace(1));
  addSOT(opL_phi, nu);
Operator opF(prob.getFeSpace(0));
  addZOT(opF, f);
Operator opL_psi(prob.getFeSpace(1), prob.getFeSpace(0));
  addSOT(opL_psi, 1.0);
Operator opM_phi(prob.getFeSpace(1), prob.getFeSpace(1));
  addZOT(opM_phi, 1.0);

// ===== add operators to problem =====
prob.addMatrixOperator(opL_phi, 0, 1);    // -nu*laplace(phi)
prob.addMatrixOperator(opL_psi, 1, 0);    // -laplace(psi)
prob.addMatrixOperator(opM_phi, 1, 1);    // phi
prob.addVectorOperator(opF,     0);       // f(x)
/// ...
```

The biharmonic equation

Source-code: see "biharmonic1.cc"

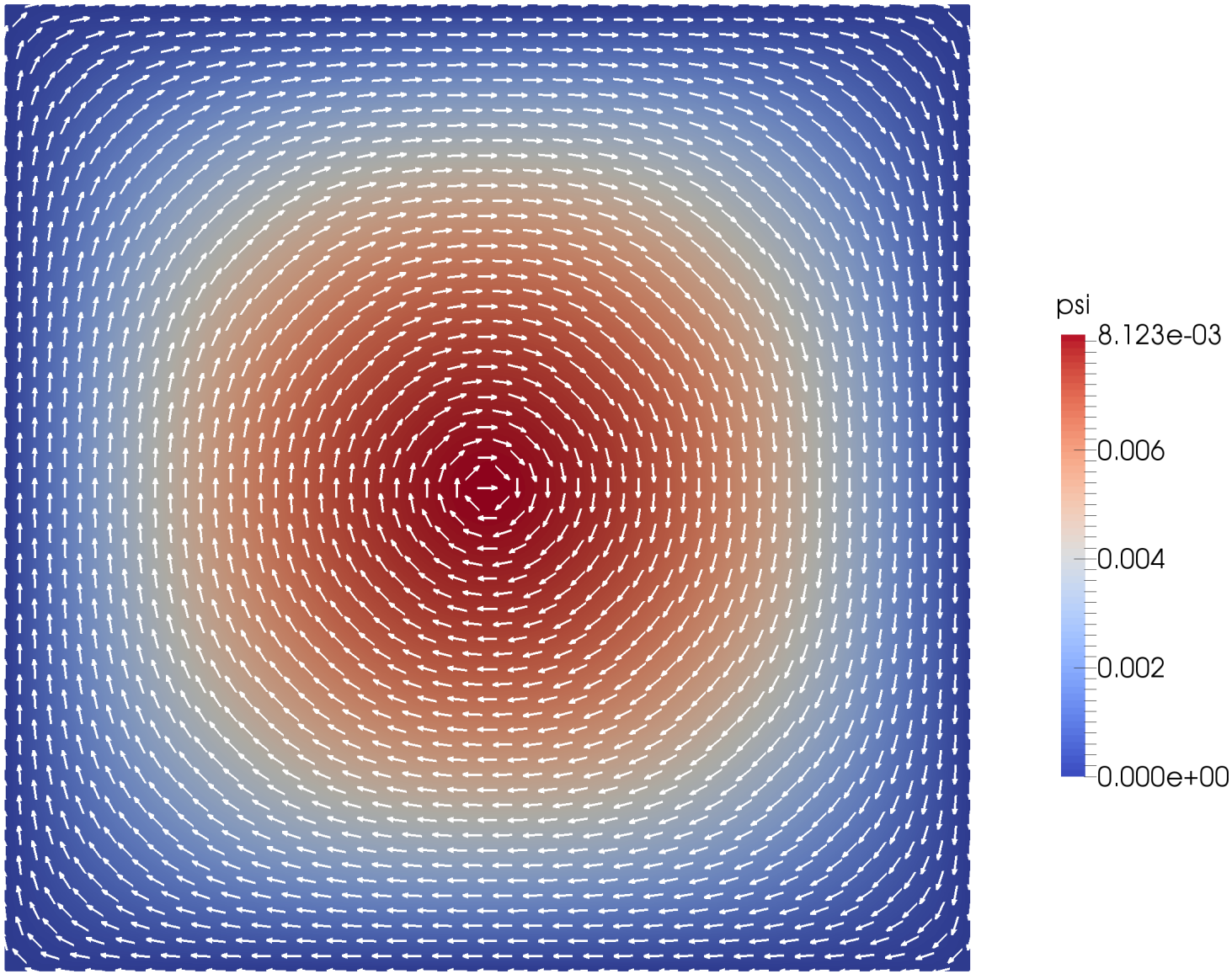
```
/// ...
BoundaryType nr = 1;
prob.addDirichletBC(nr, 1, 0, new Constant(0.0));
prob.addDirichletBC(nr, 0, 1, new Constant(0.0));

AdaptInfo adaptInfo("adapt");
AdaptStationary("adapt", prob, adaptInfo).adapt();

prob.writeFiles(&adaptInfo, true);
```

Init-file: see "biharmonic1.dat.2d"

```
biharmonic->mesh:        mesh
biharmonic->components: 2
biharmonic->feSpace[0]: P1
biharmonic->feSpace[1]: P1
biharmonic->dim:         2
biharmonic->name:        [psi,phi]
```

The biharmonic equation

Velocity field $\mathbf{v}$ and stream function ψ for $\mathbf{f} = (y, -x)^\top$



Exercise3: System of equations

Implement the 4th order PDE

$$u - \Delta(u - \epsilon\Delta u) = f(x) \quad \text{in } \Omega,$$
$$\partial_n u|_{\partial\Omega} = \partial_n(u - \epsilon\Delta u)|_{\partial\Omega} = 0$$

in a rectangular domain Ω, with $\epsilon < 1$ and $f(x) \in [-1, 1]$ (random values).

1. Formulate Splitting of equation in system of equations.
2. Assemble and solve the system.
3. Vary the value ϵ from 1 to 10^{-3}. Introduce a parameter and read it from init-file. Change the mesh resolution accordingly.

Advanced Exercise3: Error estimators

1. Modify exercise 1 to allow local mesh adaption by an error estimator.
2. Change the init-file correspondingly, i.e. add an entry for estimator and marker.
3. Choose a residual estimator/ recovery estimator and vary the estimator parameters
4. Vary the marking strategies and visualize the effect.

Some hints

1. Used functions/classes:

```
// functor that returns random values in mean +- amplitude/2
Random(mean, amplitude);

EXPRESSION: eval(&functor)
```

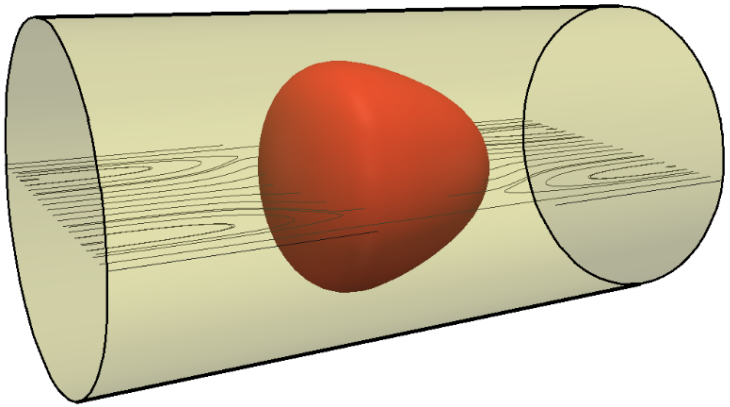
2. Parameters to modify:

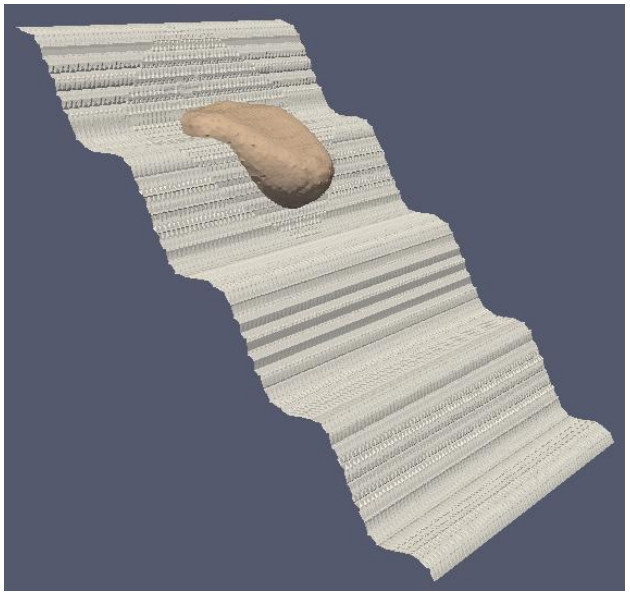
```
poisson->estimator[0]:  residual / recovery
poisson->marker[0]->strategy:  1/2/3/4
```

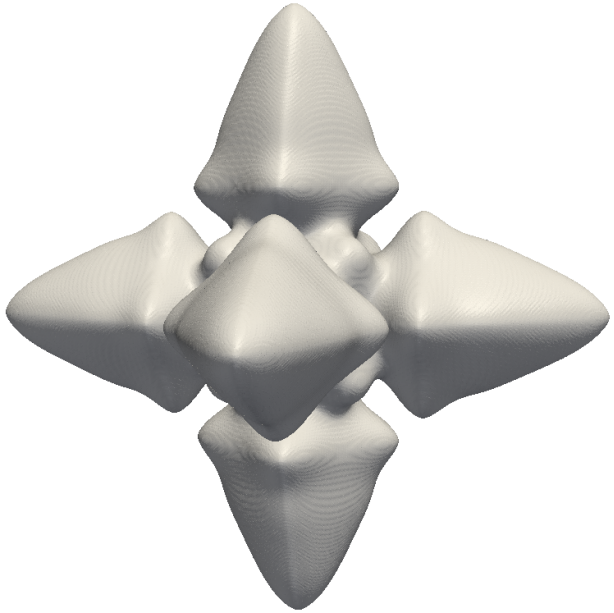
3. References:
   - ALBERT manual, Section 1.5.2 (marker strategies)

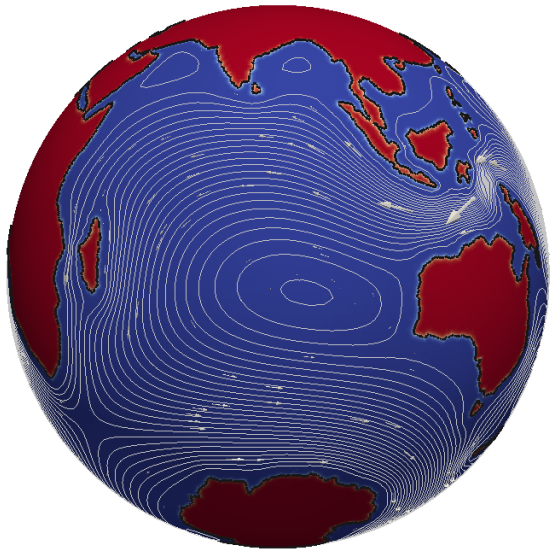
Session 4

Time-dependent problems









Time-dependent problems

Consider the instationary heat equation

$$\partial_t u - \alpha \Delta u = f, \qquad \text{in } \Omega \times (0, T],$$
$$\alpha \partial_n u|_{\partial\Omega} = g$$

with u: temperature, $\alpha > 0$: thermal diffusivity and f: heat source.

- Transform equation in a sequence of stationary equations: Let $0 = t_0 < t_1 < \ldots < t_N = T$ and $\tau_k = t_{k+1} - t_k$, $u_k \equiv u(t_k)$: For $k = 0, 1, \ldots, N-1$ find $u_{k+1} \in V$, s.t.
- Coefficients, like τ_k, α, f, g may change over time.

Time-dependent problems

Consider the instationary heat equation

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$$\frac{1}{\tau_k} u_{k+1} - \alpha \Delta u_{k+1} = \frac{1}{\tau_k} u_k + f(t_{k+1}), \quad \text{in } \Omega$$

- Coefficients, like τ_k, α, f, g may change over time.

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$$\langle \frac{1}{\tau_k} u_{k+1}, \theta \rangle + \langle \alpha \nabla u_{k+1}, \nabla \theta \rangle$$
$$= \langle \frac{1}{\tau_k} u_k + f(t_{k+1}), \theta \rangle + \langle g(t_{k+1}), \theta \rangle_{\partial\Omega}, \quad \forall \theta \in V$$

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$$\partial_t u - \alpha \Delta u = f, \quad \text{in } \Omega \times (0, T],$$
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$$\overbrace{\langle \frac{1}{\tau_k} u_{k+1}, \theta \rangle}^{\text{ZOT}} + \overbrace{\langle \alpha \nabla u_{k+1}, \nabla \theta \rangle}^{\text{SOT}}$$
$$= \underbrace{\langle \frac{1}{\tau_k} u_k + f(t_{k+1}), \theta \rangle}_{\text{ZOT}} + \underbrace{\langle g(t_{k+1}), \theta \rangle_{\partial\Omega}}_{\text{ZOT}_{\partial\Omega}}, \quad \forall \theta \in V$$

- Coefficients, like τ_k, α, f, g may change over time.

Time-dependent problems

Formulate the stationary equation, that is solved in each timestep:

```
FiniteElemSpace const* V = prob.getFeSpace(0);
DOFVector<double>* u =     prob.getSolution(0);

Operator opL(V, V);
  addZOT(opL, 1.0/var(tau) );
  addSOT(opL, A ); // e.g. A = 1.0

Operator opF(V);
  addZOT(opF, valueOf(u)/var(tau) );
  addZOT(opF, f ); // e.g. f = X()*X() + 1

Operator opG(V);
  addZOT(opG, g ); // e.g. g = X(0)+X(1)

prob.addMatrixOperator( opL, 0, 0 );
prob.addVectorOperator( opF, 0 );

prob.addBoundaryVectorOperator( nr, opG, 0 );  // <g, theta>_B
```

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prob.addMatrixOperator( opL, 0, 0 );
prob.addVectorOperator( opF, 0 );

prob.addBoundaryVectorOperator( nr, opG, 0 );  // <g, theta>_B
```

Get timestep width `tau` from `AdaptInfo`:

```
double* tau = adaptInfo.getTimestepPtr();
```

Time-dependent problems

Sequence of timesteps:

```
// ===== set initial solution =====
u << constant(0.0); // u(x,t=0) := 0

while (adaptInfo.getTime() < adaptInfo.getEndTime()) {
  adaptInfo.setTime(adaptInfo.getTime()
                    + adaptInfo.getTimestep()); // t += tau

  // ===== assemble and solve linear system =====
  prob.assemble(&adaptInfo);
  prob.solve(&adaptInfo);

  // ===== write solution to file =======
  prob.writeFiles(&adaptInfo, false);
}
```

Time-dependent problems

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  prob.assemble(&adaptInfo);
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  // ===== write solution to file =======
  prob.writeFiles(&adaptInfo, false);
}
```

Better: `AdaptInstationary`

Time-dependent problems

Sequence of timesteps: encapsulation into class:

```
ProblemInstat instat("heat", prob);
instat.initialize(INIT_NOTHING); // init (vector of) old solutions

// start adaptive solution process
AdaptInstationary("adapt", prob, adaptInfo, instat, adaptInfo)
  .adapt();
```

Nonlinear problems

- Consider the nonlinear Poisson equation

$$L(u,x) := -\Delta u + u^4 - f(x) = 0, \quad \text{in } \Omega,$$
$$u|_{\partial\Omega} = 0$$

- Newton-Method: provide Jacobian $J_u L$

$$J_u L(u,x)[\delta u] = -\Delta \delta u + 4u^3 \delta u, \quad \delta u|_{\partial\Omega} = 0$$

- Then we can write the iterative process as

$$J_u L(u_k,x)[\delta u] = -L(u_k,x),$$
$$u_{k+1} = u_k + \delta u$$

for initial solution u_0. Inserting Jacobian J and objective fct. L:

$$-\Delta \delta u + 4u_k^3 \delta u = \Delta u_k - u_k^4 + f(x)$$
$$\delta u|_{\partial\Omega} = -u_k|_{\partial\Omega}$$

Nonlinear problems

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$$-\Delta \delta u + 4u_k^3 \delta u = \Delta u_k - u_k^4 + f(x)$$
$$\delta u|_{\partial\Omega} = -u_k|_{\partial\Omega}$$

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$$u_{k+1} = u_k + \delta u$$

for initial solution u_0. Inserting Jacobian J and objective fct. L:

$$-\Delta \delta u + 4u_k^3 \delta u = \Delta u_k - u_k^4 + f(x)$$
$$\delta u|_{\partial\Omega} = -u_k|_{\partial\Omega}$$

Nonlinear problems

In weak form this can be written as

$$\langle \nabla \delta u, \nabla \theta \rangle + \langle 4u_k^3 \delta u, \theta \rangle = -\langle \nabla u_k, \nabla \theta \rangle + \langle -u_k^4 + f(x), \theta \rangle$$

Insering $\delta u = u_{k+1} - u_k$ we obtain:

Let $u_0 := u0$
FOR $k = 0, 1, 2, \ldots, M-1$
    solve:

$$\langle \nabla u_{k+1}, \nabla \theta \rangle + \langle 4u_k^3 u_{k+1}, \theta \rangle = \langle 3u_k^4 + f(x), \theta \rangle \quad \forall \theta \in V_0, \quad u_{k+1}|_{\partial\Omega} = 0$$

END-FOR
set $u := u_M$

Nonlinear problems: nonlin1.cc

Formulate the stationary equation, that is solved in each iteration:

$$\langle \nabla u_{k+1}, \nabla \theta \rangle + \langle 4u_k^3 u_{k+1}, \theta \rangle = \langle 3u_k^4 + f(x), \theta \rangle$$

```
FiniteElemSpace const* V = prob.getFeSpace(0);
DOFVector<double>& u_k =  *prob.getSolution(0);

Operator opL(V, V);
  addSOT(opL, 1.0);
  addZOT(opL, 4.0 * pow<3>(valueOf(u_k));
prob.addMatrixOperator(opL, 0, 0);

Operator opF(V);
  addZOT(opF, 3.0 * pow<4>(valueOf(u_k)) + f );
prob.addVectorOperator(opF, 0);

prob.addDirichletBC(nr, 0, 0, new Constant(0.0));
...
```

Nonlinear problems: nonlin1.cc

Formulate the stationary equation, that is solved in each iteration:

```
...
// ===== start adaption loop =====
AdaptInfo adaptInfo("adapt");
AdaptStationary adapt("adapt", prob, adaptInfo);

u_k << INITIAL_SOLUTION;
for (int k = 0; k < MAX_ITER; k++) {
  adaptInfo.reset();
  adapt.adapt();
}
```

Nonlinear problems: nonlin1.cc

Formulate the stationary equation, that is solved in each iteration:

```
...
// ===== start adaption loop =====
AdaptInfo adaptInfo("adapt");
AdaptStationary adapt("adapt", prob, adaptInfo);

u_k << INITIAL_SOLUTION;
for (int k = 0; k < MAX_ITER; k++) {
  adaptInfo.reset();
  adapt.adapt();
}
```

Better: `ProblemNonlin`

Nonlinear problems: nonlin2.cc

- Implement Newton-formulation directly:

$$J_u L(u_k, x)[\delta u] = -L(u_k, x)$$

- Iterative process implemented/ controlled by `ProblemNonlin`:

```
ProblemNonLin prob("nonlin");
prob.initialize(INIT_ALL);

FiniteElemSpace const* V = prob.getFeSpace(0);
DOFVector<double>& u_k =  *prob.getSolution(0);
...
```

Nonlinear problems: nonlin2.cc

$$-\Delta\delta u + 4u_k^3\delta u = \Delta u_k - u_k^4 + f(x)$$

```
...
// J_u L(u_k)[du]
Operator opJL(V, V);
  addSOT(opJL, 1.0);
  addZOT(opJL, 4.0 * pow<3>(valueOf(u_k)) );
prob.addMatrixOperator(opJL, 0, 0);

// -L(u_k)
Operator opL(V);
  addSOT(opL, -1.0);
opL.setUhOld(&u_k); // apply operator to DOFVector u_k
prob.addVectorOperator(opL, 0);

Operator opF(V);
  addZOT(opF, -pow<4>(valueOf(u_k)) + f );
prob.addVectorOperator(opF, 0);

prob.addDirichletBC(nr, 0, 0, new Constant(0.0));
...
```

Nonlinear problems: nonlin2.cc

$$-\Delta\delta u + 4u_k^3\delta u = \Delta u_k - u_k^4 + f(x)$$

```
...
// ===== start adaption loop =====
AdaptInfo adaptInfo("adapt");

u_k << INITIAL_SOLUTION;
prob.solve();
```

Initfile:

```
nonlin->nonlin solver:  newton % newtonArmijo
nonlin->nonlin solver->build cycle:  1
nonlin->nonlin solver->tolerance:  1e-3
nonlin->nonlin solver->max iteration:  100
...
```

(Experimental) Nonlinear problems: nonlin3.cc

$$L(u, x) := -\Delta u + u^4 - f(x)$$

Use symbolic differentiation in Jacobian:

```
// class acts as name for the unknown:
struct _U_;
...
auto coeff = pow<4>(valueOf<_U_>(u_k)) - f;

// J_u L(u_k)[du]
Operator opJL(V, V);
  addSOT(opJL, 1.0);
  addZOT(opJL, diff<_U_>(coeff) );
prob.addMatrixOperator(opJL, 0, 0);

// -L(u_k)
Operator opL(V); opL.setUhOld(&u_k);
  addSOT(opL, -1.0);
prob.addVectorOperator(opL, 0);

Operator opF(V);
  addZOT(opF, -coeff );
prob.addVectorOperator(opF, 0);
...
```

(Experimental) Nonlinear problems: nonlin3.cc

$$L(u, x) := -\Delta u + u^4 - f(x)$$

Use symbolic differentiation in Jacobian:

```
// class acts as name for the unknown:
struct _U_;
...
auto coeff = pow<4>(valueOf<_U_>(u_k)) - f;

std::cout << coeff << "\n";
std::cout << diff<_U_>(coeff) << "\n";
std::cout << diff<2, _U_>(coeff) << " --> "
          << simplify(diff<2, _U_>(coeff)) << "\n";
```

This prints:

```
(pow<4>(value(u)) - F(X))
([4] * pow<3>(value(u)))
([0] + ([12] * pow<2>(value(u)))) -->
   ([12] * pow<2>(value(u)))
```

Exercise4: Navier-Stokes equations

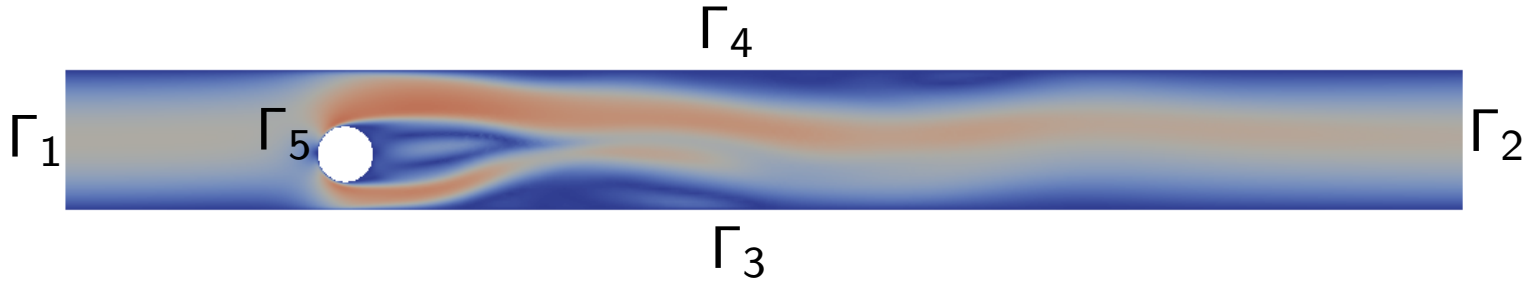
Implement the 2d Navier-Stokes equations

$$\partial_t u_i + (\mathbf{u} \cdot \nabla) u_i = \nu \Delta u_i - \partial_i p, \quad i = 1, 2$$
$$\nabla \cdot \mathbf{u} = 0$$

in a rectangular domain $\Omega = [0, 10] \times [0, 1]$ with boundary conditions

$$\mathbf{u}|_{\Gamma_3} = \mathbf{u}|_{\Gamma_4} = \mathbf{u}|_{\Gamma_5} = 0,$$
$$\mathbf{u}(x_0, x_1)|_{\Gamma_1} = \left(4u^\infty x_1(1 - x_1), 0\right)^\top$$
$$\nu \partial_n \mathbf{u} = \mathbf{n}p, \quad \text{at } \Gamma_2$$

with $\mathbf{n}$ the outer domain normal.



Exercise4: Navier-Stokes equations

1. Implement a simple semi-implicit backward Euler time-discretization
2. Linearize the non-linear term: $(\mathbf{u} \cdot \nabla) u_i \rightarrow (\mathbf{u}^{\text{old}} \cdot \nabla) u_i$
3. Take the macro-mesh `kanal.2d` from the macro directory.
4. Choose $\nu = 1, u^{\infty} = 1$ and solve for time $t \in [0, 10]$ with timestep $\tau = 0.1$.

Advanced Exercise4: Navier-Stokes equations

1. Vary the parameters ν, u^∞ and timestep τ, gridwidth h to see the karman vortex structures.
2. Change the mesh to `kanal2.2d` and modify the boundary condition accordingly.
3. Instead of a simple linearization, use a Newton-method to solve the non-linearity.
4. Implement a higher-order time-discretization. Remark: Use the `old_solution` from `ProblemInstat`

Some hints

1. Used functions/classes:

```
int dow = Global::getGeo(WORLD); // dimension of world

// AdaptInfo needs to know the nr. of components
AdaptInfo adaptInfo("adapt", prob.getNumComponents());

// partial derivative:
addFOT(OPERATOR, EXPR, i, GRD_PHI); // < EXPR d_i(u), theta >
addFOT(OPERATOR, EXPR, i, GRD_PSI); // < EXPR*u, d_i(theta) >
```

2. Parameters for Taylor-Hood element:

```
ns->components:  3
ns->feSpace[0]:  P2
ns->feSpace[1]:  P2
ns->feSpace[2]:  P1
ns->name:  [u0, u1, p]
```