Higher Order Moment Portfolio Optimization with L-Moments

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Abstract

This work presents a higher moment portfolio optimization model based on L-moments and the ordered weighted average (OWA) portfolio optimization model. In the first part, we are going to show how to model the higher L-moment portfolio problem as an utility function. In the second part, we are going to show how to create a convex risk measure combining higher L-moments. This risk measure allows to model more kind of portfolios objectives like risk constraints, minimization of risk, maximization of risk adjusted return ratio and risk parity that considers higher moments. Also, we propose four models to determinate the weights used to combine L-moments. Finally, we run some numerical examples using Python, CVXPY, Riskfolio-Lib, and MOSEK solver, in order to compare the properties of portfolios that consider higher L-moments with portfolios that not consider them.

Keywords: risk analysis, portfolio optimization, ordered weighted average, portfolio selection, l-moments, risk parity.

JEL Codes: C61, G11

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1 Introduction

Higher order moment portfolio optimization is a topic that has been studied in last three decades, mean variance skewness model was studied in Konno and ichi Suzuki (1995) and Konno and Yamamoto (2005), where they proposed a piecewise linear approximation of skewness using a mixed integer linear programming; and Lai (1991) proposed a formulation assuming that variance is one. The mean skewness kurtosis model was studied in Athayde and Flôres (2003). In the case of four moments portfolio, there are several algorithms to solve this problem due to complexity that arise from non-convexity of third and fourth moments like Jurczenko et al. (2015), Niu et al. (2019) and Zhou and Palomar (2020).

An alternative to traditional central moments, Hosking (1990) proposed linear moments or L-moments. These statistics are linear combinations of order statistics and have the advantage that are more robust in the presence of outliers than traditional moments. In the context of portfolio optimization, Yitzhaki (1982) proposed a portfolio optimization model based on the Gini mean difference that is equal to twice L-variance, however practitioners have not adopted this model due to its complexity. Cajas (2021) proposed a general model based on the ordered weighted average (OWA) operator proposed by Yager (1988) with monotonic weights and the formulation proposed by Chassein and Goerigk (2015), that allows to solve the Gini mean difference portfolio in a more efficient way. Jurczenko et al. (2008) proposed a matrix representation of portfolio L-moments based on L-comoments matrixes.

This work proposes two OWA portfolio optimization models for the case of higher L-moments optimization. The first approach is to optimize the utility function based on four L-moments. The second approach is building a risk measure that incorporates higher moments making a linear combination of L-variance, L-Skewness and L-Kurtosis. Due to the nature of L-moments, both approaches can be easily extended to moments higher than L-Kurtosis. Also, we show five techniques to calculate the weights used to combine L-moments in both approaches. Then, we are going to run some numerical examples using Python 3.9[1], CVXPY[2], Riskfolio-Lib[3] and MOSEK[4].

[1] https://www.anaconda.com/products/distribution
[2] Diamond and Boyd (2016) and Agrawal et al. (2018)
[3] Cajas (2022)
[4] ApS (2022)

2 L-Moments

L-moments or linear moments are expectations of certain linear combination of order statistics. As shown in Hosking (1990), L-moments are used to summarize the shape of a probability distribution in a similar way than central moments.

If $\{y_{[1]} < \cdots < y_{[i]} < \cdots < y_{[n]}\}$ is the ordered sample, where $y_{[i]}$ is the i-th ordered statistic. The k-th sample L-moment can be calculated using the following formula:

$$\lambda_k = \binom{T}{k}^{-1} \sum \sum \ldots \sum_{1 \leq i_1 < i_2 \cdots < i_k \leq n} \frac{1}{k} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} y_{[i_{k-j}]} \tag{1}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient, T is the number of observations and λ_k is the k-th L-moment. On the other hand, Wang (1996) proposed direct estimators for the first four L-moments in a finite sample of T observations using the following formulas:

$$\begin{aligned}
\lambda_1 &= \binom{T}{1}^{-1} \sum_{i=1}^{T} y_{[i]} \\
\lambda_2 &= \tfrac{1}{2}\binom{T}{2}^{-1} \sum_{i=1}^{T} \left[\binom{i-1}{1} - \binom{T-i}{1}\right] y_{[i]} \\
\lambda_3 &= \tfrac{1}{3}\binom{T}{3}^{-1} \sum_{i=1}^{T} \left[\binom{i-1}{2} - 2\binom{i-1}{1}\binom{T-i}{1} + \binom{T-i}{2}\right] y_{[i]} \\
\lambda_4 &= \tfrac{1}{4}\binom{T}{4}^{-1} \sum_{i=1}^{T} \left[\binom{i-1}{3} - 3\binom{i-1}{2}\binom{T-i}{1} + 3\binom{i-1}{1}\binom{T-i}{2} - \binom{T-i}{3}\right] y_{[i]}
\end{aligned} \tag{2}$$

In general, the k-th sample L-moment in a finite sample of T observations can be calculated using the following formula:

$$\begin{aligned}
\lambda_k &= \tfrac{1}{k}\binom{T}{k}^{-1} \sum_{i=1}^{T} \left[\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j}\binom{i-1}{k-1-j}\binom{T-i}{j}\right] y_{[i]} \\
\lambda_k &= \sum_{i=1}^{T} \underbrace{\tfrac{1}{k}\binom{T}{k}^{-1} \left[\sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j}\binom{i-1}{k-1-j}\binom{T-i}{j}\right]}_{w_i^k} y_{[i]} \\
\lambda_k &= \sum_{i=1}^{T} w_i^k y_{[i]}
\end{aligned} \tag{3}$$

where w_i^k is the weight associated to the i-th ordered statistic $y_{[i]}$. From the definition of the k-th L-moment in (3), we can notice that L-moments are a class of OWA operator proposed by Yager (1988).

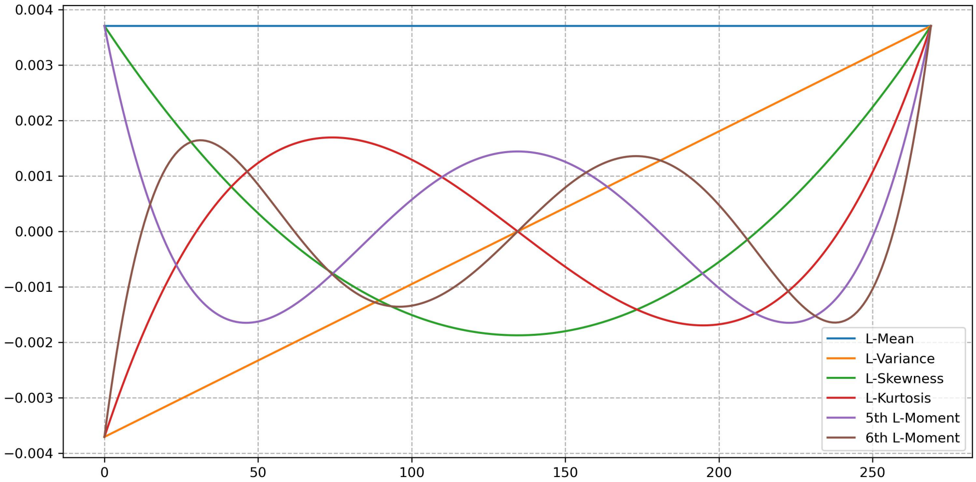

Figure 1: First Six L-Moments Weights for $T = 500$

In the context of financial time series, we can see in the following charts that traditional central moments and L-moments have a positive relationship.

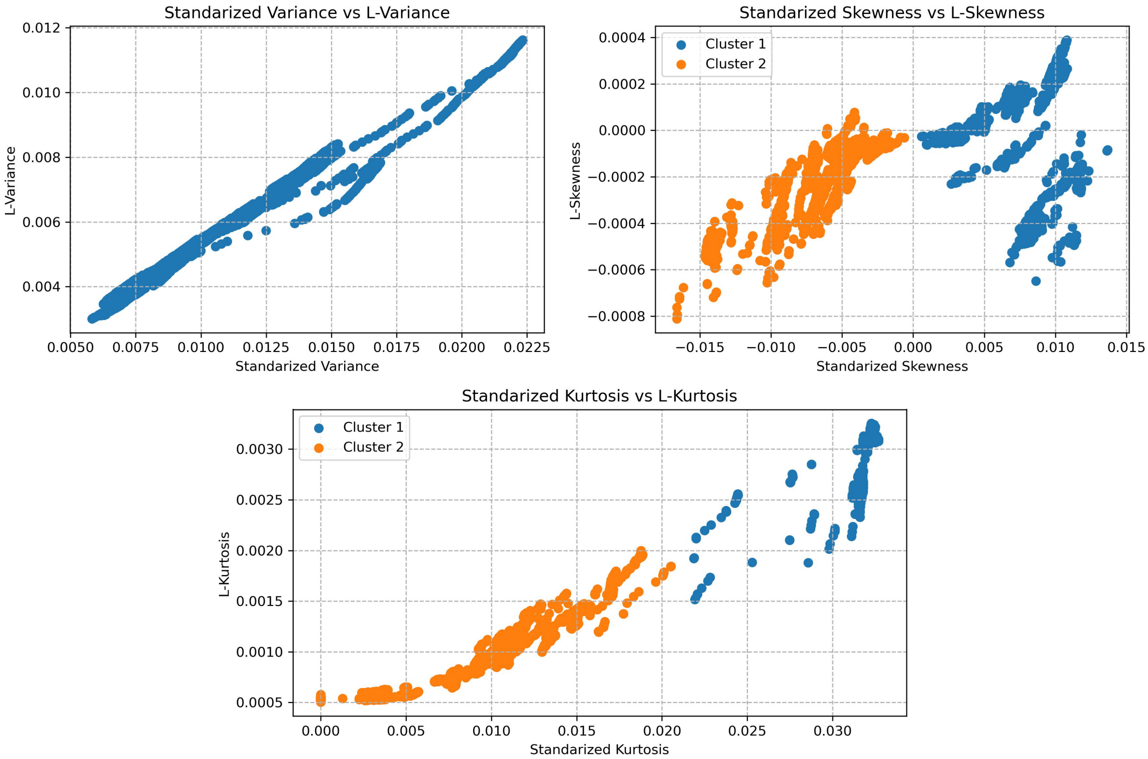

Figure 2: Correlation of Central Moments and L-moments

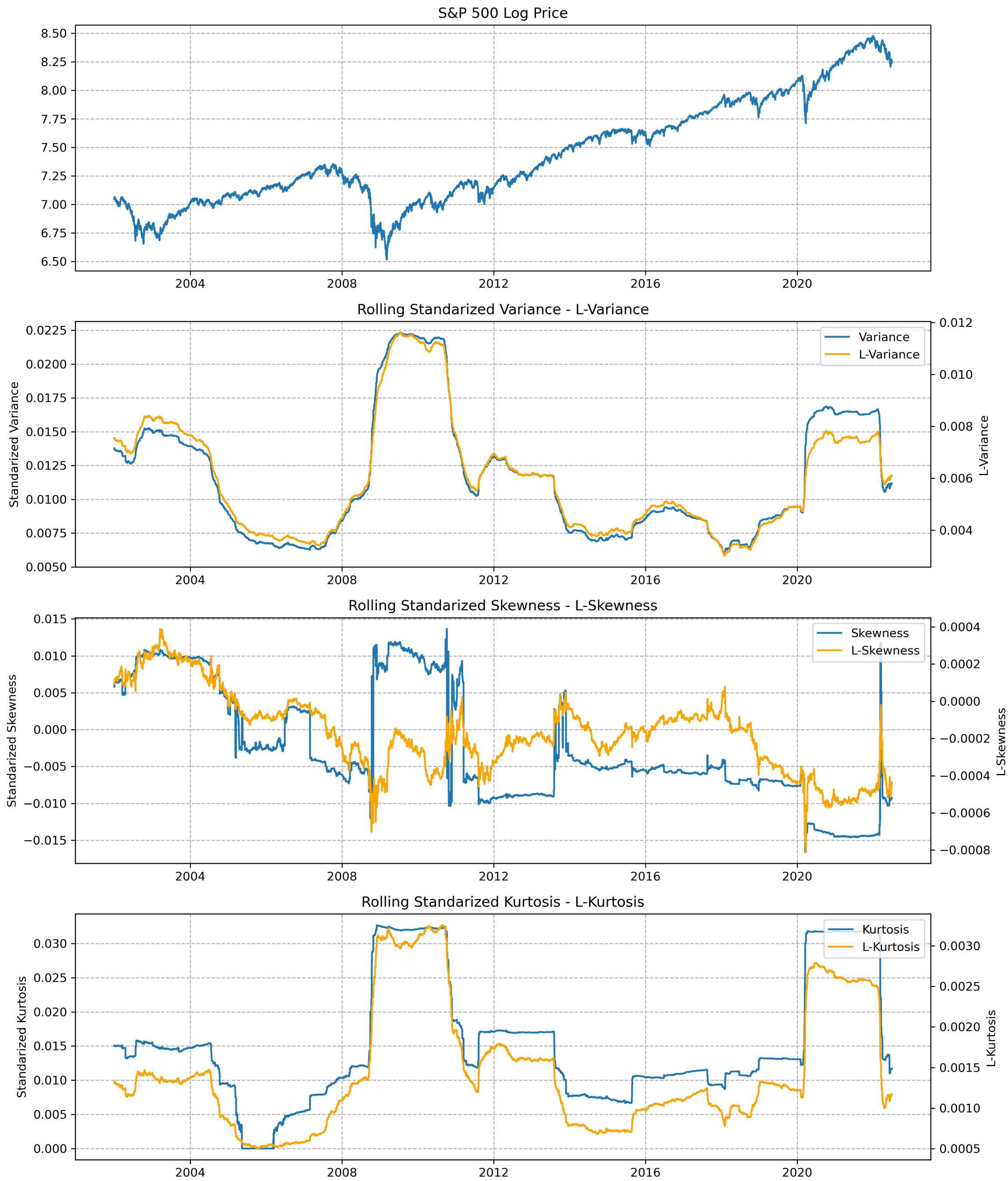

Figure 3: Rolling Window Central Moments and L-moments of S&P500 Index

3 Higher L-Moments Portfolio Optimization

3.1 Formulation as an Utility Function

The objective function of the utility function formulatio based on higher order L-moments can be posed as:

$$
\begin{aligned}
U(x) &= \phi_1\lambda_1(rx) - \phi_2\lambda_2(rx) + \phi_3\lambda_3(rx) - \phi_4\lambda_4(rx) + \dots \\
U(x) &= \sum_{k=1}^{\infty}(-1)^{k-1}\phi_k\lambda_k(rx) \\
U(x) &= \sum_{k=1}^{\infty}(-1)^{k-1}\phi_k\sum_{i=1}^{T} w_i^k(rx)_{[i]} \\
U(x) &= \sum_{i=1}^{T}\underbrace{\left[\sum_{k=1}^{\infty}(-1)^{k-1}\phi_k w_i^k\right]}_{\theta_i}(rx)_{[i]} \\
U(x) &= \sum_{i=1}^{T}\theta_i(rx)_{[i]}
\end{aligned} \tag{4}
$$

where $U(x)$ is the utility function, ϕ_i is the risk aversion coefficients of the i-th L-moment, θ_i is the weight associated to the i-th ordered portfolio return observation $(rx)_{[i]}$, r_i is the row vector of returns of observation i and x is the column vector of portfolio weights. We can notice that the utility function is equivalent to an OWA operator. Also, the advantage of the L-moments formulation of the utility function is that we can approximate the function using the first k L-moments without increase the complexity of the problem:

$$
\begin{aligned}
U(x) &\approx \sum_{i=1}^{T}\underbrace{\left[\sum_{k=1}^{K}(-1)^{k-1}\phi_k w_i^k\right]}_{\theta_i}(rx)_{[i]} \\
U(x) &\approx \sum_{i=1}^{T}\theta_i(rx)_{[i]}
\end{aligned} \tag{5}
$$

In order to use the OWA portfolio model proposed by Cajas (2021) to maximize the utility function, we need that the weights θ_i be non-increasing (monotonic) due to the signs of L-moments in the utility function. To build non-increasing weights we have several alternatives like:

- **Weights based on constant relative risk aversion (CRRA)**: following Martellini and Ziemann (2010) weights are defined as:

$$
\begin{aligned}
\phi_1 &= \gamma \\
\phi_2 &= \frac{\gamma}{2} \\
\phi_3 &= \frac{\gamma(1+\gamma)}{6} \\
\phi_4 &= \frac{\gamma(1+\gamma)(1+\gamma)}{24}
\end{aligned} \tag{6}
$$

where $\gamma < 1$ is a risk aversion factor. It could be good for comparison to normalize weights to sum 1.

- **Weights that maximize entropy:** to obtain non-increasing weights that maximizes the entropy of absolute values of weights θ_i, we can solve the following problem:

$$
\begin{aligned}
\max_{\psi,\,\theta} \quad & \sum_{i=1}^{T} \ln(\psi_i) \\
\text{s.t.} \quad & \psi \geq \theta \\
& \psi \geq -\theta \\
& \theta_i = \sum_{k=1}^{K} (-1)^{k-1} \phi_k w_i^k \\
& \sum_{k=1}^{K} \phi_k = 1 \\
& \phi_k \leq 0.5 \\
& \phi_k \geq 0 \\
& \phi_{k+1} \leq \phi_k \ \forall k \ \in [1, \ldots, K-1] \\
& \theta_{i+1} \leq \theta_i \ \forall i \ \in [1, \ldots, T-1]
\end{aligned} \tag{7}
$$

where ψ_i is an auxiliary variable that represents the absolute value of θ_i, K is the order of the highest L-moment and T is the number of observations.

- **Weights that minimize sum of squares:** to obtain non-increasing weights that minimize the sum of squares of weights θ_i, we can solve the following problem:

$$
\begin{aligned}
\min_{\psi,\theta} \quad & \sum_{i=1}^{T} \theta_i^2 \\
\text{s.t.} \quad & \theta_i = \sum_{k=1}^{K} (-1)^{k-1} \phi_k w_i^k \\
& \sum_{k=1}^{K} \phi_k = 1 \\
& \phi_k \leq 0.5 \\
& \phi_k \geq 0 \\
& \phi_{k+1} \leq \phi_k \ \forall k \ \in [1, \dots, K-1] \\
& \theta_{i+1} \leq \theta_i \ \forall i \ \in [1, \dots, T-1]
\end{aligned} \tag{8}
$$

- **Weights that minimize square distance:** to obtain non-increasing weights that minimizes the square distance of weights, we can solve the following problem:

$$
\begin{aligned}
\max_{\psi,\theta} \quad & \sum_{i=1}^{T-1} (\theta_{i+1} - \theta_i)^2 \\
\text{s.t.} \quad & \theta_i = \sum_{k=1}^{K} (-1)^{k-1} \phi_k w_i^k \\
& \sum_{k=1}^{K} \phi_k = 1 \\
& \phi_k \leq 0.5 \\
& \phi_k \geq 0 \\
& \phi_{k+1} \leq \phi_k \ \forall k \ \in [1, \dots, K-1] \\
& \theta_{i+1} \leq \theta_i \ \forall i \ \in [1, \dots, T-1]
\end{aligned} \tag{9}
$$

Finally, the optimization problem that maximizes the utility function based on the first K higher L-moments can be posed as:

$$
\begin{aligned}
\max_{\alpha,\beta,x,y} \quad & \sum_{i=1}^{T} \alpha_i + \beta_i \\
\text{s.t.} \quad & rx = y \\
& \alpha_i + \beta_j \leq \theta_i y_{[j]} \quad \forall \ i,j = 1, \dots, T \\
& \sum_{i=1}^{N} x_i = 1 \\
& x_i \geq 0 \ ; \ \forall \ i = 1, \dots, N
\end{aligned} \tag{10}
$$

3.2 Formulation as a Convex Risk Measure

In this case we split the utility function between expected return and the rest of higher L-moments. Then we group the higher L-moments as a new risk measure:

$$\begin{aligned}
\rho(x) &= \phi_2\lambda_2(rx) - \phi_3\lambda_3(rx) + \ldots + (-1)^K\phi_K\lambda_K(rx) \\
\rho(x) &= \sum_{k=2}^{K}(-1)^k\phi_k\lambda_k(rx) \\
\rho(x) &= \sum_{i=1}^{T}\underbrace{\left[\sum_{k=2}^{K}(-1)^k\phi_k w_i^k\right]}_{\eta_i}(rx)_{[i]} \\
\rho(x) &= \sum_{i=1}^{T}\eta_i(rx)_{[i]}
\end{aligned} \tag{11}$$

To make sure that the new measure of risk will be convex, we need to build monotonic weights η_i, however, in this case we need non-decreasing (monotonic) due to the signs of L-moments in the risk measure. So, in order to get weights that meet our requirements, we have to modify the methods of previous section:

- **Weights based on constant relative risk aversion (CRRA)**:

$$\begin{aligned}
\phi_2 &= \frac{\gamma}{2} \\
\phi_3 &= \frac{\gamma(1+\gamma)}{6} \\
\phi_4 &= \frac{\gamma(1+\gamma)(1+\gamma)}{24}
\end{aligned} \tag{12}$$

  where $\gamma < 1$ is a risk aversion factor. It could be good for comparison to normalize weights to sum 1.

- **Weights that maximize entropy:** to obtain non-decreasing weights that maximizes the entropy of absolute values of weights η_i, we can solve the following problem:

$$
\begin{aligned}
\max_{\psi,\,\eta} \quad & \sum_{i=1}^{T} \ln(\psi_i) \\
\text{s.t.} \quad & \psi \geq \eta \\
& \psi \geq -\eta \\
& \eta_i = \sum_{k=2}^{K} (-1)^k \phi_k w_i^k \\
& \sum_{k=2}^{K} \phi_k = 1 \\
& \phi_k \leq 0.5 \\
& \phi_k \geq 0 \\
& \phi_{k+1} \leq \phi_k \ \forall k \ \in [1, \ldots, K-1] \\
& \eta_{i+1} \geq \eta_i \ \forall i \ \in [1, \ldots, T-1]
\end{aligned}
\tag{13}
$$

where ψ_i is an auxiliary variable that represents the absolute value of θ_i, K is the order of the highest L-moment and T is the number of observations.

- **Weights that minimize sum of squares:** to obtain non-decreasing weights that minimize the sum of squares of weights η_i, we can solve the following problem:

$$
\begin{aligned}
\min_{\psi,\,\eta} \quad & \sum_{i=2}^{T} \eta_i^2 \\
\text{s.t.} \quad & \eta_i = \sum_{k=2}^{K} (-1)^k \phi_k w_i^k \\
& \sum_{k=1}^{K} \phi_k = 1 \\
& \phi_k \leq 0.5 \\
& \phi_k \geq 0 \\
& \phi_{k+1} \leq \phi_k \ \forall k \ \in [1, \ldots, K-1] \\
& \eta_{i+1} \geq \eta_i \ \forall i \ \in [1, \ldots, T-1]
\end{aligned}
\tag{14}
$$

- **Weights that minimize square distance:** to obtain non-decreasing weights that minimizes the square distance of weights η_i, we can solve the following problem:

$$
\begin{aligned}
\max_{\psi,\,\eta} \quad & \sum_{i=1}^{T-1} (\eta_{i+1} - \eta_i)^2 \\
\text{s.t.} \quad & \eta_i = \sum_{k=2}^{K} (-1)^k \phi_k w_i^k \\
& \sum_{k=1}^{K} \phi_k = 1 \\
& \phi_k \leq 0.5 \\
& \phi_k \geq 0 \\
& \phi_{k+1} \leq \phi_k \ \forall k \ \in [1, \ldots, K-1] \\
& \eta_{i+1} \geq \eta_i \ \forall i \ \in [1, \ldots, T-1]
\end{aligned}
\tag{15}
$$

Finally, this formulation is more flexible because it allows to use all OWA portfolio optimization models proposed by Cajas (2021) like risk constraints, minimization of risk, maximization of risk adjusted return ratio and risk parity. For example, the portfolio that maximizes the risk adjusted return using a risk measure that consider first k-th L-moments can be posed as:

$$
\begin{aligned}
\min_{\alpha,\,\beta,\,y,\,z} \quad & \sum_{i=1}^{T} \alpha_i + \beta_i \\
\text{s.t.} \quad & \mu z^{\tau} - r_f k = 1 \\
& r z^{\tau} = y \\
& \alpha_i + \beta_j \geq \eta_i y_{[j]} \quad \forall \, i, j = 1, \ldots, T \\
& \sum_{i=1}^{N} z_i = k \\
& z_i \geq 0 \; ; \; \forall \, i = 1, \ldots, N
\end{aligned}
\tag{16}
$$

where the optimal portfolio is obtained making the transformation $x = z / \sum_{i=1}^{N} z_i$.

4 Numerical Examples

We select 20 assets (i.e., stocks AIG, AKAM, AMT, APA, AXP, BKNG, EQT, F, GLW, HST, IDXX, INCY, MAC, MGM, NKTR, NTAP, REGN, SBAC, SEE and TTWO) from the S&P 500 (NYSE) and download monthly adjusted closed prices from Yahoo Finance for the period from December 30, 1999 to June 30, 2022. Then, we calculated monthly returns building a returns matrix of size $T = 270$ and $N = 20$. To calculate the portfolios we use Python 3.9, CVXPY, Riskfolio-Lib and MOSEK solver.

4.1 Utility Function Example

In this section, we calculate the portfolio that maximize the utility function based on first four L-moments using the four methods proposed to build monotonic weights:

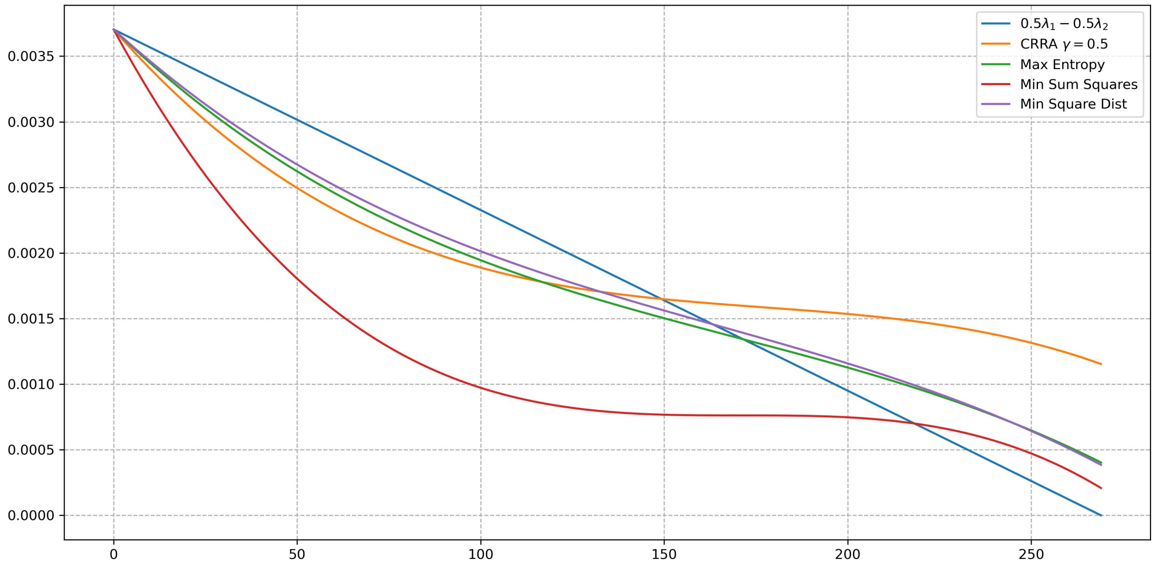

Figure 4: Weights of Utility Function for $T = 270$

Weight	CRRA $\gamma = 0.5$	Max Entropy	Min Sum Squares	Min Square Distance
ϕ_1	52.46%	47.76%	31.90%	48.82%
ϕ_2	26.23%	38.81%	31.90%	39.25%
ϕ_3	13.11%	7.62%	20.90%	6.38%
ϕ_4	8.20%	5.81%	15.29%	5.55%

Table 1: Weights of L-Moments in the Utility Function per Weighting Method

We can see that max entropy and min square distance weighting methods have similar compositions. Now we are going to calculate the portfolios that maximize the utility function for each weighting method, then we calculate the L-moments and central moments and finally we plot the portfolios in expected return L-variance, and L-skewness L-kurtosis planes.

Weighting Method	λ_1	λ_2	λ_3	λ_4	λ_1/λ_2	λ_3/λ_4
$0.5\lambda_1 - 0.5\sigma_2$	1.67%	3.26%	0.02%	0.77%	51.26%	2.57%
$0.5\lambda_1 - 0.5\lambda_2$	1.90%	3.40%	0.13%	0.88%	55.77%	15.26%
CRRA $\gamma = 0.5$	2.29%	3.98%	0.36%	1.03%	57.53%	34.72%
Max Entropy	2.04%	3.56%	0.20%	0.92%	57.19%	21.68%
Min Sum Squares	1.96%	3.49%	0.22%	0.92%	56.29%	23.74%
Min Square Distance	2.04%	3.57%	0.20%	0.92%	57.22%	21.36%

Table 2: L-Moments per Weighting Method

Weighting Method	λ_1	σ_2	σ_3	σ_4	λ_1/σ_2	σ_3/σ_4
$0.5\lambda_1 - 0.5\sigma_2$	1.67%	6.26%	5.31%	9.13%	26.67%	58.13%
$0.5\lambda_1 - 0.5\lambda_2$	1.90%	6.94%	8.65%	13.59%	27.33%	63.61%
CRRA $\gamma = 0.5$	2.29%	9.05%	15.11%	23.57%	25.29%	64.10%
Max Entropy	2.04%	7.52%	10.57%	16.45%	27.09%	64.28%
Min Sum Squares	1.96%	7.33%	10.05%	15.40%	26.79%	65.25%
Min Square Distance	2.04%	7.52%	10.58%	16.47%	27.12%	64.21%

Table 3: Standardized Central Moments per Weighting Method

where λ_1 is the L-mean or mean, λ_2 is the L-variance, λ_3 is the L-skewness, λ_4 is the L-Kurtosis, σ_2 is the standard deviation, σ_3 is the cubic root of skewness and σ_4 is the fourth root kurtosis. We can see in tables 2 and 3 that the utility function that incorporates higher L-moments gives us an improvement in L-skewness and skewness. Also, we can see that the ratios of mean-L-variance and mean-variance decrease while the ratios L-skewness-L-kurtosis and skewness-kurtosis increase respect to the portfolios that not incorporate higher L-moments.

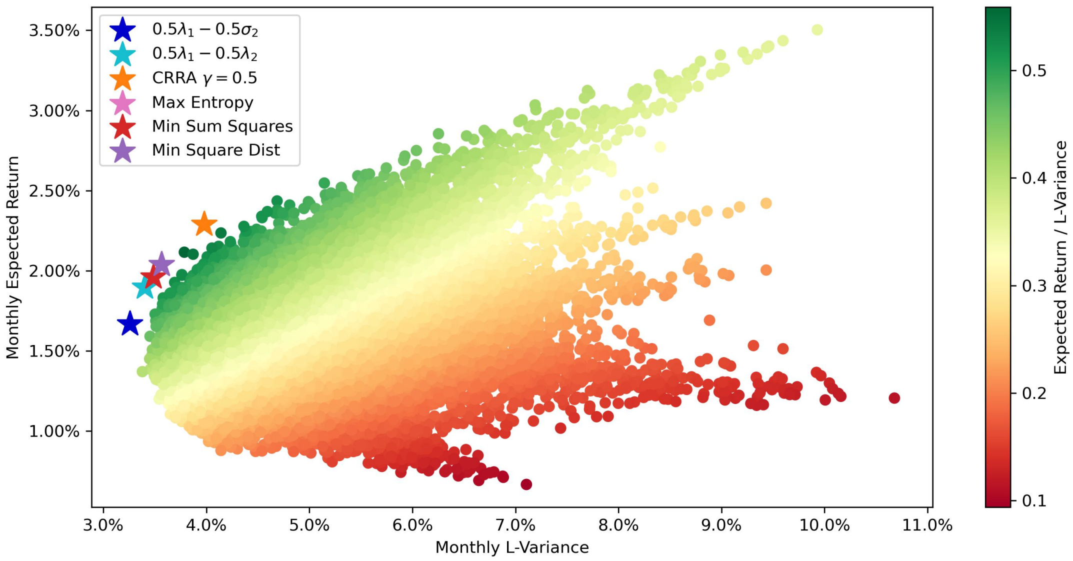

Figure 5: Optimal Portfolios in Plane Expected Return / L-Variance

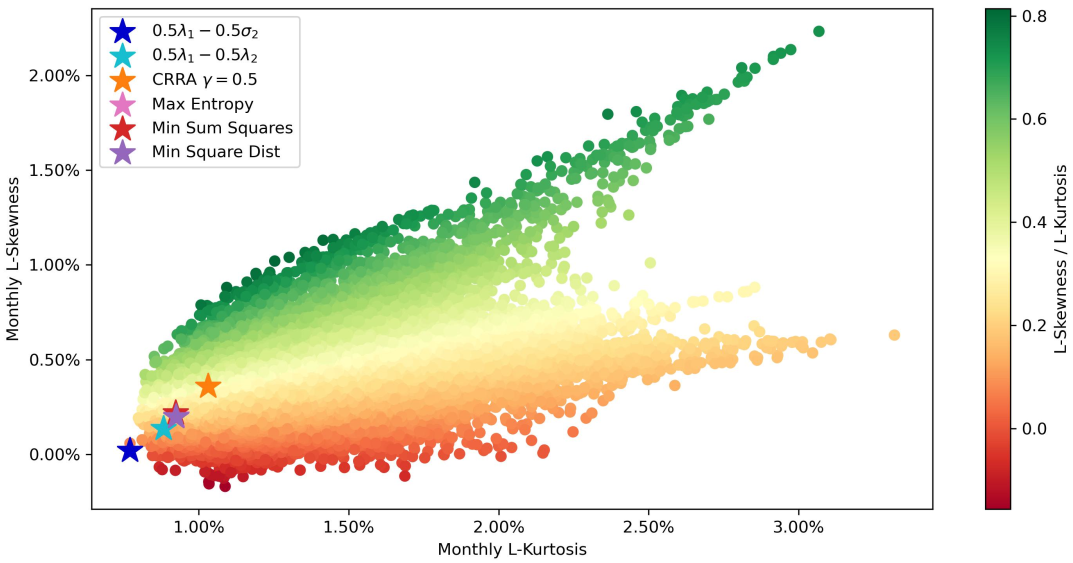

Figure 6: Optimal Portfolios in Plane L-Skewness / L-Kurtosis

In figures 5 and 6, we can notice that the incorporation of L-skewness and L-kurtosis in the utility function increase the L-skewness of the portfolios and also increase the ratio L-skewness-L-kurtosis respect to the utility function that only considers the first two L-moments.

4.2 Convex Risk Measure Example

In this section, we calculate the portfolio that maximize the utility function based on first four L-moments using the four methods proposed to build monotonic weights:

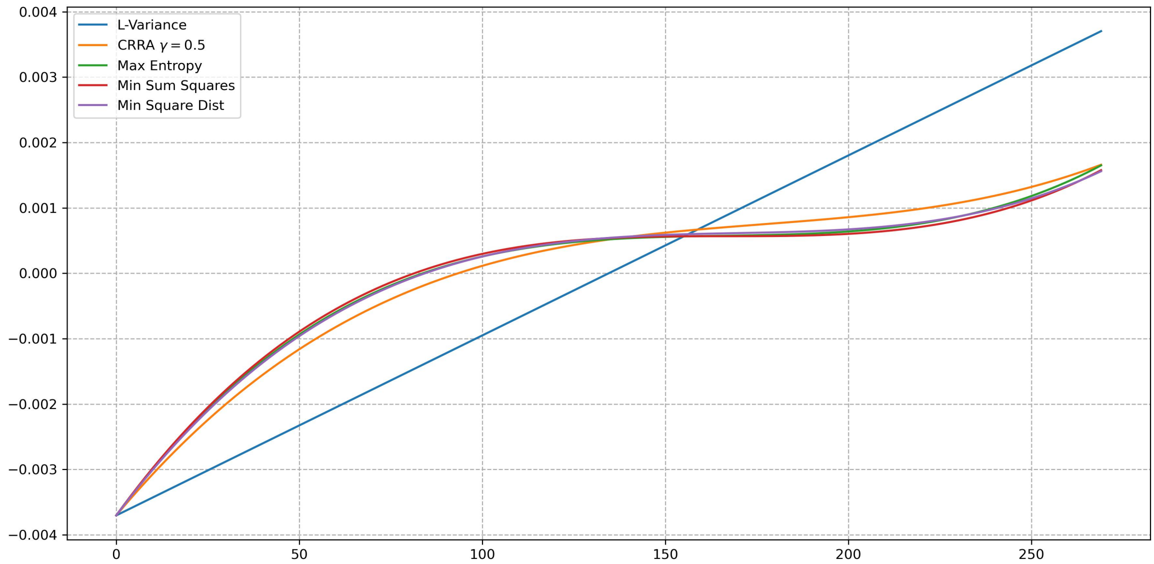

Figure 7: Weights of the Convex Risk Measure for $T = 270$

Weight	CRRA $\gamma = 0.5$	Max Entropy	Min Sum Squares	Min Square Distance
ϕ_2	55.17%	48.82%	47.12%	48.96%
ϕ_3	27.59%	27.70%	28.71%	28.92%
ϕ_4	17.24%	23.48%	24.17%	22.12%

Table 4: Weights of L-Moments in the Convex Risk Measure per Weighting Method

We can see that all methods have similar compositions. Now we are going to calculate the portfolios that maximizes the ratio of return-risk for each weighting method, then we calculate the L-moments and central moments for each portfolio and finally we plot the portfolios in expected return L-variance, and L-skewness L-kurtosis planes.

Weighting Method	λ_1	λ_2	λ_3	λ_4	λ_1/λ_2	λ_3/λ_4
Variance	2.04%	3.68%	0.07%	0.83%	55.45%	8.65%
L-Variance	2.22%	3.82%	0.24%	0.95%	57.97%	24.74%
CRRA $\gamma = 0.5$	2.25%	3.90%	0.33%	1.00%	57.72%	32.61%
Max Entropy	2.25%	3.89%	0.32%	1.00%	57.72%	32.58%
Min Sum Squares	2.25%	3.90%	0.33%	1.00%	57.67%	33.33%
Min Square Distance	2.25%	3.90%	0.34%	1.00%	57.66%	33.55%

Table 5: L-Moments per Weighting Method

Weighting Method	λ_1	σ_2	σ_3	σ_4	λ_1/σ_2	σ_3/σ_4
Variance	2.04%	7.13%	6.99%	11.59%	28.62%	60.27%
L-Variance	2.22%	8.35%	12.96%	20.32%	26.53%	63.78%
CRRA $\gamma = 0.5$	2.25%	8.74%	14.22%	22.15%	25.73%	64.22%
Max Entropy	2.25%	8.73%	14.18%	22.07%	25.75%	64.23%
Min Sum Squares	2.25%	8.77%	14.30%	22.25%	25.67%	64.26%
Min Square Distance	2.25%	8.78%	14.34%	22.32%	25.63%	64.27%

Table 6: Standardized Central Moments per Weighting Method

We can see in 5 and 6 that the incorporation of higher L-moments in the convex risk measure gives us an improvement in L-skewness and skewness. Also, we can see that the ratios of mean-L-variance and mean-variance decrease while the ratios L-skewness-L-kurtosis and skewness-kurtosis increase respect to the portfolios that not incorporate higher L-moments.

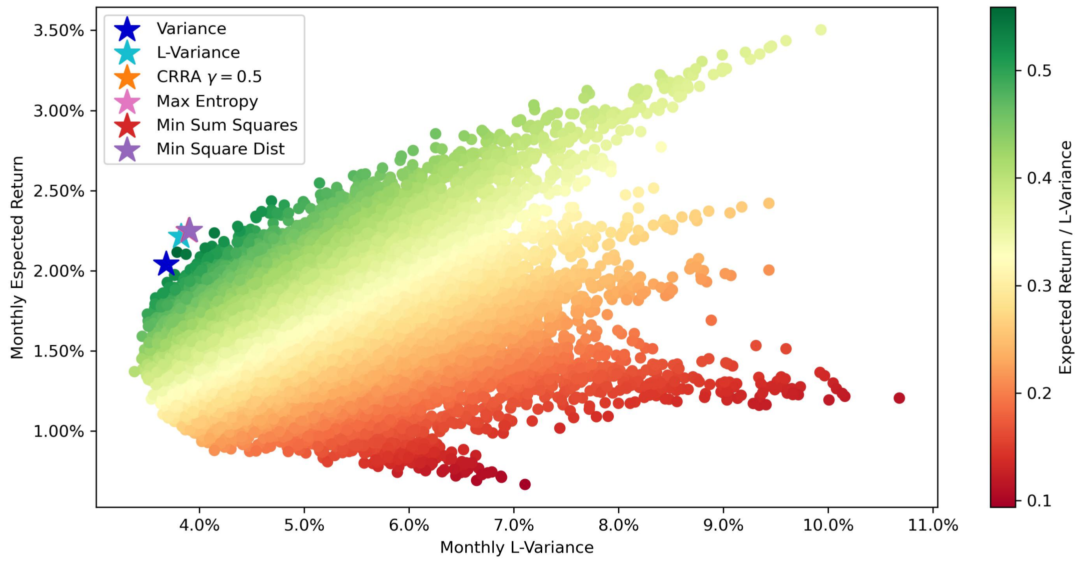

Figure 8: Optimal Portfolios in Plane Expected Return / L-Variance

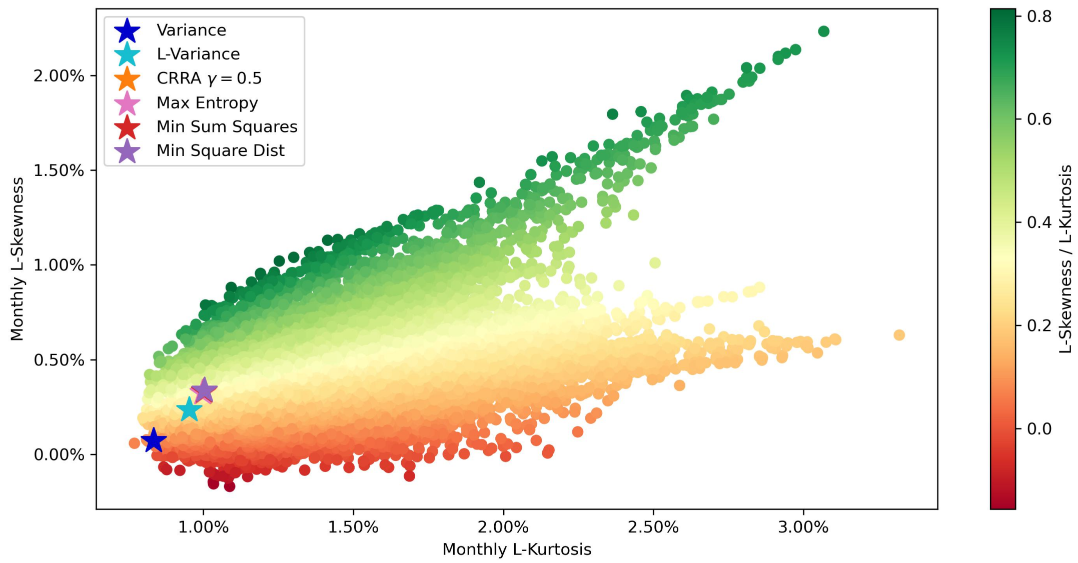

Figure 9: Optimal Portfolios in Plane L-Skewness / L-Kurtosis

In figures 8 and 9, we can notice that the incorporation of L-skewness and L-kurtosis in the convex risk measures increase the L-skewness of the portfolios and also increase the ratio L-skewness-L-kurtosis respect to case when the convex risk measure is the L-variance.

5 Conclusions

This work presents a disciplined convex programming formulation of higher order moments portfolio optimization based on L-moments. We propose two formulations to include L-moments in the portfolio optimization process. The first approach is to maximize an utility function based on higher L-moments, this approach is commonly used with central moments but has the disadvantage that is not enough flexible to model different portfolio features. The second approach is to combine L-moments higher than two in order to build a convex risk measure, this approach has the advantage that we have the flexibility to model several objective functions like risk constraints, maximization of risk adjusted return ratio, minimization of risk and risk parity. We run some numerical examples that show how our formulations allow to increase L-skewness and skewness at the cost of an increase in L-variance and variance. Finally, our proposed models has the advantage that can be modeled with state of art solvers, making them accessible to most users.

References

Agrawal, A., R. Verschueren, S. Diamond, and S. Boyd (2018). A rewriting system for convex optimization problems. *Journal of Control and Decision 5*(1), 42–60.

ApS, M. (2022). *MOSEK Optimizer API for Python 10.0.38.*

Athayde, G. M. D. and R. G. Flôres (2003). Chapter 10 - incorporating skewness and kurtosis in portfolio optimization: A multidimensional efficient set. In S. Satchell and A. Scowcroft (Eds.), *Advances in Portfolio Construction and Implementation*, Quantitative Finance, pp. 243–257. Oxford: Butterworth-Heinemann.

Cajas, D. (2021, 12). Owa portfolio optimization: a disciplined convex programming framework. *SSRN Electronic Journal*.

Cajas, D. (2022). Riskfolio-lib (4.0.3).

Chassein, A. and M. Goerigk (2015). Alternative formulations for the ordered weighted averaging objective. *Information Processing Letters 115*(6), 604–608.

Diamond, S. and S. Boyd (2016). CVXPY: A Python-embedded modeling language for convex optimization. *Journal of Machine Learning Research 17*(83), 1–5.

Hosking, J. R. M. (1990). L-moments: Analysis and estimation of distributions using linear combinations of order statistics. *Journal of the Royal Statistical Society. Series B (Methodological) 52*(1), 105–124.

Jurczenko, E., B. Maillet, and P. Merlin (2008, October). Efficient Frontier for Robust Higher-order Moment Portfolio Selection. Documents de travail du Centre d'Economie de la Sorbonne 2008.62 - ISSN : 1955-611X.

Jurczenko, E., B. Maillet, and P. Merlin (2015, aug). Hedge fund portfolio selection with higher-order moments: A nonparametric mean-variance-skewness-kurtosis efficient frontier. In *Multi-moment Asset Allocation and Pricing Models*, pp. 51–66. John Wiley & Sons, Inc.

Konno, H. and K. ichi Suzuki (1995). A mean-variance-skewness portfolio optimization model. *Journal of the Operations Research Society of Japan 38*(2), 173–187.

Konno, H. and R. Yamamoto (2005, June). A mean-variance-skewness model: Algorithm and applications. *International Journal of Theoretical and Applied Finance 08*(04), 409–423.

Lai, T.-Y. (1991, July). Portfolio selection with skewness: A multiple-objective approach. *Review of Quantitative Finance and Accounting 1*(3), 293–305.

Martellini, L. and V. Ziemann (2010, 04). Improved estimates of higher-order comoments and implications for portfolio selection. *Review of Financial Studies 23*, 1467–1502.

Niu, Y.-S., Y.-J. Wang, H. A. L. Thi, and D. T. Pham (2019). High-order moment portfolio optimization via an accelerated difference-of-convex programming approach and sums-of-squares.

Wang, Q. J. (1996, December). Direct sample estimators of l moments. *Water Resources Research 32*(12), 3617–3619.

Yager, R. (1988). On ordered weighted averaging aggregation operators in multicriteria decisionmaking. *IEEE Transactions on Systems, Man, and Cybernetics 18*(1), 183–190.

Yitzhaki, S. (1982, 01). Stochastic dominance, mean variance, and gini's mean difference. *American Economic Review 72*, 178–85.

Zhou, R. and D. P. Palomar (2020). Solving high-order portfolios via successive convex approximation algorithms.