SOLVED NUMERICAL EXAMPLES

1. **Evaluate the Fourier series coefficients of the continuous time periodic signal**

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

Solution:

On expanding x(t) in terms of complex exponentials, we get,

$$x(t) = 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} - e^{-j\omega_0 t}] + \frac{1}{2}\left[(e^{j\left(2\omega_0 t + \frac{\pi}{4}\right)} + e^{-j\left(2\omega_0 t + \frac{\pi}{4}\right)}\right]$$

Collecting terms, we get,

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 + \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{\frac{j\pi}{4}}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-\frac{j\pi}{4}}\right)e^{-j2\omega_0 t}$$

Thus, the Fourier series coefficients for this x(t) are

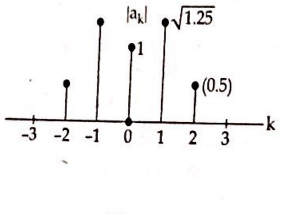
$a_0 = 1$

$a_1 = 1 + \frac{1}{2j} = 1 - j\frac{1}{2}$ $\qquad a_{-1} = 1 - \frac{1}{2j} = 1 + j\frac{1}{2}$

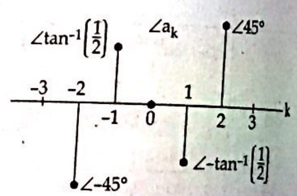
$a_2 = \frac{1}{2}e^{j\frac{\pi}{4}} = \frac{1}{2}\left[\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right] = \frac{1}{2\sqrt{2}}(1 + j)$

$a_{-2} = \frac{1}{2}e^{-j\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}(1 - j)$ $\qquad a_k = 0\,;\ |k| > 2$

Plots of magnitude and phase of the Fourier series coefficients of x(t) are shown in figure (a) and (b) respectively.

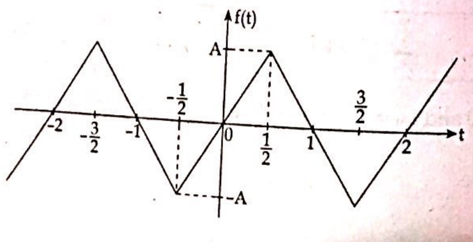


(a)



(b)

2. **Find the trigonometric Fourier series for the triangular periodic signal f(t) as shown in figure.**



Solution:

$$f(t) = a_0 + \sum_{k=1}^{\infty}[A_k \cos k\omega_0 t + B_k \sin k\omega_0 t]$$

Here, $T = 2$, hence $\omega_0 = \frac{2\pi}{2} = \pi$

Here it will be advantageous to choose the interval of integration from $\frac{-1}{2}$ to $\frac{3}{2}$ rather than other like 0 to 2.

$a_0 = \frac{1}{T}\int_T f(t)\,dt = 0$, since average or dc value of f(t) is zero.

Now,

$$f(t)\begin{cases} 2At & |t| \le \frac{1}{2} \text{ or } \frac{-1}{2} \le t \le \frac{1}{2} \\ 2A(1-t) & \frac{1}{2} < t \le \frac{3}{2} \end{cases}$$

$$A_k = \frac{2}{T}\int_T f(t)\cos k\omega_0 t\,dt = \frac{2}{2}\int_{\frac{-1}{2}}^{\frac{3}{2}} f(t)\cos k\pi t\,dt$$

$$= \int_{\frac{-1}{2}}^{\frac{1}{2}} 2At\cos k\pi t\,dt + \int_{\frac{-1}{2}}^{\frac{3}{2}} 2A(1-t)\cos k\pi t\,dt$$

$$= 0$$

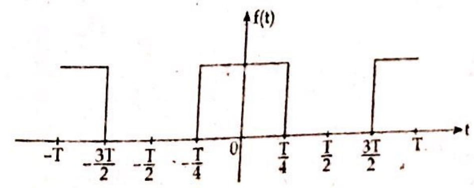
(Also, we know that $A_k = 0$, since given function is an odd function.

and, $B_k = \int_{\frac{-1}{2}}^{\frac{1}{2}} 2At\sin k\pi t\,dt + \int_{\frac{-1}{2}}^{\frac{3}{2}} 2A(1-t)\sin k\pi t$

$$B_k = \frac{8A}{k^2\pi^2}\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 0 & ;\ k \text{ even} \\ \frac{8A}{k^2\pi^2} & ;\ k = 1, 5, 9, 13 \\ \frac{-8A}{k^2\pi^2} & ;\ k = 3, 7, 11, \ldots\ldots\ldots \end{cases}$$

Hence, $f(t) = \frac{8A}{\pi^2}\left[\sin \pi t \frac{-1}{9}\sin 3\pi t + \frac{1}{25}\sin 5\pi t \frac{-1}{49}\sin 7\pi t + \ldots\ldots\ldots\right]$

3. **Obtain the trigonometric Fourier series representation of the given periodic rectangular wave form.**



Solution:

The given wave form for one period may be written as,

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{T}{4} \\ A & \text{for} \quad -\frac{T}{4} < t < \frac{T}{4} \\ 0 & \text{for} \quad \frac{T}{4} < t < \frac{T}{2} \end{cases}$$

From figure it is clear that given waveform is an even function of time, hence, $B_k = 0$.

Time period = T and fundamental frequency, $\omega_0 = \frac{2\pi}{T}$ or $\omega_0 T = 2\pi$.

$$f(t) = a_0 + \sum_{k=1}^{\infty} [A_k \cos k\omega_0 t + B_k \sin \omega_0 kt]$$

$$= a_0 + \sum_{k=1}^{\infty} A_k \cos k\,\omega_0\, t$$

$$a_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)\,dt = \frac{1}{T}\int_{\frac{-T}{4}}^{\frac{T}{4}} A\,dt$$

$$= \frac{1}{T}\cdot A\cdot t\Big|_{\frac{-T}{4}}^{\frac{T}{4}} = \frac{A}{T}\left[\frac{T}{4} - \left(\frac{-T}{4}\right)\right] = \frac{A}{T}\cdot\frac{T}{2} = \frac{A}{2}$$

Now,

$$A_k = \frac{2}{T}\int_T f(t)\cos k\,\omega_0 t$$

$$= \frac{2}{T}\int_{\frac{-T}{4}}^{\frac{T}{4}} A\cos k\,\omega_0 t\,dt = \frac{2A}{T}\left[\frac{\sin k\,\omega_0 t}{k\,\omega_0}\right]_{\frac{-T}{4}}^{\frac{T}{4}}$$

$$= \frac{2A}{Tk\,\omega_0}\left[\sin\frac{k\,\omega_0 T}{4} - \sin\left(\frac{-k\,\omega_0 T}{4}\right)\right]$$

$$= \frac{2A}{Tk\,\omega_0}\cdot 2\sin\frac{k\,\omega_0 T}{4} = \frac{4A}{k\cdot 2\pi}\sin\frac{k\cdot 2\pi}{4} = \frac{2A}{k\pi}\cdot\sin\left(\frac{k\pi}{2}\right)$$

Putting k = 1, 2, 3, ...

$$A_1 = \frac{2A}{\pi}\sin\frac{\pi}{2} = \frac{2A}{\pi}$$

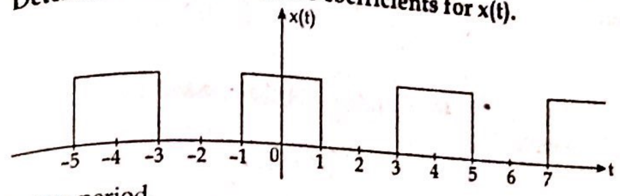
$$A_2 = \frac{2A}{2\pi}\cdot\sin\left(\frac{2\pi}{2}\right) = 0$$

$$A_3 = \frac{2A}{3\pi}\cdot\sin\frac{3\pi}{2} = \frac{-2A}{3\pi}$$

and, so on.

$$\therefore \quad f(t) = \frac{A}{2} + \frac{2A}{\pi}\left[\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \text{.........}\right]$$

4. **Determine the Fourier series coefficients for x(t).**



For over one period,

$$x(t) = \begin{cases} 1 \;; & |t| < 1 \quad \text{or} \quad -1 < t < 1 \\ 0 \;; & |t| > 1 \quad \text{or} \quad t < -1 \text{ and } t > 1 \end{cases}$$

Time period, T = 4 hence $\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

Because of the symmetry of x(t) about vertical axis or origin and it is convenient to choose $-1 \le t \le 1$ as the interval over which the integration is performed, although any interval of length T is equally valid and thus will lead to the same result.

$$a_0 = \frac{1}{T}\int_T x(t)\,dt = \frac{1}{4}\int_{-1}^{1} 1\cdot dt = \frac{1}{4}t\Big|_{-1}^{1} = \frac{1}{4}[1-(-1)] = \frac{1}{2}$$

$$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt = \frac{1}{4}\int_{-1}^{1} 1\cdot e^{\frac{-jk\pi_0 t}{2}}\,dt$$

$$= \frac{1}{4}\left[\frac{e^{\frac{-jk\pi_0 t}{2}}}{\frac{-jk\pi}{2}}\right]_{-1}^{1} = \frac{-1}{2jk\pi}\left[e^{\frac{-jk\pi}{2}} - e^{\frac{-jk\pi}{2}}\right] = \frac{1}{k\pi}\left[\frac{e^{\frac{jk\pi}{2}} - e^{\frac{-jk\pi}{2}}}{2j}\right] = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi};\ k \neq 0$$

$a_0 = 2$

$$a_1 = a_{-1} = \frac{1}{\pi}$$

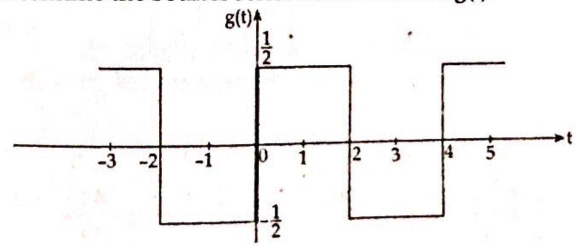
$$a_3 = a_{-3} = \frac{-1}{3\pi}$$

$$a_5 = a_{-5} = \frac{1}{5\pi}$$

$a_2 = a_4 = a_6 = 0$ and so on

and, $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{\frac{jk\pi t}{2}}$

5. **Determine the Fourier series coefficients for g(t)**



Solution:

$T = 4;\ \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

For a time interval -2 to -2, g(t) can be expressed in terms of x(t) of previous example as,

$g(t) = x(t-1) - \frac{1}{2}$

Now we calculate the Fourier series coefficients of g(t). The time shift property indicates that, if the Fourier series coefficients of x(t) are denoted by a_k, the Fourier series coefficients of $x(t - t_0)$ may be expressed as,

$B_k = a_k e^{-jk\omega_0 t_0} = a_k e^{\frac{-jk\pi}{2}}$

$\left(\text{Since } \omega_0 = \frac{\pi}{2} \text{ and } t_0 = 1\right)$

The Fourier series coefficients of term $-\frac{1}{2}$ are given by

$$C_k = \begin{cases} 0 & ;\ k \neq 0 \\ -\frac{1}{2} & ;\ k = 0 \end{cases}$$

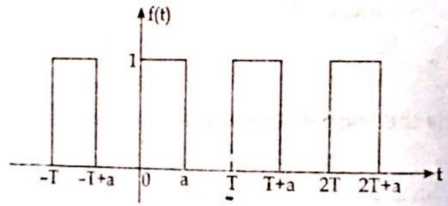
Applying the linearity property,

$$d_k = B_k + C_k = \begin{cases} a_k e^{\frac{-jk\pi}{2}} & ;\ k \neq 0 \\ a_0 - \frac{1}{2} & ;\ k = 0 \end{cases}$$

where, each a_k may now be replaced by the corresponding expressions, gives

$$d_k = \begin{cases} \dfrac{\frac{\sin \pi k}{2}}{k\pi} \cdot e^{\frac{-jk\pi}{2}} & ;\ k \neq 0 \\ 0 & ;\ k = 0 \end{cases}$$

6. **Determine the exponential form of Fourier series expansion for the periodic wave form.**



Solution:

$$f(t) = \begin{cases} 1 & ;\ 0 < t < a \\ 0 & ;\ a < t < T \end{cases}$$

$a_0 = \frac{1}{T}\int_T^0 f(t)\,dt = \frac{1}{T}\int_0^a dt = \frac{a}{T}$

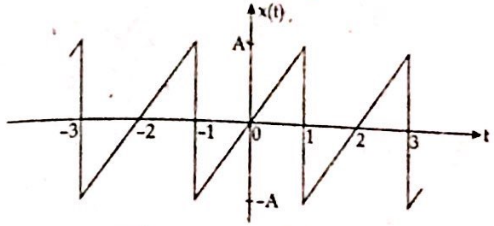
$a_k = \frac{a}{T}\int_T^0 f(t)\, e^{-jk\omega_0 t}\,dt = \frac{1}{2}\int_0^a e^{-jk\omega_0 t} = \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^a$

$= \frac{1}{T}\left[\frac{e^{-jk\omega_0 t} - 1}{-jk\omega_0}\right]$ and $a_{-k} = \frac{1}{T}\left[\frac{e^{-jk\omega_0 a_{-1}}}{-jk\omega_0 a}\right]$

so, $x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k e^{-jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right)$

$= \frac{a}{T} + \sum_{k=1}^{\infty}\left[\frac{1}{T}\left(\frac{e^{-jk\omega_0 t} - 1}{-jk\omega_0}\right)e^{-jk\omega_0 A} + \frac{1}{T}\left(\frac{e^{-jk\omega_0 t}}{jk\omega_0 a}\right)e^{-jk\omega_0 t}\right]$

8. **Find the Fourier series representation of waveform given in figure using trigonometric Fourier series.**



Solution:

Since the given wave form is an odd function, hence,

$a_0 = 0$ and $A_k = 0$

$$x(t) = \begin{cases} At & ;\ -1 < t < 1 \\ 0 & ;\ \text{otherwise} \end{cases}$$

$T = 2,\ \omega_0 = \frac{2\pi}{2} = \pi$

$B_k = \frac{2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x(t) \sin k\,\omega_0 t\,dt = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin k\,\omega_0 t\,dt$

$= \frac{2}{4}\int_0^1 x(t)\sin k\,\omega_0 t\,dt = 2\int_0^1 At \cdot \sin k\,\omega_0 t\,dt$

$= 2A\left[t \cdot \frac{\cos k\omega_0 t}{-k\omega_0}\Big|_0^1 + \frac{1}{k\omega_0}\int_0^1 \cos k\omega_0 t\,dt\right]$

$= 2A\left[\frac{\cos k\omega_0}{-k\omega_0} + \frac{1}{k^2\omega_0^2}\sin k\omega_0 t\Big|_0^1\right] - \frac{2A}{k\omega_0}\left[\frac{\sin k\omega_0}{k\omega_0} - \cos k\omega_0\right]$

Since $\omega_0 = \pi$

$B_k = \frac{2A}{k\pi}[0 - \cos k\pi] = \frac{-2A}{k\pi}(-1)^k$

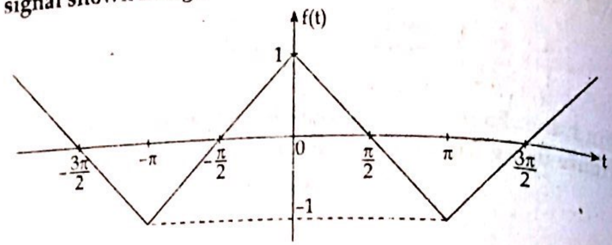
$= \frac{-2A}{k\pi}$; if k is even

$= \frac{2A}{k\pi}$; if k is odd

Hence, $x(t) = \sum_{k=1}^{\infty} B_k \sin k\omega_0 t$

$$= \frac{2A}{\pi}\left[\sin \pi t - \frac{1}{2}\sin 2\pi t + \frac{1}{3}3\pi t - \frac{1}{4}\sin 4\pi t + \dots\dots\right]$$

9. **Obtain the trigonometric Fourier series representation for the signal shown in figure.**



Solution:

$$f(t) = \frac{2t}{\pi} + 1\ ;\ -\pi < t < 0$$

$$= \frac{-2t}{\pi} + 1\ ;\ 0 < t < \pi$$

Since function f(t) is even function, hence, $B_k = 0$. Also, the average value of f(t) over an interval is zero, *i.e.,*

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,dt = 0$$

Now,

$$A_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos k\omega_0 t\,dt$$

$$T = 2\pi$$

$$\omega_0 = \frac{2\pi}{2\pi} = 1$$

$$A_k = \frac{4}{T}\int_{-\frac{T}{2}}^{0} f(t)\cos k\omega_0 t\,dt$$

or, $$A_k = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos k\omega_0 t\,dt = \frac{4}{2\pi}\int_{-\pi}^{0}\left(1 + \frac{2t}{\pi}\right)\cos kt\,dt$$

$$= \frac{4}{2\pi}\left[\frac{\sin kt}{k}\bigg|_{-\pi}^{0} + \frac{2}{\pi}\left\{\frac{t\sin kt}{k}\bigg|_{-\pi}^{0} - \frac{1}{k}\int_{-\pi}^{0}\sin kt\,dt\right\}\right]$$

$$= \frac{2}{\pi}\left[0 + 0 - \frac{2}{k\pi}\left\{\frac{\cos kt}{k}\bigg|_{-\pi}^{0}\right\}\right]$$

$\therefore$ $$A_k = \frac{4}{k^2\pi^2}[1 - (-1)^k]$$

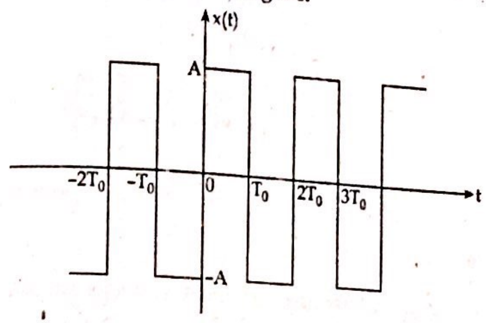
Hence, $A_k = 0$; if k is even, *i.e.,* 2, 4, 6, ...

$$= \frac{8}{k^2\pi^2}\ ;\ \text{if k is odd, i.e., } 1, 3, 5, 7, \dots\dots$$

$$f(t) = \sum_{k=1}^{\infty}\frac{8}{k^2\pi^2}\cos kt\ ;\ \text{k is odd}$$

$$= \frac{8}{\pi^2}\left[\cos t + \frac{1}{9}\cos 3t + \frac{1}{25}\cos 5t + \dots\dots\right]$$

10. **Obtain the trigonometric Fourier series representation of the periodic signal as shown in figure.**



Solution:

$$x(t) = -A \text{ for } -T_0 < t < 0$$

$$= A \text{ for } 0 < t < T_0$$

Time period, $T = 2\,T_0$

$$\omega_0 = \frac{2\pi}{2T_0} = \frac{\pi}{T_0}$$

Since the function is an odd function, hence $A_k = 0$ and $a_0 = 0$.

$$B_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,\text{sin}k\,\omega_0 t\,dt$$

$$= \frac{4}{T}\int_{-\frac{T}{2}}^{0} x(t)\sin k\,\omega_0 t\,dt$$

$$= \frac{2}{2T_0}\int_{-T_0}^{T_0} x(t)\sin\frac{\pi}{T_0}t\,dt$$

$$= \frac{1}{T_0}\left[\int_{-T_0}^{0} - A\,\text{sink}\frac{\pi}{T_0}t\,dt + \int_0^{T_2} A\,\text{sink}\frac{\pi}{T_0}t\,dt\right]$$

$$= \frac{2}{T_0}\left[A\int_0^{T_0}\text{sink}\frac{\pi}{T_0}t\,dt\right]$$

$$= \frac{2A}{T_0}\ -\frac{\cos k\frac{\pi}{T_0}t}{\frac{k\pi}{T_0}}\Bigg|_0^{T_0}$$

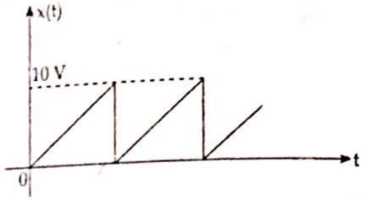
$\therefore \quad B_k = \frac{2A}{k\pi}\left\{-[\cos k\pi - \cos_0]\right\} = \frac{2A}{k\pi}\{1-(-1)^k\}$

Hence, $B_k = 0$; if k is even

$= \frac{4A}{k\pi}$; if k is odd

$\therefore \quad x(t) = \frac{4A}{\pi}\left[\sin\frac{\pi}{T_0}t + \frac{1}{3}\sin\frac{3\pi}{T_0}t + \frac{1}{5}\sin\frac{5\pi}{T_0}t + \ldots\ldots\right]$

11. **Find the trigonometric Fourier series for the waveform shown figure.**



Solution:

$x(t) = 10\,t \quad ; 0 < t < 1$

$T = 1;\ \omega_0 = 2\pi$

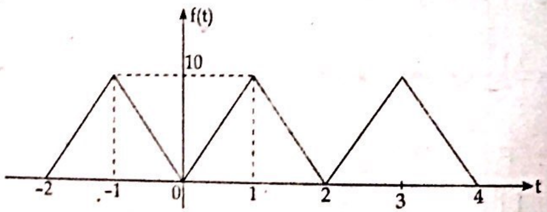
$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \int_0^1 10\,t\,dt = 10\left.\frac{t^2}{2}\right|_0^1 = 5$

$A_k = \frac{2}{1}\int_0^2 10\,t \cdot \cos 2\pi kt\,dt = 0$

$B_k = \frac{2}{1}\ 10\,t \cdot \sin 2\pi kt\,dt = \frac{-10}{k\pi}$

Hence, $x(t) = 5 - \frac{10}{\pi}\sum_{k=1}^{\infty} \sin 2\pi kt$

12. **Find the trigonometric Fourier series for the waveform shown in figure.**



Solution:

Since given function is an even function.

Hence, $B_k = 0$

$A_k \quad = \frac{20}{k^2\pi^2}[(-1)^k - 1] \leftarrow \frac{2}{T}\int_T x(t)\cdot\cos\omega t\,dt$

$= \frac{4}{T}\int_{\frac{T}{2}} x(t)\cos\omega t\,dt = \frac{4}{2}\int_0^1 10\,t\cos\pi kt\,dt$

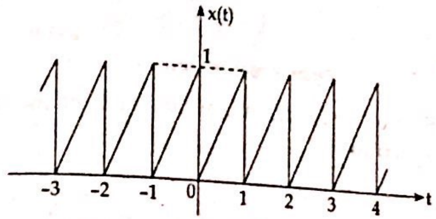
$= 20\int_0^1 t\cos\pi kt\,dt = 0$; k is even

$= \frac{-40}{k^2\pi^2}$; k is odd

$a_0 = 5$

$\therefore \quad x(t) = 5 - \frac{-40}{\pi^2}\sum_{k=1}^{\infty}\left[\frac{\cos k\pi t}{k^2}\right]$; k is odd $= 5\frac{-40}{\pi^2}\sum_{k=1}^{\infty}\cos\left[\frac{(2k-1)\pi t}{(2k-1)^2}\right]$

13. **Calculate the Fourier series coefficients of the continuous time periodic signal shown in figure.**



Solution:

Over one period, the signal x(t) may be expressed as,

$x(t) = t$ for $0 < t < 1$

$T = 1,\ \omega_0 = \frac{2\pi}{1} = 2\pi$

$a_k = \frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\,dt = \int_0^1 te^{-jk2\pi t}\,dt = \left.\frac{t^2}{2}\right|_0 = \frac{-1}{j2\pi k}$; $k \neq 0$

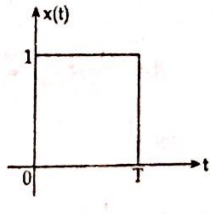
$a_0 = \frac{1}{T}\int_T x(t)\,dt = \int_0^1 t\,dt$

For frequency spectrum,

Amplitude, $|a_k| = \sqrt{0 + \left(\frac{1}{2\pi k}\right)^2} = \frac{1}{2\pi k}$

$\angle a_k = \tan^{-1}\left(\frac{\frac{1}{2\pi k}}{0}\right) = 90°$; $k > 0 = -90°$; $k < 0$

14. **Find the Fourier transform of the rectangular pulse shown in figure.**



Solution:

$$X(j\omega) = T e^{\frac{-j\omega T}{2}} \text{since}\left(\frac{\omega T}{2}\right)$$

Amplitude spectrum, $\angle X(j\omega) = -\left(\frac{\omega T}{2}\right)$

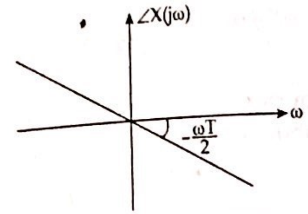


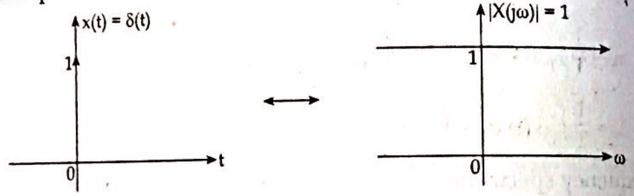
Figure: Amplitude spectrum

15. Find the Fourier transform of an impulse function $x(t) = \delta(t)$. Also draw the spectrum.

Solution:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = e^{-j\omega t}\big|_{t=0} = 1$$

Hence the Fourier transform of an unit impulse function is unity. From figure, it is clear that an unit impulse contains the entire frequency components having identical magnitude. This means that the bandwidth of unit impulse function is infinite.



Also, since the $X(j\omega) = 1$ is real, only magnitude spectrum is required. The phase spectrum $\theta(\omega)$ or $\angle X(j\omega) = 0$ which means that all frequency components are in the same phase.

16. Find the inverse Fourier transform of $\delta(\omega)$.

Solution:

The inverse Fourier transform is expressed as,

$$F^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\therefore \quad F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} e^{j\omega t}\big|_{\omega=0} = \frac{1}{2\pi}$$

Hence, $F\left[\frac{1}{2\pi}\right] = \delta(\omega)$ or $\frac{1}{2\pi} \leftrightarrow \delta(\omega)$

or, $1 \leftarrow 2\pi\, \delta(\omega)$

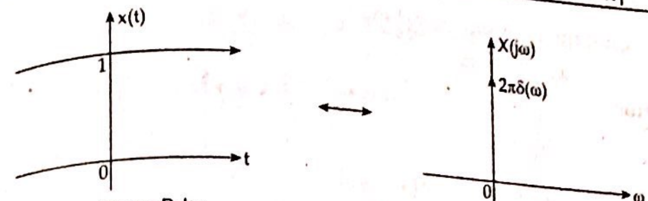


Figure: Pulse

Figure: Amplitude spectrum

17. Find the Fourier transform of the sinusoidal $\cos \omega_0 t$.

Solution:

We know,

$$e^{j\theta} = \cos\theta + j\sin\theta$$

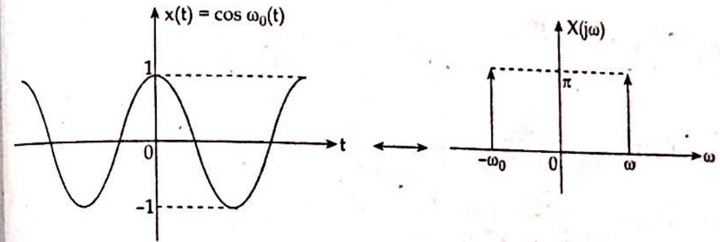
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Hence, $\cos\theta = \frac{1}{2}[e^{j\theta} + e^{-j\theta}]$

or, $\cos\omega_0 t = \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}]$

or, $F[\cos(\omega_0 t)] = \frac{1}{2}[2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)]$

$$= \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$



18. Find the Fourier transform of $x(t) = \sin \omega_0 t$.

Solution:

The Fourier series coefficients for this signal are,

$$a_1 = \frac{1}{2j}$$

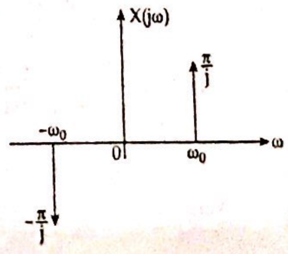
$$a_{-1} = \frac{-1}{2j}$$

$$a_k = 0;\; k \neq 1 \text{ or } -1$$

Thus, $X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\omega_0)$

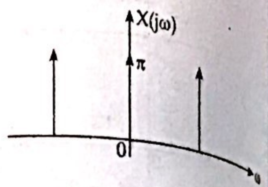
$$= 2\pi\left[\frac{-1}{2j}\delta(\omega + \omega_0) + \frac{1}{2j}\delta(\omega - \omega_0)\right]$$

$$= \frac{\pi}{j}[-\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$



19. **Find the Fourier transform of**

$$x(t) = \cos \omega_0 t$$

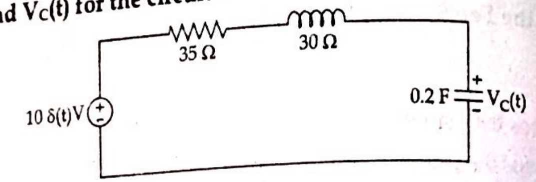


Solution:

$$a_1 = a_{-1} = \frac{1}{2}$$

Hence, $X(j\omega) = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

20. **Find $V_C(t)$ for the circuit shown in figure.**



Solution:

Let i(t) be the current in the circuit, then applying KVL,

$$10\,\delta(t) = 35\,i(t) + 30\frac{di(t)}{dt} + \frac{1}{0.2}\int_{-\infty}^{t} i(t)\,dt$$

Taking Fourier transform, we have,

$$10 = 35\,I(j\omega) + 30\,I(j\omega) + 5\left[\frac{I(j\omega)}{j\omega} + \pi\,I(0)\,\delta(\omega)\right]$$

or, $$I(j\omega) = \frac{10}{35 + 30j\omega + \frac{5}{j\omega}}$$

or, $$I(j\omega) = \frac{2j\omega}{30(j\omega)^2 + 35\,j\omega + 5}$$

or, $$I(j\omega) = \frac{2j\omega}{6(j\omega)^2 + 7j\omega + 1}$$

or, $$I(j\omega) = \frac{2j\omega}{(6j\omega + 1) + (j\omega + 1)}$$

or, $$V_C(j\omega) = \frac{1}{0.2j\omega} \cdot I(j\omega) = \frac{10}{(6j\omega + 1)(j\omega + 1)}$$

or, $$V_C(j\omega) = \frac{\frac{5}{3}}{\left(j\omega + \frac{1}{6}\right)(j\omega + 1)}$$

or, $$V_C(j\omega) = \frac{2}{j\omega + \frac{1}{6}} - \frac{2}{j\omega + 1}$$

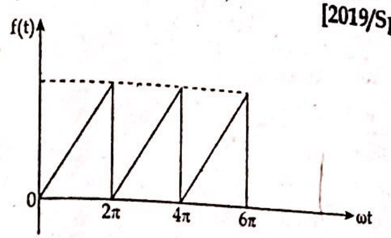
Taking inverse Fourier transform, we get,

$$v_C(t) = 2\,(e^{-t/8} - e^{-t})\,U(t)\text{ V}$$

BOARD EXAMINATION SOLVED QUESTIONS

1. **Find the exponential Fourier series for the signal below.**

[2019/Spring, 2017/Fall]



Solution:

Given the period is;

$\Rightarrow$ $\omega t = 2\pi$

or, $\frac{2\pi}{T}t = 2\pi$

$\therefore$ $t = T$ [$\because$ T is the period]

Let, y = f(t), then using slope equation,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $$y - 0 = \frac{5 - 0}{T - 0}(t - 0)$$

or, $$y = \frac{5}{T}t$$

We know,

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

a_k = Fourier series coefficients.

ω_0 = Fundamental frequency = $\frac{2\pi}{T}$

We can also write,

$$f(t) \doteq a_0 + \sum_{k=1}^{\infty} [a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}]$$

where, a_k^* is complex conjugate of a_k.

Here, $$a_k = \frac{1}{T}\int_0 f(t)\,e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T \frac{5}{T} t e^{-jk\omega_0 t}\,dt$$

$$= \frac{5}{T^2}\int_0^T t e^{-jk\omega_0 t}\,dt$$

$$= \frac{5}{T^2}\left[t\int_0^T te^{-jk\omega_0 t}\,dt - \int_0^T \frac{dt}{dt}\int e^{-jk\omega_0 t}\,dt\right]$$

$$= \frac{5}{T^2}\left[\left(t\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right)_0^T - \int_0^T \frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]$$

$$= \frac{5}{T^2}\left[\frac{Te^{-jk\omega_0 T}}{-jk\omega_0} - \left(\frac{e^{-jk\omega_0 T}}{(-jk\omega_0)^2}\right)_0^T\right]$$

$$= \frac{5}{T^2}\left[\frac{Te^{-jk\omega_0 T}}{-jk\omega_0} + \frac{e^{-jk\omega_0 T}}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}\right]$$

$$= \frac{5}{T^2}\left[\frac{Te^{-jk\omega_0 T}}{-jk\omega_0} + \frac{e^{jk\omega_0 T}}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}\right]$$

We know, $\omega_0 = \frac{2\pi}{T}$, so,

$$a_k = \frac{5}{T^2}\left[\frac{Te^{-jk2\pi}}{-jk\frac{2\pi}{T}} + \frac{e^{-jk2\pi}}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}\right]$$

Since, k is integer, we have, $e^{-jk2\pi} = 1$. Thus,

$$a_k = \frac{5}{T^2}\left[\frac{T^2}{-jk2\pi} + \frac{1}{k^2\omega_0^2} - \frac{1}{k^2\omega_0^2}\right]$$

$$= \frac{5}{-j2\pi k} = \frac{5j}{2\pi k}$$

We know,

$$a_0 = \frac{1}{T}\int_0 f(t)\,dt = \frac{1}{T}\int_0^T \frac{5}{T}t\,dt = \frac{5}{T^2}\left[\frac{T^2}{2}\right]_0^T = \frac{5}{2}$$

Hence,

$$a_0 = \frac{5}{2}$$

$$a_1 = \frac{5j}{2\pi} = 0.796\angle 90°$$

$$a_{-1} = -\frac{5j}{2\pi} = 0.796\angle -90°$$

$$a_2 = \frac{5j}{4\pi} = 0.398\angle 90°$$

$$a_{-2} = -\frac{5j}{4\pi} = 0.398\angle -90°$$

and so on.

$$\text{Thus, } a_k = \begin{Bmatrix} \frac{5}{2}, & k = 0 \\ \frac{5j}{2\pi k}, & k \neq 0 \end{Bmatrix}, \text{ k is an integer.}$$

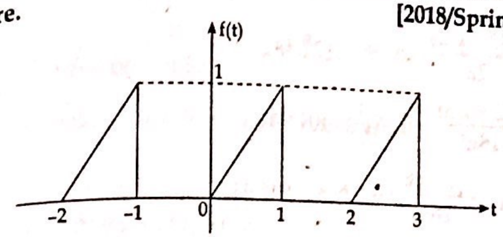
Hence, Fourier series representation is,

$$f(t) = \frac{5}{2} + \sum_{k=-\infty}^{\infty} \frac{5j}{2\pi k} e^{-jk\omega_0 t}$$

$$= -\frac{5j}{6\pi}e^{-j3\omega_0 t} - \frac{5j}{4\pi}e^{-j2\omega_0 t} + \frac{5j}{2\pi}e^{-j\omega_0 t} + \frac{5}{2} + \frac{5}{2} + \frac{5j}{2\pi}e^{-j\omega_0 t} + \frac{5j}{4\pi}e^{-j2\omega_0 t}$$

$$+ \frac{5}{6\pi}e^{-j3\omega_0 t} + ...$$

2. **Obtain the Fourier series for the periodic function shown in the figure.** **[2018/Spring, 2020/Fall]**



Solution:

Exponential Fourier series representation:

Here; the period is 2 – 0 = 2

$$f(t) = \frac{1-0}{1-0} \times t$$

$$\therefore \quad f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

We know,

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \text{ } a_k \text{ is Fourier series coefficients.}$$

where, $a_k = \frac{1}{T}\int_0 f(t)\,e^{-jk\omega_0 t}\,dt$, $\omega_0 = \frac{2\pi}{T}$ = Fundamental frequency

$$= \frac{1}{2}\int_0^2 f(t)\,e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{2}\int_0^1 te^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{2}\left[\left(\frac{te^{-jk\omega_0 t}}{-jk\omega_0}\right)_0^1 - \int_0^1 \frac{1 \times e^{-jk\omega_0 t}}{-jk\omega_0}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0} - \left(\frac{e^{-jk\omega_0 t}}{(-jk\omega_0)^2}\right)_0^1\right]$$

$$= \frac{1}{2}\left[\frac{e^{-jk\omega_0}}{-jk\omega_0} + \frac{e^{-jk\omega_0}}{k^2\omega_0^2} - \frac{1}{k^2\omega_0}\right]$$

We have,

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

The signal is uniquely defined in time domain as well as in frequency domain. The spectrum $X(j\omega)$ is continuous in frequency unlike Fourier series representation of periodic signal where the spectrum is discrete.

For a real valued function x(t), its conjugate $x^*(t) = x(t)$

$$x^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt \qquad \text{[since } x^*(t) = x(t)]$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega t)}\, dt$$

$$= X(-j\omega)$$

Thus it follow that for real valued function x(t).

$$|X(-j\omega)| = |X(j\omega)|$$

and, $\text{Arg}\,[X(-j\omega)] = -\text{Arg}[X(j\omega)]$

Therefore,

i) Amplitude spectrum of a real valued signal x(t) is an even function.

ii) Phase spectrum of a real valued signal x(t) is an odd function.

Combining the above two statements, it can be stated that spectrum of a real valued signal is conjugate symmetric.

6.5 EXISTENCE OF FOURIER TRANSFORM

A function x(t) is said to be Fourier transformable, if it is satisfied following Dirichlet conditions.

a) The function x(t) is absolutely integrable *i.e.,* $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

b) The function has finite number of maxima and minima, if any, within a finite interval of time.

c) The function is single valued and has finite number of discontinuities, if any, within a finite interval of time.

6.6 FOURIER TRANSFORM OF PERIODIC SIGNALS

We can also develop Fourier transform representations for periodic signals, directly from its Fourier series representation. The resulting transform consists of a train of impulses in the frequency domain.

To suggest the general result, let us consider a signal x(t) with fourier transform $X(j\omega)$ that it is a single impulse of area 2π at $\omega = \omega_0$ *i.e.,*

$$X(j\omega) = 2\pi\, \delta(\omega - \omega_0)$$

To determine the signal x(t) for which this is the fourier transform, we can apply the inverse transform relation to obtain,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0)\, e^{j\omega t}\, d\omega = e^{j\omega_0 t}$$

More generally if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency *i.e.,*

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\omega_0)$$

Hence, $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{-jk\omega_0 t}$

We see that the above equation corresponds exactly to the Fourier series representation of a periodic signal with Fourier series coefficients a_k.

6.7 TABLE OF FOURIER TRANSFORM PAIRS

S.N.	x(t)	X(jω)
1.	$e^{-at}u(t)$	$\frac{1}{a + j\omega}$
2.	$e^{-at}u(-t)$	$\frac{1}{a - j\omega}$
3.	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2 + \omega^2}$
4.	$t\, e^{-at}u(t)$	$\frac{1}{(a + j\omega)^2}$
5.	$\delta(t)$	1
6.	$\delta(t \pm t_0)$	$e^{\pm j\omega t_0}$
7.	1	$2\pi\delta(\omega)$
8.	$e^{j\omega_0 t}$	$2\pi\delta(\omega_0)$
9.	$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
10.	$\sin \omega_0 t$	$j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
11.	u(t)	$\pi\delta(\omega) + \frac{1}{j\omega}$
12.	$e^{-at}\cos \omega_0 t\, u(t)$	$\frac{a + j\omega}{(a + j\omega)^2 + \omega_0^2}$
13.	$e^{-at}\sin \omega_0 t\, u(t)$	$\frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}$
14.	$\text{rect}\left(\frac{t}{T}\right)$	$T\, \text{sin c}\left(\frac{\omega T}{2}\right)$
15.	$\Delta\left(\frac{t}{T}\right)$	$\frac{T}{2}\, \text{sin c}^2\left(\frac{\omega T}{4}\right)$
16.	$\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$\omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$

Here all $a > 0$ and $\omega_0 \neq \frac{2\pi}{T}$; where T-fundamental period of the x(t).

nature, *i.e.*, appear only at the discrete in nature *i.e.*, appear only at the discrete frequencies. They are therefore referred to as discrete frequency spectra.

For a real valued periodic signal x(t)

$a_{-k} = a_k^*$ Thus,

$|a_{-k}| = |a_k|$ and $\angle a_{-k} = -\angle a_k$

$\therefore$

i) Magnitude spectrum of a real valued period signal x(t) is an even function of ω or k.

ii) Phase spectrum of a real valued periodic signal x(t) is an odd function of ω or k.

Combining the above two statements, it can be stated that spectra of a real valued periodic signal x(t) is conjugate symmetric.

6.3 EFFECTS OF SYMMETRY

The Fourier series of only even periodic function f(t) consists of cosine terms only and the Fourier series for any odd periodic function f(t) consists of sine terms only.

$$a_0 = \frac{1}{T}\int_T f(t)\, dt$$

$$A_k = \frac{2}{T}\int_T (t) \cos k\omega_0 t\, dt$$

$$B_k = \frac{2}{T}\int_T f(t) \sin k\omega_0 t\, dt$$

Recall $\cos k\omega_0 t$ is an even function while $\sin k\omega_0 t$ is an odd function.

a) If f(t) is an even function of t_1 then f(t). $\cos k\omega_0 t$ is also an even function and f(t). $\sin k\omega_0 t$ is an odd function of t.

Hence, $B_k = 0$

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)\, dt$$

$$A_0 = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\, dt \sin k\omega_0 t\, dt$$

6.4 FOURIER TRANSFORM

A non-periodic signal may be assumed as a limiting case of a periodic signal where the period of the signal approaches infinity. We can use this approach to develop the frequency domain representation of a non periodic signal over an entire interval.

For a continuous-time periodic signal

$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t} \quad (A)$$

where, $\omega_0 = \frac{2\pi}{T} \rightarrow$ Angular frequency (fundamental)

$T \rightarrow$ Time period (fundamental)

and, $a_k = \frac{1}{T}\int_T x_p(t) \cdot e^{-jk\omega_0 t}\, dt$

Let us define $X(jk\,\omega_0) = \frac{2\pi}{\omega_0} \cdot a_k = \int_T x_p(t)\, e^{-jk\omega_0 t}\, dt$ (B)

or, $a_k = \frac{\omega_0}{\pi} X(jk\omega_0 t)$

Hence, from equation (A), we get,

$$x_p(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0 \quad (C)$$

A non-periodic continuous time signal x(t) can be viewed as periodic signal $x_p(t)$ with time period $T \rightarrow \infty$ and frequency $\omega_0 \rightarrow 0$.

As $T \rightarrow \infty$, $x_p(t) \rightarrow x(t)$

Also, $k\omega_0 \rightarrow \omega$ (continuous variable)

$\omega_0 \rightarrow d\omega$ (differential variable)

From equation (B), we get,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad (1)$$

and, from equation (C),

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \quad (2)$$

Equation (1) is known as Fourier transform or Fourier integral of non-periodic signal x(t) and equation (2) is known as inverse Fourier transform of $X(j\omega)$. Mathematically, .

$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and, $x(t) = F^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

Therefore, $X(j\omega)$ is the frequency domain representation of time domain function x(t). This means that we are converting a time domain function into its frequency domain representation with the help of Fourier transform, and vice-versa.

Hence, (t) and $X(j\omega)$ are a Fourier transform pair. Symbolically, this maybe expressed as $x(t) \leftrightarrow X(j\omega)$.

Here, double ended arrow means that Fourier transform of x(t) is $X(j\omega)$ and inverse Fourier transform of $X(j\omega)$ is x(t).

In general, Fourier transform $X(j\omega)$ is complex function of ω and may be expressed as,

$$X(j\omega) = |X(j\omega)| \cdot e^{j\theta(\omega)}$$

where, $|X(j\omega)|$ = Amplitude spectrum (plot of $|X(j\omega)|$ versus ω)

Arg $[X(j\omega)] = \theta(\omega)$ = Phase spectrum (plot of Arg $(X(j\omega)]$ versus ω.

Thus the linear combination of harmonically related compels exponentials of the form,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{-2\pi t}{T}} \quad \text{(A)}$$

$k = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$ is also periodic with period T.

The terms for $k = \pm 1 \rightarrow$ both have fundamental frequency equal to ω_0 and referred to as the fundamental components or first harmonic components. The terms $k = \pm 2$ are periodic with half the period or (equivalent, twice the frequency) of the fundamental components and are referred to as the second harmonic components. Therefore generally, the components for $k = \pm N$ are referred to as the N^{th} harmonic components.

6.2 FOURIER SERIES

6.2.1 Fourier Series Representation

The representation of a periodic signal in the form of equation (A) is referred to as the Fourier series representation. There are some alternative forms of Fourier series of real periodic signals. Suppose that x(t) is real and can be represented in the form of equation (A). Then,

$$x(t) = x^*(t) \quad \text{[Since x(t) is real]}$$

$$= \sum_{k=-\infty}^{\infty} a_k^* e^{jk\omega_0 t}$$

Replacing k by –k in the summation, we have,

$$x(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

Which by comparison with equation (A), we get,

$$a_k = a_{-k}^*$$

or, $$a_k^* = a_{-k} \quad (1)$$

From equation (A), we get,

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[e_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

Replacing a_k^* for a_{-k} from equation (1), we obtain,

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[e_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right]$$

Since the two terms inside the summation are complex conjugates of each other. Hence,

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\text{Re}\left\{a_k e^{jk\omega_0 t}\right\}$$

If a_k is expressed in polar form as,

$$a_k = A_k' e^{j\theta_k}$$

Then, $$x(t) = a_0 + \sum_{k=1}^{\infty} 2\text{Re}\left(A_k' e^{j\theta_k} \cdot e^{jk\omega_0 t}\right)$$

i.e., $$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k' \cos(k\omega_0 t + \theta k) \quad (2)$$

Another form is obtained by writing a_k in rectangular form as,

$a_k = B_k' + jC_k' \cos$ (where B_k' and C_k' are both real).

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\text{Re}\,(B_k' + jC_k')\, e^{jk\omega_0 t})$$

i.e., $$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k' \cos k\,\omega_0 t - C_k' \sin k\,\omega_0 t] \quad (3)$$

Thus, for real periodic functions, the Fourier series in terms of complex exponentials as given in equation (1), is mathematically equivalent to either of the two forms in equation (2) and (3) that are trigonometric functions.

6.2.2 Determination of the Fourier Coefficients

a) For complex exponential functions

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} x(t)\, e^{-jk\omega_0 t}\, dt$$

The coefficient a_0 is the dc or constant component of x(t) and is given by above equation with k = 0 *i.e.,*

$$a_0 = \frac{1}{T}\int_T x(t)\,(t)\,dt$$

(Which is simply the average value of x(t) over one period)

b) For trigonometric functions

Equation (3) is identical to the equation,

$$x(t) = a_0 + \sum_{k=1}^{\infty} [A_k \cos k\,\omega_0 t + B_k \sin k\,\omega_0 t]$$

(where, $A_k = B_k'$ and $B_k = -C_k'$; another constant)

$$A_k = \frac{2}{T}\int_T x(t) \cos k\,\omega_0 t$$

$$B_k = \frac{2}{T}\int_T x(t) \sin k\,\omega_0 t$$

and, $a_0 = \frac{1}{T}\int_T x(t)\,dt$; same as in previous case

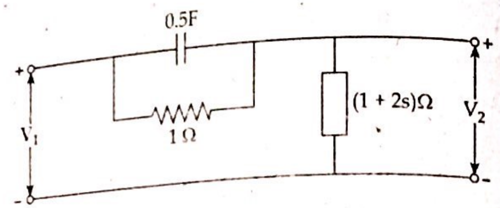
6.2.3 Magnitude and Phase Spectra of a Periodic Signal

Let the complex Fourier coefficients a_k be expressed as,

$$a_k = |a_k|\, e^{j\theta_k}$$

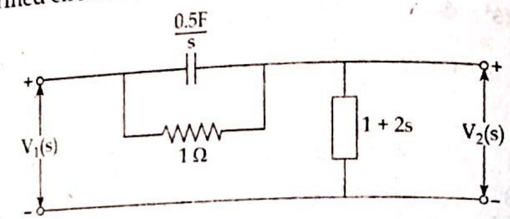
where, $|a_k|$ is the magnitude and θ_k is the phase of k^{th} harmonic component. A plot of $|a_k|$ versus the angular frequency ω(or k) is called the magnitude spectrum of the periodic signal x(t), and a plot of θ_k of $\angle a_k$ versus ω or k is called the phase spectrum of x(t). Since here k assumes only integers, the magnitude and phase spectra are not continuous curves but discrete in

23. **Find the driving point impedance $Z_{11}(s)$ and voltage transfer function $G_{21}(s)$ in laplace domain in the circuit given below.** [2020/Fall]



Solution:

The transformed circuit is,



$$Z_1(s) = \frac{\frac{0.5}{s} \times 1}{\frac{0.5}{s} + 1} = \frac{0.5}{s + 0.5} = \frac{1}{1 + 2s}$$

$$Z_2(s) = 1 + 2s$$

Now,

$$V_1(s) = I_1(s) \times [Z_1(s) + Z_2(s)]$$

$$= \left(\frac{1}{1+2s} + 1 + 2s\right) I_1(s)$$

$$= \frac{(4s^2 + 4s + 2)}{1 + 2s} I_1(s) = \frac{2\,(2s^2 + 2s + 1)}{1 + 2s} I_1(s)$$

$$V_2(s) = Z_2(s) \times I_1(s) = (1 + 2s)\, I_1(s)$$

So, driving point impedance,

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = \frac{\frac{2(2s^2 + 2s + 1)}{(1 + 2s)}}{I_1(s)} \times I_1(s) = \frac{2(2s^2 + 2s + 1)}{(1 + 2s)}$$

and, Voltage transfer function,

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{(1 + 2s)}{\frac{2(2s^2 + 2s + 1)}{(1 + 2s)}} I_1(s) = \frac{4s^2 + 4s + 1}{4s^2 + 4s + 2}$$

✡✡✡

Chapter 6

FOURIER SERIES TRANSFORM

6.1 BASIC CONCEPT OF FOURIER SERIES AND ANALYSIS INTRODUCTION

A signal is periodic if, for some positive value of T. $x(t) = x(t + T)$ for all t.

Fundamental or time period (T)

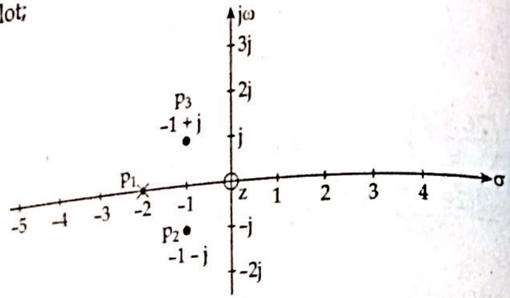
Minimum positive, non-zero value of T, for which above equation is satisfied. Fundamental frequency $(\omega_0) = \frac{2\pi}{T}$

As we know, there are two basic periodic signals:

a) The sinusoidal signal; $x(t) = \cos \omega_0 t$ or $\sin \omega_0 t$
b) The complex exponential; $x(t) = e^{j\omega_0 t}$

Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $(T) = \frac{2\pi}{\omega_0}$.

Pole-zero plot;



Here;

$$T(s) = \frac{3s}{(s+2)(s+1+j)(s+1-j)} = \frac{A}{s+2} + \frac{B}{s+1+j} + \frac{C}{s-1-j}$$

For the pole, $p_1 = -2$

$$\therefore \quad A = 3\frac{(p_1 - z)}{(p_1 - p_2)(p_1 - p_3)}$$

$$= 3\frac{(-2-0)}{(-2+1+j)(-2+1-j)} = -3 \times \frac{-2}{(-1+j)(-1-j)}$$

$$= -3 \times \frac{-2}{(-1)^2 - (j)^2} - \frac{-6}{-1-(-1)}$$

For the pole, $p_2 = -1 - j$

$$\therefore \quad B = 3 \times \frac{(-1-j-0)}{(-1-j+2)(-1-j+1-j)} = 3\frac{(-1-j)}{(1-j)(-2j)} = -3 \times \frac{(1+j)}{-2(1+j)} = \frac{3}{2}$$

$$\therefore \quad C = B^{*} = \frac{3}{2}$$

Hence, $T(s) = \frac{A}{s+2} + \frac{B}{s+1+j} + \frac{C}{s+1-j}$

or, $T(s) = \frac{-3}{s+2} + \frac{3}{2(s+1+j)} + \frac{3}{2(s+1-j)}$

Taking laplace inverse,

$$T(t) = -3\,e^{-2t} + \frac{3}{2}e^{-(1+j)t} + \frac{3}{2}e^{-(1-j)t}$$

or, $T(t) = -3\,e^{-2t} + \frac{3}{2}e^{-t}(e^{-jt} + e^{jt})$

$$= -3\,e^{-2t} + \frac{3}{2}e^{-t}(2\cos t) \qquad \left[\because \cos t = \frac{e^{jt} + e^{-jt}}{2}\right]$$

$\therefore \quad T(t) = -3\,e^{-2t} + 3\,e^{-t}\cos t = -3\,e^{-t}(e^{-t} - \cos t)$

16. **Form the routh array for the following characteristic equation and find whether the system is stable or not.**
$s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 6s + 4 = 0$ **[2011/Spring]**

Solution:

Routh array is:

s^6	1	3	5	8	4
s^5	3	9	6		
s^4	2	6	4		
s^3	0	0			
s^2					
s^1					
s^0					

The elements of fourth row are zero, so required auxiliary equation,

$$A(s) = 2s^4 + 6s^2 + 4$$

or, $\frac{dA(s)}{ds} = 8s^3 + 12s$

Now,

s^6	1	5	8	4
s^5	3	9	6	
s^4	2	6	4	
s^3	8	12		
s^2	3	4		
s^1	$\frac{4}{3}$			
s^0	4			

Since there is no sign change in the first column. Hence, system is stable.

17. **Define transfer function.** **[2011/Fall, 2015/Fall, 2020/Fall]**
Solution: See the topic 5.1.

18. **What does the Routh criteria state?** **[2011/Spring, 2018/Fall]**
Solution: See the topic 5.6.

19. **Write short notes on poles and zero of network function.** **[2017/Fall]**
Solution: See the topic 5.4.

20. **Write short notes on time domain behaviour from pole zero location.** **[2017/Spring]**
Solution: See the topic 5.7.

21. **Write short notes on transfer function.** **[2018/Fall]**
Solution: See the topic 5.1.

22. **From the pole zero plot, obtain the time domain response of given function, $I(s) = \frac{s(s+1)}{(s+4)(s^2+6s+8)}$** **[2020/Fall]**
Solution: See the 'solved example' question number 12.

Solution:

Given that;

$Q(s) = s^4 + s^3 + s^2 s + K$

The Routh-Hurwitz table is;

s^4	1	1	K
s^3	1	1	0
s^2	0	K	
s^1			
s^0			

Since the element of first column, third row is 0, replacing 0 with ϵ,

s^4	1	1	K
s^3	1	1	0
s^2	ϵ	K	0
s^1	$\frac{\epsilon - K}{\epsilon}$	0	
s^0	K		

For system to be stable,

$\frac{\epsilon - K}{\epsilon} > 0$ and $K > 0$

or, $1 - \frac{K}{\epsilon} > 0$

or, $1 > \frac{K}{\epsilon}$

$\therefore$ $K < \epsilon$

If $K < 1$, the term $1 - \frac{K}{\epsilon}$ is positive and there is no change in signs

If $K > 1$, the term $1 - \frac{K}{\epsilon}$ is negative, so number of roots in right hand side is 2. Due to change in sign, system is unstable.

13. **Check the stability of the system whose characteristic equation is given by, $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$ using routh Hurwitz criteria.** [2013/Fall]

Solution:

$P(s) = s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5$

The R-H table is

s^5	1	2	3
s^4	1	2	5
s^3	0	-2	0
s^2			
s^1			
s			

Since element in the first column, third row is 0, replacing it with ϵ. Then,

s^5	1	2	3
s^4	1	2	5
s^3	ϵ	-2	0
s^2	$\left(\frac{2\epsilon + 2}{\epsilon}\right)$	5	0
s^1	$\frac{-2(2\epsilon + 2) - 5\epsilon^2}{2\epsilon + 2}$	0	
s^0	5		

For stability,

$\frac{2\epsilon + 2}{\epsilon} > 0$ and $\left(\frac{-4\epsilon - 4 - 5\epsilon^2}{2\epsilon + 2}\right) > 0$

or, $2 + \frac{2}{\epsilon} > 0$

or, $2 > \frac{-2}{\epsilon}$

$\therefore$ $\epsilon > -1$

As $\epsilon \to 0$, first term of fifth row has value -2, thus there are sign changes in the first column making system unstable.

14. **Check the stability of the system given below expressed in polynomial as $Q(s) = s^3 + 2s^2 + 2s + 40$ using routh Hurwitz criteria.** [2012/Fall]

Solution:

$Q(s) = s^3 + 2s^2 + 2s + 40$

The R-H table is

s^3	1	2
s^2	2	40
s^1	-18	0
s	40	

Since first column of R-H table consists the element with change in sign (2 to -18 and -18 to 40) the system is unstable.

15. **Plot the poles and zeros in s-plane and obtain T(t) for the transfer function of a network given by,**

$$T(s) = \frac{3s}{(s + 2)(s^2 + 2s + 2)}$$ [2012/Fall, 2013/Fall]

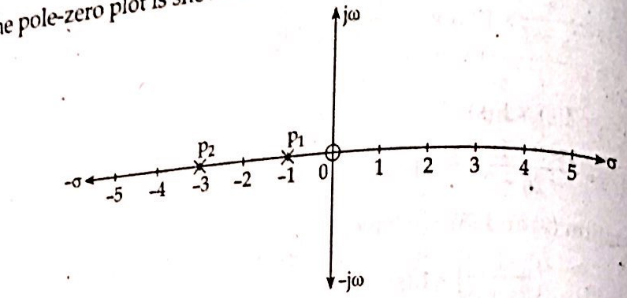
Solution:

$$T(s) = \frac{3s}{(s + 2)(s^2 + 2s + 2)} = \frac{3s}{(s + 2)(s + 1 + j)(s + 1 - j)}$$

Here; one zero is at $z = 0$

Poles are at; $p_1 = -2$, $p_2 = -1 - j$ and, $p_3 = -1 + j$

The pole-zero plot is shown below



Let, $I(s) = \frac{4s}{(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$

For the pole $p_1 = -1$,

$$A = 4 \times \frac{(p_1 - z)}{(p_1 - p_2)} = 4 \times \frac{(-1-0)}{[-1-(-3)]} = \frac{-4}{2} = -2$$

For the pole, $p_2 = -3$,

$$B = 4 \times \frac{(p_1 - z)}{(p_1 - p_2)} = 4 \times \frac{(-3-0)}{[-3-(-1)]} = \frac{-12}{-3+1} = 6$$

Thus, $I(s) = \frac{-2}{(s+1)} + \frac{6}{(s+3)}$

Taking inverse laplace transform, we get,

$$i(t) = -2\,e^{-t} + 6\,e^{-3t}\text{ A}$$

Hence time response of I(s) is $i(t) = -2\,e^{-t} + 6\,e^{-3t}$ A

10. **Plot the poles and zero in s-plane for the network function.**

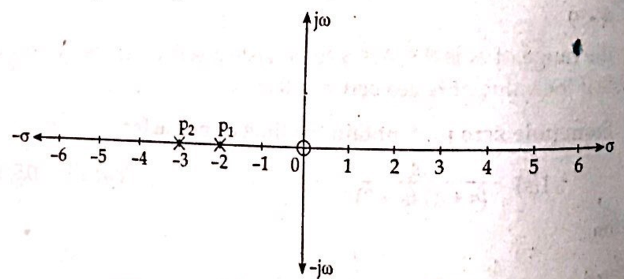
$$V(s) = \frac{10s}{(s+2)(s+3)}$$ **[2011/Spring, 2013/Spring, 2014/Spring]**

Solution:

$$V(s) = \frac{10s}{(s+2)(s+3)}$$

There is one zero at z = 0 and two poles at $p_1 = -2$ and $p_2 = -3$

The pole-zero plot on s-plane is:



Here; $V(s) = \frac{10s}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)}$

For the pole, $p_1 = -2$,

$$A = 10 \times \frac{(p_1 - z)}{(p_1 - p_2)} = 10 \times \frac{(-2-0)}{[-2-(-3)]} = \frac{-20}{1} = -20$$

For the pole, $p_2 = -3$,

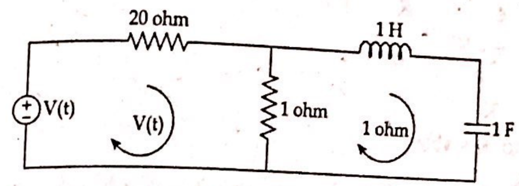
$$B = 10 \times \frac{(p_2 - z)}{(p_2 - p_1)} = 10 \times \frac{(-3-0)}{[-3-(-2)]} = \frac{-30}{-1} = 30$$

Thus, $V(s) = \frac{-20}{(s+2)} + \frac{30}{(s+3)}$

Taking inverse laplace transform, we get,

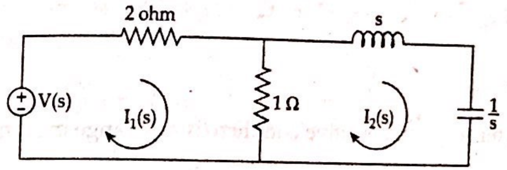
$$V(t) = -20\,e^{-2t} + 30\,e^{-3t}\text{ V}$$

11. **For the network shown below, determine the driving point impedance.** **[2015/Fall]**



Solution:

The transformed circuit is,



Diving point impedance, $Z_{11}(s) = \frac{V(s)}{I_1(s)}$

$Z_{11}(s)$ is also the equivalent impedance of circuit

$$\therefore \quad Z_{11}(s) = 2 + 1 \times \frac{\left(s + \frac{1}{s}\right)}{1 + s + \frac{1}{s}} = 2 + \frac{\frac{(s^2+1)}{s}}{\frac{(s^2+s+1)}{s}}$$

$$= 2 + \frac{s^2+1}{s^2+s+1} = \frac{2s^2+2s+2+s^2+1}{s^2+s+1}$$

$$= \frac{3s^2+2s+3}{s^2+s+1}$$

12. **Determine the range of K for which the system is stable using R-H criteria,** **[2012/Spring, 2013/Spring, 2014/Spring]**

$$Q(s) = s^4 + s^3 + s^2 + s + K$$

so, $a_k = \left[\frac{e^{-jk\pi}}{-jk\pi} + \frac{e^{-jk\pi}}{\pi^2k^2} - \frac{1}{\pi^2k^2}\right]$

$= \frac{1}{2}\left[\frac{-2}{\pi^2k^2} - \frac{j}{k\pi}\right] = \frac{1}{2\pi^2k^2}[-2 - jk\pi]$

$= -\frac{1}{2\pi^2k^2}[2 + jk\pi]$

Thus, $a_1 = \frac{-(2+j\pi)}{2\pi^2} = 0.189 \angle -122.482^\circ$

$a_{-1} = \frac{-(2+j3\pi)}{2\pi^2} = 0.189 \angle 122.482^\circ$

$a_3 = \frac{-(2+j3\pi)}{18\pi^2} = 0.054 \angle -101.981^\circ$

$a_{-3} = a_3^* = \frac{-(2-j3\pi)}{18\pi^2} = 0.054 \angle 101.981^\circ$

and so on.

If k is even, then $e^{-jk\pi} = 1$

$$a_{k\,even} = \frac{1}{2}\left[\frac{1}{-jk\pi} + \frac{1}{\pi^2k^2} - \frac{1}{\pi^2k^2}\right] = \frac{1}{2}\left[\frac{j}{k\pi}\right] = \frac{j}{2k\pi}$$

$\therefore$ $a_2 = \frac{j}{4\pi} = 0.079 \angle -90^\circ$

$a_{-2} = a_2^* = \frac{-j}{4\pi} = 0.079 \angle 90^\circ$

$a_4 = \frac{j}{8\pi} = 0.04 \angle 90^\circ$

$a_{-4} = a_4^* = \frac{-j}{8\pi} = 0.04 \angle -90^\circ$

and so on.

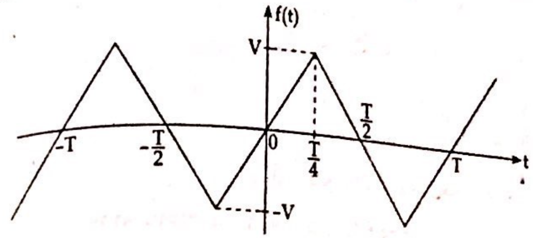
$$a_0 = \frac{1}{T}\int_T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)\,dt$$

$$= \frac{1}{2}\int_0^1 t\,dt = \frac{1}{2}\left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Thus,

$$a_k = \begin{cases} \frac{1}{4} & ; \quad k = 0 \\ -\frac{(2+j\pi k)}{2\pi^2k^2} & ; \quad k \text{ is odd} \\ -\frac{j}{2\pi k} & ; \quad k \text{ is even} \end{cases}$$

3. **For the following waveform, find the trigonometric Fourier series expansion.** **[2018/Fall]**



Solution:

From figure, the time period is T.

For ease of calculation, we take period from $-\frac{T}{4}$ to $\frac{3T}{4}$.

Let, $y = f(t)$

For $-\frac{T}{4}$ to $\frac{T}{4}$, the expression for f(t) is,

$$[f(t) - (-V)] = \frac{V - (-V)}{\frac{T}{4} - \left(-\frac{T}{4}\right)}\left[t - \left(-\frac{T}{4}\right)\right]$$

or, $f(t) + V = \frac{2V}{\frac{T}{2}}\left(t + \frac{T}{4}\right)$

$\therefore$ $f(t) + V = \frac{4V}{T}t$

For $\frac{T}{4}$ to $\frac{3T}{4}$, the expression for f(t) is,

$$[f(t) - V] = \frac{-V - V}{\frac{3T}{4} - \frac{T}{4}}\left(t - \frac{T}{4}\right)$$

or, $f(t) - V = \frac{-2V}{\frac{T}{2}}\left(t - \frac{T}{4}\right)$

or, $f(t) - V = \frac{-4Vt}{T} + V = 2V - \frac{4V}{T}t = 2V\left(t - \frac{2t}{T}\right)$

so, $f(t) = \begin{cases} \frac{4V}{T}t & ; \quad -\frac{T}{4} \le t \le \frac{T}{4} \\ 2V\left(1 - \frac{2t}{T}\right) & ; \quad -\frac{T}{4} \le t \le \frac{3T}{4} \end{cases}$

Trigonometric Fourier series expansion is given by,

$$f(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t$$

where, $A_k = \frac{2}{T}\int f(t)\cos k\omega_0 t\, dt$

$$B_k = \frac{2}{T}\int_T f(t)\sin k\omega_0 t\, dt$$

$$a_0 = \frac{1}{T}\int_T f(t)\, dt$$

$$\omega_0 = \frac{2\pi}{T} = \text{Fundamental frequency}$$

From figure, we know that f(t) is an odd function, so we can write, $a_0 = A_k = 0$. Hence,

$$B_k = \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{3T}{4}} f(t)\sin k\omega_0 t\, dt$$

$$= \frac{2}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}}\left(\frac{4Vt}{T}\right)\sin k\omega_0 t\, dt + \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}} 2V\left(1-\frac{2t}{T}\right)$$

$$= \frac{8V}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} t\sin k\omega_0 t\, dt + \frac{4V}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}}\left(1-\frac{2t}{T}\right)\sin k\omega_0 t\, dt$$

We know,

$t \times \sin k\omega_0 t$ is an even function. Thus, we can write,

$$B_k = \frac{8V}{T^2}\times 2\int_0^{\frac{T}{4}} t\sin k\omega_0 t\, dt + \frac{4V}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}}\left(1-\frac{2t}{T}\right)\sin k\omega_0 t\, dt \quad (1)$$

Calculating the two integrals of equation (1) separately,

$$\int t\sin k\omega_0 t\, dt = \left[\frac{t(-\cos k\omega_0 t)}{k\omega_0} - \int\frac{dt}{dt}\left(\frac{-\cos k\omega_0 t}{k\omega_0}\right)dt\right]$$

$$= \left[\frac{-t\cos k\omega_0 t}{k\omega_0} + \frac{\sin k\omega_0 t}{k^2\omega_0^2}\right]$$

Now, $\int_0^{\frac{T}{4}} t\sin k\omega_0 t\, dt = \left[\frac{-t\cos k\omega_0 t}{k\omega_0} + \frac{\sin k\omega_0 t}{k^2\omega_0}\right]_0^{\frac{T}{4}}$

$$= \left(\frac{\frac{-T}{4}\cos k\omega_0\frac{T}{4}}{k\omega_0} + \frac{\sin k\omega_0\frac{T}{4}}{k^2\omega_0^2}\right) - (0+0)$$

We know, $\omega_0 = \frac{2\pi}{T}$, so

$$= \frac{\left(\frac{-T}{4}\cos k\times\frac{2\pi}{T}\times\frac{T}{4}\right)}{k\times\frac{2\pi}{T}} + \left(\frac{\sin k\times\frac{2\pi}{T}\times\frac{T}{4}}{k^2\times\left(\frac{2\pi}{T}\right)^2}\right)$$

$$= \frac{-T^2}{8k\pi}\cos k\frac{\pi}{2} + \frac{T^2}{4\pi^2k^2}\sin k\frac{\pi}{2}$$

Similarly,

$$\int\left(1-\frac{2}{T}t\right)\sin k\omega_0 t\, dt$$

$$= \left[\left(1-\frac{2}{T}t\right)\left(-\frac{\cos k\omega_0 t}{k\omega_0}\right) - \int\frac{d}{dt}\left(1-\frac{2t}{T}\right)\left(-\frac{\cos k\omega_0 t}{k\omega_0}\right)dt\right]$$

$$= \left[\left(\frac{2t}{T}-1\right)\frac{\cos k\omega_0 t}{k\omega_0} - \frac{2}{Tk^2\omega_0^2}\sin k\omega_0 t\right]$$

Now,

$$\int_{\frac{T}{4}}^{\frac{3T}{4}}\left(1-\frac{2t}{T}\right)\sin k\omega_0 t\, dt = \left[\left(\frac{2t}{T}-1\right)\frac{\cos k\omega_0 t}{k\omega_0} - \frac{2}{Tk^2\omega_0^2}\sin k\omega_0 t\right]_{\frac{T}{4}}^{\frac{3T}{4}}$$

$$= \left[\left(\frac{2\times\frac{3T}{4}}{T}-1\right)\frac{\cos k\omega_0\frac{3T}{4}}{k\omega_0} - \frac{2}{Tk^2\omega_0^2}\sin k\omega_0\frac{3T}{4}\right.$$

$$\left.\left(\frac{2\times\frac{T}{4}}{T}-1\right)\frac{\cos k\omega_0\frac{T}{4}}{k\omega_0} - \frac{2}{Tk^2\omega_0^2}\sin k\omega_0\frac{T}{4}\right]$$

We know, $\omega_0 = \frac{2\pi}{T}$ so

$$= \left[\left(\frac{3}{2}-1\right)\frac{\cos k\times\frac{2\pi}{T}\times\frac{3T}{4}}{k\times\frac{2\pi}{T}} - \frac{2}{Tk^2\left(\frac{2\pi}{T}\right)^2}\sin k\times\frac{2\pi}{T}\times\frac{3T}{4}\right.$$

$$\left.-\left(\frac{1}{2}-1\right)\frac{\cos k\times\frac{2\pi}{T}\times\frac{T}{4}}{k\times\frac{2\pi}{T}} - \frac{2}{Tk^2\left(\frac{2\pi}{T}\right)^2}\sin k\times\frac{2\pi}{T}\times\frac{T}{4}\right]$$

$$= \frac{T}{4\pi k}\cos k\frac{3\pi}{2} - \frac{T}{2\pi^2k^2}\sin k\frac{3\pi}{2} + \frac{T}{4\pi k}\cos\frac{k\pi}{2} + \frac{T}{2\pi^2k^2}\sin\frac{k\pi}{2}$$

$$= \frac{T}{4\pi k}\left[\cos k\left(2\pi-\frac{\pi}{2}\right) + \cos k\frac{\pi}{2}\right]$$

$$- \frac{T}{2\pi^2k^2}\left[\sin k\left(2\pi-\frac{\pi}{2}\right) - \sin k\frac{\pi}{2}\right]$$

$$= \frac{T}{4\pi k}\left[\cos k\frac{\pi}{2} + \cos k\frac{\pi}{2}\right] - \frac{T}{2\pi^2k^2}\left[-\sin k\frac{\pi}{2} - \sin k\frac{\pi}{2}\right]$$

$$= \frac{T}{4\pi k}\cos k\frac{\pi}{2} + \frac{T}{\pi^2k^2}\sin k\frac{\pi}{2}$$

Replacing value of integrals in equation (1), we get,

$$\therefore \quad B_k = \frac{16V}{T^2} \times \left(\frac{-T^2}{8\pi k} \cos k\frac{\pi}{2} + \frac{T^2}{4\pi^2k^2} \sin k\frac{\pi}{2} \right) + \frac{4V}{T}\left(\frac{T}{2\pi k} \cos\frac{k\pi}{2} + \frac{T}{\pi^2k^2} \sin\frac{k\pi}{2} \right)$$

$$= \frac{-2V}{\pi k} \cos k\frac{\pi}{2} + \frac{4V}{\pi^2k^2} \sin k\frac{\pi}{2} + \frac{2V}{\pi k} \cos k\frac{\pi}{2} + \frac{4V}{\pi^2k^2} \sin k\frac{\pi}{2}$$

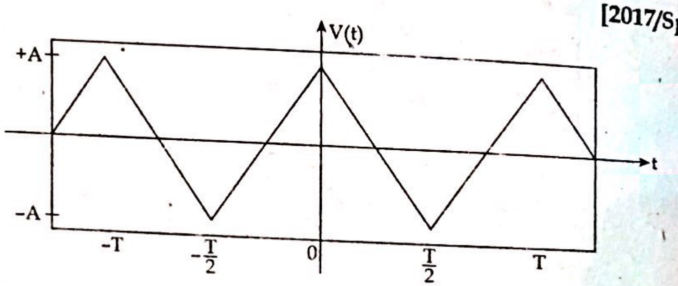
$$= \frac{8V}{\pi^2k^2} \sin k\frac{\pi}{2}$$

$$\therefore \quad B_k = \begin{cases} 0 & ; \ k \text{ is even} \\ \dfrac{8V}{\pi^2k^2} & ; \ k = 1, 5, 9, 13, \ldots\ldots \\ \dfrac{-8V}{\pi^2k^2} & ; \ k = 3, 7, 11, \ldots\ldots \end{cases}$$

$$\therefore \quad f(t) = \sum_{k=1}^{\infty} B_k \sin k\omega_0 t = \sum_{k=1}^{\infty} \frac{8V}{\pi^2k^2} \sin k\frac{\pi}{2} \sin k\omega_0 t$$

$$= \frac{8V}{\pi^2}\left[\sin \omega_0 t - \frac{1}{9} \sin 3\omega_0 t + \frac{1}{25} \sin 5\omega_0 t - \frac{1}{49} \sin \omega_0 t + \ldots\ldots \right]$$

4. **Obtain the Fourier series representation of the voltage waveform shown.** **[2017/Spring]**



Solution:

From figure, time period = T

For ease of calculation, we take time period from $-\frac{T}{2}$ to $\frac{T}{2}$.

For 0 to $\frac{T}{2}$, the expression for V (t) is:

$$\text{or,} \quad V(t) - A = \frac{-A - A}{\frac{T}{2} - 0}(t - 0)$$

$$\text{or,} \quad V(t) - A = -\frac{4A}{T} t$$

$$\therefore \quad V(t) = A\left(1 - \frac{4}{T}t\right)$$

We know, Fourier series representation is;

$$V(t) = a_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$$

From figure, we see V(t) is even function. So, $B_k = 0$

$$\text{and,} \quad A_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} V(t) \cos k\omega_0 t \, dt$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} A\left(1 - \frac{4t}{T}\right) \cos k\omega_0 t \, dt$$

$$= \frac{4A}{T}\left[\left(1 - \frac{4t}{T}\right)\left(\frac{\sin k\omega_0 t}{k\omega_0}\right) - \int \frac{d}{dt}\left(1 - \frac{4t}{T}\right) \times \frac{\sin k\omega_0 t}{k\omega_0} dt \right]$$

$$= \frac{4A}{T}\left[\left(1 - \frac{4t}{T}\right)\left(\frac{\sin k\omega_0 t}{k\omega_0}\right) - \left(-\frac{4}{T}\right) \times \left(\frac{-\cos k\omega_0 t}{k^2\omega_0^2}\right) \right]_0^{\frac{T}{2}}$$

$$= \frac{4A}{T}\left[\left(1 - \frac{4t}{T}\right)\left(\frac{\sin k\omega_0 t}{k\omega_0}\right) - \frac{4}{T}\frac{\cos k\omega_0 t}{k^2\omega_0^2} \right]_0^{\frac{T}{2}}$$

$$= \frac{4A}{T}\left[\left(\left(1 - \frac{4 \times \frac{T}{2}}{T}\right) \frac{\sin k\omega_0 \frac{T}{2}}{k\omega_0} - \frac{4}{T}\frac{\cos k\omega_0 \frac{T}{2}}{k^2\omega_0^2} \right) - \left(\left(1 - \frac{4 \times 0}{T}\right) \frac{\sin k\omega_0 \times 0}{k\omega_0} - \frac{4}{T}\frac{\cos k\omega_0 \times 0}{k^2\omega_0^2} \right) \right]$$

We know, $\omega_0 = \frac{2\pi}{T}$. so,

$$= \frac{4A}{T}\left[\left(\frac{(-1) \sin k\frac{2\pi}{T} \times \frac{T}{2}}{k \times \frac{2\pi}{T}} - \frac{4}{T}\frac{\cos k \times \frac{2\pi}{T} \times \frac{T}{2}}{k^2 \times \left(\frac{2\pi}{T}\right)^2} \right) - \left(0 - \frac{4}{T} \times \frac{1}{k^2 \times \left(\frac{2\pi}{T}\right)^2} \right) \right]$$

$$= \frac{4A}{T}\left[\frac{-T \sin k\pi}{2\pi k} - \frac{T \cos k\pi}{\pi^2k^2} + \frac{T}{\pi^2k^2} \right]$$

Since, $\sin k\pi = 0$ for k = Integer, so,

$$= \frac{4A}{T}\left[0 - \frac{T \cos k\pi}{\pi^3k^2} + \frac{T}{\pi^2k^2} \right]$$

$$= \frac{4A}{\pi^2k^2}[1 - \cos k\pi]$$

For odd k, $\cos k\pi = -1$, so,

$$A_{k,odd} = \frac{4A}{\pi^2k^2}[1 - (-1)] = \frac{4A}{\pi^2k^2} \times 2 = \frac{8A}{\pi^2k^2}$$

For even k, $\cos k\pi = 1$, so,

$$A_{k,even} = \frac{4A}{\pi^2 k^2}(1-1) = 0$$

Hence, $$A_k = \begin{cases} 0 & ; \text{ k even } i.e., k = 2, 4, 6, 8, \text{.........} \\ \frac{8A}{\pi^2 k^2} & ; \text{ k odd } i.e., k = 1, 3, 5, 7, 9, \text{.........} \end{cases}$$

Now,

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} V(t)\, dt$$

$$= \frac{2}{T}\int_0^{\frac{T}{2}} A\left(1 - \frac{4t}{T}\right) dt = \frac{2A}{T}\left[t - \frac{2t^2}{T}\right]_0^{\frac{T}{2}}$$

$$= \frac{2A}{T}\left(\frac{T}{2} - \frac{2 \times \frac{T^2}{4}}{T}\right) = \frac{2A}{T}\left(\frac{T}{2} - \frac{T}{2}\right) = 0$$

Thus, fourier expression of V(t) is;

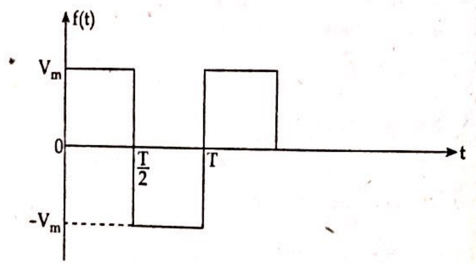
$$V(t) = a_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$$

$$= 0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + 0)$$

$$= \frac{8A}{\pi^2 k^2} \cos \omega_0 t + 0 + \frac{8A}{9\pi^2} \cos 3\omega_0 t + 0 + \frac{8A}{25\pi^2} \cos 5\omega_0 t + \text{.........}$$

$$= \frac{8A}{\pi^2}\left[\cos \omega_0 t + \frac{1}{9} \cos 3\omega_0 t + \frac{1}{25} \cos 5\omega_0 + \text{.........}\right]$$

5. **Obtain the Fourier series representation of the waveform shown.**
[2016/Spring, 2015/Spring, 2015/Fall]



Solution:

Given that;

$$f(t) = \begin{cases} V_m & ; \quad 0 \le t \le \frac{T}{2} \\ -V_m & ; \quad \frac{T}{2} \le t \le T \end{cases}$$

Fourier series representation of f(t) is,

$$f(t) = \sum_{k=-\infty}^{\infty} A_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + A_k^* e^{-jk\omega_0 t})$$

where, $$A_k = \frac{1}{T}\int_T f(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T f(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^{\frac{T}{2}} V_m e^{-jk\omega_0 t}\, dt + \frac{1}{T}\int_{\frac{T}{2}}^{T} -V_m e^{-jk\omega_0 t}\, dt$$

$$= \frac{V_m}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^{\frac{T}{2}} - \frac{V_m}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{\frac{T}{2}}^{T}$$

$$= \frac{V_m}{T}\left[\frac{e^{-jk\omega_0 T/2} - e^{-0}}{-jk\omega_0}\right] - \frac{V_m}{T}\left[\frac{e^{-jk\omega_0 T} - e^{jk\omega_0 T/2}}{-jk\omega_0}\right]$$

We know, $\omega_0 = \frac{2\pi}{T}$, so,

$$= \frac{V_m}{T}\left[\frac{e^{-jk\frac{2\pi}{T}\times\frac{T}{2}} - 1}{-jk\omega_0}\right] - \frac{V_m}{T}\left[\frac{e^{-jk\frac{2\pi}{T}\times T} - e^{-jk\frac{2\pi}{T}\times\frac{T}{2}}}{-jk\omega_0}\right]$$

$$= \frac{V_m}{T}\left[\frac{e^{-jk\pi} - 1}{-jk\omega_0}\right] - \frac{V_m}{T}\left[\frac{e^{-jk2\pi} - e^{jk\pi}}{-jk\omega_0}\right]$$

$$= \frac{V_m}{-jk\omega_0 T}(e^{-jk\pi} - 1 - e^{-j2\pi k} + e^{-jk\pi})$$

We know, $e^{-j2\pi k} = 1$. Thus,

$$= \frac{V_m}{-jk\frac{2\pi}{T}T}(e^{-jk\pi} - 2) = \frac{jV_m}{\pi k}(e^{-jk\pi} - 1)$$

For odd, $e^{-jk\pi} = 1$,

$$A_{k(odd)} = \frac{jV_m}{\pi k}(-1-1) = \frac{-2V_m j}{\pi k}$$

For k = even, $e^{-jk\pi} = 1$

$$A_k(\text{even}) = \frac{jV_m}{\pi k}(1-1) = 0$$

so, $$A_k = \begin{cases} 0 & ; \quad k = \text{even} \\ -j\frac{2V_m}{\pi k} & ; \quad k = \text{odd} \end{cases}$$

Also, $$a_0 = \frac{1}{T}\int_T f(t)\, dt$$

$$= \frac{1}{T}\int_0^{\frac{T}{2}} V_m\, dt + \frac{1}{T}\int_{\frac{T}{2}}^{T} -V_m\, dt$$

$$= \frac{V_m}{T}[t]_0^{\frac{T}{2}} - \frac{V_m}{T}[t]_{\frac{T}{2}}^{T}$$

$$= \frac{V_m}{T}\left(\frac{T}{2} - 0\right) - \frac{V_m}{T}\left(T - \frac{T}{2}\right)$$

$$= \frac{V_m}{2} - \frac{V_m}{2} = 0$$

Hence, $a_0 = 0$

$$a_1 = \frac{-j2V_m}{\pi} \qquad a_{-1} = \frac{j2V_m}{\pi}$$

$$a_2 = 0 \qquad a_{-2} = 0$$

$$a_3 = \frac{-j2V_m}{3\pi} \qquad a_{-3} = \frac{j2V_m}{3\pi}$$

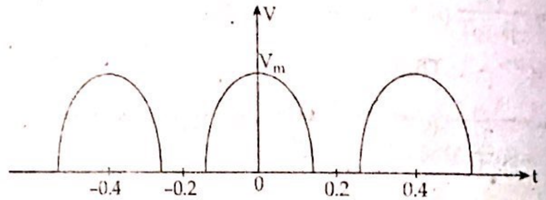
$$a_4 = 0 \qquad a_{-4} = 0$$

and so on ....

Thus, Fourier series representation is

$$f(t) = \ldots\ldots + \frac{j2V_m}{5\pi}e^{-j5\omega_0 t} + \frac{j2V_m}{3\pi}e^{-j3\omega_0 t} + \frac{j2V_m}{\pi}e^{-j\omega_0 t} - \frac{j2V_m}{\pi}e^{j\omega_0 t} - \frac{j2V_m}{3\pi}e^{j3\omega_0 t} - \frac{j2V_m}{5\pi}e^{j5\omega_0 t} \ldots\ldots$$

6. **Find the Fourier series for the voltage obtained at the output of a half wave rectifier shown in the following figure.** [2016/Fall]



Solution:

Time period, $T = 0.2 - (0.2) = 0.4$

From figure, we see the wave is cosine wave. If f(t) represents the given figure, we have,

$$f(t) = \begin{cases} V_m \cos \omega_0 t & ; \quad |t| \le t_0 \\ 0 & ; \quad t_0 \le |t| \le 0.2 \end{cases}$$

For value of t, we have,

$$V_m \cos \omega_0 t = 0$$

or, $\cos \frac{2\pi}{0.4} t = \cos \frac{\pi}{2}$

or, $5\pi t = \frac{\pi}{2}$

$\therefore$ $t = 0.1$

Hence, $f(t) = \begin{cases} V_m \cos 5\pi t & ; \quad |t| \le 0.1 \\ 0 & ; \quad 0.1 \le |t| \le 0.2 \end{cases}$

Fourier series representation of f(t) is;

$$f(t) = a_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$$

From figure, we see f(t) is even function, so $B_k = 0$.

We need to find only a_0 and A_k.

We know,

$$A_k = \frac{2}{T}\int_{-0.2}^{0.2} f(t) \cos k\omega_0 t \, dt$$

$$= \frac{2}{0.4} \times 2\int_0^{0.2} f(t) \cos 5\pi kt \, dt$$

$$= 10\int_0^{0.1} V_m \cos 5\pi t \cos 5\pi kt \, dt$$

$$= 10V_m \int_0^{0.1} \frac{\cos 5\pi(1-k)t + \cos 5\pi(1+k)t}{2} dt$$

$$= 5V_m \left[\frac{\sin 5\pi(1-k)t}{5\pi(1-k)} + \frac{\sin 5\pi(1+k)t}{5\pi(1+k)}\right]_0^{0.1}$$

$$= 5V_m \left[\frac{\sin 5\pi(1-k) \times 0.1 - \sin 0}{5\pi(1-k)} + \frac{\sin (1+k) \times 0.1 - \sin 0}{5\pi(1+k)}\right]$$

$$= 5V_m \left[\frac{\sin (1-k)\frac{\pi}{2}}{5\pi(1-k)} + \frac{\sin 5\pi(1+k)\frac{\pi}{2}}{5\pi(1+k)}\right]$$

$$= \frac{5V_m}{5\pi}\left[\frac{\sin\left(\frac{\pi}{2} - \frac{k\pi}{2}\right)}{(1-k)} + \frac{\sin\left(\frac{\pi}{2} + \frac{k\pi}{2}\right)}{(1+k)}\right]$$

$$= \frac{V_m}{\pi}\left[\frac{\cos\frac{k\pi}{2}}{(1-k)} + \frac{\cos\frac{k\pi}{2}}{(1+k)}\right]$$

$$= \frac{V_m}{\pi(1-k^2)} \cos\frac{k\pi}{2}(1+k+1-k)$$

$$= \frac{2V_m}{\pi(1-k^2)} \cos\frac{k\pi}{2}$$

and, $a_0 = \frac{1}{T}\int_{-0.2}^{0.2} f(t) \, dt$

$$= \frac{1}{0.4} \times 2\int_0^{0.1} V_m \cos 5\pi t \, dt$$

$$= 5V_m \left[\frac{\sin 5\pi t}{5\pi}\right]_0^{0.1}$$

$$= \frac{1}{\pi}(\sin 0.5\pi - \sin 0) = \frac{V_m}{\pi}$$

so, $a_0 = \frac{1}{\pi}$

$$A_k = \frac{2V_m}{\pi(1-k^2)}\cos\frac{k\pi}{2}$$

For k = odd, $\cos\frac{k\pi}{2} = 0$

For k = 2, 6, 10, 14, ..., $\cos\frac{k\pi}{2} = -1$

For k = 4, 8, 12, 16, ..., $\cos\frac{k\pi}{2} = 1$

Thus, $A_k = \begin{cases} 0 & ; \text{ k is odd} \\ \frac{-2V_m}{\pi(1-k^2)} & ; \text{ k} = 2, 6, 10, 14, \ldots\ldots\ldots \\ \frac{2V_m}{\pi(1-k^2)} & ; \text{ k} = 4, 8, 12, 16, \ldots\ldots\ldots \end{cases}$

Hence, fourier series representation is,

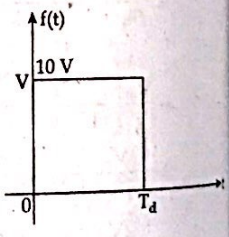
$$f(t) = a_0 + \sum_{k=1}^{\infty}(A_k \cos k\omega_0 t)$$

$$= \frac{V_m}{\pi} + \sum_{k=1}^{\infty}\left(\frac{2V_m}{\pi(1-k^2)}\cos\frac{k\pi}{2}\right)\cos 5\pi kt$$

$$= \frac{1}{\pi} + \frac{2V_m}{3\pi}\cos 10\pi t - \frac{2V_m}{15\pi}\cos 20\pi t + \frac{2V_m}{35\pi}\cos 30\pi t$$

$$- \frac{2V_m}{63\pi}\cos 40\pi t - \ldots\ldots\ldots$$

7. **Determine the Fourier transform of a pulse of duration T_d = 5 sec and amplitude V = 10 V as shown in figure below. [2011/Fall, 2019/Fall]**



Solution:

Given that;

T_d = 5 sec

V = 10 volts

The Fourier transform of above function f(t) is,

$$\therefore \quad F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^{T_d} V\, e^{-j\omega t}\, dt = 10\int_0^{5} e^{-j\omega t}\, dt$$

$$= 10\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^5 = \frac{10}{-j\omega}(e^{j5\omega} - e^{-0})$$

$$= \frac{10}{-j\omega}(e^{-j5\omega} - 1) = \frac{10}{-j\omega}e^{-j\frac{5}{2}\omega}\left(e^{-j\frac{5}{2}\omega} - e^{j\frac{5}{2}\omega}\right)$$

$$= \frac{2 \times 10 \times e^{j\frac{5}{2}\omega}}{\omega}\left(\frac{e^{j\frac{5}{2}\omega} - e^{-j\frac{5}{2}\omega}}{2j}\right)$$

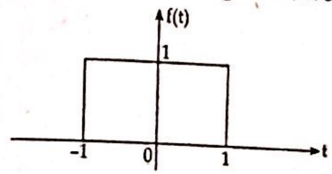
$$= \frac{20\, e^{-j\frac{5}{2}\omega}\sin\frac{5}{2}\omega}{\omega} = 20\, e^{-j\frac{5}{2}\omega}\frac{\sin\frac{5}{2}\omega}{\frac{5}{2}\omega} \times \frac{5}{2}$$

$$= 50\, e^{-j\frac{5}{2}\omega}\, \text{sin c}\left(\frac{5}{2}\omega\right) \qquad \left[\because \text{sin c}(x) = \frac{\sin x}{x}\right]$$

Here, since $\left(\frac{5}{2}\omega\right) = 0$ when $\frac{5}{2}\omega = \pm n\pi$

$\therefore \quad \omega = \pm\frac{2}{5}n\pi$

8. **Find the Fourier transform of the given waveform. [2014/Spring]**



Solution:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-1}^{1} 1 \times e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-1}^{1}$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega} - e^{j\omega}\right] = \frac{1}{j\omega}(e^{j\omega} - e^{-j\omega})$$

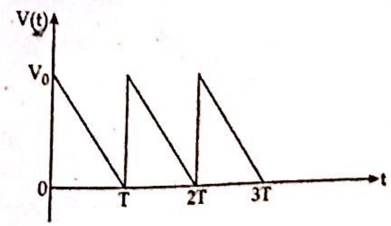
$$= \frac{2}{\omega}\left(\frac{e^{j\omega} - e^{-j\omega}}{2j}\right) = \frac{2}{\omega}\sin\omega = 2\frac{\sin\omega}{\omega}$$

$$= 2\,\text{sin c}(\omega) \qquad \left[\because \text{sin c}(\omega) = \frac{\sin\omega}{\omega}\right]$$

Here, sin c(ω) = 0 when $\omega = \pm n\pi$

Hence, $F(j\omega) = \begin{bmatrix} 0 & ; & \omega = \pm n\pi, \text{ n = integer} \\ 2\,\text{sin c}(\omega) & ; & \text{elsewhere} \end{bmatrix}$

9. **For the following periodic waveform, find the Fourier series expansion. [2014/Fall]**



Solution:

Time period is T.

V(t) is given by;

$$V(t) - V_0 = \frac{0 - V_0}{T - 0}(t - 0)$$

or, $$V(t) - V_0 = -\frac{V_0 t}{T}$$

$\therefore$ $$V(t) = V_0\left(1 - \frac{t}{T}\right)$$

Fourier series representation of V(t) is;

$$V(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad , \quad \omega_0 = \frac{2\pi}{T}$$

$$= a_0 + \sum_{k=1}^{\infty} (a_k e^{jk\omega_0 t} + a_k^{*} e^{-jk\omega_0 t})$$

We know,

$$a_k = \frac{1}{T}\int_0^T V(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_0^T V_0\left(1 - \frac{t}{T}\right) e^{-jk\omega_0 t}\, dt$$

$$= \frac{V_0}{T}\int_0^T \left(1 - \frac{t}{T}\right) e^{-jk\omega_0 t}\, dt$$

$$= \frac{V_0}{T}\left[\left(1 - \frac{t}{T}\right)\frac{e^{-jk\omega_0 t}}{-jk\omega_0} - \int \frac{dt}{dt}\left(1 - \frac{t}{T}\right)\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\, dt\right]$$

$$= \frac{V_0}{T}\left[\left(1 - \frac{t}{T}\right)\frac{e^{-jk\omega_0 t}}{-jk\omega_0} - \left(\frac{-1}{T}\right)\frac{e^{-jk\omega_0 t}}{j^2k^2\omega_0^2}\right]_0^T$$

$$= \frac{V_0}{T}\left[\left(1 - \frac{t}{T}\right)\frac{e^{-jk\omega_0 t}}{-jk\omega_0} - \frac{e^{-jk\omega_0 t}}{Tk^2\omega_0^2}\right]_0^T$$

$$= \frac{V_0}{T}\left[\left(\left(1 - \frac{t}{T}\right)\frac{e^{-jk\omega_0 T}}{-jk\omega_0} - \frac{e^{-jk\omega_0 T}}{Tk^2\omega_0^2}\right)\left(\left(1 - \frac{0}{T}\right)\frac{e^{-0}}{-jk\omega_0} - \frac{e^{-0}}{Tk^2\omega_0^2}\right)\right]$$

$$= \frac{V_0}{T}\left[-\frac{e^{-jk\omega_0 t}}{Tk^2\omega_0^2} - \frac{1}{-jk\omega_0} + \frac{1}{Tk^2\omega_0^2}\right]$$

We know, $\omega_0 = \frac{2\pi}{T}$, so,

$$a_k = \frac{V_0}{T}\left[\frac{e^{-jk\frac{2\pi}{T}\times T}}{Tk^2\left(\frac{2\pi}{T}\right)^2} + \frac{1}{jk\frac{2\pi}{T}} + \frac{1}{Tk^2\left(\frac{2\pi}{T}\right)^2}\right]$$

$$= \frac{V_0}{T}\left[-\frac{Te^{-j2\pi k}}{4\pi^2k^2} + \frac{T}{j2\pi k} + \frac{T}{4\pi^2k^2}\right]$$

We know,

$e^{-j2\pi k} = 1$,

$$= V_0\left(\frac{-1}{4\pi^2k^2} - \frac{j}{2\pi k} + \frac{1}{4\pi^2k^2}\right) = \frac{-jV_0}{2\pi k}$$

so,

so, $a_1 = \frac{-jV_0}{2\pi}$; $a_{-1} = a_1^{*} = \frac{jV_0}{2\pi}$

$a_2 = \frac{-jV_0}{4\pi}$; $a_{-2} = \frac{jV_0}{4\pi}$

$a_3 = \frac{-jV_0}{6\pi}$; $a_{-3} = \frac{jV_0}{6\pi}$

Also, $$a_0 = \frac{1}{T}\int_0^T V(t)\, dt$$

$$= \frac{1}{T}\int_0^T V_0\left(1 - \frac{t}{T}\right) dt = \frac{V_0}{T}\left[t - \frac{t^2}{2T}\right]_0^T$$

$$= \frac{V_0}{T}\left[\left(T - \frac{T^2}{2T}\right) - \left(0 - \frac{0}{2T}\right)\right]$$

$$= \frac{V_0}{T}\left(\frac{T}{2}\right) = \frac{V_0}{2}$$

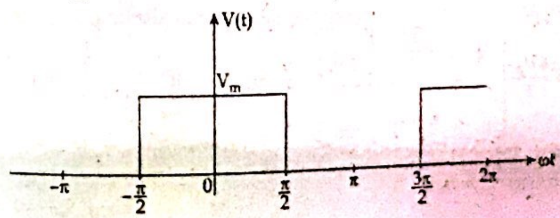
$\therefore$ Fourier series expansion is,

$$V(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \ldots\ldots + \frac{jV_0}{6\pi}e^{-j3\omega_0 t} + \frac{jV_0}{4\pi}e^{-j2\omega_0 t} + \frac{jV_0}{2\pi}e^{-j\omega_0 t}$$

$$+ \frac{V_0}{2} - \frac{jV_0}{2\pi}e^{j\omega_0 t} - \frac{jV_0}{4\pi}e^{j2\omega_0 t} - \frac{jV_0}{6\pi}e^{j3\omega_0 t} - \ldots\ldots$$

10. **Obtain the trigonometric form of the fouerier series for the following.** **[2013/Spring]**



Solution:

Time period in radian frequency is 2π.

Here; $\omega t = 2\pi$

or, $\frac{2\pi}{T}t = 2\pi$

$\therefore$ t = T, T is time period.

For ease of calculation, we take period from $-\pi$ to π. Here 2π is T, so π is $\frac{T}{2}$.

Now,

$$V(t) = \begin{cases} V_m & ; \quad |t| \le \frac{\pi}{2} \text{ or } \frac{T}{4} \\ 0 & ; \quad \frac{T}{4} \le |t| \le \frac{T}{2} \end{cases}$$

Fourier series of V(t) in trigonometric form is,

$$V(t) = a_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t); \ \omega_0 = \frac{2\pi}{T}$$

From figure, we see that V(t) is even function, so $B_k = 0$.

Hence, $V(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos k\omega_0 t$

We know,

$$A_k = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} V(t) \cos k\omega_0 t \, dt$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} V(t) \cos k\omega_0 t \, dt$$

$$= \frac{4}{T}\int_0^{\frac{T}{4}} V_m \cos k\omega_0 t \, dt$$

$$= \frac{4V_m}{T}\left[\frac{\sin k\omega_0 t}{k\omega_0}\right]_0^{\frac{T}{4}}$$

$$= \frac{4V_m}{T}\left[\frac{\sin k\omega_0 \times \frac{T}{4} - \sin k\omega_0 \times 0}{k\omega_0}\right]$$

We know, $\omega_0 = \frac{2\pi}{T}$, so,

$$= \frac{4V_m}{T}\left[\frac{\sin k \times \frac{2\pi}{T} \times \frac{T}{4} - 0}{k \times \frac{2\pi}{T}}\right] = \frac{2V_m}{\pi k}\left(\sin k\frac{\pi}{2}\right)$$

For k = even, $\sin\frac{k\pi}{2} = 0$

For k = 1, 5, 9, ..., $\sin\frac{k\pi}{2} = 1$

so, $A_k = \frac{2V_m}{\pi k}$

For k = 3, 7, 11, ..., $\sin\frac{k\pi}{2} = -1$

so, $A_k = \frac{-2V_m}{\pi k}$

$$\therefore \quad A_k = \begin{cases} 0 & ; \quad k \text{ is even} \\ \frac{2V_m}{\pi k} & ; \quad k = 1, 5, 9, \ldots\ldots \\ \frac{-2V_m}{\pi k} & ; \quad k = 3, 7, 11, \ldots\ldots \end{cases}$$

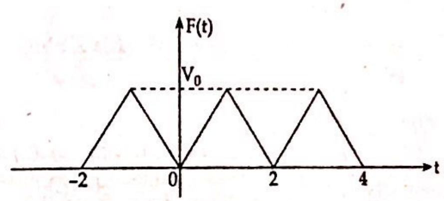
and, $a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} V(t)\,dt = \frac{2}{T}\int_0^{\frac{T}{4}} V_m \, dt = \frac{2V_m}{T}[t]_0^{T/4}$

$$= \frac{2V_m}{T}\left[\frac{T}{4} - 0\right] = \frac{V_m}{T}$$

$\therefore$ Trigonometric form of V(t) is,

$$V(t) = \frac{V_m}{2} + \frac{2V_m}{\pi}\cos\omega_0 t - \frac{2V_m}{3\pi}\cos 3\omega_0 t + \frac{2V_m}{5\pi}\cos 5\omega_0 t - \frac{2V_m}{7\pi}\cos 7\omega_0 t + \frac{2V_m}{9\pi}\cos 9\omega_0 t - \ldots\ldots$$

11. **Find the trigonometric Fourier series for the waveform shown.** **[2013/Fall]**



Solution:

Here; T = 2

$$\omega_0 = \frac{2\pi}{T} = \frac{2 \times \pi}{2} = \pi$$

For ease of calculation, we take period from -1 to 1. From -1 to 0, f(t) is given by,

$$f(t) - V_0 = \frac{0 - V_0}{0 - (-1)}[t - (-1)]$$

or, $f(t) - V_0 = -V_0(t + 1)$

or, $f(t) = -V_0 t - V_0 + V_0$

$\therefore$ $f(t) = -V_0 t$

From 0 to 1, f(t) is given by,

$$f(t) - 0 = \frac{V_0 - 0}{1 - 0}(t - 0)$$

$\therefore$ $f(t) = V_0 t$

Hence, $f(t) = \begin{cases} -V_0 t & , \quad -1 \le t \le 0 \\ V_0 t & , \quad 0 \le t \le 1 \end{cases}$

The trigonometric Fourier series of f(t) is,

$$f(t) = a_0 + \sum_{k=1}^{\infty} (A_k \cos k\omega_0 t + B_k \sin k\omega_0 t)$$

From figure, we see f(t) is even function, so $B_k = 0$.

We know, $\omega_0 = \pi$, so,

$$f(t) = a_0 + \sum_{k=1}^{\infty} A_k \cos k\pi t$$

We have,

$$A_k = \frac{2}{T}\int_{-1}^{1} f(t) \cos k\omega_0 t \, dt$$

$$= \frac{2}{2} \times 2\int_0^1 V_0 \, t \cos k\pi t \, dt$$

$$= 2V_0 \int_0^1 t \cos k\pi t \, dt$$

$$= 2V_0 \left[\frac{t \sin k\pi t}{k\pi} - \int \frac{dt}{dt} \times \frac{\sin k\pi t}{k\pi} dt\right]$$

$$= 2V_0 \left[\frac{t \sin k\pi t}{k\pi} + \frac{\cos k\pi t}{k^2\pi^2}\right]_0^1$$

$$= 2V_0 \left[\frac{1 \times \sin k\pi}{k\pi} + \frac{\cos k\pi}{k^2\pi^2} - \left(\frac{0 \times \sin 0}{k\pi} + \frac{\cos k\pi \times 0}{k^2\pi^2}\right)\right]$$

$$= 2V_0 \left[\frac{\sin k\pi}{k\pi} + \frac{\cos k\pi}{k^2\pi^2} - \frac{1}{k^2\pi^2}\right]$$

We know, $\sin k\pi = 0$, so,

$$A_k = 2V_0 \left(\frac{\cos k\pi - 1}{k^2\pi^2}\right) = \frac{2V_0}{k^2\pi^2}(\cos k\pi - 1)$$

For k = odd, $\cos k\pi = -1$,

$$A_k = \frac{2V_0}{k^2\pi^2}(-1-1) = \frac{-4V_0}{k^2\pi^2}$$

For k = even, $\cos k\pi = 1$, so,

$$A_k = \frac{2V_0}{k^2\pi^2}(1-1) = 0$$

$$\therefore \quad A_k = \begin{cases} 0 & ; \ k \text{ even} \\ -\dfrac{4V_0}{k^2\pi^2} & ; \ k \text{ odd} \end{cases}$$

and, $a_0 = \frac{1}{T}\int_{-1}^{1} f(t)\, dt = \frac{1}{2} \times 2\int_0^1 V_0 t \, dt$

$$= V_0\left[\frac{t^2}{2}\right]_0^1 = V_0\left(\frac{(1)^2}{2} - 0\right) = \frac{V_0}{2}$$

So, trigonometirc Fourier series of f(t) is,

$$f(t) = \frac{V_0}{2} + \sum_{k=1}^{\infty} A_k \cos k\pi t$$

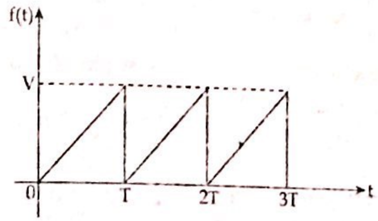
$$= \frac{V_0}{2} - \frac{4V_0}{\pi^2}\cos \pi t - \frac{4V_0}{9\pi^2}\cos 3\pi t - \frac{4V_0}{25\pi^2}\cos 5\pi t - \ldots\ldots\ldots$$

12. **Compare Fourier series and Fourier transform.** [2019/Fall]

Solution:

Fourier series is an expansion of periodic signal as a linear combination of sines and cosine while Fourier transform is the process or function used to convert signals from time domain into frequency domain. Fourier series is defined for periodic signals and the Fourier transform can be applied to a periodic (occurring without periodicity) signals.

13. **Find the trigonometric Fourier series for the waveform shown below.** [2012/Spring]



Solution;

Here;

T = T − 0 = T sec

Let, Time period, T = f(t), then using slope equation,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or, $y - 0 = \left(\frac{V - 0}{T - 0}\right)(t - 0)$

$\therefore \quad y = \left(\frac{V}{T}\right)t$

$\therefore \quad f(t) = \left(\frac{V}{T}\right)t$ for $0 < t < T$

Trigonometric Fourier series representation is,

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t$$

where, $a_0 = \frac{1}{T}\int_0^T f(t)\, dt = \frac{1}{T}\int_0^T \left(\frac{V}{T}\right) t \, dt$

$$= \frac{V}{T^2} \times \frac{t^2}{2}\Big|_0^T = \frac{V}{T^2} \times \frac{T^2}{2}$$

$$\therefore \quad a_0 = \frac{V}{2}$$

and, $$A_k = \frac{2}{T}\int_0^T \left(\frac{V}{T}\right) t \cos k\omega_0 t \, dt$$

$$= \frac{2V}{T^2}\int_0^T \left(\frac{V}{T}\right) t \cos k\omega_0 t \, dt$$

$$= \frac{2V}{T^2}\left[t \times \frac{1}{k\omega_0}\cdot \sin k\omega_0 t\Big|_0^T - \int_0^T t \cdot \frac{1}{k\omega_0} \sin k\omega_0 t \, dt\right]$$

$$= \frac{2V}{T^2}\left[\frac{T}{k\omega_0}\sin k\omega_0 T - 0\left(\frac{1}{k\omega_0}\right)\sin 0 - \left(\frac{t}{k\omega_0} - \frac{\cos k\omega_0 t}{k\omega_0}\right)_0^T\right]$$

$$= \frac{2V}{T^2}\left[\frac{T}{k\omega_0}\sin k\omega_0 T + \frac{T}{k^2\omega_0^2}(\cos k\omega_0 T - \cos 0)\right]$$

$$= \frac{2V}{T^2}\left[\frac{T}{k\omega_0}\sin k\omega_0 T + \frac{T\cos k\omega_0 T}{k^2\omega_0^2} - \frac{T}{k^2\omega_0^2}\right]$$

Since $\omega_0 = 2\pi f_0$, for all integer value of k, $\sin k\omega_0 T = 0$

$$= \frac{2V}{T^2}\left[\frac{T\cos k\omega_0 T}{n^2\omega_0^2} - \frac{T}{k^2\omega_0^2}\right] = \frac{2V}{T^2}\times T\left[\frac{1}{k^2\omega_0^2}(\cos k\omega_0 T - 1)\right]$$

$$= \frac{2V}{T}\left[\frac{1}{k^2\omega_0^2}(\cos 2\pi k - 1)\right]\left[\because \omega_0 = \frac{2\pi}{T}\right] = 0 \text{ for all values of k}$$

Also, $$B_k = \frac{2}{T}\int_0^T \left(\frac{V}{T}\right) t \sin k\omega_0 t \, dt$$

$$= \frac{2V}{T^2}\left[\frac{1}{k^2\omega_0^2}\sin k\omega_0 t - \frac{t}{k\omega_0}\cos k\omega_0 t\right]\Bigg|_0^T$$

$$= \frac{2V}{T^2}\left[\frac{1}{k^2\omega_0^2}\times 0 - \frac{T}{k\omega_0}\cos 2\pi k\right] \qquad \left[\because \omega_0 = 2\pi f_0 \text{ and, } f_0 = \frac{1}{T}\right]$$

$$= \frac{2V}{T^2}\left(0 - \frac{T}{k\omega_0}\cos 2\pi k\right) = 0 - \frac{2V}{T \times k \times \frac{2\pi}{T}}\cos 2\pi k$$

$$\therefore \quad B_k = \frac{-V}{\pi k} \qquad [\because \cos 2\pi = 1]$$

Hence, $$f(t) = \frac{V}{2} - \frac{V}{2}\sum_{k=1}^{\infty}\frac{1}{k}\sin k\omega_0 t$$

$$= \frac{V}{2} - \frac{V_m}{\pi}\sin \omega_0 t - \frac{V}{2\pi}\sin 2\omega_0 t - \frac{V}{3\pi}\sin 3\omega_0 t + \ldots\ldots\ldots$$

✿✿✿

Chapter 7

FREQUENCY RESPONSE OF NETWORK

7.1 INTRODUCTION

The response given by the system when input frequency ω is changed over a certain range is called frequency response of the system.

7.2 ANGLE AND MAGNITUDE OF A FUNCTION

A frequency response can be obtained by expressing the system function H(s) in the frequency domain. The frequency domain transfer function can be obtained by replacing the complex frequency variable 's' by 'jω'. It is denoted as H(jω), *i.e.*, $H(j\omega) = H(s)|_{s=j\omega}$

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Such a frequency domain transfer function can be expressed in the polar form as,

$$H(j\omega) = |H(j\omega)| \angle H(j\omega) = M_r \angle \phi_r$$

where, M_r = Resultant magnitude which is function of ω

ϕ_r = Resultant phase angle which is function of ω

For example:

$$H(s) = \frac{20}{(s+1)(s+3)}$$

$$\therefore \quad H(j\omega) = \frac{20}{(1+j\omega)(3+j\omega)} = M_r < \phi_r$$

where, $$M_r = \frac{20}{(\sqrt{1+\omega^2}) \cdot (\sqrt{3^2+\omega^2})}$$

and, $$\phi_r = 0^\circ - \tan^{-1}\frac{\omega}{1} - \tan^{-1}\frac{\omega}{3}$$

7.3 BODE PLOT OR BODE DIAGRAMS

The scientist H.W. bode suggested a specific method to obtain the frequency response in which logarithmic values of ω.

a) Magnitude plot in which logarithmic values of M_r are plotted against logarithmic values of ω.

b) Phase angle plot in which ϕ_r in degress are plotted against logarithmic values of ω.

7.3.1 Magnitude Plot

Bode suggested to express M_r in decibel (dB) to plot against $\log_{10}\omega$. Such a dB value of Mr can be obtained as,

$$|H(j\omega)| \text{ in dB} = M_r \text{ in dB} = 20 \log_{10} |M_r|$$

Such dB values are obtained for various values of ω from 0 to ∞ and are plotted against $\log_{10}\omega$. The magnitude plot is shown in figure 7.1.

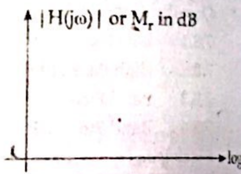

Figure 7.1: Magnitude plot

7.3.2 Phase Angle Plot

The $\angle H(j\omega)$ or ϕ_r is obtained in degrees for various values of ω and are plotted against $\log_{10}\omega$. The phase plot is shown in figure 7.2. $\angle H(j\omega)$ or ϕ_r in degrees.

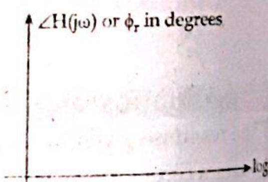

Figure 7.2: Phase angle plot

7.3.3 Logarithmic Scales (Semi Log Papers)

To plot the magnitude in dB and phase angle in degrees against $\log_{10}\omega$ the logarithmic scale is used. This is available on semi log paper. In such paper the X-axis is divided into a logarithmic scale which is non-linear one. While Y-axis is divided into linear scale and hence it is called semi log paper.

The interesting part about X-axis is the distance between 1 and 2 is greater than distance between 2 and 3 and so on. Similarly, on such scale, distance between 1 and 10 is equal to the distance between 10 and 100 or between 100 and 1000 and so on. The distance is called 1 decade. So X-axis is available which is divided into two, three or four cycles *i.e.*, decades.

As $\log_{10} 0 = -\infty$ it is obvious that X-axis cannot be calibrated from 0 but as per requirement the smallest frequency may be selected as starting frequency may be selected as starting frequency like 0.001, 0.01, 0.1 etc. This hardly affects the result of the frequency response. The Y-axis is divided into linear scale, similar to standard graph paper.

To clear the idea of semi log paper and decade, the semi log paper is shown in the figure 7.3.

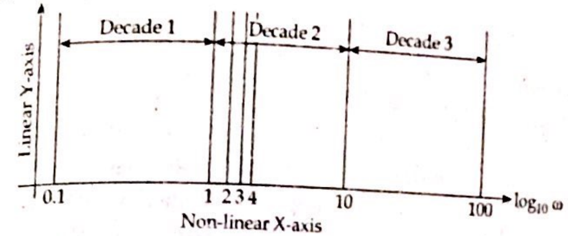

Figure 7.3:

The main advantage using the logarithmic representation is that the multiplication and division of magnitudes get replaced by the addition and subtraction respectively. The another advantages of the logarithmic scale is wide range of frequencies can be accommodated on a single paper.

7.4 STANDARD FORM OF H (jω)

The frequency domain transfer function may be written as the ratio of ploynomials as (in time constant form)

$$H(j\omega) = \frac{K(1+T_1 j\omega)(1+T_2 j\omega)\ldots\ldots}{(j\omega)^P (1+T_a j\omega)(1+T_b j\omega)\ldots\ldots}$$

The above transfer function shows that the numerator and denominator have four basic types of factors. They are,

i) Resultant system gain K, constant factor (when $H(j\omega)$ is expressed in time constant form.

ii) Poles or zeros at the origin $(j\omega)^{\mp P}$, P = 1, 2, ...

iii) simple poles and zeros are also called fist order factors of the form $(1 + T j\omega)^{\pm 1}$.

iv) Quadratic factors which cannot be factorized into real factors, of the form.

$$\left[1 + 2\frac{\zeta}{\omega_n} j\omega + \frac{1}{\omega_n^2}(j\omega)^2\right]^{\mp 1}$$

7.5 BODE PLOTS OF BASIC FACTORS OF H(jω)

For each factor, procedure to obtain its bode plot can be divided into the following steps.

Step 1: Replace 's' by 'jω' to convert it to frequency domain.

Step 2: Find its magnitude as a function of 'ω'.

Step 3: Express the magnitude in dB by $20 \log_{10} |H(j\omega)|$.

Step 4: Find phase angle by using $\tan^{-1}\left[\frac{\text{Imaginary part}}{\text{Real part}}\right] = \phi$ in degrees.

Step 5: With required approximation, plot magnitude in dB and phase angle in degrees against $\log_{10} \omega$ by varying ω from 0 to ∞.

7.5.1 System Gain 'K'

$$H(s) = K, \quad H(j\omega) = K + j0, \quad H(j\omega) = \sqrt{K^2 + 0} = K$$

and, $\angle H(j\omega) = \tan^{-1}\frac{0}{K} = 0°$

Magnitude plot

Magnitude in dB = $20 \log_{10} K$ dB

As gain 'K' is constant, $20 \log_{10} K$ is always constant though ω is varied from 0 to ∞. So its magnitude plot for K > 1 is a line parallel to X-axis at a distance $20 \log_{10} K$ above 0 dB reference line. While K < 1 it is at a distance of $20 \log_{10} K$ below 0 dB reference line This means effect of 'K' is constant equal to $20 \log_{10} K$ dB for all frequencies. This 'K' shifts the magnitude plot $|H(j\omega)|$ by a distance $20 \log_{10} H$ dB upwards if K > 1 and downwards if K < 1 as shown in figure 5.3(a).

Phase angle Plot

As $\angle H(j\omega) = 0°$

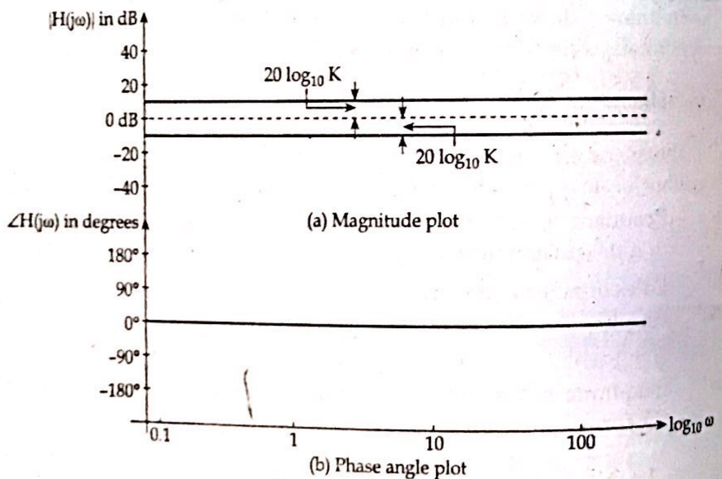

Figure 7.4: Contribution by K

So it does not affect the phase angle plot as its contribution to phase angle plot is 0° as shown in figure 7.4(b).

7.5.2 Poles or Zeros at The Origin $(j\omega)^{\pm p}$

Let us consider for simplicity single pole at the origin,

$$H(s) = \frac{1}{s}$$

$$H(j\omega) = \frac{1}{j\omega} = \frac{1}{0 + j\omega}$$

Magnitude plot: $|H(j\omega)| = \frac{1}{\sqrt{(0)^2 + \omega^2}} = \frac{1}{\omega}$

Magnitude in dB $= 20 \log_{10} = \frac{1}{\omega} = -20 \log_{10} \omega$

$= 0$ dB ; ω = 1

$= -20$ dB ; ω = 10

$= -40$ dB ; ω = 100

$= +20$ dB ; ω = 0.1

Now 10 times change in frequency range is called 1 decade described earlier *i.e.,* 1 pole reduces the $|H(j\omega)|$ at the rate of -20 dB decade. So magnitude plot for 1 pole at origin is a straight line of slope -20 dB/decade.

Now at ω= 1, $|H(j\omega)| = 0$ dB this line intersects the reference 0 dB line at ω = 1.

Consider two poles at the origin,

$$H(s) = \frac{1}{s^2}$$

$$H(j\omega) = \frac{1}{(j\omega)^2}$$

$$|H(j\omega)| = \frac{1}{\omega^2}$$

$$|H(j\omega)| \text{ in dB} = 20 \log_{10} \frac{1}{\omega^2} = -40 \log_{10} \omega$$

So it is a straight line slope = –40 dB/decade

Also at ω = 1, $|H(j\omega)| = 0$ dB *i.e.,* this line though has slope –40 dB/decade, intersects 0 dB line at ω = 1.

Similarly, for P number of poles at origin.

$$H(s) = \frac{1}{s^P}$$

$$H(j\omega) = \frac{1}{(j\omega)^P}, \ |H(j\omega)| = \frac{1}{\omega^P}$$

$$|H(j\omega)| \text{ in dB} = 20 \log_{10} \frac{1}{(\omega)^P} = -20 \times p \log_{10} \omega$$

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So this is a straight line of slope-20 × P dB/decade but again intersecting with 0 dB line at $\omega = 1$. Therefore, magnitude plot for P poles at the origin gives a family of lines passing through intersection of $\omega = 1$ and 0 dB line having slope $-20 \times P$ dB/decade as shown in figure 7.4(c).

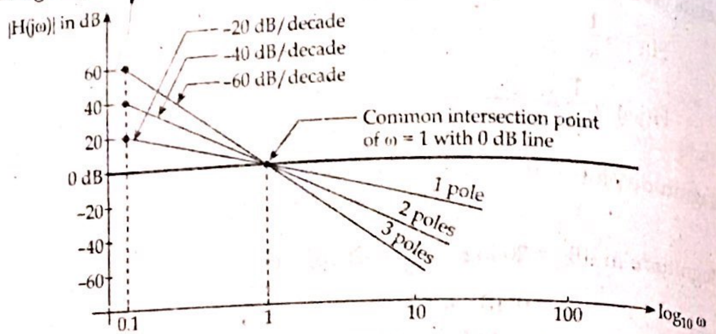


Figure 7.5: (a)

Now if there is a zero at the origin,

$H(s) = s$ *i.e.*, $H(j\omega) = 0 + j\omega$

Magnitude in dB = $20 \log_{10} \omega$

This is equation of a straight line whose slope is +20 dB/decade. The only change is the sign of the slope, for a pole it is –20 dB/decade while for a zero it is +20 dB/deacde, but for both intersection of line with 0 dB occurs at $\omega = 1$. In general, for P numbers of zeros at the origin,

$H(s) = s^P$, $H(j\omega) = (j\omega)^P$

Magnitude in dB = $20 \times P \log_{10} \omega$

i.e., slope = $+20 \times P$ dB/decade

So it gives family of lines with slope as +20, +40, ..., +20 × P dB/decade passing through intersection point of $\omega = 1$ with 0 dB line as shown in the figure 7.5 (b).

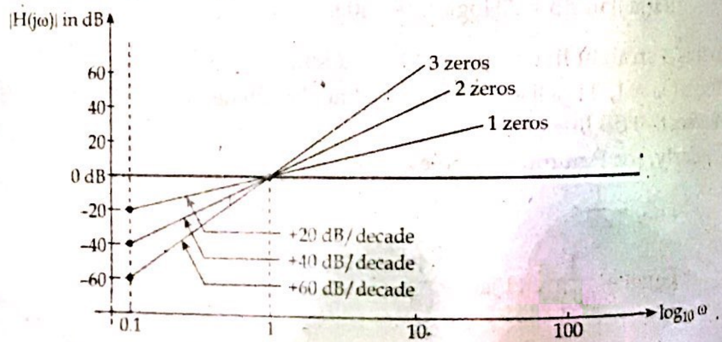


Figure 7.5: (b) Contribution by zero at origin

Phase Angle Plot

Consider single pole at origin

$$H(s) = \frac{1}{s}$$

$$H(j\omega) = \frac{1}{j\omega}$$

$$\angle H(j\omega) = \angle\frac{1}{j\omega} = -\angle j\omega = -90°$$

This is independent of 'ω'. So phase angle plot of a pole at origin is line parallel to X-axis contributing -90° to phase angle.

For 2 poles at origin, $H(s) = \frac{1}{s^2}$

$$\angle H(j\omega) = \angle\frac{1}{(j\omega)^2} = -2\angle j\omega = -2 \times 90° = -180°$$

(angle gets added to each other).

In generals P number of poles at the origin contribute 90° × P angle to overall phase angle plot. This contribution is irrespective of 'ω'.

Similarly for zeros, the contribution is same as that of pole, the only change is its sign. In general, 'P' numbler of zeros at the origin, the total angle contribution is +90° × P, irrespective of 'ω'. This is shown in figure 7.6.

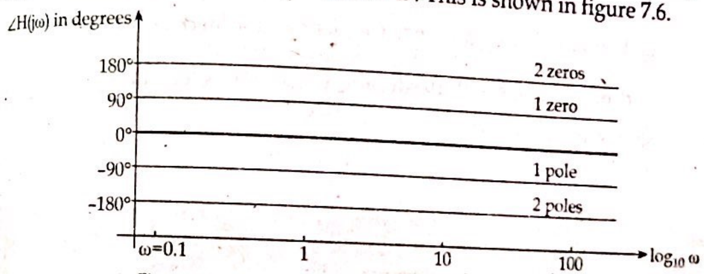


Figure 7.6: Angle contribution by poles and zeros at origin

7.5.3 Simple Poles or Zeros $(1 + T j\omega)^{\pm 1}$

First let us consider a simple pole

$$H(s) = (1 + Ts)^{-1} = \frac{1}{1 + Ts}$$

$$H(j\omega) = \frac{1}{1 + T\omega}$$

$$H(j\omega) = \frac{1}{\sqrt{(1 + \omega T)^2}} = \left[\sqrt{1 + (\omega T)^2}\right]^{-1}$$

Magnitude in dB $= 20 \log_{10} \left[\sqrt{1 + (\omega T)^2}\right]^{-1}$

$= -20 \log_{10} \sqrt{1 + (\omega T)^2}$

Now, instead of sketching magnitude plot exactly according to this expression we can approximate this into two regions and can draw straight line approximated magnitude plot. The approximation is

i) For low frequency range $\omega \ll \frac{1}{T}$ *i.e.,* $\omega^2T^2 \ll 1$ hence can be neglected.

$\therefore$ Magnitude in dB = $-20 \log_{10} 1 = 0$

So for low frequencies it is straight line of 0 dB only. Thus the contribution by such factors can be completely neglected for low frequency rangle.

ii) For high frequency range $\omega \gg \frac{1}{T}$ *i.e.,* $\omega^2T^2 \gg 1$.

$\therefore$ Magnitude in dB = $-20 \log_{10} \omega T$

i.e., it is straight line of slope −20 dB/decade. But the intersection of this line with 0 dB line will give us range high frequency and low frequency *i.e.,* two lines, 0 dB line for low 'ω' and line with slope -20dB/ decade for high 'ω' are going to intersect when,

$-\log_{10} \omega T = 0$ or $\omega T = 1$

or, $\omega = \frac{1}{T}$

This frequency at which change of slope from 0 dB to −20 dB/decade occurs is called corner frequency, denoted by ω_c . *i.e.,* $\omega_c = \frac{1}{T}$ and line of slope −20 dB/decade when $\omega > \omega_c$ *i.e.,* above $\omega_c = \frac{1}{T}$ as shown in figure 7.7.

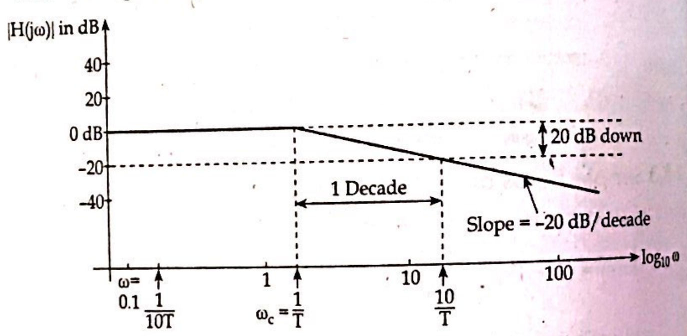


Figure 7.7: Contribution by simple pole

Similarly for a simple zero,

$H(s) = (1 + Ts)$, $H(j\omega) = 1 + T j\omega$

$H(j\omega) = \sqrt{1 + (\omega T)^2}$

Magnitude in dB = $20 \log_{10} \sqrt{1 + (\omega T)^2}$

Therefore, the magnitude plot for a simple zero is a straight line of 0 dB up to $\omega_c = \frac{1}{T}$ and then straight line of slope +20 dB/decade for all frequencies more than corner frequency $\omega_c = \frac{1}{T}$ as shown in figure 7.8.

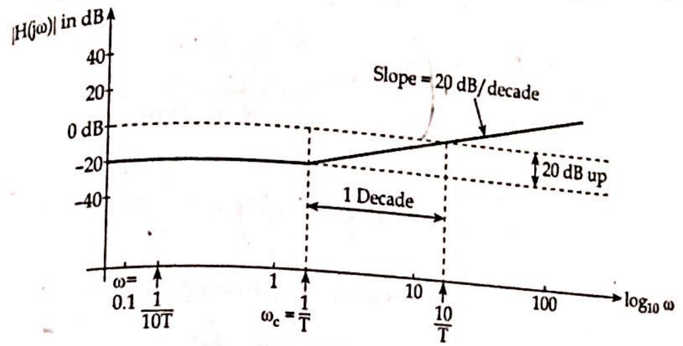


Figure 7.8: Contribution by simple zero

Phase Angle Plot

For a simple pole, $H(s) = (1 + Ts)^{-1}$

$$H(j\omega) = (1 + T j\omega)^{-1} = \frac{1}{1 + Tj\omega}$$

$$\angle H(j\omega) = 0 - \tan^{-1}\frac{\omega T}{1} = -\tan^{-1} \omega T$$

while for a simple zero,

$H(s) = (1 + Ts)$ $H(j\omega) = (1 + Tj\omega)$

$\therefore$ $\angle H(j\omega) = +\tan^{-1} \omega T$

So the shape remains the same. Only sign of the angles changes from negative to positive when factor changes from pole to zero. Such plot is to draw by actually calculating angles for different frequencies. So make a table as shown below.

ω	$\pm \tan^{-1} \omega T$ (+for zero, − for pole)
$0.1\,\omega_c = \frac{1}{10\,T}$	± 5.71°
$0.5\,\omega_c = \frac{1}{2T}$	± 26.6°
$\omega_c = \frac{1}{T}$	±45°
$2\omega_c = \frac{2}{T}$	±63.4°
$10\,\omega_c = \frac{10}{T}$	±84.3

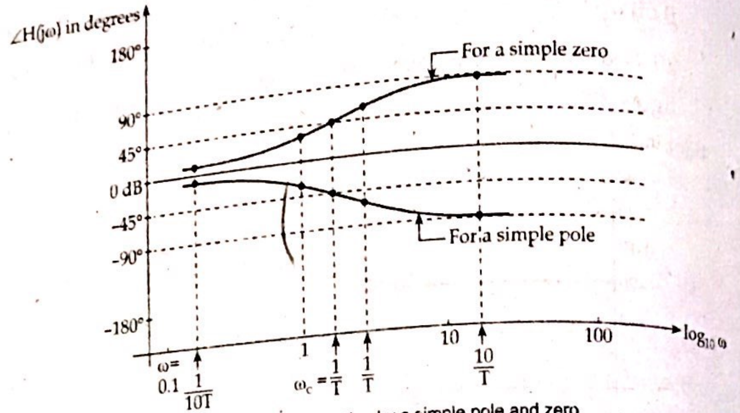


Figure 7.9: Angle contribution by a simple pole and zero.

7.5.4 Quadratic Factors $\left[1 + 2\frac{\zeta}{\omega_n} j\omega + \frac{1}{\omega^2}(j\omega)^2\right]^{\pm 1}$

Generally, we have the quadratic factor is in the denominator of the transfer function i.e., the transfer function of the form,

$$H(s) = n\frac{1}{1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}} \equiv \frac{\omega_n^2}{s^2 + 2\zeta\,\omega_n s + \omega_n^2}$$

$$H(j\omega) = \frac{1}{1 + j2\zeta\frac{\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2}$$

where, ω is variable and ω_n is constant

$$H(j\omega) = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j\,2\zeta\frac{\omega}{\omega_n}}$$

$$|H(j\omega)| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta\left(\frac{\omega}{\omega_n}\right)^2}}$$

$$\text{Magnitude in dB} = -20\log_{10}\frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta\left(\frac{\omega}{\omega_n}\right)^2}}$$

i) For low frequency, $\omega \ll \omega_n$, $\left(\frac{\omega}{\omega n}\right)^2 \ll 1$, hence neglected.

∴ Magnitude in dB = $-20\log_{10} 1 = 0$

Thus similar to simple pole, quadratic poles also negligible till its corner frequency occurs.

ii) For high frequency, $\omega \gg \omega_n$, $4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2 \ll \left(\frac{\omega}{\omega_n}\right)^4$ as ζ is very low.

∴ Magnitude in dB $= -20\log_{10}\sqrt{\left[\left(\frac{\omega}{\omega_n}\right)^2\right]^2}$

$$= -40\log_{10}\frac{\omega}{\omega_n} = -40\log_{10}\omega + 40\log_{10}\omega_n$$

This is equation of straight line of slope–40 dB/decade. Hence in general, magnitude plot for quadratic poles is 0 dB line till corner frequency and then a straight line of slope –40 dB decade.

To find corner frequency ω_c,

$$-40\log_{10}\frac{\omega}{\omega_n} = 0$$

or, $\frac{\omega}{\omega_n} = 1$

∴ $\omega = \omega_n$ or $\omega_c = \omega_n$

So ω_n is the corner frequency for quadratic poles.

The magnitude plot for above transfer function is shown in figure 7.10.

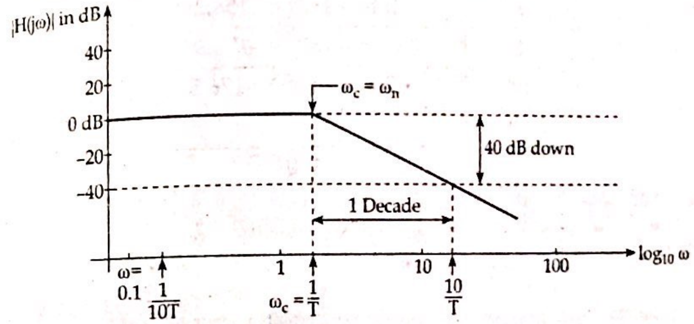


Figure 7.10: Contribution by quadratic poles

Phase Angle Plot

For the quadratic poles,

$$\angle H(j\omega) = -\tan^{-1}\left[\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

The phase angle table for ζ = 0.3 is shown below.

$$\angle H(j\omega) = \phi = -\tan^{-1}\left[\frac{2 \times 0.3\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$\frac{\omega}{\omega_n}$	ϕ
0.1	-3.46°
0.5	-21.8°
1	-90°

so at $\frac{\omega}{\omega_n} = 1$ *i.e.*, $\omega = \omega_c = \omega_n$, it contributes –90° and hence must approaches to –180° as $\frac{\omega}{\omega_n} \to \infty$. But according to above formula when $\frac{\omega}{\omega_n} > 1$, ϕ becomes positive, but in such case the angle contribution is obtained by subtracting 180° from the positive ϕ. This happens because, behaviour of $\tan^{-1}$ function for the compels quantities with real part negative or imaginary part negative cannot be identified on calculator. Hence phase angle table becomes,

$\frac{\omega}{\omega_n}$	ϕ
0.1	-3.46°
0.5	-21.8°
1	-90°
2	+21.8° – 180° = –158.2°
4	+10.9° – 180° = –170.9°
10	+3.46° – 180° = –176.5
...	....
∞	-180°

NOTE:

i) For quadratic zeros, sign of the angle should be made positive.

ii) The above discussion is applicable only when the roots of a quadratic factor are complex conjugate of each other. If roots are real, factorize it and consider its two components independently as simple factor rather than quadratic.

This can be shown in figure 7.11.

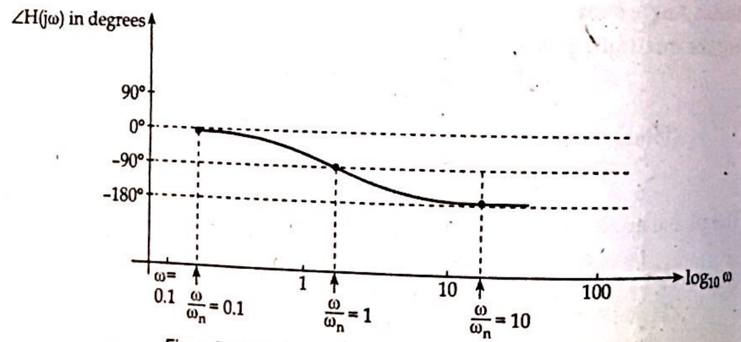


Figure 7.11: Angle contribution by the quadratic poles

7.6 STEPS TO SKETCH THE BODE PLOT

a) Express given H(s) into time constant form.

b) Draw a line of $20 \log_{10} K$ dB.

c) Draw a line of appropriate slope representing poles or zeros at origin, passing through intersection point of $\omega = 1$ and 0 dB.

d) Shift this intersection point on $20 \log_{10} K$ line and draw parallel line to the line draw in step (c). This is resultant of constant K and poles or zeros at the origin.

e) Change the slope of this line at various corner frequencies by appropriate values *i.e.*, depending upon which factor is occurring at corner frequency. For a simple pole, slope must be changed by -20 dB/ decade, for a simple zero by +20 dB/decade and for the quadratic poles by -40 dB/decade. Do not draw these individual lines.

f) Prepare the phase angle table and obtain the table of ω and resultant phase angle ϕ_r by actual calculation, plot these points and draw the smooth curve obtaining the necessary phase angle plot.

7.7 BASIC CONCEPT OF FILTERS

A filter is a device or process that removes some unwanted components or features from a signal. Filtering is a class of signal processing, the defining features of filters being the complete or partial suppression of some aspect of the signal. Filter networks are widely used in communication systems to separate various voice channels in carrier frequency telephone circuits. Filters also find applications in instrumentation, telemetry equipment etc where it is necessary to transmit or attenuate a limited range of frequencies.

7.7.1 Parameters of a Filter

The following parameters characterize a typical filter.

a) Characteristic impedance Z_c or Z_0

The characteristic impedance of a filter must be chosen such that the filter may fit into a given line or between two types of the equipment.

b) Pass band

Band, in which idea filters have to pass all frequencies without reduction in magnitude are referred to as pass band.

c) Stop band

Band in which ideal filters have to attenuate (or stop) frequencies are referred to as a stop band.

d) Cut-off frequency f_c

The frequency which separates the pass-band and the stop band is defined as the cut-off frequency of the filter.

7.8 CLASSIFICATION OF FILTERS

The filters may, in principle, have any number of pass bands separated by attenuation bands. However, they are classified into four common types as follows.

7.8.1 Low Pass Filters

These filters reject all frequencies above cut-off frequency, f_c. The attenuation characteristic of an ideal low pass filter is shown in figure 7.12 (a). Thus the pass band or transmission band for the low pass filter is the frequency range 0 to f_c and the stop band or attenuation band is the frequency range of f_c.

7.8.2 High Pass Filters

These filters reject all frequencies below cut-off frequency, f_c. Thus the pass band and stop band of the high pass filter are the frequency range above f_c and below f_c respectively. The attenuation characteristic of a HP filter is shown in figure 7.12(b)

7.8.3 Band Pass Filters

These filters allow transmission of frequencies between two designated cut off frequencies and reject all other frequencies. As shown in figure a band pass filter has two cut off frequencies and will have the pass band $f_{c_2} - f_{c_1}$. f_{c_1} is called as the lower cut-off frequency while f_{c_2} is called the upper cut-off frequency.

7.8.4 Band Stop or Band Elimination Filters

These filters pass all frequencies lying outside a certain range, while it attenuates all frequencies between the two frequencies f_{c_1} and f_{c_2}.

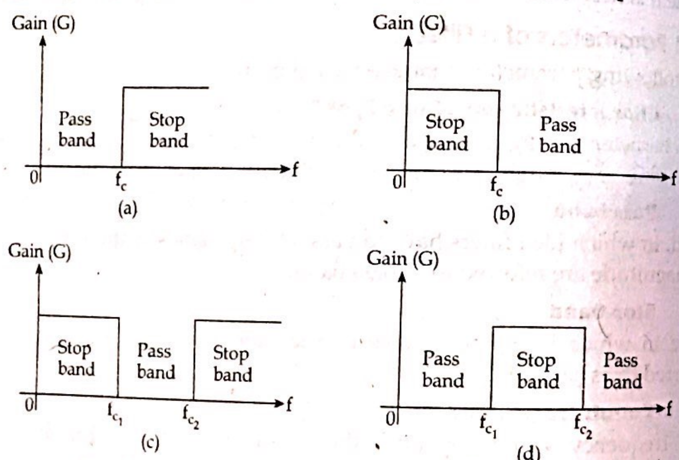


Figure 7.12: Ideal response of (a) low pass (L.P), (b) High-pass (H.P), (c) Band–pass (B.P) and (d) Band stop (B.S) filters.

Here, $G = \frac{V_0}{V_i}$ represents the gain of the filter and attenuation is inverse of the gain of the filters.

7.9 BLOCK DIAGRAM REPRESENTATION OF THE FILTERS

The block diagram of low pass and high pass filters are shown in figure 7.13 (a) and (b) respectively.

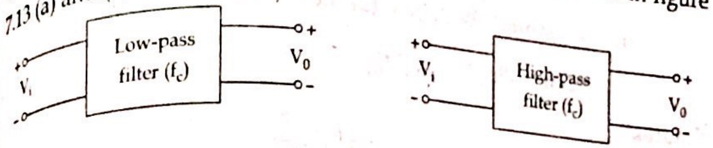


Figure 7.13: Block diagram representation

A band pass filter, is generally obtained by series or cascade connection of a low-pass filter and high-pass fitter as shown in figure 7.14 (a) where cut-off frequency f_{c_1} of high-pass filter is less than the cut-off frequency f_{c_2} of low pass filter. While a band-stop filter is obtained by parallel connection of a low-pass filter and high-pass filter as shown in figure 7.13 (b) where cut-off frequency f_{c_2} of high-pass filter is greater than the cut-off frequency f_{c_1} of low-pass filter.

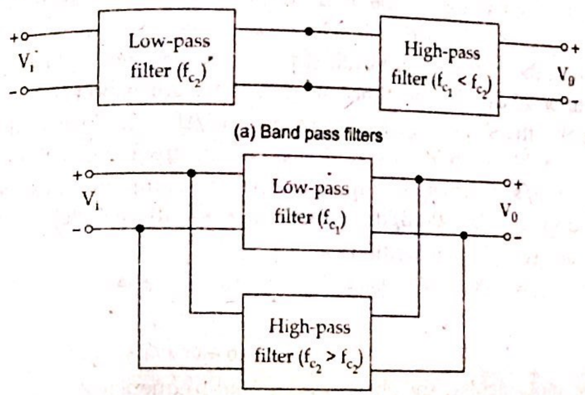


Figure 7.14: Block diagram representation

Mathematically, Band-pass filter is intersection of low-pass and high-pass filters *i.e.*, B.P = (L.P)∩(H.P) while the band-stop filter is union of them *i.e.*, B.S = (L.P)∪(H.P).

Types of filters	Pass band	Attenuation band
Low-pass	$0 \to f_c$	$f_c \to \infty$
High-pass	$f_c \to \infty$	$0 \to f_c$
Band pass	$f_{c_1} \to f_{c_2}$	$0 \to f_{c_1}, f_{c_2} \to \infty$
Band stop	$0 \to f_{c_1}, f_{c_2} \to \infty$	$f_{c_1} \to f_{c_2}$

SOLVED NUMERICAL EXAMPLES

1. Draw asymptotic Bode plot for the following functions.

$$G(s) = 20\frac{(s+2)}{s(s^2+4s+16)}$$

Solution:

$$G(s) = \frac{20\left(1+\frac{s}{2}\right)\cdot 2}{s\left[\frac{s^2}{16}+\frac{4s}{16}+1\right]\cdot 16} = \frac{2.5\left(1+\frac{s}{2}\right)}{s\left(\frac{s^2}{16}+\frac{s}{4}+1\right)}$$

Comparing $\frac{s^2}{16}+\frac{s}{4}+1$ with $\frac{s^2}{\omega_n^2}+2\frac{\zeta^2}{\omega_n}s+1$,

$\omega_n^2 = 16$ $\quad\therefore\quad$ $\omega_n = 4$

$\frac{2\zeta}{\omega_n} = \frac{1}{4}$ $\quad\therefore\quad$ $\zeta = 0.5$

We start with the factor 2.5/jω corresponds to the pole at origin. Its magnitude plot having a slope of –20 dB/decade and passes through the point 20 $\log_{10}$ 2.5 = 7.96 dB at ω = 1, meets the y-axis. This is the starting point. From the starting point to the first comer frequency (2 rad/sec) the slope will be 20 dB/decade. Since ω = 2 rad/sec corresponding to a zero, at this point the slope is increased by +20 dB/decade. Hence, the slope will be –20 + 20 = 0 dB/decade up to next corner frequency *i.e.*, 4 rad/sec. At ω = 4 rad/sec corresponding to the pair of complex poles the slope must be decreased by-40 dB/decade (0 – 40 = –40 dB/decade).

Now the phase of G(ω) is written as,

$$\angle G(j\omega) = \angle 20 + \angle(2+j\omega) - \angle j\omega - \angle(16+\omega^2) + j4\,\omega$$

$$= 0 + \tan^{-1}\left(\frac{\omega}{2}\right) - 90° - \tan^{-1}\left(\frac{4\omega}{16-\omega^2}\right)$$

For phase plot calculate the phases at different frequencies:

ω	$\tan^{-1}\frac{\omega}{2}$	-90°	$-\tan^{-1}\left(\frac{4\omega}{16-\omega^2}\right)$	Resultant ∠G(jω)
0.1	2.86°	-90°	-1.43°	-88.57°
0.2	5.71°	-90°	-2.87°	-87.16°
0.5	14.04°	-90°	-7.23°	-83.19°
1.0	26.56°	-90°	-14.93°	-78.37°
2.0	45.0°	-90°	-33.69°	-78.69°
4.0	63.43°	-90°	-90°	-116.57°
10	78.69°	-90°	-154.54°	-165.57°
20	84.28°	-90°	-168.23°	-173.95°
50	87.71°	-90°	-175.4°	-177.69°
100	88.85	-90°	-177.71°	-178.86°

The Bode plot is shown in figure below;

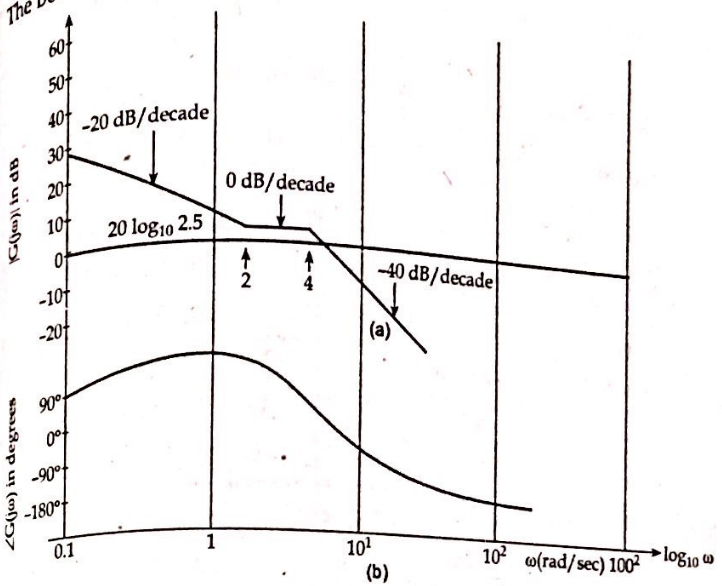


Figure: Bode plot

2. Draw the Bode log-magnitude and phase plots for the following system $G(s) = \frac{s+3}{s(s+1)(s+2)}$.

Solution:

Step 1: Arrange G(s) in the time-constant form

$$G(s) = \frac{3\left(1+\frac{s}{3}\right)}{2s(1+s)\left(1+\frac{s}{2}\right)} = \frac{3\left(1.5+\frac{s}{3}\right)}{2s(1+s)\left(1+\frac{s}{2}\right)}$$

Step 2: Factor and analysis

i) K = 15 hence contribution by K is 20 log 1.5 = 3.52 dB.

ii) One pole at the origin *i.e.*, $\frac{1}{s}$. It contribute a straight line of slope -20 dB/decade passing through the intersection of 0 dB and ω = 1.

iii) Shift this intersection point on 3.52 dB line and draw parallel line to the line drawn is step ii. This is addition of constant $K = \frac{3}{2}$ and one pole at the origin.

iv) Simple pole $\frac{1}{1+s}$. Comparing with $\frac{1}{1+T_1 s}$, $T_1 = 1$, $\omega_{c1} = 1$ rad/sec. It contribute straight line of slope –20 dB/decade for ω > 1.

v) Simple pole $\frac{1}{1+\frac{s}{2}}$. Comparing with $\frac{1}{1+T_2 s}$, $T_2 = \frac{1}{2}$, $\omega_{c2} = 2$ rad/sec.

It contribute straight line of slope –20 dB/decade for ω>2.

vi) Simple zero $\left(1+\frac{3}{3}\right)$ comparing with $(1 + T_3\ s)$, $T_3 = \frac{1}{3}$, $\omega_{c3} = 3$ rad/sec.

Range of ω	Factor	Resultant slope
Starting	Pole at origin	-20 dB/decade
$0 < \omega < 1$	-	-20 dB/decade
$1 < \omega < 2$	Pole at $\omega_{c_1} = 1$	-20 – 20 = –40 dB/decade
$2 < \omega < 3$	Pole at $\omega_{c_2} = 2$	–40 – 20 = –60 dB/decade
$3 < \omega < \infty$	Zero at $\omega_{c_3} = 3$	-60 + 20 = –40 dB/decade

Step 3: Phase angle table

$$G(j\omega) = \frac{1.5\left(1+\frac{j\omega}{3}\right)}{j\omega\,(1+\omega)\left(1+\frac{j\omega}{2}\right)}$$

ω	$\frac{1}{j\omega}$	$1+\frac{j\omega}{3} = \tan^{-1}\frac{\omega}{3}$	$1+\frac{j\omega}{3}$ $= \tan^{-1}$	$\frac{1}{\left(1+\frac{\omega}{2}\right)} = -\tan^{-1}\left(\frac{\omega}{2}\right)$	Resultant ; ϕ
0.1	-90°	+1.91°	-5.71°	-2.86°	-96.66°
1	-90°	+18.43°	-45°	-26.56°	-143.13°
2	-90°	+33.69°	-63.43°	-45°	-164.74°
3	-90°	+45	-71.56°	-56.31°	-172.87°
10	-90°	+73.3°	-84.29°	-78.69°	-179.68°
100	-90°	+88.28°	-89.42°	-88.85°	-179.99°
∞	-90°	+90°	-90°	-90°	-180°

Step 4: Bode plot

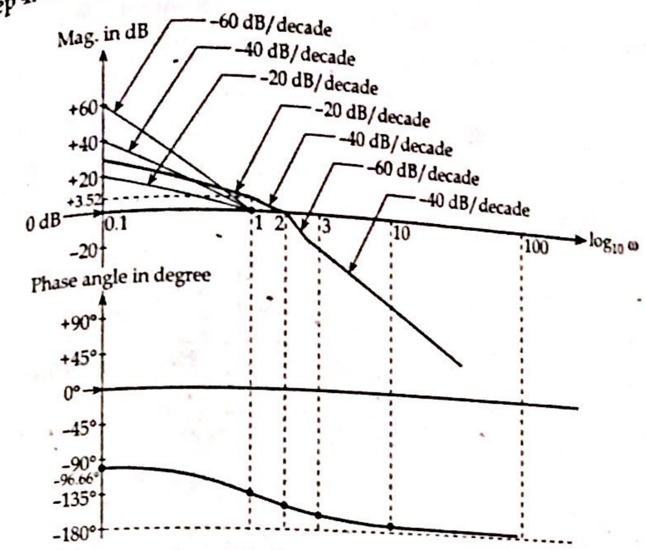


Figure: Bode plot

3. **Draw the bode plot for $G(s) = \frac{20}{s(s+2)(s+10)}$.**

Solution:

Step 1: Arrange G(s) in the time constant form

$$G(s) = \frac{1}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{10}\right)}$$

Step 2: Factors and analysis

i) K = 1 hence contribution by K is $20 \log_{10} 1 = 0$ dB.

ii) One pole at the origin *i.e.*, $\frac{1}{s}$. It contributes a straight line of slope - 20 dB/decade passing through the intersection of 0 dB and $\omega = 1$.

iii) One simple pole $\frac{1}{1+\frac{s}{2}}$. It contributes a straight line of slope -20 dB/decade for $\omega > 2$.

iv) Another simple pole $\frac{1}{1+\frac{s}{10}}$. It contributes a straight line of slope - 20 dB/decade for $\omega > 10$.

The resultant slope table is

Range of ω	Factor	Resultant slope
Starting	Pole at origin	-20 dB/decade
$0 \angle \omega \angle 2$	-	-20 dB/decade
$2 \angle \omega \angle 10$	Pole at $\omega_{c_1} = 2$	-20 – 20 = -40 dB/decade
$10 \angle \omega \angle \infty$	Pole at $\omega_{c_2} = 10$	-40 -20 = -60 dB/decade

Step 3: Phase angle table

$$G(j\omega) = \frac{1}{j\omega\left(1+\frac{j\omega}{2}\right)\left(1+\frac{j\omega}{10}\right)}$$

ω	$\frac{1}{j\omega}$	$\frac{1}{1+\frac{j\omega}{2}}$	$\frac{1}{1+\frac{j\omega}{\omega}}$	Resultant, ϕ_r
0.1	-90°	-2.86°	-0.57°	-93.43°
1	-90°	-26.56°	-5.71°	-122.27°
2	-90°	-45°	-11.31°	-146.31°
10	-90°	-78.69°	-45°	-213.69°
100	-90°	-88.85°	-84.29°	-263.14°
∞	-90°	-90°	-90°	-270°

Step 4: Bode plot.

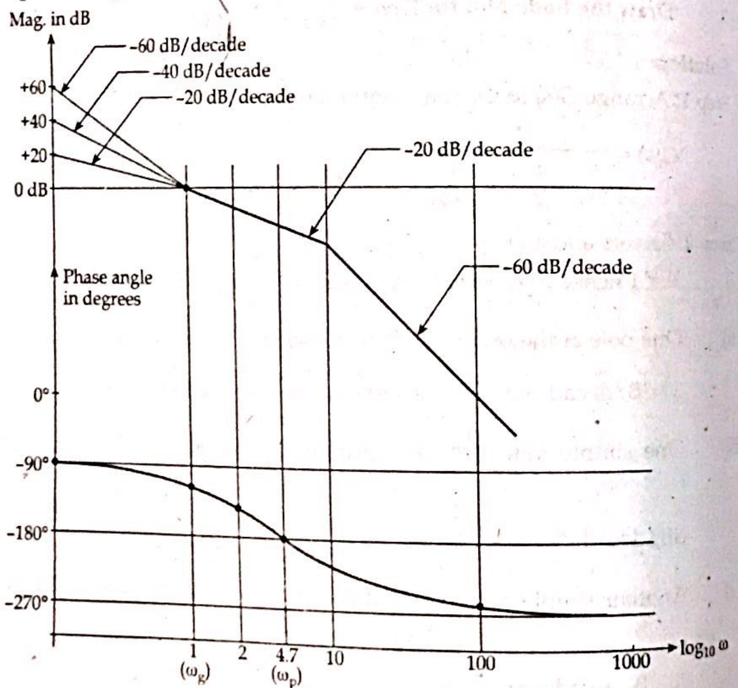


Figure: Bode plot

BOARD EXAMINATION SOLVED QUESTIONS

1. **Construct the asymptotic magnitude bode plot for the given transfer function.** **[2019/Spring]**

$$H(s) = \frac{10(s+1)(s+10)}{s^2(s+5)(s+100)}$$

Solution:

Given that;

$$H(s) = \frac{10(s+1)(s+10)}{s^2(s+5)(s+10)}$$

First converting each term to their standard form,

$$H(s) = \frac{10(1+s)\times 10\left(1+\frac{s}{10}\right)}{s^2\times 5\left(1+\frac{s}{5}\right)\times 100\left(1+\frac{s}{100}\right)}$$

$$= \frac{\frac{1}{5}(1+s)\left(1+\frac{s}{10}\right)}{s^2\left(1+\frac{s}{5}\right)\left(1+\frac{s}{100}\right)}$$

Magnitude Plot:

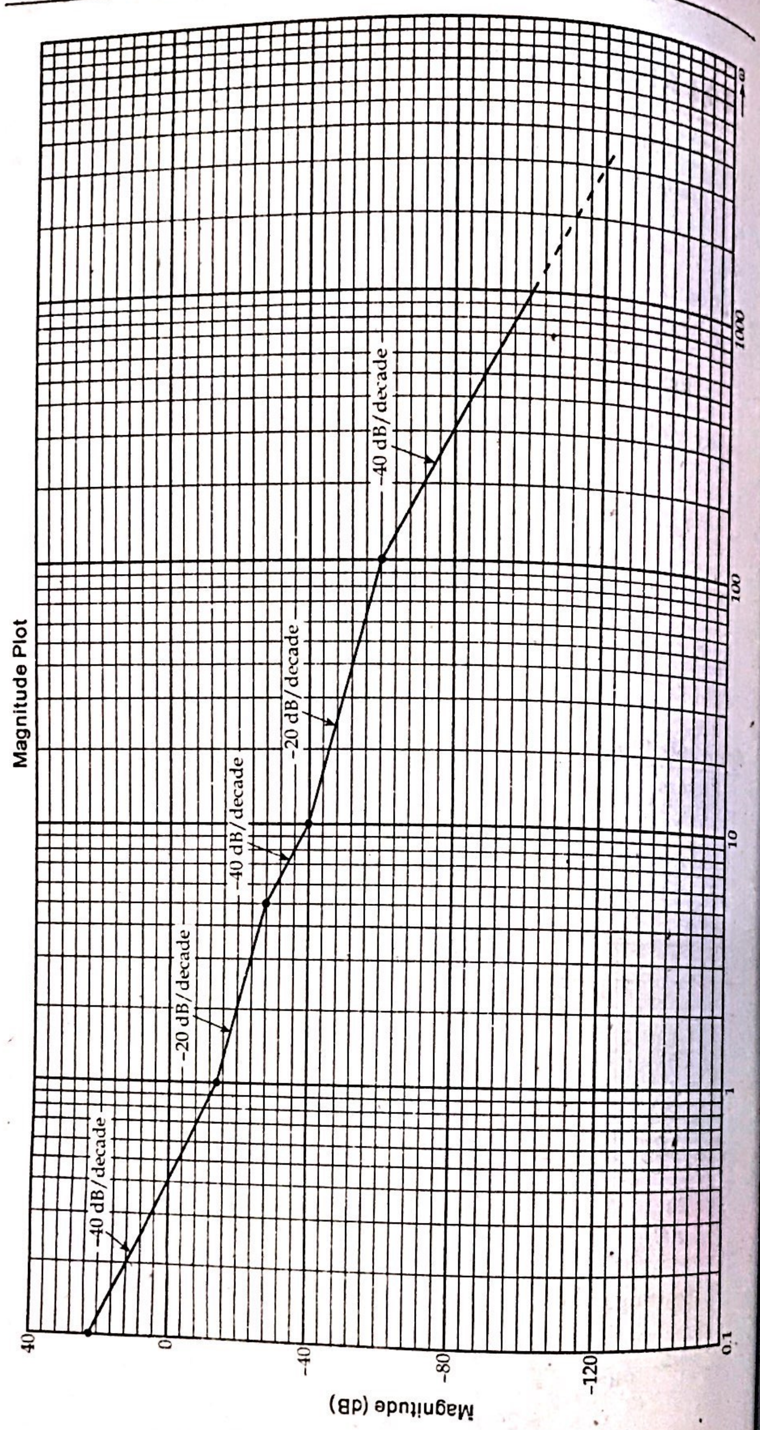
Factors	Corner frequency	Slope	Cumulative slope
$\left(\frac{1}{5}\right)$	-	-	-
$\frac{1}{s^2}$	-	-40 dB/decade	-40 dB/decade
$(1+s)$	1	20 dB/decade	-20 dB/decade
$\frac{1}{\left(1+\frac{s}{5}\right)}$	5	-20 dB/decade	-40 dB/decade
$\left(1+\frac{s}{10}\right)$	10	20 dB/decade	-20 dB/decade
$\frac{1}{\left(1+\frac{s}{100}\right)}$	100	-20 dB/decade	-40 dB/decade

∴ Starting point $= 20\log\left(\frac{1}{5}\right) - 20\times 2\log(0.1)$

$= -13.98 + 40$

$= 26.02$ dB

$\simeq 26$ dB



2. **Plot the straight line asymptotic magnitude and phase response for**

$$G(s) = \frac{(s+10)(s+1)}{s(s+50)}$$

[2019/Spring]

Solution:

Given that;

$$G(s) = \frac{(s+10)(s+1)}{s(s+50)}$$

Representing each factor in standard form,

$$G(s) = \frac{10\left(1+\frac{s}{10}\right)(1+s)}{s \times 50\left(1+\frac{s}{10}\right)} = \frac{\frac{1}{5}(1+s)\left(1+\frac{s}{10}\right)}{s\left(1+\frac{s}{50}\right)}$$

Magnitude Plot:

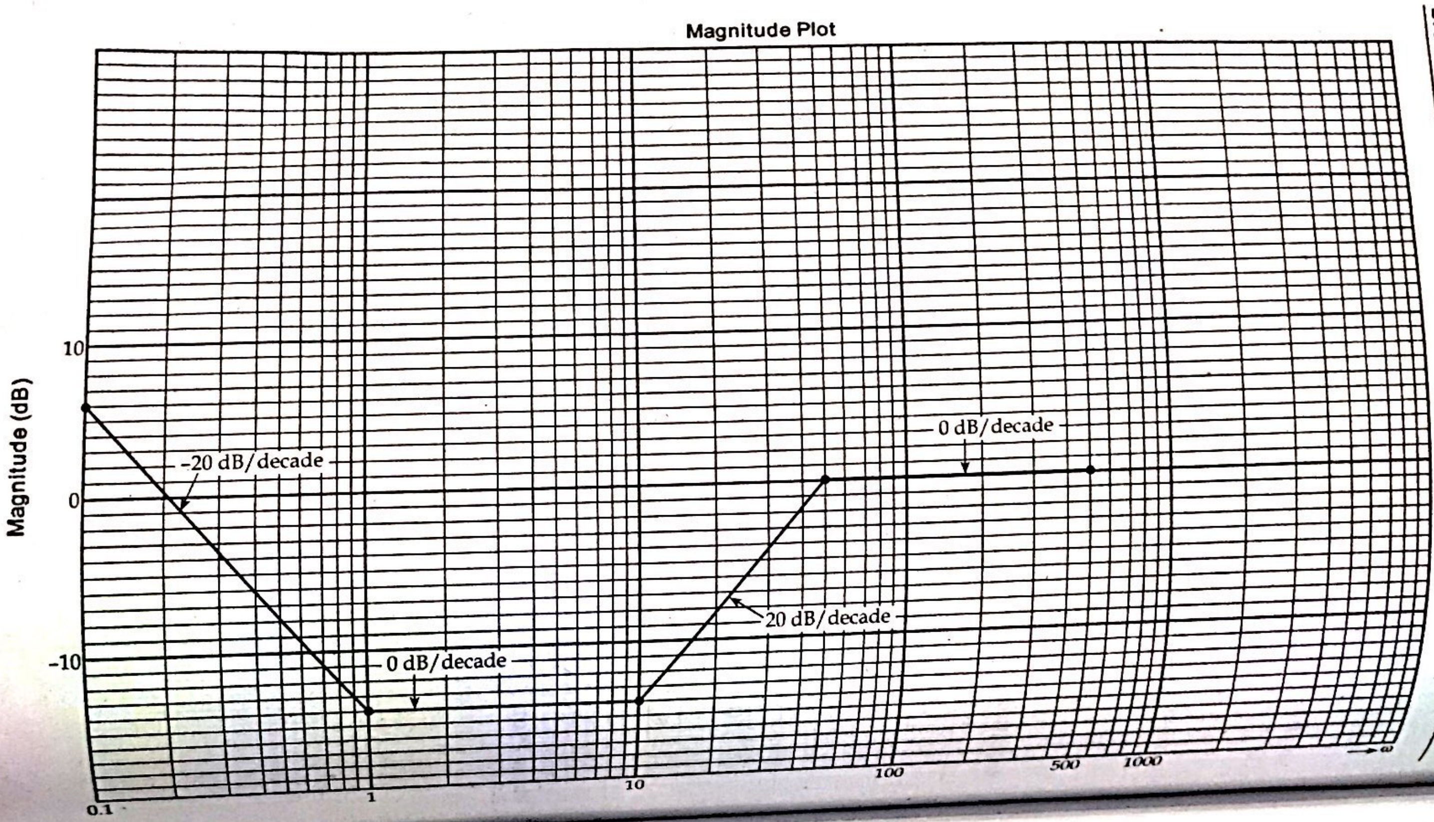
Factors	Corner frequency	Slope	Cumulative slope
$\left(\frac{1}{5}\right)$	-	-	-
$\frac{1}{s}$	-	-20 dB/decade	-20 dB/decade
$(1+s)$	1	20 dB/decade	0 dB/decade
$\left(1+\frac{s}{10}\right)$	10	20 dB/decade	20 dB/decade
$\frac{1}{\left(1+\frac{s}{100}\right)}$	50	-20 dB/decade	0 dB/decade

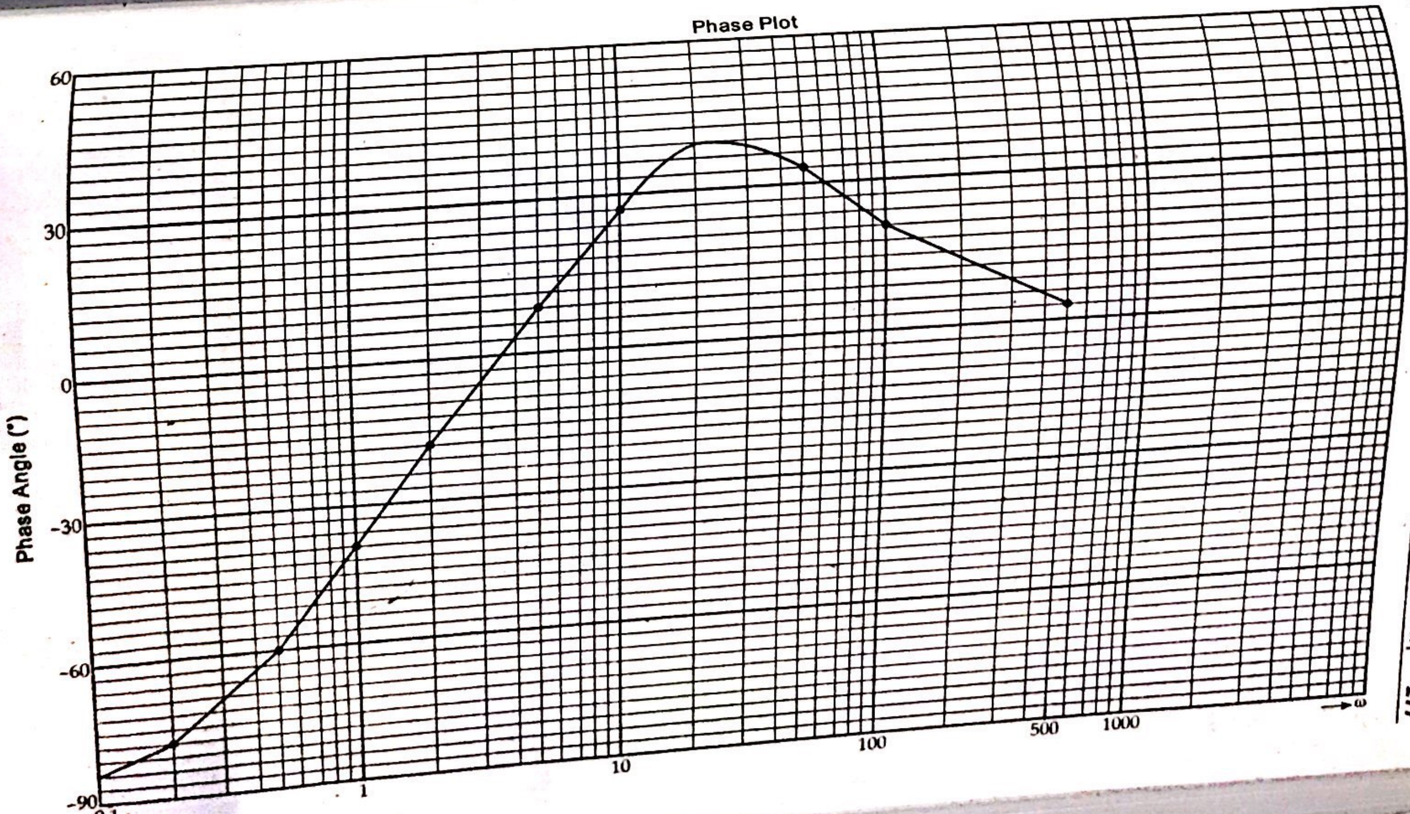
∴ Starting point $= 20 \log\left(\frac{1}{5}\right) - 20 \times 1 \log(0.1) = 6.02 \text{ dB} \simeq 6 \text{ dB}$

Phase Plot:

$$\phi = \tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{10}\right) - 90° - \tan^{-1}\left(\frac{\omega}{50}\right)$$

Frequency (ω)	Phase angle (ϕ)
0.1	-83.83°
0.2	-77.77°
0.5	-61.15°
1	-40.44°
2	-17.55°
5	9.54°
10	27.98°
20	38.77°
50	32.54°
100	20.28°
500	4.45°


Magnitude Plot
Magnitude (dB)
10
0
-10
-20 dB/decade
0 dB/decade
20 dB/decade
0 dB/decade
0.1
1
10
100
500
1000
ω


Phase Plot
Phase Angle (°)
60
30
0
-30
-60
-90
0.1
1
10
100
500
1000
ω

3. **Plot the straight line asymptotic magnitude and phase response for,**

$$G(s) = \frac{(s+10)}{s(s+0.5)(s+50)}$$ **[2019/Fall]**

Solution: Given that; $G(s) = \frac{(s+10)}{s(s+0.5)(s+50)}$

Representing each factor in standard form,

$$G(s) = \frac{10\left(1+\frac{s}{10}\right)}{s \times 0.5\left(1+\frac{s}{0.5}\right) \times 50 \times \left(1+\frac{s}{50}\right)} = \frac{\frac{2}{5}\left(1+\frac{s}{10}\right)}{s\left(1+\frac{s}{0.5}\right)\left(1+\frac{s}{50}\right)}$$

Magnitude Plot:

Factors	Corner frequency	Slope	Cumulative slope
$\left(\frac{2}{5}\right)$	-	-	-
$\frac{1}{s}$	-	-20 dB/decade	-20 dB/decade
$\frac{1}{\left(1+\frac{s}{0.5}\right)}$	0.5	-20 dB/decade	-40 dB/decade
$\left(1+\frac{s}{10}\right)$	10	20 dB/decade	-20 dB/decade
$\frac{1}{\left(1+\frac{s}{100}\right)}$	50	-20 dB/decade	-40 dB/decade

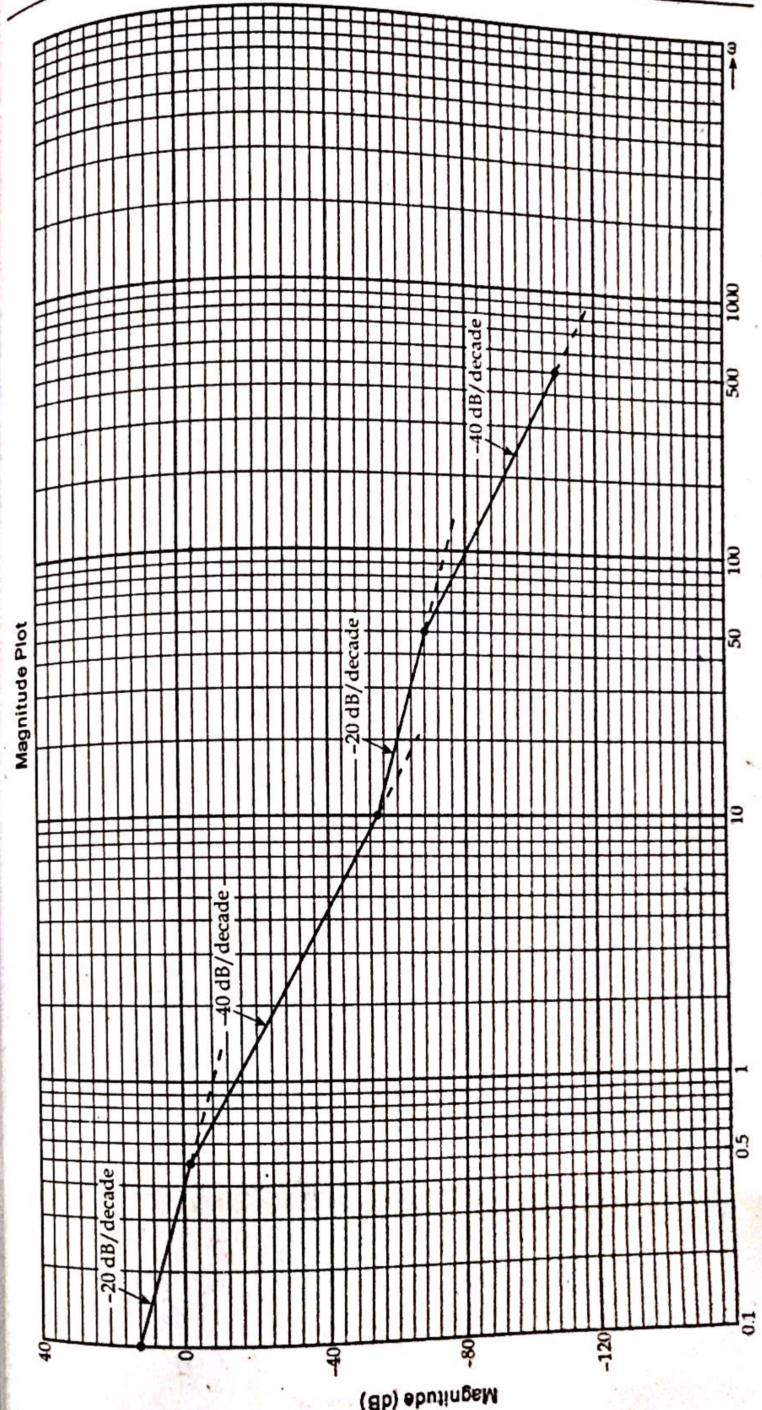
∴ Starting point $= 20\log\left(\frac{2}{5}\right) - 20\,\cancel{20}\log(0.1) = 12.04\text{ dB} = 12\text{ dB}$

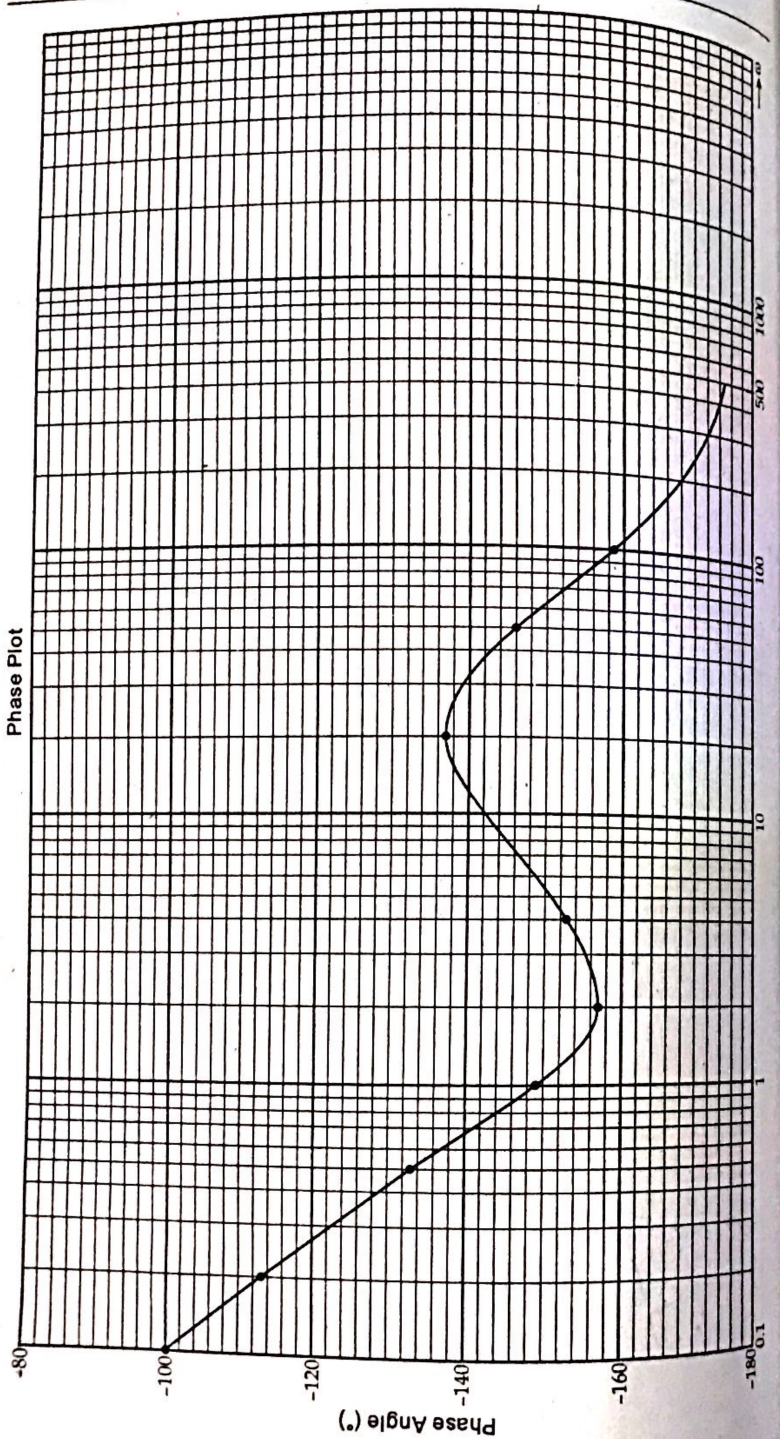
Phase Plot:

$$\phi = \tan^{-1}\left(\frac{\omega}{10}\right) - 90^\circ - \tan^{-1}\left(\frac{\omega}{0.5}\right) - \tan^{-1}\left(\frac{\omega}{50}\right)$$

$$= \tan^{-1}(0.1\,\omega) - 90^\circ - \tan^{-1}(2\,\omega) - \tan^{-1}(0.02\,\omega)$$

Frequency (ω)	Phase angle (ϕ)
0.1	-100.85°
0.2	-110.88°
0.5	-132.71°
1	-148.87°
2	-156.94°
5	-153.43°
10	-143.45°
20	-136.93°
50	-145.74°
100	-158.86°
500	-175.38°





4. **Sketch the bode plot for the following transfer function.[2017/Fall]**

$$G(s) = \frac{2s^2}{(1 + 0.25s)(1 + 0.02s)}$$

Solution:

$$G(s) = \frac{2s^2}{(1 + 0.25s)(1 + 0.02s)}$$

Here all factors are in standard form, so,

Magnitude Plot:

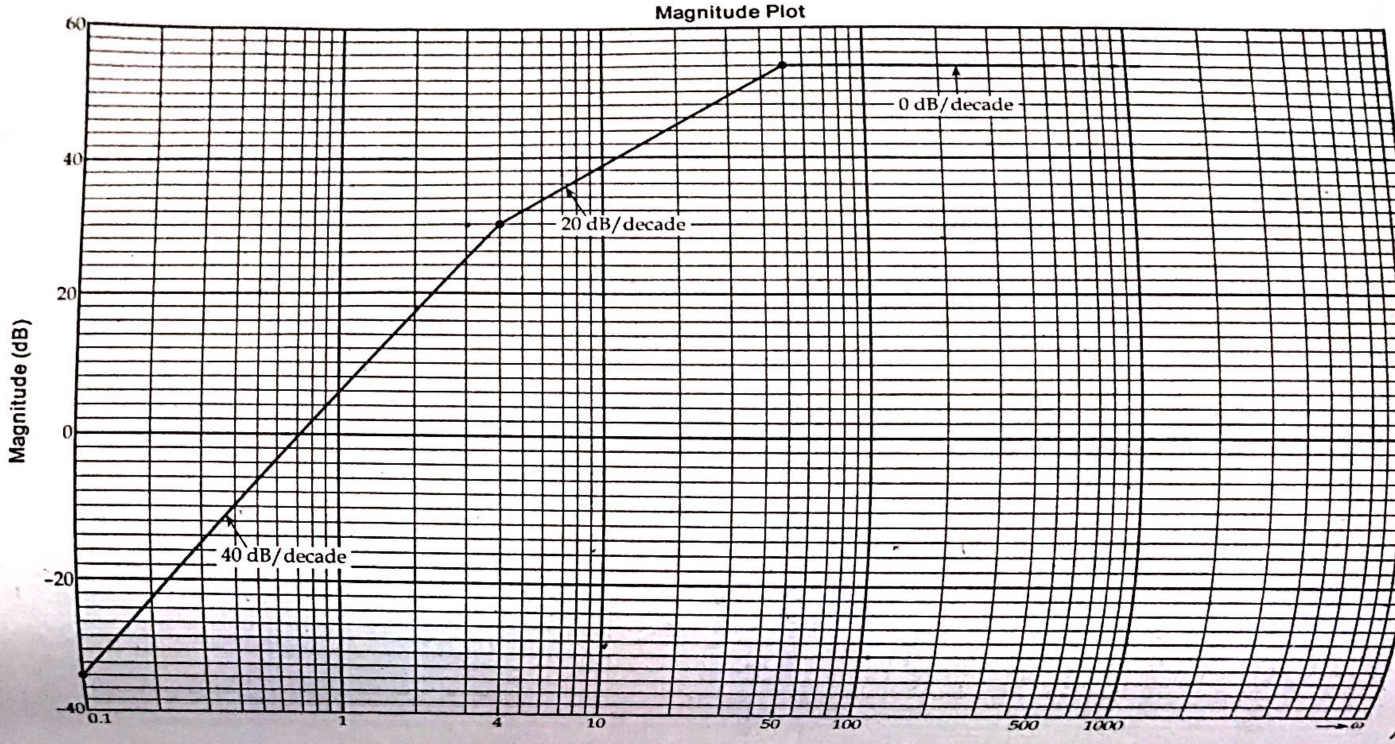
Factors	Corner frequency	Slope	Cumulative slope
2	-	-	-
s^2	-	40 dB/decade	40 dB/decade
$\frac{1}{(1+0.25s)}$	4	-20 dB/decade	20 dB/decade
$\frac{1}{(1+0.02s)}$	50	-20 dB/decade	0 dB/decade

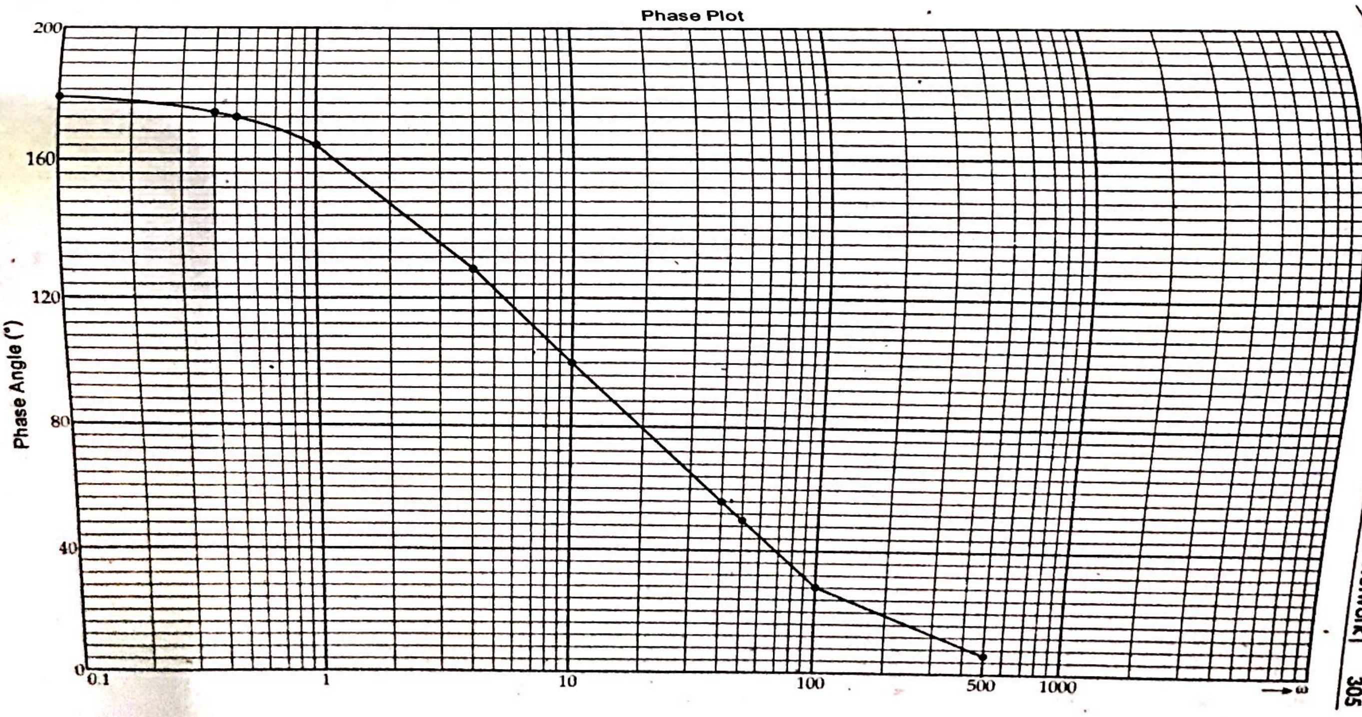
∴ Starting point = 20 log (2) + 20 × 2 log (0.1) = –33.98 dB ≃ 34 dB

Phase Plot:

$$\phi = 0 + 90° \times 2 - \tan^{-1}(0.25\omega) - \tan^{-1}(0.02\,\omega)$$
$$= 180° - \tan^{-1}(0.25\,\omega) - \tan^{-1}(0.02\,\omega)$$

Frequency (ω)	Phase angle (ϕ)
0.1	178.45°
0.4	173.83°
0.5	172.30°
1	164.82°
4	130.43°
5	122.95°
10	100.49°
40	57.05°
50	49.57°
100	28.85°
500	6.17°


Magnitude Plot
Magnitude (dB)
60
40
20
0
-20
-40
0.1
1
4
10
50
100
500
1000
ω
0 dB/decade
20 dB/decade
40 dB/decade


Phase Plot
Phase Angle (°)
200
160
120
80
40
0
0.1
1
10
100
500
1000
ω

5. Plot the straight line asymptotic magnitude response and phase of $G(s) = \dfrac{(2s + 10)}{(0.5s + 1)(2s + 2)}$ **[2016/Spring]**

Solution:

Given that;

$$G(s) = \frac{(2s + 10)}{(0.5s + 1)(2s + 2)}$$

Representing each fctor in standard form,

$$G(s) = \frac{10\left(1 + \frac{s}{5}\right)}{(1 + 0.5s) \times 2(1 + s)} = \frac{5\left(1 + \frac{s}{5}\right)}{(1 + s)\left(1 + \frac{s}{2}\right)}$$

Magnitude Plot:

Factors	Corner frequency	Slope	Cumulative slope
5	–	–	–
$\frac{1}{(1+s)}$	1	-20 dB/decade	-20 dB/decade
$\frac{1}{\left(1+\frac{s}{2}\right)}$	2	-20 dB/decade	-40 dB/decade
$\left(1+\frac{s}{10}\right)$	5	20 dB/decade	-20 dB/decade

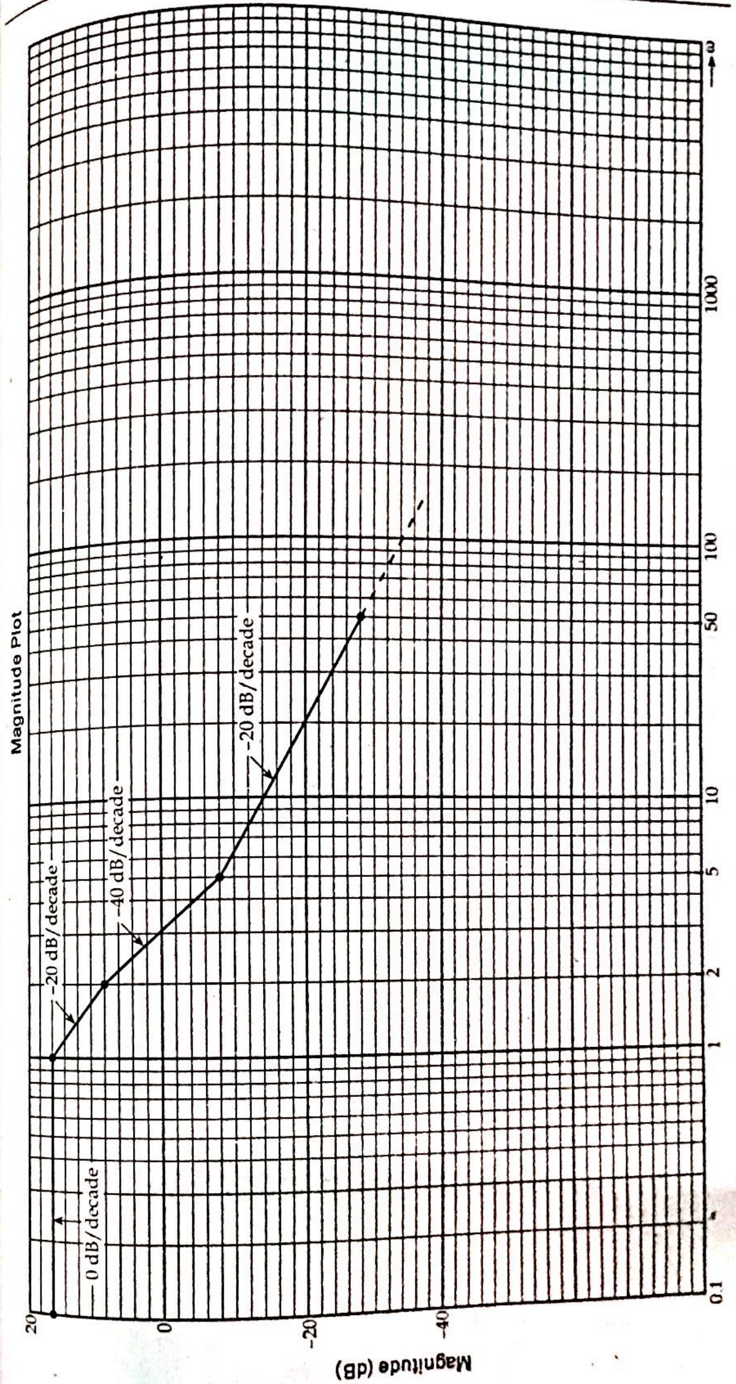
$\therefore$ Starting point = 20 log (5) = 13.98 dB ≃ 14 dB

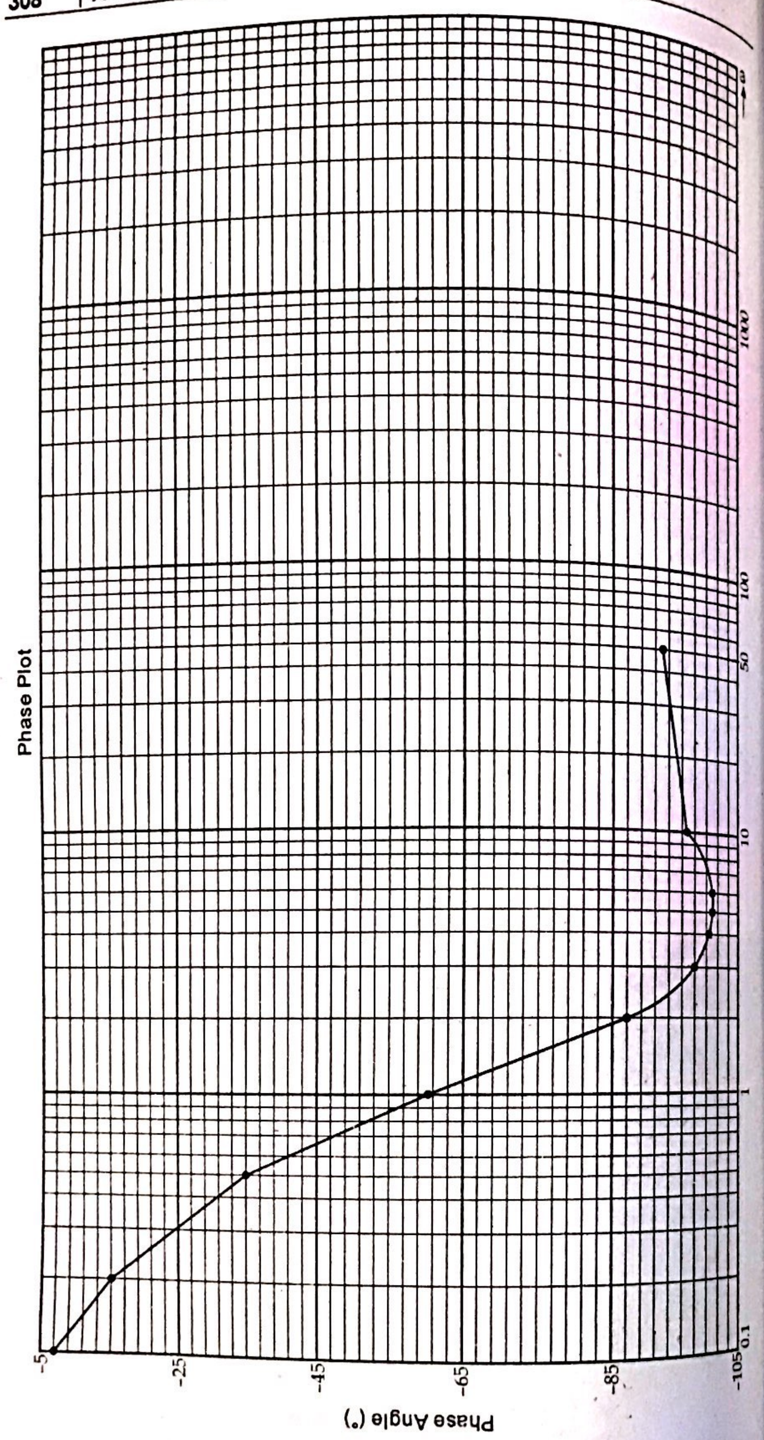
Phase Plot:

$$\phi = 0 + \tan^{-1}\left(\frac{\omega}{5}\right) - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{2}\right)$$

$$= \tan^{-1}(0.2\,\omega) - \tan^{-1}(\omega) - \tan^{-1}(0.5\,\omega)$$

Frequency (ω)	Phase angle (ϕ)
0.1	-7.43°
0.2	-14.73°
0.5	-34.89°
1	-60.25°
2	-86.63°
3	-96.91°
4	-100.74°
5	-101.89°
6	-101.91°
10	-99.54°
50	-92.27°





6. **Draw magnitude and phase plot for the given transfer function.**

$$G(s) = \frac{100(s+10)}{s(s+2)(s+5)}$$

[2015/Spring, 2015/Fall]

Solution:

Given that;

$$G(s) = \frac{100(s+10)}{s(s+2)(s+5)}$$

Representing each factor in standard form,

$$G(s) = \frac{100\left(1+\frac{s}{10}\right)}{s \times 2\left(1+\frac{s}{2}\right) \times 5\left(1+\frac{s}{5}\right)} = \frac{100\left(1+\frac{s}{10}\right)}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{5}\right)}$$

Magnitude Plot:

Factors	Corner frequency	Slope	Cumulative slope
5	–	–	–
$\frac{1}{s}$	–	-20 dB/decade	-20 dB/decade
$\frac{1}{\left(1+\frac{s}{2}\right)}$	2	-20 dB/decade	-40 dB/decade
$\frac{1}{\left(1+\frac{s}{5}\right)}$	5	-20 dB/decade	-60 dB/decade
$\left(1+\frac{s}{10}\right)$	10	20 dB/decade	-40 dB/decade

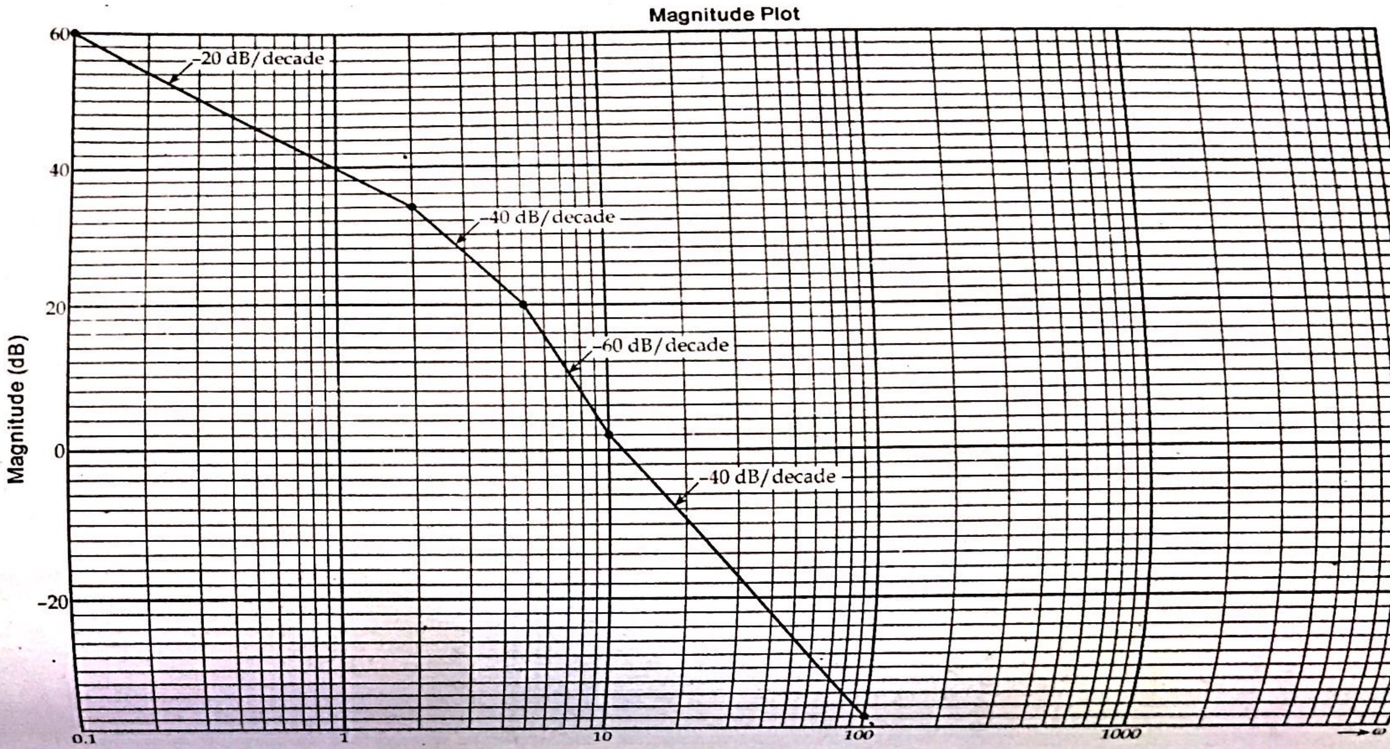
∴ Starting point = 20 log (100) – 20 log (0.1) = 60 dB

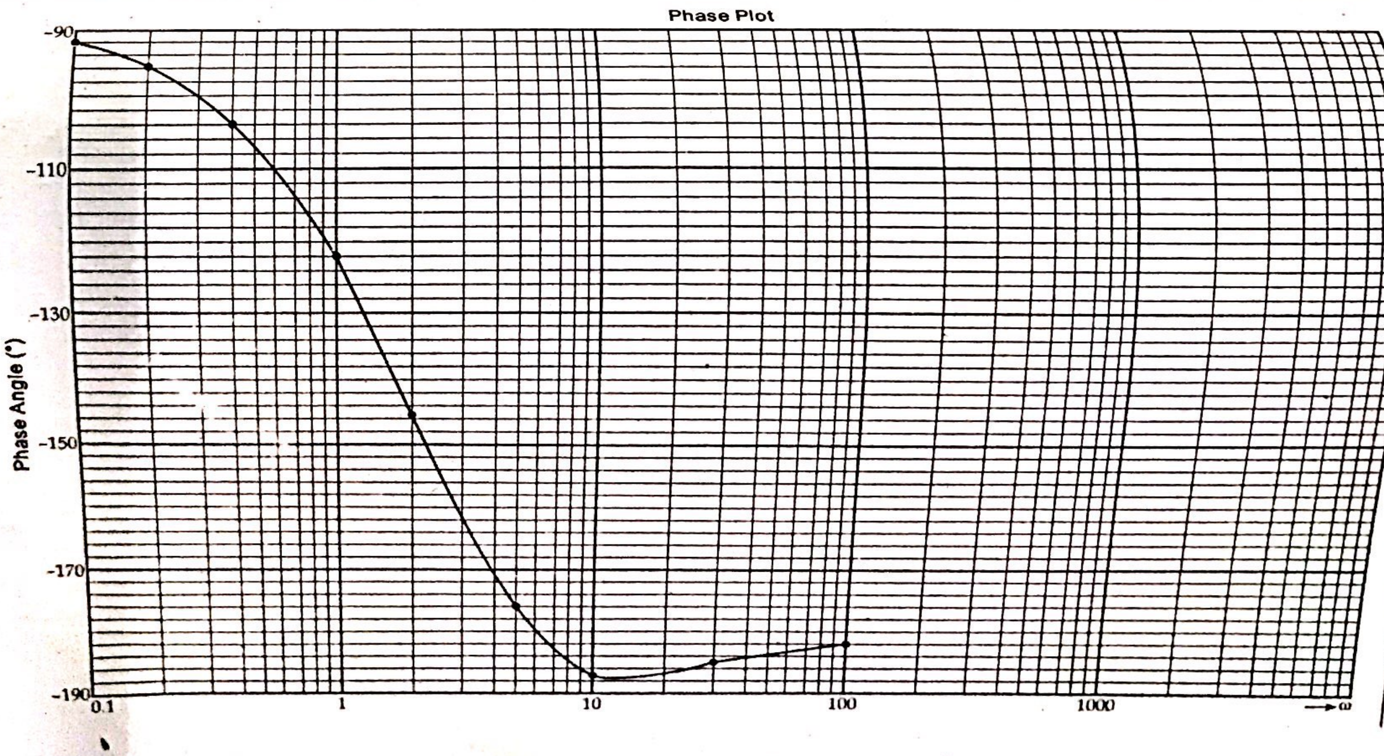
Phase Plot:

$$\phi = 0 + \tan^{-1}\left(\frac{\omega}{10}\right) - 90° - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{5}\right)$$

$$= \tan^{-1}(0.1\,\omega) - 90° - \tan^{-1}(0.5\,\omega) - \tan^{-1}(0.2\,\omega)$$

Frequency (ω)	Phase angle (ϕ)
0.1	-93.43°
0.3	-100.25°
0.5	-106.88°
1	-122.16°
2	-145.49°
3	-160.57°
5	-176.63°
7	-183.52°
10	-187.12°
30	-185.16°
100	-181.70°


Magnitude Plot
Magnitude (dB)
60
40
20
0
-20
–20 dB/decade
–40 dB/decade
–60 dB/decade
–40 dB/decade
0.1
1
10
100
1000
ω


Phase Plot
Phase Angle (°)
-90
-110
-130
-150
-170
-190
0.1
1
10
100
1000
ω

7. **Sketch the magnitude plot for the transfer function.**

$$G(s) = \frac{10(0.2s + 1)}{s(0.1s + 1)}$$

[2017/Spring]

Solution:

Given that;

$$G(s) = \frac{10(0.2s + 1)}{s(0.1s + 1)}$$

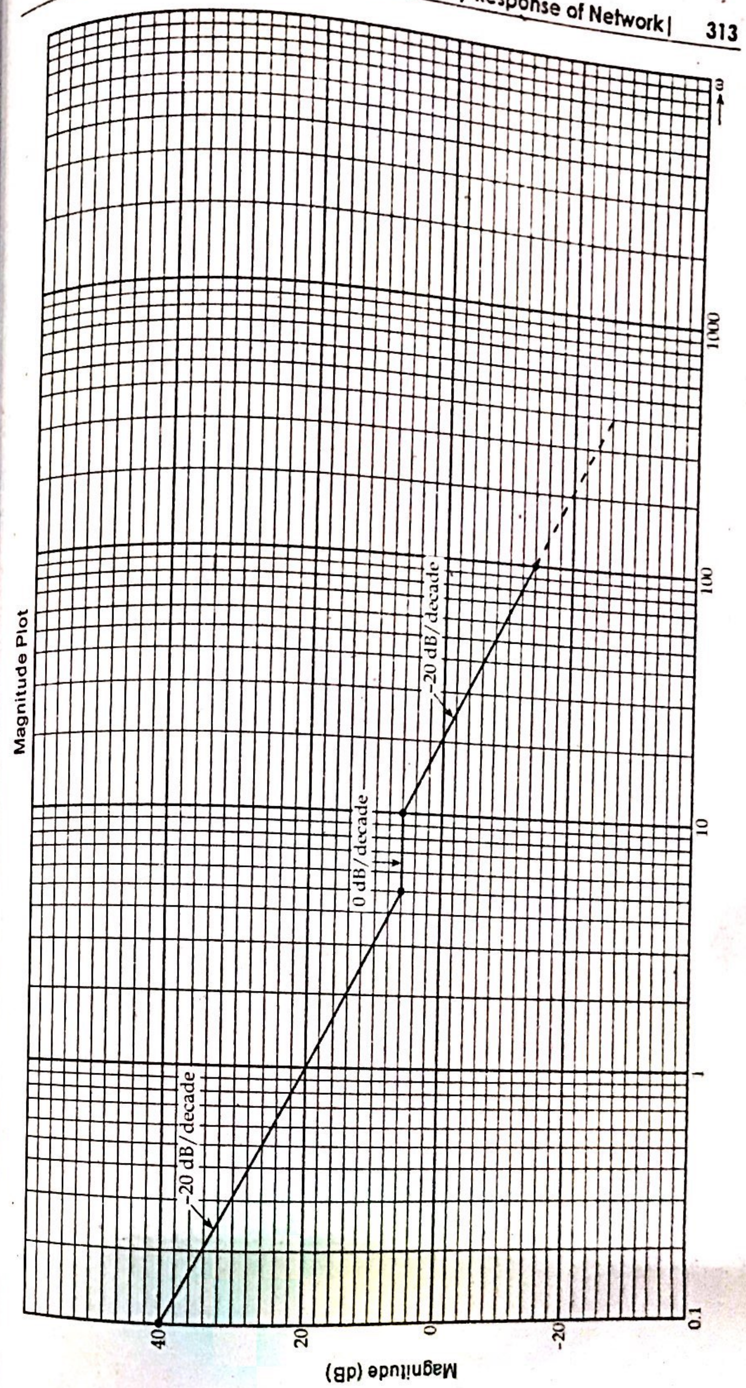
Here, the factors are in standard form

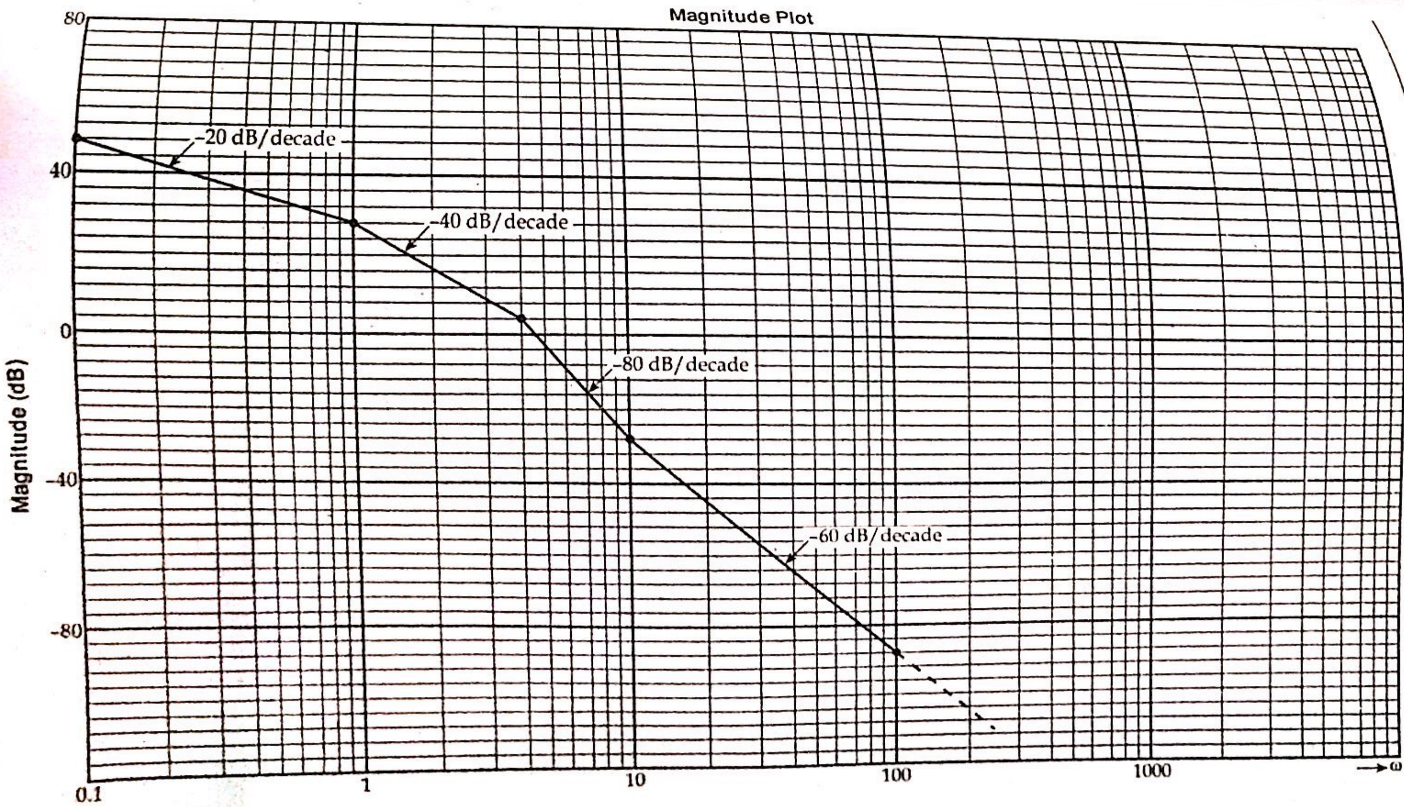
Magnitude Plot:

Factors	Corner frequency	Slope	Cumulative slope
10	–	–	–
$\frac{1}{s}$	–	-20 dB/decade	-20 dB/decade
(1 + 0.2s)	5	20 dB/decade	0 dB/decade
$\frac{1}{(1 + 0.1s)}$	10	-20 dB/decade	-20 dB/decade

$\therefore$ Starting point = 20 log (10) – 20 log (0.1)

= 40 dB





Magnitude Plot
80
40
0
-40
-80
Magnitude (dB)
-20 dB/decade
-40 dB/decade
-80 dB/decade
-60 dB/decade
0.1
1
10
100
1000
ω

9. Plot the straight line asymptotic magnitude response for

$$G(s) = \frac{40\,(s+100)}{s(s+10)\,(s^2+70s+1000)}$$ [2018/Spring]

Solution:

Given that;

$$G(s) = \frac{40\,(s+100)}{s(s+10)\,(s^2+70s+1000)}$$

$$= \frac{40\,(s+100)}{s(s+100)\,(s+20)\,(s+50)}$$

$$= \frac{40\times 100\left(1+\frac{s}{100}\right)}{s\times 10\left(1+\frac{s}{10}\right)\times 20\left(1+\frac{s}{20}\right)\times 50\left(1+\frac{s}{50}\right)}$$

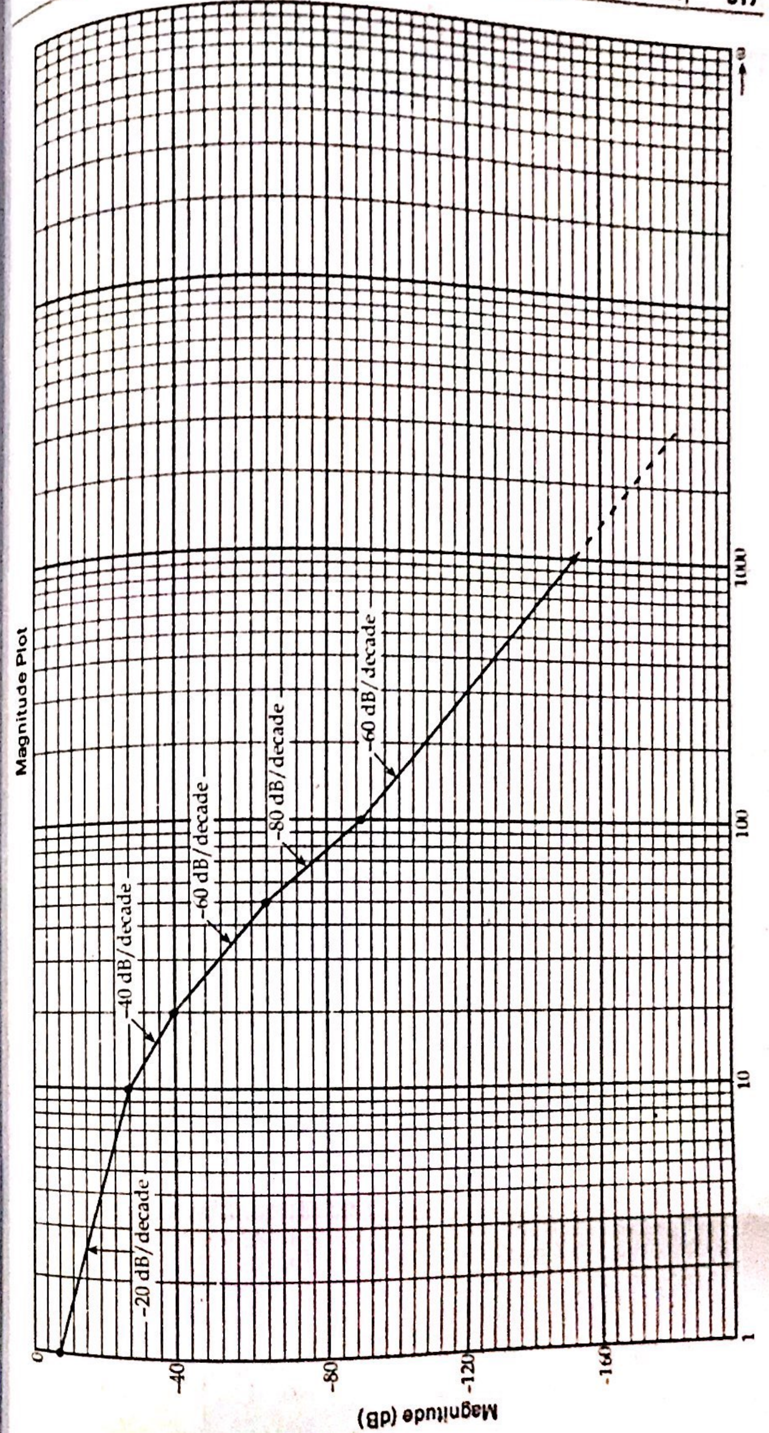
$$= \frac{0.4\times\left(1+\frac{s}{100}\right)}{s\left(1+\frac{s}{10}\right)\left(1+\frac{s}{20}\right)\left(1+\frac{s}{50}\right)}$$

Magnitude Plot:

Factors	Corner frequency	Slope	Cumulative slope
0.4	-	-	-
$\frac{1}{s}$	-	-20 dB/decade	-20 dB/decade
$\frac{1}{\left(1+\frac{s}{10}\right)}$	10	-20 dB/decade	-60 dB/decade
$\frac{1}{\left(1+\frac{s}{20}\right)}$	20	-20 dB/decade	-60 dB/decade
$\frac{1}{\left(1+\frac{s}{50}\right)}$	50	-20 dB/decade	-80 dB/decade
$\frac{1}{\left(1+\frac{s}{100}\right)}$	100	20 dB/decade	-60 dB/decade

$\therefore$ Starting point $= 20\log(0.4) - 20\log(1)$

$= -7.96\text{ dB} \simeq -8\text{ dB}$



10. **Sketch the bode plot for the transfer function**

$$G(s) = \frac{20(1+5s)}{2(1+0.5s)(1+0.05s)(1+50s)}$$ **[2014/Spring]**

Solution:

Given that;

$$G(s) = \frac{20(1+5s)}{2(1+0.5s)(1+0.05s)(1+50s)} = \frac{20\left(1+\frac{s}{0.2}\right)}{s\left(1+\frac{s}{2}\right)\left(1+\frac{s}{20}\right)\left(1+\frac{s}{0.02}\right)}$$

All factors are in standard form.

Here we start from $\omega = 0.01$ and our corner frequency includes 0.02.

Magnitude Plot:

Factors	Corner frequency	Slope	Cumulative slope
20	–	–	–
$\frac{1}{s}$	–	–20 dB/decade	–20 dB/decade
$\frac{1}{\left(1+\frac{s}{0.02}\right)}$	0.02	–20 dB/decade	–40 dB/decade
$\frac{1}{\left(1+\frac{s}{0.02}\right)}$	0.2	20 dB/decade	–20 dB/decade
$\frac{1}{\left(1+\frac{s}{2}\right)}$	2	–20 dB/decade	–40 dB/decade
$\frac{1}{\left(1+\frac{s}{20}\right)}$	20	–20 dB/decade	–60 dB/decade

$\therefore$ Starting point = 20 log (20) – 20 log (0.01) = –66.02 dB ≃ 66 dB

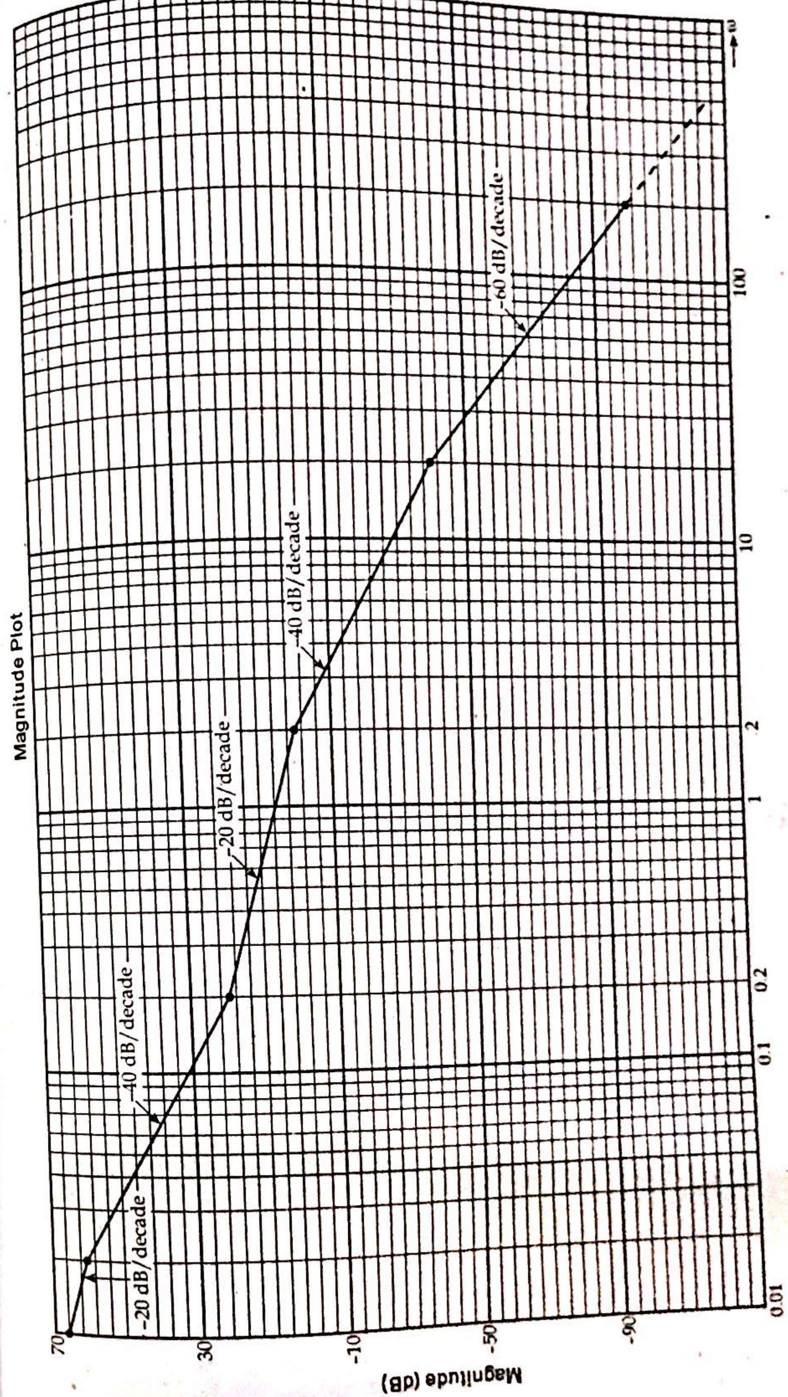
Phase Plot:

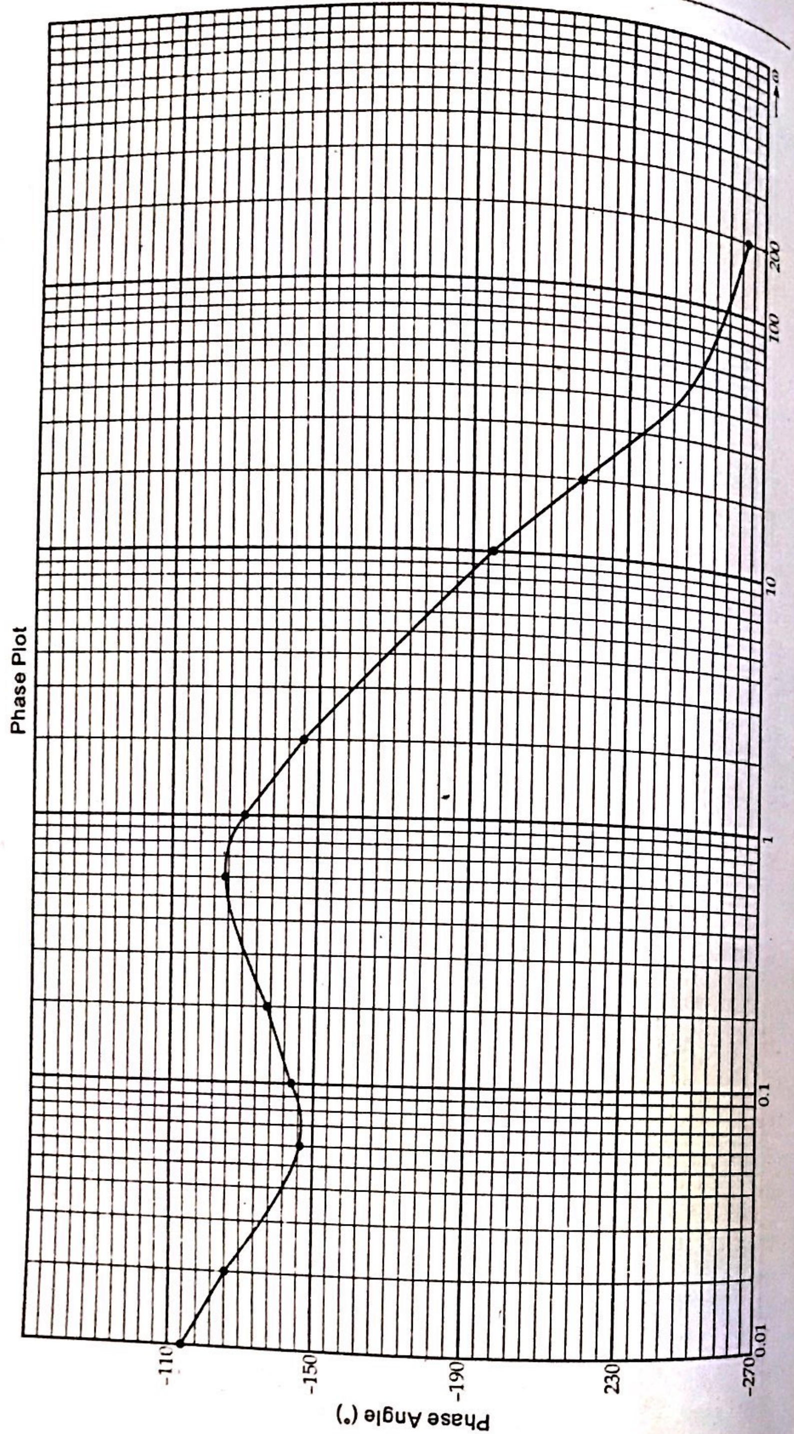
$$\phi = 0 + \tan^{-1}\left(\frac{\omega}{0.2}\right) - 90° - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{20}\right) - \tan^{-1}\left(\frac{\omega}{0.02}\right)$$

$$= \tan^{-1}(5\,\omega) - 90° - \tan^{-1}(0.05\,\omega) - \tan^{-1}(0.5\,\omega) - \tan^{-1}(50\,\omega)$$

Frequency (ω)	Phase angle (ϕ)
0.01	–114.02°
0.02	–129.92°
0.06	–146.76°
0.1	–145.27°
0.2	–135.57°
0.6	–124.94°

Frequency (ω)	Phase angle (ϕ)
1	–129.59°
2	–145.85°
6	–179.98°
10	–196.29°
20	–219.8°
60	–249.83°
200	–263.77°





11. **Draw the bode diagram for the given network function.[2014/Fall]**

$$H(s) = \frac{2,000\,(s+4)}{(s+16)\,(s+100)}$$

Solution:

Given that;

$$H(s) = \frac{2000\,(s+4)}{(s+16)\,(s+100)}$$

Representing each factor in the standard in the standard form

$$H(s) = \frac{2000 \times 4 \times \left(1+\frac{s}{4}\right)}{16 \times \left(1+\frac{s}{16}\right) \times 100\left(1+\frac{s}{100}\right)} = \frac{5\left(1+\frac{s}{4}\right)}{\left(1+\frac{s}{16}\right)\left(1+\frac{s}{100}\right)}$$

Magnitude Plot:

Factors	Corner frequency	Slope	Cumulative slope
5	-	-	-
$\frac{1}{\left(1+\frac{s}{4}\right)}$	4	20 dB/decade	20 dB/decade
$\frac{1}{\left(1+\frac{s}{16}\right)}$	16	-20 dB/decade	0 dB/decade
$\frac{1}{\left(1+\frac{s}{100}\right)}$	100	-20 dB/decade	-20 dB/decade

∴ Starting point = 20 log (5) = 13.98 dB ≃ 14 dB

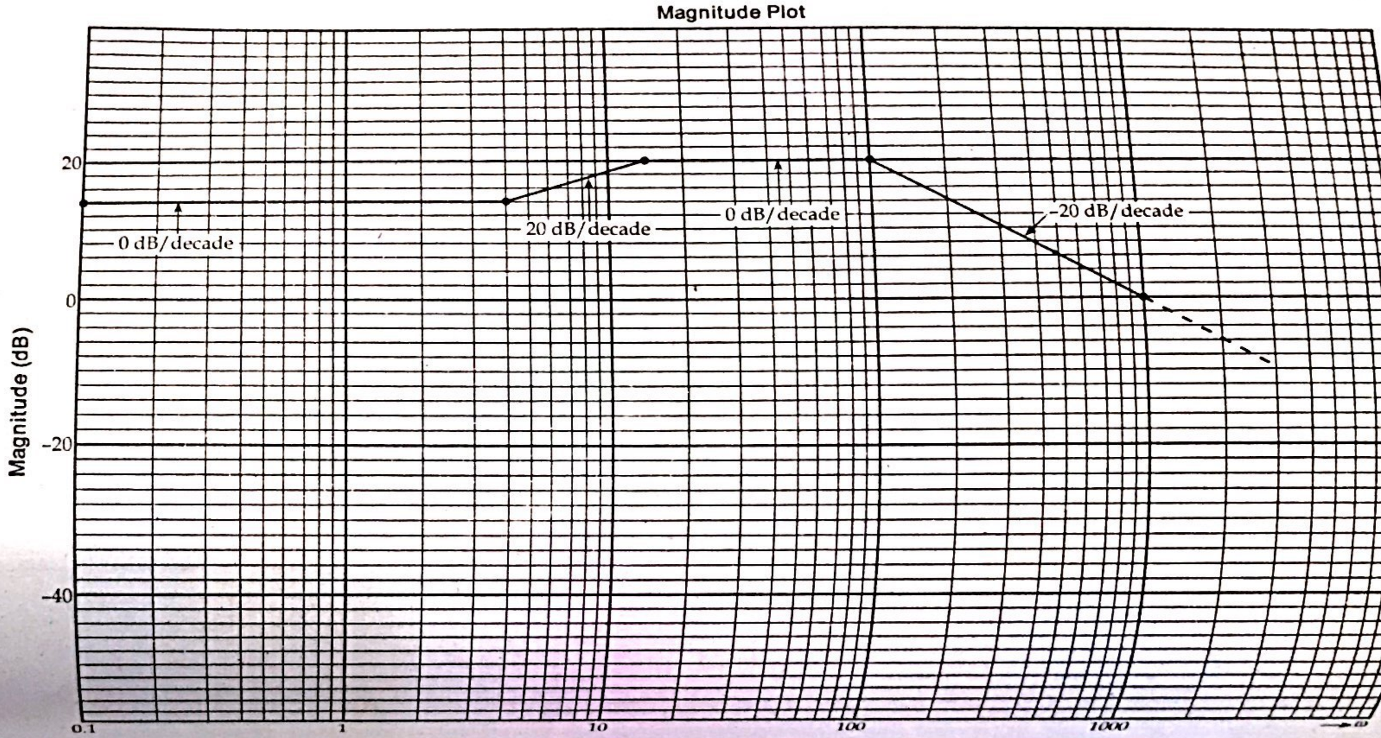
Phase Plot:

$$\phi = 0 + \tan^{-1}\left(\frac{\omega}{4}\right) - \tan^{-1}\left(\frac{\omega}{16}\right) - \tan^{-1}\left(\frac{\omega}{100}\right)$$

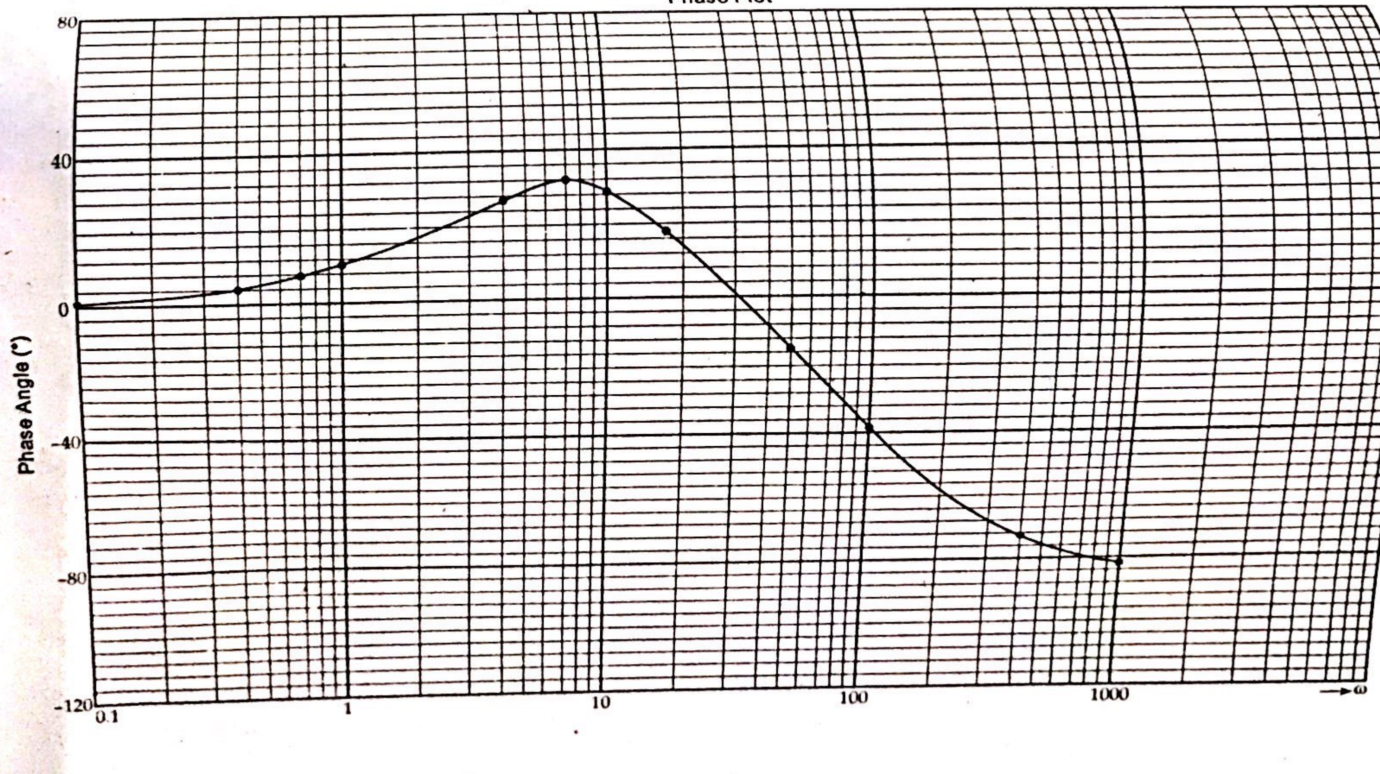
$$= \tan^{-1}(0.25\,\omega) - \tan^{-1}(0.0625\,\omega) - \tan^{-1}(0.01\,\omega)$$

Frequency (ω)	Phase angle (ϕ)
0.1	1.02°
0.4	4.05°
0.7	7.02°
1	9.89°
4	28.67°
7	32.62°
10	30.48°
16	21.87°
50	-13.39°
100	-38.2°
400	-74.25°
1000	-83.6°

Magnitude Plot



Phase Plot



Scanned with CamScanner

12. **Sketch the asymptotic bode plot for the transfer function given by,**

$$G(s) = \frac{20\,(s+5)}{s(s^2+21s+20)\,(s+10)}$$ **[2013/Fall]**

Solution:

Given that;

$$G(s) = \frac{20\,(s+5)}{s(s^2+21s+20)\,(s+10)} = \frac{20\,(s+5)}{s(s+1)\,(s+20)\,(s+10)}$$

$$= \frac{20 \times 5\left(1+\frac{s}{5}\right)}{s(1+s) \times 20\left(1+\frac{s}{20}\right) \times 10\left(1+\frac{s}{10}\right)} = \frac{0.5\left(1+\frac{s}{5}\right)}{s(1+s)\left(1+\frac{s}{10}\right)\left(1+\frac{s}{20}\right)}$$

Magnitude Plot:

Factors	Corner frequency	Slope	Cumulative slope
0.5	-	-	-
$\frac{1}{s}$	-	-20 dB/decade	-20 dB/decade
$\frac{1}{(1+s)}$	1	-20 dB/decade	-40 dB/decade
$\left(1+\frac{s}{5}\right)$	5	20 dB/decade	-20 dB/decade
$\frac{1}{\left(1+\frac{s}{10}\right)}$	10	-20 dB/decade	-40 dB/decade
$\frac{1}{\left(1+\frac{s}{20}\right)}$	20	-20 dB/decade	-60 dB/decade

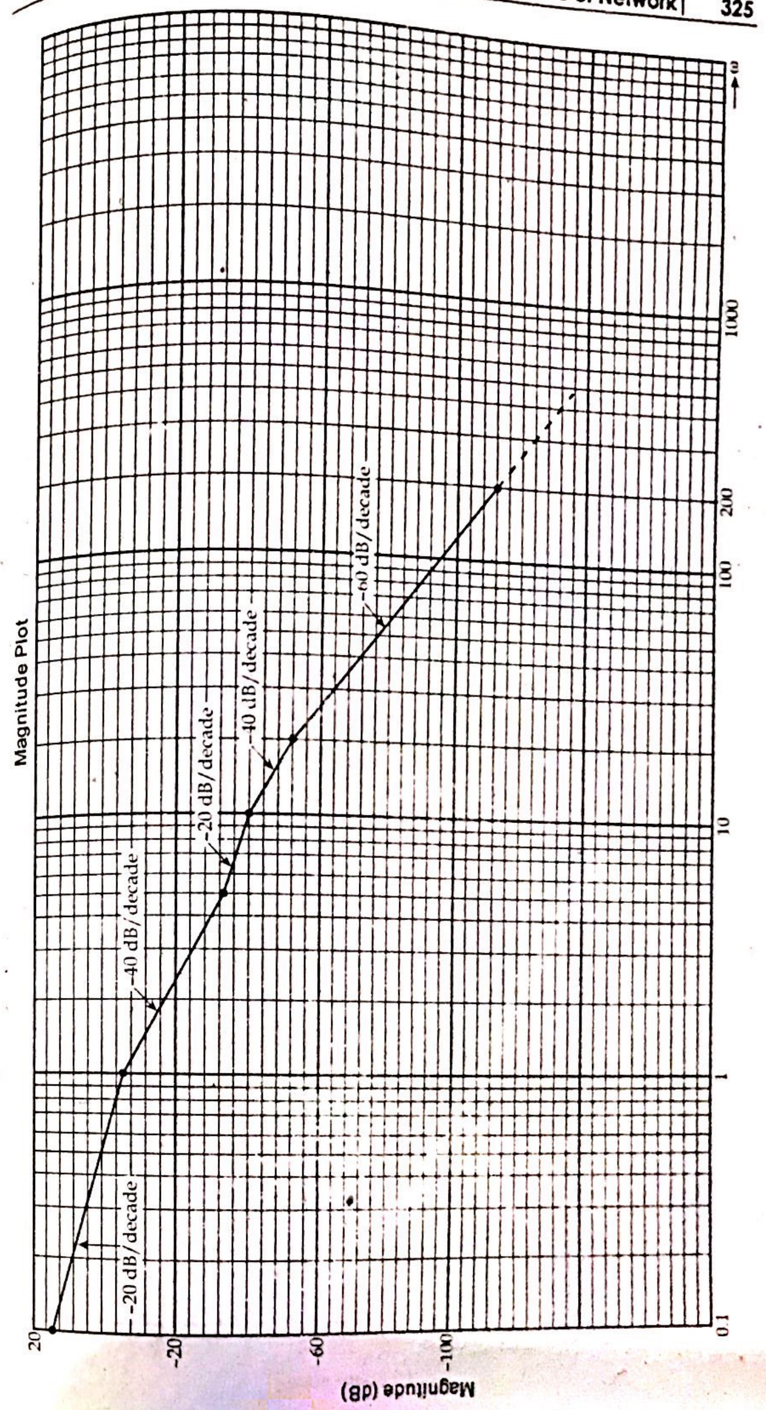
∴ Starting point = 20 log (0.5) – 20 log (0.1) = 13.98 dB ≃ 14 dB

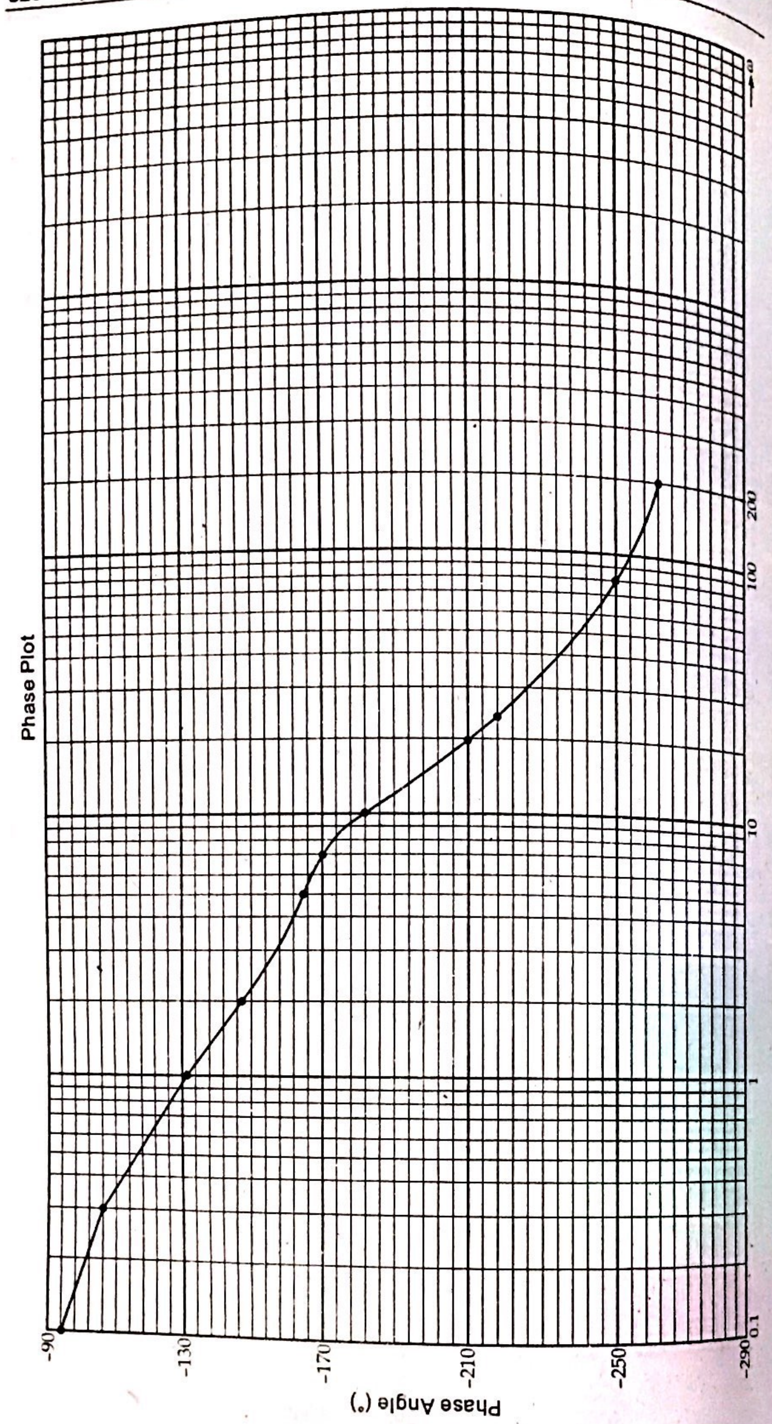
Phase Plot:

$$\phi = \tan^{-1}\left(\frac{\omega}{5}\right) - 90^\circ - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{20}\right)$$

$$= \tan^{-1}(0.2\,\omega) - 90^\circ - \tan^{-1}(\omega) - \tan^{-1}(0.1\,\omega) - \tan^{-1}(0.02\,\omega)$$

Frequency (ω)	Phase angle (ϕ)
0.1	-95.42°
0.3	-105.84°
1	-132.26°
2	-148.65°
5	-164.29°
7	-171.69°
10	-182.42°
20	-209.61°
80	-251.7°
200	-262.57°





13. **Sketch the bode plot:**

$$Z(s) = \frac{200(s+20)}{s(2s+1)(s+40)}$$

[2013/Spring]

Solution:

Given that;

$$Z(s) = \frac{200(s+20)}{s(2s+1)(s+40)}$$

Representing each factor in the standard form,

$$Z(s) = \frac{200 \times 20 \times \left(1+\frac{s}{20}\right)}{s \times 40 \times \left(1+\frac{s}{40}\right) \times (1+2s)} = \frac{100\left(1+\frac{s}{20}\right)}{s(1+2s)\left(1+\frac{s}{40}\right)}$$

Magnitude Plot:

Factors	Corner frequency	Slope	Cumulative slope
100	-	-	-
$\frac{1}{s}$	-	-20 dB/decade	-20 dB/decade
$\frac{1}{(1+2s)}$	0.5	-20 dB/decade	-40 dB/decade
$\left(1+\frac{s}{20}\right)$	20	20 dB/decade	-20 dB/decade
$\frac{1}{\left(1+\frac{s}{40}\right)}$	40	-20 dB/decade	-40 dB/decade

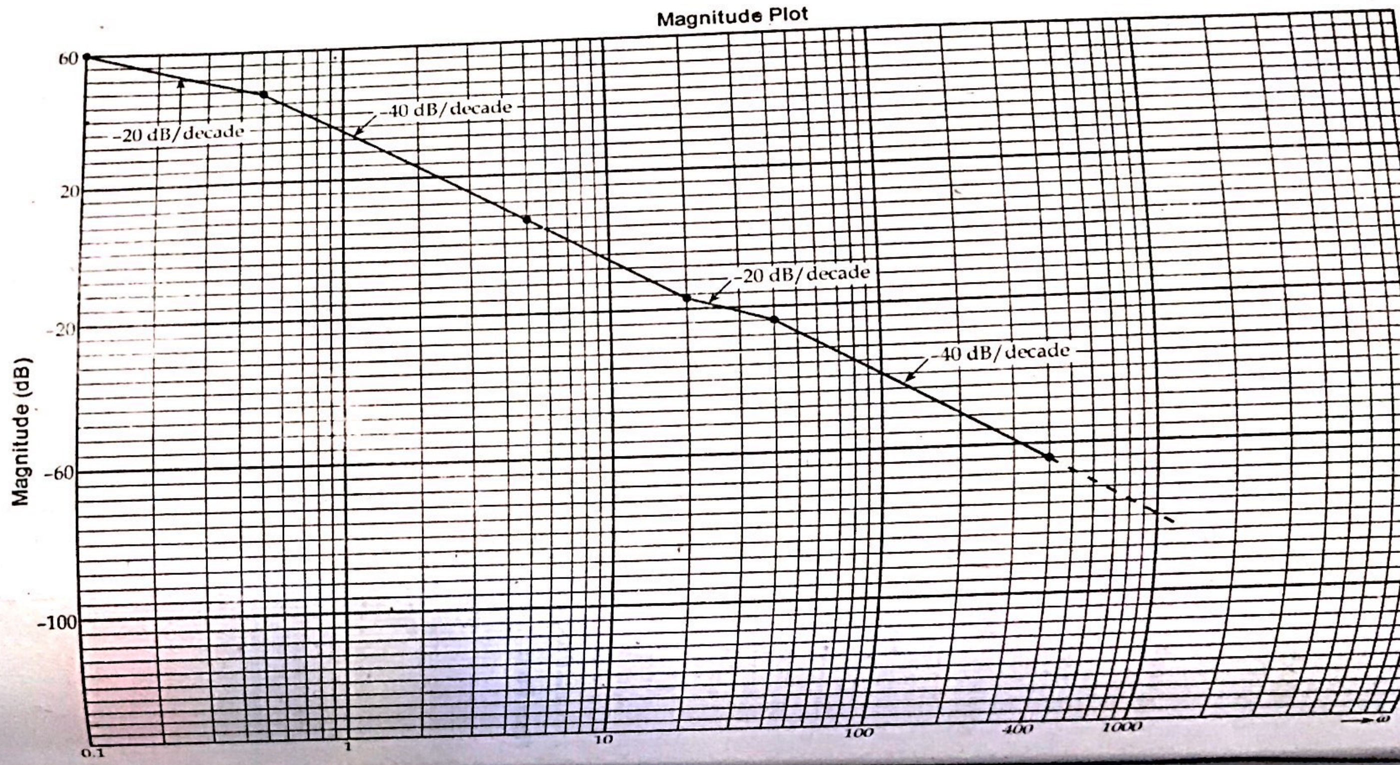
∴ Starting point = 20 log (100) – 20 log (0.1) = 60 dB

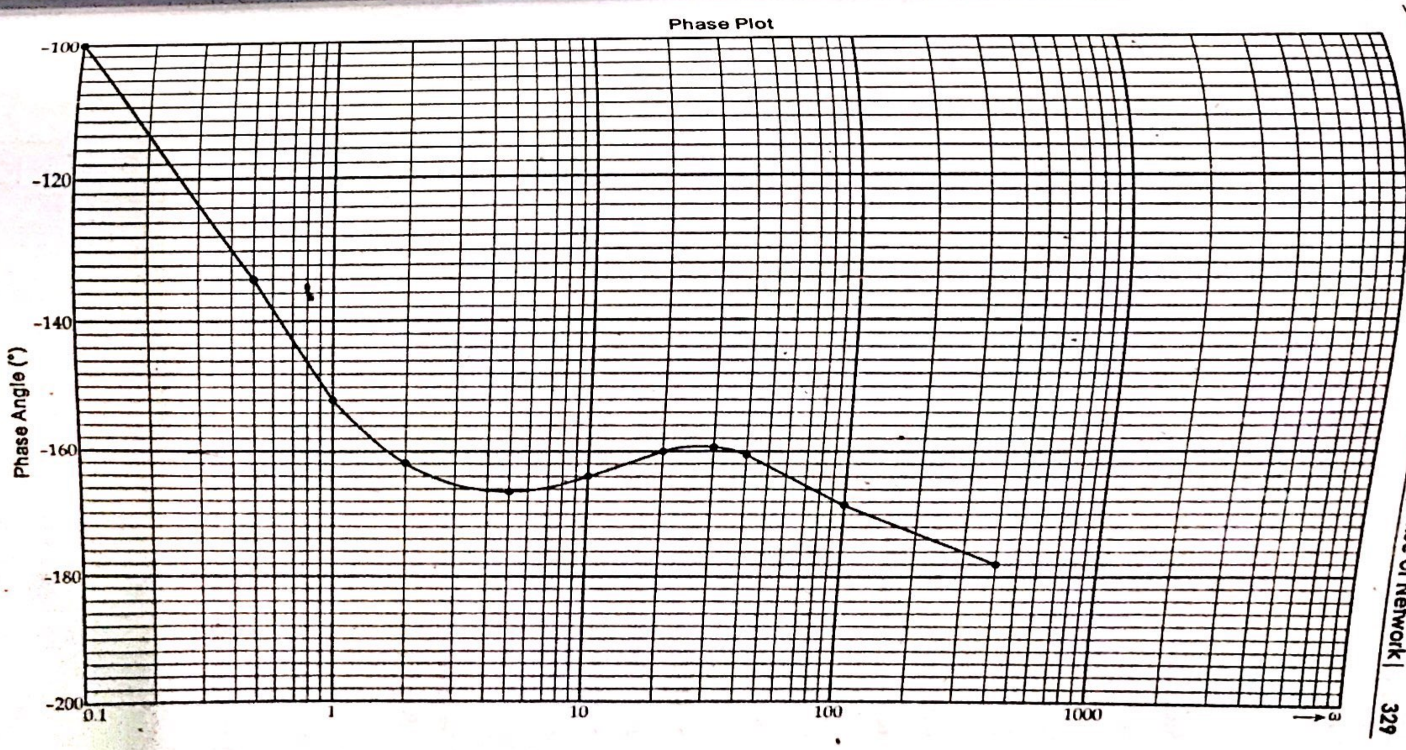
Phase Plot:

$$\phi = 0 + \tan^{-1}\left(\frac{\omega}{20}\right) - 90° - \tan^{-1}(2\omega) - \tan^{-1}\left(\frac{\omega}{40}\right)$$

$$= \tan^{-1}(0.05\omega) - 90° - \tan^{-1}(2\omega) - \tan^{-1}(0.025\omega)$$

Frequency (ω)	Phase angle (ϕ)
0.1	-101.17°
0.5	-134.28°
1	-152°
2	-163.12°
5	-167.38°
10	-164.61°
20	-160.13°
30	-159.61°
40	-160.85°
100	-169.22°
400	-177.08°


Magnitude Plot
Magnitude (dB)
60
20
-20
-60
-100
-20 dB/decade
-40 dB/decade
-20 dB/decade
-40 dB/decade
0.1
1
10
100
400
1000
ω


Phase Plot
Phase Angle (°)
-100
-120
-140
-160
-180
-200
0.1
1
10
100
1000
ω

14. Write short notes on filter. **[2017/Fall, 2019/Spring]**

Solution: See the topic 7.7, 7.8 and 7.9.

15. Define filters in electric network. Classify filter according to their ranges of their bands with example. **[2019/Fall]**

Solution: See the topic 7.7 and 7.8.

16. Explain the concept of high pass, low pass and band pass filter with neat sketch. **[2016/Spring, 2017/Spring, 2018/Spring]**

Solution: See the topic 7.8.1, 7.8.2 and 7.8.3.

17. Define filter example the concept of band pass filters with magnitude plot. **[2018/Fall]**

Solution: See the topics 7.7 and 7.8.3.

18. Write short notes on band pass and band stop filters. **[2011/Spring, 2016/Fall]**

Solution: See the topic 7.8.3 and 7.8.4.

19. Write short notes on types of filter. **[2011/Fall, 2012/Fall, 2013/Spring, 2015/Spring]**

Solution: See the topic 7.8.

20. Write short notes on bode plot. **[2015/Spring]**

Solution: See the topic 7.3, 7.3.1 and 7.3.2.

21. Sketch the bode magnitude plot for the given transfer function.

$$G(s) = \frac{300\,(s+4)}{5(2s+1)\,(3s^2+11s+12)\,(s+10)}$$ **[2020/Fall]**

Solution:

Given that;

$$G(s) = \frac{300\,(s+4)}{5\,(2s+1)\,(3s^2+11s+12)\,(s+10)}$$

Expressing each term in standard form,

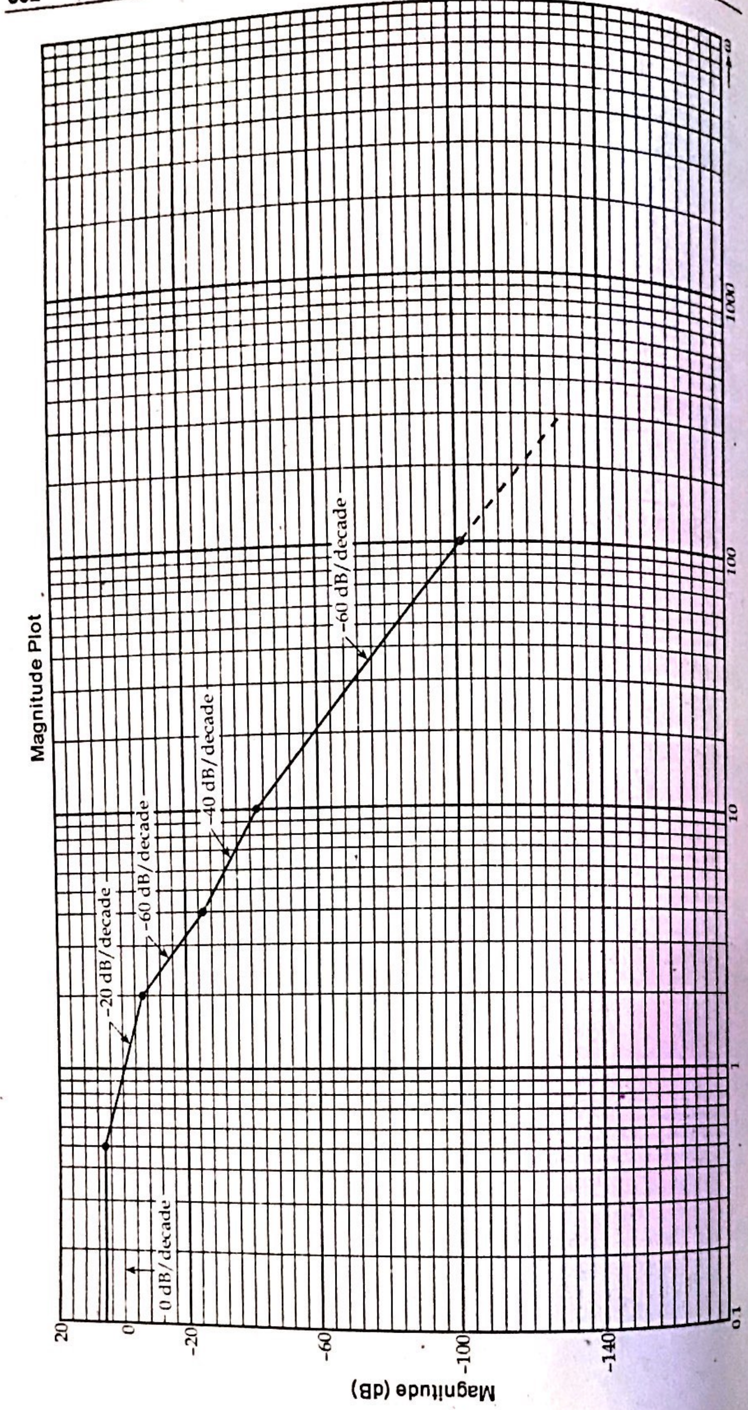
$$= \frac{300 \times 4\left(1+\frac{s}{4}\right)}{5\,(1+2s) \times 12\left(1+\frac{11s}{12}+\left(\frac{s}{2}\right)^2\right)\left(1+\frac{s}{10}\right)}$$

$$= \frac{2\left(1+\frac{s}{4}\right)}{\left(1+\frac{s}{0.5}\right)\left(1+\frac{11}{12}+\left(\frac{s}{2}\right)^2\right)\left(1+\frac{s}{10}\right)}$$

Magnitude plot

S.N	Factors	Corner frequency	Slope	Cumulative frequency
1	2	-	-	-
2	$\frac{1}{\left(1+\frac{s}{0.5}\right)}$	0.5	-20 dB/dec	-20 dB/dec
3.	$1+\frac{11}{2}s+\left(\frac{s}{2}\right)^2$	2	-40 dB/dec	-60 dB/dec
4	$\left(1+\frac{s}{4}\right)$	4	20 dB/dec	-40 dB/dec
5	$\frac{1}{\left(1+\frac{s}{10}\right)}$	10	-20 dB/dec	-60 dB/dec

$\therefore$ Starting point = 20 log (2) = 6.02 dB ≃ 6 dB



Chapter 8 | ONE-PORT PASSIVE NETWORK

8.1 ELEMENTS OF REALIZATION THEORY

The starting point for any network synthesis problem is the network function which is the ratio of response R(s) to the excitation E(s) *i.e.*,

$$T(s) = \frac{R(s)}{E(s)}$$

Our task is to synthesize a network from a given network function.

8.1.1 Causality and Stability

The first step in a synthesis procedure is to determine whether T(s) can be realized as a physical passive network. There are two important considerations causality and stability. By causality, we mean that a voltage cannot appear between any pair of terminals in the network before a current is impressed or vice-versa. In other words, the response of the network must be zero for $t < 0$.

In order for network to be stable, the following three conditions on its network function T(s) must be satisfied.

a) T(s) cannot have poles in the right half s-plane.
b) T(s) cannot have multiple poles in the imaginary ($j\omega$) axis.
c) The degree of numerator of T(s) cannot exceed the degree of denominator by more than unity.

8.1.2 Hurwitz Polynomial

Another element of reliability is a class of polynomial known as Hurwitz polynomial which is, in fact, the denominator polynomial of the network function satisfying certain conditions. A polynomial P(s) is said to be Hurwitz if the following conditions are satisfied.

a) P(s) is real when s is real.
b) The roots of P(s) have real parts which are zero or negative.

Properties of Hurwitz Polynomial

Hurwitz polynomial P(s) have the following properties:

a) If the polynomial P(s) can be written as,

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots\dots + a_1 s + a_0$$

Then, all the coefficients a_i must be positive. A corollary is that between the highest order term in s and the lowest order term, none of the coefficients may zero unless the polynomial is even or odd. In other words, $a_{n-1}, a_{n-2}, \dots\dots, a_2, a_1$ must not be zero if the polynomial is neither even or odd.

b) Both the odd and even parts of a Hurwitz polynomial P(s) have roots on the $j\omega$-axis only. If we denote the even parts of P(s) as M(s) and odd parts as N(s), so that P(s) = M(s) + N(s), then M(s) and N(s) both have roots on the $j\omega$-axis only.

c) As a result of property (b), if P(s) is either even or odd, all its roots on the $j\omega$-axis (including origin).

d) The continued fraction expansion of the ratio [$\psi(s)$] of the odd to even parts (N(s)/M(s)) or the even to odd parts (M(s)/N(s)) of a Hurwitz polynomial yields all positive quotient terms. As,

$$\psi(s) = \frac{N(s)}{M(s)} \text{ or } \frac{M(s)}{N(s)} = q_1 s + \cfrac{1}{q_2 s + \cfrac{1}{q_3 s + \cfrac{1}{\dots\dots + \cfrac{1}{q_n s}}}}$$

where, the quotients $q_1, q_2, \dots\dots, q_n$ must be positive if the polynomial P(s) = M(s) + N(s) is Horwitz.

e) If P(s) is Hurwitz polynomial and W(s) is a multiplicative factor, then $P_1(s) = P(s)$. W(s) is also Hurwitz polynomial, if W(s) is hurwitz polynomial.

f) In case the polynomial is either only even or only odd, it is not possible to obtain the continued fractional expansion in such cases the polynomial P(s) is Hurwitz if the ratio of P(s) and its derivative P(s) gives a continued fraction expansion.

Procedure for Obtaining the Continued Fraction Expansion

To obtain the continued fraction expansion, we must perform a series of long division. Suppose $\psi(s) = \frac{M(s)}{N(s)}$ where, M(s) is of one higher degree than N(s). Then, we obtain a single quotient and a remainder $\psi(s) = q_1 s + \frac{R_1(s)}{N(s)}$.

The degree of the term $R_1(s)$ is one lower than the degree of N(s). Hence if we invert the second term and divide, we have

$$\frac{N(s)}{R_1(s)} = q_2 s + \frac{R_2(s)}{R_1(s)}$$

Inverting and dividing again, we obtain,

$$\frac{R_1(s)}{R_2(s)} = q_3 s + \frac{R_3(s)}{R_2(s)} \text{ and so on.}$$

8.1.3 Positive real Functions (PRF)

A function $T(s) = \frac{N(s)}{D(s)}$ is positive real if the following conditions are satisfied.

a) T(s) is real for a real *i.e.,* T (σ) is purely real.
b) D(s) is Hurwitz polynomial.
c) T(s) may have poles on the $j\omega$-axis. These poles are simple and the residues there of are real and positive.
d) The real part of T(s) is greater than or equal to zero for the real part of s is greater than or equal to zero *i.e.,*

Re [T(s)] $\geq$ 0 for Real s $\geq$ 0

or, Re [T(s) $\geq$ 0 for Real s = 0

and, Re [T(s)] $\geq$ 0 for Real s > 0

Hence, Re [T($j\omega$)] $\geq$ 0 for all ω

A simplification of condition (d) is possible: Let,

$$T(s) = \frac{N(s)}{D(s)} = \frac{M_1(s) + N_1(s)}{M_2(s) + N_2(s)}$$

where, M_i (s) is an even function and $N_i(s)$ an odd function.

Rationalising,

$$T(s) = \frac{M_1 + N_2}{M_2 + N_2} \times \frac{M_2 - N_2}{M_2 - N_2} = \frac{M_1 M_2 - N_1 N_2}{M_2^2 - N_2^2} + \frac{N_1 M_2 - M_1 N_2}{M_2^2 - N_2^2}$$

Here, M_1 M_2 and N_1 N_2 are even functions while N_1 M_2 and M_1 N_2 are odd functions. Hence the even part of T(s) is,

$$\text{Even } [T(s)] = \frac{M_1 M_2 - N_1 N_2}{M_2^2 - N_2^2}$$

and, odd part of T(s) is,

$$\text{odd}\,[T(s)] = \frac{N_1 M_2 - M_1 N_2}{M_2^2 - N_2^2}$$

If we let $s = j\omega$ (since $\sigma = 0$), we see that the even part of any polynomial is real, while the odd part of the polynomial is imaginary *i.e.*,

$$\text{Re}\,[T(j\omega)] = \text{Even}\,[T(s)]|_{s=j\omega}$$

and, $$j\,I_m\,[T(j\omega)] = \text{odd}\,[T(s)]|_{s=j\omega}$$

It is clear that;

$$T(j\omega) = \text{Re}\,[T(j\omega)] + j\,\text{Im}\,[T(j\omega)]$$

Therefore, to test for the condition (d) for positive realness, we determine the real part of T (jω) by finding the even part of T(s) and then putting s = jω. We then check to see whether Re [T (jω)] ≥ 0 for all ω.

The denominator of Re [T(jω)] is always a positive quantity because $[M_2(j\omega)]^2 - [N_2\,(j\omega)]^2 = M_2(\omega^2) + N_2(\omega^2) \geq 0$ *i.e.*, there is an extra j or imaginary term in N_2 (jω), which when squared gives -1, so that denominator of Re[T(jω)] is the sum of two squared numbers and is always positive. Therefore, our task resolves into the problem of determining whether

$$A(\omega^2) \equiv M_1(j\omega)\,M_2\,(j\omega) - N_1\,(j\omega)\,N_2(j\omega) \geq 0$$

8.1.3.1 Properties of Positive Real Function (P.R.F)

a) The poles and zeros of a P.R.F are real or occur in conjugate pairs.

b) The highest powers of the numerators N(s) and denominator D(s) polynomial may differ at most by unity. This condition prohibits multiple poles or zeros at s = ∞.

c) The lowest power of D(s) and N(s) polynomials may differ by at most unity. This condition prevents the possibility of multiple poles or zeros at s = 0.

d) If T(s) is positive real, then $\frac{1}{T(s)}$ is also positive real. This property implies that if a driving point impedance Z(s) is positive real, then its reciprocal $\left(\frac{1}{Z(s)}\right)$, the driving point admittance Y(s) is also positive real.

e) The sum of positive real functions is positive real from an imittance stand point, we see that if two impedances are connected in series or two admittances are connected in parallel, the resultant impedance or admittance is positive real. (Note that the difference of two positive real function is not necessarily positive real).

f) The poles and zeros of a PRF cannot have positive real parts *i.e.*, they cannot be in the right half of the s-plane. In addition to this, only simple poles with real positive residues can exist on the jω-axis.

8.1.3.2 The Necessary and Sufficient Conditions for a Rational Function(s) with Real Coefficients to be Positive Real

a) Condition (1)

T(s) must have no poles in the right half of s-plane *i.e.*, denominator D(s) must be Hurwitz polynomial. This condition can check through a continued fraction expansion of the odd to even parts or even to odd parts of the D(s) in which quotients must be positive.

b) Condition (2)

This condition is checked only when the poles of T(s) are on the jω-axis, otherwise not, (If denominator D(s) of T(s) has a factor of the types $s^2 + a$; where a is positive real constant then we can say, the poles of T(s) are on the jω-axis). T(s) may have only simple poles on the imaginary axis (jω-axis) with real and positive residues.

NOTE:
This condition is tested by making a partial fraction expansion of T(s) and checking whether the residues of the poles on the jω-axis are positive are real.

c) Condition (3)

$$\text{Re}\,[T(j\omega)] \geq 0, \text{ for all } \omega$$

or, $$A(\omega^2) = M_1(j\omega)\cdot M_2\,(j\omega) - N_1(j\omega)\cdot N_2\,(j\omega) \geq; \text{ for all } \omega$$

OR,

A polynomial P(s) is said to be Hurwitz if the following conditions are satisfied:

a) P(s) is real when s is real.

b) The roots of P(s) have real parts which are zero or negative.

As a result of these conditions, if P(s) is a Hurwitz polynomial given by

$$P(s) = a_n\,s^n + a_{n-1}\,s^{n-1} + \dots\dots + a_1 s + a_0$$

Then all the coefficient a_i must be real; if $s_i = \alpha_i + j\beta$ is a root of P(s), then α_i must be negative. The polynomial $P(s) = (s + 1)\,(s + 1 + j\sqrt{2})\,(s + 1 - j\sqrt{2})$ is Hurwitz because all of its roots have negative real parts. On the other hand, $G(s) = (s - 1)\,(s + 2)\,(s + 3)$ is not Hurwitz because of the root s = 1, which has a positive real part.

Hurwitz polynomials have the following properties

a) All the coefficients a_i are, non-negative. This is readily seen by examining the three types of roots that a Hurwitz polynomial might have. These are,

$a = -\gamma_i$ γ_i real and positive

$s = \pm j\omega_i$ ω_i real

$s = -\alpha_i \pm j\beta_i$ α_i real and positive

The polynomial; P(s) which contains these roots can be written as,

$$P(s) = (s + \gamma_i)\,(s^2 + \omega_i^2)\,[(s + \alpha_i)^2 + \beta_i^2]$$

Since P(s) is the product of terms with only positive coefficients, it follows that the coefficients of P(s) must be positive. A corollary is that between the highest order term in S and the lowest order term, none of the coefficients may be zero unless the polynomial is even or odd. In other words, $a_{n-1}, a_{n-2}, \dots\dots, a_2, a_1$ must not be zero if the polynomial is neither even nor odd. This is readily seen because the absence of a term a_i implies cancellation brought about by a root $s-\gamma_i$ with a positive real part.

b) Both the odd and even parts of a Hurwitz polynomial P(s) have roots on the jω axis only. If we denote the odd part of P(s) as N(s) and the even part M(s), so that,

$$P(s) = N(s) + M(s)$$

Then M(s) and N(s) both have roots on the jω axis only

c) As a result of property (b), if P(s) is either even or odd, all its roots are on the jω axis.

d) The continued fraction expansion of the ratio of the odd to even parts or the even to odd parts of a Hurwitz polynomial yields all positive quotient terms. Suppose we denote the ratios as

$$\psi(s) = \frac{N(s)}{M(s)} \text{ or } \psi(s) = \frac{M(s)}{N(s)}$$

Then, the continued fraction expansion of ψ(s) can be written as,

$$\psi(s) = q_1 s + \cfrac{1}{q_2 s + \cfrac{1}{q_3 s + \cfrac{1}{\dots\dots\dots + \cfrac{1}{q_n s}}}}$$

where, the quotients $q_1, q_2, \dots\dots, q_n$ must be positive if the polynomial P(s) = N(s) + M(s) is Hurwitz. To obtain the continued fraction expansion, we must perform a series of long divisions.

Suppose $\psi(s) = \frac{M(s)}{N(s)}$

where, M(s) is of one higher degree than N(s). Then if we divide N(s) into M(s), we obtain a single quotient and a reminder.

$$\psi(s) = q_1 s + \frac{R_1(s)}{N(s)}$$

The degree of the term $R_1(s)$ is one lower than the degree of N(s). Therefore, if we invert the remainder term and divide, we have,

$$\frac{N(s)}{R_1(s)} = q_2 s + \frac{R_2(s)}{R_1(s)}$$

Inverting and dividing again, we get,

$$\frac{R_1(s)}{R_2(s)} = q_3 s + \frac{R_3(s)}{R_2(s)}$$

Inverting and dividing again, we get,

$$\frac{R_1(s)}{R_2(s)} = q_3 s + \frac{R_3(s)}{R_2(s)}$$

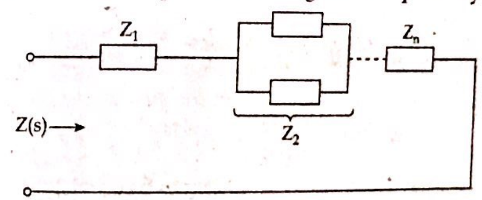
We see that the process of obtaining the continued fraction expansion of ψ(s) simply involves division division and inversion. At each step, we obtain a quotient term q.s and a remainder term, $\frac{R_{i+1}}{R_i(s)}$. We then invert the remainder term and divide $R_{i+1}(s)$ in to $R_i(s)$ to obtain a new quotient. There is a theorem in the theory of continued fractions which states that the continued fraction expansion of the even to odd or odd to even parts of a polynomial must be finite in length. Another theorem states that, if the continued fraction expansion of the odd to even or even to odd parts of a polynomial yields positive quotient terms, then the polynomial must be Hurwitz to within a multiplicative factor W(s). That is, if we write F(s) = W(s) F_1(s), then, F(s) is Hurwitz, if W(s) and F_1(s) are Hurwitz.

8.2 SYNTHESIS OF ONE PORT NETWORK WITH TWO KINDS OF ELEMENTS

There are a number of methods of synthesizing (or realizing) a one port network. Only consider the four basic forms as follows:

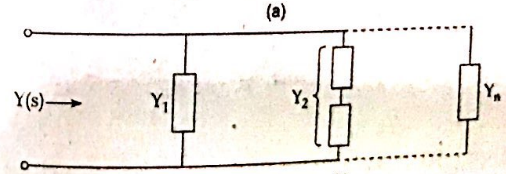
a) Foster-I or Foster series form as shown in figure 8.1 (a).
b) Foster-II or Foster parallel form, as shown in figure 8.1 (b).
c) Cauer I form.
d) Cauer-II form as shown in figure 8.1 (c).

The foster I and II forms are obtained by partial expansion of Z(s) and Y(s) respectively and the cauer I and II forms by continued-fraction expansion of admittance function by arranging both numerator and denominator in descending and ascending order respectively.



$Z(s) = Z_1 + Z_2 + \dots\dots + Z_1$

(a)



$Y(s) = Y_1 + Y_2 + \dots\dots + Y_1$

(b)

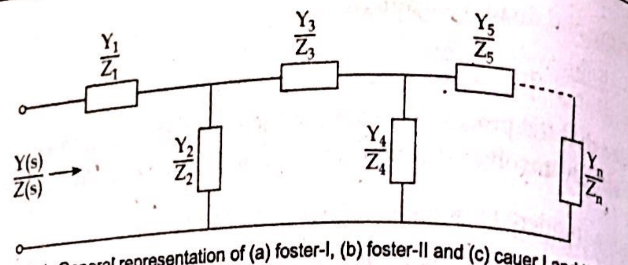


Figure 8.1: General representation of (a) foster-I, (b) foster-II and (c) cauer I and II forms

8.2.1 L-C Immittance Function

Consider the impedance Z(s) of a passive one port network. Let us represent Z(s) as $Z(s) = \frac{M_1(s) + N_1(s)}{M_2(s) + N_2(s)}$

where, M_1, M_2 are even parts of the numerator and denominator and N_1, N_2 are odd parts. The average power dissipated by the one-port passive network is average power $= \frac{1}{2} Re\,[Z\,(j\omega)] \cdot |I|^2$ where, I is the input current.

For a pure L-C (reactive) network, it is known that the power dissipated is zero. Hence, real part of Z (jω) is zero *i.e.*, Re [Z (jω)] = 0

Since we know that (from condition of PRF)

$$Re\,(Z\,(j\omega)] = Even\,[Z\,(j\omega)]$$
$$= \frac{M_1(j\omega) \cdot M_2(jw) - N_1(j\omega) \cdot N_2(j\omega)}{M_2^2\,(j\omega) - N_2^2\,(j\omega)}$$

In order to Re [Z (jω)] = 0

$$M_1(j\omega) \cdot M_2\,(j\omega) - N_1\,(j\omega) \cdot N_2\,(j\omega) = 0$$

For existence of the function Z(s), either of the following cases must hold

a) $M_1 = 0$ and $N_2 = 0$

b) $M_2 = 0$ and $N_1 = 0$

In case (a), $Z(s) = \frac{N_1(s)}{M_2(s)}$

and, In case (b), $Z(s) = \frac{M_1(s)}{N_2(s)}$

Case I: When L and C are in series as shown in figure 8.2 (a).

$$Z(s) = sL + \frac{1}{sC} = \frac{s^2LC + 1}{sC} = \frac{s^2 + \frac{1}{LC}}{\frac{1}{L}s}$$

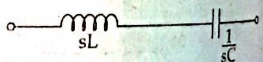


Figure 8.2: (a)

$$\text{and,}\quad Y(s) = \frac{1}{Z(s)} = \frac{\frac{1}{L}s}{s^2 + \frac{1}{LC}}$$

Case II: When L and C are in parallel as shown in figure 8.1 (b)

$$Y(s) = sC + \frac{1}{sL} = \frac{s^2LC + 1}{sL} = \frac{s^2 + \frac{1}{LC}}{\frac{1}{C}s}$$

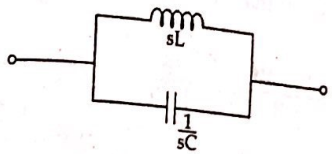


Figure 8.2: (b)

$$\text{and,}\quad Z(s) = \frac{1}{Y(s)} = \frac{\frac{1}{C}s}{s^2 + \frac{1}{LC}}$$

8.2.2 R-C Impedance or R-L Admittance Function

R-C Impedance or R-L Admittance function has following properties

a) The poles and zeros lie on the negative real axis (included origin) of the complex s-plane.

b) The poles and zeros interlace or alternate along the negative real axis.

c) i) The residues of the poles of $X_{R\text{-}C}(s)$ or $Y_{R\text{-}L}(s)$ must be real and positive.

ii) The residues of the poles of the $Y_{R\text{-}C}(s)$ or $Z_{R\text{-}L}(s)$ are real and negative, however, the residues of the poles of $\frac{Y_{R\text{-}C}(s)}{s}$ or $\frac{Z_{R\text{-}L}(s)}{s}$ must be real and positive.

d) The singularity nearest to (or at) the origin must be a pole *i.e.*, function $Z_{R\text{-}C}(s)$ or $Y_{R\text{-}L}(s) \to \infty$ with $s \to 0$.

e) The singularity nearest to (or at) the minus infinity (−∞) must be a zero *i.e.*, function $Z_{RC}(s)$ or $Y_{R\text{-}L}(s) \to 0$ with $s \to \infty$.

An R-C impedance $Z_{R\text{-}C}(s)$ also can be realized as an R-L admittance, $Y_{R\text{-}L}(s)$. All the properties of R-L admittance are the same as the properties of R-C impedances. It is therefore important to specify whether a function is to be realized as an RC impedance or an R-L admittance.

Case I: When R and C are in series as shown in figure 8.3 (a).

$$Z(s) = R + \frac{1}{sC} = \frac{sRC + 1}{sC} = \frac{s^2 + \frac{1}{LC}}{\frac{1}{R}s}$$

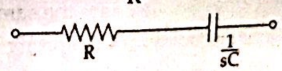


Figure 8.3: (a)

and, $Y(s) = \dfrac{\frac{1}{R}s}{s^2 + \frac{1}{RC}}$

Case II: When R and C are in parallel as shown in figure 8.3 (b).

$$Y(s) = \frac{1}{R} + sC = \frac{1 + sRC}{R} = \frac{s^2 + \frac{1}{LC}}{\frac{1}{C}}$$

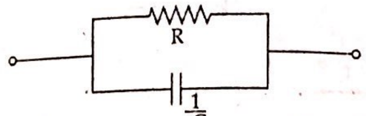


Figure 8.3: (b)

and, $Z(s) = \dfrac{\frac{1}{C}}{s + \frac{1}{RC}}$

8.2.3 R-L Impedance or R-C Admittance Function

The R-L impedance or R-C admittance function has following properties:

a) The poles and zeros lie on the negative real axis (included origin) of the complex s-plane.

b) The poles and zeros interlace along the negative real axis.

c)

i) The residues of the poles of $Z_{R-L}(s)$ or $Y_{R-C}(s)$ are real and negative. However, the residues of the poles of $\frac{Z_{R-L}(s)}{s}$ or $\frac{Y_{R-C}(s)}{s}$ must be real and positive.

ii) The residues of the poles of $Y_{R-L}(s)$ or $Z_{R-C}(s)$ must be real and positive.

d) The singularity nearest to (or at) the origin must be a zero *i.e.*, function $Z_{R-L}(s)$ or $Y_{R-C}(s) \to 0$ with $s \to 0$.

e) The singularity nearest to (or at) the minus infinity ($-\infty$) must be a pole *i.e.*, the function $Z_{R-L}(s)$ or $Y_{R-C}(s) \to \infty$ with $s \to \infty$.

An R-L impedance $Z_{R-L}(s)$ also can be realized as an R-C admittance, $Y_{R-C}(s)$. All the properties of R-C admittances are the same as the properties of R-L impedances. It is therefore important to specify whether a function is to be realized as an R-L impedance or R-C impedance.

Case I: When R and L are in series as shown in figure 8.4 (a)

$$Z(s) = R + sL = L\left(s + \frac{R}{L}\right)$$

Figure 8.4: (a)

and, $Y(s) = \dfrac{\frac{1}{L}}{s + \frac{R}{L}}$

Case II: When R and L are in parallel as shown in figure 8.4 (b)

$$Y(s) = \frac{1}{R} + \frac{1}{sL} = \frac{sL + R}{sLR} = \frac{1}{R} + \frac{s + \frac{R}{L}}{s}$$

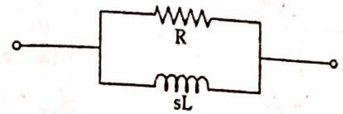


Figure 8.4: (b)

and, $Z(s) = \dfrac{Rs}{s + \frac{R}{L}}$

8.3 FOSTER'S REACTANCE THEOREM

Network synthesis involves the methods used to determine an electric circuit that satisfy certain specifications. Given an impulse response, there are many technique that can be use to synthesize a circuit with the specified response. Different methods may also be used to synthesize circuits, all of which may be optimal. Hence the solution to a network synthesis problem is never unique.

For a foster-I realization the component value are given by the partial fraction expansion

$$Z(s) = K_{inf}\,s + \frac{K_0}{s} + \frac{K_1 s}{s^2 + \omega_1^2} + \frac{K_2 s}{s^2 + \omega_2^2} + \dots\dots\dots + \frac{K_n s}{s^2 + \omega_n^2}$$

While the foster II form the values are given by the alternative partial fraction expansion,

$$Y(s) = K'_{inf}\,s + \frac{K'_0}{s} + \frac{K'_1 s}{s^2 + \omega_1'^2} + \frac{K'_2 s}{s^2 + w_1'^2} + \dots\dots\dots + \frac{K'_n s}{s^2 + \omega_n'^2}$$

For the cauer I realization the component values are given by a continued fraction expansion around infinity.

$$Z(s) = K_{inf}\,s + \cfrac{1}{K_2 s + \cfrac{1}{K_3 s + \dots\dots\dots}}$$

The cauer II values are given by a continued fraction expansion around zero.

$$Z(s) = \frac{K_1'}{s} + \cfrac{1}{\cfrac{K_2'}{s} + \cfrac{1}{\cfrac{K_3'}{s} + \ldots\ldots}}$$

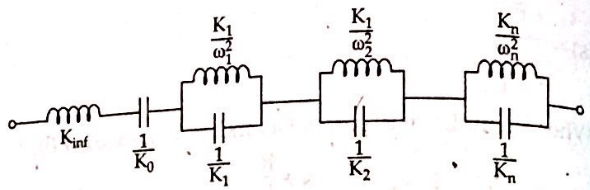


Figure 8.5: (a) Foster-I form

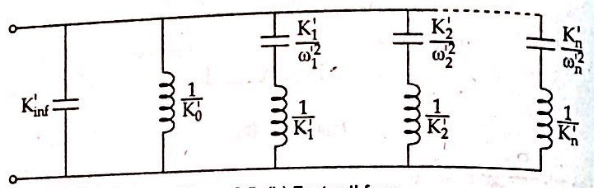


Figure 8.5: (b) Foster-II form

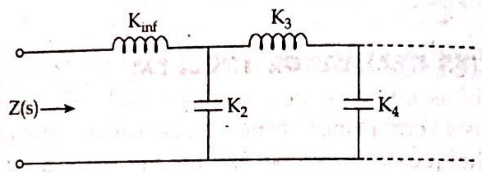


Figure 8.5: (c) Cauer-I form

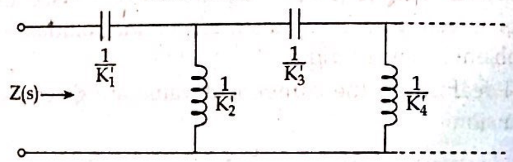


Figure 8.5: (d) Cauer II form

SOLVED NUMERICAL EXAMPLES

1. **Check whether the given polynomial $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ is Hurwitz or not.**

Solution:

Condition 1: Since all coefficients of P(s) are positive, so P(s) is real for s real.

Condition 2: The even and odd parts of P(s) are

$$M(s) = s^4 + 5s^2 + 4$$
$$N(s) = s^3 + 3s$$

Continued fraction expansion of $\psi(s) = \frac{M(s)}{N(s)}$ is given by

$$s^3 + 3s \,\big)\, s^4 + 5s^2 + 4 \,\big(\, s$$
$$\underline{-s^4 + 3s^2}$$
$$2s^2 + 4 \,\big)\, s^2 + 3s \,\big(\, \tfrac{1}{2}s$$
$$\underline{-s^3 + 2s}$$
$$s \,\big)\, 2s^2 + 4 \,\big(\, 2s$$
$$\underline{-2s^2}$$
$$4 \,\big)\, s \,\big(\, \tfrac{1}{4}s$$
$$\underline{-s}$$
$$\times$$

so that the continued fraction expansion $\psi(s)$ is,

$$\psi(s) = \frac{M(s)}{N(s)} = s + \cfrac{1}{\cfrac{s}{2} + \cfrac{1}{2s + \cfrac{1}{\cfrac{s}{4}}}}$$

Since all the quotient terms of the continued fraction expansion are positive, P(s) is Hurwitz.

2. **Check whether the given polynomial is Hurwitz or not.**

$$P(s) = s^4 + s^2 + 2s^3 + 4s + 1$$

Solution:

Condition 1: Since all coefficients of P(s) are positive, P(s) is real for s real.

Condition 2: Even and odd parts of P(s) are

$$M(s) = s^4 + 2s^2 + 1$$
$$N(s) = s^3 + 4s$$

So continued fraction expansion of $\psi(s) = \frac{M(s)}{N(s)}$ is given as,

$$s^3+4s \,\big)\, s^4+2s^2+1 \,\big(\, s$$
$$\underline{-s^4+4s^2}$$
$$-2s^2+1 \,\big)\, s^3+4s \,\big(\, -\tfrac{1}{2}s$$
$$\underline{-s^3-\tfrac{s}{2}}$$
$$\tfrac{9}{2}s \,\big)\, -2s^2+1 \,\big(\, -\tfrac{2}{9}\cdot 2s = -\tfrac{4}{9}s$$
$$\underline{-\,-2s^2}$$
$$1 \,\big)\, \tfrac{9}{2}s \,\big(\, \tfrac{9}{2}s$$
$$\underline{-\tfrac{9}{2}s}$$
$$\times$$

The given polynomial is not Hurwitz because of presence of the negative quotient terms in the continued fraction expansion.

3. **Check whether the given polynomial is Hurwitz or not**
   $P(s) = s^3 + 2s^2 + 3s + 6$

Solution:

Condition 1: Satisfied (since all a_i are positive)

Condition 2: Even and odd parts of P(s) are

$M(s) = 2s^2 + 6$

$N(s) = s^3 + 3s$

So continued fraction expansion of $\psi(s) = \dfrac{N(s)}{M(s)}$ is given as,

$$2s^2+6 \,\big)\, s^3+3s \,\big(\, \tfrac{1}{2}s$$
$$\underline{-s^3+3s}$$
$$\times$$

We see that the division has been terminated suddenly by a common factor $s^3 + 3s$. Thus,

$$P(s) = (s^3+3s)\left(1+\frac{2}{s}\right) = W(s).\,P_1(s)$$

We know that the term $P_1(s) = \left(1+\dfrac{2}{s}\right)$ is Hurwitz

Now check for

$W(s) = s^3 + 3s = s(s^2+3) = s\,(s + j\sqrt{3})$ is Hurwitz.

Now,
$$3s^2+3 \,\big)\, s^3+3s \,\big(\, \tfrac{1}{3}s$$
$$\underline{-s^3+s}$$
$$2s \,\big)\, 3s^2+3 \,\big(\, \tfrac{3}{2}s$$
$$\underline{-3s^2}$$
$$3 \,\big)\, 2s \,\big(\, \tfrac{2}{3}s$$
$$\underline{-2s}$$
$$\times$$

Since $\omega(s)$ is Hurwitz, P(s) is Hurwitz.

4. **Check whether given polynomial is Hurwitz or not. $P(s) = s^7 + 2s^6 + 2s^5 + s^4 + 4s^3 + 8s^2 + 8s + 4$**

Solution:

Condition 1: Satisfied since all a_i are positive.

Condition 2: The continued fraction expansion of $\psi(s)$ is given as,

$$2s^6+s^4+8s^2+4 \,\big)\, s^7+2s^5+4s^3+8s \,\big(\, \tfrac{1}{2}s$$
$$\underline{-s^7+\tfrac{1}{2}s^5+4s^3+2s}$$
$$\tfrac{3}{2}s^5+6s \,\big)\, 2s^6+s^4+8s^2+4s \,\big(\, \tfrac{2}{3}\cdot 2s = \tfrac{4}{3}s$$
$$\underline{-2s^6+8s^2}$$
$$s^4+4 \,\big)\, \tfrac{3}{2}s^5+6s \,\big(\, \tfrac{3}{2}s$$
$$\underline{-\tfrac{3}{2}s^5+6s}$$
$$\times$$

Thus, we see that the term $(s^4 + 4)$ which can be factored into

$(s^4+4) = (s^2+2)^2 - 4s^2 = (s^2+2+2s)\,(s^2+2-2s)$

First factor $(s^2 + 2 + 2s)$ is Hurwitz and second factor is not Hurwitz. As,

$s^2 + 2 - 1s = (s-1)^2 + (1)^2 = (s-1-j1)\,(s-1+j1)$

or, $s = 1 + j1, 1 - j1$ (roots le on right half of s-plane)

Hence, P(s) is not Hurwitz.

5. **Find the range of values of a so that $P(s) = s^4 + s^3 + as^2 + 2s + 3$ is Hurwitz.**

Solution:

Condition 1: All a_i must be positive, hence $a > 0$.

Condition 2: Forming $\dfrac{M(s)}{N(s)}$ and obtaining the continued fraction expansion and requiring the quotients to be positive.

$$s^3+2s \,\big)\, s^4+as^2+3 \,\big(\, s$$
$$\underline{-s^4+2s^2}$$
$$(a-2)s^2+3 \,\big)\, s^3+2s \,\big(\, \tfrac{1}{a-2}s$$
$$\underline{-s^3+\tfrac{3}{a-2}s}$$
$$\left(2-\tfrac{3}{a-2}\right)s \,\big)\, (a-2)s^2+3 \,\big(\, ..$$
$$\underline{..............}$$
$$....\,\big)\, ..\,\big(\, ..$$
$$\underline{..}$$
$$\times$$

Hence, $a - 2 > 0$ and $2 - \frac{3}{a-2} > 0$

or, $a > 2$ and $a > 3.5$

Thus the range of a is $a > 3.5$

6. **Check whether the given polynomial is Hurwitz or not.**

$$P(s) = s^5 + 7s^4 + 6s^3 + 9s^2 + 8s$$

Solution:

The polynomial P(s) is not Hurwitz because a root of P(s) lies at the origin which is not permitted according to condition of Hurwitz polynomial.

7. **Determine whether the function, $Z(s) = \frac{2s^2+5}{s(s^2+1)}$ is PRF or not.**

Solution:

Condition 1: $s^3 + s$, then $D'(s) = 3s^2 + 1$

$$\begin{array}{l} 3s^2+1 \,\big)\, s^3 + s \,\big(\, \frac{1}{3}s \\ \quad\quad\quad -s^3 + \frac{1}{3}s \\ \hline \quad\quad\quad \frac{2}{3}s \,\big)\, 3s^2 + 1 \,\big(\, \frac{3}{2} \cdot 3s = \frac{9}{2}s \\ \quad\quad\quad\quad -3s^2 \\ \hline \quad\quad\quad\quad 1 \,\big)\, \frac{2}{3}s \,\big(\, \frac{2}{3}s \\ \quad\quad\quad\quad\quad -\frac{2}{3}s \\ \hline \quad\quad\quad\quad\quad \times \end{array}$$

Hence, D(s) is Hurwitz polynomial

Condition 2: Z(s) has a pair of poles at $s = \pm j1$.

The partial expansion of Z(s) is,

$$Z(s) = \frac{-3s}{s^2+1} + \frac{5}{s}$$

which shows that the residue of the poles at $s = \pm j1$ is negative. Hence Z(s) is not PRF.

8. **Show that the function $F(s) = \frac{(s+2)(s+4)}{(s+1)(s+3)}$ is P.R.F.**

Solution:

$$F(s) = \frac{s^2+6s+8}{s^2+4s+3} = \frac{N(s)}{D(s)}$$

Condition 1: Since all roots of D(s) lies on the negative real axis or left half of s-plane, hence D(s) is Hurwitz polynomial.

Condition 2: Does not exist.

Condition 3: $M_1 = s^2 + 8$, $\quad M_2 = s^2 + 3$

$N_1 = 6s$, $\quad N_2 = 4s$

$$M_1 M_2 - N_1 N_2 \geq 0$$

$$(s^2+8) \cdot (s^2+3) - (6s) \cdot (4s) \geq 0$$

$$s^4 - 13s^2 + 24 \geq 0$$

or, $A(\omega^2) = \omega^4 + 13\omega^2 + 24 \geq 0$ for all ω

Hence, F(s) is PRF.

9. **Test for positive real $F(s) = \frac{2s^2+2s+1}{s^3+2s^2+s+2}$**

Solution:

Condition 1: $M(s) = 2s^2 + 2$, $N(s) = s^3 + s$

$$\begin{array}{l} 2s^2+2 \,\big)\, s^3 + s \,\big(\, \frac{s}{2} \\ \quad\quad\quad -s^3 + s \\ \hline \quad\quad\quad\quad \times \end{array}$$

The process terminates prematurely. The divisor $(2s^2 + 2)$ is a factor of M(s) and N(s) of this denominator polynomial.

Condition 2: $D(s) = (2s^2+2)\left(\frac{s}{2}+1\right) = (s^2+1)(s+2)$

D(s) has a root on $j\omega$ axis at $s = \pm j$

We have to determine the residue,

$$F(s) = \frac{As+B}{s^2+1} + \frac{C}{s+2}$$

By the partial fraction expansion, we have

$$F(s) = \frac{s}{s^2+1} + \frac{1}{s+2}$$

So, we can say residue is positive

Condition 3:

$$M_1(s) = 2s^2 + 1 \,;\, N_1(s) = 2s$$

$$M_2(s) = 2s^2 + 2 \,;\, N_2(s) = s^3 + s$$

$$\begin{aligned} A(\omega^2) &= M_1 M_2 - N_1 N_2 \geq 0 \\ &= (2s^2+1)(2s^2+2) - (2s)(s^3+s) \geq 0 \\ &= (2s^2+1)\,2\,(s^2+1) - 2s^2(s^2+1) \geq 0 \\ &= 2(s^2+1)(2s^2+1-s^2) \geq 0 \\ &= 2(s^2+1)^2 \geq 0 \\ &= 2(-\omega^2+1)^2 \geq 0 \text{ (By putting } j\omega) \\ &= 2(1-\omega^2)^2 \geq 0 \text{ for all } \omega \end{aligned}$$

Hence, F(s) is positive real function

10. **Test whether the following function is a PRF or not.**

$$F(s) = \frac{s^3+s^2+3s+5}{s^2+6s+8}$$

Solution:

Condition 1: $M(s) = s^2 + 8$

$N(s) = 6s$

Hence, D(s) is Hurwitz polynomial.

Condition 2: There are no poles of given function F(s) lie on $j\omega$-axis. So the condition is ignored.

Condition 3: $M_1 = s^2 + 5$, $M_2 = s^2 + 8$, $N_1 = s^3 + 3s$, $N_2 = 6s$

$$A(\omega^2) = M_1M_2 - N_1N_2 \geq 0$$

$$(s^2+5)(s^2+8) - (s^3+3s)(6s) \geq 0$$

or, $$s^4 + 13s^2 + 40 - 6s^4 - 18s^2 \geq 0$$

or, $$-5s^4 - 5s^2 + 40 \geq 0$$

or, $$-5\omega^4 + 5\omega^2 + 40 \geq 0 \text{ (By putting } s = j\omega)$$

Since all coefficients of $A(\omega^2)$ are not positive.

Using sturm's test as,

$$A_0(x) = -5x^2 + 5x + 40$$

$$A_1(x) = \frac{dA_0(x)}{dx} = -10x + 5$$

$$\frac{A_0(x)}{A_1(x)} = (K_1x - K_0) - \frac{A_2(x)}{A_1(x)}$$

or,
$$\begin{array}{r|l} -10x+5 & -5x^2 + 5x + 40 \;\left(\frac{x}{2} - \frac{1}{4}\right. \\ & \underline{-5x^2 + \frac{5}{2}x} \\ & \quad \frac{5}{2}x + 40 \\ & \quad \underline{-\frac{5}{2}x - \frac{5}{2}} \\ & \qquad \frac{165}{4} \end{array}$$

Hence,

$$\frac{A_0(x)}{A_1(x)} = \left(\frac{x}{2} - \frac{1}{4}\right) + \frac{\left(\frac{165}{4}\right)}{-10x+5} = \left(\frac{x}{2} - \frac{1}{4}\right) - \frac{\left(\frac{-165}{4}\right)}{-10x+5}$$

$$A_2(x) = \frac{-165}{4}$$

A_0	A_1	A_2	No. of sign changes
x = 0 +	+	–	$s_0 = 1$
x = ∞ –	–	–	$s_\infty = 0$

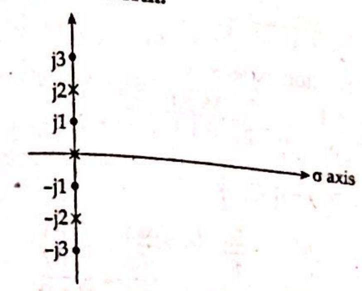
Now, $s_\infty - s_0 = 1$

Hence, $A(\omega^2) \geq 0$ for all ω.

11. An impedance function has the pole zero pattern shown in the figure below. If $Z(-2) = \frac{-130}{16}$, synthesize the impedance in,

a) Foster I and II form.

b) Cauer I and II form.



Solution:

From the pole zero diagram,

$$Z(s) = \frac{K(s^2+1)(s^2+9)}{s(s^2+4)}$$

By putting $Z(-2) = \frac{-130}{16} = \frac{K \times 5 \times 13}{-2 \times 8}$ gives K = 2

Hence, $$Z(s) = 2\frac{(s^2+1)(s^2+9)}{s(s^2+4)}$$

a) **Foster -I form**

$$\begin{array}{r|l} s^3 + 4s & 2s^4 + 20s^2 + 18 \;(2s \\ & \underline{-2s^4 + 8s^2} \\ & \quad 12s^2 + 18 \end{array}$$

$$Z(s) = 2s + \frac{12s^2 + 18}{s(s^2+4)}$$

Using partial fraction expansion,

$$\frac{12s^2+18}{s(s^2+4)} = \frac{A}{s} + \frac{Bs + C}{s^2+4}$$

∴ $$A = \frac{9}{2},\ C = 0 \text{ and } B = \frac{15}{2}$$

Therefore, $$Z(s) = 2s + \frac{\left(\frac{9}{2}\right)}{s} + \frac{\left(\frac{15}{2}\right)}{s^2+4}$$

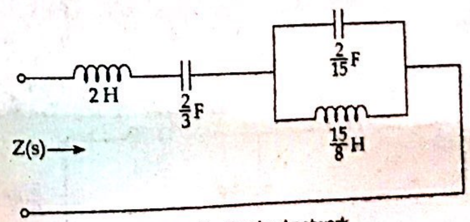


Figure: Synthesized network

b) **Foster-II form**

$$Y(s) = \frac{s(s^2+4)}{2(s^2+1)(s^2+9)}$$

Using partial fraction expansion,

$$\frac{s(s^2+4)}{2(s^2+1)(s^2+9)} = \frac{1}{2}\left[\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9}\right]$$

$\therefore \quad A = \frac{3}{8}, B = D = 0, C = \frac{5}{8}$

$$\therefore \quad Y(s) = \frac{1}{2}\left[\frac{\frac{3s}{8}}{s^2+1} + \frac{\frac{5s}{8}}{s^2+9}\right] = \frac{\left(\frac{3s}{16}\right)}{s^2+1} + \frac{\left(\frac{5s}{16}\right)}{s^2+9}$$

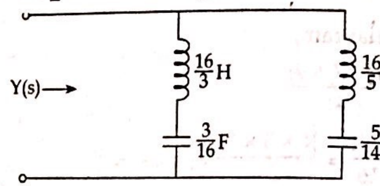


Figure: Synthesized network

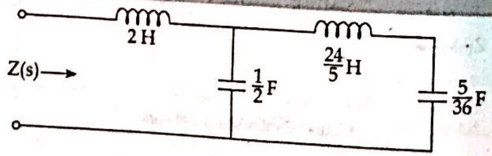
c) **Cauer-I form**

$$Z(s) = \frac{2(s^2+1)(s^2+9)}{s(s^2+4)} = \frac{2s^4+20s^2+18}{s^3+4s}$$

The continued fraction expansion is,

$$\begin{array}{l}
s^3+4s \,\big)\, 2s^4+20s^2+18 \;\left(2s \leftrightarrow Z_1\right. \\
\quad -2s^4+8s^2 \\
\qquad 12s^2+18 \,\big)\, s^3+4s \;\left(\frac{1}{12}s \leftrightarrow Y_2\right. \\
\qquad\quad -s^3+\frac{3}{2}s \\
\qquad\qquad \frac{5}{2}s \,\big)\, 12s^2+18 \;\left(\frac{2}{5}\times 12s = \frac{24}{5} \leftrightarrow Z_3\right. \\
\qquad\qquad\quad -12s^2 \\
\qquad\qquad\qquad 18 \,\big)\, \frac{5}{2}s \;\left(\frac{1}{18}\cdot\frac{5}{2}s = \frac{5}{36}s \leftrightarrow Y_4\right. \\
\qquad\qquad\qquad\quad -\frac{5}{2}s \\
\qquad\qquad\qquad\qquad \times
\end{array}$$

NOTE: We know that the quotients of the continued fraction expansion gives the elements of the network in cauer I and II forms. Because the continued fraction expansion always inverts each remainder and divides, the successive alternate between Z and X and then Z again, as shown in the proceeding expansion. If the initial function is an impedance, the first quotient must necessarily be an impedance. When the initial function is an admittance, the first function is an admittance, the first quitient is an admittance.



d) **Cauer-II form**

$$Z(s) = \frac{2s^4+20s^2+18}{s^3+4s} = \frac{18+20s^2+2s^4}{4s+s^3}$$

The continued fraction expansion is,

$$\begin{array}{l}
4s+s^3 \,\big)\, 18+20s^2+2s^4 \;\left(\frac{18}{4s} = \frac{9}{2s} \leftrightarrow Z_1\right. \\
\quad -18+\frac{5}{2}s^2 \\
\qquad \frac{31}{2}s^2+2s^4 \,\big)\, 4s+s^3 \;\left(\frac{2}{31}\cdot\frac{4}{s} = \frac{8}{31}s \leftrightarrow Y_2\right. \\
\qquad\quad -4s+\frac{16}{31}s^3 \\
\qquad\qquad \frac{15}{31}s^3 \,\big)\, \frac{31}{2}s^2+2s^4 \;\left(\frac{31}{15}\cdot\frac{31}{2s} = \frac{961}{30}s \leftrightarrow Z_3\right. \\
\qquad\qquad\quad -\frac{31}{2}s^2 \\
\qquad\qquad\qquad 2s^4 \,\big)\, \frac{15}{31}s^3 \;\left(\frac{1}{2}\cdot\frac{15}{31}s = \frac{15}{62}s \leftrightarrow Y_4\right. \\
\qquad\qquad\qquad\quad -\frac{15}{31}s^3 \\
\qquad\qquad\qquad\qquad \times
\end{array}$$

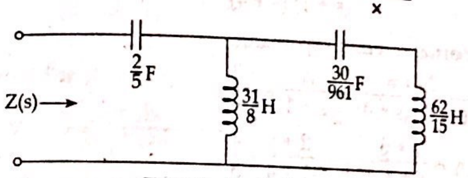


Figure: Synthesized network

12. **An impedance function is given by,** $Z(s) = \frac{(s+1)(s+4)}{s(s+2)(s+5)}$

Find the R-C representation of,

a) Foster-I and II forms **b) Cauer-I and II forms**

Solution:

a) **Foster-I form**

Using partial fraction expansion,

$$Z(s) = \frac{(s+1)(s+4)}{s(s+2)(s+5)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5}$$

$$A = s.\,Z(s)\,|_{s=0} = \frac{1\times 4}{2\times 5} = \frac{2}{5}$$

$$B = (s+2)\,Z(s)\,|_{s=-2} = \frac{(-1)\times 2}{(-2)\times 3} = \frac{1}{3}$$

$$C = (s+5)\,Z(s)\,|_{s=-5} = \frac{(-4)\times(-1)}{(-5)\times(-3)} = \frac{4}{15}$$

$$\text{Hence, } Z(s) = \frac{\left(\frac{2}{5}\right)}{s} + \frac{\left(\frac{1}{3}\right)}{s+2} + \frac{\left(\frac{4}{15}\right)}{s+5}$$

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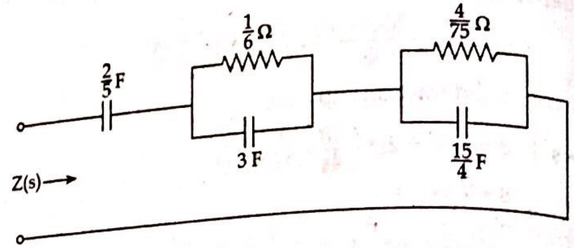


b) **Foster II form**

$$Y(s) = \frac{s(s+2)(s+5)}{(s+1)(s+4)} = \frac{s^3+7s^2+10s}{s^2+5s+4}$$

$$\frac{Y(s)}{s} = \frac{s^2+7s+10}{s^2+5s+4}$$

$$s^3+5s+4 \overline{\big)\, s^4+7s+10\,}\big(1$$
$$\underline{-s^3+5s+4}$$
$$2s+6$$

$$= 1 + \frac{2s+6}{(s+1)(s+4)}$$

Using partial fraction expansion,

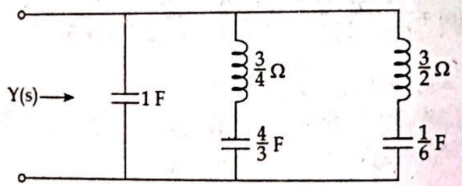
$$\frac{2s+6}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$$A = \frac{2s+6}{s+4}\Big|_{s=-1} = \frac{-2+6}{-1+4} = \frac{4}{3}$$

$$B = \frac{2s+6}{s+1}\Big|_{s=-4} = \frac{-8+6}{-4+1} = \frac{2}{3}$$

Then, $\frac{Y(s)}{s} = 1 + \frac{\left(\frac{4}{3}\right)}{s+1} + \frac{\left(\frac{2}{3}\right)}{s+4}$

or, $Y(s) = s + \frac{\left(\frac{4}{3}\right)s}{s+1} + \frac{\left(\frac{2}{3}\right)s}{s+4}$



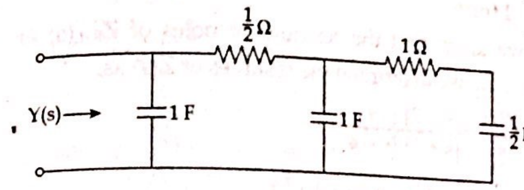
c) **Cauer-I form**

$$Z(s) = \frac{(s+1)(s+4)}{s(s+2)(s+5)} = \frac{s^2+5s+4}{s^3+7s^2+10s}$$

or, $Y(s) = \frac{s^3+7s^2+10s}{s^2+5s+4}$

The continued fraction expansion is,

$$s^2+5s+4 \overline{\big)\, s^3+7s^2+10s\,}\big(s \leftrightarrow Y_1$$
$$\underline{-s^3+5s^2+4s}$$
$$2s^2+6s \overline{\big)\, s^2+5s+4\,}\big(\tfrac{1}{2} \leftrightarrow Z_2$$
$$\underline{-s^2+3s}$$
$$2s+4 \overline{\big)\, 2s^2+6s\,}\big(s \leftrightarrow Y_3$$
$$\underline{-2s^2+4s}$$
$$2s \overline{\big)\, 2s+4\,}\big(1 \leftrightarrow Z_4$$
$$\underline{-2s}$$
$$4 \overline{\big)\, 2s\,}\big(\tfrac{1}{2}s \leftrightarrow Y_5$$
$$\underline{-2s}$$
$$\times$$

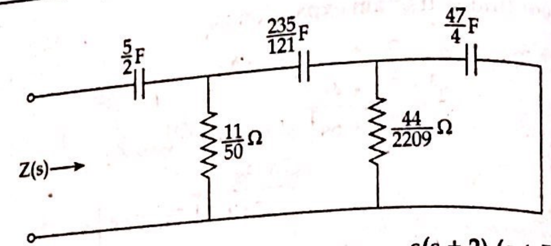


d) **Cauer -II form**

$$Z(s) = \frac{s^2+5s+4}{s^3+7s^2+10s} = \frac{4+5s+s^2}{10s+7s^2+s^3}$$

The continued fraction expansion is,

$$10s+7s^2+s^3 \overline{\big)\, 4+5s+2s^2\,}\Big(\frac{4}{10s} = \frac{2}{5s} \leftrightarrow Z_1$$
$$\underline{-4+\frac{15}{5}+s\frac{7}{5}s^2}$$
$$\frac{11}{5}s+\frac{3}{5}s^2 \overline{\big)\, 10s+7s^2+s^3\,}\Big(\frac{5}{11}\cdot 10 = \frac{50}{11} \leftrightarrow Y_2$$
$$\underline{-10s+\frac{30}{11}s^2}$$
$$\frac{47}{11}s^2+s^3 \overline{\big)\, \frac{11}{5}s+\frac{3}{5}s^2\,}\Big(\frac{11}{47}\cdot\frac{11}{5}s = \frac{121}{44} \leftrightarrow Z_3$$
$$\underline{-\frac{11}{5}s+\frac{121}{235}s^2}$$
$$\frac{20}{235}s^2 \overline{\big)\, \frac{47}{11}s^2+s^3\,}\Big(\frac{235}{20}\cdot\frac{47}{11} = \frac{2209}{44} \leftrightarrow Y_4$$
$$\underline{-\frac{47}{11}s^2}$$
$$s^3 \overline{\big)\, \frac{20}{235}s^2\,}\Big(\frac{20}{235s} = \frac{4}{47s} \leftrightarrow Z_5$$
$$\underline{-\frac{20}{235}s^2}$$
$$\times$$



13. An impedance function is given by $Z(s) = \frac{s(s+2)(s+5)}{(s+1)(s+4)}$. Find the R-L representation of a) Foster I and II forms and b) cauer I and II forms.

Solution:

a) **Foster -I form**

Since we know that the residues of poles of $Z_{R\text{-}L}(s)$ are real and negative. So we determine the residues of Z(s) as,

$$\frac{Z(s)}{s} = \frac{(s+2)(s+5)}{(s+1)(s+4)}$$

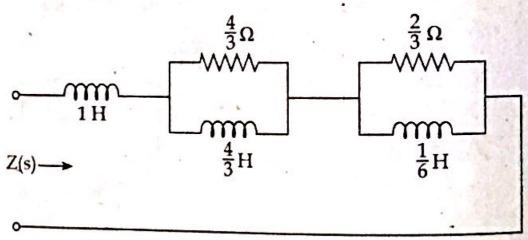
$$s^2+5s+4 \,\big)\, s^2+7s+10 \,\big(\, 1$$
$$\underline{-\,s^2+5s+4}$$
$$2s+6$$

$$= 1 + \frac{2s+6}{(s+1)(s+4)}$$

Using partial fraction expansion,

$$\frac{Z(s)}{s} = 1 + \frac{\left(\frac{4}{3}\right)}{s+1} + \frac{\left(\frac{2}{3}\right)}{s+4}$$

or, $$Z(s) = s + \frac{\left(\frac{4}{3}\right)s}{s+1} + \frac{\left(\frac{2}{3}\right)s}{s+4}$$

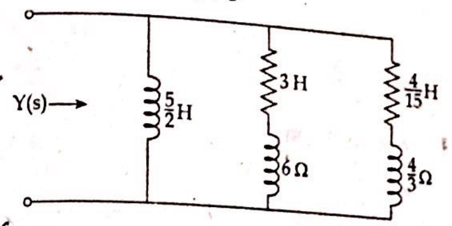


b) **Foster II form**

$$Y(s) = \frac{(s+1)(s+4)}{s(s+2)(s+5)}$$

Using partial fraction expansion, we have,

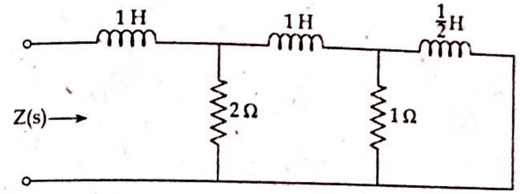
$$Y(s) = \frac{\left(\frac{2}{5}\right)}{s} + \frac{\left(\frac{1}{3}\right)}{s+2} + \frac{\left(\frac{4}{15}\right)}{s+5}$$



c) **Cauer-I form**

$$Z(s) = \frac{s^3+7s^2+10s}{s^2+5s+4}$$

∴ $Z_1 = s,\ Y_2 = \frac{1}{2},\ Z_3 = s,\ Y_4 = 1,\ Z_5 = \frac{1}{2}s$

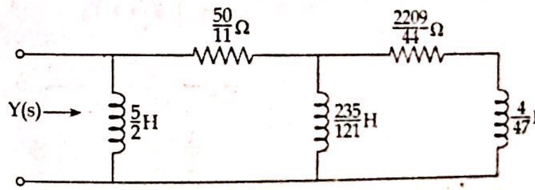


d) **Cauer-II form**

$$Z(s) = \frac{s^3+7s^2+10s}{s^2+7s+4} = \frac{10s+7s^2+s^3}{4+5s+s^2}$$

or, $$Y(s) = \frac{4+5s+s^2}{10s+7s^2+s^3}$$

$$Y_1 = \frac{2}{5s},\ Y_3 = \frac{121}{235\,s},\ Z_2 = \frac{50}{11},\ Z_4 = \frac{2209}{44},\ Y_5 = \frac{4}{475}$$



14. Which of the following function are L-C driving point impedances? Why?

$$Z_1(s) = \frac{s(s^2+4)(s^2+16)}{(s^2+9)(s^2+25)}$$

$$Z_2(s) = \frac{(s^2+1)(s^2+8)}{s(s^2+4)}$$

and, synthesize the reliable impedances in a foster and cauer form.

Solution:

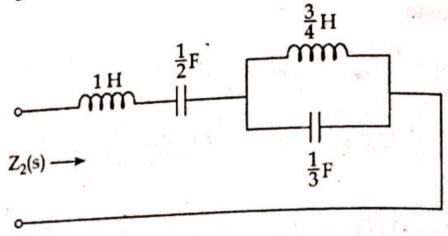
$Z_1(s)$ is not L-C driving point impedance, since poles and zeros are not interlacing on the $j\omega$-axis. On the other hand, $Z_2(s)$ is L-C driving point impedance.

Foster-I form

$$Z_2(s) = \frac{(s^2+1)(s^2+8)}{s(s^2+4)}$$

Using partial fraction expansion,

$$Z_2(s) = s + \frac{2}{s} + \frac{3s}{s^2+4}$$

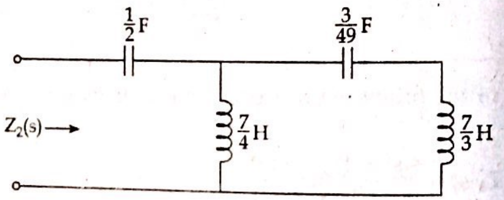


Cauer-II form

$$Z_2(s) = \frac{8 + 9s^2 + s^4}{4s + s^3}$$

The continued fraction expansion is,

$$4s + s^3 \,\big)\, 8 + 9s^2 + s^4 \,\Big(\frac{2}{s} \leftrightarrow Z_1$$
$$\underline{-8 + 2s^2}$$
$$7s^2 + s^4 \,\big)\, 4s + s^3 \,\Big(\frac{4}{7s} \leftrightarrow Y_2$$
$$\underline{-4s + \frac{4}{7}s^3}$$
$$\frac{3}{7}s^3 \,\big)\, 7s^2 + s^4 \,\Big(\frac{7}{3}\cdot\frac{7}{s} = \frac{49}{3s} \leftrightarrow Z_3$$
$$\underline{-7s^2}$$
$$s^4 \,\big)\, \frac{3}{7}s^3 \,\Big(\frac{3}{7s} \leftrightarrow Y_4$$
$$\underline{-\frac{3}{7}s^3}$$
$$\times$$



15. **Realize the function $Z(s) = \dfrac{s(s^2+4)}{2(s^2+1)(s^2+9)}$ in both foster forms LC network.**

Solution:

Foster-I form

Using partial fraction expansion,

$$Z(s) = \frac{s(s^2+4)}{2(s^2+1)(s^2+9)} = \frac{K_1 s}{(s^2+1)} + \frac{K_2 s}{(s^2+9)}$$

$$K_1 = \left.\frac{(s^2+4)}{2(s^2+9)}\right|_{s^2=-1} = \frac{3}{16}$$

$$K_2 = \left.\frac{(s^2+4)}{2(s^2+1)}\right|_{s^2=-9} = \frac{5}{16}$$

$$\therefore\ Z(s) = \frac{\left(\frac{3}{16}\right)s}{(s^2+1)} + \frac{\left(\frac{5}{16}\right)s}{(s^2+9)}$$

$$Z(s) = \frac{\left(\frac{1}{C_1}\right)s}{s^2 + \frac{1}{L_1C_1}} + \frac{\left(\frac{1}{C_2}\right)s}{-s^2 + \frac{1}{L_2C_2}}$$

$$C_1 = \frac{16}{3}\,\text{F}, \qquad \frac{1}{L_1C_1} = 1 \Rightarrow L_1 = \frac{3}{16}\,\text{H}$$

$$C_2 = \frac{16}{5}, \qquad L_2C_2 = \frac{1}{9} \Rightarrow L_2 = \frac{5}{144}\,\text{H}$$

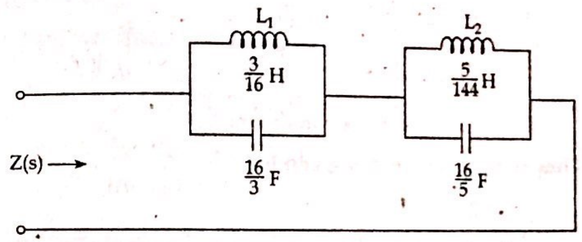


Figure: Foster-I form

Foster-II form

$$Y(s) = \frac{2(s^2+1)(s^2+9)}{s(s^2+4)} = \frac{2s^4 + 20s^2 + 18}{s^3 + 4s}.$$

$$s^2 + 4s \,\big)\, 2s^4 + 20s^2 + 18 \,\Big(2s$$
$$\underline{-2s^4 + 8s^2}$$
$$12s^2 + 18$$

$$Y(s) = 2s + \frac{12s^2 + 18}{s(s^2+4)}$$

Using partial fraction expansion,

$$\frac{12s^2+18}{s(s^2+4)} = \frac{K_1}{s} = \frac{K_2 s}{s^2+4}$$

$$K_1 = \frac{12s^2+18}{(s^2+4)}\bigg|_{s=0} = \frac{9}{2}$$

$$K_2 = \frac{12s^2+18}{s^2}\bigg|_{s^2=-4} = \frac{15}{2}$$

$$\therefore \quad Y(s) = 2s + \frac{\left(\frac{9}{2}\right)}{s} + \frac{\left(\frac{15}{2}\right)s}{s^2+4}$$

$$Y(s) = C_1 s + \frac{1}{L_2 s} + \frac{\left(\frac{1}{L_3}\right)s}{s^2 + \frac{1}{L_3C_3}}$$

$\therefore \quad C_1 = 2\text{ F}$

$\therefore \quad L_2 = \frac{2}{9}\text{ H}$

$\therefore \quad L_3 = \frac{2}{15}\text{ H}$

$L_3C_3 = \frac{1}{4} \Rightarrow C_3 = \frac{15}{8}\text{ F}$

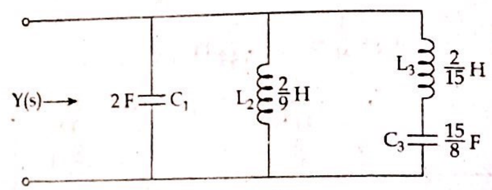


Figure: Foster-II form

16. An impedance function is given by,

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+4)}$$

Solution:

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+4)}$$

or, $$\frac{Z(s)}{s} = \frac{2(s+1)(s+3)}{s(s+2)(s+4)}$$

Using partial fraction expansion,

$$\frac{Z(s)}{s} = \frac{\left(\frac{3}{4}\right)}{s} + \frac{\left(\frac{1}{2}\right)}{s+2} + \frac{\left(\frac{3}{4}\right)}{s+4}$$

or, $$Z(s) = \frac{3}{4} + \frac{\left(\frac{1}{2}\right)s}{s+2} + \frac{\left(\frac{3}{4}\right)s}{s+4}$$

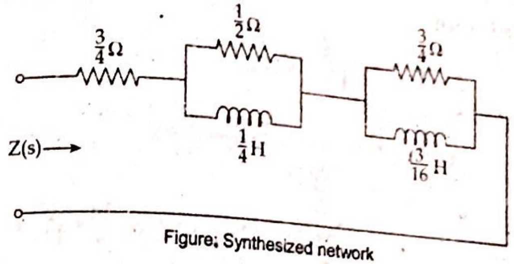


Figure: Synthesized network

17. **An impedance is given by,**

$$Z(s) = \frac{8(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)}$$

Realize the network in,

a) Foster-I form

b) Cauer-II form

Solution:

$$Z(s) = \frac{8(s^2+1)(s^2+3)}{s(s^2+2)(s^2+4)} = \frac{8(s^4+4s^2+3)}{s(s^2+2)(s^2+4)}$$

a) **Foster-I form**

Using partial fraction expansion,

$$Z(s) = \frac{A}{s} + \frac{Bs}{s^2+2} + \frac{Cs}{s^2+4}$$

$$8s^4 + 32s^2 + 24 = A(s^2+2)(s^2+4) + B(s^2+4) + Cs^2(s^2+2)$$

Equating coefficients,

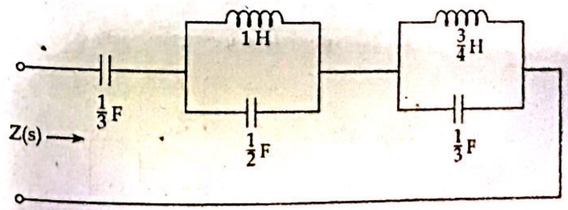
$s^4 \rightarrow A + B + C = 8$

$s^2 \rightarrow 6A + 4B + 2C = 32$

Constant $\rightarrow 8A = 24$

Gives, $A = 3, B = 2\ C = 3$

$$\therefore \quad Z(s) = \frac{3}{s} + \frac{2s}{s^2+2} + \frac{3s}{s^2+4}$$



b) Cauer-II form

$$8s+6s^3+s^5\,\big)\,24+32s^2+8s^4\,\Big(\frac{3}{s}\leftrightarrow Z_1$$
$$-\,24+18s^2+3s^4$$
$$14s^2+5s^4\,\big)\,8s+6s^3+s^5\,\Big(\frac{8}{14s}=\frac{4}{7s}\leftrightarrow Y_2$$
$$-\,8s+\frac{20}{7}s^3$$
$$\frac{22}{7}s^3+s^5\,\big)\,14s^2+5s^4\,\Big(\frac{7}{22}\cdot\frac{14}{s}=\frac{49}{11s}\leftrightarrow Z_3$$
$$-\,14s^2+\frac{49}{11}s^4$$
$$\frac{6}{11}s^4\,\big)\,\frac{22}{7}s^3+s^5\,\Big(\frac{11}{6}\cdot\frac{22}{7s}=\frac{121}{21s}\leftrightarrow Y_4$$
$$-\,\frac{22}{7}s^3$$
$$s^5\,\big)\,\frac{6}{11}s^4\,\Big(\frac{6}{11s}\leftrightarrow Z_5$$
$$-\,\frac{6}{11}s^4$$
$$\times$$

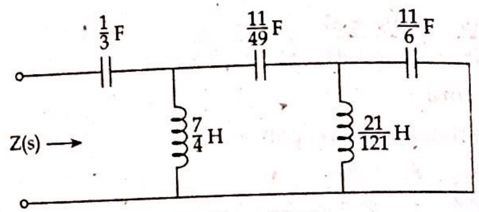


Figure: Cauer-II form

18. **Realize the following functions in first and second foster form**

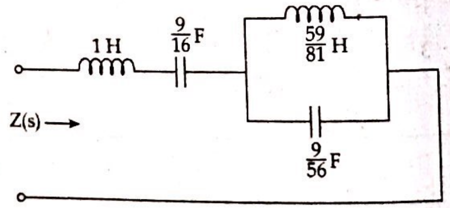
a) $Z(s)=\dfrac{(s^2+1)(s^2+16)}{s(s^2+9)}$ b) $Z(s)=\dfrac{(s+2)(s+5)}{(s+1)(s+4)}$

Solution:

a) **First foster form**

$$Z(s)=\frac{(s^2+1)(s^2+16)}{s(s^2+9)}=s+\frac{8s^2+16}{s(s^2+9)}$$

$$s+\frac{A}{s}+\frac{Bs}{s^2+9}=s+\frac{\left(\frac{16}{9}\right)}{s}+\frac{\left(\frac{56}{9}\right)s}{s^2+9}$$



Second foster form

$$Y(s)=\frac{s(s^2+9)}{(s^2+1)(s^2+16)}$$
$$=\frac{As}{s^2+1}+\frac{Bs}{s^2+16}$$
$$=\frac{\left(\frac{8}{15}\right)s}{s^2+1}=\frac{\left(\frac{7}{15}\right)s}{s^2+16}$$

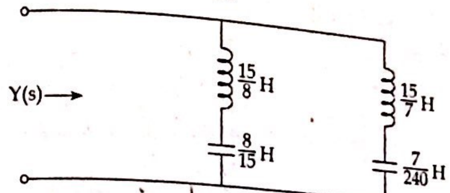


Figure: Second foster form

b) **First foster form**

$$Z(s)=\frac{(s+2)(s+5)}{(s+1)(s+4)}=1+\frac{2s+6}{(s+1)(s+4)}$$

$$=1+\frac{A}{s+1}+\frac{B}{s+4}=1+\frac{\left(\frac{4}{3}\right)}{s+1}+\frac{\left(\frac{2}{3}\right)}{s+4}$$

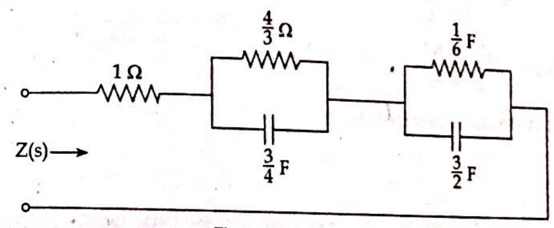
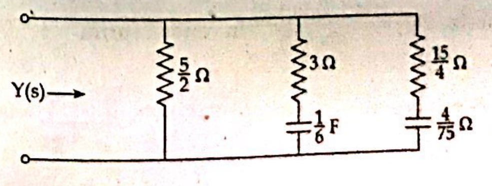


Figure: First foster form

Second foster form

$$\frac{Y(s)}{s}=\frac{(s+1)(s+4)}{s(s+2)(s+5)}$$
$$=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+5}$$

$$Y(s)=\frac{2}{s}+\frac{\left(\frac{1}{3}\right)s}{s+2}+\frac{\left(\frac{4}{15}\right)s}{s+5}$$



19. Synthesize $Z(s) = \dfrac{(s+5)}{(s+1)(s+6)}$ in foster II form

Solution:

$$Z(s) = \frac{s+5}{(s+1)(s+6)}$$

$$Y(s) = \frac{(s+1)(s+6)}{(s+5)}$$

$$\begin{array}{r|l} s+5 & s^2+7s+6 \;\{ s+2 \\ & \underline{-s^2+5s} \\ & \quad 2s+6 \\ & \underline{\;-2s+10} \\ & \qquad -4 \end{array}$$

$$Y(s) = s + 2 + \frac{(-4)}{s+5}$$

or, $\dfrac{Y(s)}{s} = \dfrac{s^2+7s+6}{s+5}$

Then,

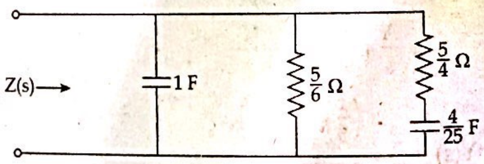
$$\begin{array}{r|l} s^2+5 & s^2+7s+6 \;\{ 1 \\ & \underline{-s^2+5s} \\ & \quad 2s+6 \end{array}$$

$\therefore$ $\dfrac{Y(s)}{s} = 1 + \dfrac{(2s+6)}{s(s+5)}$

Using partial fraction expansion,

$$\frac{Y(s)}{s} = 1 + \frac{\left(\frac{6}{5}\right)}{s} + \frac{\left(\frac{4}{5}\right)}{s+5}$$

$\therefore$ $Y(s) = s + \dfrac{6}{5} + \dfrac{\left(\frac{4}{5}\right)s}{s+5}$

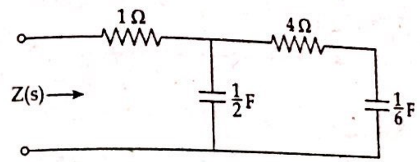


20. Synthesize $Z(s) = \dfrac{(s+1)(s+3)}{s(s+2)}$ in cauer-I form.

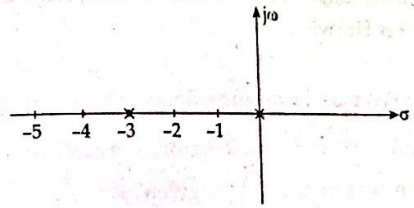
Solution:

$$Z(s) = \frac{(s+1)(s+3)}{s^2+2s}$$

$$\begin{array}{l} s^3+2s \,\big)\, s^2+4s+3 \;\{ 1 \leftrightarrow Z_1 \\ \qquad\quad \underline{-s^2+2s} \\ \qquad\qquad 2s+3 \,\big)\, s^2+2s \;\{ \frac{s}{2} \leftrightarrow Y_2 \\ \qquad\qquad\qquad \underline{-s^3+\frac{3}{2}s} \\ \qquad\qquad\qquad \frac{1}{2}s \,\big)\, 2s+3 \;\{ 4 \leftrightarrow Z_3 \\ \qquad\qquad\qquad\qquad \underline{-2s} \\ \qquad\qquad\qquad\qquad 3 \,\big)\, \frac{1}{2}s \;\{ \frac{1}{6}s \leftrightarrow Y_4 \\ \qquad\qquad\qquad\qquad\quad \underline{-\frac{1}{2}s} \\ \qquad\qquad\qquad\qquad\qquad \times \end{array}$$



21. An impedance function has the pole-zero plot as shown in figure. If $Z(-2) = 3$, synthesize the impedance in foster second form.



Solution:

From the pole-zero diagram,

$$Z(s) = \frac{K(s+1)(s+5)}{s(s+3)}$$

and, $Z(-2) = 3 = \dfrac{K(-1)(3)}{(-2)(1)} = \dfrac{3}{2}K$

or, $K = 2$

$\therefore$ $Z(s) = \dfrac{2(s+1)(s+5)}{s(s+3)}$

Since singularity nearest to origin is a pole, therefore, given function is RC impedance function.

$$Y(s) = \frac{s(s+3)}{2(s+1)(s+5)}$$

or, $\dfrac{Y(s)}{s} = \dfrac{\frac{(s+3)}{2}}{(s+1)(s+5)}$

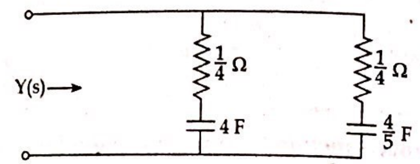
Using partial fraction expansion

$$\frac{Y(s)}{s} = \frac{A}{s+1} + \frac{B}{s+5}$$

$$A = \left.\frac{\frac{(s+3)}{2}}{s+5}\right|_{s=-1} = \frac{1}{4}$$

$$B = \left.\frac{\frac{(s+3)}{2}}{s+1}\right|_{s=-5} = \frac{-1}{-4} = \frac{1}{4}$$

$$\text{Hence, } Y(s) = \frac{\frac{1}{4}s}{(s+1)} + \frac{\frac{1}{4}s}{(s+5)}$$



22. Determine the range of β such that the polynomial $P(s) = s^4 + s^2 + 4s^2 + \beta s + 3$ is Hurwitz.

Solution:

All coefficients of P(s) must be positive, hence $\beta > 0$.

$$M(s) = s^4 + 4s^2 + 3,\ N(s) = s^3 + \beta s$$

Continued fraction expansion of $\frac{M(s)}{N(s)}$ is given as,

$$s^3 + \beta s \,\big)\, s^4 + 4s^2 + 3s \,\big(\, s$$
$$-s^4 + \beta s^2$$
$$(4-\beta)s^2 + 3 \,\big)\, s^3 + \beta s \,\Big(\, \frac{s}{4-\beta}$$
$$-s^3 + \frac{3}{4-\beta}s$$
$$\left(\beta - \frac{3}{4-\beta}\right)s$$

All quotients must be positive *i.e.,*

i) $4 - \beta > 0$ or $\beta < 4$

ii) $\beta - \frac{3}{4-\beta} > 0$ or $-\beta^2 + 4\beta - 3 > 0$

or, $\beta^2 - 4\beta + 3 < 0$

or, $(\beta - 1)(\beta - 3) < 0$

i.e., $1 < \beta < 3$

Hence, the required range of β is $1 < \beta < 3$.

23. Test whether the given function F(s) represents a PRF

$$F(s) = \frac{s+3}{s^2 + 5s + 1}$$

Solution:

Condition 1: $M(s) = s^2 + 1,\ N(s) = 5s$

$$5s \,\big)\, s^2 + 1 \,\big(\, s \leftrightarrow Z_1$$
$$-s^2$$
$$1 \,\big)\, 5s \,\big(\, 5s$$
$$-5s$$
$$\times$$

All quotients are positive. Hence, denominator of F(s) is Hurwitz polynomial.

Condition 2: There are no poles of given function F(s) lie on jω-axis, so this condition does not exist.

Condition 3: $M_1 = 3,\ M_2 = s^2 + 1,\ N_1 = s,\ N_2 = 5s$

$$A(\omega^2) \equiv M_1 M_1 - N_1 N_2 \geq 0$$

or, $3(s^2 + 1) - s(5s) \geq 0$

or, $-2s^2 + 3 \geq 0$

or, $2\omega^2 + 3 \geq 0$

Hence, $A(\omega) \geq 0$ for all ω

Therefore, given function F(s) is a positive real function.

24. Test whether the following polynomial is Hurwitz.

$$P(s) = s^3 + 2s^2 + s + 2$$

Solution:

Condition 1: is satisfied (since all coefficients of P(s)) *i.e.,* a_i are positive.

Condition 2: Even and odd parts of P(s) are;

$$M(s) = 2s^2 + 2,\ N(s) = (5)^3 + s$$

So continued fraction expansion of $\psi(s) = \frac{N(s)}{M(s)}$ is given as,

$$2s^2 + 2 \,\big)\, s^3 + s \,\Big(\, \frac{1}{2}s$$
$$-s^3 + s$$
$$\times$$

We see that the division has been terminated prematurely. Thus $(2s^2 + 2)$ factor is common in M(s) and N(s).

$$\text{As } P(s) = (2s^2 + 2)\left(1 + \frac{1}{s}\right) = \omega(s)\, P_1(s)$$

We know that the term $P_1(s)$ is Hurwitz and W(s) is also Hurwitz. Hence P(s) is Hurwitz polynomial.

25. **Realize the R-C admittance in cauer-I and Foster II form.**

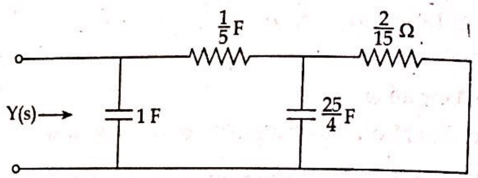
$$Y(s) = \frac{s^2 + 7s + 6}{s + 2}$$

Solution:

a) **Cauer-I form**

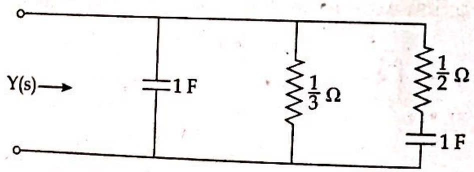
The continued fraction expansion is,

$$s + 2 \overline{)\, s^2 + 7s + 6 \,} \left(s \leftrightarrow Z_1 \right.$$
$$\underline{-s^2 + 2s}$$
$$5s + 6 \overline{)\, s + 2 \,} \left(\frac{1}{5} \leftrightarrow Z_2 \right.$$
$$\underline{-s + \frac{6}{5}s}$$
$$\frac{4}{5} \overline{)\, 5s + 6 \,} \left(\frac{25}{4} s \leftrightarrow Y_3 \right.$$
$$\underline{-5s}$$
$$6 \overline{)\, \frac{4}{5} \,} \left(\frac{2}{15} \leftrightarrow Z_4 \right.$$
$$\underline{-\frac{4}{5}}$$
$$\times$$



b) **Foster-II form**

$$Y(s) = \frac{s^2 + 7s + 6}{s + 2}$$
$$= \frac{s + 5s + 6}{s + 2}$$
$$= s + 3 + \frac{2s}{s + 2}$$



BOARD EXAMINATION SOLVED QUESTIONS

1. **Determine whether the following is a positive real function:**

$$\frac{(2s^2 + s + 1)}{(s^2 + s + 2)}$$

[2011/Fall]

Solution:

Let, $A(s) = s^2 + 1$

$B(s) = s$

Now, $s \overline{)\, s^2 + 1 \,} \left(s \Rightarrow \text{+ve real} \right.$
$\underline{-s^2}$
$s \overline{)\, s \,} \left(1 \Rightarrow \text{+ve real} \right.$
$\underline{-s}$
$\times$

i) Poles are at $s = -0.5 - 1.5j$ and $s = -0.5 + 1.5j$, as there are pole on imaginary axis, hence residue test is to be done.

ii) Finding residue;

$$\frac{As + B}{s^2 + s + 1} = \frac{2s^2 + s + 1}{s^2 + s + 1}$$

or, $As + B = 2s^2 + s + 1$

Comparing coefficients of like terms on both sides,

$A = 1$ and $B = 1$

so, $\frac{2s^2 + s + 1}{s^2 + s + 1} = \frac{1s + 1}{s^2 + s + 1} = \frac{s + 1}{s^2 + s + 1}$

Hence, residue test is positive.

iii) $M_1(s) = 2s^2 + 1$

$M_2(s) = s^2 + 1$

$N_1(s) = s$

$N_2(s) = s$

Hence, $A(j\omega) = M_1M_2 - N_1N_2 \geq 0$

$= (2s^2 + 1)(s^2 + 1) - s \cdot s \geq 0$

$= 2s^4 + 2s^2 + s^2 + 1 - s^2 \geq 0$

$= 2s^4 + 2s^2 + 1 \geq 0$

$= 2(j\omega)^4 + 2(j\omega)^2 + 1 \geq 0$

$= 2 \times \omega^4 + 2 \times (-\omega)^2 + 1 \geq 0$

$\therefore \quad A(j\omega) = 2\omega^4 - 2\omega^2 + 1 \geq 0$ for all ω

Hence, Z(s) is positive real function.

2. **Check whether the given function is positive real function (PRF) or not.** [2014/Spring]

$$F(s) = \frac{s^2 + 2s + 6}{s(s + 3)}$$

Solution:

$$F(s) = \frac{s^2 + 2s + 6}{s(s + 3)}$$

i) Poles are at s = 0 and s = –3 *i.e.*, no poles lie on the right half s-plane.

ii) s = 0, there is a pole on imaginary axis, hence residue test is to be done.

so, $\left.\frac{s^2 + 2s + 6}{s(s + 3)} \times s\right|_{s=0} = \frac{6}{3} = 2 \Rightarrow$ +ve and real

iii) $F(j\omega) = M_1(j\omega) \times M_2(j\omega) - N_1(j\omega) \times N_2(j\omega) \geq 0$

$= [(s^2 + 6) \times s^2 - (2s \times 3s)]$

$= [((j\omega)^2 + 6) \times (j\omega)^2] - [2(j\omega) \times (j\omega)]$

$= \omega^4 + 6\,\omega^2 - 6\,\omega^2 = \omega^4 \geq 0$ for all ω

Hence F(s) is positive real function.

3. **Show that the function given below is positive real function.**

$$G(s) = \frac{2s^2 + 2s + 1}{s^3 + 2s^2 + s + 2}$$ **[2012/Spring, 2013/Spring, 2016/Fall]**

Solution:

$$G(s) = \frac{2s^2 + 2s + 1}{s^3 + 2s^2 + s + 2} = \frac{2s^2 + 2s + 1}{(s^2 + 1)(s + 2)}$$

i) $M(s) = 2s^2 + 2$

$N(s) = s^3 + s$

Now,

$$2s^2 + 2 \overline{\big)\, s^3 + s\,} \left(\frac{s}{2} \Rightarrow \text{+ve real} \right.$$
$$\underline{-s^3 + s}$$
$$\times$$

ii) Since there are poles on imaginary axis, s = ± jω, residue test is necessary. For finding residue test,

$$F(s) = \frac{As + B}{s^2 + 1} + \frac{C}{s + 2}$$

or, $2s^2 + 2s + 1 = (As + B)(s + 2) + C(s^2 + 1)$

Put s = –2,

or, $2(-2)^2 + 2 \times 2 + 1 = [A \times (-2) + B](-2 + 2) + C[(-2)^2 + 1]$

or, $5 = 0 + 5C$

∴ $C = 1$

Put s = 0,

or, $1 = B + C$

or, $1 = B + 1$

∴ $B = 0$

Put s = 1,

$2 \times 1^2 + 2 \times 1 + 1 = (A + 0)(1 + 2) + 1(1^2 + 1)$

or, $5 = (A + 0) \times 3 + 2$

or, $\frac{5 - 2}{3} = A$

∴ $A = 1$

Hence, $F(s) = \frac{s}{s^2 + 1} + \frac{1}{s + 2}$

Hence residue test is positive

iii) $N_1(s) = 2s$

$N_2(s) = s^3 + s$

$G(\omega^2) = M_1M_2 - N_1N_2 \geq 0$

$= (2s^2 + 1)(2s^2 + 2) - (2s)(s^3 + s) \geq 0$

$= 2(s^2 + 1)(2s^2 + 1 - s^2) - 2s^2(s^2 + 1) \geq 0$

$= 2(s^2 + 1)(2s^2 + 1 - s^2) \geq 0 = 2(s^2 + 1)^2 \geq 0 = 2(-\omega^2 + 1)^2 \geq 0$

∴ $G(\omega^2) = 2(1 - \omega^2)^2 \geq 0$ for all ω

Hence F(s) is positive real function.

4. **Check for PRF when $F(s) = \frac{(s + 2)(s + 4)}{(s + 1)(s + 3)}$**

[2014/Fall, 2015/Fall, 2015/Spring, 2018/Fall, 2019/Fall]

OR,

Check either the given function is PRF or not.

$$F(s) = \frac{s^2 + 6s + 8}{s^2 + 4s + 3}$$ **[2012/Fall, 2018/Spring]**

Solution:

$$F(s) = \frac{(s + 2)(s + 4)}{(s + 1)(s + 3)} = \frac{s^2 + 6s + 8}{s^2 + 4s + 3}$$

i) Poles are at s = –1 and s = –3 *i.e.*, no poles on right half s-plane. It is Hurwitz polynomial.

ii) As there is no pole on imaginary axis, hence residual test is not be done.

iii) $M_1 = s^2 + 8$

$M_2 = s^2 + 3$

$N_1 = 6s$

$N_2 = 4s$

Now, $M_1M_2 - N_1N_2 \geq 0$

or, $(s^2 + 8)(s^2 + 3) - 6s \times 4s \geq 0$

or, $s^4 + 11s^2 + 24 - 24s^2 \geq 0$

or, $s^4 + 11s^2 - 24s^2 + 24 \geq 0$

or, $s^4 - 13s^2 + 24 \geq 0$

Then, $F(j\omega) = (j\omega)^4 + 13(j\omega)^2 + 24 = \omega^4 + 13\omega^2 + 24 \geq 0$ for all ω

so, F(s) is positive real function.

5. Test whether the given function F(s) represent a PRF or not?

$$F(s) = \frac{s+3}{s^2+5s+1}$$

[2016/Spring]

Solution:

$$F(s) = \frac{s+3}{s^2+5s+1} = \frac{s+3}{(s+0.2087)(s+4.7912)}$$

i) Poles are at s = –0.2087 and s = –4.7912 *i.e.*, no poles on right half s-plane. Hence it is Hurwitz polynomial.

ii) Since there is no pole on imaginary axis, hence residue test is not to be done.

iii) Here; $M_1(s) = 3$

$M_2(s) = s^2 + 1$

$N_1(s) = s$

$N_2(s) = 5s$

Now, $M_1M_2 - N_1N_2 \geq 0$

or, $3 \times (s^2+1) - s(5s) \geq 0$

or, $3 + 3s^2 - 5s^2 \geq 0$

or, $3 - 2s^2 \geq 0$

and, $F(j\omega) = 3 + 2\omega^2 \geq 0$ for all ω

so, F(s) is a positive real function

6. Check whether the given function is PRF or not. [2017/Fall]

$$Z(s) = \frac{s(s^2+2)}{(s^2+1)(s^2+3)}$$

Solution:

$$Z(s) = \frac{s^3+2s}{(s^2+1)(s^2+3)} = \frac{s^3+2s}{s^4+4s^2+3} = \frac{s\left(\sqrt{2}i - s\right)\left(\sqrt{2}i + s\right)}{(s+i)(s-i)\left(\sqrt{3}i - s\right)\left(\sqrt{3}i + s\right)}$$

i) Poles are at s = –i, + i, $s = \sqrt{3}i$ and $s = -\sqrt{3}i$ *i.e.*, no poles on right half s-plane. Hence, it is Hurwitz polynomial.

ii) All poles lies on imaginary axis, hence residue test is to be done. For finding residue,

$$Z(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+3}$$

or, $$\frac{s(s^2+2)}{(s^4+4s^2+3)} = \frac{(As+B)(s^2+3) + (Cs+D)(s^2+1)}{(s^4+4s^2+3)}$$

or, $s(s^2+2) = (As+B)(s^2+3) + (s^2+1)(Cs+D)$ (a)

Put s = 0,

or, $0 = B \times 3 + 1 \times D$

∴ $3B + D = 0$ (1)

Put s = 1,

or, $3 = (A+B) \times 4 + (1^2+1)(C+D)$

or, $3 = (A+B) + 2(C+D)$

or, $3 = 4A + 4B + 2C + 2D$

or, $3 = 4A + 4 \times \left(\frac{-D}{3}\right) + 2C + 2D$

or, $9 = 12A - 4D + 6C + 6D$

or, $9 = 12A + 6C + 2D$ (2)

Put s = –1,

or, $-3 = (-A+B) \times 4 + 2 \times (-C+D)$

or, $-3 = -4A + 4B - 2C + 2D$

or, $-3 = -4A + \frac{-4}{3}D - 2C + 2D$

or, $-9 = -12A - 4D - 6C + 6D$

or, $9 = 12A - 2D + 6C$ (3)

Adding (3) and (2), we get,

$24A + 12C = 18$

$8A + 4C = 6$

$4A + 2C = 3$ (4)

Subtracting (3) and (4), we get,

$9 = 12A - 2D + 6C$

$-9 = 12A + 2D + 6C$

$0 = -4D$

∴ $D = 0$

From equation (1), B = 0

Put s = 2, B = D = 0 in equation (a)

or, $2(2^2+2) = (2A+0)(2^2+3) + (2^2+1)(C \times 2 + 0)$

or, $12 = 2A \times 7 + 5 \times 2 \times C$

or, $12 = 14A + 10C$

or, $6 = 7A + 5C$ (5)

Solving equation (4) and (5), we get,

$20A + 10C = 15$

$-14A + 10C = 12$

$6A = 3$

∴ $A = \frac{1}{2}$ and $C = \frac{1}{2}$

Hence, $Z(s) = \frac{1}{2}\frac{s}{(s^2+1)} + \frac{1}{2}\frac{s}{(s^2+3)}$

Hence, residue is positive.

iii) $M_1(s) = 2s$

$M_2(s) = s^4 + 4s^2 + 3$

$N_1(s) = s^3$

$N_2(s) = 0$

Now, $Z(\omega^2) = M_1M_2 - N_1N_2 \geq 0$

$= (2s)(s^4 + 4s^2 + 3) - 0 \times s^3$

$= 2s^5 + 8s^3 + 6s - 0$

$\therefore \quad Z(\omega^2) = 2(\omega^5 - 4\omega^3 + 3\omega) \geq 0$

so, Z(s) is positive real function

7. Check the function for PRF **[2019/Spring]**

$$H(s) = \frac{s^2 + 9s + 3}{s^2 + 7s + 9}$$

Solution:

$$H(s) = \frac{s^2 + 9s + 3}{s^2 + 7s + 9} = \frac{s^2 + 9s + 3}{(s + 1.697)(s + 5.302)}$$

i) Since all the coefficients of numerator and denominator polynomial are +ve, hence H(s) is real if s is real.

ii) The poles of H(s) are $\frac{-7 \pm \sqrt{13}}{2} = -1.697$ and -5.302. All the poles and zeros lies in the left hand of s-plane.

iii) $$\text{Re}[H(j\omega)] = \text{Re}\left[\frac{-\omega^2 + 9j\omega + 3}{-\omega^2 + 7j\omega + 9}\right]$$

$$= \text{Re}\left[\frac{(3 - \omega)^2 + 9j\omega}{(9 - \omega^2) + 7j\omega} \times \frac{(9 - \omega^2) - 7j\omega}{(9 - \omega^2) - 7j\omega}\right]$$

$$= \text{Re}\left[\frac{56\omega^2 + (-3 - \omega^2)(9 - \omega^2) + j[\omega(9 - \omega^2) - \omega(3 - \omega^2)]}{(9 - \omega^2)^2 + 49\omega^2}\right]$$

$$= \frac{56\omega^2 + (3 - \omega^2)(9 - \omega^2)}{(9 - \omega^2)^2 + 49\omega^2} = \frac{56\omega^2 + 27 - 3\omega^2 - 9\omega^2 + \omega^4}{(9 - \omega^2)^2 + 49\omega^2}$$

$$= \frac{\omega^4 + 44\omega^2 + 27}{(9 - \omega^2)^2 + 49\omega^2} \geq 0 \text{ for all values of } \omega$$

$\therefore \quad \text{Re}[H(j\omega)] \geq 0$ for all ω

Thus the given function H(s) is PRF.

8. Synthesize the given impedance function in foster I and II forms:

$$Z(s) = \left[\frac{8(s^2 + 4)(s^2 + 25)}{[s(s^2 + 16)]}\right]$$ **[2011/Fall, 2014/Spring, 2016/Fall]**

Solution:

Given that;

$$Z(s) = \left[\frac{8(s^2 + 4)(s^2 + 25)}{(s^3 + 16s)}\right]$$

or, $$s^3 + 16s \overline{)\, 8s^4 + 232s^2 + 800\,} \left(8s \right.$$
$$\frac{-8s^4 + 138s^2}{104s^2 + 800}$$

so, $$Z(s) = 8s + \frac{800 + 104s^2}{s(s^2 + 16)}$$

Let, $$\frac{10s\, s^2 + 800}{s(s^2 + 16)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 16}$$

or, $104s^2 + 800 = A(s^2 + 16) + Bs^2 + Cs$

Put s = 0,

$800 = \frac{A}{16}$

or, $A = 50$

$\therefore$ Comparing $Cs = 0 \therefore C = 0$

and, $Bs^2 = 104s^2$

$\therefore \quad B = 104$

Then, $$Z(s) = 8s + \frac{50}{s} + \frac{54s}{s^2 + 16}$$

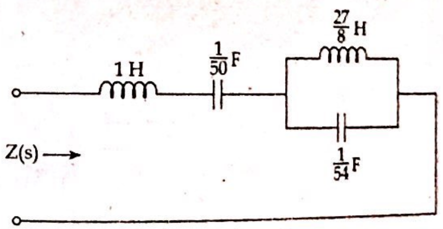
Now the synthesized network

or, $Ls = 8s \Rightarrow L = 8\text{ H}$

and, $\frac{1}{Cs} = \frac{50}{s} \Rightarrow C = \frac{1}{50}\text{ F}$

Also, $$\frac{\frac{s}{C}}{s^2 + \frac{1}{LC}} = \frac{54s}{s^2 + 16} \Rightarrow C = \frac{1}{54}\text{ F, and, } L = \frac{27}{8}\text{ H}$$

Hence, Foster-I form



Foster-II form

$$Y(s) = \frac{s(s^2 + 16)}{8(s^2 + 4)(s^2 + 25)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 25}$$

or, $$\frac{s(s^2 + 16)}{8} = (As + B)(s^2 + 25) + [(Cs + D)(s^2 + 4)]$$

or, $$\frac{(s^3 + 16s)}{8} = As^3 + Bs^2 + 25As + 25B + (Cs^3 + Ds^2 + 4D + 4sC)$$

Comparing coefficients,

$\therefore \quad A + C = \frac{1}{8}$

$\therefore \quad B = 0$

$\therefore \quad 4C + 25A = \frac{16}{8}$

or, $\quad 4C + 25A = 2$

or, $\quad 4 \times \frac{1}{8} + 25\,A = 2$

$\therefore \quad A = \frac{1}{14}$

Thus, $C = \frac{3}{56}$

Hence, $Y(s) = \frac{\left(\frac{1}{14}\right)s}{s^2 + 4} + \frac{3}{56}\left[\frac{s}{(s^2 + 25)}\right]$

Comparing with, $Y(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$

or, $\frac{1}{L} = \frac{1}{14} \Rightarrow L = 14\text{ H}$

Also, $\frac{1}{LC} = 4 \Rightarrow \frac{1}{C} = 4 \times 14$

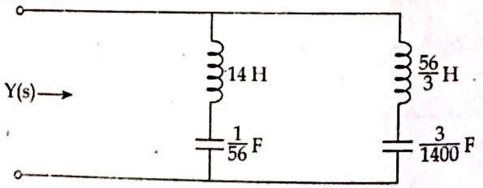
$\therefore \quad C = \frac{1}{56}\text{ F}$

and, $\frac{3}{56} = \frac{1}{L} \Rightarrow L = \frac{56}{3}\text{ H}$

$\frac{1}{LC} = 25 \Rightarrow \frac{1}{C} = 25 \times \frac{56}{3}$

$\therefore \quad C = \frac{3}{1400}\text{ F}$

Synthesized network



9. **Realize the following RC driving point impedance function in;**
   a) Foster I form
   b) Cauer I form

$$G(s) = \frac{s^2 + 6s + 8}{s^2 + 4s + 3}$$

[2013/Spring, 2012/Fall]

Solution:

a) **Foster I form**

$$Z(s) = \frac{s^2 + 6s + 8}{s^2 + 4s + 3}$$

or, $\quad s^2 + 4s + 3 \overline{)\, s^2 + 6s + 8\,}(1$
$\qquad\qquad\qquad\; -s^2 + 4s + 3$
$\qquad\qquad\qquad\quad 2s + 5$

$$Z(s) = 1 + \frac{2s + 5}{s^2 + 4s + 3} = 1 + \frac{2s + 5}{(s + 1)(s + 3)}$$

Let, $\frac{2s + 5}{(s + 1)(s + 3)} = \frac{A}{s + 1} + \frac{B}{s + 3}$

or, $\quad 2s + 5 = A(s + 3) + B(s + 1)$

Put $s = -3$,

or, $\quad -6 + 5 = 0 + B(-3 + 1)$

$\therefore \quad B = \frac{1}{2}$

Put $s = -1$,

or, $\quad -2 + 5 = A(-2 + 3) + 0$

$\therefore \quad A = \frac{1}{2}$

Hence, $\frac{2s + 5}{(s + 1)(s + 3)} = \frac{3}{2(s + 1)} + \frac{1}{2(s + 3)}$

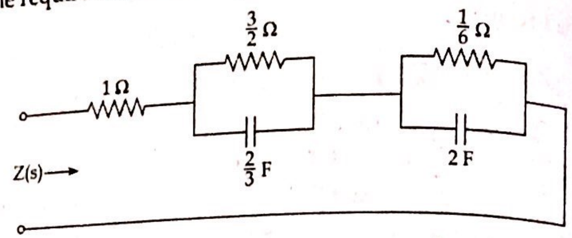
and, $Z(s) = 1 + \frac{3}{2(s + 1)} + \frac{1}{2(s + 3)}$

Comparing with $Z(s) = \frac{\frac{1}{C}}{s + \frac{1}{RC}}$

1st element = Resistance = 1 Ω

2nd element $= \frac{3}{2(s + 1)}$	3rd element $= \frac{1}{2(s + 3)}$
Thus, $C_1 = \frac{2}{3}\text{ F}$	Thus, $C_2 = 2\text{F}$
$\frac{1}{R_1C_1} = 1$	$\frac{1}{R_1C_2} = 3$
or, $\frac{1}{R_1} = 1 \times \frac{2}{3}$	or, $\frac{1}{R_2} = 3 \times 2$
$\therefore \quad R_1 = \frac{3}{2}\,\Omega$	$\therefore \quad R_2 = \frac{1}{6}\,\Omega$

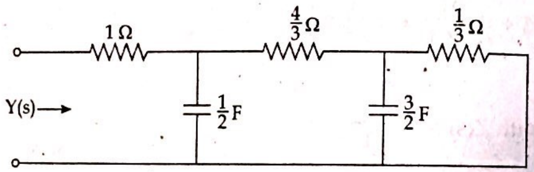
Hence the required circuit is;



b) Cauer-I form

$$s^2+4s+3\,\big)\,s^2+6s+8\,\big(1 \rightarrow Z_1$$
$$\underline{-s^2+4s+3}$$
$$2s+5\,\big)\,s^2+4s+3\,\big(\tfrac{1}{2}s \rightarrow Y_2$$
$$\underline{-s^2+\tfrac{5}{2}s}$$
$$\tfrac{3}{2}s+3\,\big)\,5s+5\,\big(\tfrac{4}{3}s \rightarrow Z_3$$
$$\underline{-2s+4}$$
$$1\,\big)\,\tfrac{3}{2}s+3\,\big(\tfrac{3}{2}s \rightarrow Y_4$$
$$\underline{-\tfrac{3}{2}s}$$
$$3\,\big)\,1\,\big(\tfrac{1}{3} \rightarrow Z_5$$
$$\underline{-1}$$
$$\times$$

Hence the required circuit is;



10. Synthesize the foster–I and cauer–II forms of RC driving function.

$$Z_{RC} = \frac{(s+1)(s+4)}{s(s+2)(s+5)}$$ **[2015/Spring]**

Solution:

Given that;

$$Z_{RC} = \frac{(s+1)(s+4)}{s(s+2)(s+5)}$$

a) Foster-I form

Using partial fraction method,

$$Z(s) = \frac{(s+1)(s+4)}{s(s+2)(s+5)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5}$$

Now,

$$A = s\cdot Z(s)\big|_{s=0} = \frac{s\times(s+1)(s+4)}{s(s+2)(s+5)}\bigg|_{s=0} = \frac{1\times 4}{2\times 5} = \frac{2}{5}$$

$$B = (s+2)\cdot Z(s)\big|_{s=-2} = \frac{(-1)(2)}{(-2)(3)} = \frac{1}{3}$$

$$C = (s+5)\cdot Z(s)\big|_{s=-5} = \frac{(-4)(-1)}{(-5)(-3)} = \frac{4}{15}$$

$$\therefore \quad Z(s) = \frac{\frac{2}{5}}{s} + \frac{\frac{1}{3}}{s+2} + \frac{\frac{4}{15}}{s+5}$$

Synthesized network:

First term, $\frac{1}{Cs} = \frac{\frac{2}{5}}{s}$

$\therefore \quad C = \frac{5}{2}F$

Also, 2nd term $= \frac{1}{C} = \frac{1}{3} \Rightarrow C' = 3F$

$$s + \frac{1}{RC} s + 2 \Rightarrow \frac{1}{RC} = 2$$

or, $RC = \frac{1}{2} \Rightarrow R = \frac{1}{2\times 3} = \frac{1}{6}\Omega$

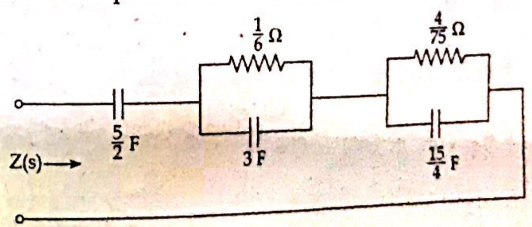
and, 3rd term

$$\frac{1}{C} = \frac{4}{15} \Rightarrow C = \frac{15}{4}F$$

$$s + \frac{1}{RC} = s+5 \Rightarrow \frac{1}{RC} = 5$$

or, $RC = \frac{1}{5} \Rightarrow R = \frac{1}{5\times\left(\frac{15}{4}\right)} = \frac{4}{75}\Omega$

Hence the required foster –I circuit is



b) Foster -II form

$$Y(s) = \frac{1}{Z(s)} = \frac{s(s+2)(s+5)}{(s+1)(s+4)} = \frac{s^3+7s^2+10s}{s^2+5s+4}$$

or, $\frac{Y(s)}{s} = \frac{s^3 + 7s + 10}{s^2 + 5s + 4} = 1 + \frac{2s + 6}{(s + 1)(s + 4)}$

so, $s^2 + 5s + 4 \overline{)\, s^2 + 7s + 10\,}\,(1$
$\quad \underline{-\,s^2 + 5s + 4}$
$\quad 2s + 6$

$$= 1 + \frac{2s + 6}{(s + 1)(s + 4)}$$

Using partial fraction expansion,

Let, $\frac{2s + 6}{(s + 1)(s + 4)} = \frac{A}{s + 1} + \frac{B}{s + 4}$

On solving,

$$A = \frac{2s + 6}{(s + 1)(s + 4)} \times (s + 1)\Big|_{s=-1} = \frac{4}{3}$$

$$B = \frac{2s + 6}{(s + 1)(s + 4)} \times (s + 4)\Big|_{s=-4} = \frac{2}{3}$$

Then, $\frac{Y(s)}{s} = 1 + \frac{\left(\frac{4}{3}\right)}{s + 1} + \frac{\left(\frac{2}{3}\right)}{s + 4}$

$\therefore \quad Y(s) = s + \frac{\left(\frac{1}{3}\right)s}{s + 1} + \frac{\left(\frac{2}{3}\right)s}{s + 4}$

Thus synthe sized network becomes,

Since, $Y(s) = \frac{\left(\frac{2}{3}\right)s}{s + \frac{1}{RC}}$

First term = 1

$\therefore \quad C = 1$

2nd term $= \frac{\left(\frac{4}{3}\right)s}{s + 1}$

$$\frac{1}{R}s = \frac{4}{3}s \text{ and, } \frac{1}{RC} = 1$$

$\therefore \quad R = \frac{3}{4}\,\Omega$ or, $\frac{1}{\left(\frac{3}{4}\right)C} = 1 \Rightarrow C = \frac{4}{3}\,F$.

3rd term $= \frac{\left(\frac{2}{3}\right)s}{s + 4}$

$$\frac{1}{R}s = \frac{2}{3}s \text{ and, } \frac{1}{RC} = 4$$

$\therefore \quad R = \frac{3}{2}\,\Omega$ or, $\frac{1}{C} = 4 \times \frac{3}{2} = 6$

$\therefore \quad C = \frac{1}{6}\,F$

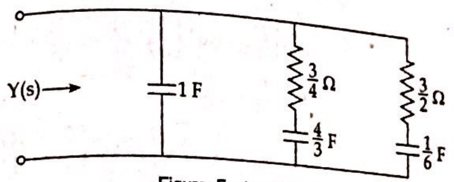


Figure: Foster -II circuit

c) **Cauer -I form**

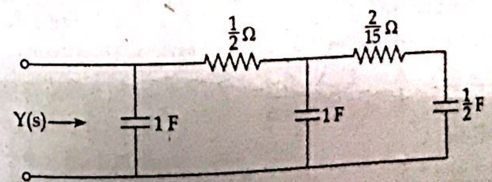
$$Z(s) = \frac{s^2 + 5s + 4}{s^3 + 7s + 10s}$$

or, $\quad Y(s) = \frac{s^2 + 7s^2 + 10s}{s^3 + 5s + 4}$

The continued fraction expansion is,

$s^2 + 5s + 4 \overline{)\, s^3 + 7s^2 + 10s\,}\,(s \leftrightarrow Y_1$
$\quad \underline{-\,s^3 + 5s^2 + 4s}$
$\quad 2s^2 + 6s \overline{)\, s^2 + 5s + 4\,}\,\left(\frac{1}{2} \leftrightarrow Z_2\right.$
$\quad\quad \underline{-\,s^2 + 3s}$
$\quad\quad 2s + 4 \overline{)\, 2s^2 + 6s\,}\,(s \leftrightarrow Y_3$
$\quad\quad\quad \underline{-\,2s^2 + 4s}$
$\quad\quad\quad 2s \overline{)\, 2s + 4\,}\,(1 \leftrightarrow Z_4$
$\quad\quad\quad\quad \underline{-\,2s}$
$\quad\quad\quad\quad 4 \overline{)\, 2s\,}\,\left(\frac{1}{2}s \leftrightarrow Y_5\right.$
$\quad\quad\quad\quad\quad \underline{-\,2s}$
$\quad\quad\quad\quad\quad \times$

The synthesized network is:



d) **Cauer-II form**

$$Z(s) = \frac{s^2 + 7s^2 + 10s}{s^3 + 7s^2 + 10s} = \frac{4 + 5s + s^2}{10s + 7s^2 + s^3}$$

The continued fraction expansion is,

$$10s+7s^2+s^3\overline{\Big)\,4+5s+s^2}\quad\left(\frac{4}{10s}=\frac{2}{5s}\leftrightarrow Z_1\right.$$

$$-11+\frac{14}{5}s+\frac{2}{5}s^2$$

$$\frac{3}{5}s^2\Big)\,10s+7s^2+s^3\quad\left(\frac{5}{11}\times 10=\frac{50}{11}\leftrightarrow Y_2\right.$$

$$-10s+\frac{30}{11}s^2$$

$$\frac{47}{11}s^2+s^3\Big)\,\frac{11}{5}s+\frac{3}{5}s^2\quad\left(\frac{11}{47}\cdot\frac{11}{5s}=\frac{121}{235s}\leftrightarrow Z_3\right.$$

$$-\frac{11}{5}s+\frac{121}{235}s^2$$

$$\frac{20}{235}s^2\Big)\,\frac{47}{11}s^2+s^3\quad\left(\frac{235}{20}\cdot\frac{47}{11}=\frac{2209}{44}\leftrightarrow Y_4\right.$$

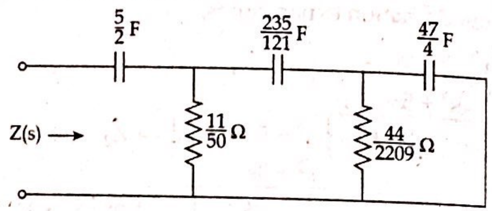
$$-\frac{47}{11}s^2$$

$$s^5\Big)\,\frac{20}{235}s^2\quad\left(\frac{20}{235s}=\frac{4}{47s}\leftrightarrow Z_5\right.$$

$$-\frac{20}{235}s^2$$

$$\times$$

The synthesized network is:



11. **Synthesize the RL network for the driving point impedance. Use Foster-I method.**

$$Z(s)=\frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

[2017/Spring]

Solution:

$$Z(s)=\frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

or, $$\frac{Z(s)}{s}=\frac{2(s+1)(s+3)}{s(s+2)(s+6)}$$

Using partial fraction expansion,

$$\frac{2(s+1)(s+3)}{s(s+2)(s+6)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s+6}$$

Solving for A, B and C,

$\therefore$ $$A=\frac{2(s+1)(s+3)}{s(s+2)(s+6)}\times(s+1)\Big|_{s=0}=\frac{2(s+1)(s+3)}{s(s+2)(s+6)}\Big|_{s=0}=\frac{2\times 3}{2\times 6}=\frac{1}{2}$$

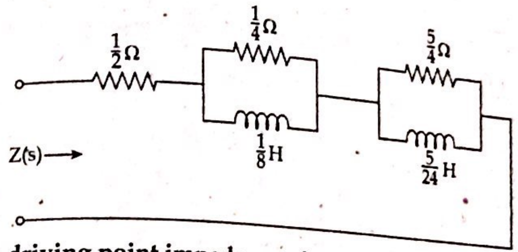
$\therefore$ $$B=\frac{2(s+1)(s+3)}{s(s+2)(s+6)}\times(s+6)\Big|_{s=-2}=\frac{2\times(-1)\times 1}{-2\times 4}=\frac{1}{4}$$

$\therefore$ $$C=\frac{2(s+1)(s+3)}{s(s+2)(s+6)}\times(s+6)\Big|_{s=-6}=\frac{2\times(-5)\times(-3)}{-6\times(-4)}=\frac{5}{4},$$

Hence, $$\frac{Z(s)}{s}=\frac{\left(\frac{1}{2}\right)}{s}+\frac{\left(\frac{1}{4}\right)}{s+2}+\frac{\left(\frac{5}{4}\right)}{s+6}$$

$\therefore$ $$Z(s)=\left(\frac{1}{2}\right)+\frac{\left(\frac{1}{4}\right)s}{s+2}+\frac{\left(\frac{5}{4}\right)s}{s+6}$$

Hence the synthesized network is:



12. **The driving point impedance of a LC network is given by,**

$$Z(s)=\frac{10(s^2+4)(s^2+10)}{s(s^2+9)}$$

Obtain foster I form realization.

[2018/Fall]

Solution:

Given that;

$$Z(s)=\frac{10(s^2+4)(s^2+10)}{s(s^2+9)}=\frac{10s^4+140s^2+400}{s^3+9s}$$

The continued fraction expansion is,

$$s^3+9s\Big)\,10s^4+140s^2+400\,\Big(10s$$

$$-10s^4+90s^2$$

$$50s^2+400$$

Thus, $$Z(s)=10s+\frac{50s^2+400}{s(s^2+9)}$$

Using partial fraction expansion, we have,

$$\frac{50s^2+400}{s(s^2+9)}=\frac{A}{s}+\frac{Bs+C}{s^2+9}$$

or, $50s^2+400=As^2+9A+Bs^2+Cs$

or, $50s^2+400=s^2(A+B)+9A+Cs$

Comparing coefficients,

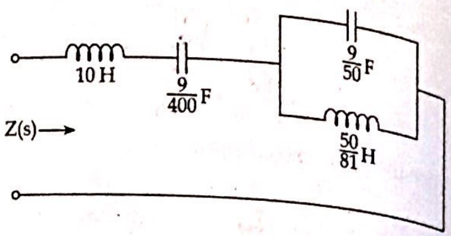
$A+B=50$ and, $9A=400$

$\therefore$ $$A=\frac{400}{9}$$

and, $$B=\frac{50}{9}$$

Hence, $Z(s) = 10s + \dfrac{\left(\frac{400}{9}\right)}{s} + \dfrac{\left(\frac{400}{9}\right)s}{s}$

The synthesized LC network becomes,



First term = 10s

or, $Ls = 10s$

$\therefore$ $L = 10\ H$

2^{nd} term $= \dfrac{\left(\frac{400}{9}\right)}{s}$

or, $\dfrac{1}{Cs} = \dfrac{\left(\frac{400}{9}\right)}{s}$

$\therefore$ $C = \dfrac{9}{400}F$

Last term $= \dfrac{\left(\frac{50}{9}\right)s}{s^2+9}$ or, $\dfrac{\frac{s}{C}}{s^2+\frac{1}{LC}} = \dfrac{\left(\frac{50}{9}\right)s}{s^2+9}$

$\therefore$ $C = \dfrac{9}{400}F$

and, $\dfrac{1}{LC} = 9 \Rightarrow L = \dfrac{1}{9\times\left(\frac{9}{50}\right)} = \dfrac{50}{81}H$

13. **Synthesize the given LC network function in Foster-I and cauer-I forms.** $Z(s) = \dfrac{(s^2+4)(s^2+25)}{s(s^2+16)}$ **[2014/Fall, 2018/Spring, 2019/Spring]**

Solution:

Given that;

$$Z(s) = \frac{(s^2+4)(s^2+25)}{s(s^2+16)}$$

a) **Foster -I form**

$$Z(s) = \frac{s^4+4s^2+100+25s^2}{s^3+16s} = \frac{s^4+29s^2+100}{s^3+16s}$$

or,
$$\begin{array}{r|l} s^3+16s & s^4+29s^2+100 \;(\, s \\ & \underline{-s^4+16s^2} \\ & \quad 13s^2+100 \end{array}$$

so, $Z(s) = s + \dfrac{13s^2+100}{s^3+16s} = s + \dfrac{13s^2+100}{s(s^2+16)}$

Let, $\dfrac{13s^2+100}{s(s^2+16)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+16}$

or, $13s^2+100 = A(s^2+16) + Bs^2 + Cs$

Put s = 0,

or, $100 = 16\ A$

$\therefore$ $A = \dfrac{25}{4}$

Comparing coefficients,

$Cs = 0 \Rightarrow C = 0$

and, $Bs^2 = 13s^2 \Rightarrow B = 13$

Then, $Z(s) = s + \dfrac{\left(\frac{25}{4}\right)}{s} + \dfrac{13s}{s^2+16}$

Synthesized network:

First term = s

or, $Ls = s$

$\therefore$ $L = 1\ H$

2^{nd} term $= \dfrac{\left(\frac{25}{4}\right)}{s}$

or, $\dfrac{1}{Cs} = \dfrac{\left(\frac{25}{4}\right)}{s}$

$\therefore$ $C = \dfrac{4}{25}F$

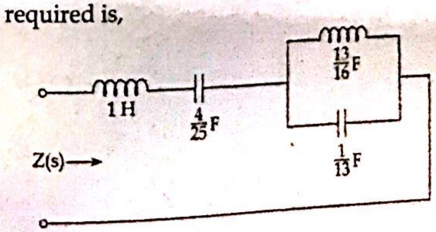
3^{rd} term $= \dfrac{13s}{s^2+16}$

or, $\dfrac{\frac{s}{C}}{s^2+\frac{1}{LC}} = \dfrac{13s}{s^2+16}$

$\therefore$ $C = \dfrac{1}{13}F$ and, $\dfrac{1}{L\times\left(\frac{1}{13}\right)} = 16$

$\therefore$ $L = \dfrac{13}{16}H$

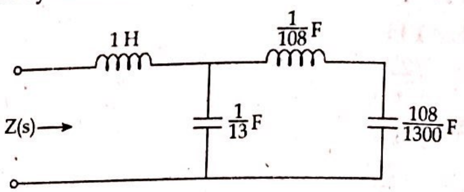
Thus, required is,



b) Cauer-I form

$$s^3+16s \overline{\big)\, s^4+29s^2+100} \;\left(s \leftrightarrow Z_1\right.$$
$$\underline{-s^4+16s^2}$$
$$13s^2+100 \overline{\big)\, s^3+16s} \;\left(\frac{s}{13} \rightarrow Y_2\right.$$
$$\underline{-s^3+\frac{100}{13}s}$$
$$\frac{108}{13}s \overline{\big)\, 13s^2+100} \;\left(s \rightarrow Z_3\right.$$
$$\underline{-13s^2+108}$$
$$100 \overline{\big)\, \frac{108}{13}s} \;\left(\frac{108}{13}s \times \frac{1}{100} \rightarrow Y_4\right.$$
$$\underline{-\frac{108}{13}s}$$
$$\times$$

Hence the synthesized network circuit becomes,



14. The driving point impedance of a LC network is given by,

$$Z(s)=\frac{(s^2+1)(s^2+8)}{s(s^2+4)}$$ **. Obtain cauer-II form.** [2017/Fall]

Solution:

Given that;

$$Z(s)=\frac{(s^2+1)(s^2+8)}{s(s^2+4)}=\frac{s^4+8s^2+8+s^2}{s^3+4s}=\frac{s^4+9s^2+8}{s^3+4s}$$

$$\therefore \quad Y(s)=\frac{1}{Z(s)}=\frac{4s+s^3}{8+9s^2+s^4}=\frac{s^4+4s}{s^4+9s^2+8}$$

Continued fraction expansion is,

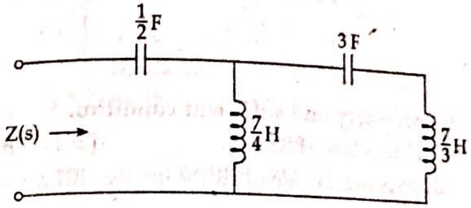
$$4s+s^3 \overline{\big)\, 8+9s^2+s^4} \;\left(\frac{2}{s} \rightarrow Y_1\right.$$
$$\underline{-8+2s^2}$$
$$7s^2+s^4 \overline{\big)\, 4s+s^3} \;\left(\frac{1}{7}\cdot\frac{4}{s}=\frac{4}{7s} \rightarrow Z_2\right.$$
$$\underline{-4s+\frac{4}{7}s^3}$$
$$\frac{3}{7}s^3 \overline{\big)\, 7s^2+s^4} \;\left(\frac{1}{3s} \rightarrow Y_3\right.$$
$$\underline{-7s^2}$$
$$s^4 \overline{\big)\, \frac{3}{7}s} \;\left(\frac{3}{7s} \rightarrow Z_4\right.$$
$$\underline{-\frac{3}{7}s}$$
$$\times$$

Thus, $$Z(s)=\left(\frac{2}{s}\right)+\cfrac{1}{\left(\frac{4}{7s}\right)+\cfrac{1}{\left(\frac{1}{3s}\right)+\cfrac{1}{\left(\frac{3}{7s}\right)}}}$$

Comparing,

$$C_1=\frac{1}{2}F,\ L_2=\frac{7}{4}F,\ C_3=3F,\ L_4=\frac{7}{3}H$$

The second cauer network is shown below,



15. Synthesize the network Z(s) for the driving point impedance Z(s) using Foster-II form. [2019/Fall]

$$Z(s)=\frac{s^2+5s+4}{s(s+2)}$$

Solution:

$$Z(s)=\frac{s^2+5s+4}{s(s+2)}$$

or, $$Y(s)=\frac{1}{Z(s)}=\frac{s(s+2)}{s^2+5s+4}=\frac{s(s+2)}{(s+1)(s+4)}$$

Using partial fraction expansion,

$$\frac{Y(s)}{s}=\frac{(s+2)}{(s+1)(s+4)}=\frac{A}{(s+1)}+\frac{B}{(s+4)}$$

or, $(s+2)=A(s+4)+B(s+1)$

Put $s=-4$,

$$(-4+2)=0+B(-4+1)$$

$$\therefore \quad B=\frac{2}{3}$$

Put $s=-1$,

$$-1+2=A(-1+4)+0$$

$$\therefore \quad A=\frac{1}{3}$$

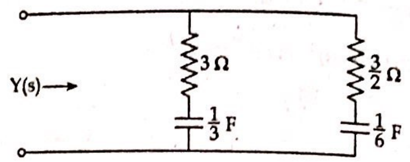
Hence, $$\frac{Y(s)}{s}=\frac{1}{3(s+1)}+\frac{2}{3(s+4)}$$

$$\therefore \quad Y(s) = \frac{\left(\frac{1}{3}\right)s}{(s+1)} + \frac{\left(\frac{2}{3}\right)s}{(s+4)}$$

Thus, $R_1 = 3\,\Omega$ and, $C_1 = \left(\frac{1}{3}\right)$ F

$R_2 = \frac{3}{2}$ and, $C_2 = \frac{1}{6}$ F

The foster-II network is shown below;



16. **Write the necessary and sufficient condition for a function to be positive real function (PRF).** **[2011/Spring, 2012/Fall, 2015/Spring, 2018/Fall, 2018/Spring, 2019/Spring, 2020/Fall]**

Solution: See the topic 8.1.2.2.

17. **Write short notes on properties of positive real function.** **[2015/Fall]**

Solution: See the topic 8.1.3.1.

18. **Write short notes on properties of RC impedance function.** **[2012/Spring, 2013/Spring]**

Solution: See the topic 8.2.2.

19. **Write short notes on positive real function.** **[2012/Fall, 2013/Fall]**

Solution: See the topic 8.1.3.

20. **Write short notes on RL and RC network.** **[2011/Fall]**

Solution: See the topic 8.2.2 and 8.2.3.

21. **Check the given function for PRF.** **[2020/Fall]**

$$H(s) = \frac{s^3+5s^2+9s+3}{s^3+4s^2+7s+9}$$

Solution:

Given that;

$$H(s) = \frac{s^3+5s^2+9s+3}{s^3+4s^2+7s+9}$$

$$= \frac{s^3+5s^2+9s+3}{(s+2\,2.6398)(s+0.680+1.716i)(s+0.680-1.716i)}$$

i) Poles are at $s = -2.6398$, $s = -0.680 - 1.716i$ and $s = -0.680 + 1.716i$ *i.e.*, no poles on right half s-plane Hence, it is Hurwitz polynomial,

ii) Here, all the coefficients in the numerator and denominator polynomial are positive hence H(s) is real if s is real.

iii) To find Re $[H(j\omega)] \geq 0$,

Let, $M_1(s) = 5s^2 + 3$

$M_2(s) = 4s^2 + 9$

$N_1(s) = s^3 + 9s$

$N_2(s) = s^3 + 7s$

Hence, $(s) = M_1(s)\,M_2(s) - N_1(s)\,N_2(s)$

$= (5s^2+3)(4s^2+9) - (s^3+9s)(s^3+7s)$

$= 20s^4 + 45s^2 + 12s^2 + 27 - s^6 - 7s^4 - 9s^4 - 63s^2$

$= -s^6 + 4s^4 - 6s^2 + 27$

$\therefore \quad (j\omega) = \omega^6 + 4\omega^4 + 6\omega^2 + 27 \geq 0$ for any value of ω

Hence, Re $[H(j\omega)] \geq$ for any value of ω.

Thus the given function H(s) is PRF.

22. **Synthesize the given transfer function using foster-I and cauer-I method.** **[2020/Fall]**

$$Y(s) = \frac{s(s+2)(s+4)}{(s+1)(s+3)}$$

Solution:

Given that;

$$y(s) = \frac{s(s+2)(s+4)}{(s+1)(s+3)}$$

or, $$Z(s) = \frac{1}{y(s)} = \frac{(s+1)(s+3)}{s(s+2)(s+4)}$$

a) **Foster -I form**

Using partial fraction expansion,

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

Now, $A = s \cdot Z(s)|_{s=0} = \frac{1\times 3}{2\times 4} = \frac{3}{8}$

$B = (s+2)\,Z(s)|_{s=-2} = \frac{(-1)(1)}{(-2)(2)} = \frac{1}{4}$

$C = (s+5)\,Z(s)|_{s=-4} = \frac{(-3)(-1)}{(-4)(-2)} = \frac{3}{8}$

Thus, $$Z(s) = \frac{\left(\frac{3}{8}\right)}{s} + \frac{\left(\frac{1}{4}\right)}{s+2} + \frac{\left(\frac{3}{8}\right)}{s+4}$$

First term $= \frac{\left(\frac{3}{8}\right)}{s}$

or, $\frac{1}{Cs} = \frac{\left(\frac{3}{8}\right)}{s}$

$\therefore \quad C = \frac{8}{3} F$

$$\text{2nd term} = \frac{\left(\frac{1}{4}\right)}{s+2}$$

$\therefore \quad C = 4\,F$ and, $RC = \frac{1}{2}$ $\qquad \left[\because R = \frac{1}{8}\Omega\right]$

$$\text{3rd term} = \frac{\left(\frac{3}{8}\right)}{(s+4)}$$

or, $\quad \dfrac{\left(\frac{1}{C}\right)}{s + \frac{1}{RC}} = \dfrac{\left(\frac{3}{8}\right)}{s+4}$

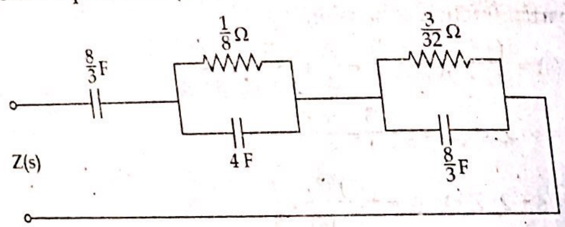
$\therefore \quad C = \frac{8}{3} F$

and, $\quad \frac{1}{RC} = 4$

or, $\quad \frac{1}{R} = 4 \times \frac{8}{3}$

$\therefore \quad R = \frac{3}{32}\Omega$

Thus the foster-I representation is,



b) **Cauer-I method**

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)(s+4)} = \frac{s^2+4s+3}{s(s^2+6s+8)} = \frac{s^2+4s+3}{s(s^3+6s^2+8s)}$$

$$\therefore \quad y(s) = \frac{1}{Z(s)} = \frac{s^3+6s^2+8s}{(s^2+4s+3)}$$

Proceed as solved numerical examples number 12 (b).

☆☆☆

Chapter 9

TWO PORT PASSIVE NETWORK

9.1 DEFINITION OF TWO-PORT NETWORK

If the current entering one terminal of a pair is equal and opposite to the current leaving the other terminal of the pair, then this type of terminal pair is called as a "port". A two-port network is shown in figure 9.1. The two-port network is represented by a black box with four variables, namely, two voltages (V_1, V_2) and two currents (I_1, I_2) which are available for measurements and are relevant for the analysis of two port networks. Out of these four variables, which two variables may be considered 'independent' and which two 'dependent' is generally decided by the the problem under consideration.

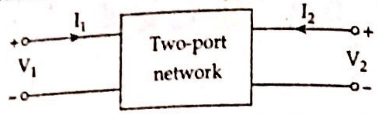


Figure 9.1: A two-port network with standard reference directions for the voltages and currents indicated.

By analogy with transmission networks, one of the ports is called the input port, while the other is termed as the output port. Some different forms of two-ports networks are shown in figure 9.2.

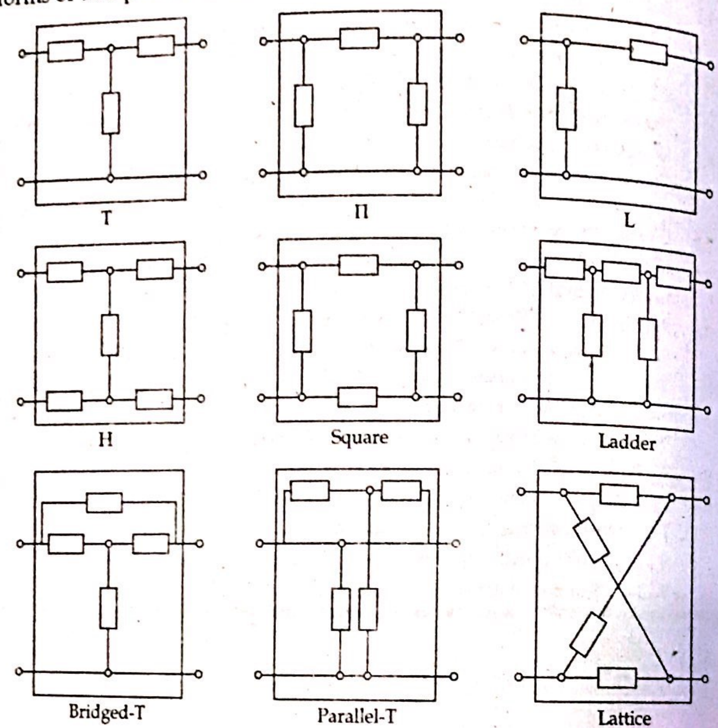


Figure 9.2: Different forms of two-port network

9.1.1 Characterization of Linear Time-Invariant (LTI) Two Port Networks

A two port network is a special case of multi-port network. Each port consists of two terminals, one for entering the current and the other for leaving. In order to describe the relationships between the port voltages and port current of a linear two port network, two linear equations are required among the four variables.

9.1.2 Relationships of Two Port Variable

The number of possible combinations generated by the four variable, taken two at a time, is six, Thus, there are total six combinations given in table, by which we analyse any type of the two port network.

Table: Two-port parameters

Name	Function		Matrix equation
	Express	In terms of	
Open-circuit impedance [Z]	V_1, I_1	I_1, I_2	$\begin{bmatrix}V_1\\V_2\end{bmatrix}=\begin{bmatrix}Z_{11} & Z_{12}\\Z_{21} & Z_{22}\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}$
Short-circuit admittance [Y]	I_1, I_2	V_1, V_2	$\begin{bmatrix}I_1\\I_2\end{bmatrix}=\begin{bmatrix}Y_{11} & Y_{12}\\Y_{21} & Y_{22}\end{bmatrix}\begin{bmatrix}V_1\\V_2\end{bmatrix}$
Transmission or chain [T]	V_1, I_1	$V_2, -I_2$	$\begin{bmatrix}V_1\\V_2\end{bmatrix}=\begin{bmatrix}A & B\\C & D\end{bmatrix}\begin{bmatrix}V_1\\-I_2\end{bmatrix}$
Inverse transmission [T']	V_2, I_2	$V_1, -I_1$	$\begin{bmatrix}V_2\\I_2\end{bmatrix}=\begin{bmatrix}A' & B'\\C' & D'\end{bmatrix}\begin{bmatrix}V_1\\-I_1\end{bmatrix}$
Hybrid (h)	V_1, I_2	I_1, V_2	$\begin{bmatrix}V_1\\I_2\end{bmatrix}=\begin{bmatrix}h_{11} & h_{12}\\h_{21} & h_{22}\end{bmatrix}\begin{bmatrix}I_1\\V_2\end{bmatrix}$
Inverse hybrid (g)	I_1, V_2	V_1, I_2	$\begin{bmatrix}I_1\\V_2\end{bmatrix}=\begin{bmatrix}g_{11} & g_{12}\\g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}V_1\\I_2\end{bmatrix}$

9.2 OPEN-CIRCUIT IMPEDANCE (Z) PARAMETERS

Expressing two-port voltages in terms of two-port currents *i.e.*,

$$(V_1, V_2) = f(I_1, I_2)$$

$$\begin{bmatrix}V_1\\V_2\end{bmatrix}=\begin{bmatrix}Z_{11} & Z_{12}\\Z_{21} & Z_{22}\end{bmatrix}\begin{bmatrix}I_1\\I_2\end{bmatrix}$$

or, $[V] = [Z]\,[I]$

where, [Z] is the open circuit impedance matrix of the two port network and impedances Z_{ij} are the open circuit impedance (Z) parameters.

$$V_1 = Z_{11}I_1 + Z_{12}I_2 \quad \text{(a)}$$

$$V_5 = Z_{21}I_1 + Z_{22}I_2 \quad \text{(b)}$$

From equations (a) and (b), we draw the two voltage source equivalent of two-port network may be obtained in terms of Z-parameters as shown in figure 9.3, where $Z_{12}I_2$ and $Z_{21}I_1$ are current-controlled voltage sources (CCVS).

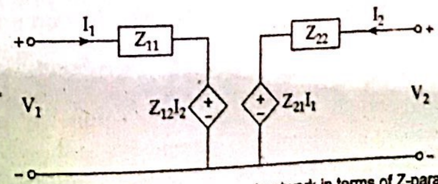


Figure 9.3: Equivalent circuit of a two-port network in terms of Z-parameters.

Determination of Z-parameters

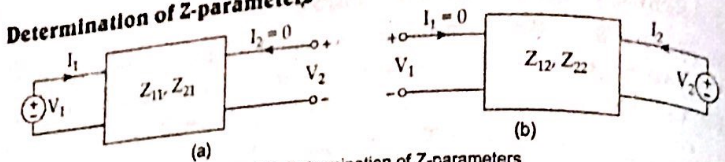


Figure 9.4: Determination of Z-parameters

In order to determine the Z-parameters, open the output port and applied some voltage V_1 to input port as shown in figure 9.4 (b). We determine I_1 and V_2 to obtain Z_{11} and Z_{21}. Then, the input port is open circuited and the output port is excited with the voltage V_2 as shown in figure 9.4 (a). The circuit is analysed to determine I_2 and V_2, so as to obtain Z_{12} and Z_{22}.

Mathematically,

Case I: $V_1 = V_1$, $I_1 = ?$, $V_2 = ?$, $I_2 = 0$ [Output port open circuited as shown in figure 9.4 (a)].

$$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}$$ [Input driving point impedance with the output port open circuited]

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}$$ [Forward transfer impedance with the output port open circuited]

Case II: $V_2 = V_2$, $I_2 = ?$, $V_1 = ?$, $I_1 = 0$

[Input port open circuited as shown in figure 9.4 (b)]

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$

[Reverse transfer impedance with the input port open circuited]

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0}$$

[Output driving point impedance with the input port open circuited]

9.3 SHORT CIRCUIT ADMITTANCE PARAMETERS

Expressing two-port current in terms of two-port voltages *i.e.,*

$$(I_1, I_2) = f(V_1, V_2)$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

or, $[I] = [Y]\,[V]$

where, [Y] is the short circuit admittance matrix of the two port network and admittance Y_{ij} are the short circuit admittance (Y)-parameters.

$$I_1 = Y_{11}V_1 + Y_{12}V_2 \quad (1)$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2 \quad (2)$$

From equations (1) and (2), two current source equivalent of a two-port network may be obtained in terms of Y-parameters as shown in figure 9.5; where $Y_{12}V_2$ and $Y_{21}V_1$ are voltage controlled current source (VCCS).

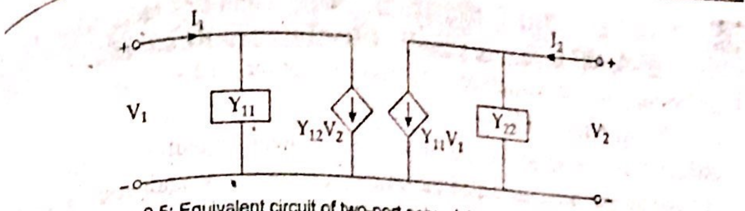


Figure 9.5: Equivalent circuit of two-port network in terms of Y-parameters.

Determination of Y-Parameters

In order to determine the Y-parameters, short the output port and apply some voltage V_1 to input port as shown in figure 9.6 (a). We determine I_1 and I_2 to obtain Y_{11} and Y_{21}. Then, the input port is short circuited and the output port is excited with the voltage V_2 as shown in figure 9.6 (b). The circuit is analysed to determine I_2 and I_2, so as to obtain Y_{12} and Y_{22}.

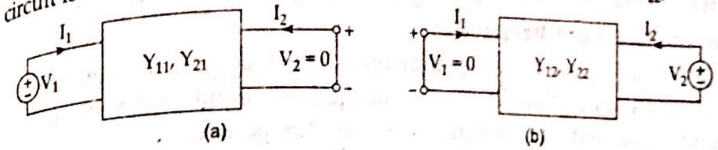


Figure 9.6: Determination of Y-parameters

Case I: $V_1 = V_1$, $I_1 = ?$, $I_2 = ?$, $V_2 = 0$

[Output port short circuited as shown in figure 9.6 (a)]

$$Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$$

[Input driving point admittance with the output port short circuited]

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$$

[Forward transfer admittance with the output port short circuit]

Case II: $V_2 = V_2$, $I_2 = ?$, $I_1 = ?$, $V_1 = 0$

[In put port short circuit as shown in figure 9.6 (b)]

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$$

[Reverse transfer admittance with the input port short circuited]

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$$

[Output driving point admittance with the input port short circuited]

NOTE:

a) For two port passive network, Y_{12} and Y_{21} are always negative, since I_2 is negative, when output port is short circuited and similarly I_1 is negative, when input port is short circuited.

b) [Short circuit admittance matrix] = [Open circuit impedance matrix]$^{-1}$

or, $[Y] = [Z]^{-1}$

and, $Y_{ij} \neq \frac{1}{Z_{ij}}$ i.e., $Y_{11} \neq \frac{1}{Z_{11}}$ and $Y_{12} \neq \frac{1}{Z_{12}}$ etc

9.4 TRANSMISSION SHORT CIRCUIT ADMITTANCE PARAMETERS OR CHAIN OR ABCD PARAMETERS

Expressing one port variables in terms of the other port variable.

i.e., $(V_1, I_1) = f(V_2, -I_2)$

T-parameters are used in the analysis of power transmission line. The input and output ports are called the sending and receiving ends respectively.

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_1 \\ -I_2 \end{bmatrix}$$

This matrix equation defines A, B, C, D parameters, where matrix is known as the transmission (T) or ABCD matrix.

$$V_1 = AV_2 + B(-I_2) \quad (1)$$

$$I_1 = CV_2 + D(-I_2) \quad (2)$$

NOTE: The equivalent circuit of a two-port network is not possible in terms of T-parameters.

Determination of T-Parameters

In order to determine the T-parameters, open and short the output port (receiving end) and applied some voltage V_1 to input port as shown in figures 9.7 (a) and (b) to obtain A, C and B, D respectively.

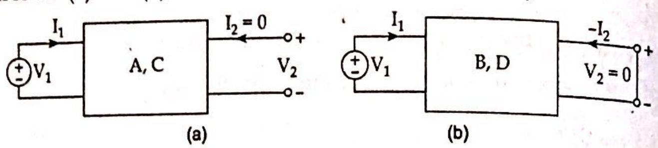


Figure 9.7: Determination of T-parameters

Case I: $V_1 = V_1$, $I_1 = ?$, $V_2 = ?$, $I_2 = 0$

[Output port open circuited as shown in figure 7.7 (a)]

$$A = \left.\frac{V_1}{V_2}\right|_{V_2=0}$$

[Reverse voltage ratio with the receiving end open circuited]

$$C = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

[Reverse transfer admittance with the receiving end open circuited]

Case II: V_1, $I_1 = ?$, $I_2 = ?$, $V_2 = 0$

[Output port short circuited as shown in figure 9.7 (b)]

$$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0}$$

[Reverse transfer impedance with the receiving end short circuited]

$$D = \left.\frac{I_1}{-I_2}\right|_{V_2=0}$$

[Reverse current ratio with the receiving end short circuited]

NOTE: For passive network the all four T-parameters are positive, since I_2 is itself negative or $-I_2$ is positive.

9.5 INVERSE TRANSMISSION (T') PARAMETERS

Expressing output port variables in terms of input port variables *i.e.*,

$(V_2, I_2) = f(V_1 - I_1)$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_2 \end{bmatrix}$$

where, matrix consisting of A', B', C', D is called as inverse transmission matrix.

$$V_2 = A'V_1 + B'(-I_1) \quad (1)$$

$$I_2 = C'V_1 + D'(-I_1) \quad (2)$$

Determination of T'-parameters

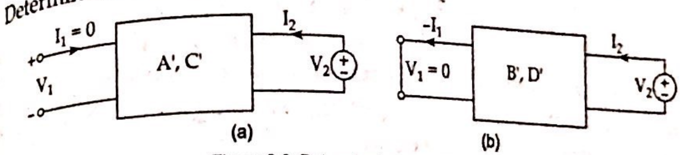


Figure 9.8: Determination of T'-parameters

Case I: $V_2 = V_2$; $I_2 = ?$, $V_1 = ?$, $I_1 = 0$

[Input port is open circuited as shown in figure 7.8 (a)]

$$A' = \left.\frac{V_2}{-V_1}\right|_{I_1=0}$$

[Forward voltage ratio with sending end open circuited]

$$C' = \left.\frac{I_2}{-V_1}\right|_{I_1=0}$$

[Transfer admittance with sending end open circuited]

Case II: $V_2 = V_2$, $I_1 = ?$, $I_2 = ?$, $V_1 = 0$

[Input port is short circuited as shown in figure 9.8 (b)]

$$B' = \left.\frac{V_2}{-I_1}\right|_{V_1=0}$$

[Transfer impedance with sending end short circuited]

$$D = \left.\frac{I_2}{-I_1}\right|_{V_1=0}$$

[Forward current ratio with sending end short circuited]

NOTE: For passive network, in this case also all for T' parameters are positive, as I_1 is itself negative or $-I_1$ is positive.

9.6 HYBRID (H) PARAMETERS

The hybrid parameters are wide usage in electronic circuits, especially in constructing models for transistors. In this case, voltage of the input port and the current of the output port are expressed in terms of the current of the input port and the voltage of the output part. Due to this reason, these parameters are called as "hybrid" parameters, *i.e.*,

$$(V_1, I_2) = f(I_1, V_2)$$

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

The matrix equation defines h-parameters, and h-parameters matrix is known as hybrid matrix.

$$V_1 = h_{11}I_1 + h_{12}V_2 \quad (1)$$
$$I_2 = h_{21}I_1 + h_{22}V_2 \quad (2)$$

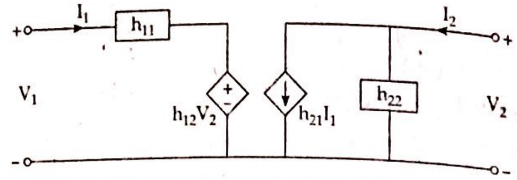


Figure 9.9: Equivalent circuit of a two-port network in terms of h-parameters.

From equations (1) and (2), we draw the one voltage and one current source equivalent of a two port network may be obtained in terms of h-parameters as shown in figure 9.9 where $h_{12}V_2$ and $h_{21}I_1$ are voltage controlled voltage and current controlled current sources respectively.

Determination of h-Parameters

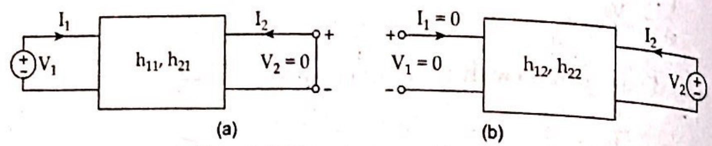


Figure 9.10: Determination of h-parameters

Case I: $V_1 = V_1, I_1 = ?, I_2 = ?, V_2 = 0$

[Output port short circuited as shown in figure 9.10 (a)]

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}$$

[Input impedance with the output port short circuited]

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

[Forward current gain with the output port short circuited]

Case II: $V_2 = V_2, I_2 = ?, V_1 = ?, I_1 = 0$

[Input port open circuited as shown in figure 9.10 (b)]

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{V_1=0}$$

[Reverse voltage gain with the input port open circuited].

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

[Output admittance with the input port open circuited]

NOTE: For passive network, h_{21} is always negative, since I_2 is negative in this case.

9.7 INVERSE HYBRID (g) PARAMETERS

The hybrid parameter and inverse hybrid parameter are duel of each other like Z and Y parameters,

$$[g] = [h]^{-1}$$

i.e.,
$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

This matrix equation defines g-parameters and g-parameter matrix is known as inverse hybrid matrix.

$$I_1 = g_{11}V_2 + g_{12}I_2 \quad (1)$$
$$V_2 = g_2V_1 + g_{22}I_2 \quad (2)$$

From equations (1) and (2), we can draw the equivalent circuit of a two-port network as shown in figure 9.11.

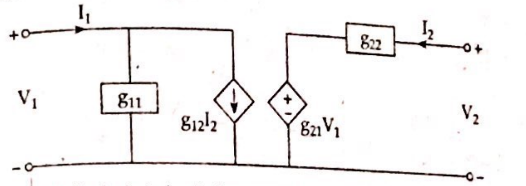


Figure 9.11: Equivalent circuit of a two port network in terms of g-parameters

Determination of g-parameters

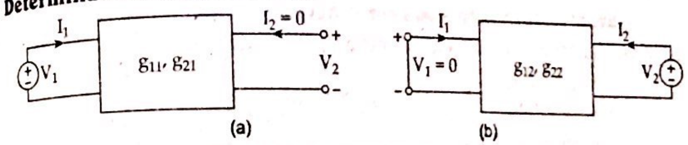


Figure 9.12: Determination of g-parameters

Case I: $V_1 = V_1, I_1 = ?, V_2 = ?, I_2 = 0$

[Output port open circuited as shown in figure 9.12 (a)]

$$g_{11} = \left.\frac{I_2}{V_1}\right|_{I_2=0}$$

[Input admittance with output port open circuited]

$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0}$$

[Forward voltage gain with output port open circuited]

Case II: $V_2 = V_2, I_2 = ?, V_1 = 0$

[Input port short circuited as shown in figure 9.12 (b)]

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{V_1=0}$$

[Reverse current gain with input port short circuited]

$$g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0}$$

[Output impedance with input port short circuited]

NOTE: For passive network g_{12} is always negative since I_1 is negative in this case.

9.8 RELATIONSHIP AND TRANSFORMATIONS BETWEEN SETS OF PARAMETERS

It is a simple matter to find the relationships of the sets of parameters as follows: "If we want to express α-parameters in terms of β-parameters, we have to write β parameter equations and then, by algebraic manipulation, rewrite the equations as needed for α-parameters.

NOTE: The equivalences involve a factor

$$\Delta x = x_{11}x_{22} - x_{12}x_{21}$$

where, x is either Z, Y, T, T', h or g

9.8.1 Z-parameters in Terms of Other Parameters

a) Z-parameters in terms Y-parameters

We know,

$$[I] = [Y][V]$$

and, $[V] = [Z][I]$

Hence, $[Z] = [Y]^{-1}$

or, $$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta Y}\begin{bmatrix} Y_{22} & -Y_{12} \\ -Y_{21} & Y_{11} \end{bmatrix}$$

or, $$Z_{11} = \frac{Y_{22}}{\Delta Y}; Z_{12} = \frac{-Y_{12}}{\Delta Y}; Z_{21} = \frac{-Y_{21}}{\Delta Y}; Z_{22} = \frac{Y_{11}}{\Delta Y}$$

b) Z-parameters in terms of T-parameters

The transmission parameter equations are,

$$V_1 = AV_2 + B(-I_2) \quad (1)$$

$$I_1 = CV_2 + D(-I_2) \quad (2)$$

Rewriting equation (2), we get,

$$V_2 = \frac{1}{C} I_1 + \frac{D}{C} I_2$$

Comparing with equation,

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

∴ $$Z_{21} = \frac{1}{C} \text{ and } Z_{22} = \frac{D}{C}$$

Also, $V_1 = AV_2 + B(-I_2)$

$$V_1 = A\left[\frac{1}{C}\cdot I_1 + \frac{D}{C}\cdot I_2\right] + B(-I_2)$$

$$= \frac{A}{C}\cdot I_1 + \left(\frac{AD}{C} - B\right)\cdot I_2$$

Comparing with equation $V_1 = Z_{11}I_1 + Z_{12}I_2$

∴ $$Z_{11} = \frac{A}{C}; \quad Z_{12} = \frac{AD - BC}{C} \text{ or } Z_{12} = \frac{\Delta T}{C}$$

c) Z-parameters in terms of T' Parameters

As similar to above case:

$$Z_{11} = \frac{D'}{C'}, Z_{12} = \frac{1}{C'}, Z_{21} = \frac{\Delta T'}{C'} \text{ and } Z_{12} = \frac{A'}{C'}$$

d) Z-parameters in terms of h-parameters

The h-parameter equations are,

$$V_1 = h_{11}I_1 + h_{12}V_2 \quad (1)$$

$$I_1 = h_{21}I_1 + h_{22}V_2 \quad (2)$$

Rewriting equation (2), *i.e.*,

$$V_2 = -\frac{h_{21}}{h_{11}} I_1 + \frac{1}{h_{22}} I_2$$

Comparing with equation

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

∴ $$Z_{21} = \frac{-h_{21}}{h_{22}} \text{ and } Z_{22} = \frac{1}{h_{22}}$$

Also, $$V_1 = h_{11}I_1 + h_{12}\left[\frac{-h_{21}}{h_{22}} I_1 + \frac{1}{h_{22}} I_2\right]$$

$$= \left[h_{11} - \frac{h_{21}h_{12}}{h_{22}}\right] I_1 + \frac{h_{12}}{h_{22}} I_2$$

∴ $$Z_{11} = \frac{\Delta h}{h_{12}} \text{ and } Z_{12} = \frac{h_{12}}{h_{22}}$$

e) Z-parameters in terms of g parameters

As similar to above case,

$$Z_{11} = \frac{1}{g_{11}}, Z_{12} = \frac{-g_{12}}{g_{11}}, Z_{21} = \frac{g_{21}}{g_{11}} \text{ and } Z_{22} = \frac{\Delta g}{g_{11}}$$

9.8.2 Y-parameters in Terms of Other Parameters

a) Y-parameters in terms of Z-parameters

As we know, $[Y] = [Z]^{-1}$

Hence, $$Y_{11} = \frac{Z_{22}}{\Delta Z}, Y_{12} = \frac{-Z_{12}}{\Delta Z}, Y_{21} = \frac{-Z_{21}}{\Delta Z} \text{ and } Y_{22} = \frac{Z_{11}}{\Delta Z}$$

b) Y-parameters in terms of T-parameters

The transmission parameter equations are,

$$V_1 = AV_2 + B(-I_2)$$

$$I_1 = CV_2 + D(-I_2)$$

or, $$I_2 = \frac{-1}{B} V_1 + \frac{A}{B} V_2$$

Comparing with $I_2 = Y_{21}V_1 + Y_{22}V_2$

$$Y_{21} = \frac{-1}{B} \text{ and } Y_{22} = \frac{A}{B}$$

Also, $I_1 = CV_2 + D(-I_2)$

$$I_1 = CV_2 + D\left[\frac{1}{B}V_1 - \frac{A}{B}V_2\right] = \frac{D}{B}V_1 + \left(C - \frac{AD}{B}\right)V_2$$

Comparing with equation,

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$\therefore \quad Y_{11} = \frac{D}{B} \text{ and } Y_{12} = \frac{-\Delta T}{B}$$

c) Y-parameter in terms of T' parameters

As similar to above case,

$$Y_{11} = \frac{A'}{B'},\ Y_{12} = \frac{-1}{B'},\ Y_{21} = \frac{-\Delta T'}{B'} \text{ and } Y_{22} = \frac{D'}{B'}$$

d) Y-parameters in terms of h-parameters

The h-parameter equations are,

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

$$\text{or,} \quad I_1 = \frac{1}{h_{11}}V_1 - \frac{h_{12}}{h_{11}}V_2$$

Comparing with equation,

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$

$$Y_{11} = \frac{1}{h_{11}} \text{ and } Y_{12} = \frac{-h_{12}}{h_{11}}$$

Again, we have,

$$I_2 = h_{21}I_1 + h_{22}V_2 = h_{21}\left[\frac{1}{h_{11}}V_1 \ \frac{-h_{12}}{h_{11}}V_2\right] + h_{22}V_2 = \frac{h_{22}}{h_{11}}V_1 + \frac{\Delta h}{h_{11}}V_2$$

Comparing with equation,

$$I_2 = Y_{21}V_1 + Y_{22}V_2$$

$$\therefore \quad Y_{21} = \frac{h_{21}}{h_{11}} \text{ and } Y_{22} = \frac{\Delta h}{h_{11}}$$

e) Y-parameters in terms of g-parameters

As similar to above case,

$$Y_{11} = \frac{\Delta g}{\Delta_{g_{22}}},\ Y_{12} = \frac{g_{12}}{g_{22}},\ Y_{21} = \frac{-g_{21}}{g_{22}} \text{ and } Y_{22} = \frac{1}{g_{22}}$$

9.8.3 T-parameters in Terms of Other Parameters

a) T-parameters in terms of Z-parameters

The Z-parameter equation are,

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

$$\text{or,} \quad I_1 = \frac{1}{Z_{11}}V_2 + \frac{Z_{22}}{Z_{11}}(-I_2)$$

Comparing with equation $I_1 = CV_2 + D(-I_2)$

$$\therefore \quad C = \frac{1}{Z_{21}} \text{ and } D = \frac{Z_{22}}{Z_{21}}$$

Also, we have,

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

$$V_1 = Z_{11}\left(\frac{1}{Z_{21}}V_2 + \frac{Z_{22}}{Z_{21}}(-I_2)\right) + Z_{12}I_2 = \frac{Z_{11}}{Z_{21}}V_2 + \left(\frac{Z_{11}Z_{22}}{Z_{21}} - Z_{12}\right)(-I_2)$$

Comparing with equation $V_1 = AV_2 + B(-I_2)$

$$\therefore \quad A = \frac{Z_{11}}{Z_{21}} \text{ and } B = \frac{\Delta Z}{Z_{21}}$$

b) T-parameters in terms of Y-parameters

As similar to above case,

$$A = \frac{-Y_{22}}{Y_{21}},\ B = \frac{-1}{Y_{21}},\ C = \frac{-\Delta Y}{Y_{21}} \text{ and } D = \frac{-Y_{11}}{Y_{21}}$$

c) T-parameters in terms of T'-parameters

The T' parameter equation in matrix form,

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ -I_1 \end{bmatrix}$$

Rewriting the above equation *i.e.*,

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} A' & -B' \\ -C' & D' \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

$$\text{or,} \quad \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A' & -B' \\ -C' & D' \end{bmatrix}^{-1} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$\text{Hence, } [T] = \begin{bmatrix} A' & -B' \\ -C' & D' \end{bmatrix}^{-1} = \frac{1}{\Delta T'} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}$$

$$\therefore \quad A = \frac{D'}{\Delta T'},\ B = \frac{B'}{\Delta T'},\ C = \frac{C'}{\Delta T'} \text{ and } D = \frac{A'}{\Delta T'}$$

d) T-parameters in terms of h-parameters

The h-parameters equations are,

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$$I_2 = h_{21}I_1 + h_{22}V_2$$

$$\text{or,} \quad I_1 = \frac{-h_{22}}{h_{21}}V_2 + \left(\frac{-1}{h_{21}}\right)(-I_2)$$

Comparing with equation $I_1 = CV_2 + D(-I_2)$

$$C = \frac{-h_{22}}{h_{21}} \text{ and } D = \frac{-1}{h_{21}}$$

Also, $V_1 = h_{11}I_1 + h_{12}V_2$

$$V_1 = h_{11}\left[\frac{-h_{22}}{h_{21}}V_2 + \left(\frac{-1}{h_{21}}\right)(-I_2)\right] + h_{12}V_2 = \frac{-\Delta h}{h_{21}}V_2 + \left(\frac{-h_{11}}{h_{21}}\right)(-I_2)$$

Comparing with equation $V_1 = AV_2 + B(-I_2)$

$$\therefore \quad A = \frac{-\Delta h}{h_{21}} \text{ and } B = \frac{-h_{11}}{h_{21}}$$

e) T-parameters in terms of g-parameters

As similar to above case,

$$A = \frac{1}{g_{21}},\ B = \frac{g_{22}}{g_{11}},\ C = \frac{g_{11}}{g_{21}} \text{ and } D = \frac{\Delta g}{g_{21}}$$

9.8.4 h-Parameters in Terms of Other Parameters

a) h-parameters in terms of Z-parameters

The Z-parameter equations are;

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{11}I_2$$

or, $$I_2 = \frac{-Z_{21}}{Z_{22}} I_1 + \frac{1}{Z_{22}} V_2$$

Comparing with equation $I_2 = h_{21}I_1 + h_{22}V_2$

$$h_{21} = \frac{Z_{21}}{Z_{22}} \text{ and } h_{22} = \frac{1}{Z_{22}}$$

Again, we have,

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$

or, $$V_1 = Z_{11} + I_1 + Z_{12}\left[\frac{-Z_{21}}{Z_{22}} I_1 + \frac{1}{Z_{22}} V_2\right]$$

or, $$V_1 = \left(Z_{11} \frac{-Z_{12}Z_{21}}{Z_{22}}\right) I_1 + \frac{Z_{12}}{Z_{22}} V_2$$

Comparing with equation $V_1 = h_{11}I_1 + h_{12}V_2$

$$h_{11} = \frac{\Delta Z}{Z_{22}} \text{ and } h_{12} = \frac{Z_{12}}{Z_{22}}$$

b) h-parameters in terms of Y-parameters

As similar to above case,

$$h_{11} = \frac{1}{Y_{11}},\ h_{12} = \frac{-Y_{12}}{Y_{11}},\ h_{21} = \frac{Y_{21}}{Y_{11}} \text{ and } h_{22} = \frac{\Delta Y}{Y_{11}}$$

c) h-parameters in terms of T-parameters

The T-parameter equations are,

$$V_1 = AV_2 + B(-I_2)$$
$$I_1 = CV_2 + D(-I_2)$$

or, $$I_2 = \frac{-1}{D} I_1 + \frac{C}{D} V_2$$

Comparing with equation, $I_2 = h_{21}I_1 + h_{22}V_2$

$$h_{21} = \frac{-1}{D} \text{ and } h_{22} = \frac{C}{D}$$

Also, $V_1 = AV_2 + B(-I_2)$

$$V_1 = AV_2 + B\left[\frac{1}{D} I_1 - \frac{C}{D} V_2\right] = \frac{B}{D} I_1 + \left(A - \frac{BC}{D}\right) V_2$$

Comparing with equation,

$$V_1 = h_{11}I_1 + h_{12}V_2$$

$\therefore$ $$h_{11} = \frac{B}{D} \text{ and } h_{12} = \frac{\Delta T}{D}$$

d) h-parameters in terms of T'-parameters

As similar to above case,

$$h_{11} = \frac{B'}{A'},\ h_{12} = \frac{1}{A'},\ h_{21} = \frac{-\Delta T'}{A'} \text{ and } h_{22} = \frac{C'}{A'}$$

e) h-parameters in terms of g-parameters

As we know $[h] = [g]^{-1}$

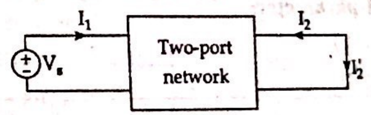
$$h_{11} = \frac{g_{22}}{\Delta g},\ h_{22} = \frac{-g_{12}}{\Delta g},\ h_{21} = \frac{-g_{12}}{\Delta g} \text{ and } h_{22} = \frac{g_{11}}{\Delta g}$$

9.9 RECIPROCITY

A two port network is said to be reciprocal if the ratio of the excitation to response is invariant to an interchange of the positions of the excitations and response in the network. Networks containing resistors, inductors and capacitors are generally reciprocal. Networks that additionally have dependent sources are generally non- reciprocal. Mathematically, we can say from figure 9.13 (a) and (b).

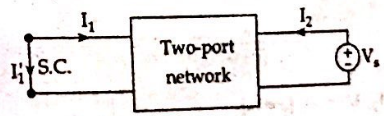
$$\frac{V_s}{I_2'} = \frac{V_s}{I_1'}$$

or, $$I_2' = I_1'$$



(a)

$(V_2 = V_s, I_2 = I_2, V_1 = 0, I_1 = -I_1')$



(b)

$(V_1 = V_s, I_1 = I_1, V_2 = 0, I_2 = -I_2')$

Figure 9.13: Determination the condition for reciprocity

a) In terms of Z-parameters

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

As in figure (a), $V_1 = V_s, I_1 = I_1, V_2 = 0, I_2 = -I_2'$

Hence, $V_s = Z_{11}I_1 - Z_{12}I_2'$

and, $I_2' = \frac{V_s Z_{21}}{Z_{11}Z_{22} - Z_{21}Z_{12}}$

As in figure (b),

$V_2 = V_s, I_2 = I_2, V_1 = 0, I_1 = -I_1'$

$\therefore \quad 0 = -Z_{11}I_1' + Z_{12}I_2$

and, $V_s = -Z_{21}I_1' + Z_{22}I_2$

Hence, $I_1' = \frac{V_s Z_{12}}{Z_{11}Z_{22} - Z_{21}Z_{12}}$

Comparing I_2' and I_1', we get,

$Z_{12} = Z_{21}$

(This is the condition of reciprocity in terms of Z-parameters).

b) In terms of Y-parameters

$I_1 = Y_{11}V_1 + Y_{12}V_2$

$I_2 = Y_{21}V_1 + Y_{22}V_2$

As in figure (a),

$V_1 = V_s, I_1 = I_1, V_2 = 0, I_2 = -I_2', I_2' = -Y_{12}V_s$

As in figure (b), we get,

$V_2 = V_s, I_2 = I_2, V_1 = 0, I_1 = -I_1', I_1' = Y_{12}V_s$

Comparing I_2' and I_1', then,

$Y_{12} = Y_{21}$

(This is the condition of reciprocity in terms of Y-parameters).

c) In terms of T-parameters

$V_1 = AV_2 + B(-I_2)$

$I_1 = CV_2 + D(-I_2)$

As in figure (a);

$V_1 = V_s, I_1 = I_1, V_2 = 0, I_2 = -I_2'$

$I_2' = \frac{V_s}{B}$

As figure (b);

$V_1 = V_s, I_2 = I_2, V_1 = 0, I_1 = -I_1'$

$\therefore \quad I_1' = V_s\left(\frac{AD - BC}{B}\right)$

$I_2' = V_s\left(\frac{AD - BC}{B}\right)$

Above discussion leads to the condition of reciprocity,

$AD - BC = 1$ or $\Delta T = 1$

or, $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$

d) In terms of T' parameters

Condition of reciprocity in case of T'-parameters is similar to as in case of T-parameters *i.e.*,

$A'D' - B'C' = 1$ or $\Delta T' = 1$

or, $\begin{vmatrix} A' & B' \\ C' & D' \end{vmatrix} = 1$

e) In terms of h-parameters

$V_1 = h_{11}I_1 + h_{12}V_2$

$I_2 = h_{21}I_1 + h_{22}V_2$

As in figure (a); $V_1 = V_s, I_1 = I_1, V_2 = 0, I_2 = -I_2'$

$I_2' = -V_s\frac{h_{21}}{h_{11}}$

As in figure (b); $V_2 = V_s, I_2 = I_2, V_1 = 0, I_1 = -I_1'$

$I_1' = -V_s\frac{h_{12}}{h_{11}}$

From the definition of reciprocity, $I_2' = I_1'$ leads to $h_{12} = -h_{21}$

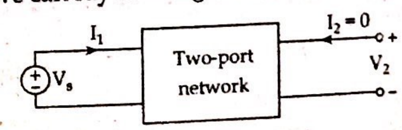
f) In terms of g-parameters

In this case, the condition of reciprocity is similar to as in case of h-parameters *i.e.*,

$g_{12} = -g_{21}$

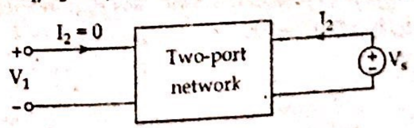
9.10 CONDITION FOR SYMMETRY

A two port network is said to be symmetrical if the ports can be interchanged without changing the port voltages and currents. Mathematically, we can say from figure 9.14 (a) and (b).



(a)

$(V_1 = V_s, I_1 = I_1, I_2 = 0, V_2 = V_2)$



(b)

$(V_2 = V_s, I_2 = I_2, I_1 = 0, V_1 = V_1)$

Figure 9: 14 Determination of condition for symmetry

$\left.\frac{V_s}{I_s}\right|_{I_2=0} = \left.\frac{V_s}{I_2}\right|_{I_1=0}$

a) In terms of Z-parameters

As in figure (a);

$V_1 = V_s, I_1 = I_1, I_2 = 0, V_2 = V_2$

$V_s = Z_{11} I_1$

$\left.\frac{V_s}{I_1}\right|_{I_2=0} = Z_{11}$

As in figure (b);

$V_2 = V_s, I_1 = I_2, I_1 = 0, V_1 = V_1$

$V_s = Z_{22} I_2 = Z_{22}$

$\left.\frac{V_s}{I_2}\right|_{I_1=0}$

From the definition of symmetry,

$\left.\frac{V_s}{I_2}\right|_{I_2=0} = \left.\frac{V_s}{I_2}\right|_{I_1=0}$ leads to $Z_{11} = Z_{22}$

b) In terms of Y-parameters

As in figure (a);

$V_1 = V_s, I_1 = I_1, I_2 = 0, V_2 = V_2$

$I_1 = Y_{11} V_s + Y_{12} V_2$

$0 = Y_{21} V_s + Y_{22} V_2$

$I_1 = Y_{11} V_s + Y_{12} \left\{-\frac{Y_{21}}{Y_{22}}\right\} V_s$

$\frac{V_s}{I_1} = \frac{Y_{22}}{Y_{11} Y_{12} - Y_{12} Y_{21}}$

And as in figure (b);

$V_2 = V_s, I_2 = I_2, I_1 = 0, V_1 = V_1$

$0 = Y_{11} V_1 + Y_{12} V_s$

$\frac{V_s}{I_2} = \frac{Y_{11}}{Y_{11} Y_{22} - Y_{12} Y_{21}}$

$\left.\frac{V_s}{I_1}\right|_{I_2=0} = \left.\frac{V_s}{I_2}\right|_{I_1=0}$

$Y_{11} = Y_{22}$

c) In terms of T-parameters

As in figure (a);

$V_1 = V_s, I_1 = I_1, I_2 = 0, V_2 = V_2$

$V_s = AV_2$

$I_1 = CV_2$

Then, $\frac{V_s}{I_1} = \frac{A}{C}$

As in figure (b);

$V_2 = V_s, I_2 = I_2, I_2 = 0, V_1 = V_1$

$V_1 = AV_s - BI_2$

$0 = CV_s - DI_2$ or $\frac{V_s}{I_2} = \frac{D}{C}$

Hence, condition for symmetry; $\left.\frac{V_s}{I_1}\right|_{I_2=0} = \left.\frac{V_s}{I_2}\right|_{I_1=0}$ leads to $A = D$

d) In terms of T'-parameters

Condition of symmetry in case of T'-parameters is similar to as in case of T-parameters *i.e.*, $A' = D'$.

e) In terms of h-parameters

As in figure (a);

$V_1 = V_s, I_1, I_2 = 0, V_2 = V_2$

$V_s = h_{11} I_1 + h_{12} V_2$

$0 = h_{21} I_2 + h_{22} V_2$

$\frac{V_s}{I_1} = \frac{h_{11} h_{22} - h_{12} h_{21}}{h_{22}}$

As in figure (b);

$V_2 = V_s, I_2 = I_2, I_1 = 0, V_1 = V_1$

$V_1 = h_{12} V_s$

$I_2 = h_{22} V_s$

or, $\frac{V_s}{I_2} = \frac{1}{h_{22}}$

∴ The condition for symmetry; $\left.\frac{V_s}{I_1}\right|_{I_2=0} = \left.\frac{V_s}{I_2}\right|_{I_1=0}$ leads to $h_{11} h_{22} - h_{12} h_{21} = 1$

or, $\Delta h = 1$

or, $\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = 1$

f) In terms of g-parameters

Condition of symmetry in case of g-parameters is similar to as in case of h-parameters *i.e.*,

$g_{11} g_{12} - g_{12} g_{21} = 1$

or, $\Delta g = 1$

or, $\begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = 1$

Table: condition under which two port network is reciprocal and symmetrical

Parameters	Condition for reciprocity	Condition for symmetry
[Z]	$Z_{12} = Z_{21}$	$Z_{11} = Z_{22}$
[Y]	$Y_{12} = Y_{21}$	$Y_{11} = Y_{22}$
[T] or [ABCD]	$\Delta T = AD - BC = 1$	$A = D$
[T'] or [A'B'C'D']	$\Delta T' = A'D' - B'C' = 1$	$A' = D'$
[h]	$h_{12} = -h_{21}$	$\Delta h = h_{11} h_{22} - h_{12} h_{21} = 1$
[g]	$g_{12} = -g_{21}$	$\Delta_g = g_{11} g_{22} - g_{12} g_{21} = 1$

9.11 T-π TRANSFORMATION

T and π networks are the two networks which are used to represent the equivalence of transmission lines, filters etc.

a) The T-network is represented by the model shown in figure 9.15 (a). For which the Z-parameters are,

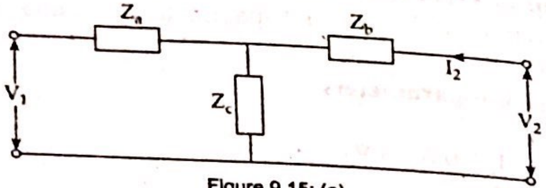


Figure 9.15: (a)

$$Z_{11} = Z_a + Z_a \quad (a)$$
$$Z_{12} = Z_{21} = Z_c \quad (b)$$
$$Z_{22} = Z_b + Z_c \quad (c)$$

From equations (a) to (c) the elements of T-network, Z_a, Z_b and Z_c can be expressed in terms of Z-parameters.

$$Z_a = Z_{11} - Z_{12}$$
$$Z_b = Z_{22} - Z_{12}$$
$$Z_c = Z_{12} = Z_{21}$$

Hence, if Z-parameters of a network are given, the equivalent T-network model can be constructed.

b) The π-network is represented by the model shown in figure 9.15(b). For which the Y-parameters are,

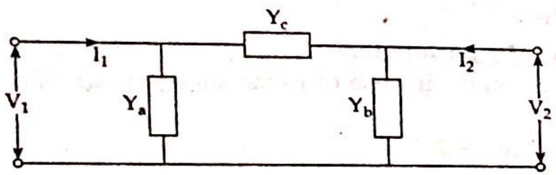


Figure 9.15: (b)

$$Y_{11} = Y_a + Y_c \quad (d)$$
$$Y_{12} = Y_{21} = -Y_c \quad (e)$$
$$Y_{22} = Y_b + Y_c \quad (f)$$

Using equation (d) to (e), we can express Y_a, Y_b and Y_c in terms of Y-parameters. Thus,

$$Y_a = Y_{11} + Y_{12}$$
$$Y_b = Y_{22} + Y_{12}$$
$$Y_c = -Y_{12} = -Y_{21}$$

Hence, if Y-parameters of a network are given then it is convenient to construct am equivalent π model rather an equivalent T-model.

9.11.1 π to T transformation

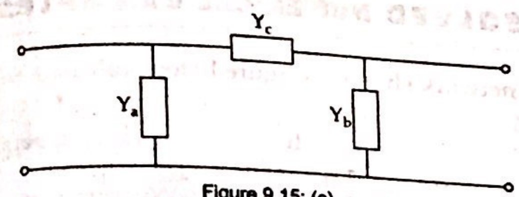


Figure 9.15: (c)

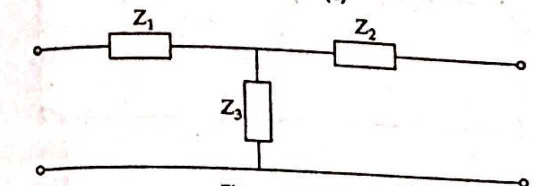


Figure 9.15: (d)

Admittances Y_a, Y_b and Y_c of the figure 9.15 (c) are known and it is required to calculate equivalent impedance Z_1, Z_2 and Z_3 of the figure 9.15 (d).

Using delta to star conversion, we have,

$$Z_1 = \frac{\frac{1}{Y_a}\frac{1}{Y_c}}{\frac{1}{Y_a} + \frac{1}{Y_b} + \frac{1}{Y_c}} = \frac{Y_b}{Y_aY_b + Y_bY_c + Y_cY_a}$$

Similarly,

$$Z_2 = \frac{Y_a}{Y_aY_b + Y_bY_c + Y_cY_a}$$

$$Z_3 = \frac{Y_c}{Y_aY_b + Y_bY_c + Y_cY_a}$$

9.11.2 T to π Transformation

In this case, the impedances, Z_1, Z_2 and Z_3 of the figure 9. (d) are known and it is required to calculate the equivalent admittances Y_a, Y_b and Y_c of the figure 9. (c).

Using star to delta conversion, we have,

$$\frac{1}{Y_a} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2}$$

or, $$Y_a = \frac{Z_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}$$

Similarly,

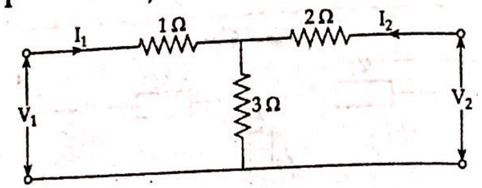
$$Y_b = \frac{Z_1}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}$$

$$Y_c = \frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}$$

SOLVED NUMERICAL EXAMPLES

1. **For the network shown in figure below, calculate**

   i) Z ii) Y iii) T
   iv) T v) h vi) g-parameters



Solution:

The loop equations become

$V_1 = 1 \cdot I_1 + 3 \cdot (I_1 + I_2)$

or, $V_1 = 4\,I_1 + 3\,I_2$ ... (a)

$V_2 = 2\,I_2 + 3 \cdot (I_1 + I_2)$

or, $V_2 = 3I_1 + 5\,I_2$ ... (b)

a) Z-parameters

$(V_1, V_2) = f(I_1, I_2)$

Case I: When $I_2 = 0$, from equation (a) and (b), we get,

$V_1 = 4\,I_1,\ V_2 = 3\,I_1$

Hence, $Z_{11} = \left.\dfrac{V_1}{I_1}\right|_{I_2=0} = \dfrac{V_1}{\left(\dfrac{V_1}{4}\right)} = 4\,\Omega$

$Z_{21} = \left.\dfrac{V_2}{I_1}\right|_{I_2=0} = 3\,\Omega$

Case II: When $I_1 = 0$, from equation (a) and (b), $V_1 = 3\,I_2$ and $V_2 = 5\,I_2$

$Z_{12} = \left.\dfrac{V_1}{I_2}\right|_{I_1=0} = 3\,\Omega$

$Z_{22} = \left.\dfrac{V_2}{I_2}\right|_{I_1=0} = 5\,\Omega$

b) Y-parameters

$(I_1, I_2) = f(V_1, V_2)$

Case I: When $V_2 = 0$, from equation (a) and (b),

$3I_1 = -5\,I_2$

and, $V_1 = 4\,I_1 + 3\,I_2$

Then, $V_1 = \dfrac{11}{5} I_1$ and $V_1 = -\dfrac{11}{3} I_2$

$Y_{11} = \left.\dfrac{I_1}{V_1}\right|_{V_2=0} = \dfrac{5}{11}\mho$

$Y_{21} = \left.\dfrac{I_2}{V_1}\right|_{V_2=0} = \dfrac{-3}{11}\mho$

Case II: When $V_1 = 0$, from equation (a) and (b), we get,

$4\,I_1 = -3I_2$

$V_2 = 3\,I_1 + 5\,I_2$

Then, $V_2 = \dfrac{-11}{3} I_2$ and $V_2 = \dfrac{11}{4} I_2$

$Y_{12} = \left.\dfrac{I_1}{V_2}\right|_{V_1=0} = \dfrac{-3}{11}\mho$

$Y_{22} = \left.\dfrac{I_2}{V_2}\right|_{V_1=0} = \dfrac{4}{11}\mho$

c) T-parameters

$(V_1, I_1) = f(V_2, -I_2)$

Case I: When $I_2 = 0$, from equation (a) and (b),

$V_1 = 4\,I_1$

and, $V_2 = 3\,I_1$

$A = \left.\dfrac{V_1}{V_2}\right|_{I_2=0} = \dfrac{4}{3}$

$C = \left.\dfrac{I_1}{I_2}\right|_{I_2=0} = \dfrac{1}{3}\mho$

Case II: When $V_2 = 0$, from equation (a) and (b), we get,

$3I_1 = -5\,I_2,\ V_1 = 4\left(\dfrac{-5}{3} I_2\right) + 3\,I_2 = \dfrac{-11}{3} I_2$

$B = \left.\dfrac{V_1}{-I_2}\right|_{V_2=0} = \dfrac{11}{3}\mho$

$D = \left.\dfrac{I_1}{-I_2}\right|_{V_2=0} = \dfrac{5}{3}$

d) T' parameters

$(V_2, I_2) = f(V_1, -I_1)$

Case I: When $I_1 = 0$, from equation (a) and (b), we get,

$V_1 = 3\,I_2$

$V_2 = 5\,I_2$

$A' = \left.\dfrac{V_2}{V_1}\right|_{I_1=0} = \dfrac{5}{3}$

$C' = \left.\dfrac{I_2}{V_1}\right|_{I_1=0} = \dfrac{1}{3}\mho$

Case II: when $V_1 = 0$, from equation (a) and (b), we get,

$4\,I_1 = -3\,I_2$

$V_2 = 3I_1 + 5\left(\dfrac{-4I_1}{3}\right) = \dfrac{-11}{3} I_1$

$$B' = \left.\frac{I_2}{I_1}\right|_{V_1=0} = \frac{11}{3}\,\Omega$$

$$D' = \left.\frac{I_2}{-I_1}\right|_{V_1=0} = \frac{4}{3}$$

e) h-parameters

$(V_1, I_2) = f(I_1, V_2)$

Case I: when $V_2 = 0$, from equation (a) and (b), we get,

$3I_1 = -5\,I_2$

$$V_1 = 4\,I_1 + 3\left(\frac{-3}{5}I_1\right) = \frac{11}{5}I_1$$

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{11}{5}\,\Omega$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{-3}{5}$$

Case II: when $I_1 = 0$, from equation (a) and (b), we get,

$V_1 = 3\,I_2,\ V_2 = 5\,I_2$

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} \frac{3}{5}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{1}{5}\,\mho$$

f) g-parameters

$(I_1, V_2) = f(V_1, I_2)$

Case I: When $I_2 = 0$, from equation (a) and (b), we get,

$V_1 = 4\,I_1,\ V_2 = 3\,I_1$

$$g_{11} = \left.\frac{I_1}{V_1}\right|_{I_2=0} = \frac{1}{4}\,\mho$$

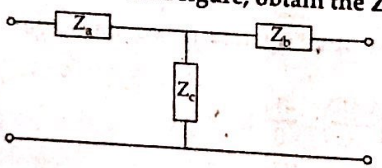
$$g_{21} = \left.\frac{V_2}{V_1}\right|_{I_2=0} = \frac{3}{4}$$

Case II: When $V_1 = 0$, from equation (a) and (b), we get,

$$4\,I_1 = -3\,I_2,\ V_2 = 3\left(\frac{-3}{4}I_2\right) + 5\,I_2 = \frac{11}{4}I_2$$

$$g_{12} = \left.\frac{I_1}{I_2}\right|_{I_1=0} = \frac{-3}{4} \quad ; \quad g_{22} = \left.\frac{V_2}{I_2}\right|_{V_1=0} = \frac{11}{4}\,\Omega$$

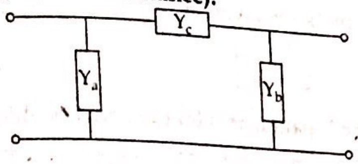
2. For the T network shown in figure, obtain the Z-parameters.



Solution:

$$Z\text{-parameter matrix, } Z = \begin{bmatrix} Z_a + Z_c & Z_c \\ Z_c & Z_b + Z_c \end{bmatrix}$$

3. For the π-network shown in figure, obtain the Y-parameters (where Y_a, Y_b, Y_c are admittance).



Solution:

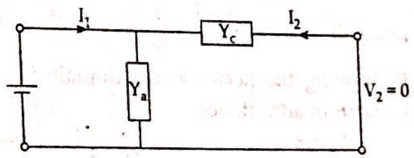
For Y-parameters;

$(I_1, I_2) = f(V_1 + V_2)$

i.e., $Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0}$; $Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0}$

$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0}$; $Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0}$

Case I: When $V_2 = 0$,



$$V_1 = (I_1 + I_2)\frac{1}{Y_a}$$

Also, $V_1 = \dfrac{-I_2}{Y_c}$

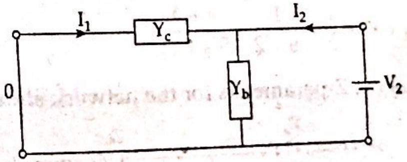
Then, $V_1 = [I_1 + V_1(-Y_c)] \cdot \dfrac{1}{Y_a}$

or, $(Y_a + Y_c)\,V_1 = I_2$

Hence, $Y_{11} = \left.\dfrac{I_1}{V_1}\right|_{V_2=0} = Y_a + Y_c$

and, $Y_{21} = \left.\dfrac{I_2}{V_1}\right|_{V_2=0} = -Y_c$

Case II: $V_1 = 0$



$$V_2 = (I_1 + I_2)\cdot\frac{1}{Y_b}$$

Also, $V_2 = \dfrac{-I_1}{Y_c}$

Then, $V_2 = (-V_2\,Y_c + I_2)\cdot\dfrac{1}{Y_b}$

or, $V_2\,(Y_b + Y_c) = I_2$

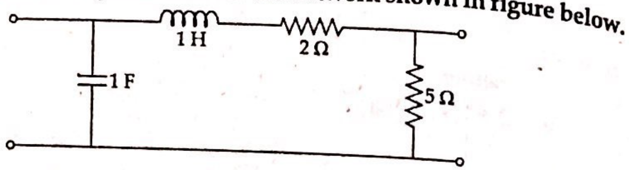
Hence, $Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -Y_c$

and, $Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = Y_b + Y_c$

Thus, Y-parameter matrix is given by,

$$Y = \begin{bmatrix} Y_a + Y_c & -Y_c \\ -Y_c & Y_b + Y_c \end{bmatrix}$$

4. **Determine the Y-parameters for the network shown in figure below.**

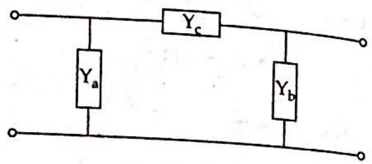


Solution:

We know, impedance, $Z = R = L_s = \frac{1}{C_s}$

and, admittance, $Y = \frac{1}{R} = \frac{1}{L_s} = C_s$

Redrawing the network in s-domain in terms of admittance,



$Y_a = s$

$Y_b = \frac{1}{5}$

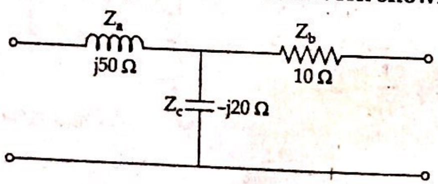
$Y_c = \frac{1}{s+2}$ Since $Z = s + 2$

$Y_{11} = Y_a + Y_c = s + \frac{1}{s+2} = \frac{s^2 + 2s + 1}{s+2}$

$Y_{12} = Y_{21} = -Y_c = \frac{-1}{s+2}$

$Y_{22} = Y_b + Y_c = \frac{1}{5} + \frac{1}{s+2} = \frac{s+2}{5(s+2)}$

5. **Calculate the Z-parameters for the network shown in figure.**



Solution:

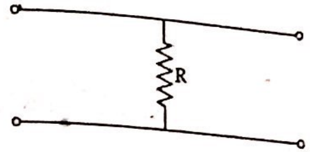
We know,

$Z_{11} = Z_a + Z_c = j30\ \Omega$

$Z_{12} = Z_{21} = Z_c = -j20\ \Omega$

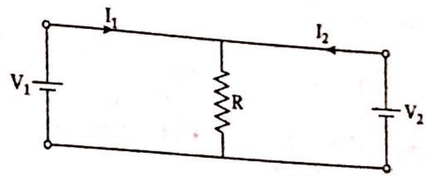
$Z_{22} = Z_b + Z_c = (10 - j120)\ \Omega$

6. **Determine the (i) Z, (ii) Y, (iii) T and (iv) h parameters of the network shown in figure.**



Solution:

Redrawing figure,



The loop equation becomes,

$V_1 = (I_1 + I_2)R$

$V_2 = (I_1 + I_2)R$

i.e., $V_1 = V_2$

a) **Z-parameters**

$Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = R$; $Z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = R$

$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = R$; $Z_{22} = \left.\frac{V_2}{I_{22}}\right|_{I_1=0} = R$

∴ Hence, Z-parameter matrix, $[Z] = \begin{bmatrix} R & R \\ R & R \end{bmatrix}$

b) **Y-parameters**

They exist, since denominator of all Y-parameters is $V_1 = V_2 = 0$.

c) **T-parameters**

$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = 1$; $C = \left.\frac{I_1}{V_1}\right|_{I_2=0} = \frac{1}{R}$

$B = \left.\frac{V_1}{-I_2}\right|_{V_2=0} = 0$; $D = \left.\frac{I_1}{-I_2}\right|_{V_2=0} = 1$

so, T-parameter, $[T] = \begin{bmatrix} 1 & 0 \\ \frac{1}{R} & 1 \end{bmatrix}$

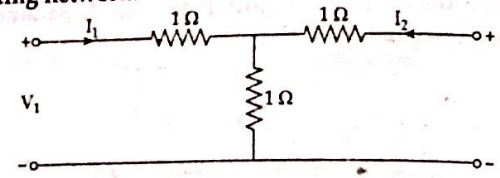
d) **h-parameters**

$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = 0$; $h_{21} = \left.\frac{I_2}{-I_1}\right|_{V_2=0} = -1$

$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = 1$; $h_{22} = \left.\frac{I_2}{-V_2}\right|_{I_1=0} = \frac{1}{R}$

∴ h-parameter matrix, $[h] = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{R} \end{bmatrix}$

7. **Two identical sections of the network shown in figure are cascaded. Calculate the transmission (ABCD) parameters of the resulting network.**



Solution:

Loop equation are,

$V_1 = 1 \cdot I_1 + 1 \cdot (I_1 + I_2)$

or, $V_2 = 2I_1 + I_2$ (a)

and, $V_2 = 1 \cdot I_2 + 1 \cdot (I_2 + I_1)$

$V_2 = I_1 + 2I_2$ (b)

Hence,

$$A = \frac{V_1}{V_2}\bigg|_{I_2=0} = 2 \quad , \quad C = \frac{I_1}{V_2}\bigg|_{I_2=0} = 1$$

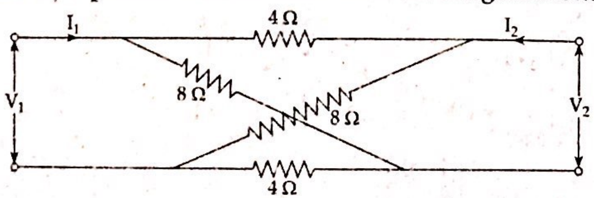
$$B = \frac{-V_1}{I_2}\bigg|_{V_2=0} = 3 \quad , \quad D = \frac{-V_1}{I_2}\bigg|_{V_2=0}$$

Thus, overall T-parameters of the cascaded network is given by,

$$[T] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & 6+6 \\ 2+2 & 3+4 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

8. **Find the T-parameter for the lattice network of figure shown below.**



Solution:

$\Delta Z = Z_{11}Z_{12} - Z_{12}Z_{21} = 36 - 4 = 32$

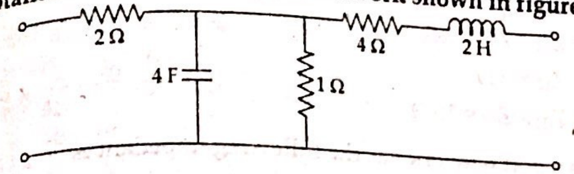
$$A = \frac{Z_{11}}{Z_{21}} = \frac{6}{2} = 3$$

$$B = \frac{\Delta Z}{Z_{21}} = \frac{32}{2} = 16\ \Omega$$

$$C = \frac{1}{Z_{21}} = \frac{1}{2}\ \Omega$$

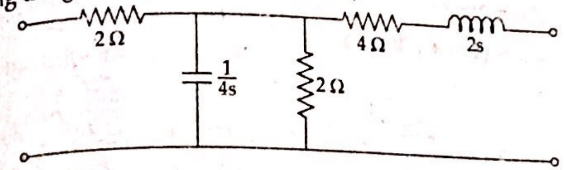
$$D = \frac{Z_{22}}{Z_{21}} = 3$$

9. **Obtain the Z-parameters of the network shown in figure below.**



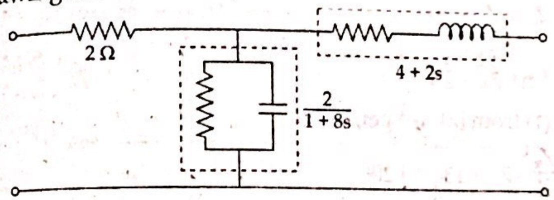
Solution:

Converting the given network in s-domain,



Since $\frac{1}{4s}$ and 2 Ω are in parallel, so equivalent impedance $= \dfrac{\frac{1}{4s}\cdot 2}{\frac{1}{4s}+2} = \dfrac{2}{1+8s}$

Now, redrawing the network as,



As, $Z_a' = 2\ \Omega$

$Z_b = 4 + 2s$

$$Z_c = \frac{2}{1+8s}$$

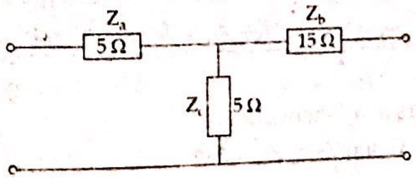
Hence, $Z_{11} = Z_a + Z_c = 2 + \dfrac{2}{1+8s} = \dfrac{4+16s}{1+8s}$

$$Z_{12} = Z_{21} = Z_c = \frac{2}{1+8s}$$

$$Z_{22} = Z_b + Z_c = (4+2s) + \frac{2}{1+8s} = \frac{6+34s+16s^2}{1+8s}$$

10. **The Z-parameters of a two-port network are $Z_{11} = 10\ \Omega$, $Z_{22} = 20\ \Omega$, $Z_{12} = Z_{21} = 5\ \Omega$. Find its equivalent T-network.**

Solution:



Using relationship,

$Z_a = Z_{11} - Z_{12} = 5\ \Omega$

$Z_c = Z_{12} = 5\ \Omega$

$Z_b = Z_{22} - Z_{12} = 15\ \Omega$

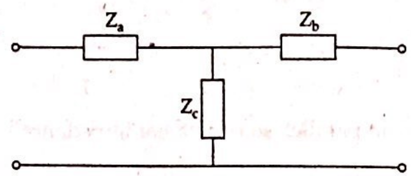
11. **A two port network has the following impedances:**

$Z_{10} = (250 + j\,100)\ \Omega$

$Z_{20} = 200\ \Omega$

$Z_{1s} = (400 + j\,300)\ \Omega$

Solution:



$$Z_{10} = Z_a + Z_c = 250 + j\,100 \quad (1)$$

$$Z_{20} = Z_b + Z_c = 200 \quad (2)$$

$$Z_{1s} = Z_a + \frac{Z_b \cdot Z_c}{Z_b + Z_c} \quad (3)$$

Subtraction (1) from (2), we get,

$$\frac{Z_b \cdot Z_c}{Z_b + Z_c} - Z_c = 150 + j\,200$$

or, $Z_b \cdot Z_c - Z_b \cdot Z_c - Z_c^2 = (Z_b + Z_c)\,(150 + j\,200)$

or, $Z_c^2 = -200\,(150 + j\,200)$ [From equation (2)]

$= 10^4\,(-3-j4) = 10^4\,(1 - j2)^2$

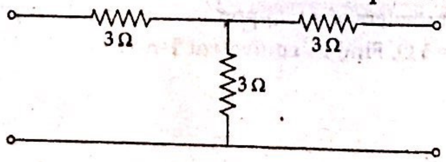
Hence,

$Z_c = (100 - j200)\ \Omega$

$Z_b = (100 + j200)\ \Omega$

$Z_a = (150 + j300)\ \Omega$

12. **Obtain the image impedance for a T-network as shown in figure, for which the resistance of three arms are equal to Ω.**



Solution:

Z-parameters of a T-network are

$Z_{11} = Z_{22} = 6\ \Omega$ and $Z_{12} = Z_{21} = 3\ \Omega$

Hence, $A = \dfrac{Z_{11}}{Z_{21}} = 2$

$B = \dfrac{\Delta Z}{Z_{21}} = \dfrac{36 - 9}{3} = 9\ \Omega$

$C = \dfrac{1}{Z_{21}} = \dfrac{1}{3}\ \mho$

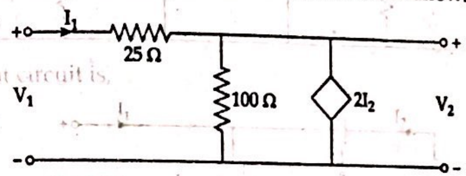
$D = \dfrac{Z_{22}}{Z_{21}} = \dfrac{6}{3} = 2$

The image impedance is then,

$$Z_{i1} = \sqrt{\frac{AB}{CD}}$$

$$\therefore \quad Z_{i2} = \sqrt{\frac{BD}{AC}} = \sqrt{\frac{18}{\left(\frac{2}{3}\right)}} = \sqrt{27} = 5.2\ \Omega$$

13. **Find the short circuit parameters of the circuit shown in figure**



Solution:

Applying KCL,

$$I_1 + I_2 = \frac{V_2}{100} + 2\,I_1$$

or, $I_1 = \dfrac{-V_2}{100} + I_2$ (1)

Also, $I_1 = \dfrac{V_1 - V_2}{25}$ (2)

or, $I_1 = \dfrac{1}{25}V_1 - \dfrac{1}{25}V_2$ (3)

From equation (1) and (2), we get,

$$\frac{V_1 - V_2}{25} = \frac{-V_2}{100} + I_2$$

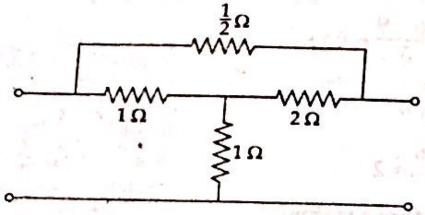
or, $I_2 = \dfrac{1}{25}V_1 - \dfrac{3}{100}V_2$ (4)

Comparing equation (3) and (4) with the Y-parameters equations,

$\therefore \quad Y_{11} = \dfrac{1}{25}\ \mho \qquad \therefore \quad Y_{21} = \dfrac{1}{25}\ \mho$

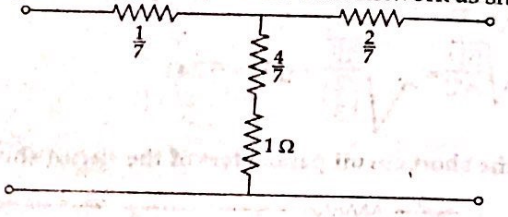
$Y_{12} = \dfrac{-1}{25}\ \mho \qquad \therefore \quad Y_{22} = \dfrac{-3}{100}\ \mho$

14. **Apply the Y-Δ transformation to the network of the figure to obtain an equivalent (a) T network (b) π-network.**



Solution:

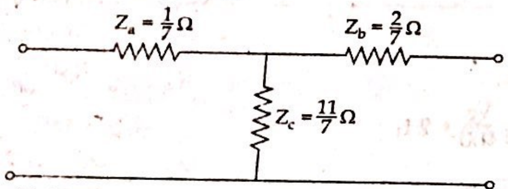
a) Applying Δ-Y conversion, we have the network as shown in figure.



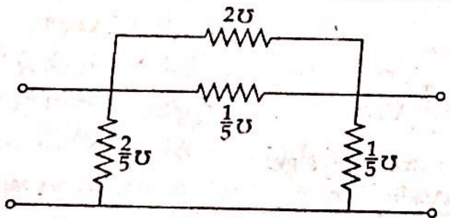
Now,

$$Z_a = \frac{1}{7}\Omega;\ Z_b = \frac{2}{7}\Omega;\ Z_c = \frac{4}{7} + 1 = \frac{11}{7}\Omega$$

Hence the equivalent T-network is,

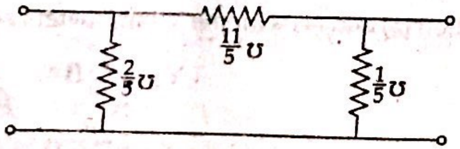


b) Applying Y-Δ conversion, we have the network as shown in figure.



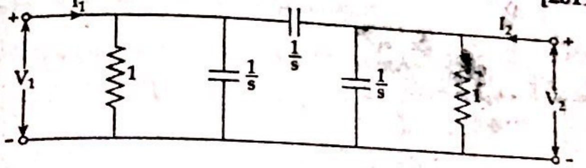
$$Y_a = \frac{2}{5}℧;\quad Y_b = \frac{1}{5}℧;\quad Y_c = 2 + \frac{1}{5} = \frac{11}{5}℧$$

Hence the equivalent π-network is,



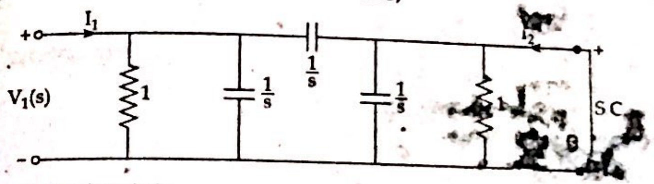
BOARD EXAMINATION SOLVED QUESTIONS

1. **Obtain ABCD parameters for the given network. Also check the reciprocity and symmetry of the given network.** [2019/Spring]

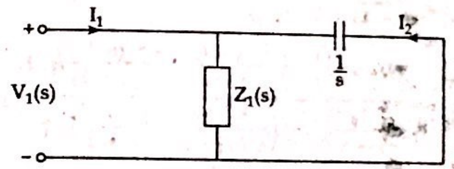


Solution;

With V_2 short circuited, the circuit becomes,



It's equivalent circuit is,



where, $Z_1(s) = \dfrac{1 \times \left(\frac{1}{s}\right)}{1 + \left(\frac{1}{s}\right)} = \dfrac{1}{s+1}$

so, $I_2(s) = -\dfrac{V_1(s)}{\left(\frac{1}{s}\right)} = -s\,V_1(s)$

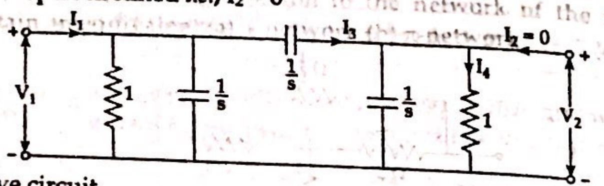
$\therefore\ B = \dfrac{-V_1}{I_2}\Big|_{V_2=0} = \dfrac{-V_1}{-sV_1} = \dfrac{1}{s}$

Now,

$$I_1 = \frac{V_1}{Z_1 \parallel \left(\frac{1}{s}\right)} = \frac{V_1}{\dfrac{\left(\frac{1}{s}\right) \times \left(\frac{1}{s+1}\right)}{\left(\frac{1}{s}\right) + \left(\frac{1}{s+1}\right)}} = \frac{V_1}{\left(\frac{1}{2s+1}\right)}$$

so, $D = \dfrac{-I_1}{I_2}\Big|_{V_2=0} = \dfrac{-(2s+1)V_1}{-sV_1} = \dfrac{2s+1}{s}$

With port 2 open circuited i.e., $I_2 = 0$



From above circuit,

$$\frac{V_1 - V_2}{\left(\frac{1}{s}\right)} = I_3$$

or, $$V_1 = V_2 + \frac{I_3}{s}$$

and, $$I_4 = \left[\frac{\left(\frac{1}{s}\right)}{\frac{1}{s}+1}\right] I_3 \qquad \text{[current divider rule]}$$

∴ $$I_4 = \frac{I_3}{s+1}$$

$$V_2 = I_4 \times 1 = \frac{I_3}{(s+1)}$$

Hence, $$A = \frac{V_1}{V_2} = \frac{\left(V_2 + \frac{I_3}{s}\right)}{\left(\frac{I_3}{s+1}\right)} = \frac{\frac{I_3}{s+1} + \frac{I_3}{s}}{\left(\frac{I_3}{s+1}\right)} = \frac{\frac{s+(s+1)}{s(s+1)} \cdot I_3}{\left(\frac{1}{s+1}\right) \cdot I_3} = \frac{2s+1}{s}$$

Also, $$I_1 = \frac{V_1}{1} + \frac{V_1}{\left(\frac{1}{s}\right)} + I_3 = V_1 (1+s) + I_3 = \left(V_2 + \frac{I_3}{s}\right)(1+s) + I_3$$

$$= \left(\frac{I_3}{s+1} + \frac{I_3}{s(s+1)}\right)(1+s) + I_3 = \frac{(s+1)}{s(s+1)}(s+1) I_3 + I_3$$

$$= \left[\frac{2s+1}{s}\right] I_3$$

Hence, $$C = \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{\left(\frac{2s+1}{s}\right) I_3}{\left(\frac{1}{s+1}\right) I_3} = \frac{(2s+1)(s+1)}{s} = \frac{2s^2+3s+2}{s}$$

Thus, A, B, C, D parameters are,

$$A = \frac{2s+1}{s};\ B = \frac{1}{s};\ C = \frac{2s^2+3s+2}{s};\ D = \frac{2s+1}{s}$$

For reciprocity, the condition is,

$$AD - BC = 1$$

or, $$\frac{(2s+1)}{s}\left(\frac{s+1}{s}\right) - \frac{1}{s}\frac{(s^2+3s+2)}{s} = 1$$

or, $$\frac{4s^2+4s+1}{s^2} - \frac{(s^2+3s+2)}{s^2} = 1$$

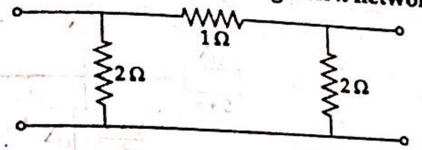
Not equal. So, the circuit is not reciprocal

For symmetry, the condition is, A = D

∴ $$\frac{2s+1}{s} = \frac{2s+1}{s}$$

Hence the network is symmetric

2. **Find the equivalent T-network for given π network. [2011/Spring]**



Solution:

Let the equivalent T-network have impedance in the shunt arm and impedance Z_a and Z_b in series arms. Then,

We have,

$$Z_1 = \frac{1}{Y_1} = \frac{1}{2} = 0.5\ \Omega$$

$$Z_2 = \frac{1}{Y_2} = \frac{1}{2} = 0.5\ \Omega$$

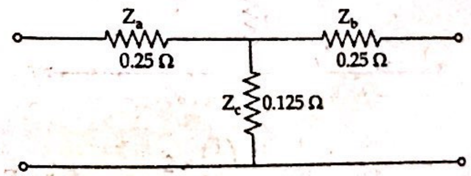
$$Z_3 = \frac{1}{Y_3} = \frac{1}{1} = 1\ \Omega$$

Hence, $$Z_a = \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3} = \frac{0.5 \times 1}{0.5 + 0.5 + 1} = 0.25\ \Omega$$

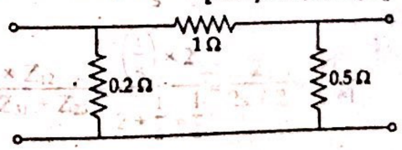
$$Z_b = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3} = \frac{0.5 \times 1}{0.5 + 0.5 + 1} = 0.25\ \Omega$$

$$Z_C = \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3} = \frac{0.5 \times 0.5}{0.5 + 0.5 + 1} = 0.125\ \Omega$$

Hence equivalent T-network is,



3. **Find the equivalent T-network for given π-network.**
[2011/Fall, 2014/Spring, 2015/Spring]



Solution:

Let the equivalent T-network have impedance Z_c in the shunt arm and impedances Z_a, Z_b in series arms. Then, we have,

$Z_1 = \frac{1}{Y_1} = \frac{1}{0.2} = 5\,\Omega$, $\quad Z_2 = \frac{1}{Y_2} = \frac{1}{0.5} = 2\,\Omega$

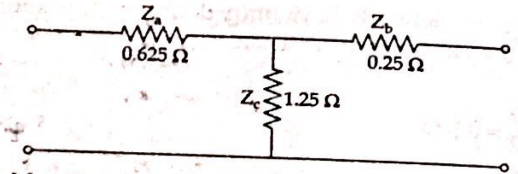
$Z_3 = \frac{1}{Y_3} = \frac{1}{1} = 1\,\Omega$

Hence, $Z_a = \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3} = \frac{5 \times 1}{5 + 2 + 1} = 0.625\,\Omega$

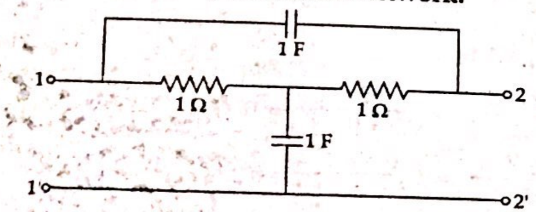
$Z_b = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3} = \frac{2 \times 1}{5 + 2 + 1} = 0.25\,\Omega$

$Z_c = \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3} = \frac{5 \times 2}{5 + 2 + 1} = 1.25\,\Omega$

Thus the equivalent T-network is,

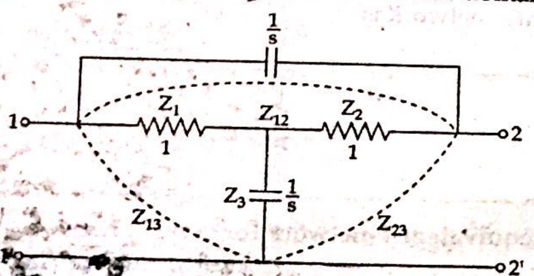


4. For a bridge T-network, construct π-network. [2014/Fall]



Solution:

Redrawing the given figure of bridge- T network in s-domain,



Applying Y-Δ transformation in the network,

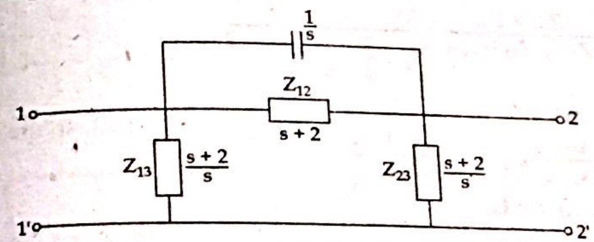
$\therefore \quad Z_{12} = Z_1 + Z_2 + \frac{Z_1 \cdot Z_2}{Z_3} = 1 + 1 + \frac{1 \times 1}{\left(\frac{1}{s}\right)} = 2 + s$

$Z_{23} = Z_2 + Z_3 + \frac{Z_2 \cdot Z_3}{Z_1}$

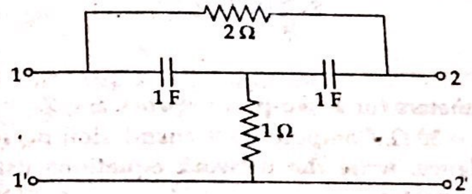
$\therefore \quad = 1 + \frac{1}{s} + \frac{1}{1} \times \left(\frac{1}{s}\right) = \frac{2 + s}{s}$

$\therefore \quad Z_{31} = Z_3 + Z_1 + \frac{Z_3 \cdot Z_1}{Z_2} = \frac{1}{s} + 1 + \frac{1}{s} \times \frac{1}{1} = \frac{s + 2}{s}$

Hence the required π network is,

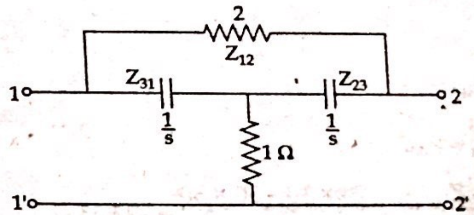


5. **Find the equivalent T-network for the given π-network from the given network.** [2014/Fall, 2016/Spring, 2018/Fall]



Solution:

Redrawing the given figure in s-domain,

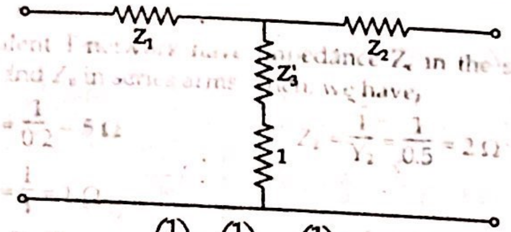


Now,

$$Z_1 = \frac{Z_{12} \times Z_{31}}{Z_{12} + Z_{31} + Z_{23}} = \frac{2 \times \left(\frac{1}{s}\right)}{2 + \frac{1}{s} + \frac{1}{s}} = \frac{2}{2s + 2} = \frac{1}{s + 1}$$

$$Z_2 = \frac{Z_{23} \times Z_{12}}{Z_{12} + Z_{31} + Z_{23}} = \frac{\left(\frac{1}{s}\right) \times 2}{2 + \frac{1}{s} + \frac{1}{s}} = \frac{2}{2s + 2} = \frac{1}{s + 1}$$

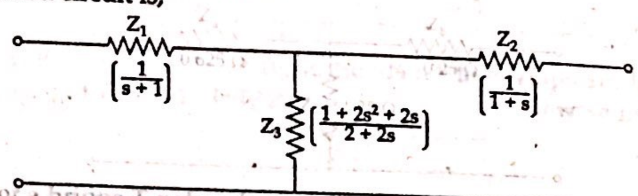
Scanned with CamScanner



$$Z_3' = \frac{Z_{31} \times Z_{23}}{Z_{12} + Z_{31} + Z_{23}} = \frac{\left(\frac{1}{s}\right) \times \left(\frac{1}{s}\right)}{2 + \frac{1}{s} + \frac{1}{s}} = \frac{\left(\frac{1}{s}\right)}{2s + 2}$$

Thus, $Z_3 = \frac{\left(\frac{1}{s}\right)}{2s+2} + 1 = \frac{1 + 2s^2 + 2s}{2s + 2}$

The required circuit is,



6. The Z-parameters for a two-port network are; $Z_{11} = 20\ \Omega$, $Z_{22} = 10\ \Omega$, $Z_{12} = Z_{21} = 30\ \Omega$. Compute the transmission parameters for the network. Hence, write the network equations using these two types of parameters. **[2013/Fall]**

Solution:

Given that;

$Z_{11} = 20\ \Omega$

$Z_{22} = 10\ \Omega$

$Z_{12} = Z_{21} = 30\ \Omega$

so, $A = \frac{Z_{11}}{Z_{21}} = \frac{20}{30} = 0.67$

$$B = \frac{Z_{11} Z_{22} - Z_{12} Z_{21}}{Z_{22}} = \frac{20 \times 10 - 30 \times 30}{30} = -23.33\ \Omega$$

$C = \frac{1}{Z_{21}} = \frac{1}{30} = 0.033\ S$ $\qquad$ $D = \frac{Z_{22}}{Z_{21}} = \frac{10}{30} = 0.33$

The transmission parameters for the network are;

$A = 0.67$ $\qquad$ $B = -23.33\ \Omega$

$C = 0.033\ S$ $\qquad$ $D = 0.33$

Network equation using Z-parameters:

$\therefore$ $V_1 = 20\ I_1 + 30\ I_2$

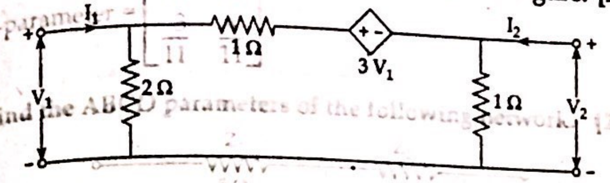
$\therefore$ $V_2 = 30\ I_1 + 10\ I_2$

Network equation using ABCD parameters:

$V_1 = AV_2 - BI_2 = 0.67\ V_2 + 23.33\ I_2$

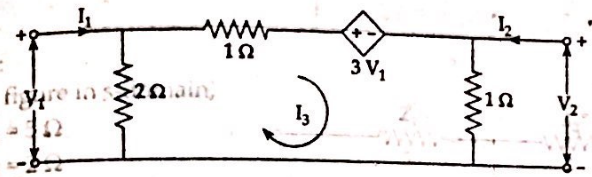
$\therefore$ $I_1 = CV_2 - DI_2 = 0.033\ V_2 - 0.33\ I_2$

7. Find the Z and Y parameters for the network of figure. [2013/Fall]



Solution:

Let I_3 be the middle loop current,



Applying KVL, we get,

$V_1 = 2(I_1 - I_3)$ (1)

$2(I_3 - I_1) + 1 \times I_3 + 3V_1 + 1 \times (I_3 - I_2) = 0$

or, $4I_3 = 2\ I_1 - I_2 - 3V_1$ (2)

$V_2 = 1(I_2 + I_3)$ (3)

From equation (1), (2) and (3), we have,

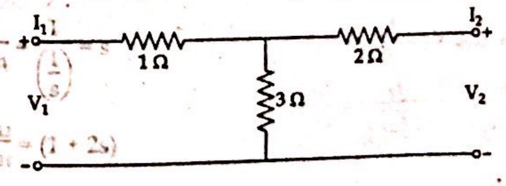
$$V_1 = 2I_1 - 2\left(\frac{I_1}{2} - \frac{I_2}{4} - \frac{3V_1}{4}\right) = -2I_1 - I_2 \tag{4}$$

and, $V_2 = I_2 + \frac{1}{4}(2I_1 + I_2 + 6I_1 + 3I_2) = 2\ I_1 + \frac{3}{2} I_2$ (5)

Hence, Z parameter $= \begin{bmatrix} -2 & -1 \\ 2 & \frac{3}{2} \end{bmatrix}$

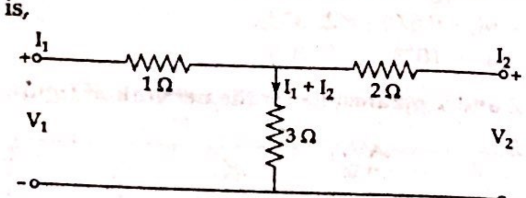
Now, $Y = [Z]^{-1} = \begin{bmatrix} -2 & -1 \\ 2 & \frac{3}{2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{-3}{2} & -1 \\ 2 & 2 \end{bmatrix}$

8. For the network shown, find the Z and Y parameters.
[2011/Fall, 2018/Spring, 2019/Fall]



Solution:

Given figure is,



$V_1 = 1 \times I_1 + 3\,(I_1 + I_2)$

$\therefore \quad V_1 = 4\,I_1 + 3\,I_2$

and, $V_2 = 2\,I_2 + 3\,(I_1 + I_2)$ (1)

$\therefore \quad V_2 = 3\,I_1 + 5\,I_2$

Hence, $Z_{11} = \left.\dfrac{V_1}{I_1}\right|_{I_2=0} = \dfrac{V_1}{\left(\frac{V_1}{4}\right)} = 4\,\Omega$ (2)

$Z_{22} = \left.\dfrac{V_2}{I_2}\right|_{I_1=0} = \dfrac{V_2}{\left(\frac{V_2}{5}\right)} = 5\,\Omega$

$Z_{21} = \left.\dfrac{V_2}{I_1}\right|_{I_2=0} = \dfrac{3I_1}{I_1} = 3\,\Omega$

$Z_{12} = \left.\dfrac{V_1}{I_2}\right|_{I_1=0} = \dfrac{3I_2}{I_2} = 3\,\Omega$

When $V_2 = 0$ in equation (1) and (2), we get,

$3I_1 = -5I_2$

$V_1 = 4I_1 + 3I_2$

Then, $V_1 = \dfrac{11}{5} I_1$

and, $V_1 = -\dfrac{11}{3} I_2$

$\therefore \quad Y_{11} = \left.\dfrac{I_1}{V_1}\right|_{V_2=0} = \dfrac{3I_1}{\left(\frac{11}{5}\right)I_2} = \dfrac{5}{11}\,\mho$

$\therefore \quad Y_{21} = \left.\dfrac{I_2}{V_1}\right|_{V_2=0} = \dfrac{I_2}{\left(\frac{-11}{3}\right)I_2} = \dfrac{-3}{11}\,\mho$

When $V_1 = 0$ in equation (1) and (2), we get,

$V_2 = 3I_1 + 5I_2$

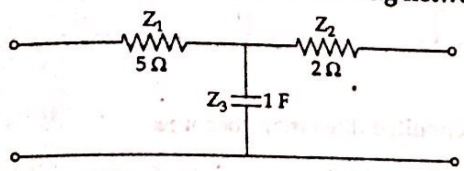
$V_2 = \dfrac{-3}{11} I_1$ and $V_2 = \dfrac{11}{4} I_2$

$\therefore \quad Y_{12} = \left.\dfrac{I_2}{V_2}\right|_{V_1=0} = \dfrac{I_2}{\left(\frac{-11}{3}\right)I_2} = \dfrac{-3}{11}\,\mho$

$\therefore \quad Y_{22} = \left.\dfrac{I_2}{V_2}\right|_{V_1=0} = \dfrac{I_2}{\left(\frac{11}{4}\right)I_2} = \dfrac{4}{11}\,\mho$

$\therefore \quad$ y-parameter $= \begin{bmatrix} \frac{5}{11} & \frac{-3}{11} \\ \frac{-3}{11} & \frac{4}{11} \end{bmatrix}$

9. **Find the ABCD parameters of the following network. [2017/Fall]**



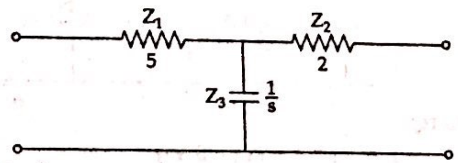
Solution:

Drawing figure in s-domain,

$Z_1 = 5\,\Omega$

$Z_2 = 2\,\Omega$

$Z_3 = 1\,F$



Here; $Z_1 = 5;\quad Z_2 = 2;\quad Z_3 = \dfrac{1}{s}$

$\therefore \quad Z_{11} = Z_1 + Z_3 = 5 + \dfrac{1}{s} = \dfrac{5s+1}{s}$

$\therefore \quad Z_{12} = Z_{21} = Z_3 = \dfrac{1}{s}$

$\therefore \quad Z_{22} = Z_2 + Z_3 = 2 + \dfrac{1}{s}$

$\therefore \quad Z_{22} = Z_2 + Z_3 = 2 + \dfrac{1}{s} = \dfrac{1+2s}{s}$

$\therefore \quad \Delta Z = Z_{11}Z_{22} - Z_{21}Z_{12} = \dfrac{(5s+1)}{s} \times \dfrac{(1+2s)}{s} - \dfrac{1}{s} \times \dfrac{1}{s} = \dfrac{10s^2+7s}{s^2} = \dfrac{10s+7}{s}$

Now, $A = \dfrac{Z_{11}}{Z_{21}} = \dfrac{5s+1}{s \times \frac{1}{s}} = 5s+1$

$\therefore \quad B = \dfrac{\Delta Z}{Z_{21}} = \dfrac{(10s+7)}{s \times \frac{1}{s}} = 10s+7$

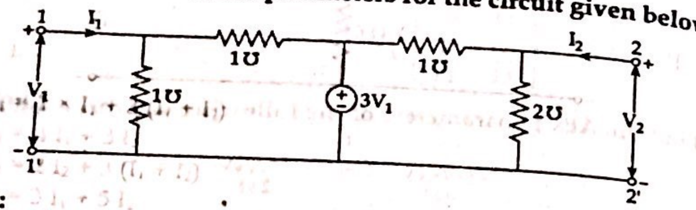
$\therefore \quad C = \dfrac{1}{Z_{21}} = \dfrac{1}{\left(\frac{1}{s}\right)} = s$

$\therefore \quad D = \dfrac{Z_{22}}{Z_{21}} = (1+2s)$

Hence, transmission parameter,

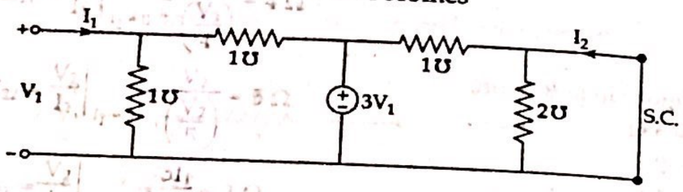
$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1+5s & 10s+7 \\ s & 2s+1 \end{bmatrix}$$

10. Determine admittance parameters for the circuit given below.



Solution:

With port 2 short circuited, the circuit becomes



Here;

$$I_1 = V_1 \times 1 + (V_1 - 3V_1) \times 1$$

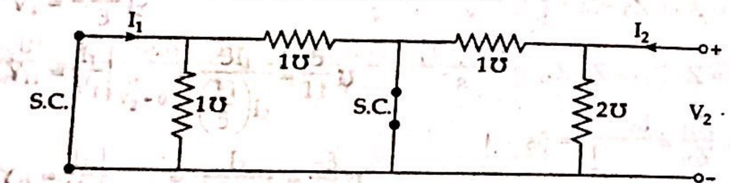
or, $I_1 = -V_1$

Hence, $Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{-V_1}{V_1} = -1$

and, $I_2 = -3V_1 \times 1$

Hence, $Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-3V_1}{V_1} = -3$

with port-1 short circuited, the circuit becomes



Here; $V_1 = 0$, $V_2 = \frac{-I_1}{1} = -I_1$

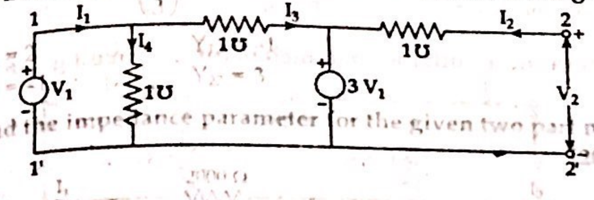
so, $Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{I_1}{-I_1} = -1$

$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{V_2 \times (1+2)}{V_2} = 3$

Hence, Y parameters $= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -3 & 3 \end{bmatrix}$

Alternative Method:

With port 2 short circuited, the circuit reduces to the following form;



Here; $\frac{-I_2}{1s} = 3V_1$ (1)

$\frac{I_3}{1s} + 3V_1 = V_1$

or, $2V_1 = -I_3$ (2)

Also, $I_1 = I_4 + I_3$ (3)

and, $V_1 = \frac{I_3}{1s}$ (4)

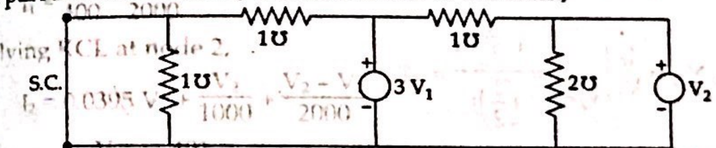
From equation (1) and (4), we get,

$I_1 = V_1 + I_3 = V_1 - 2V_1 = -V_1$

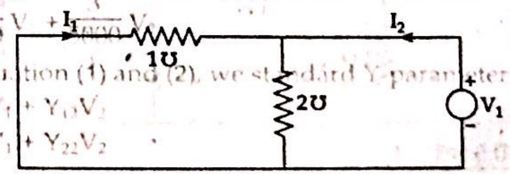
Hence, $Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = -1s$

$Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-3V_1}{V_1} = -3s$

with part 1 short circuited, circuit reduces to the form,



Here, $V_1 = 0$, the voltage source $3V_1$ reduces to a short circuit. Hence, the circuit reduces to



Here, 2s in shunt with 1s results in total admittance of 3s at port 2.

Hence, $Y_{22} = 3s$

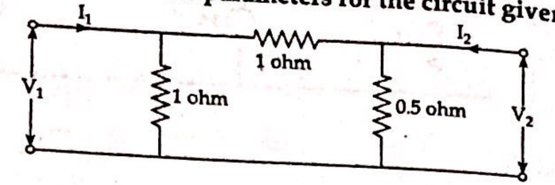
Further, $\frac{-I_1}{1s} = V_2$

so, $Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{I_1}{\left(\frac{-I_1}{1s}\right)} = -1s$

In this case, $Y_{12} = -Y_{21}$, since the network includes controlled voltage source.

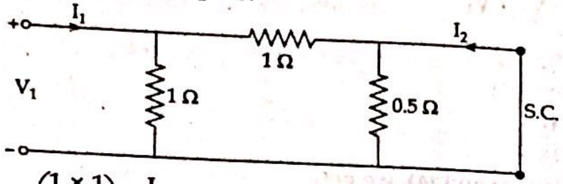
so, Y-parameters $= \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -3 & 3 \end{bmatrix}$

11. Determine admittance parameters for the circuit given below.



Solution:

With port 2 short circuited, $V_2 = 0$.



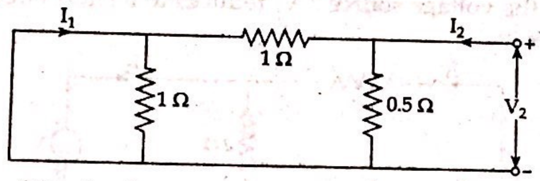
Here, $V_1 = I_1 \times \left(\frac{1 \times 1}{1+1}\right) = \frac{I_1}{2}$

so, $Y_{11} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{I_1}{\left(\frac{I_1}{2}\right)} = 2$

$I_2 = \left(\frac{-1}{1+1}\right) \times I_1 = \frac{-1}{2} I_1$

Hence, $Y_{21} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{\left(\frac{-1}{2}\right)I_1}{\left(\frac{1}{2}\right)I_1} = -1$

with part 1 short circuited, $V_1 = 0$.



Here, $V_2 = I_2 \times \frac{0.5 \times 1}{0.5 + 1} = \frac{I_2}{3}$

$I_1 = \left(\frac{0.5}{1+0.5}\right) I_2 = \frac{-I_2}{3}$

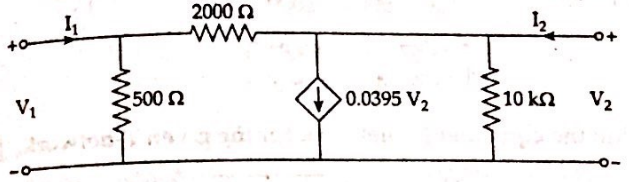
so, $Y_{12} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{\left(\frac{-I_2}{3}\right)}{\left(\frac{I_2}{3}\right)} = -1$

$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{I_2}{\left(\frac{I_2}{3}\right)} = 3$

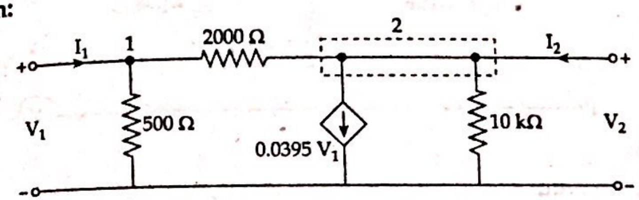
so, $Y_{11} = 2$; $Y_{12} = -1$

$Y_{21} = -1$; $Y_{22} = 3$

12. Find the impedance parameter for the given two part network. **[2017/Spring]**



Solution:



Applying KCL at node 1,

$I_1 = \frac{V_1}{500} + \frac{V_1 - V_2}{2000}$

or, $I_1 = \frac{V_1}{400} - \frac{V_2}{2000}$ **(1)**

Applying KCL at node 2,

$I_2 = 0.0395\, V_1 + \frac{V_2}{1000} + \frac{V_2 - V_1}{2000}$

or, $I_2 = -\frac{V_1}{2000} + \frac{401}{1000}$

or, $I_2 = \frac{39}{1000} V_1 + \frac{3}{5000} V_2$ **(2)**

Comparing equation (1) and (2), we standard Y-parameter

$I_1 = Y_{11}V_1 + Y_{12}V_2$

$I_2 = Y_{21}V_1 + Y_{22}V_2$

We get,

$Y_{11} = \frac{1}{400}$; $Y_{12} = \frac{-1}{2000}$

$Y_{21} = \frac{39}{1000}$; $Y_{22} = \frac{3}{5000}$

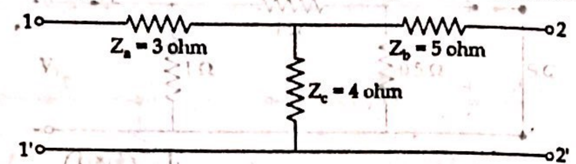
Now, for Z-parameters, we know the relation,

$$Z = [Y]^{-1}$$

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} \frac{3}{5000} & \frac{1}{2000} \\ \frac{-39}{1000} & \frac{1}{400} \end{bmatrix}}{\frac{1}{400} \times \frac{3}{5000} - \frac{39}{1000} \times \frac{-1}{2000}} = \frac{\begin{bmatrix} \frac{3}{5000} & \frac{1}{2000} \\ \frac{-39}{1000} & \frac{1}{400} \end{bmatrix}}{\left(\frac{21}{10^6}\right)}$$

$$= \begin{bmatrix} 28.57 & 23.81 \\ 1857.14 & 119.05 \end{bmatrix}$$

13. **Find the equivalent π-network for the given T-network. [2017/Fall]**



Solution:

$Z_a = 3$ ohm

$Z_b = 5$ ohm

$Z_c = 4$ ohm

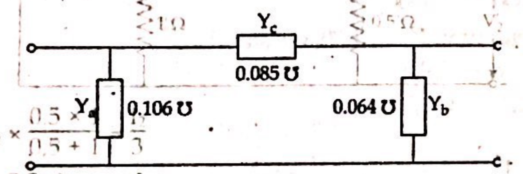
We have,

$$Y_a = \frac{Z_b}{Z_aZ_b + Z_bZ_c + Z_aZ_c} = \frac{5}{3 \times 4 + 4 \times 5 + 5 \times 3} = 0.106\ \mho$$

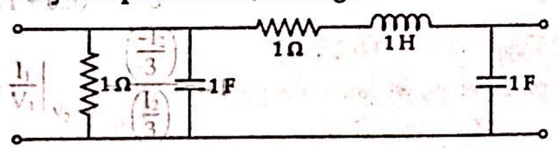
$$Y_b = \frac{Z_a}{Z_aZ_b + Z_bZ_c + Z_aZ_c} = \frac{3}{3 \times 4 + 4 \times 5 + 5 \times 3} = 0.064\ \mho$$

$$Y_c = \frac{Z_c}{Z_aZ_b + Z_bZ_c + Z_aZ_c} = \frac{4}{3 \times 4 + 4 \times 5 + 5 \times 3} = 0.085\ \mho$$

Thus the equivalent p-network is,

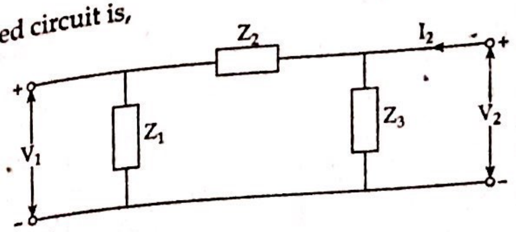


14. **Obtain hybrid parameters for the given network. [2016/Spring]**



Solution:

The transformed circuit is,

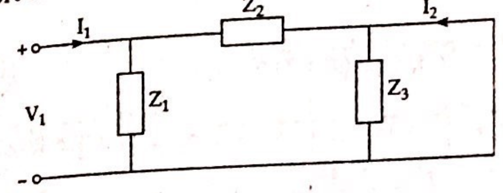


Here; $Z_1 = \frac{1 \times \frac{1}{s}}{1 + \frac{1}{s}} = \frac{1}{1+s}$

$Z_2 = 1 + s$

$Z_3 = \frac{1}{s}$

with port 2 short circuited, $V_2 = 0$



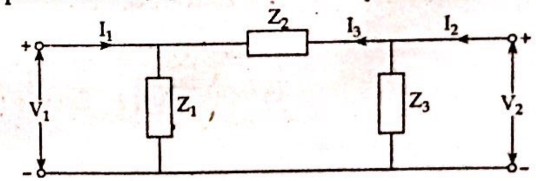
$$V_1 = (Z_1 \| Z_2) \times I_1 = \frac{\frac{1}{(s+1)} \times (1+s)}{\frac{1}{(s+1)} + (1+s)} \times I_1 = \frac{s+1}{s^2+2s+2} \times I_1$$

$$I_2 = -\frac{Z_1}{Z_1+Z_2} \times I_1 = -\frac{\frac{1}{(s+1)}}{\frac{1}{(s+1)} + (1+s)} \times I_1 = -\frac{1}{s^2+2s+2} \times I_1$$

so, $$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{\frac{s+1}{s^2+2s+2} \times I_1}{I_1} = \frac{s+1}{s^2+2s+2}$$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{-\frac{1}{s^2+2s+2} \times I_1}{I_1} = \frac{-1}{s^2+2s+2}$$

with port 1 open circuited, $I_1 = 0$



Now,

$$V_2 = I_2 \times (Z_3 \parallel (Z_1 + Z_2))$$

$$= I_2\left[\frac{1}{s} \parallel \left(\frac{s}{s+1}\right) + (s+1)\right]$$

$$= I_2 \times \left[\frac{\frac{1}{s} \times \left(\frac{s^2+2s+2}{s+1}\right)}{\frac{1}{s} + \frac{s^2+2s+2}{s+1}}\right]$$

$$= I_2 \times \frac{s^2+2s+2}{s+1+s(s^2+2s+2)} = \frac{s^2+2s+2}{s^3+2s^2+3s+1}$$

and, $$I_3 = \frac{Z_3}{Z_1+Z_2+Z_3} \times I_2 = \frac{\frac{1}{s}}{\frac{1}{(s+1)} + (s+1) + \frac{1}{s}} \times I_2$$

$$= \frac{s+1}{s + s(s+1)^2 + (s+1)} \times I_2$$

$$V_1 = I_3 \times Z_1 = \frac{s+1}{2s+1+s(s^2+1)} I_2 \times \frac{1}{s+1}$$

$$= \frac{1}{2s+1+s(s^2+1)} \times \frac{1}{(s+1)} = \frac{1}{s^3+2s^2+3s+1} \times I_2$$

Hence, $$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0} = \frac{\left(\frac{1}{s^3+2s^2+3s+1}\right) \times I_2}{\left(\frac{s^3+2s+2}{s^3+2s^2+3s+1}\right) \times I_2} = \frac{1}{s^2+2s+2}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{I_2}{\frac{s^3+2s+2}{s^3+2s^2+3s+1} \times I_2}$$

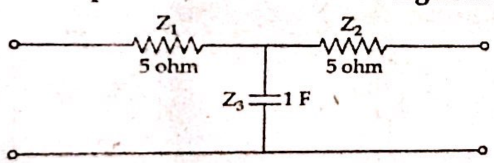
$\therefore$ $$h_{11} = \frac{s+1}{s^2+2s+2}$$

$\therefore$ $$h_{21} = \frac{-1}{s^2+2s+2}$$

$\therefore$ $$h_{12} = \frac{1}{s^2+2s+2}$$

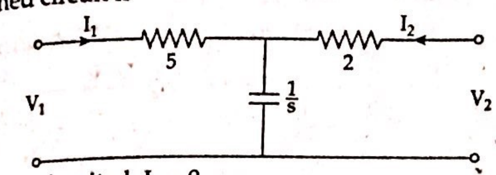
$\therefore$ $$h_{22} = \frac{s^3+2s^2+3s+1}{s^2+2s+2}$$

15. Find the ABCD parameters of the following network. [2017/Fall]



Solution:

The transformed circuit is



with port 2 open circuited, $I_2 = 0$

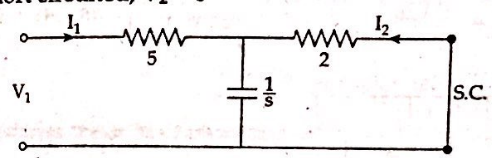
$$V_1 = I_1 \times \left(5 + \frac{1}{s}\right) = \left(\frac{5s+1}{s}\right) \times I_1$$

$$V_2 = I_1 \times \frac{1}{s} = \frac{I_1}{s}$$

Now, $$A = \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{\left(\frac{5s+1}{s}\right) \times I_1}{\frac{1}{s} \times I_1} = 5s+1$$

$$C = \left.\frac{I_1}{V_1}\right|_{I_2=0} = \frac{I_1}{\left(\frac{I_1}{s}\right)} = s$$

with port 2 short circuited, $V_2 = 0$



Equivalent impedance,

$$Z_{eq} = 5 + \frac{2 \times \frac{1}{s}}{2 + \frac{1}{s}} = 5 + \frac{2}{2s+1} = \frac{10s+7}{2s+1}$$

$$I_1 = \frac{V_1}{Z_{eq}} = \left(\frac{2s+1}{10s+7}\right) V_1$$

$$I_2 = \frac{\frac{-1}{s}}{2+\frac{1}{s}} \times I_1 = \frac{-1}{(2s+1)} \times V_1 \times \frac{(2s+1)}{(10s+7)} = \frac{-1}{(10s+7)} \times V_1$$

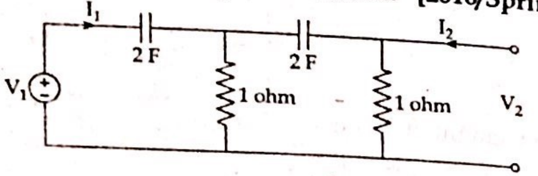
Hence, $$B = \left.\frac{-V_1}{I_2}\right|_{V_2=0} = \frac{-V_1}{\frac{-1}{10s+7} \times V_1} = 10s+7$$

$$D = \left.\frac{-I_1}{I_2}\right|_{V_2=0} = \frac{-\left(\frac{2s+1}{10s+7}\right) \times V_1}{-\frac{1}{(10s+7)} \times V_1} = 2s+1$$

$\therefore$ $A = 5s + 1$; $B = 10s + 7$

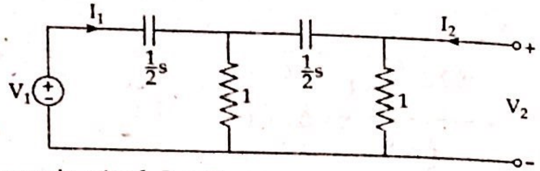
$\therefore$ $C = s$; $D = 2s + 1$

16. **Find Z-parameters of the given network. [2016/Spring, 2016/Fall]**



Solution:

The transformed network is,



with port 2 open circuited, $I_2 = 0$

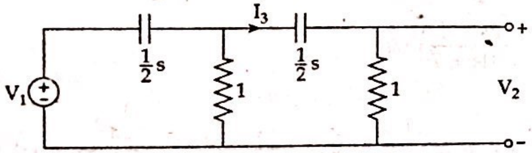
Equivalent impedance,

$$Z_{eq} = \frac{1}{2s} + \left[1 \,\|\, \left(\frac{1}{2s} + 1\right)\right] = \frac{1}{2s} + \frac{\left(1 \times \frac{2s+1}{2s}\right)}{1 + \left(\frac{2s+1}{2s}\right)} = \frac{1}{2s} + \frac{2s+1}{4s+1}$$

$$= \frac{4s + 1 + 4s^2 + 2s}{2s(4s+1)} = \frac{4s^2 + 6s + 1}{2s(4s+1)}$$

$$V_1 = I_1 Z_{eq} = \frac{4s^2 + 6s + 1}{2s(4s+1)} \times I_1$$

$$I_3 = \frac{1}{1 + \frac{1}{\frac{1}{2s} + 1}} \times I_1 = \frac{2s}{4s+1} \times I_1$$

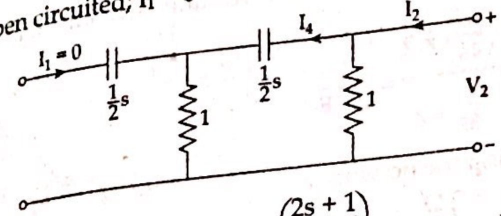


$$V_2 = I_3 \times 1 = \frac{2s}{4s+1} \times I_1$$

$$\text{Hence, } Z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2 = 0} = \frac{\frac{4s^2 + 6s + 1}{2s(4s+1)}}{I_1} \times I_1 = \frac{4s^2 + 6s + 1}{2s(4s+1)}$$

$$Z_{21} = \left.\frac{V_2}{I_1}\right|_{V_2 = 0} = \frac{2s}{\left(\frac{4s+1}{I_1}\right)} \times I_1 = \frac{2s}{4s+1}$$

with port 1 open circuited; $I_1 = 0$



$$V_2 = I_2 \times \left[1 \,\|\, \left(\frac{1}{2s} + 1\right)\right] = \frac{I_2 \times 1 \times \left(\frac{2s+1}{2s}\right)}{1 + \left(\frac{2s+1}{2s}\right)}$$

$$I_4 = \frac{1}{1 + 1 + \frac{1}{2s}} \times I_2 = \frac{2s}{4s+1} \times I_2$$

$$V_1 = I_4 \times 1 = \frac{2s}{4s+1} I_2$$

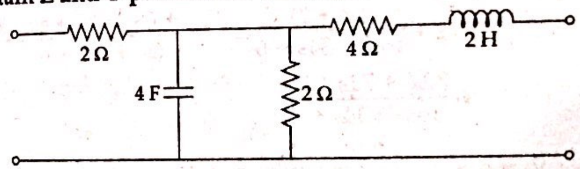
$$\text{Hence, } Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1 = 0} = \frac{\left(\frac{2s}{4s+1}\right) I_2}{I_2} = \frac{2s}{4s+1}$$

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1 = 0} = \frac{\left(\frac{2s+1}{4s+1}\right) I_2}{I_2} = \frac{2s+1}{4s+1}$$

Hence, $Z_{11} = \frac{4s^2 + 6s + 1}{2s(4s+1)}$; $Z_{12} = \frac{2s}{4s+1}$

$Z_{21} = \frac{2s}{4s+1}$; $Z_{22} = \frac{2s+1}{4s+1}$

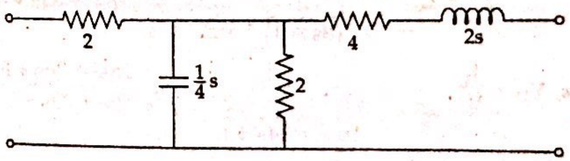
17. **Obtain Z and Y-parameters of the network shown below.[2015/Fall]**



Solution:

a) **Z-parameters**

Converting into s-domain,

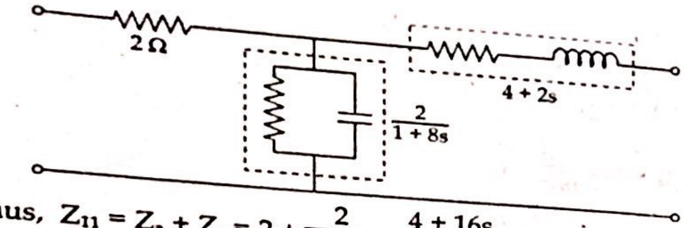


Here, $\frac{1}{4s}$ and 2 are in parallel, so equivalent impedance

$$= \frac{\left(\frac{1}{4s}\right) \times 2}{\frac{1}{4s} + 2} = \frac{2}{1 + 8s}$$

Redrawing the network as,

$$Z_a = 2\ \Omega$$
$$Z_b = 4 + 2s$$
$$Z_c = \frac{2}{1 + 8s}$$



Thus, $Z_{11} = Z_a + Z_c = 2 + \frac{2}{1 + 8s} = \frac{4 + 16s}{1 + 8s}$

$$Z_{12} = Z_{21} = Z_c = \frac{2}{1 + 8s}$$

$$Z_{22} = Z_b + Z_c = (4 + 2s) + \frac{2}{1 + 8s} = \left(\frac{6 + 34s + 16s^2}{1 + 8s}\right)$$

b) **Y-parameters**

With port 2 short circuited, $V_2 = 0$

Hence, $Z_{eq} = Z_a + (Z_b \parallel Z_c) = 2 + \frac{\frac{2}{(8s + 1)} \times (2s + 4)}{\frac{2}{(8s + 1)} + (2s + 4)}$

$$= 2 + \frac{4(s + 2)}{16s^2 + 34s + 6} = \frac{32s^2 + 68s + 12 + 4s + 8}{16s^2 + 34s + 6}$$

$$= \frac{32s^2 + 72s + 20}{16s^2 + 34s + 6}$$

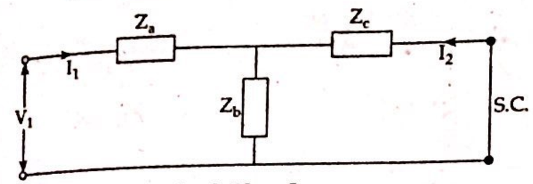
Now, $V_1 = I_1 \times Z_{eq} = \frac{32s^2 + 72s + 20}{16s^2 + 34s + 6} \times I_1$

$$I_2 = \frac{-Z_b}{Z_b + Z_c} \times I_1 = \frac{-\frac{2}{(8s + 1)}}{\frac{2}{(8s + 1)} + (2s + 4)} = \frac{-2}{16s^2 + 34s + 6} \times I_1$$

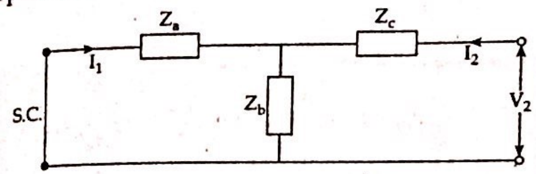
Thus, $Y_{11} = \left.\frac{I_1}{V_2}\right|_{V_2 = 0} = \frac{I_1}{\frac{32s^2 + 72s + 20}{16s^2 + 34s + 6} \times I_1} = \frac{16s^2 + 34s + 6}{32s^2 + 72s + 20}$

$$Y_{21} = \left.\frac{I_1}{V_1}\right|_{V_2 = 0} = \frac{\frac{-2}{(16s^2 + 34s + 6)} \times I_1}{\left(\frac{32s^2 + 72s + 20}{16s^2 + 34s + 6}\right) \times I_1} = \frac{-2}{32s^2 + 72s + 20}$$

$$= \frac{-1}{16s^2 + 36s + 10}$$



With port 1 short circuited, $V_1 = 0$



Now, $Z_{eq} = Z_c + \frac{Z_a + Z_b}{Z_a + Z_b} = 2s + 4 + \frac{2 \times \frac{2}{(8s + 1)}}{2 + \frac{2}{(8s + 1)}}$

$$= 2s + 4 + \frac{4}{16s + 4} = 2s + 4 + \frac{1}{4s + 1}$$

$$= \frac{8s^2 + 18s + 4 + 1}{4s + 1} = \frac{8s^2 + 18s + 5}{4s + 1}$$

$$V_2 = I_2 \times Z_{eq} = \frac{8s^2 + (8s + 5)}{4s + 1} \times I_2$$

$$I_1 = -\frac{Z_b}{Z_a + Z_b} \times I_2 = \frac{\frac{-2}{(8s + 1)}}{2 + \frac{2}{(8s + 1)}} \times I_2$$

$$= -\frac{2}{16s + 4} \times I_2 = \frac{-1}{2(4s + 1)} \times I_2$$

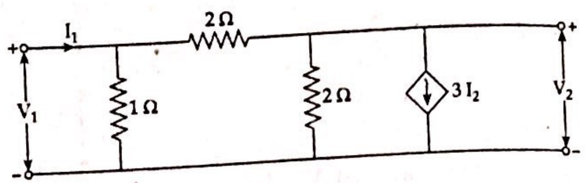
Thus, $Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1 = 0} = \frac{\frac{-1}{2(4s + 1)}}{\left(\frac{8s^2 + 18s + 5}{4s + 1}\right)} \times I_2 = \frac{-1}{16s^2 + 36s + 10}$

$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1 = 0} = \frac{I_2}{\frac{(8s^2 + 18s + 5)}{(4s + 1)} \times I_2} = \frac{4s + 1}{8s^2 + 18s + 5}$$

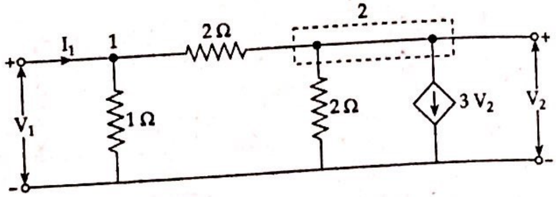
$\therefore \quad Y_{11} = \dfrac{16s^2 + 34s + 6}{32s^2 + 72s + 20} \quad ; \quad Y_{12} = \dfrac{-1}{16s^2 + 36s + 10}$

$\therefore \quad Y_{21} = \dfrac{-1}{16s^2 + 36s + 10} \quad ; \quad Y_{22} = \dfrac{4s + 1}{8s^2 + 18s + 5}$

18. For the following network, find the open circuit parameters. **[2014/Fall]**



Solution:



Applying KCL at node 1,

$$I_1 = \frac{V_1}{1} + \frac{V_1 - V_2}{2}$$

$$I_1 = 3V_1 - V_2 \qquad (1)$$

Applying KCL at node 2,

$$I_2 = 3I_1 + \frac{V_2}{2} + \frac{V_2 - V_1}{2}$$

$$I_2 = 3(3V_1 - V_2) + \frac{2V_2 - V_1}{2}$$

$$= \frac{9V_1 - 6V_2 + 2V_2 - V_1}{2} = \frac{8V_1 - 4V_2}{2}$$

$$\therefore \quad I_2 = 4V_1 - 2V_2 \qquad (2)$$

Comparing equation (1) and (2) with

$I_1 = Y_{11}V_1 + Y_{12}V_2$

$I_2 = Y_{21}V_1 + Y_{22}V_2$

We get,

$Y_{11} = 3 \qquad Y_{12} = -1$

$Y_{21} = 4 \qquad Y_{22} = -2$

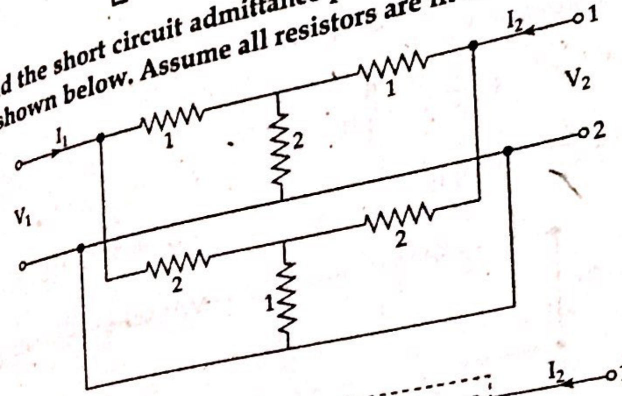
We know, for Z-parameters,

$Z = [Y]^{-1}$

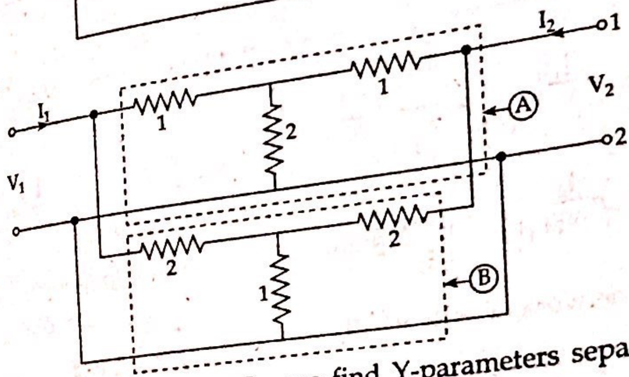
$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}^{-1} = \frac{\ldots}{Y_{11}Y_{22} - Y_{12}\ldots}$$

$$= \begin{bmatrix} \frac{-2}{-2} & \frac{1}{-2} \\ \frac{-4}{-2} & \frac{3}{-2} \end{bmatrix} = \begin{bmatrix} 1 & -0.5 \\ 2 & -1.5 \end{bmatrix}$$

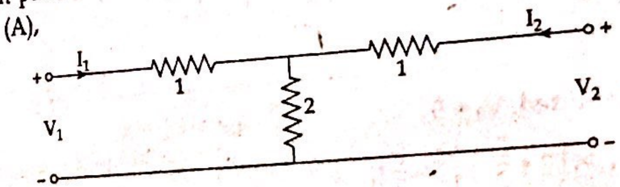
19. *Find the short circuit admittance parameters of two port network as shown below. Assume all resistors are in ohm.* **[2013/Spring]**



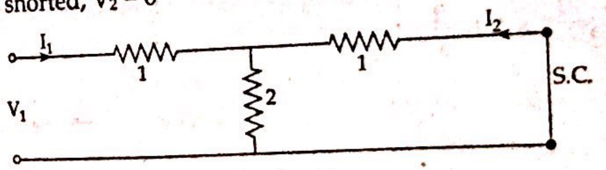
Solution:



This is in parallel connection. So, we find Y-parameters separately. For network (A),



with port 2 shorted, $V_2 = 0$



$$V_1 = I_1 \times \left(1 + \frac{1 \times 2}{1 + 2}\right) = I_1 \times \frac{3 + 2}{3} = \frac{5}{3} I_1$$

$$I_2 = \frac{-2}{2+1} \times I_1 = \frac{-2}{3} I_1$$

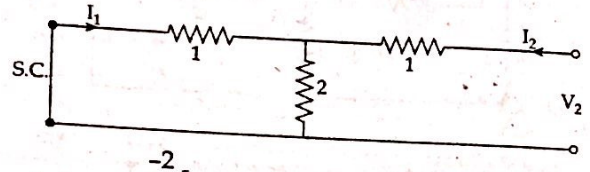
Hence, $Y_{11A} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{I_1}{\frac{5}{3} I_1} = \frac{3}{5}$

$$Y_{21A} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{\frac{-2}{3} I_1}{\frac{5}{3} I_1} = \frac{-2}{5}$$

with port 1 shorted, $V_1 = 0$

$$V_2 = I_2 \times \left(1 + \frac{1 \times 2}{1+2}\right) = \frac{5}{3} I_2$$

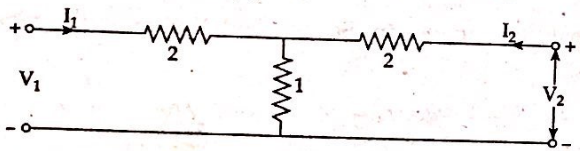
$$I_1 = \frac{-2}{2+1} I_2 = -\frac{2}{3} I_2$$



so, $Y_{12A} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{\frac{-2}{3} I_2}{\frac{5}{3} I_2} = \frac{-2}{5}$

$$Y_{22A} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{I_2}{\frac{5}{3} I_2} = \frac{3}{5}$$

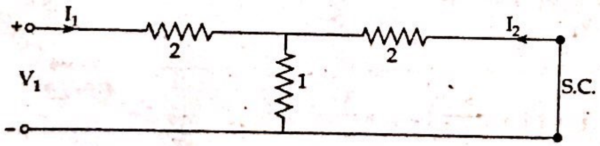
For lower network or network (B),



with port 2 shorted, $V_2 = 0$

$$V_1 = I_1 \times \left(2 + \frac{2 \times 1}{2+1}\right) = \frac{8}{3} I_1$$

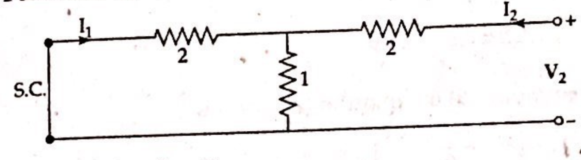
$$I_2 = \frac{-1}{1+2} \times I_1 = \frac{-1}{3} I_1$$



Hence, $Y_{11B} = \left.\frac{I_1}{V_1}\right|_{V_2=0} = \frac{I_1}{\frac{8}{3} I_1} = \frac{3}{8}$

$$Y_{21B} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = \frac{-\frac{1}{3} I_1}{\frac{8}{3} I_1} = \frac{-1}{8}$$

with port 1 shorted, $V_1 = 0$



$$V_2 = I_2 \times \left(2 + \frac{2 \times 1}{2+1}\right) = \frac{8}{3} I_2$$

$$I_2 = \frac{-1}{1+2} \times I_2 = \frac{-1}{3} I_2$$

$$Y_{12B} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{\frac{-1}{3} I_2}{\frac{8}{3} I_2} = \frac{-1}{8}$$

$$Y_{22B} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = \frac{I_2}{\frac{8}{3} I_2} = \frac{3}{8}$$

Now, overall Y-parameter of network is sum of individual parameters of network A and B, so,

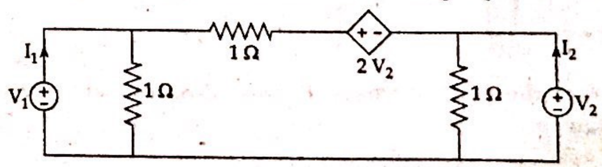
$$Y_{11} = Y_{11A} + Y_{11B} = \frac{3}{5} + \frac{3}{8} = \frac{39}{40}$$

$$Y_{12} = Y_{12A} + Y_{12B} = \frac{-2}{5} + \left(\frac{-1}{8}\right) = \frac{-21}{40}$$

$$Y_{21} = Y_{21A} + Y_{21B} = \frac{-2}{5} - \frac{1}{8} = \frac{-21}{40}$$

$$Y_{22} = Y_{22A} + Y_{22B} = \frac{3}{5} + \frac{3}{8} = \frac{39}{40}$$

20. **Obtain the Z-parameters for the given two port network. Also draw the general equivalent circuit showing Z-parameters.[2019/Spring]**



Solution:

Applying KVL in 1st loop,

$$V_1 = 1(I_1 - I_3) = I_1 - I_3 \quad (1)$$

Applying KVL in 3rd loop,

$$V_2 = 1(I_2 - I_3) = I_2 - I_3 \quad (2)$$

Applying KVL in 2nd loop,

$$1 \times (I_3 - I_1) + 1 \times I_3 + 2V_2 + 1(I_3 + I_2) = 0$$

or, $3I_3 = I_1 - I_2 - 2V_2$

or, $I_3 = \dfrac{I_1 - I_2 - 2V_2}{3} \quad (3)$

Replacing equation (3) in equation (2), we get,

$$V_2 = I_2 + \left(\frac{I_1 - I_2 - 2V_2}{3}\right)$$

or, $V_2 + \dfrac{2}{3}V_2 = \dfrac{1}{3}I_1 + I_2 - \dfrac{I_2}{3}$

or, $\dfrac{5}{3}V_2 = \dfrac{1}{3}I_1 + \dfrac{2}{3}I_2$

or, $V_2 = \dfrac{1}{5}I_1 + \dfrac{2}{5}I_2 \quad (4)$

Replacing equation (3) in (1), we get,

$$V_1 = I_1 - \left(\frac{I_1 - I_2 - 2V_2}{3}\right)$$

or, $V_1 = I_1 - \dfrac{I_1}{3} + \dfrac{I_2}{3} + \dfrac{2V_2}{3}$

Replacing value of V_2 from equation (4), we get,

$$V_1 = \frac{2}{3}I_1 + \frac{I_2}{3} + \frac{2}{3}\left(\frac{1}{5}I_1 + \frac{2}{5}I_2\right)$$

or, $V_1 = \dfrac{4}{5}I_1 + \dfrac{3}{5}I_2 \quad (5)$

Comparing equation (4) and (5) with standard Z-parameter equation,

$$V_1 = Z_{11}I_1 + Z_{12}I_2$$
$$V_2 = Z_{21}I_1 + Z_{22}I_2$$

We get,

$$Z_{11} = \frac{4}{5} \quad ; \quad Z_{12} = \frac{3}{5}$$

$$Z_{21} = \frac{1}{5} \quad ; \quad Z_{22} = \frac{2}{5}$$

21. Define the port networks. Express Z-parameters in terms of T-parameter. **[2019/Fall]**

Solution: See topic 9.1 and 9.8.1 'b'.

22. Explain how equivalent T and π section representation are performed in any network. **[2018/Spring]**

Solution: See topic 9.11, 9.11.1 and 9.11.2.

23. Write short notes on reciprocity and symmetry of a two port network. **[2016/Spring, 2016/Fall, 2011/Fall, 2011/Spring, 2014/Fall, 2014/Spring, 2020/Fall]**

Solution: See topic 9.9 and 9.10.

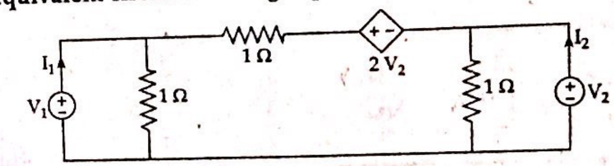
24. Write short notes on hybrid parameter representation of two port network. **[2015/Fall/ 2012/Spring]**

Solution: See topic 9.6.

25. Write short notes on network symmetry. **[2013/Spring]**

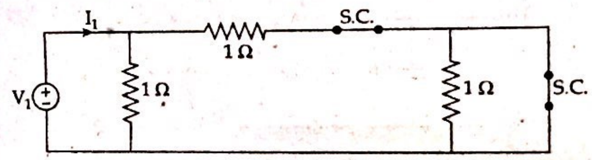
Solution: See topic 9.10.

26. Define Z-parameters for two port network. Obtain the Y-parameter for the given two port network. Also, draw the general equivalent circuit showing Y-parameters. **[2020/Fall]**



Solution:

With port 2 short circuited, $V_2 = 0$



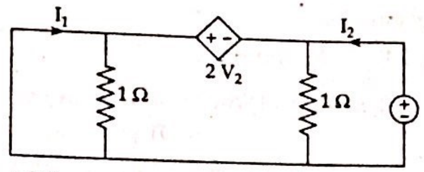
$$V_1 = \left(\frac{1 \times 1}{1+1}\right) \times I_1 = \frac{1}{2}I_1$$

$$I_2 = \left(\frac{-1}{1+1}\right) I_1 = \frac{-1}{2}I_1$$

so, $Y_{11} = \left.\dfrac{I_2}{V_1}\right|_{I_2=0} = \dfrac{I_1}{\frac{1}{2}I_1} = 2$

$$Y_{21} = \left.\frac{I_2}{V_1}\right|_{I_2=0} = \frac{\left(\frac{-1}{2}\right)I_2}{\left(\frac{1}{2}\right)I_1} = -1$$

With port 1 short circuited,



$$I_2 = \frac{V_2}{1} + \frac{V_2 + 2V_2}{1} = 4V_2$$

$$I_1 = -\left(\frac{V_2 + 2V_2}{1}\right) = -3V_2$$

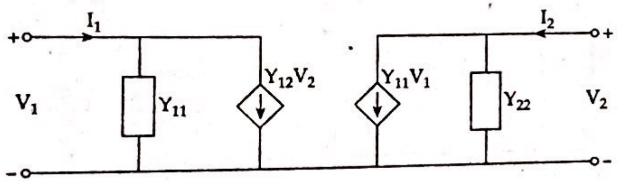
Hence, $Y_{11} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{-3V_2}{V} = -3$

$$Y_{22} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = \frac{4V_2}{V_2} = 4$$

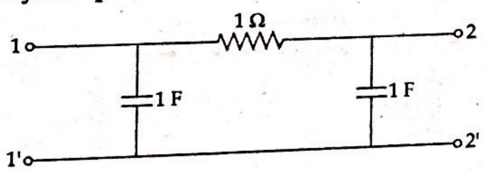
$\therefore$ $Y_{11} = 2;$ $Y_{12} = -3$

$\therefore$ $Y_{21} = -1;$ $Y_{22} = 4$

The equivalent y-parameter circuit is,

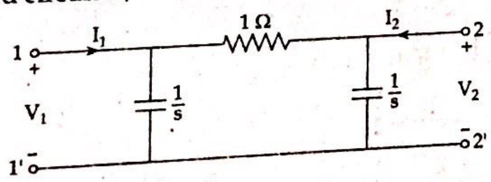


27. Find the hybrid parameters for the given RC network. [2020/Fall]



Solution:

The transformed circuit is,



With port 2 short circuited, $V_2 = 0$

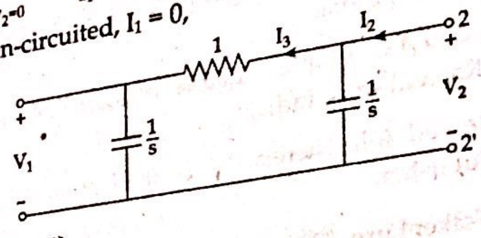
$$V_1 = \frac{\frac{1}{s} \times 1}{\frac{1}{s} + 1} \times I_1 = \left(\frac{1}{s+1}\right) I_1$$

$$I_2 = \frac{\frac{1}{s}}{\frac{1}{s} + 1} \times I_1 = \left(\frac{-1}{s+1}\right) I_1$$

Hence, $h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0} = \frac{\left(\frac{1}{s+1}\right)}{I_1} = \frac{1}{s+1}$

$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0} = \frac{\left(\frac{-1}{s+1}\right)}{I_1} \times I_1 = \frac{-1}{s+1}$$

With port 1 open-circuited, $I_1 = 0$,



$$V_2 = \frac{\frac{1}{s} \times \left(1 + \frac{1}{s}\right)}{\frac{1}{s} + \left(1 + \frac{1}{s}\right)} \times I_2 = \frac{s+1}{s(s+2)} I_2$$

$$I_3 = \frac{\frac{1}{s}}{\left(\frac{1}{s} + 1 + \frac{1}{s}\right)} I_2 = \frac{I_2}{s+2}$$

$$V_1 = I_3 \times \frac{1}{s} = \frac{1}{s(s+2)} I_2$$

Now,

$$h_{12} = \left.\frac{V_1}{I_2}\right|_{V_1=0} = \frac{\frac{1}{s(s+2)} I_2}{\frac{(s+1)}{s(s+2)} I_2} = \frac{1}{s+1}$$

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} = \frac{I_2}{\frac{(s+1)}{s(s+2)} I_2} = \frac{s(s+2)}{(s+1)}$$

$\therefore$ $h_{11} = \frac{1}{s+1};$ $h_{12} = \frac{1}{s+1}$

$\therefore$ $h_{21} = \frac{-1}{s+1};$ $h_{12} = \frac{s(s+2)}{(s+1)}$

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