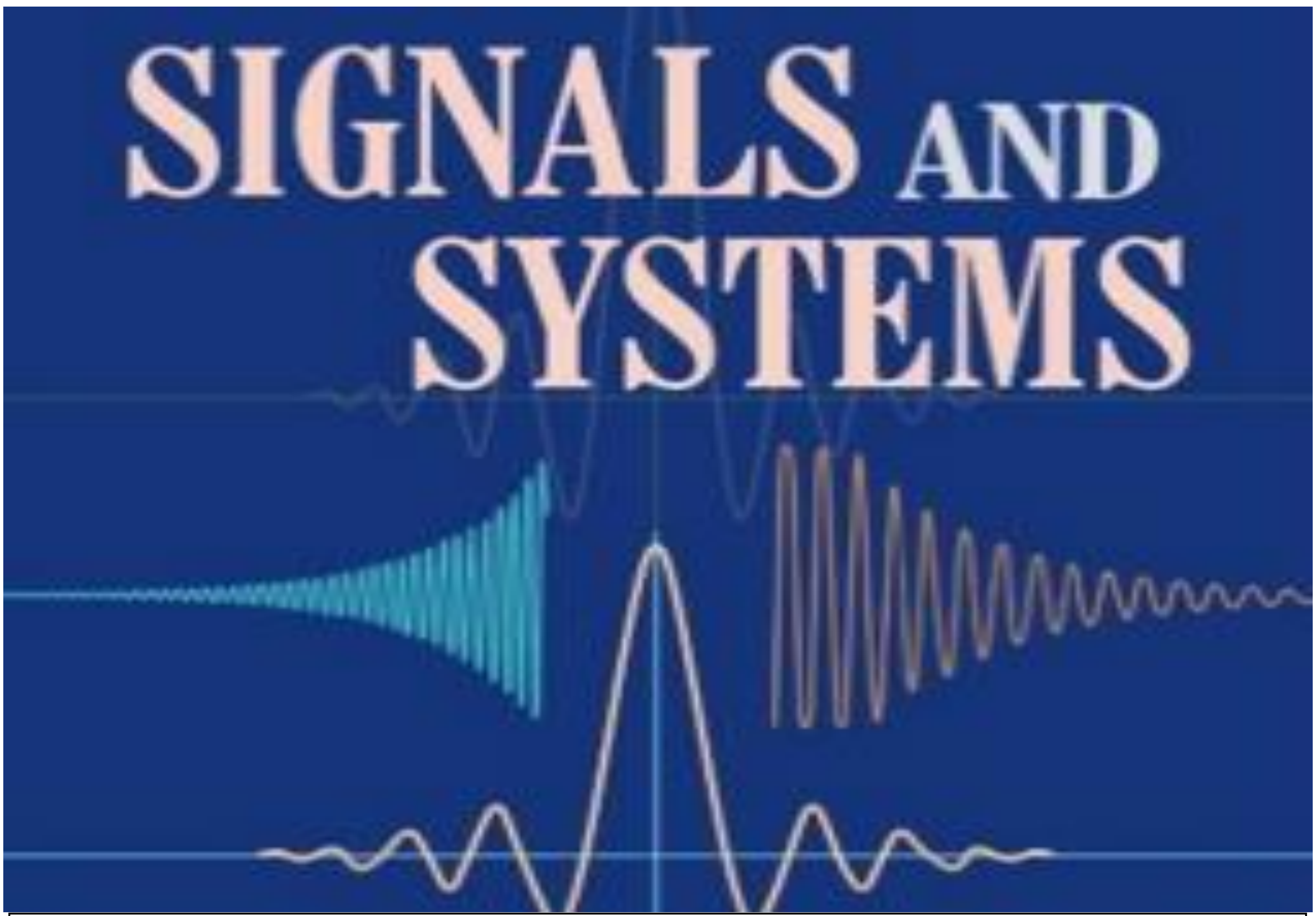
MALLA REDDY COLLEGE

OF ENGINEERING AND TECHNOLOGY

(AUTONOMOUS INSTITUTION: UGC, GOVT. OF INDIA) ELECTRONICS AND COMMUNICATION ENGINEERING



II ECE I SEM

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

II Year B.Tech ECE-I Sem	**L**	**T/P/D**	**C**
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(R17A0402) SIGNALS AND SYSTEMS

OBJECTIVES:

The main objectives of the course are:

1. Knowledge of time-domain representation and analysis concepts of basic elementary signals
2. Knowledge of Fourier Series for Continuous Time Signals
3. Knowledge of frequency-domain representation and analysis concepts F.T., L.T. & Z.T and Concepts of the sampling process.
4. Mathematical and computational skills needed to understand the principal of Linear System and Filter Characteristics of a System.
5. Mathematical and computational skills needed to understand the concepts of auto correlation and cross correlation and power Density Spectrum.

UNIT I:

INTRODUCTION TO SIGNALS: Elementary Signals**-** Continuous Time (CT) signals, Discrete Time (DT) signals, Classification of Signals, Basic Operations on signals.

FOURIER SERIES: Representation of Fourier series, Continuous time periodic signals, Dirichlet's conditions, Trigonometric Fourier Series, Exponential Fourier Series, Properties of Fourier series, Complex Fourier spectrum.

UNIT II:

FOURIER TRANSFORMS: Deriving Fourier transform from Fourier series, Fourier transform of arbitrary signal, Fourier transform of standard signals, Properties of Fourier transforms.

SAMPLING: Sampling theorem **–** Graphical and analytical proof for Band Limited Signals, impulse sampling, Natural and Flat top Sampling, Reconstruction of signal from its samples, effect of under sampling – Aliasing.

UNIT III:

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS: Introduction to Systems, Classification of Systems, Linear Time Invariant (LTI) systems, impulse response, Transfer function of a LTI system. Filter characteristics of linear systems. Distortion less transmission through a system, Signal bandwidth, System bandwidth, Ideal LPF, HPF and BPF characteristics.

UNIT IV:

CONVOLUTION AND CORRELATION OF SIGNALS: Concept of convolution in time domain, Cross correlation and auto correlation of functions, properties of correlation function, Energy density spectrum, Parseval's theorem, Power density spectrum, Relation between convolution and correlation.

UNIT V:

LAPLACE TRANSFORMS: Review of Laplace transforms, Inverse Laplace transform, Concept of region of convergence (ROC) for Laplace transforms, Properties of L.T's relation between L.T's, and F.T. of a signal.

Z–TRANSFORMS: Concept of Z- Transform of a discrete sequence. Distinction between Laplace, Fourier and Z transforms, Region of convergence in Z-Transform, Inverse Z- Transform, Properties of Z-transforms.

TEXT BOOKS:

1. "Signals & Systems", Special Edition – MRCET, McGraw Hill Publications, 2017
2. Signals, Systems & Communications - B.P. Lathi, BS Publications, 2003.
3. Signals and Systems - A.V. Oppenheim, A.S. Willsky and S.H. Nawab, PHI, 2nd Edn.
4. Signals and Systems – A. Anand Kumar, PHI Publications, 3rd edition.

REFERENCE BOOKS:

1. Signals & Systems - Simon Haykin and Van Veen,Wiley, 2nd Edition.
2. Network Analysis - M.E. Van Valkenburg, PHI Publications, 3rd Edn., 2000.
3. Fundamentals of Signals and Systems Michel J. Robert, MGH International Edition, 2008.
4. Signals, Systems and Transforms - C. L. Philips, J. M. Parr and Eve A. Riskin, Pearson education.3rd Edition, 2004.

OUTCOMES:

After completion of the course, the student will be able to:

1. Understand the basic elementary signals
2. Determine the Fourier Series for Continuous Time Signals
3. Analyze the signals using F.T, L.T & Z.T and study the properties of F.T., L.T. & Z.T.
4. Understand the principal of Linear System and Filter Characteristics of a System.
5. Understand the concepts of auto correlation and cross correlation and power Density Spectrum.

Unit I

INTRODUCTION TO SIGNALS :

1.1 INTRODUCTION

Anything that carries information can be called a signal. Signals constitute an important part of our daily life. A signal is defined as a single-valued function of one or more independent variables which contain some information A signal may also be defined as any physical quantity that varies with time, space or any other independent variable. A signal may be represented in time domain or frequency domain. Human speech is a familiar example of a signal. Electric current and voltage are also examples of signals. A signal can be a function of one or more independent variables. A signal may be a function of time, temperature, position, pressure, distance, etc. If a signal depends on only one independent variable, it is called a *one-dimensional signal*, and if a signal depends on two independent variables, it is called a *two-dimensional signal*. In this book we discuss only about one-dimensional signals. In this chapter we discuss about various basic signals available, various operations on signals and classification of signals.

1.2 REPRESENTATION OF DISCRETE-TIME SIGNALS

In general signals may be continuous-time signals or discrete-time signals. Continuous-time signals are defined for all instants of time, whereas discrete-time signals are defined only at discrete instants of time. Continuous-time signals are represented by $x(t)$ and discrete-time signals are represented by $x(n)$ where t and n are independent variables in time domain. Continuous-time signals are represented by a function or a graph.

There are four ways of representing discrete-time signals. They are:

1. Graphical representation
2. Functional representation
3. Tabular representation
4. Sequence representation

1.2.1 Graphical Representation

Consider a signal $x(n)$ with values

$$x(-2) = -3,\quad x(-1) = 2,\quad x(0) = 0,\quad x(1) = 3,\quad x(2) = 1 \quad \text{and} \quad x(3) = 2$$

This discrete-time signal can be represented graphically as shown in Figure 1.1.

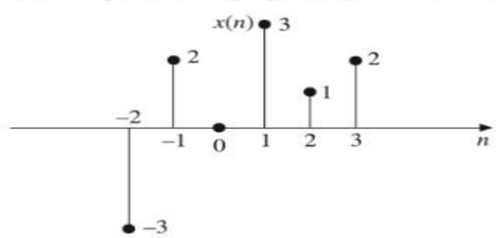


Figure 1.1 Graphical representation of discrete-time signal.

1.2.2 Functional Representation

In this, the amplitude of the signal is written against the values of n. The signal given in 1.2.1 can be represented using functional representation as given below.

$$x(n) = \begin{cases} -3 & \text{for } n = -2 \\ 2 & \text{for } n = -1 \\ 0 & \text{for } n = 0 \\ 3 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \\ 2 & \text{for } n = 3 \end{cases}$$

Another example is

$$x(n) = \begin{cases} 2^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

1.2.3 Tabular Representation

In this, the sampling instant n and the magnitude of the signal at the sampling instant are represented in tabular form. The signal given in 1.2.1 can be represented in tabular form as shown below.

n	-2	-1	0	1	2	3
$x(n)$	-3	2	0	3	1	2

1.2.4 Sequence Representation

A finite duration sequence given in 1.2.1 can be represented as:

$$x(n) = \left\{ \begin{matrix} -3, 2, \underset{\uparrow}{0}, 3, 1, 2 \end{matrix} \right\}$$

Another example is

$$x(n) = \left\{ \begin{matrix} \ldots 2, 3, \underset{\uparrow}{0}, 1, -2, \ldots \end{matrix} \right\}$$

The arrow mark $\uparrow$ denotes the $n = 0$ term. When no arrow is indicated, the first term corresponds to $n = 0$

So a finite duration sequence, that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as $x(n) = \{3, 5, 2, 1, 4, 7\}$.

1.3 ELEMENTARY SIGNALS

There are several elementary signals which play vital role in the study of signals and systems. These elementary signals serve as basic building blocks for the construction of more complex signals. Infact, these elementary signals may be used to model a large number of physical signals which occur in nature. These elementary signals are also called standard signals.

The standard signals are:

1. Unit step function
2. Unit ramp function
3. Unit parabolic function
4. Unit impulse function
5. Sinusoidal function
6. Real exponential function
7. Complex exponential function, etc.

1.3.1 Unit Step Function

The step function is an important signal used for analysis of many systems. The step function is that type of elementary function which exists only for positive time and is zero for negative time. It is equivalent to applying a signal whose amplitude suddenly changes and remains constant forever after application.

If a step function has unity magnitude, then it is called unit step function. The usefulness of the unit-step function lies in the fact that if we want a signal to start at $t = 0$, so that it may have a value of zero for $t < 0$, we only need to multiply the given signal with unit step function $u(t)$. A unit step function is useful as a test signal because the response of the system for a unit step reveals a great deal about how quickly the system responds to a sudden change in the input signal.

The continuous-time unit step function $u(t)$ is defined as:

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

From the above equation for $u(t)$, we can observe that when the argument t in $u(t)$ is less than zero, then the unit step function is zero, and when the argument t in $u(t)$ is greater than or equal to zero, then the unit step function is unity.

The shifted unit step function $u(t - a)$ is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

It is zero if the argument $(t - a) < 0$ and equal to 1 if the argument $(t - a) \geq 0$.
The graphical representations of $u(t)$ and $u(t - a)$ are shown in Figure 1.2[(a) and (b)].

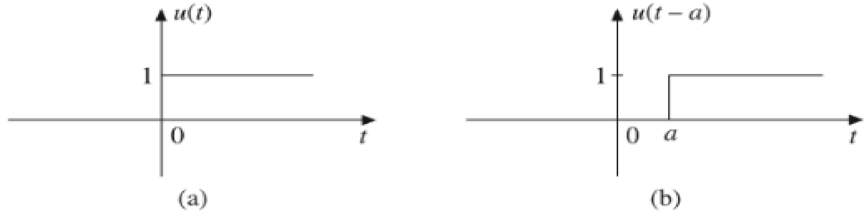


Figure 1.2 (a) Unit step function, (b) Delayed unit step function.

The discrete-time unit step sequence $u(n)$ is defined as:

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

The shifted version of the discrete-time unit step sequence $u(n - k)$ is defined as

$$u(n-k) = \begin{cases} 1 & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

The graphical representations of $u(n)$ and $u(n - k)$ are shown in Figure 1.3[(a) and (b)].

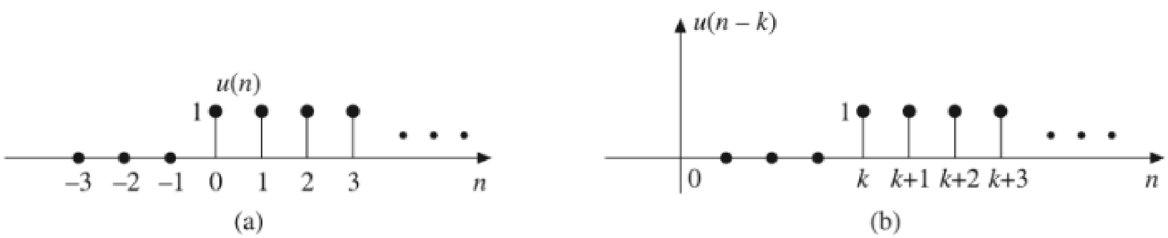


Figure 1.3 Discrete-time (a) Unit step function, (b) Shifted unit step function.

1.3.2 Unit Ramp Function

The continuous-time unit ramp function $r(t)$ is that function which starts at $t = 0$ and increases linearly with time and is defined as:

$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or

$$r(t) = t\,u(t)$$

The unit ramp function has unit slope. It is a signal whose amplitude varies linearly. It can be obtained by integrating the unit step function. That means, a unit step signal can be obtained by differentiating the unit ramp signal.

i.e.

$$r(t) = \int u(t)\,dt = \int dt = t \qquad \text{for } t \geq 0$$

$$u(t) = \frac{d}{dt} r(t)$$

The delayed unit ramp signal $r(t - a)$ is given by

$$r(t-a) = \begin{cases} t-a & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

or

$$r(t-a) = (t-a)\,u(t-a)$$

The graphical representations of $r(t)$ and $r(t - a)$ are shown in Figure 1.4[(a) and (b)].

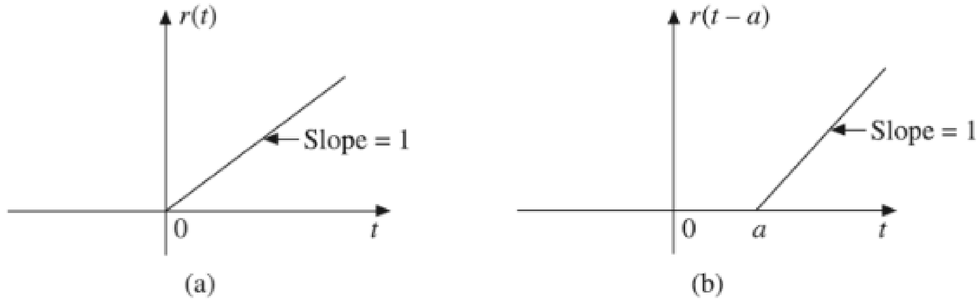


Figure 1.4 (a) Unit ramp signal, (b) Delayed unit ramp signal.

The discrete-time unit ramp sequence $r(n)$ is defined as

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or $$r(n) = n\,u(n)$$

The shifted version of the discrete-time unit-ramp sequence $r(n - k)$ is defined as

$$r(n-k) = \begin{cases} n-k & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

$$r(n-k) = (n-k)\,u(n-k)$$

The graphical representations of $r(n)$ and $r(n - 2)$ are shown in Figure 1.5[(a) and (b)].

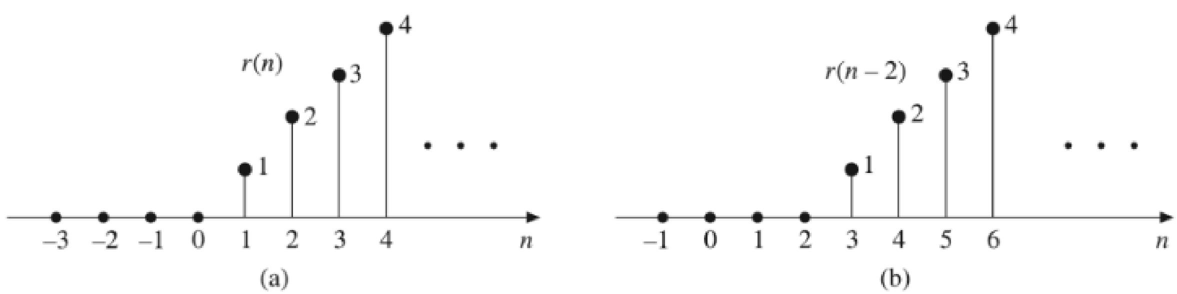


Figure 1.5 Discrete-time (a) Unit-ramp sequence, (b) Shifted-ramp sequence.

1.3.3 Unit Parabolic Function

The continuous-time unit parabolic function $p(t)$, also called unit acceleration signal starts at $t = 0$, and is defined as:

$$p(t) = \begin{cases} \dfrac{t^2}{2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or

$$p(t) = \frac{t^2}{2} u(t)$$

The shifted version of the unit parabolic sequence $p(t - a)$ is given by

$$p(t-a) = \begin{cases} \dfrac{(t-a)^2}{2} & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

or

$$p(t-a) = \frac{(t-a)^2}{2} u(t-a)$$

The graphical representations of $p(t)$ and $p(t - a)$ are shown in Figure 1.6[(a) and (b)].

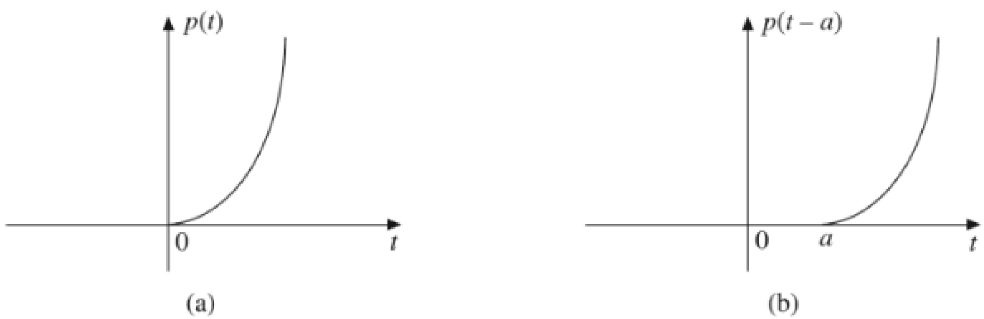


Figure 1.6 (a) Unit parabolic signal, (b) Delayed parabolic signal.

The unit parabolic function can be obtained by integrating the unit ramp function or double integrating the unit step function.

$$p(t) = \iint u(t)\, dt = \int r(t)\, dt = \int t\, dt = \frac{t^2}{2} \quad \text{for } t \geq 0$$

The ramp function is derivative of parabolic function and step function is double derivative of parabolic function

$$r(t) = \frac{d}{dt} p(t); \quad u(t) = \frac{d^2}{dt^2} p(t)$$

The discrete-time unit parabolic sequence $p(n)$ is defined as:

$$p(n) = \begin{cases} \frac{n^2}{2} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or
$$p(n) = \frac{n^2}{2} u(n)$$

The shifted version of the discrete-time unit parabolic sequence $p(n - k)$ is defined as:

$$p(n-k) = \begin{cases} \frac{(n-k)^2}{2} & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

or
$$p(n-k) = \frac{(n-k)^2}{2} u(n-k)$$

The graphical representations of $p(n)$ and $p(n - 3)$ are shown in Figure 1.7[(a) and (b)].

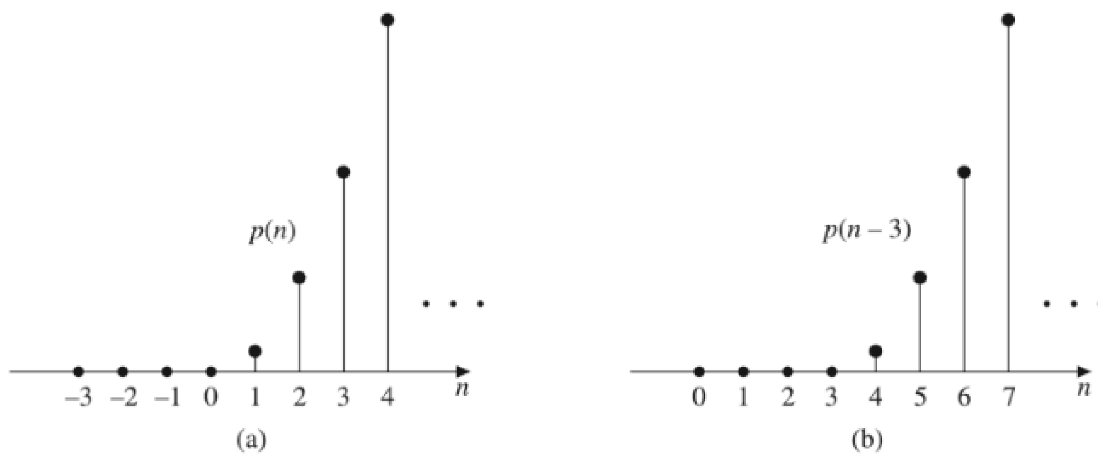


Figure 1.7 Discrete-time (a) Parabolic sequence, (b) Shifted parabolic sequence.

1.3.4 Unit Impulse Function

The unit impulse function is the most widely used elementary function used in the analysis of signals and systems. The continuous-time unit impulse function $\delta(t)$, also called Dirac delta function, plays an important role in signal analysis. It is defined as:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

and $$\delta(t) = 0 \quad \text{for } t \neq 0$$

i.e. as $$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

That is, the impulse function has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude is infinity so that the area under the curve is unity. $\delta(t)$ can be represented as a limiting case of a rectangular pulse function.

As shown in Figure 1.8(a),

$$x(t) = \frac{1}{\Delta}[u(t) - u(t - \Delta)]$$

$$\delta(t) = \underset{\Delta \to 0}{\text{Lt}}\; x(t) = \underset{\Delta \to 0}{\text{Lt}}\; \frac{1}{\Delta}[u(t) - u(t - \Delta)]$$

A delayed unit impulse function $\delta(t - a)$ is defined as:

$$\delta(t-a) = \begin{cases} 1 & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$$

The graphical representations of $\delta(t)$ and $\delta(t - a)$ are shown in Figure 1.8[(b) and (c)].

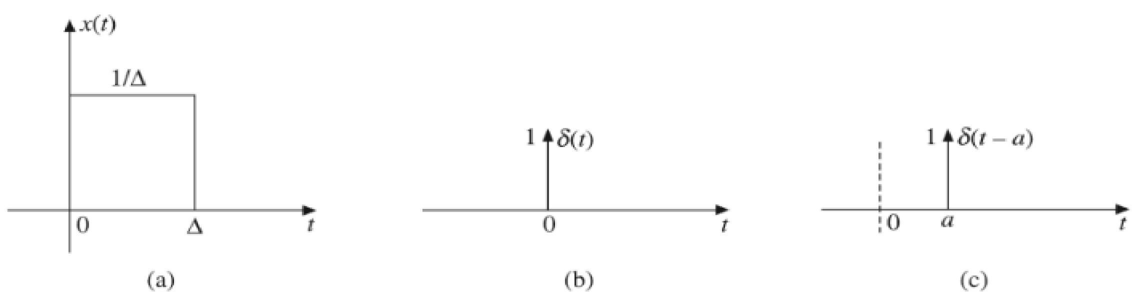


Figure 1.8 (a) $\delta(t)$ as limiting case of a pulse, (b) Unit impulse, (c) Delayed unit impulse.

If unit impulse function is assumed in the form of a pulse, then the following points may be observed about a unit impulse function.

(i) The width of the pulse is zero. This means the pulse exists only at $t = 0$.
(ii) The height of the pulse goes to infinity.
(iii) The area under the pulse curve is always unity.
(iv) The height of arrow indicates the total area under the impulse.

The integral of unit impulse function is a unit step function and the derivate of unit step function is a unit impulse function.

$$u(t) = \int_{-\infty}^{\infty} \delta(t)\, dt$$

and $$\delta(t) = \frac{d}{dt}u(t)$$

Properties of continuous-time unit impulse function

1. It is an even function of time t, i.e. $\delta(t) = \delta(-t)$

2. $\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$; $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$

3. $\delta(at) = \dfrac{1}{|a|}\delta(t)$

4. $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0) = x(t_0)$; $x(t)\,\delta(t) = x(0)\,\delta(t) = x(0)$

5. $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

The discrete-time unit impulse function $\delta(n)$, also called unit sample sequence, is defined as:

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

The shifted unit impulse function $\delta(n - k)$ is defined as:

$$\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$$

The graphical representations of $\delta(n)$ and $\delta(n - 3)$ are shown in Figure 1.9[(a) and (b)].

Figure 1.9 Discrete-time (a) Unit sample sequence, (b) Delayed unit sample sequence.

Properties of discrete-time unit sample sequence

1. $\delta(n) = u(n) - u(n-1)$

2. $\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$

3. $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$

4. $\sum_{n=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$

1.3.5 Sinusoidal Signal

A continuous-time sinusoidal signal in its most general form is given by

$$x(t) = A\sin(\omega t + \phi)$$

where

A = Amplitude
ω = Angular frequency in radians
ϕ = Phase angle in radians

Figure 1.10 shows the waveform of a sinusoidal signal. A sinusoidal signal is an example of a periodic signal. The time period of a continuous-time sinusoidal signal is given by

$$T = \frac{2\pi}{\omega}$$

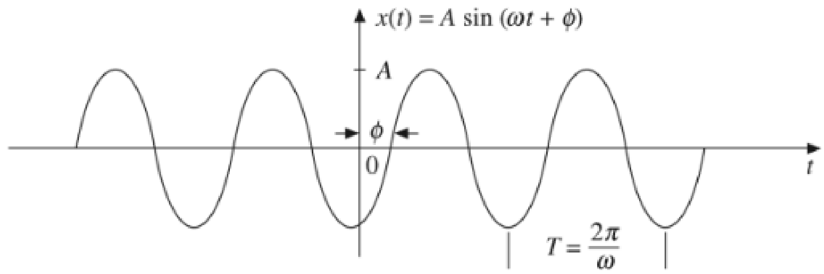


Figure 1.10 Sinusoidal waveform.

The discrete-time sinusoidal sequence is given by

$$x(n) = A \sin(\omega n + \phi)$$

where A is the amplitude, ω is angular frequency, ϕ is phase angle in radians and n is an integer.

The period of the discrete-time sinusoidal sequence is:

$$N = \frac{2\pi}{\omega} m$$

where N and m are integers.

All continuous-time sinusoidal signals are periodic but discrete-time sinusoidal sequences may or may not be periodic depending on the value of ω.

For a discrete-time signal to be periodic, the angular frequency ω must be a rational multiple of 2π.

The graphical representation of a discrete-time sinusoidal signal is shown in Figure 1.11.

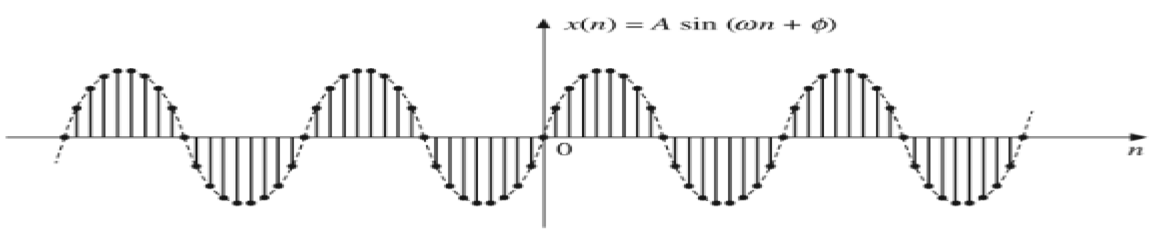


Figure 1.11 Discrete-time sinusoidal signal.

1.3.6 Real Exponential Signal

A continuous-time real exponential signal has the general form as:

$$x(t) = Ae^{\alpha t}$$

where both A and α are real.

The parameter A is the amplitude of the exponential measured at $t = 0$. The parameter α can be either positive or negative. Depending on the value of α, we get different exponentials.

1. If $\alpha = 0$, the signal $x(t)$ is of constant amplitude for all times.
2. If α is positive, i.e. $\alpha > 0$, the signal $x(t)$ is a growing exponential signal.
3. If α is negative, i.e. $\alpha < 0$, the signal $x(t)$ is a decaying exponential signal.

These three waveforms are shown in Figure 1.12[(a), (b) and (c)].

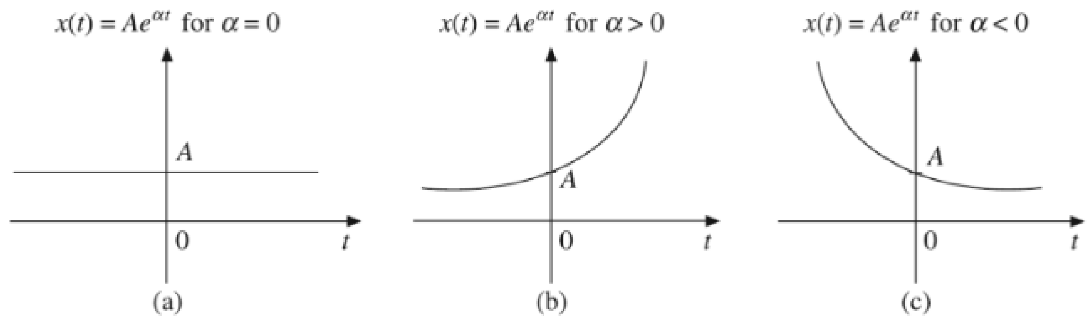


Figure 1.12 Continuous-time real exponential signals $x(t) = Ae^{\alpha t}$ for (a) $\alpha = 0$, (b) $\alpha > 0$, (c) $\alpha < 0$.

The discrete-time real exponential sequence a^n is defined as:

$$x(n) = a^n \quad \text{for all } n$$

Figure 1.13 illustrates different types of discrete-time exponential signals.

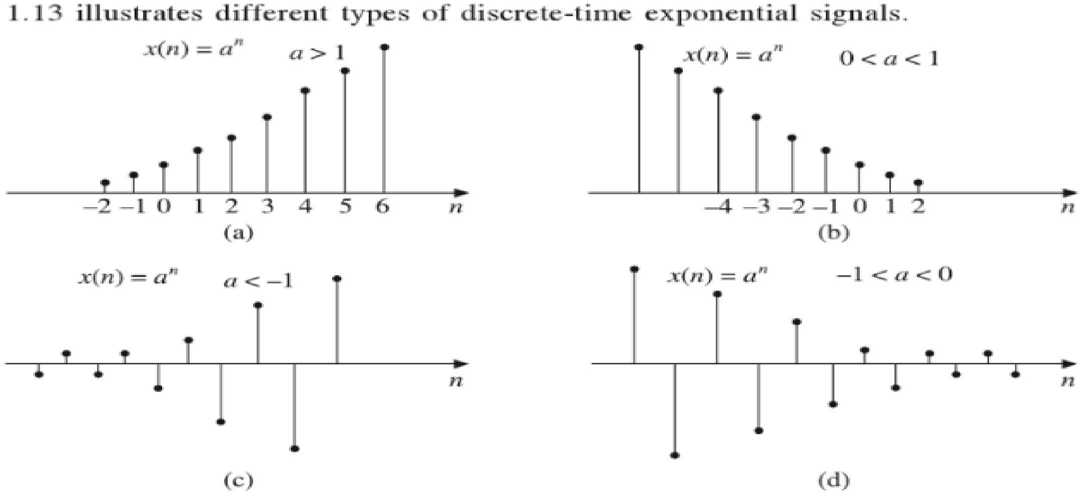


Figure 1.13 Discrete-time exponential signal a^n for (a) $a > 1$, (b) $0 < a < 1$, (c) $a < -1$, (d) $-1 < a < 0$.

When $a > 1$, the sequence grows exponentially as shown in Figure 1.13(a).

When $0 < a < 1$, the sequence decays exponentially as shown in Figure 1.13(b).

When $a < 0$, the sequence takes alternating signs as shown in Figure 1.13[(c) and (d)].

1.3.7 Complex Exponential Signal

The complex exponential signal has a general form as

$$x(t) = Ae^{st}$$

where A is the amplitude and s is a complex variable defined as

$$s = \sigma + j\omega$$

Therefore,
$$x(t) = Ae^{st} = Ae^{(\sigma + j\omega)t} = Ae^{\sigma t}e^{j\omega t}$$
$$= Ae^{\sigma t}[\cos \omega t + j \sin \omega t]$$

Depending on the values of σ and ω, we get different waveforms as shown in Figure 1.14.

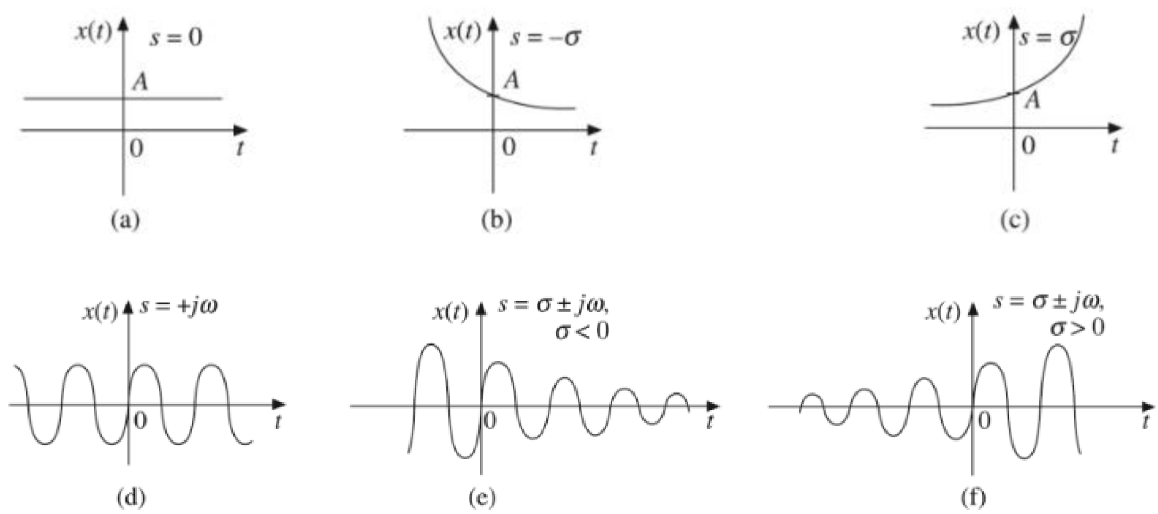


Figure 1.14 Complex exponential signals.

The discrete-time complex exponential sequence is defined as

$$x(n) = a^n e^{j(\omega_0 n + \phi)}$$
$$= a^n \cos (\omega_0 n + \phi) + ja^n \sin (\omega_0 n + \phi)$$

For $|a| = 1$, the real and imaginary parts of complex exponential sequence are sinusoidal.

For $|a| > 1$, the amplitude of the sinusoidal sequence exponentially grows as shown in Figure 1.15(a).

For $|a| < 1$, the amplitude of the sinusoidal sequence exponentially decays as shown in Figure 1.15(b)

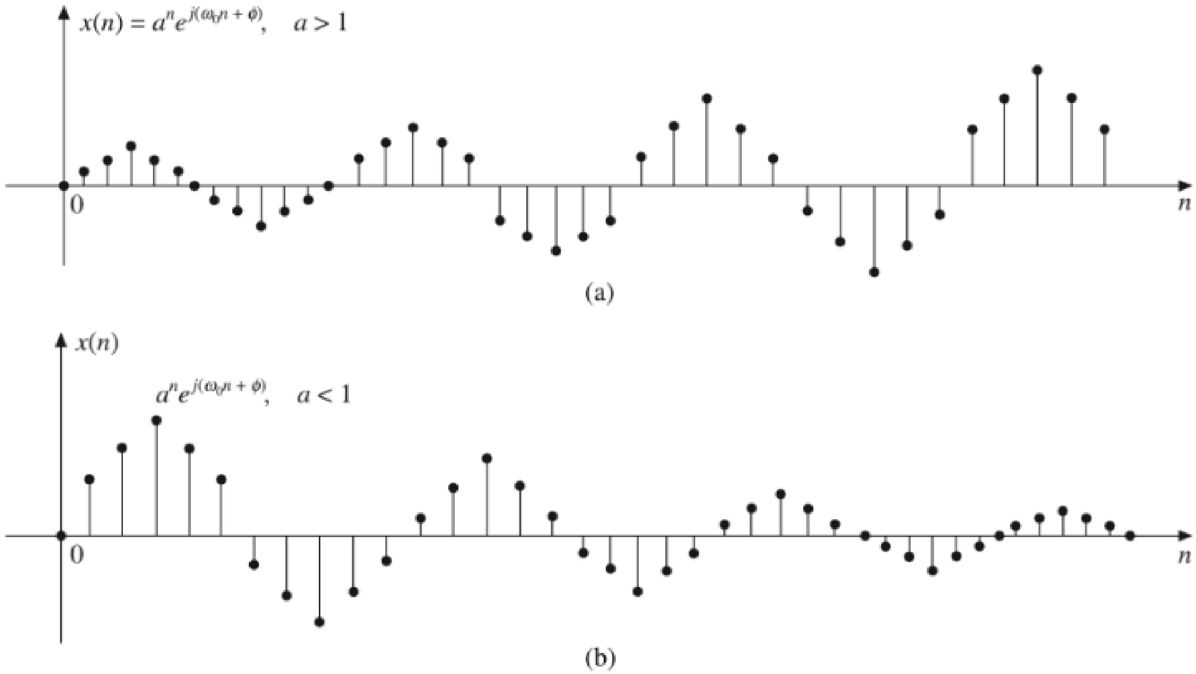


Figure 1.15 Complex exponential sequence $x(n) = a^n e^{j(\omega_0 n + \phi)}$ for (a) $a > 1$, (b) $a < 1$.

1.3.8 Rectangular Pulse Function

The unit rectangular pulse function $\Pi(t/\tau)$ shown in Figure 1.16 is defined as

$$\Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for } |t| \le \dfrac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

It is an even function of t.

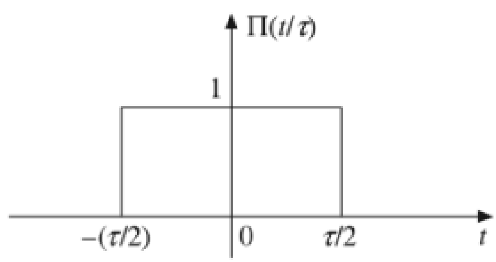


Figure 1.16 Rectangular pulse function.

1.3.10 Signum Function

The unit signum function sgn(t) shown in Figure 1.18 is defined as:

$$\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

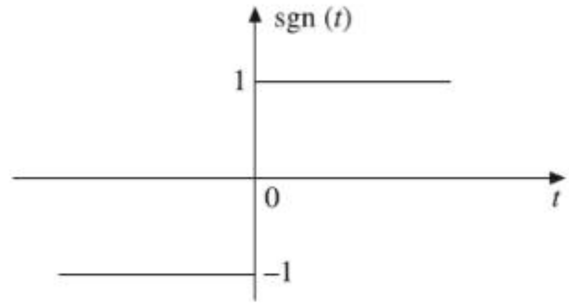


Figure 1.18 Signum function.

The signum function can be expressed in terms of unit step function as:

$$\text{sgn}(t) = -1 + 2u(t)$$

1.3.11 Sinc Function

The sinc function sinc(t) shown in Figure 1.19 is defined as:

$$\text{sinc}(t) = \frac{\sin t}{t} \quad \text{for } -\infty < t < \infty$$

The sinc function oscillates with period 2π and decays with increasing t. Its value is zero at $n\pi$, $n = \pm1, \pm2, \ldots$. It is an even function of t.

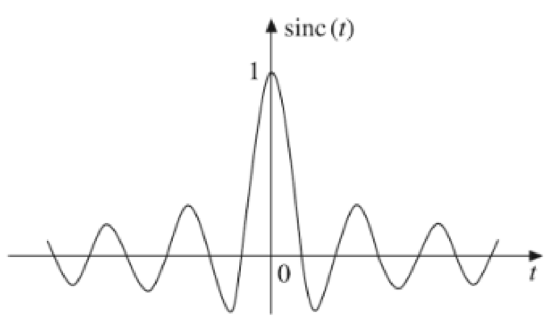


Figure 1.19 Sinc function.

1.4 BASIC OPERATIONS ON SIGNALS

When we process a signal, this signal may undergo several manipulations involving the independent variable or the amplitude of the signal. The basic operations on signals are as follows:

1. Time shifting
2. Time reversal
3. Time scaling
4. Amplitude scaling
5. Signal addition
6. Signal multiplication

The first three operations correspond to transformation in independent variable t or n of a signal. The last three operations correspond to transformation on amplitude of a signal.

1.4.1 Time Shifting

Mathematically, the time shifting of a continuous-time signal $x(t)$ can be represented by

$$y(t) = x(t - T)$$

The time shifting of a signal may result in time delay or time advance. In the above equation if T is positive the shift is to the right and then the shifting delays the signal, and if T is negative the shift is to the left and then the shifting advances the signal. An arbitrary signal $x(t)$, its delayed version and advanced version are shown in Figure 1.21[(a), (b) and (c)]. Shifting a signal in time means that a signal may be either advanced in the time axis or delayed in the time axis.

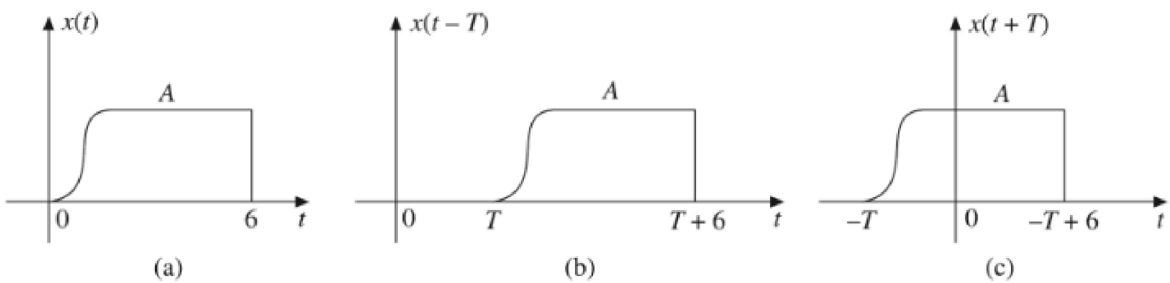


Figure 1.21 (a) Signal, (b) Its delayed version, (c) Its time advanced version.

1.4.2 Time Reversal

The time reversal, also called time folding of a signal $x(t)$ can be obtained by folding the signal about $t = 0$. This operation is very useful in convolution. It is denoted by $x(-t)$. It is obtained by replacing the independent variable t by $(-t)$. Folding is also called as the reflection of the signal about the time origin $t = 0$. Figure 1.23(a) shows an arbitrary signal $x(t)$, and Figure 1.23(b) shows its reflection $x(-t)$.

The signal $x(-t + 3)$ obtained by shifting the reversed signal $x(-t)$ to the right by 3 units (delay by 3 units) is shown in Figure 1.23(c). The signal $x(-t - 3)$ obtained by shifting the reversed signal $x(-t)$ to the left by 3 units (advance by 3 units) is shown in Figure 1.23(d).

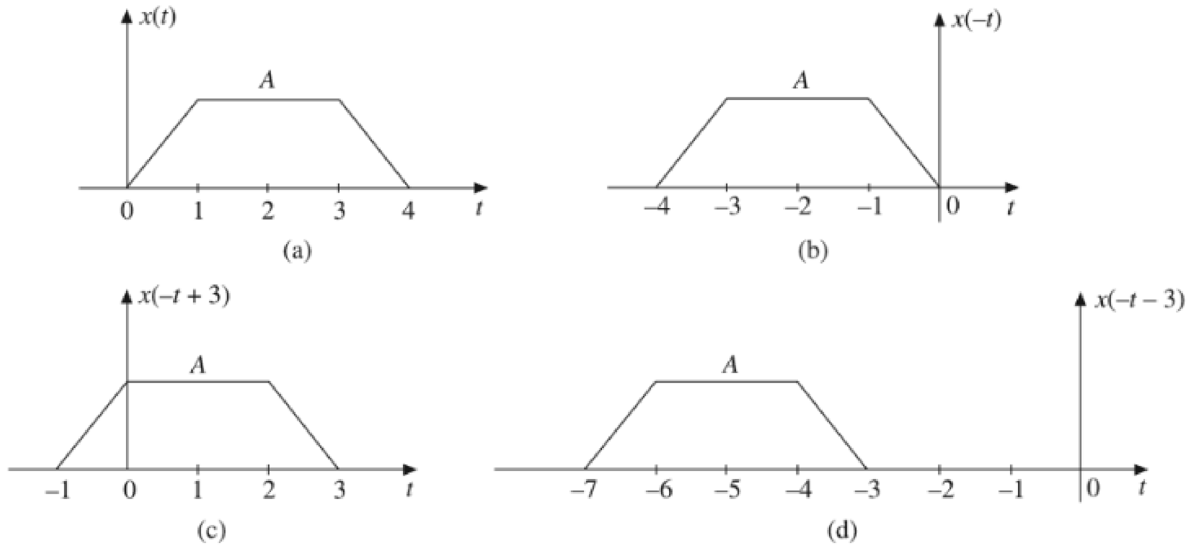


Figure 1.23 (a) An arbitrary signal $x(t)$, (b) Time reversed signal $x(-t)$, (c) Time reversed

1.4.3 Amplitude Scaling

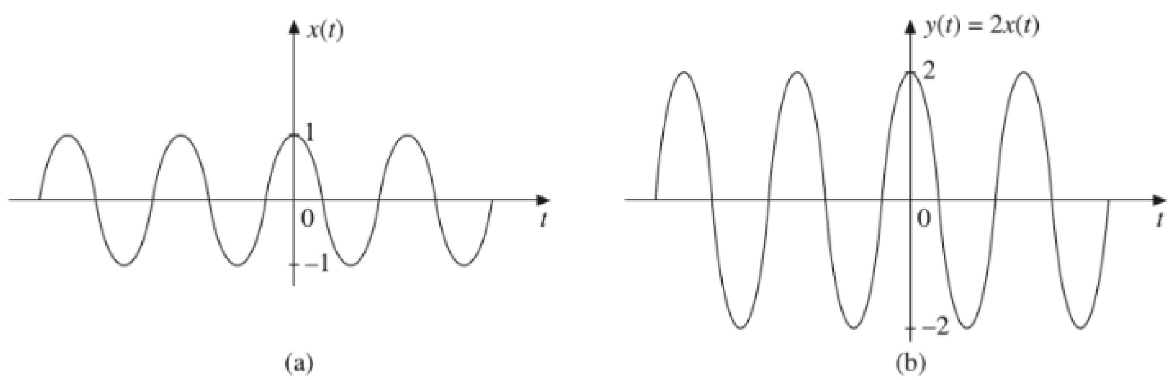
The amplitude scaling of a continuous-time signal $x(t)$ can be represented by

$$y(t) = Ax(t)$$

where A is a constant.

The amplitude of $y(t)$ at any instant is equal to A times the amplitude of $x(t)$ at that instant, but the shape of $y(t)$ is same as the shape of $x(t)$. If $A > 1$, it is amplification and if $A < 1$, it is attenuation.

Here the amplitude is rescaled. Hence the name amplitude scaling. Figure 1.35(a) shows an arbitrary signal $x(t)$ and Figure 1.35(b) shows $y(t) = 2x(t)$.



1.4.4 Time Scaling

Time scaling may be time expansion or time compression. The time scaling of a signal $x(t)$ can be accomplished by replacing t by at in it. Mathematically, it can be expressed as:

$$y(t) = x(at)$$

If $a > 1$, it results in time compression by a factor a and if $a < 1$, it results in time expansion by a factor a because with that transformation a point at 'at' in signal $x(t)$ becomes a point at 't' in $y(t)$.

Consider a signal shown in Figure 1.37(a). For a transformation $y(t) = x(2t)$, the time compressed signal is as shown in Figure 1.37(b) and for a transformation $y(t) = x(t/2)$ the time expanded signal is as shown in Figure 1.37(c).

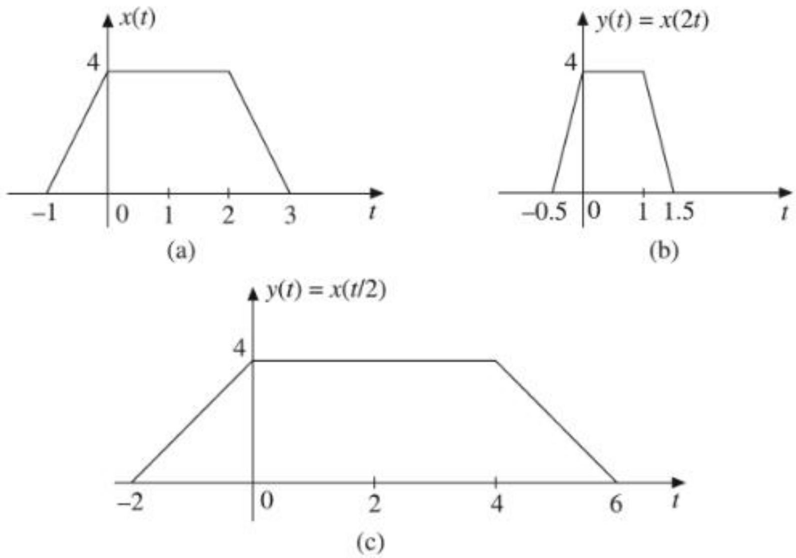


Figure 1.37 (a) Original signal, (b) Compressed signal, (c) Enlarged signal.

1.4.5 Signal Addition

The sum of two continuous-time signals $x_1(t)$ and $x_2(t)$ can be obtained by adding their values at every instant of time. Similarly, the subtraction of one continuous-time signal $x_2(t)$ from another signal $x_1(t)$ can be obtained by subtracting the value of $x_2(t)$ from that of $x_1(t)$ at every instant. Consider two signals $x_1(t)$ and $x_2(t)$ shown in Figure 1.39[(a) and (b)].

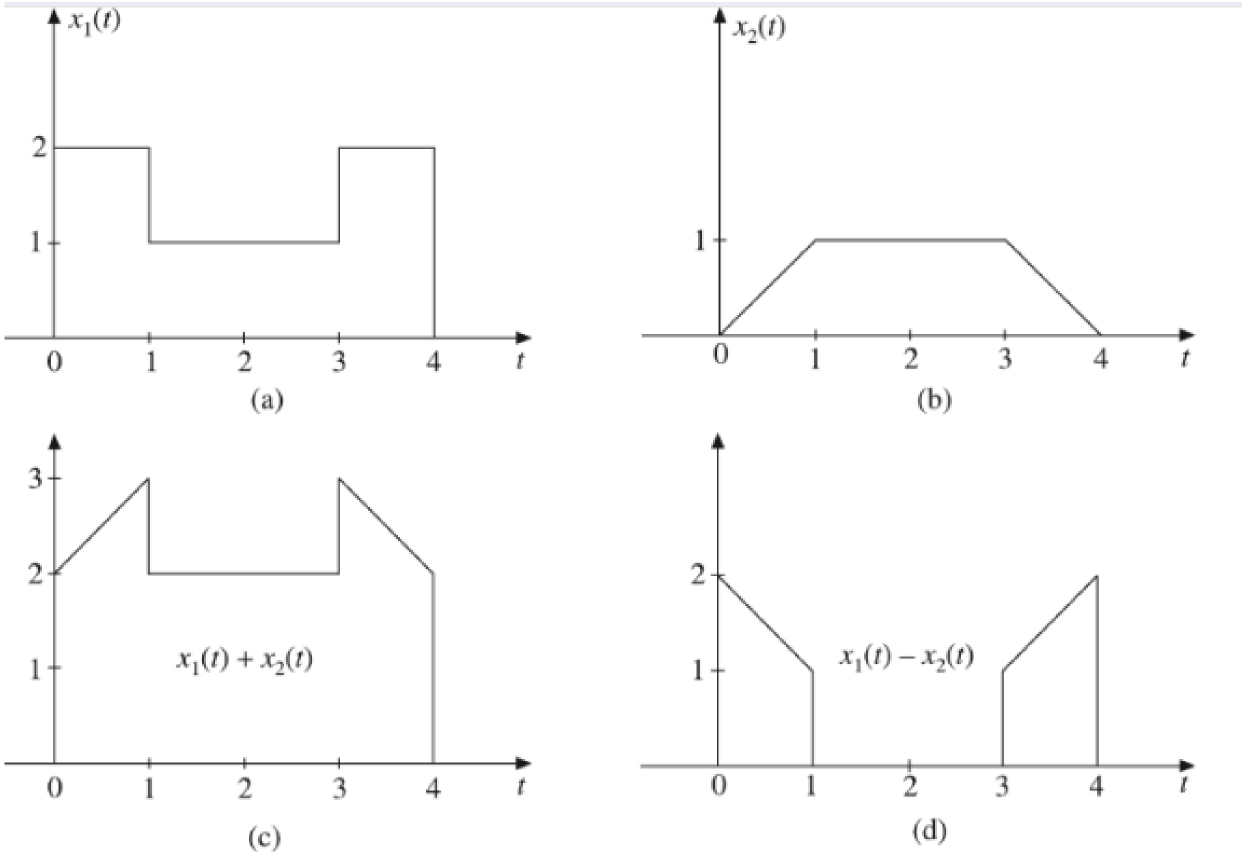


Figure 1.39 Addition and subtraction of continuous-time signals.

1.4.6 Signal Multiplication

The multiplication of two continuous-time signals can be performed by multiplying their values at every instant. Two continuous-time signals $x_1(t)$ and $x_2(t)$ shown in Figure 1.40[(a) and (b)] are multiplied as shown below to obtain $x_1(t)\,x_2(t)$ shown in Figure 1.40(c).

For $0 \le t \le 1$ $\quad x_1(t) = 2$ and $x_2(t) = 1$

Hence $x_1(t)\,x_2(t) = 2 \times 1 = 2$

For $1 \le t \le 2$ $\quad x_1(t) = 1$ and $x_2(t) = 1 + (t - 1)$

Hence $x_1(t)\,x_2(t) = (1)[1 + (t - 1)] = 1 + (t - 1)$

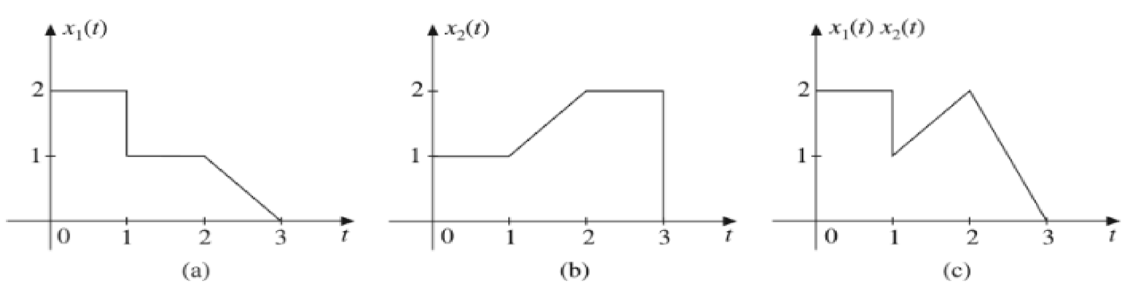


Figure 1.40 Multiplication of continuous-time signals.

1.5 CLASSIFICATION OF SIGNALS

Based upon their nature and characteristics in the time domain, the signals may be broadly classified as under

(a) Continuous-time signals

(b) Discrete-time signals

Continuous-time signals

The signals that are defined for every instant of time are known as continuous-time signals. Continuous-time signals are also called analog signals. For continuous-time signals, the independent variable is time. They are denoted by $x(t)$. They are continuous in amplitude as well as in time. Most of the signals available are continuous-time signals. Figure 1.57(a) and (b) shows the graphical representation of continuous-time signals.

Discrete-time signals

The signals that are defined only at discrete instants of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude but discrete in time. For discrete-time signals, the amplitude between two time instants is just not defined. For discrete-time signals, the independent variable is time n. Since they are defined only at discrete instants of time, they are denoted by a sequence $x(nT)$ or simply by $x(n)$ where n is an integer.

The discrete-time signals may be inherently discrete or may be discrete versions of the continuous-time signals. Figure 1.57(c) and (d) show the graphical representation of discrete-time signals.

Both continuous-time and discrete-time signals may further be classified as under

1. Deterministic and random signals
2. Periodic and non-periodic signals
3. Energy and power signals
4. Causal and non-causal signals
5. Even and odd signals

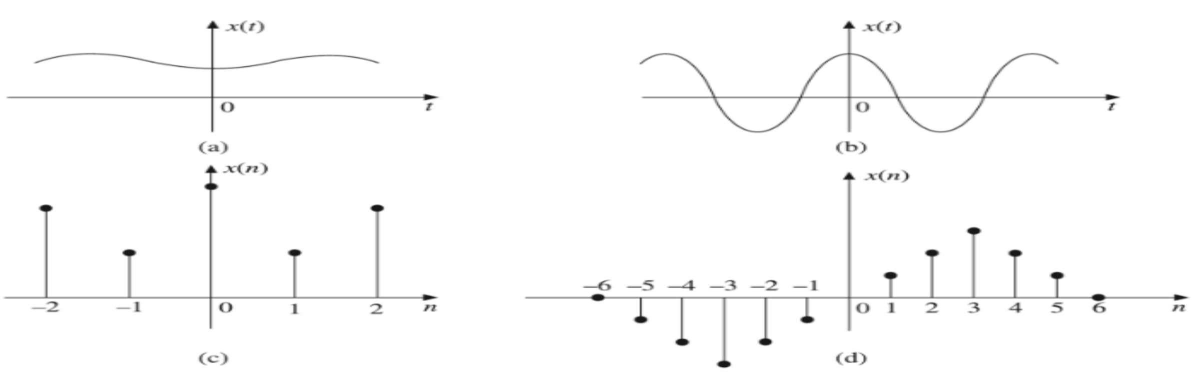


Figure 1.57 (a) and (b) Continuous-time signals, (c) and (d) Discrete-time signals.

1.5.1 Deterministic and Random Signals

A signal exhibiting no uncertainty of its magnitude and phase at any given instant of time is called deterministic signal. A deterministic signal has a regular pattern and can be completely represented by mathematical equation at any time. Its amplitude and phase at any time instant can be predicted in advance.

Examples: Sine wave, $x(t) = \cos \omega t$ or $x(n) = \cos \omega n$, Exponential signals, square wave, triangular wave, etc.

A signal characterized by uncertainty about its occurrence is called a random signal. A random signal cannot be represented by any mathematical equation. The pattern of such a signal is quite irregular. Its amplitude and phase at any time instant cannot be predicted in advance.

A typical example of non deterministic signals is thermal noise generated in an electric circuit. Such a noise has a probabilistic nature.

Figure 1.58 shows the graphical representation of deterministic and random signals.

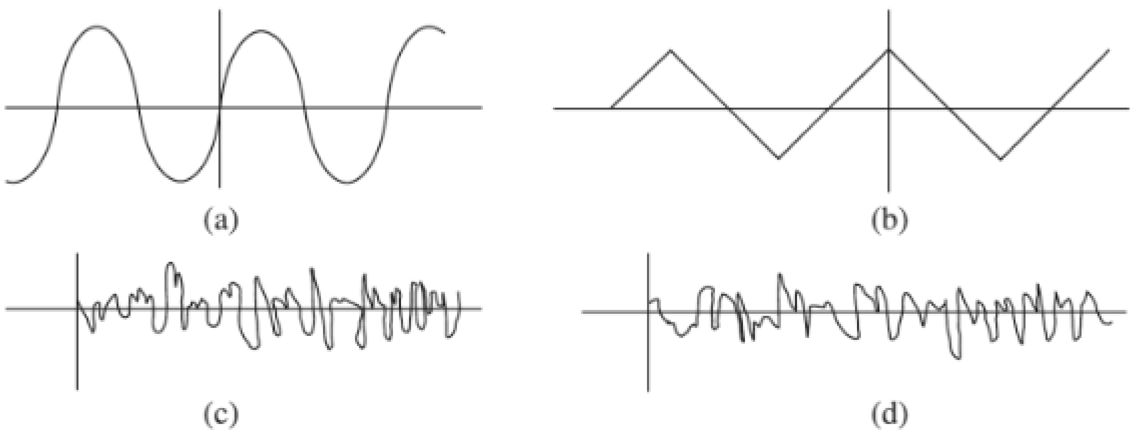


Figure 1.58 (a) and (b) Deterministic signals, (c) and (d) Random signals.

1.5.3 Energy and Power Signals

Signals may also be classified as energy signals and power signals. However there are some signals which can neither be classified as energy signals nor power signals.

- A signal is said to be an energy signal if and only if its total energy E is finite (i.e. $0 < E < \infty$). For an energy signal, average power $P = 0$. Non-periodic signals are examples of energy signals.
- A signal is said to be a power signal if its average power P is finite (i.e. $0 < P < \infty$). For a power signal, total energy $E = \infty$. Periodic signals are examples of power signals.
- Both energy and power signals are mutually exclusive, i.e. no signal can be both energy signal and power signal.
- The signals that do not satisfy the above properties are neither energy signals nor power signals.

1.5.4 Causal and Non-causal Signals

A continuous-time signal $x(t)$ is said to be causal if $x(t) = 0$ for $t < 0$, otherwise the signal is non-causal. A continuous-time signal $x(t)$ is said to be anti-causal if $x(t) = 0$ for $t > 0$.

A causal signal does not exist for negative time and an anti-causal signal does not exist for positive time. A signal which exists in positive as well as negative time is neither causal nor anti-causal. It is non-causal. $u(t)$ is a causal signal and $u(-t)$ is anti-causal signal.

Similarly, a discrete-time signal $x(n)$ is said to be causal if $x(n) = 0$ for $n < 0$, otherwise the signal is non-causal. A discrete-time signal $x(n)$ is said to be anti-causal if $x(n) = 0$ for $n > 0$.

Periodic and Non-Periodic Signals

Definition : A signal is said to be periodic if it repeats at regular intervals. Non-periodic signals do not repeat at regular intervals.

1.5.5 Even and Odd Signals

Even (symmetric) signal

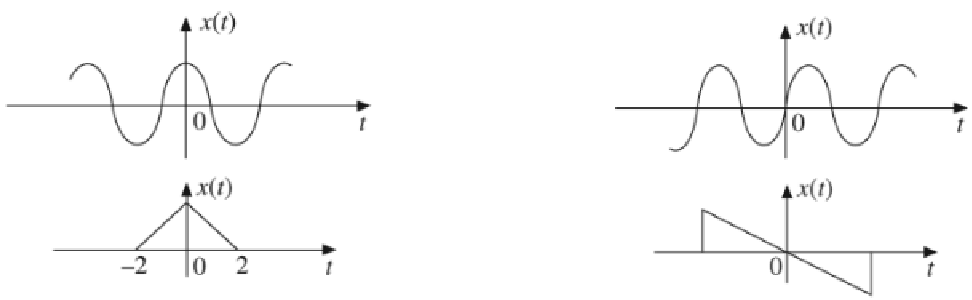
A continuous-time signal $x(t)$ is said to be an even (symmetric) signal if it satisfies the condition

$$x(t) = x(-t) \text{ for all } t$$

A discrete-time signal $x(n)$ is said to be an even (symmetric) signal if it satisfies the condition

$$x(n) = x(-n) \text{ for all } n$$

Even signals are symmetrical about the vertical axis or time origin. Hence they are also called symmetric signals: cosine wave is an example of an even signal. Some even signals are shown in Figure 1.72(a).



Odd (antisymmetric) signal

A continuous-time signal $x(t)$ is said to be an odd (antisymmetric) signal if it satisfies the condition

$$x(-t) = -x(t) \text{ for all } t$$

A discrete-time signal $x(n)$ is said to be an odd (antisymmetric) signal if it satisfies the condition

$$x(-n) = -x(n) \text{ for all } n$$

Odd signals are antisymmetrical about the vertical axis. Hence they are called antisymmetric signals. Sine wave is an example of an odd signal. For an odd signal $x(t) = 0$, $x(n) = 0$. Some odd signals are shown in Figure 1.72(b).

Any signal $x(t)$ can be expressed as sum of even and odd components. That is

$$x(t) = x_e(t) + x_o(t)$$

where $x_e(t)$ is even components and $x_o(t)$ is odd components of the signal.

Evaluation of even and odd parts of a signal

We have $$x(t) = x_e(t) + x_o(t)$$

$\therefore$ $$x(-t) = x_e(-t) + x_o(-t) = x_e(t) - x_o(t)$$

$$x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t) = 2x_e(t)$$

$\therefore$ $$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x(t) - x(-t) = [x_e(t) + x_o(t)] - [x_e(t) - x_o(t)] = 2x_o(t)$$

$\therefore$ $$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Similarly, the even and odd parts of a discrete-time signal, $x_e(n)$ and $x_o(n)$ are given by

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

and

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

The product of two even or odd signals is an even signal and the product of even signal and odd signal is an odd signal.

We can prove this as follows:

Let $$x(t) = x_1(t)\, x_2(t)$$

(i) If $x_1(t)$ and $x_2(t)$ are both even, i.e.

$$x_1(-t) = x_1(t)$$

and $$x_2(-t) = x_2(t)$$

Then $$x(-t) = x_1(-t)\, x_2(-t) = x_1(t)\, x_2(t) = x(t)$$

Therefore, $x(t)$ is an even signal.

(i) If $x_1(t)$ and $x_2(t)$ are both even, i.e.

$$x_1(-t) = x_1(t)$$

and $$x_2(-t) = x_2(t)$$

Then $$x(-t) = x_1(-t)\, x_2(-t) = x_1(t)\, x_2(t) = x(t)$$

Therefore, $x(t)$ is an even signal.

If $x_1(t)$ and $x_2(t)$ are both odd, i.e.

$$x_1(-t) = -x_1(t)$$

and $$x_2(-t) = -x_2(t)$$

Then $$x(-t) = x_1(-t)\, x_2(-t) = [-x_1(t)][-x_2(t)] = x_1(t)\, x_2(t) = x(t)$$

Therefore, $x(t)$ is an even signal.

(ii) If $x_1(t)$ is even and $x_2(t)$ is odd, i.e.

$$x_1(-t) = x_1(t)$$

and $$x_2(-t) = -x_2(t)$$

Then $$x(-t) = x_1(-t)\, x_2(-t) = -x_1(t)\, x_2(t) = -x(t)$$

Therefore $x(t)$ is an odd signal.

Thus, the product of two even signals or of two odd signals is an even signal and the product of even and odd signals is an odd signal.

Every signal need not be either purely even signal or purely odd signal, but every signal can be decomposed into sum of even and odd parts.

FOURIER SERIES:

1. Continuous Time Fourier Series

Purpose

- Fourier series used to analyze periodic signals.
- The harmonic contant of the signals is analyzed with the help of fourier series.
- Fourier series can be developed for continuous time as well as discrete time signals.

Types of Fourier series

Depending upon the representation, these are three types of fourier series

i) Trigonometric Fourier series.

ii) Compact trigonometric Fourier series or polar Fourier series.

iii) Exponential Fourier series.

which satisfy the square integrable condition,

$$\int_T |x(t)|^2 dt < \infty,$$

or Dirichlet conditions (You may find more discussions in OW § 3.4):

1. Over any period $x(t)$ must be *absolutely integrable*, that is,

$$\int_T |x(t)| dt < \infty.$$

2. In any finite interval of time $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

3. In any finite interval of time, there are only a finite number of discontinuities.

For this class of signals, we are able to express it as a linear combination of complex exponentials:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

Here, ω_0 is the fundamental frequency

$$\omega_0 = \frac{2\pi}{T},$$

and the coefficients a_k are known as the Fourier Series coefficients.

Given a periodic signal $x(t)$ that is square integrable, how do we determine the Fourier Series coefficients a_k? This is answered by the following theorem.

3.2.1 Continuous-time Fourier Series Coefficients

Theorem 8. *The continuous-time Fourier series coefficients a_k of the signal*

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

is given by

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt.$$

Fourier Series Representation

Existence of Fourier Series

In general, not every signal $x(t)$ can be decomposed as a linear combination of complex exponentials. However, such decomposition is still possible for an extremely large class of signals. We want to study one class of signals that allows the decomposition. They are the periodic signals

$$x(t+T) = x(t)$$

Trigonometric Fourier series

Defining equations

$$x(t) = a(o) + \sum_{k=1}^{\infty} a(k)\cos k\omega_0 t + \sum_{k=1}^{\infty} b(k)\sin k\omega_0 t$$

where

$$a(0) = \frac{1}{T}\int_{<T>} x(t)dt$$

$$a(k) = \frac{2}{T}\int_{<T>} x(t)\cos k\omega_0 t\, dt$$

$$b(k) = \frac{2}{T}\int_{<T>} x(t)\sin k\omega_0 t\, dt$$

1. Trigonometric Fourier Series

We know that any function $f(t)$ can be expressed as (see equation 1.6.2),

$$f(t) = \sum_{n=1}^{\infty} c_n x_n(t)$$

Here $x_n(t)$ represents orthogonal signal set. This equation is called generalized Fourier series.

We have seen that the set,

$\{1,\ \cos \omega_0 t,\ \cos 2\omega_0 t,\ \ldots,\ \cos n\omega_0 t,\ \ldots\ \sin \omega_0 t,\ \sin 2\omega_0 t,\ \ldots\ \sin n\omega_0 t,\ \ldots\}$

is orthogonal over the period T_0. Here ω_0 is called fundamental frequency. And $n\omega_0$ is called n^{th} harmonic. There is DC component of $\cos n\omega_0 t$ at $n=0$. i.e. $\cos(0\,\omega_0 t)=1$. This signal set consists of sine and cosine terms. Hence it is called *Trigonometric set*.

For this set we can write equation as,

$$f(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \ldots$$

$$+ b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \ldots$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Values of a_n and b_n can be obtained from equation (1.5.5) i.e.,

$$a_n = \frac{\int_t^{t+T_0} f(t) x_n(t)\, dt}{\int_t^{t+T_0} x_n^2(t)\, dt}$$

$$= \frac{\int_t^{t+T_0} f(t) \cos n\omega_0 t\, dt}{\int_t^{t+T_0} \cos^2 n\omega_0 t\, dt} \qquad \ldots (1.8.2)$$

Now consider denominator of above equation with $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$. i.e.,

$$\int_t^{t+T_0} \cos^2 n\omega_0 t\, dt = \frac{1}{2}\int_t^{t+T_0} (1+\cos 2n\omega_0 t)\, dt$$

$$= \frac{1}{2}\int_t^{t+T_0} dt + \frac{1}{2}\int_t^{t+T_0} \cos 2n\omega_0 t$$

Here note that second term is integration of full cycles of cosine wave over one period. It is zero. Hence,

$$\int_t^{t+T_0} \cos^2 n\omega_0 t\, dt = \frac{1}{2}[t]_t^{t+T_0} = \frac{T_0}{2}$$

Therefore equation (1.8.2) becomes,

$$a_n = \frac{2}{T_0}\int_t^{t+T_0} f(t)\cos n\omega_0 t\, dt \qquad \text{... (1.8.3)}$$

Similarly b_n can be calculated as,

$$b_n = \frac{\int_t^{t+T_0} f(t)\sin n\omega_0 t\, dt}{\int_t^{t+T_0} \sin^2 n\omega_0 t\, dt} \qquad \text{... (1.8.4)}$$

Trigonometric fourier series :

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

and

$$a_0 = \frac{1}{T_0}\int_t^{t+T_0} x(t)\, dt$$

$$a_n = \frac{2}{T_0}\int_t^{t+T_0} x(t)\cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T_0}\int_t^{t+T_0} x(t)\sin(n\omega_0 t)\, dt$$

Compact Trigonometric Fourier Series

The trigonometric Fourier series can be represented in compact form. It is also called compact or polar Fourier series.

Defining equations

$$x(t) = D(0) + \sum_{k=1}^{\infty} D(k)\cos(k\omega_0 t + \phi(k))$$

where

$$D(0) = a_0 = \frac{1}{T}\int_{<T>} x(t)dt$$

$$D(k) = \sqrt{a(k)^2 + b(k)^2} \text{ and } \phi(k) = -\tan^{-1}\left(\frac{b(k)}{a(k)}\right)$$

Exponential Fourier Series

Defining equations

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \quad \text{(synthesis equation)}$$

where

$$X(k) = \frac{1}{T}\int_{<T>} e^{-jk\omega_0 t}dt \quad \text{(analysis equation)}$$

Convergence of Fourier Series - Dirichlet Conditions

The Fourier series is convergent if the signal $x(t)$ satisfies some conditions. These conditions are called Dirichlet conditions.

i) **Single valued property :** $x(t)$ must have only one value at any time instant within the interval T_0.

ii) **Finite discontinuities :** $x(t)$ should have at the most finite number of discontinuities in the interval T_0. Because of this, the signal can be represented mathematically.

iii) **Finite peaks :** The signal $x(t)$ should have finite number of maxima and minima in the interval T_o.

iv) **Absolute integrability :** The signal $x(t)$ should be absolutely integrable, ie. $\int_{<T_0>} |x(t)| < \infty$. This is because the analysis equation integrates $x(t)$.

Explanation of Dirichlet's conditions:

The function $x(t)$ is a single-valued function, i.e. the function $x(t)$ must have a single value at any instant of time.

For example, consider Figure 4.1(a) which is not single-valued, because it has two values at time t_0.

Figure 4.1(b) represents a single-valued function as it has only one value at time t_0.

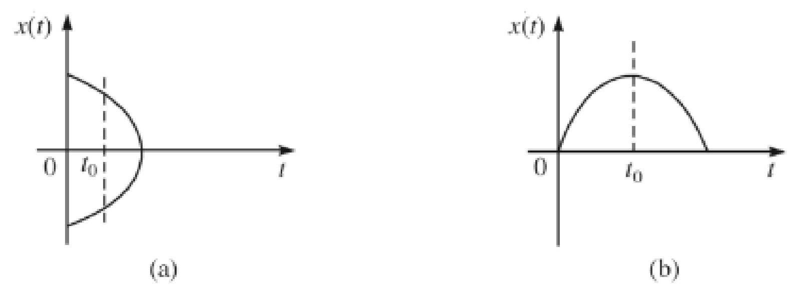


Figure 4.1 (a) Double-valued and (b) single-valued functions at t_0.

- The function $x(t)$ has a finite number of discontinuities.

  Consider Figure 4.2(a). It has no finite number of discontinuities and it is not possible to find the value of the function $x(t)$ at such a number of a discontinuities. Hence, it cannot be represented by a Fourier series.

  The function shown in Figure 4.2(b) has a finite number of discontinuities and the value of $x(t)$ at the discontinuity can be calculated by using the formula

$$x(t = T) = \frac{x(T^-) + x(T^+)}{2}$$

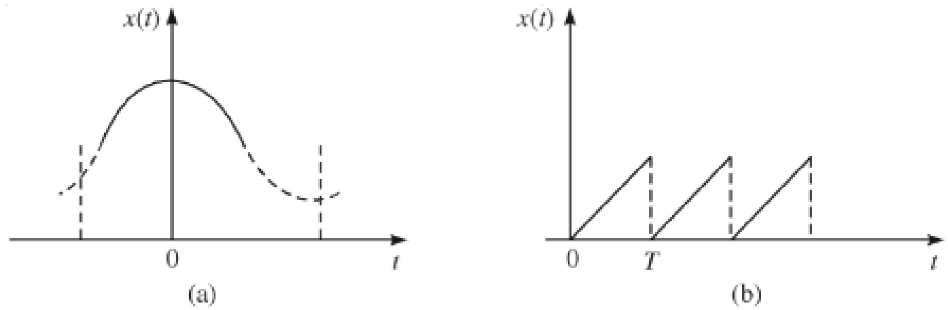


Figure 4.2 Function having (a) no finite number of discontinuities (b) finite number of discontinuities.

- The function $x(t)$ has a finite number of minima and maxima.

  The function shown in Figure 4.3(a) has no fixed number of minima and maxima. So, it cannot be represented by Fourier series.

  Whereas, the funciton shown in Figure 4.3(b) has one minimum and one maximum (finite number of minima and maxima). So, it can be represented by a Fourier series.

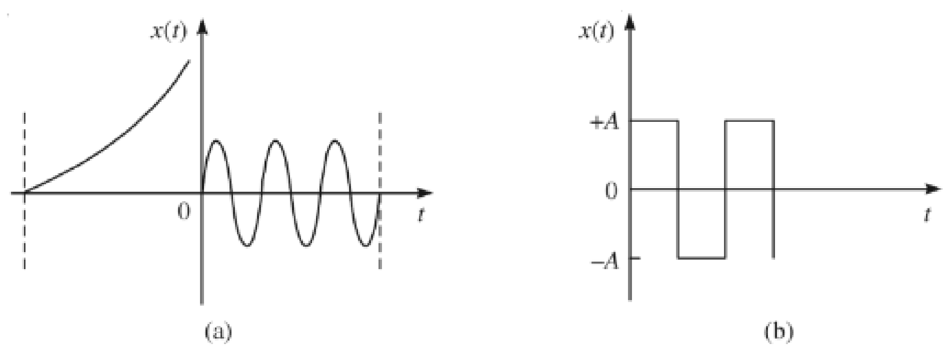


Figure 4.3 Function having (a) no fixed number of minima and maxima (b) fixed number of minima and maxima.

- The function $x(t)$ is absolutely integrable in the interval (t_1, t_2).

  The function shown in Figure 4.4(a) is not absolutely integrable within the interval (t_1, t_2) considered. Hence, it cannot be represented by a Fourier series. But the function shown in Figure 4.4(b) is absolutely integrable over $(0, T)$, i.e.

$$\int_0^T |x(t)|\,dt < \infty$$

  and hence can be represented by a Fourier series.

and hence can be represented by a Fourier series.

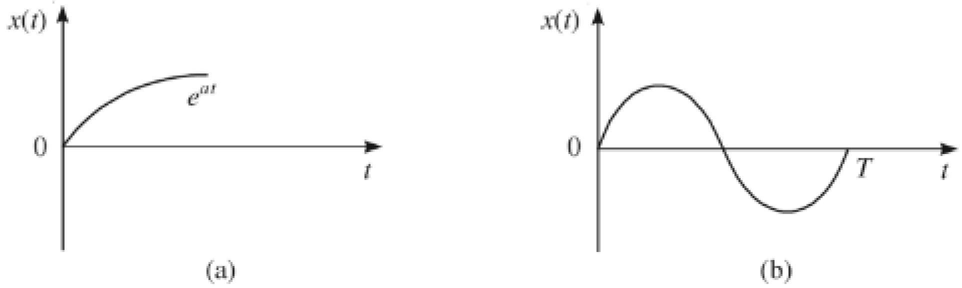


Figure 4.4 (a) Absolutely not integrable and (b) absolutely integrable function.

Properties of Fourier Series

Linearity

If $x(t) \xleftrightarrow{FS} X(k)$ and $y(t) \xleftrightarrow{FS} Y(k)$

then $\boxed{z(t) = ax(t) + by(t) \xleftrightarrow{FS} Z(k) = aX(k) + bY(k)}$

Proof : From equation (1.5.3) we can write $Z(k)$ as,

$$Z(k) = \frac{1}{T} \int_{<T>} z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_{<T>} [ax(t) + by(t)]\, e^{-jk\omega_0 t}\, dt$$

$$= a \frac{1}{T} \int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt + b \frac{1}{T} \int_{<T>} y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= aX(k) + bY(k)$$

Significance : This property is used to analyze signals which are represented as linear combination of other signals.

Time Shift or Translation

If $x(t) \xleftrightarrow{FS} X(k)$ then,

$$\boxed{z(t) = x(t - t_o) \xleftrightarrow{FS} Z(k) = e^{-jk\omega_0 t_0} X(k)}$$

Proof : Fourier coefficients of $x(t - t_0)$ will be,

$$Z(k) = \frac{1}{T} \int_{<T>} x(t - t_0)\, e^{-jk\omega_0 t}\, dt$$

Put $t - t_0 = m$. Limits of integration will shift by t_0. But again the integration is over one period. Hence limits can be kept same. i.e.,

$$Z(k) = \frac{1}{T} \int_{<T>} x(m)\, e^{-jk\omega_0 (m + t_0)}$$

$$= \left[\frac{1}{T} \int_{<T>} x(m)\, e^{-jk\omega_0 m} \right] \cdot e^{-jk\omega_0 t_0}$$

The quantity inside the square brackets is $X(k)$. Hence,

$$Z(k) = e^{-jk\omega_0 t_0} X(k)$$

Frequency Shift

If $x(t) \xleftrightarrow{FS} X(k)$ then,

$$\boxed{z(t) = e^{jk_0\omega_0 t}\, x(t) \xleftrightarrow{FS} Z(k) = X(k-k_0)}$$

$$Z(k) = \frac{1}{T}\int_{<T>} z(t)\, e^{-jk_0\omega_0 t}\, dt \text{ by definition}$$

$$= \frac{1}{T}\int_{<T>} [e^{jk_0\omega_0 t}\, x(t)]\, e^{-jk\omega_0 t} \text{ by putting for } z(t)$$

$$= \frac{1}{T}\int_{<T>} x(t)]\, e^{-j(k-k_0)\omega_0 t}$$

$$= X(k-k_0)$$

Scaling

If $x(t) \xleftrightarrow{FS} X(k)$

then, $\boxed{z(t) = x(at) \xleftrightarrow{FS} Z(k) = X(k)}$

Proof :

$$X(k) = \frac{1}{T}\int_{<T>} x(t)]\, e^{-jk\omega_0 t}\, dt$$

- Since x(t) is periodic, then z(t) = x(at) is also periodic. And if 'T' is the period of x(t), then period of **z(t)** will be $\frac{T}{a}$.
- Similarly if frequency of x(t) is ω_0. The frequency of z(t) = x(at) will be $a\omega_0$, since 't' is multiplied by factor 'a'.

Time Differentiation

If $x(t) \xleftrightarrow{FS} X(k)$

then, $$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{FS} jk\,\omega_0\,X(k)}$$

Proof :

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}$$ By definition of exponential fourier series...

Differentiating with respect to 't',

$$\frac{d\,x(t)}{dt} = \sum_{k=-\infty}^{\infty} X(k)\, jk\,\omega_0\, e^{jk\omega_0 t}$$

$$\therefore \quad \frac{d\,x(t)}{dt} = \sum_{k=-\infty}^{\infty} [jk\,\omega_0 X(k)]\, e^{jk\omega_0 t}$$

We know that $x(t) \xleftrightarrow{FS} X(k)$.

$$\frac{d\,x(t)}{dt} \xleftrightarrow{FS} jk\,\omega_0 X(k)$$

Convolution in Time

If $x(t) \xleftrightarrow{FS} X(k)$ and $y(t) \xleftrightarrow{FS} Y(k)$

then, $$\boxed{z(t) = x(t) * y(t) \xleftrightarrow{FS} Z(k) = T\,X(k)\,Y(k)}$$

Proof : We know that,

$$Z(k) = \frac{1}{T}\int_{\langle T\rangle} z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} [x(t) * y(t)]\, e^{-jk\omega_0 t}\, dt$$

$x(t) * y(t) = \int_{\langle T\rangle} x(\tau)\, y(t-\tau)\, d\tau$. This convolution is performed over one period for periodic signals. Putting this convolution in equation

$$Z(k) = \frac{1}{T}\int_{\langle T\rangle}\int_{\langle T\rangle} x(\tau)\,y(t-\tau)\,d\tau\, e^{-jk\omega_0 t}\,dt$$

Interchanging the order of integrations,

$$Z(k) = \frac{1}{T}\int_{\langle T\rangle} x(\tau)\int_{\langle T\rangle} y(t-\tau)\,e^{-jk\omega_0 t}\,d\tau\,dt$$

Put $t - \tau = m$. Therefore $dt = dm$. Since integration is over one period, this substitution will just shift the integrating limits. But it will be again over one period only. Hence we can write,

$$Z(k) = \frac{1}{T}\int_{\langle T\rangle} x(\tau)\int_{\langle T\rangle} y(m)\,e^{-jk\omega_0(\tau+m)}\,d\tau\,dm$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(\tau)\int_{\langle T\rangle} y(m)\,e^{-jk\omega_0\tau}\cdot e^{-jk\omega_0 m}\,d\tau\,dm$$

Multiplication or Modulation Theorem

If $x(t) \xleftrightarrow{FS} X(k)$ and $y(t) \xleftrightarrow{FS} Y(k)$

then, $\boxed{z(t) = x(t)\,y(t) \xleftrightarrow{FS} Z(k) = X(k) * Y(k)}$ ..

Proof :

$$Z(k) = \frac{1}{T}\int_{\langle T\rangle} z(t)\,e^{-jk\omega_0 t}\,dt \text{ By definition}$$

$$= \frac{1}{T}\int_{\langle T\rangle} [x(t)\,y(t)]\,e^{-jk\omega_0 t}\,dt \quad \text{putting for } z(t)$$

By synthesis equation, $x(t) = \sum_{k=-\infty}^{\infty} X(k)\,e^{jk\omega_0 t}$. Putting this expression for x(t) in above equation,

$$= \frac{1}{T}\int_{\langle T\rangle} x(\tau)\,e^{-jk\omega_0\tau}\,d\tau \int_{\langle T\rangle} y(m)\,e^{-jk\omega_0 m}\,dm$$

$$= \frac{1}{T}[T\,X(k)]\cdot[T\cdot Y(k)] = T\;X(k)\;Y(k)$$

$$Z(k) = \frac{1}{T}\int_{(T)} \sum_{m=-\infty}^{\infty} X(m)\,e^{jm\omega_0 t}\cdot y(t)\,e^{-jk\omega_0 t}\,dt$$

Note that index of summation is changed in above equation to differentiate between two indices of 'k' and 'm'. Interchanging the order of integration and summation,

$$Z(k) = \sum_{m=-\infty}^{\infty} X(m)\left[\frac{1}{T}\int_{(T)} y(t)\, e^{-j(k-m)\omega_0 t}\, dt\right]$$

The quantity inside the bracket indicates fourier coefficients y(k – m). Hence above equation will be,

$$Z(k) = \sum_{m=-\infty}^{\infty} X(m)\, y(k-m)$$

i.e.

$$Z(k) = X(k) * Y(k)$$

Parseval's Theorem

If x(t) is the periodic power signal with fourier coefficients X(k), then average power in the signal is given by $\sum_{k=-\infty}^{\infty} |X(k)|^2$. i.e.,

$$\boxed{\text{Power,} \quad P = \sum_{k=-\infty}^{\infty} |X(k)|^2}$$

Proof : The power in the signal is given as,

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)^2\, dt \text{ By definition}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt \text{ for periodic signal}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt \text{ for periodic signal}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt$$

We have, $$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t} \quad \text{by synthesis equation}$$

$\therefore$ $$x^*(t) = \left[\sum_{k=-\infty}^{\infty} X(k)\, e^{jk\omega_0 t}\right]^* \quad \text{by taking conjugates of both sides}$$

$$= \sum_{k=-\infty}^{\infty} X^*(k)\, e^{-jk\omega_0 t}$$

Putting above expression of $x^*(t)$ in equation (1.5.14),

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \sum_{k=-\infty}^{\infty} X^*(k)\, e^{-jk\omega_0 t}\, dt$$

Here $\int_{-T/2}^{T/2} = \int_{<T>}$ i.e. integration over one period of x(t). Interchanging the order of summation and integration,

$$P = \sum_{k=-\infty}^{\infty} X^*(k) \cdot \frac{1}{T} \int_{<T>} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} X^*(k)\, X(k) = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

Significance : Power of the signal can be obtained by squaring and adding the magnitudes of fourier coefficients.

Symmetry Properties

If x(t) is real then $X^*(k) = X(-k)$

If x(t) is imaginary then, $X^*(k) = -X(-k)$

If x(t) is real and even then, $\text{Im}\{X(k)\} = 0$

If x(t) is real and odd then, $\text{Re}\{X(k)\} = 0$

Example 1 : *Find trigonometric fourier series for the periodic signal shown in Fig. .1.*

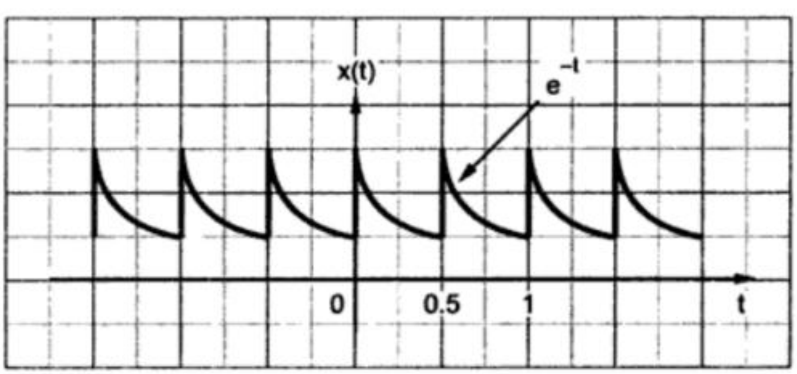


Fig. .1 Periodic exponential pulse

Solution : Here period T = 0.5. And $x(t) = e^{-t}$ over one period.

Step 1 : To calculate a(0).

$$a(0) = \frac{1}{T} \int_{<T>} x(t)\, dt,$$

$$= \frac{1}{0.5} \int_0^{0.5} e^{-t}\, dt$$

$$= \frac{1}{0.5} [-e^{-t}]_0^{0.5} = 0.7869$$

Step 2 : To calculate a(k)

$$a(k) = \frac{2}{T} \int_{<T>} x(t) \cos k\,\omega_0\, t\, dt$$

Here $$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{0.5} = 4\pi \text{ Hence,}$$

$$a(k) = \frac{2}{0.5} \int_0^{0.5} x(t) \cos(k \cdot 4\pi t)\, dt$$

$$= 4 \int_0^{0.5} e^{-t} \cos(4\pi k t)\, dt$$

Here we will use $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$ with a = – 1 and b = $4\pi k$.

Here we will use $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$ with a = – 1 and b = $4\pi k$.

Then above equation will be,

$$a(k) = 4\left\{\frac{e^{-t}}{1+(4\pi k)^2}[(-1)\cos(4\pi k)t + (4\pi k)\sin(4\pi k)t]\right\}_0^{0.5}$$

$$= 4\left\{\frac{e^{-0.5}}{1+(4\pi k)^2}[-\cos(4\pi k)\,0.5 + 4\pi k \sin(4\pi k)\,0.5]\right.$$

$$\left. - \frac{e^0}{1+(4\pi k)^2}[-\cos(4\pi k)\,0 + 4\pi k \sin(4\pi k).0]\right\}$$

$$= \frac{4}{1+(4\pi k)^2}\{0.606[-\cos(2\pi k) + 4\pi k \sin(2\pi k)]$$

$$-[-\cos(0)+4\pi k\sin(0)]\}$$

$$= \frac{4}{1+(4\pi k)^2}\{-0.606+0+1+0\}$$

$$= \frac{1.576}{1+(4\pi k)^2}$$

Step 3 : To calculate b(k)

$$b(k) = \frac{2}{T}\int_{<T>} x(t)\sin k\omega_0 t\, dt$$

$$= \frac{2}{0.5}\int_0^{0.5} e^{-t}\sin(k\cdot 4\pi t)\, dt$$

$$= 4\int_0^{0.5} e^{-t}\sin(4\pi k t)\, dt$$

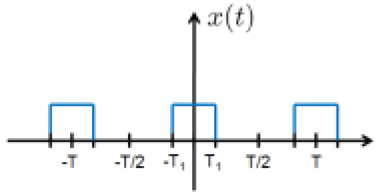
$$= \frac{6.32\pi k}{1+(4\pi k)^2}$$

Step 4 : To obtain fourier series

Putting the expressions for a(0), a(k) and b(k) in equation (1.5.1),

$$x(t) = 0.7869+\sum_{k=1}^{\infty}\frac{1.576}{1+(4\pi k)^2}\cos k\,\omega_0\, t+\sum_{k=1}^{\infty}\frac{6.32\pi k}{1+(4\pi k)^2}\sin k\,\omega_0\, t$$

Example Periodic Rectangular Wave



Let us determine the Fourier series coefficients of the following signal

$$x(t) = \begin{cases} 1 & |t| < T_1, \\ 0 & T_1 < |t| < \frac{T}{2} \end{cases}.$$

The Fourier series coefficients are ($k \neq 0$):

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt \\ &= \frac{-1}{jk\omega_0 T}\left[e^{-jk\omega_0 t}\right]_{-T_1}^{T_1} \\ &= \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right] = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}. \end{aligned}$$

If $k = 0$, then

$$a_0 = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}.$$

Example Periodic Impulse Train

Consider the signal $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$. The fundamental period of $x(t)$ is T [Why?]. The F.S. coefficients are

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)dt = \frac{1}{T},$$

for any k.

Properties of Fourier Series Coefficients

There are a number of Fourier series properties that we encourage you to read the text. The following is a quick summary of these properties.

1. Linearity: If $x_1(t) \longleftrightarrow a_k$ and $x_2(t) \longleftrightarrow b_k$, then

$$Ax_1(t) + Bx_2(t) \longleftrightarrow Aa_k + Bb_k.$$

   For DT case, we have if $x_1[n] \longleftrightarrow a_k$ and $x_2[n] \longleftrightarrow b_k$, then

$$Ax_1[n] + Bx_2[n] \longleftrightarrow Aa_k + Bb_k.$$

2. Time Shift:

$$x(t - t_0) \longleftrightarrow a_k e^{-jk\omega_0 t_0}$$
$$x[n - n_0] \longleftrightarrow a_k e^{-jk\Omega_0 n_0}$$

   To show the time shifting property, let us consider the F.S. coefficient b_k of the signal $y(t) = x(t - t_0)$.

$$b_k = \frac{1}{T}\int_T x(t - t_0)e^{-j\omega_0 t}dt.$$

Letting $\tau = t - t_0$ in the integral, we obtain

$$\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau = e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0\tau}d\tau$$

where $x(t) \longleftrightarrow a_k$. Therefore,

$$x(t-t_0) \longleftrightarrow a_k e^{-jk\omega_0 t_0}.$$

3. Time Reversal:

$$x(-t) \longleftrightarrow a_{-k}$$
$$x[-n] \longleftrightarrow a_{-k}$$

The proof is simple. Consider a signal $y(t) = x(-t)$. The F.S. representation of $x(-t)$ is

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}.$$

Letting $k = -m$, we have

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}.$$

Thus, $x(-t) \longleftrightarrow a_{-k}$.

4. Conjugation:

$$x^*(t) \longleftrightarrow a^*_{-k}$$
$$x^*[n] \longleftrightarrow a^*_{-k}$$

5. Multiplication: If $x(t) \longleftrightarrow a_k$ and $y(t) \longleftrightarrow b_k$, then

$$x(t)y(t) \longleftrightarrow \sum_{l=-\infty}^{\infty} a_k b_{k-l}.$$

6. Parseval Equality:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$
$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

UNIT II

FOURIER TRANSFORMS:

Fourier Transforms

Periodic signals which extend over the interval $(-\infty, \infty)$ can be effectively represented with the help of fourier series. A periodic signals which are strictly time limited can also be represented by fourier series. A time limited signal means it has zero value outside the specified interval. And asymptotically time limited means as time approaches to infinity (∞), the value of signal becomes zero $[i.e.\, x(t) \rightarrow 0 \; as \; |t| \rightarrow \infty]$. Such time limited signals can be more conveniently represented by *fourier transform* in frequency domain. These signals are aperiodic because their period $T_0 \rightarrow \infty$.

Fourier transform can also be found for periodic signals. It provides effective reversible transformation link between frequency domain and time domain representation of the signal. We have seen previously that for nonperiodic signals $T_0 \rightarrow \infty$. As the period of the signal $T_0 \rightarrow \infty$, $f_0 = 0$. Therefore the spacing between the spectral components becomes ınitesimal and hence the frequency spectrum appears to be *continuous*. Whereas periodic

signals has fixed period T_0. Therefore their frequency spectrum is discontinuous as we have seen in the examples in the last section.

2.2.1 Definition of Fourier Transform

Let $x(t)$ be the signal which is function of time t. The fourier transform of $x(t)$ is given as

Fourier Transform : $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$ or

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \text{since} \quad \omega = 2\pi f$$

Similarly $x(t)$ can be recovered from its fourier transform $X(f)$ by using inverse fourier transform.

Inverse Fourier Transform : $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, df \quad \dots (2.2.2)$$

The functions $x(t)$ and $X(f)$ form a fourier transform pair is written by a shorthand symbol as shown below,

$$x(t) \leftrightarrow X(f) \quad \dots (2.2.3)$$

Other shorthand notation for fourier transform is as shown below,

$$X(f) = F[x(t)] \quad \dots (2.2.4)$$

and $$x(t) = F^{-1}[X(f)] \quad \dots (2.2.5)$$

Fourier transform thus can be considered as a linear operator as shown in Fig. 2.2.1.

(a) Fourier transformation and **(b) Inverse fourier transformation**

Fig. 2.2.1

2 Fourier Transform

Purpose

- Non-periodic signals can be represented with the help of Fourier transform.
- Fourier transform provides effective reversible link between frequency domain and time domain representation of the signal.
- For non-periodic signals $T_0 \to \infty$. Hence $\omega_0 = 0$. Therefore spacing between the spectral components becomes infinitesimal and hence the spectrum appears to be continuous.

.1 Definition of Fourier Transform

The Fourier transform of $x(t)$ is defined as,

Fourier Transform :	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$ or $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$

... (2. .1)

Here '$x(t)$' is time domain representation of the signal and '$X(\omega)$' or '$X(f)$' is frequency domain representation of the signal 'ω' is the frequency.

Sometimes $X(\omega)$ is also written as $X(j\omega)$.

Similarly $x(t)$ can be obtained from $X(\omega)$ by inverse Fourier transform. i.e.,

Inverse Fourier Transform :	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega = \int_{-\infty}^{\infty} X(f) e^{-j2\pi ft}\, df$

... (2. 2)

A Fourier transform pair is represented as,

$$x(t) \xleftrightarrow{FT} X(\omega) \text{ or } x(t) \xleftrightarrow{FT} X(f)$$

.2 Existence of Fourier Transform - Dirichlet Conditions

Following conditions should be satisfied by the function $x(t)$ for Fourier transform to exist.

i) Single valued properly : $x(t)$ must have only value at any time instant over a finite time interval T.

ii) Finite discontinuities : $x(t)$ should have at the most finite number of discontinuities over a finite time interval T.

iii) Finite peaks : The signal $x(t)$ should have finite number of maxima and minima over a finite time interval T.

iv) Absolute integrability : $x(t)$ should be absolutely integrable. i.e.,

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

- These conditions are sufficient, but not necessary for the signal to be Fourier transformable.

.3 Properties of Fourier Transform

3.1 Linearity

If $x(t) \xleftrightarrow{FT} X(\omega)$ and $y(t) \xleftrightarrow{FT} Y(\omega)$.

then,

$$\boxed{z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(\omega) = aX(\omega) + bY(\omega)}$$

Meaning :

The Fourier transform of linear combination of the signals is equal to linear combination of their Fourier transforms. It is also called superposition.

Proof :

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[ax(t) + by(t)\right] e^{-j\omega t}\, dt$$

$$= a\int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt + b\int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= a\, X(\omega) + b\, Y(\omega)$$

.2 Time Shift

If $x(t) \xleftrightarrow{FT} X(\omega)$, then

$$\boxed{y(t) = x(t - t_0) \xleftrightarrow{FT} Y(\omega) = e^{-j\omega t_0} X(\omega)}$$

Meaning : A shift of 't_0' in time domain is equivalent to introducing a phase shift of $-\omega t_0$. But amplitude remains same.

Proof :

$$Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} y(t - t_0) e^{-j\omega t}\, dt$$

Put $t - t_0 = \tau$ then $t = \tau + t_0$.

$\therefore dt = d\tau$ and integration limits will remain same.

$$Y(\omega) = \int_{-\infty}^{\infty} y(\tau)\, e^{-j\omega(\tau + t_0)} d\tau$$

$$= \int_{-\infty}^{\infty} y(\tau)\, e^{-j\omega\tau} \cdot e^{-j\omega t_0} d\tau$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} y(\tau)\, e^{-j\omega\tau} d\tau$$

$$= e^{-j\omega t_0}\, Y(\omega)$$

.3 Frequency Shift

If $x(t) \xleftrightarrow{FT} X(\omega)$, then

$$\boxed{y(t) = e^{j\beta t} x(t) \xleftrightarrow{FT} Y(\omega) = X(\omega - \beta)}$$

Meaning :

It states that by shifting the frequency by 'β' in frequency domain is equivalent to multiplying the time domain signal by $e^{j\beta t}$.

Proof :

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{j\beta t} x(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\beta)t}\, dt$$

$$= X(\omega - \beta)$$

4 Time Scaling

If $x(t) \xleftrightarrow{FT} X(\omega)$, then

$$\boxed{y(t) = x(at) \xleftrightarrow{FT} Y(\omega) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$$

Meaning :

Compression of a signal in time domain is equivalent to expansion in frequency domain and vice-versa.

Proof :

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Put $at = \tau$, then $t = \frac{\tau}{a}$.

$\therefore \quad dt = \frac{1}{a} d\tau$ and limits will remain same.

$$\therefore \quad Y(\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega \cdot \frac{\tau}{a}} \cdot \frac{1}{a}\, d\tau$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

.5 Frequency-Differentiation

If $x(t) \xleftrightarrow{FT} X(\omega)$, then

$$\boxed{-jt\ x(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(\omega)}$$

Meaning :

Differentiating the frequency spectrum is equivalent to multiplying the time domain signal by complex number $-jt$.

Proof :

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$\therefore \quad \frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}\left[e^{-j\omega t}\right]dt$$

$$= \int_{-\infty}^{\infty} x(t)\,(-jt)\,e^{-j\omega t}\,dt$$

$$= -jt\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= -jt\ X(\omega)$$

.6 Time-Differentiation

If $x(t) \xleftrightarrow{FT} X(\omega)$ then,

$$\boxed{\frac{d\,x(t)}{dt} \xleftrightarrow{FT} j\omega X(\omega)}$$

Meaning : Differentiation in time domain corresponds to multiplying by $j\omega$ in frequency domain. It accentuates high frequency components of the signal.

Proof :

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\frac{d}{dt} e^{j\omega t}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, j\omega\, e^{j\omega t}\; d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega\, X(\omega)] e^{j\omega t}\, d\omega$$

.7 Convolution

if $\quad x(t) \xleftrightarrow{FT} X(\omega)$ and $\quad y(t) \xleftrightarrow{FT} Y(\omega)$.

then,

$$\boxed{z(t) = x(t) * y(t) \xleftrightarrow{FT} Z(\omega) = X(\omega) \cdot Y(\omega)}$$

Meaning :

A convolution operation is transformed to modulation in frequency domain.

Proof :

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) y(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau) e^{-j\omega t}\, dt \right] d\tau$$

Put $t - \tau = \alpha$, then $t = \tau + \alpha$.

$$dt = d\,\alpha, \text{ limits of integration will remain same.}$$

$$\therefore \qquad Z(\omega) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(\alpha) e^{-j\omega(\tau+\alpha)} d\alpha \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(\alpha) e^{-j\omega\tau} \cdot e^{-j\omega\alpha} d\alpha \right] d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau}\, d\tau \int_{-\infty}^{\infty} y(\alpha) e^{-j\omega\alpha} d\alpha$$

$$= X(\omega) \cdot Y(\omega)$$

.8 Integration

If $x(t) \xleftrightarrow{FT} X(\omega)$, then

$$\boxed{\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(\omega)}$$

Meaning :

· Integration in time represents smoothing in time domain. This smoothing in time corresponds to de-emphasizing the high frequency components of the signal.

Proof :

Let $x(t)$ be expressed as,

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$\therefore \qquad F[x(t)] = F\left\{\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]\right\}$$

By differentiation property right hand side of above equation becomes,

$$F[x(t)] = j\omega\left\{F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]\right\}$$

i.e. $$X(\omega) = j\omega\, F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

$$\therefore \qquad \frac{1}{j\omega} X(\omega) = F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]$$

or $$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(\omega)$$

.9 Modulation

If $x(t) \xleftrightarrow{FT} X(\omega)$ and $y(t) \xleftrightarrow{FT} Y(\omega)$ then,

$$\boxed{z(t) = x(t)\, y(t) \xleftrightarrow{FT} Z(\omega) = \frac{1}{2\pi}[X(\omega) * Y(\omega)]}$$

Meaning :

Modulation in time domain corresponds to convolution of spectrums in frequency domain.

Proof

$$Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} x(t)\, y(t) e^{-j\omega t}\, d\omega$$

Inverse Fourier transform states that,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)\, e^{j\lambda t}\, d\lambda$$

Putting for $x(t)$ in equation 2.4.12,

$$Z(\omega) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda)\, e^{j\lambda t}\, d\lambda \right] y(t) e^{-j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) \int_{-\infty}^{\infty} y(t) e^{-j(\omega-\lambda)t}\; dt\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) Y(\omega - \lambda)\, d\lambda$$

$$= \frac{1}{2\pi}[X(\omega) * Y(\omega)]$$

10 Duality

If $x(t) \xleftrightarrow{FT} X(\omega)$ then,

$$\boxed{X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)}$$

Proof :

Inverse Fourier transform is given as,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Interchanging t by ω we get,

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\; e^{j\omega t}\, dt$$

Interchanging t by ω we get,

$$x(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\; e^{j\omega t}\, dt$$

Interchanging ω by $-\omega$ we get,

$$x(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t)\; e^{-j\omega t}\, dt$$

i.e. $$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt$$

Right handside of above equation is Fourier transform of $X(t)$. i.e.,

$$X(t) \xleftrightarrow{FT} 2\pi x(-\omega)$$

.11 Symmetry

Let $x(t)$ be real signal and

$$X(\omega) = X_R(\omega) + jX_I(\omega)$$

then $$x_e(t) \xleftrightarrow{FT} X_R(\omega)$$

and $$x_o(t) \xleftrightarrow{FT} jX_I(\omega)$$

Here $x_e(t)$ and $x_o(t)$ are even and odd parts of $x(t)$.

Proof

We have,

$$x(t) \xleftrightarrow{FT} X_R(\omega) + jX_I(\omega)$$

Since $x(t)$ is real, $x(-t) \xleftrightarrow{FT} X^*(\omega) = X_R(\omega) - jX_1(\omega)$.

Even part is given as,

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$\therefore \quad x_e(t) \xleftrightarrow{FT} \frac{1}{2}[X(\omega) + X^*(\omega)]$$

$$\xleftrightarrow{FT} \frac{1}{2}[X_R(\omega) + jX_1(\omega) + X_R(\omega) - jX_1(\omega)]$$

$$\xleftrightarrow{FT} \frac{1}{2}X_R(\omega)$$

Odd part is given as,

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

$$x_o(t) \xleftrightarrow{FT} \frac{1}{2}[X(\omega) - X^*(\omega)]$$

$$\longleftrightarrow jX_I(\omega)$$

12 Parseval's Theorem or Rayleigh's Theorem

If $x(t) \xleftrightarrow{FT} X(\omega)$ then,

$$\boxed{E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |X(f)|^2 \, df}$$

Meaning

Energy of the signal can be obtained by interchanging its energy spectrum.

Proof :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

2.2.2 Existence of Fourier Transform

We defined fourier transform in the last subsection. Now we will see what are the conditions to be satisfied by the signal to obtain its fourier transform. In section 1.5.4 we studied Dirichlet conditions. These conditions also apply to the signals to obtain fourier transform. For the nonperiodic signals, the integration is extended to $(-\infty, \infty)$. For periodic signals the integration is over one period as we have seen earlier. The following conditions should by satisfied by the signal to obtain its fourier transform.

i) The function $x(t)$ should be single valued in any finite time interval T.

ii) The function $x(t)$ should have at the most finite number of discontinuities in any finite time interval T.

iii) The function $x(t)$ should have finite number of maxima and minima in any finite time interval T.

iv) The function $x(t)$ should be absolutely integrable i.e.

$$\int_{-\infty}^{\infty} |x(t)| \, dt < \infty \qquad \text{... (2.2.10)}$$

The condition follows from definition of fourier transform given by equation 2.2.1.

The above conditions are applied to periodic as well as nonperiodic signals. The same

Example 2.2.1 *Find the fourier transform of the decaying exponential as shown in Fig. 2.2.2.*

Solution :

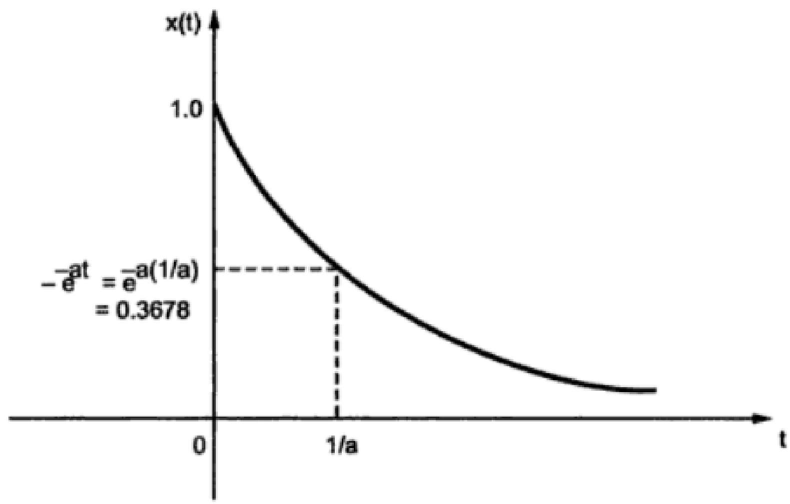


Fig. 2.2.2 Truncated decaying exponential pulse

Normally to show time delays in the function and sign of time, use of unit step function $u[(t)]$ is made. The value of unit step function is always unity i.e.

$$u(t) = 1 \qquad \text{for } t \geq 0$$

∴ The exponential pulse in Fig. 2.2.2 is represented as,

$$x(t) = e^{-at}\, u(t) \qquad \text{Here } u(t) = 1 \qquad \text{... (2.2.12)}$$

By definition of fourier transform (equation 2.2.1) we have,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-at}\, u(t) \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+j2\pi f)t}\, dt \qquad \text{... (2.2.13)}$$

The lower limit is taken '0' since $x(t) = 0$, for $t < 0$. And $u(t) = 1$ for $t \geq 0$

$$\therefore \quad X(f) = \frac{1}{-(a+j2\pi f)}\left[e^{-(a+j2\pi f)t}\right]_0^{\infty}$$

$$= \frac{1}{a+j2\pi f} \qquad \text{... (2.2.14)}$$

Thus the fourier transform pair becomes,

$$\boxed{\textbf{\textit{Decaying exponential pulse}} : e^{-at} \; u(t) \leftrightarrow \frac{1}{a+j2\pi f}} \qquad \text{... (2.2.15)}$$

To calculate magnitude and phase spectrum :

The function $X(f)$ is expressed as,

$$X(f) = A(f) + jB(f) \qquad \text{... (2.2.16)}$$

Here $A(f)$ is real part of $X(f)$ and $B(f)$ is imaginary part of $X(f)$.

Here $A(f)$ is real part of $X(f)$ and $B(f)$ is imaginary part of $X(f)$.

Therefore magnitude spectrum of $X(f)$ is given as,

$$|X(f)| = \sqrt{A^2(f) + B^2(f)} \qquad \text{... (2.2.17)}$$

And phase spectrum is given as,

$$\theta(f) = tan^{-1} \frac{B(f)}{A(f)} \qquad \text{... (2.2.18)}$$

Consider the equation 2.2.14,

$$X(f) = \frac{1}{a + j2\pi f}$$

Multiply and divide RHS by $a - j2\pi f$,

$$\therefore \quad X(f) = \frac{1}{a + j2\pi f} \times \frac{a - j2\pi f}{a - j2\pi f}$$

$$= \frac{a - j2\pi f}{a^2 + (2\pi f)^2}$$

$$= \frac{a}{a^2 + (2\pi f)^2} + j\,\frac{-2\pi f}{a^2 + (2\pi f)^2} \quad \text{... (2.2.19)}$$

$$\left.\begin{aligned} \text{Here real part } A(f) &= \frac{a}{a^2 + (2\pi f)^2} \\ \text{and imaginary part } B(f) &= \frac{-2\pi f}{a^2 + (2\pi f)^2} \end{aligned}\right\} \quad \text{... (2.2.20)}$$

$\therefore$ From equation 2.2.17 magnitude of $X(f)$ will be,

$$|X(f)| = \sqrt{\frac{a^2}{\left[a^2 + (2\pi f)^2\right]^2} + \frac{(-2\pi f)^2}{\left[a^2 + (2\pi f)^2\right]^2}}$$

$$= \sqrt{\frac{1}{a^2 + (2\pi f)^2}} \quad \text{... (2.2.21)}$$

From equation 2.2.18 phase spectrum will be,

$$\theta(f) = tan^{-1}\left\{\frac{-2\pi f / \left[a^2 + (2\pi f)^2\right]}{a / \left[a^2 + (2\pi f)^2\right]}\right\}$$

$$= tan^{-1}\left(\frac{-2\pi f}{a}\right)$$

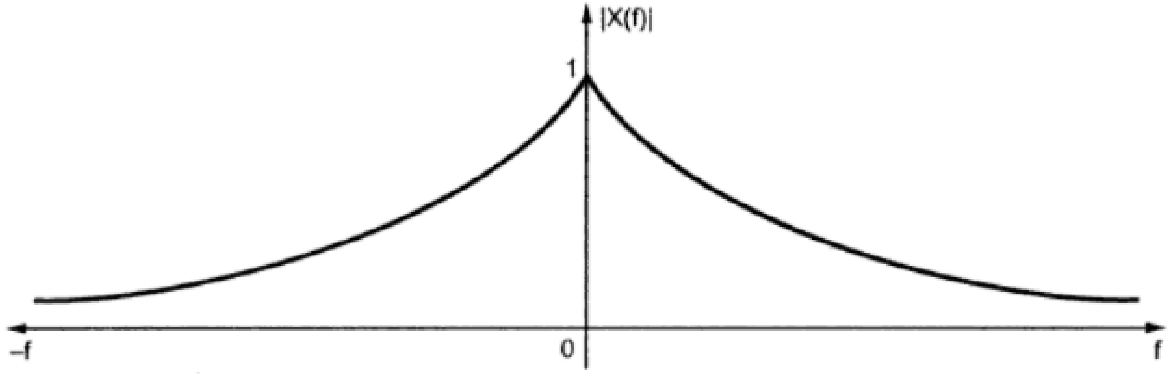


Fig. 2.2.3 (a) Amplitude spectrum of decaying exponential pulse of Fig. 2.2.2. Here a = 1 (assumed). It is even function of frequency.

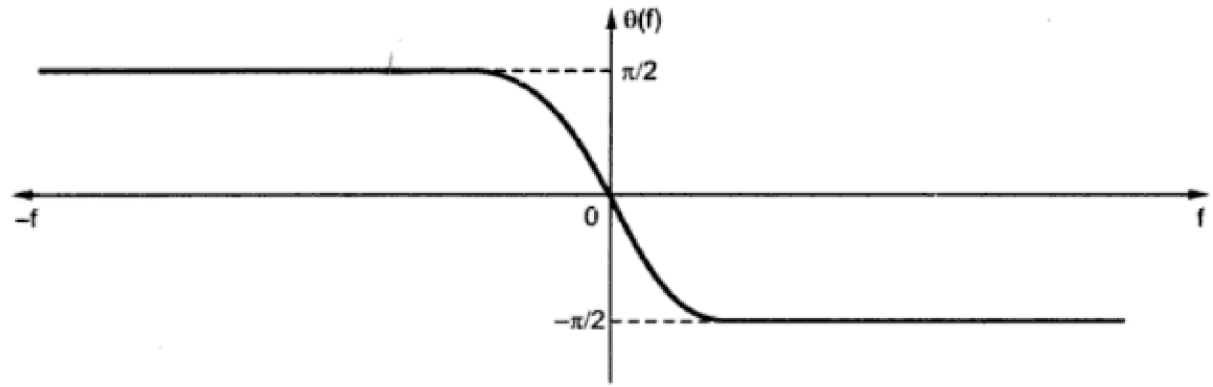


Fig. 2.2.3 (b) Phase spectrum. It is odd function of frequency.

Example 2.2.7 : *Obtain the fourier transform of rectangular pulse of duration T and amplitude `A' as shown in Fig. 2.2.11 below.*

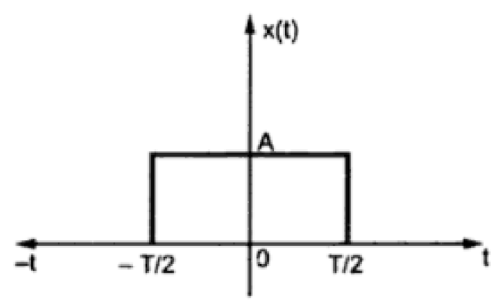


Fig. 2.2.11 Rectangular pulse

Solution : This rectangular pulse is defined as,

$$rect\left(\frac{t}{T}\right) = \begin{cases} A & for \ -\frac{T}{2} < t < \frac{T}{2} \\ 0 & elsewhere \end{cases} \qquad \text{... (2.2.40)}$$

$$\therefore \qquad x(t) = A \ rect\left(\frac{t}{T}\right)$$

FT of $x(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t)\ e^{-j2\pi ft}\ dt \qquad \text{by equation 2.2.1}$$

$$= \int_{-T/2}^{T/2} A\ e^{-j2\pi ft}\ dt \qquad \text{by equation 2.2.40 above.}$$

$$= \frac{A}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{-T/2}^{T/2}$$

$$= \frac{A}{-j2\pi f}\left[e^{-j\pi fT} - e^{j\pi fT}\right]$$

$$= \frac{A}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right]$$

$$= \frac{A}{\pi f}\ sin\ (\pi f T) \qquad \text{By Euler's theorem.}$$

$$= A\,T\ \frac{sin(\pi f T)}{\pi f T} \qquad \text{By rearranging the equation.}$$

$$= A\,T\ sinc(f\,T) \qquad \text{Since } sinc\ x = \frac{\sin(\pi x)}{\pi x}$$

Fig. 2.2.12 shows the amplitude spectrum in (a) and phase spectrum in (b). In the spectrum shown below, the negative values of amplitude $|X(f)|$ are made positive by phase shift of ± 180 in the phase spectrum $\theta(f)$.

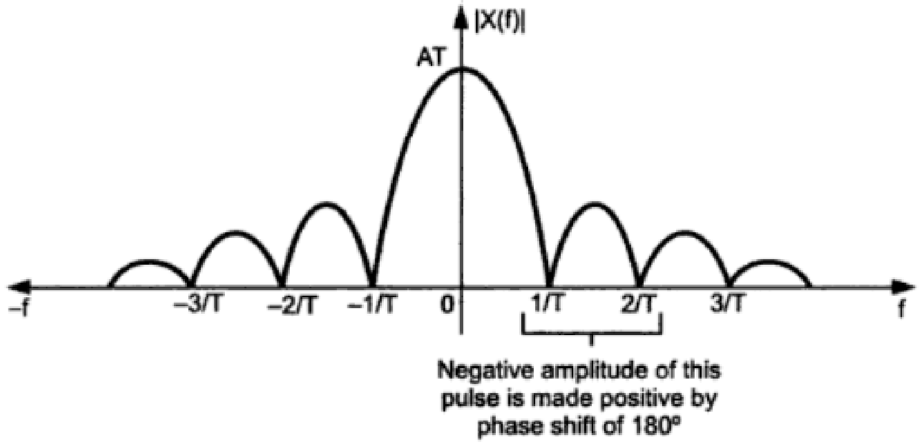


Fig. 2.2.12 (a) Amplitude spectrum of rectangular pulse

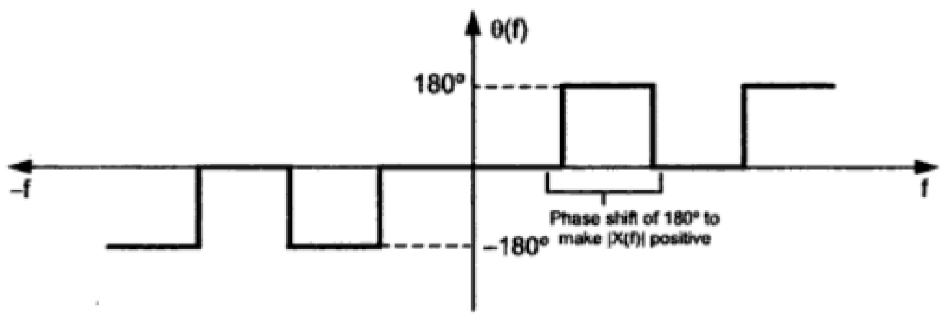


Fig. 2.2.12 (b) Phase spectrum of rectangular pulse

The fourier transform pair of sinc and rectangular function is,

$$\boxed{\begin{array}{c} A\, rect\left(\frac{t}{T}\right) \leftrightarrow AT\, sinc\,(fT) \\ \\ \textit{Rectangular pulse} \leftrightarrow \textit{sinc pulse.} \end{array}}$$

i.e. ... (2.2.41)

Example 2.2.11 : *Obtain the fourier transform of the impulse function shown in Fig. 2.2.15 below.*

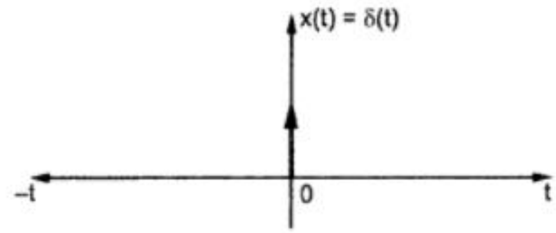


Fig. 2.2.15 Delta function

Solution : By definition of FT,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt \qquad \ldots (2.2.58)$$

The sifting property of impulse function is given as,

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-t_0) = f(t_0)$$

Here $f(t) = e^{-j2\pi ft}$ and $t_0 = 0$

$\therefore$ $X(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft}\, \delta(t-0)\, dt$

By rearranging equation 2.2.58,

$= e^{-2\pi f \cdot 0}$ By applying sifting property.

$= 1$

Delta Function : $\boxed{\delta(t) \leftrightarrow 1}$ ... (2.2.59)

Fig. 2.2.16 shows the amplitude spectrum of delta function. It shows that delta function or unit impulse contains all the frequencies with same amplitude in its spectrum.

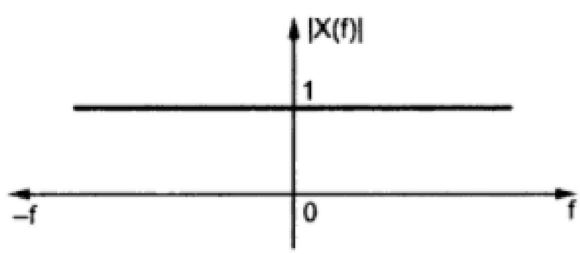


Fig. 2.2.16 Amplitude spectrum of impulse (delta) function is unity & independent of frequency. All frequencies are present with equal amplitudes

2.2.4 Fourier Transform of a Periodic Function

Fourier transform is the limiting case of fourier series if period of a periodic function becomes infinite. Then the spectrum given by fourier series will be the same as that of fourier transform, that is continuous. The reverse way is also possible that the fourier series is first a limiting case of fourier transform. Thus fourier transform of both periodic and non–periodic signals can be obtained. This is extremely useful in signal analysis. We know that fourier transform of a periodic function extending from $-\infty$ to $+\infty$ cannot be obtained by using direct principles studied in the last section. This is because such function is not absolutely integrable. i.e.,

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \infty \qquad \textit{for a periodic function.}$$

But for periodic functions fourier transform can be obtained over the interval $(-T/2, T/2)$; i.e. one time period. Such periodic function can be expressed by fourier series. The fourier transform of the function is given by summing fourier transforms of the individual components of fourier series.

The exponential fourier series of a periodic function of period T_0 can also be expressed as,

$$^{*}\, x_p(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{j 2\pi n t/T_0} \quad \text{and} \quad C_n = \frac{1}{T_0} \int_t^{t+T_0} x_p(t)\, e^{-j 2\pi n t/T_0}$$

* **$x_p(t)$ indicates the periodic signal with period T_0.**
$x(t)$ indicates nonperiodic signal in this section.

Since $\frac{1}{T_0} = f_0$, the above equations will be,

$$\textbf{\textit{Exponential fourier series :}}\ x_p(t) = \sum_{n=-\infty}^{\infty} C_n\ e^{j2\pi n f_0 t}$$

and

$$C_n = \frac{1}{T_0} \int_t^{t+T_0} x_p(t)\ e^{-j2\pi n f_0 t}\ dt \qquad \text{... (2.2.73)}$$

Here we have written $x_p(t)$ to indicate that it is periodic with period T_0.

By taking fourier transform of both sides of $x(t)$ in above equation,

$$F[x(t)] = F\left[\sum_{n=-\infty}^{\infty} C_n\ e^{j2\pi n f_0 t}\right]$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_n\ e^{j2\pi n f_0 t}\ e^{-j2\pi f t}\ dt$$

$$= \sum_{n=-\infty}^{\infty} C_n \int_{-\infty}^{\infty} e^{-j2\pi(f-nf_0)t}\ dt \qquad \text{... (2.2.74)}$$

$$= \sum_{n=-\infty}^{\infty} C_n\ \delta(f-nf_0) \quad \text{from equation 2.2.61} \qquad \text{... (2.2.75)}$$

Thus,

$$\textbf{\textit{Fourier transform of a periodic signal x(t):}}\ X(f) = \sum_{n=-\infty}^{\infty} C_n\ \delta(f-nf_0)$$

$$\textbf{Here, } C_n = \frac{1}{T_0} \int_t^{t+T_0} x_p(t)\ e^{-j2\pi n f_0 t}\ dt$$

... (2.2.76)

Thus, from this result we can state that the fourier transform of a periodic function consists of impulses weighed by C_n. These impulses are located at harmonic frequencies of fundamental frequency f_0. This result clearly resembles with our discussion at the start of this section that periodic function has discrete frequency spectrum.

The signal in time domain can be obtained from its fourier transform given by equation 2.2.73. By definition of IFT,

$$x_p(t) = F^{-1}[X(f)]$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_n\ \delta(f-nf_0)\ e^{j2\pi f t}\ dt$$

$$= \sum_{n=-\infty}^{\infty} C_n \int_{-\infty}^{\infty} \delta(f - n f_0) \, e^{j 2\pi f t} \, dt$$

$$= \sum_{n=-\infty}^{\infty} C_n \, e^{j 2\pi n f_0 t}$$

The result is obtained i.e. sifting property of delta function.

Thus,

Inverse fourier transform of periodic signal x(t)	$: x_p(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{j 2\pi n f_0 t}$
Here C_n is the coefficient of $\delta(f - n f_0)$ in the given fourier transform equation.	

... (2.2.77)

Thus the equation 2.2.76 and equation 2.2.77 represent fourier transforms and inverse fourier transforms of a periodic signal $x_p(t)$. Since $x_p(t)$ is periodic, its fourier transform $X(f)$ is defined only at $n f_0$; i.e. harmonics of fundamental frequency f_0. Hence spectrum is not continuous.

➠ *Find the fourier transform of the signum function shown in Fig. 2.2.9*

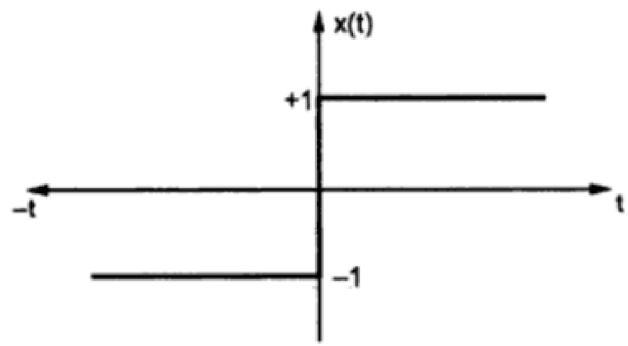


Fig. 2.2.9 Signum function

Here, $\lim_{a \to 0} \{ e^{-a|t|} \ sgn(t) \} = sgn(t)$

$\therefore$ The same limit can be applied to $X(f)$.

$$\therefore \qquad F\{sgn(t)\} = F\left\{ \lim_{a \to 0} \left[e^{-a|t|} \ sgn(t) \right] \right\}$$

$$= \lim_{a \to 0} \frac{-j\,4\,\pi f}{a^2 + (2\,\pi f)^2} = \frac{-j\,4\,\pi f}{4\,\pi^2 f^2} = \frac{-j}{\pi f}$$

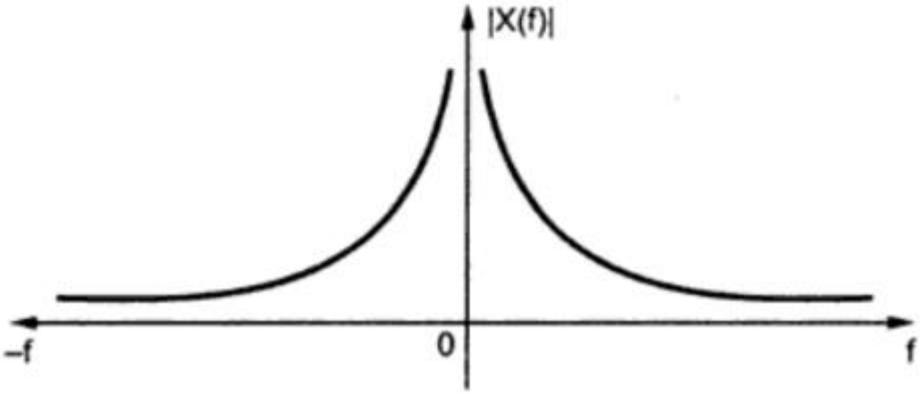
This forms a fourier transform pair,

$$\boxed{\textit{Signum function} : sgn(t) \leftrightarrow \frac{-j}{\pi f}}$$

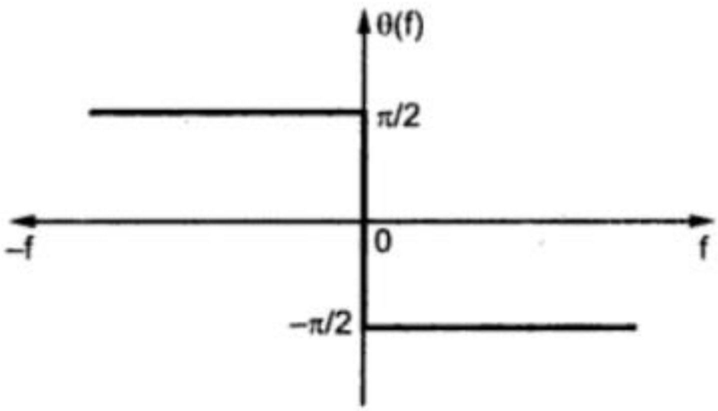
$$|X(f)| = \frac{1}{\pi f} \quad \textit{and}$$

$$\theta(f) = -tan^{-1}\left(\frac{\pi f}{0}\right) = \begin{cases} -\frac{\pi}{2} & \textit{for } f > 0 \\ \frac{\pi}{2} & \textit{for } f < 0 \end{cases}$$

Fig. 2.2.10 (a) and (b) shows the amplitude and phase spectrum of signum function.



(a) Amplitude spectrum of signum function



(b) Phase spectrum of signum function

Fig. 2.2.10

➠ *Obtain the fourier transform of rectangular pulse of duration T and amplitude `A' as shown in Fig. 2.2.11 below.*

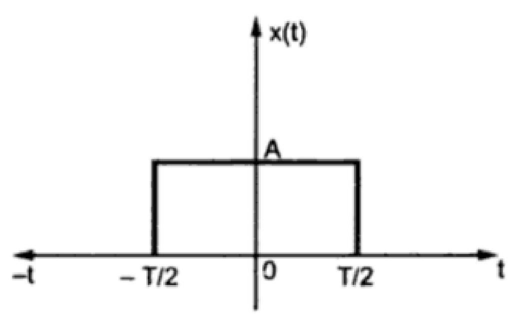


Fig. 2.2.11 Rectangular pulse

Solution : This rectangular pulse is defined as,

$$rect\left(\frac{t}{T}\right) = \begin{cases} A & for \quad -\frac{T}{2} < t < \frac{T}{2} \\ 0 & elsewhere \end{cases}$$

$$\therefore \qquad x(t) = A \; rect\left(\frac{t}{T}\right)$$

FT of $x(t)$ $\quad X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt$ $\quad$ by equation 2.2.1

$= \int_{-T/2}^{T/2} A\, e^{-j2\pi ft}\, dt$ $\quad$ by equation 2.2.40 above.

$$= \frac{A}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{-T/2}^{T/2}$$

$$= \frac{A}{-j2\pi f}\left[e^{-j\pi fT} - e^{j\pi fT}\right]$$

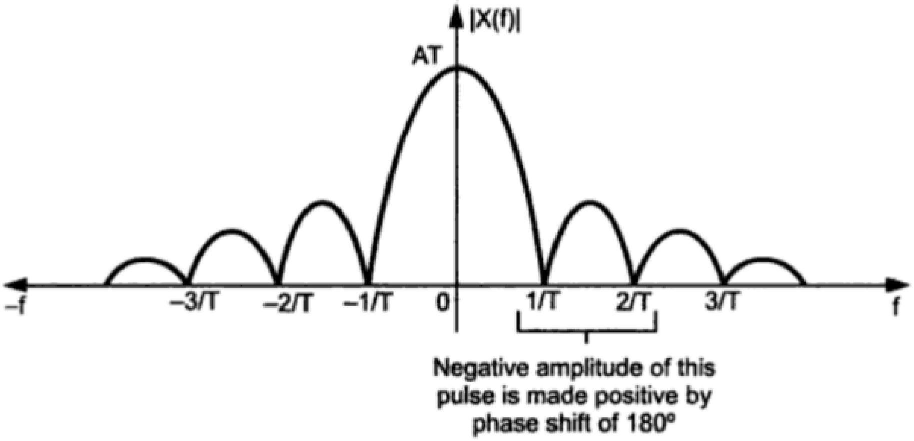
$$= \frac{A}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right]$$

$= \frac{A}{\pi f} \sin(\pi f T)$ $\quad$ By Euler's theorem.

$= AT\,\frac{\sin(\pi f T)}{\pi f T}$ $\quad$ By rearranging the equation.

$= AT\, sinc\,(fT)$ $\quad$ Since $sinc\; x = \frac{\sin(\pi x)}{\pi x}$

Fig. shows the amplitude spectrum in (a) and phase spectrum in (b). In the spectrum shown below, the negative values of amplitude $|X(f)|$ are made positive by phase shift of ± 180 in the phase spectrum $\theta(f)$.



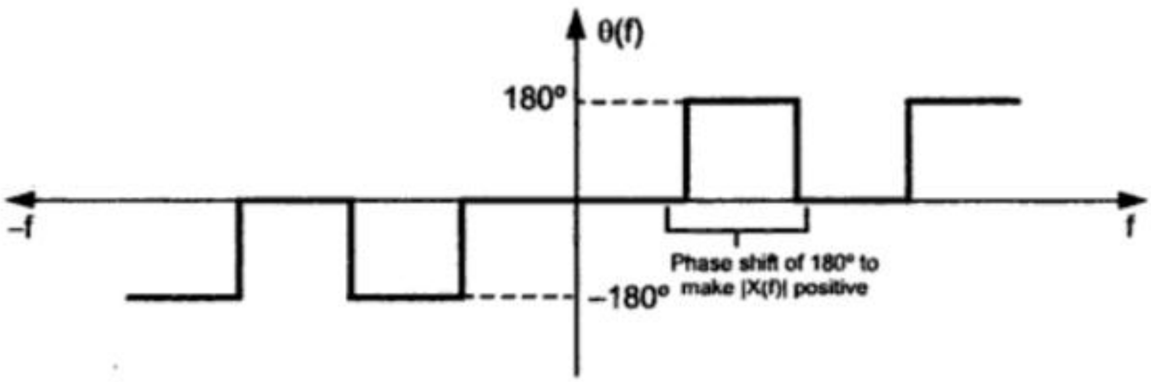


Fig. 2.2.12 (b) Phase spectrum of rectangular pulse

The fourier transform pair of sinc and rectangular function is,

$$A\, rect\left(\frac{t}{T}\right) \leftrightarrow AT\, sinc\,(f\,T)$$

i.e. *Rectangular pulse* $\leftrightarrow$ *sinc pulse.*

SAMPLING:

The first operation in digital communications is the sampling. Almost all the natural signals exist in analog form. For example voice, any moving scene, environmental data etc. exist in analog or continuous form. Such information can be transmitted from one place to another by continuous modulation of the suitable carrier. Then this type of communication is called Analog Communication. In digital communication, the data to be transmitted is sampled at regular intervals. Such samples are then transmitted directly or through the modulation of some carrier. There are various sampling techniques discussed in this chapter. The sampling techniques affect spectral content of the signal.

Sampling Theorem

Sampling of the signals is the fundamental operation in digital communication. A continuous time signal is first converted to discrete time signal by sampling process. The sufficient number of samples of the signal should be taken so that the original signal is represented in its samples completely. Also it should be possible to recover or reconstruct the signal completely from its samples. The number of samples to be taken depends on maximum signal frequency present. Sampling theorem gives the complete idea about the sampling of signals. Different types of samples are also taken i.e. Flat top samples, regular samples, instantaneous samples etc. Let us first discuss the sampling theorem and then we will see different types of sampling processes.

Sampling Theorem For Low Pass Signals In Time Domain

1) *A band limited signal of finite energy, which has no frequency components higher than W Hertz, is completely described by specifying the values of the signal at instants of time separated by* $\frac{1}{2W}$ *seconds and*

2) *A band limited signal of finite energy, which has no frequency components higher than W Hertz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second.*

The above statement of sampling theorem stated in two parts can be combined. The first part represents the representation of the signal in its samples and minimum sampling rate required to represent a continuous time signal into its samples. The second part of the theorem represents reconstruction of the original signal from its samples. It gives sampling rate required for satisfactory reconstruction of signal from its samples. The theorem can be combined and alternately stated as follows :

"A continuous time signal can be completely represented in its samples and ·ecovered back if the sampling frequency $f_s \geq 2$ W. Here f_s is sampling frequency and W is the maximum frequency present in the signal".

Proof of sampling theorem :

Let $x(t)$ be the continuous time signal as shown in Fig. 5.1.1 (a). Let this signal be of finite energy and infinite duration. And suppose that $x(t)$ is strictly band limited. i.e., $x(t)$ does not contain any frequency components higher than 'W' Hertz. A sampling function samples this signal regularly at the rate of f_s samples per second.

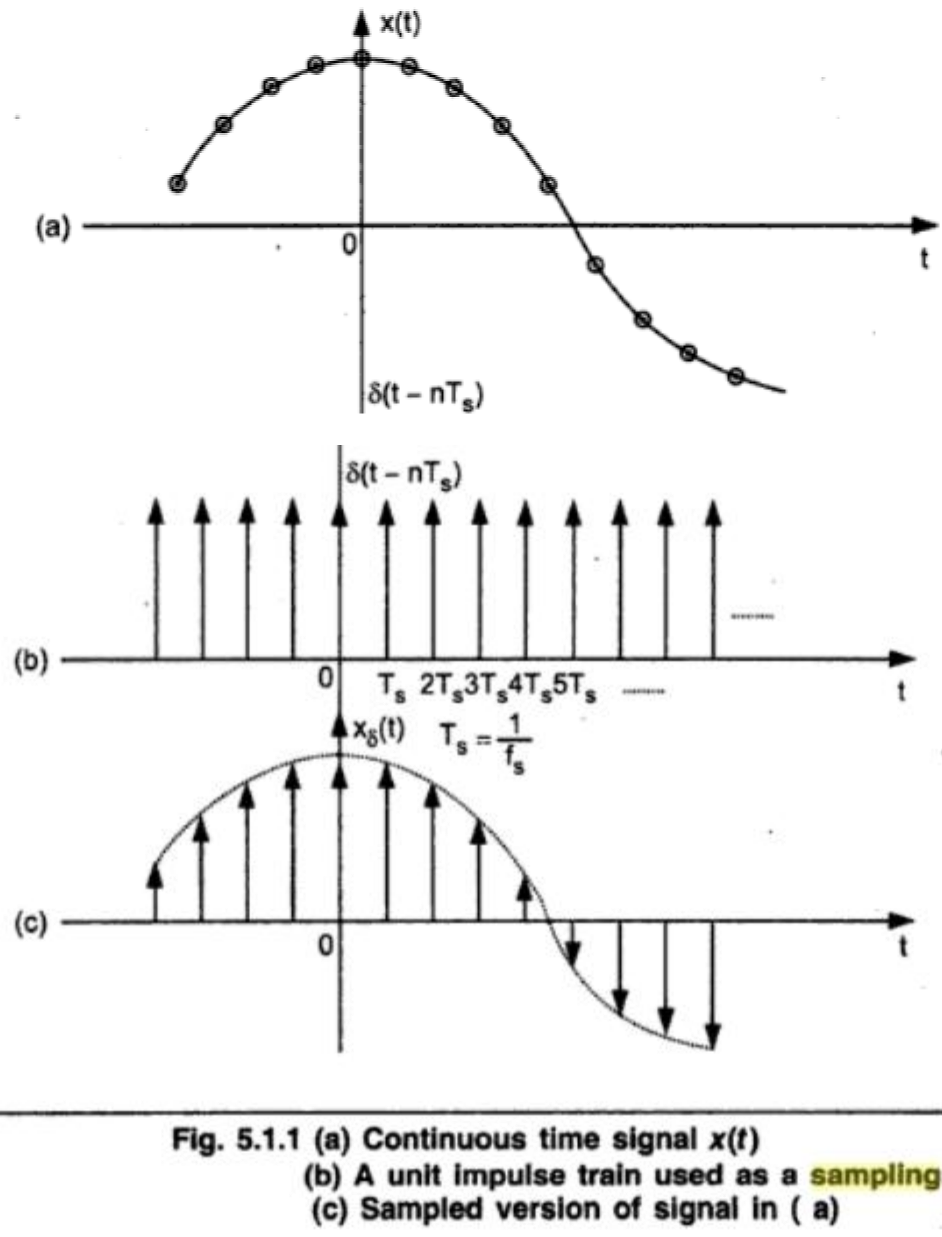


Fig. 5.1.1 (a) Continuous time signal $x(t)$
(b) A unit impulse train used as a sampling function
(c) Sampled version of signal in (a)

Then

$$T_s = \frac{1}{f_s} = \text{represents sampling period} \qquad \dots (5.1.1)$$

The time space between any two successive samples is T_s seconds. Fig. 5.1.1 (b) shows the impulse train of frequency equal to sampling frequency f_s, and Fig. 5.1.1 (c) shows an instantaneously sampled version of signal $x(t)$.

The impulse train of pulses in Fig. 5.1.1 (b) can be expressed as,

$$\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad \dots (5.1.2)$$

Let $x(nT_s)$ represent the instantaneous amplitude of signal $x(t)$ at instant $t = T_s$. This amplitude is shown by encircled dots in Fig. 5.1.1 (a). Each impulse in Fig.5.1.1(b) has amplitude equal to 1. Therefore we can say that the waveform in Fig. 5.1.1 (c) is obtained by multiplying unit impulse with instantaneous value of $x(t)$ i.e. $x(nT_s)$. Therefore waveform of Fig. 5.1.1 (c) can be represented mathematically as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)\, x(nT_s) \qquad \dots (5.1.3)$$

Thus $x_\delta(t)$ is represented by multiplying equation 5.1.2 by $x(nT_s)$. Since the width of the impulse in $x_\delta(t)$ approaches to zero, it represents only instantaneous value. Therefore this method is called *instantaneous sampling*. It is also called *ideal sampling*.

Here $x_\delta(t)$ represents the sampled version of continuous time signal $x(t)$. The Fourier transform of impulse train of equation 5.1.2 is given as

$$X(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s), \text{ Here } f_s = \frac{1}{T_s}$$

Therefore Fourier transform of waveform of Fig. 5.1.1 (c) can be written from above equation as,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad (5.1.4)$$

Here $X(f)$ is the Fourier transform of the original signal $x(t)$. This equation shows that a process of uniformly sampling a continuous time signal results in a periodic spectrum with period equal to simply rate f_s. That is in the equation 5.1.4. Fourier transform of signal $x(t)$ results in $X(f - nf_s)$.

i.e. $\quad X(f - nf_s) \quad = X(f)$ at $f = 0, \pm f_s, \pm 2f_s, \pm 3f_s, \dots$

Thus the same spectrum $X(f)$ appears at $f = 0, f = \pm f_s, f = \pm 2f_s$ etc. This means that a periodic spectrum with period equal to f_s is generated in frequency domain because of sampling $x(t)$ in time domain. Otherwise if $x(t)$ would not have been sampled, then there would be only one spectrum $X(f)$ around $f = 0$.

Equation 5.1.4 can be written as,

$$X_\delta(f) = f_s X(f) + f_s X(f \pm f_s) + f_s X(f \pm 2f_s) + f_s X(f \pm 3f_s) + f_s X(f \pm 4f_s) + \ldots \quad \ldots (5.1.5)$$

This expansion shows that every term in the sum is the same spectrum at multiple of sampling frequency f_s.

Equation 5.1.4 can also be written as,

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - n f_s) \quad \ldots (5.1.6)$$

In this equation, first term represents spectrum that would have been obtained without sampling and rest of the terms under summation represents spectrums repeating at multiple frequencies of sampling frequency f_s.

By definition of a Fourier transform we know that, Fourier transform of continuous time signal $x(t)$ is given as,

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \quad \text{By definition of FT.}$$

If $x(t)$ in the above equation becomes discontinuous in time 't', then integration becomes summation. Fourier transform of equation 5.1.3 becomes,

$$FT[x_\delta(t)] = \sum_{n=-\infty}^{\infty} x(nT_s)\, e^{-j2\pi f n T_s} \quad \ldots (5.1.7)$$

The above equation gives Fourier transform of discrete time signal. Hence it is also called *Discrete Fourier Transform*. In the above equation t is replaced by nT_s.

Let us consider that signal $x(t)$ is strictly bandlimited, with no frequency components higher than W Hertz. That is, the Fourier transform $X(f)$ of $x(t)$ has the property that,

$$|X(f)| = 0 \quad \text{for} \quad |f| \geq W \quad \ldots (5.1.8)$$

The spectrum of such signal is shown in Fig. 5.1.2 (a). The shape of the spectrum is just arbitrary and is taken because it is convenient for explanation. Let the sampling frequency be exactly equal to twice of the maximum frequency in $x(t)$ i.e.,

$$f_s = 2W \text{ or } T_s = \frac{1}{2W}$$

Here W is maximum frequency in $x(t)$.

Consider the spectrum of $x_\delta(t)$ given by equation 5.1.6 i.e.,

$$\boxed{X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - nf_s)} \quad \ldots (5.1.9)$$

This equation shows that same $X(f)$ will be reproduced at $f = 0, f = +f_s$ and $-f_s$,

$$f = +2f_s \text{ and } -2f_s \text{ etc.}$$

Since $f_s = 2W$ i.e. $f_s - W = W$ and

$$f_s + W = 3W ;$$

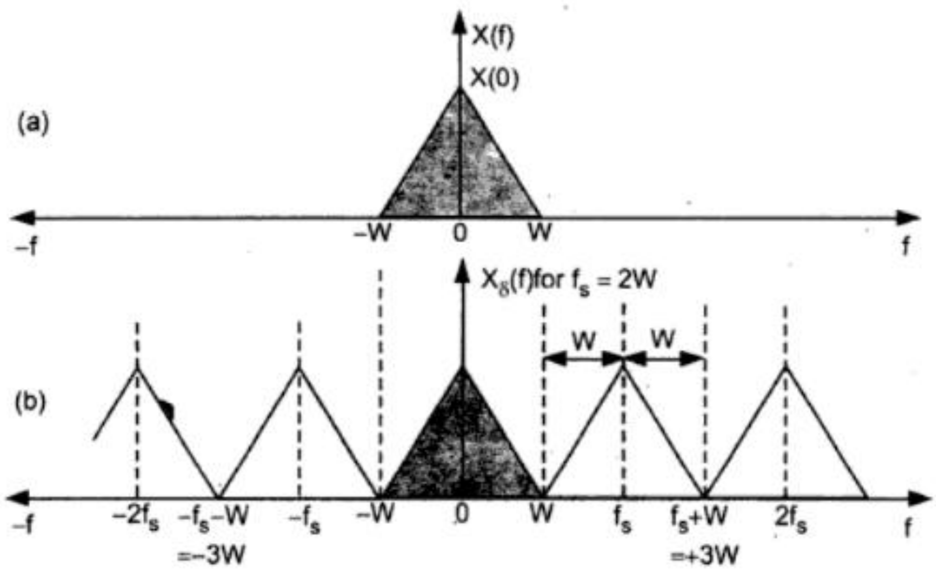


Fig. 5.1.2 **(a) Spectrum of bandlimited signal $x(t)$**
(b) Spectrum of sampled version of $x(t)$ with $T_s = \frac{1}{2W}$ becomes periodic in f_s

Thus we see that the periodic spectrums $X(f)$ just touch each other at $+W, \pm 3W, \pm 5W, \ldots$ etc. It is shown in Fig. 5.1.2 (b).

Now let us see whether the sampled signal completely represents $x(t)$. This will be true if the sample values $x(nT_s)$ are specified not only at $t = T_s$, but for all the time. We know from equation 5.1.9 that,

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - nf_s) \qquad \ldots (5.1.10)$$

We have made two assumptions at the start of this discussion.

1. $X(f) = 0 \quad$ for $|f| \geq W$ by equation 5.1.8
2. $f_s = 2W$ Sampling rate.

Equation 5.1.10 can be written as,

$$f_s X(f) = X_\delta(f) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - nf_s)$$

$$X(f) = \frac{1}{f_s} X_\delta(f) - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} X(f - nf_s)$$

$\therefore f_s = 2W$, above equation will be,

This equation can also be expanded like equation 5.1.17. Therefore we get,

$$\begin{aligned} x(t) = {} & x(0)\, sinc\,(2Wt) + x(\pm T_s)\, sinc\ 2W\,(t \pm T_s) \\ & + x(\pm 2T_s)\, sinc\ 2W\,(t \pm 2T_s) + x(\pm 3T_s)\, sinc\ 2W\,(t \pm 3T_s) \\ & + x(\pm 4T_s)\, sinc\ 2W\,(t \pm 4T_s) + \ldots \end{aligned} \qquad \ldots (5.1.18)$$

This equation is similar to equation 5.1.17 but written in other form. A sinc function given by $sinc\,[2Wt]$ is multiplied by sample value $x(nT_s)$. Therefore amplitude of the sinc pulse changes in accordance with the sample value $x(nT_s)$.

Fig. 5.1.3 (a) shows the sampled version of signal i.e. $x_\delta(t)$ and Fig. 5.1.3 (b) shows the sinc pulses of varying amplitude located at the sampling instants. The peaks of the sinc pulses represent the amplitudes of samples. This signal given by equation 5.1.18 can be passed through a low-pass reconstruction filter to get smooth $x(t)$. Here we have assumed that minimum sampling frequency should be equal to 2W; i.e. twice of maximum sampling frequency. Thus the statement of sampling theorem is proved here that a signal can be completely represented and recovered from its samples if the sampling frequency is twice the maximum signal frequency i.e. $f_s \geq 2W$.

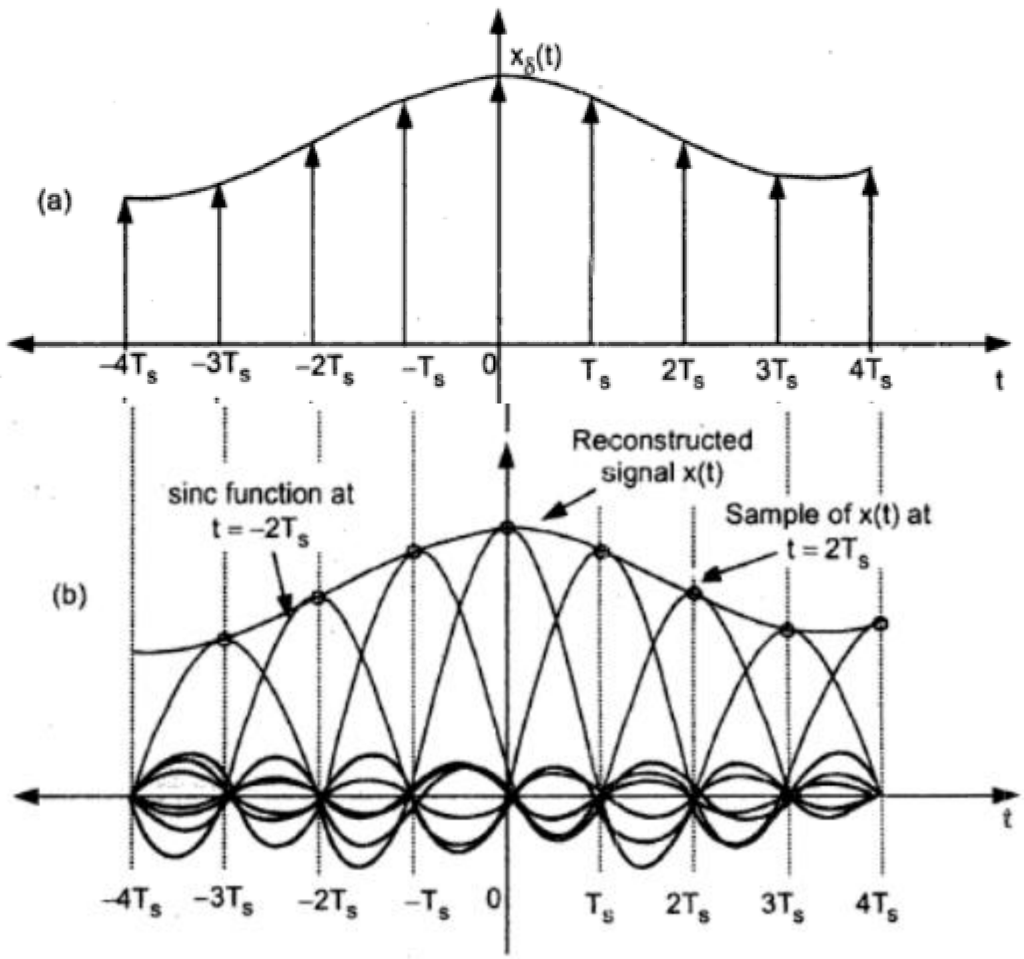


Fig. 5.1.3 (a) Sampled version of signal *x* (*t*)
(b) Reconstruction of *x* (*t*) from its samples

5.1.2 Aliasing

In the discussion of proof of sampling theorem we assumed that the signal $x(t)$ is strictly bandlimited. However in practice an information signal can contain wide range of frequencies and cannot be strictly bandlimited. Therefore the maximum frequency 'W' in the signal $x(t)$ cannot be predictable. Hence it is not possible to select suitable sampling frequency f_s.

Sampling theorem states that, $f_s \geq 2W$

Let us see the two conditions such that sampling frequency is selected to be less than 2W and greater than 2W.

When $f_s < 2W$:

We know that spectrum of a sampled signal is given by equation 5.1.10 as,

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - n f_s) \quad \dots (5.1.19)$$

Let us assume that f_s is slightly less than 2W. According to equation 5.1.19, the spectrums $X(f)$ are placed at

$$f = 0, \pm f_s, \pm 2f_s, \pm 3f_s, \pm \dots..$$

That is $X(f)$ is periodic in f_s. The spectrum shown in Fig. 5.1.2 will be changed as shown in Fig. 5.1.4 (b).

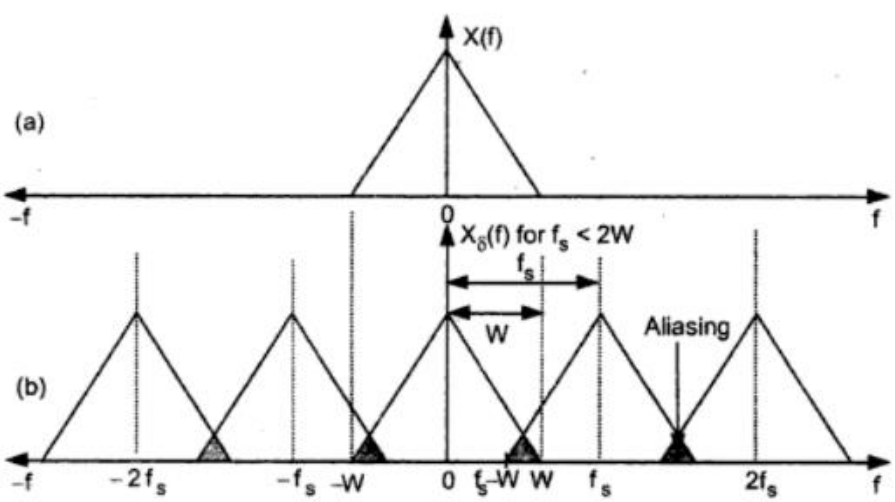


Fig. 5.1.4 (a) Spectrum of continuous time signal $x(t)$
(b) Spectrum of sampled version of $x(t)$ with $f_s < 2W$

Fig. 5.1.4 (b) shows an effect of taking sampling rate less than twice of maximum sampling frequency. The spectrum of a sampled signal consists of spectrums repeating at $\pm n f_s$. As can be seen from Fig. 5.1.4 (b), each spectrum extends to W Hz around the sampling frequency.

$$\text{Since} \quad f_s < 2W$$

$$\text{or} \quad W > \frac{f_s}{2},$$

The spectrums interfere with each other. This is called *aliasing*. Because of aliasing high frequency component in the spectrum of the signal takes identity of lower frequency in the spectrum of its sampled version. This is illustrated in Fig. 5.1.5.

The signal $x_1(t)$ actually has higher frequency F_1 but it is sampled at lower rate, i.e., $f_s < 2W$. When the samples are reconstructed, we get a signal $x_2(t)$ which is of lower frequency F_2. This happens because of aliasing. This shows how high frequency signal takes an identity of low frequency because of aliasing. In the Fig. 5.1.5, the signal $x_1(t)$ has maximum frequency equal to $F_1 = \frac{7}{8}$ Hz. Therefore

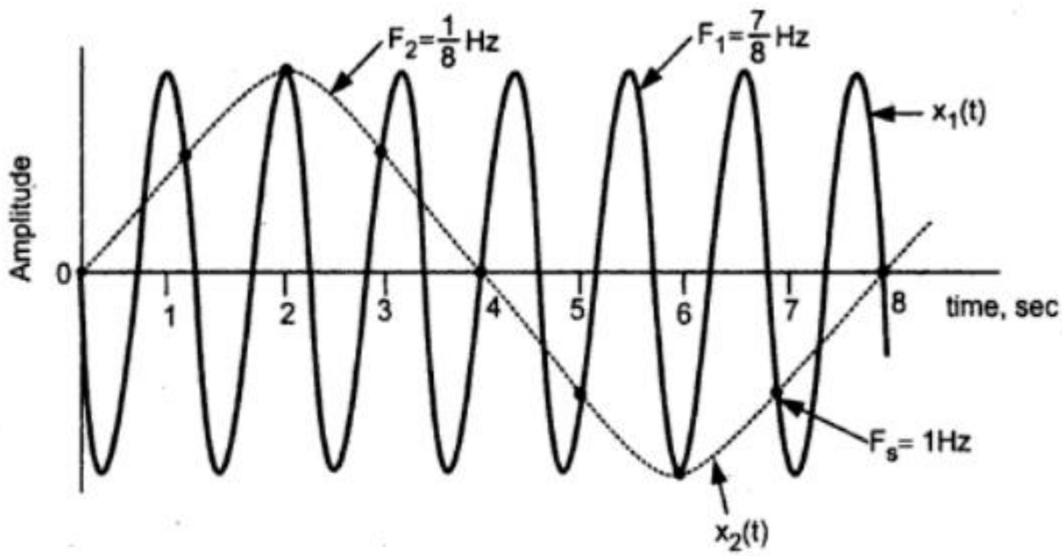


Fig. 5.1.5 Illustration of aliasing

$$W = F_1 = \frac{7}{8} Hz$$

Sampling frequency $f_s = 1\,Hz$

$$\because \quad f_s < 2W \text{ or}$$

$$1\,Hz < 2 \times \frac{7}{8}$$

Hence aliasing occurs in the sampling process and high frequency components are converted to low frequencies. Therefore because of aliasing, the signal is not represented properly with all its contents in its sampled version. Therefore sampling theorem needs to be satisfied.

When $f_s > 2W$:

Now we will consider when sampling frequency is more than twice the maximum signal frequency. Here again we recall equation 5.1.19

easy to decide sampling frequency, since maximum frequency is fixed at 'W' Hz. The low-pass filter is then called *prealias filter,* since it is used to prevent aliasing effect. In other words we can say that to overcome aliasing :

1. Prealias filter should be used to limit band of frequencies of the signal to 'W' Hz.
2. Sampling frequency 'f_s' should be selected such that,

$$f_s > 2W$$

Nyquist Rate : When the sampling rate becomes exactly equal to '2W' samples per second, for a signal bandwidth of W Hertz, then it is called Nyquist rate. The Nyquist interval is then obtained as,

$$\boxed{\begin{aligned} \textbf{Nyquist interval} &= \frac{1}{2W} \textbf{ seconds} \qquad \dots (5.1.22) \\ \textbf{Nyquist rate} &= 2W \textbf{ Hz.} \qquad \dots (5.1.23) \end{aligned}}$$

5.1.3 Reconstruction filter (lowpass filter or interpolation filter)

This is also called interpolation filter. This filter is used to recover original signal from its sampled version. Let us consider that the signal $x(t)$ is passed through a

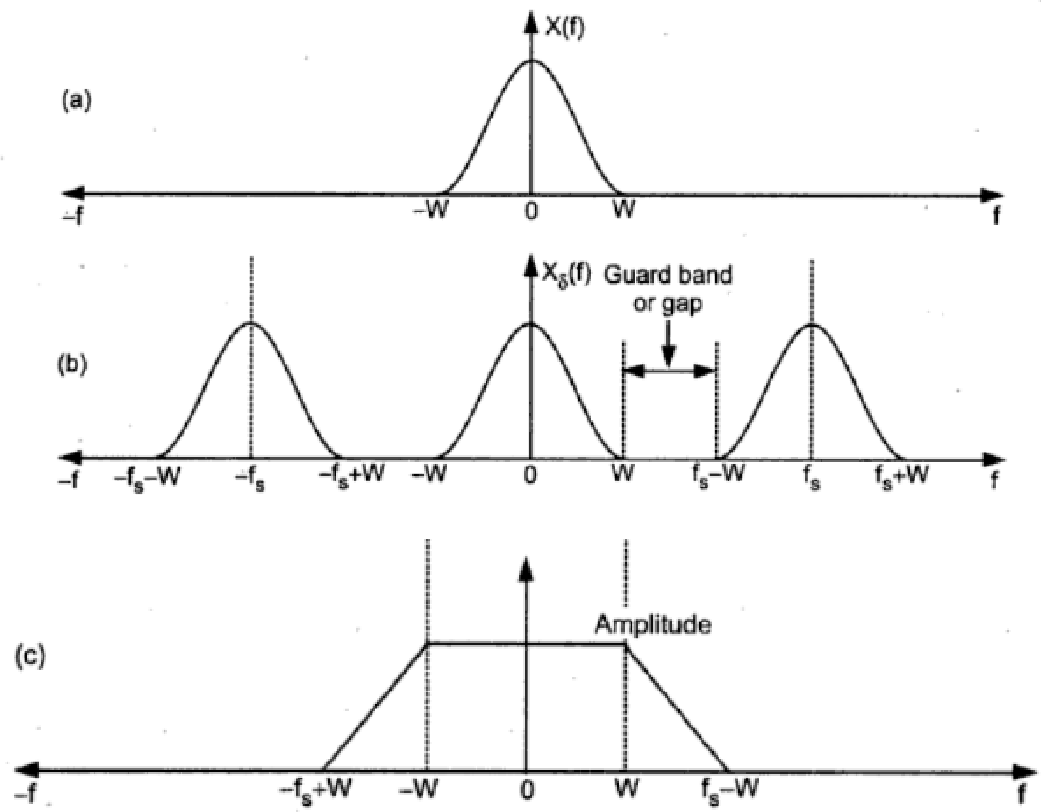


Fig. 5.1.7 (a) Spectrum of information signal after passing signal through prealias filter
(b) Spectrum of sampled version of signal i.e. $X_\delta(f)$ Here $f_s > 2W$
(c) Required amplitude response of the reconstruction filter

prealias filter of cutoff frequency 'W' Hz. Then $x(t)$ will have maximum frequency of 'W' Hz. Let the sampling frequency be selected higher than Nyquist rate i.e.,

$$f_s > 2W$$

The spectrums of $x(t)$ and sampled signal $x_\delta(t)$ are shown in Fig. 5.1.7 (a) and (b).

Fig. 5.1.7 (a) and (b) shows the spectrums, the shapes of which are different from that we have considered previously. This is also an arbitrary shape we have assumed. Normally for an information signal, the spectrum slowly reduces and becomes minimum at $\pm W$ Hz. In the Fig. 5.1.7 (a) and (b) it is assumed to be zero. Now since sampling frequency is greater than Nyquist rate ($f_s > 2W$), a guard band or Gap is produced between neighbouring spectrums as shown in Fig. 5.1.7 (b). That is in the overall spectrum of sampled signal, the spectrums $X(f)$ are periodic in f_s. Because of the guard band there is no chance of aliasing.

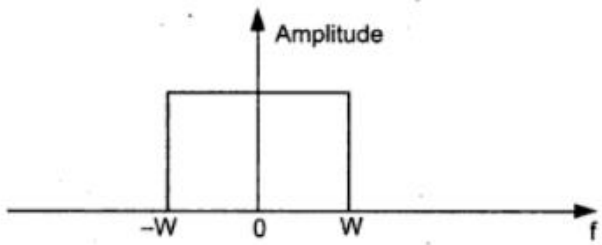


Fig. 5.1.8 Ideal low-pass filter

A reconstruction filter is basically a low-pass filter. This filter should pass all the frequencies between (-W, W), since original signal was having maximum frequency of 'W' Hz. Therefore cut-off frequency of this low-pass reconstruction filter will be 'W' Hz.

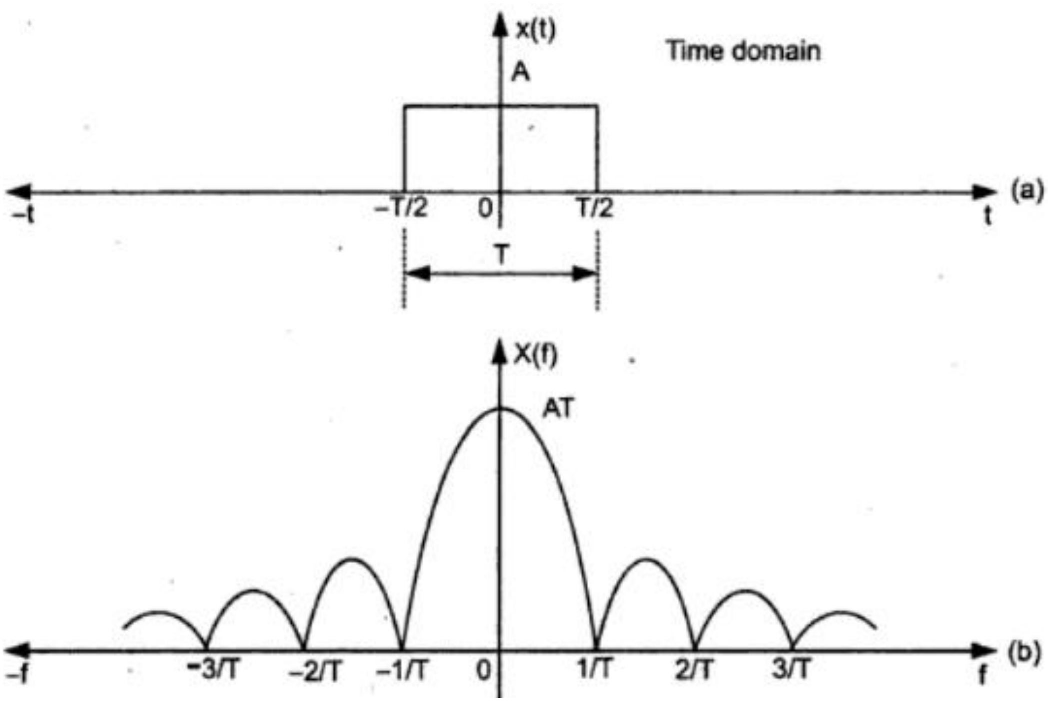
Therefore expected frequency response of the reconstruction filter is as shown in Fig. 5.1.8 . That is, it is an ideal low-pass filter. Fig. 5.1.8 shows frequency response of an ideal low pass filter. As we have seen that an ideal low-pass filter having frequency response given in Fig. 5.1.8 is not physically realizable. Therefore the frequency response of reconstruction filter is as shown in Fig. 5.1.7 (c). That is from W to $(f_s - W)$ Hz a transition band is required.

5.1.4 Sampling Theorem in Frequency Domain

We have seen that if the bandlimited signal is sampled at the rate of $(f_s > 2W)$ in time domain, then it can be fully recovered from its samples. This is sampling theorem in time domain. A dual of this also exists and it is called sampling theorem in frequency domain. It states that, "A timelimited signal which is zero for $|t| > T$ is uniquely determined by the samples of its frequency spectrum at intervals less than $\frac{1}{2T}$ Hertz apart".

Thus the spectrum is sampled at $f_s < \frac{1}{2T}$ in the frequency domain. T is the maximum time limit above which signal $x(t)$ goes to zero. 'f_s' represents the sampling frequency interval in the frequency spectrum of the signal. Note that here f_s does not represent number of samples taken per second. But it represents the frequency interval at which the samples are separated in frequency domain.

Fig. 5.1.9 illustrates the sampling theorem in frequency domain. We can see from 5.1.9 (a) that a rectangular pulse is time limited to $\pm\frac{T}{2}$ seconds i.e., $x(t) = A$ for $-\frac{T}{2} \leq t < \frac{T}{2}$. The spectrum of rectangular pulse is shown in Fig. 5.1.9 (b). This spectrum $X(f)$ of Fig. 5.1.9 (b) is sampled at the uniform intervals less than $\frac{1}{2T}$ Hz. The sampled version of this spectrum is shown in Fig. 5.1.9 (c) and called $X_\delta(f)$. Thus each frequency sample of $x_\delta(f)$ is separated by 'f_s' Hz with respect to the neighbouring frequency samples.



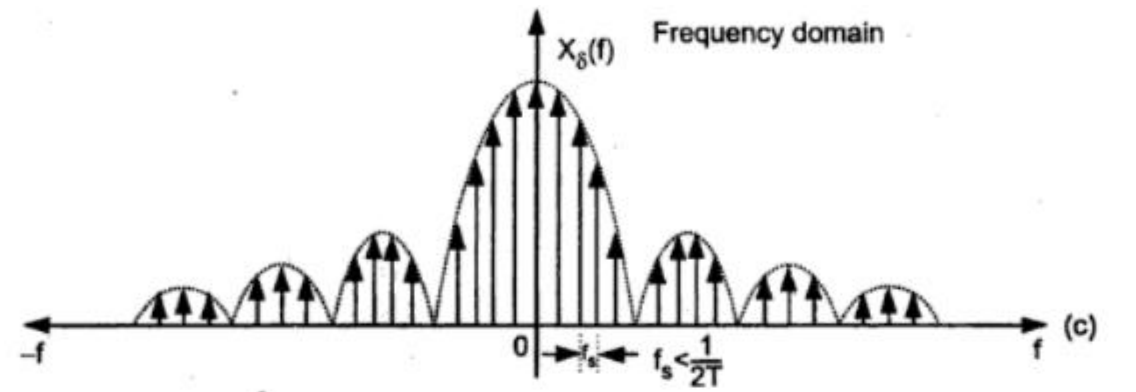


Fig. 5.1.9 **(a) Signal $x(t)$ time limited to $\pm \frac{T}{2}$**
(b) Continuous spectrum of $x(t)$
(c) Sampled spectrum $X_\delta(f)$

The spectrum is centered around frequency f_c. The bandwidth is 2W. Thus the frequencies in the bandpass signal are from $f_c - W$ to $f_c + W$. That is the highest frequency present in the bandpass signal is $f_c + W$. Normally the centre frequency $f_c > W$.

This bandpass signal is first represented in terms of its inphase and quadrature components.

Let $x_I(t)$ = Inphase component of $x(t)$

and $x_Q(t)$ = Quadrature component of $x(t)$

Then we can write $x(t)$ in terms of inphase and quadrature components as,

$$x(t) = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t) \quad \dots (5.1.25)$$

The inphase and quadrature components are obtained by multiplying $x(t)$ by $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ and then suppressing the sum frequencies by means of low-pass filters. Thus inphase $x_I(t)$ and quadrature $x_Q(t)$ components contain only low frequency components. The spectrum of these components is limited between -W to +W. This is shown in Fig. 5.1.11.

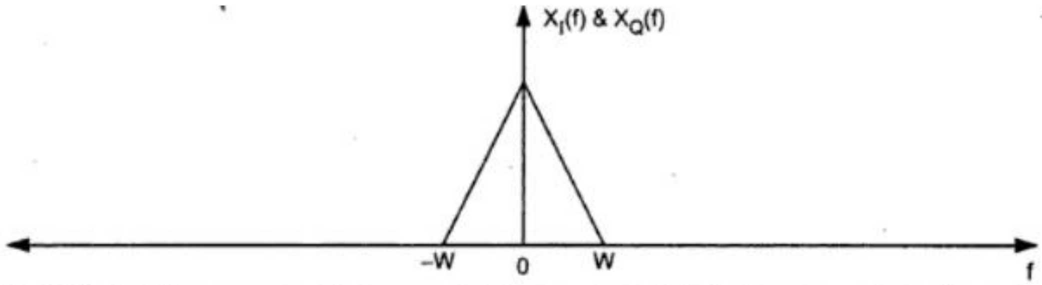


Fig. 5.1.11 Spectrum of inphase and quadrature components of bandpass signal x(t)

After some mathematical manipulations, on equation 5.1.25. We obtain the reconstruction formula as,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{4W}\right) \operatorname{sinc}\left(2Wt - \frac{n}{2}\right) \cos\left[2\pi f_c\left(t - \frac{n}{4W}\right)\right] \quad \ldots (5.1.26)$$

Compare this reconstruction formula with that of low-pass signals given by equation (5.1.16). It is clear that $x(t)$ is represented by $x\left(\frac{n}{4W}\right)$ completely. Here,

$$x\left(\frac{n}{4W}\right) = x(nT_s) = \text{Sampled version of bandpass signal}$$

and $$T_s = \frac{1}{4W}$$

Thus if 4W samples per second are taken, then the bandpass signal of band-width 2W can be completely recovered from its samples.

Thus, for bandpass signals of bandwidth 2W,

Minimum sampling rate = Twice of bandwidth

= 4W samples per second

Since $\text{sinc}\ \theta = \frac{\sin \pi\theta}{\pi\theta}$, above equation becomes,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) sinc\,(2\,Wt - n) \qquad -\infty < n < \infty$$

Reconstruction of signal from samples :

Consider equation 5.1.36,

$$x(t) = IFT\left\{\frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi fn/W}\right\}$$

By definition of Inverse Fourier Transform (IFT) the above equation becomes,

$$x(t) = \int_{-W}^{W} \frac{1}{2W} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) e^{-j\pi fn/W} e^{j2\pi ft}\, dt$$

Interchanging the order of summation of integration,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{1}{2W} \int_{-W}^{W} e^{j2\pi f\left(t - \frac{n}{2W}\right)}\, dt$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{\sin\,(2\pi\,Wt - n\pi)}{(2\,\pi\,Wt - n\pi)}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{\sin\,\pi(2\,Wt - n)}{\pi(2\,Wt - n)}$$

Since $\text{sinc } \theta = \frac{\sin \pi\theta}{\pi\theta}$, above equation becomes,

$$x(t) = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \text{sinc}\,(2\,Wt - n) \qquad -\infty < n < \infty$$

This is interpolation formula to reconstruct $x(t)$ from its samples $x(nT_s)$.

Thus the above discussion shows that the signal can be completeiy represented into and recovered from its samples if the spacing between the successive samples is $\frac{1}{2W}$ seconds. i.e. $f_S = 2W$ samples per second.

Sampling frequency for bandpass signal :

The spectral range of the bandpass signal is 20 to 82 kHz.

$$\text{Bandwidth} = 2W = 82 \text{ kHz} - 20 \text{ kHz} = 62 \text{ kHz}$$

$$\begin{aligned}\text{Minimum Sampling rate} &= 2 \times \text{ Bandwidth} \\ &= 2 \times 62 \text{ kHz} \\ &= 124 \text{ kHz}\end{aligned}$$

Normally the range of minimum sampling frequencies is specified for bandpass signals. It lies between 4 W to 8 W samples per second.

$\therefore$ Range of minimum sampling frequencies

$$\begin{aligned}&= (2 \times \text{Bandwidth}) \text{ to } (4 \times \text{Bandwidth}) \\ &= 2 \times 62 \text{ kHz to } 4 \times 62 \text{ kHz}\end{aligned}$$

$= 124$ kHz to 248 kHz

UNIT III:
SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

Systems

> *Definition :*
>
> ***A system is a set of elements or functional blocks that are connected together and produce an output in response to an input signal.***

Classification

- There are two types of systems : (i) continuous time and (ii) discrete time systems.
- Continuous time (CT) systems handle continuous time signals. Analog filters, amplifiers, attenuators, analog transmitters and receivers etc are examples of continuous time systems.
- Discrete time (DT) systems handle discrete time signals. Fig. 1.6.1 (b) shows such system. Computers, printers, microprocessors, memories, shift registers etc are examples of discrete time systems. They operate only on discrete time signals.

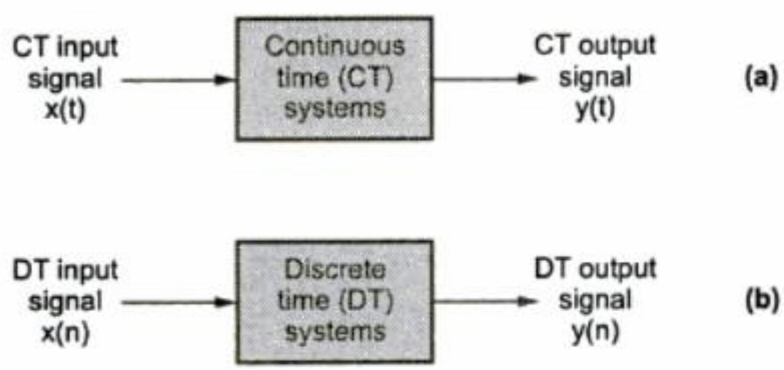


Fig. 1 Two types of systems based on signals they handle

Continuous as well as discrete time systems can be further classified based on their properties. These properties are as follows :

i) Dynamicity property : Static and dynamic systems.

ii) Shift invariance : Time invariant and time variant systems.

iii) Linearity property : Linear and non-linear systems.

iv) Causality property : Causal and non-causal systems.

v) Stability property : Stable and unstable systems.

vi) Invertibility property : Inversible and non-inversible systems.

1 Static and Dynamic Systems (Systems with Memory or without Memory)

> *Definition : The continuous time system is said to be static or (memoryless, instantaneous) if its output depends upon the present input only.*

The discrete time systems can also be static or dynamic. If output of the discrete time system depends upon the present input sample only, then it is called static or memoryless or instantaneous system. For example,

$$y(n) = 10 \cdot x(n)$$

or

$$y(n) = 15 \cdot x^2(n) + 10x(n)$$

are the static systems. Here the $y(n)$ depends only upon n^{th} input sample. Hence such systems do not need memory for its operation. A system is said to be dynamic if the output depends upon the past values of input also. For example,

$$y(n) = x(n) + x(n-1)$$

2 Time Invariant and Time Variant Systems

Definition : *A continuous time system is time invariant if the time shift in the input signal results in corresponding time shift in the output.*

Let $y(t) = f[x(t)]$ i.e. $y(t)$ is response for $x(t)$. Then if $x(t)$ is delayed by time t_1, then output $y(t)$ will also be delayed by the same time. i.e.,

$$f[x(t-t_1)] = y(t-t_1) \qquad \text{... (1.6.1)}$$

The time variant system do not satisfy above relation. The time invariant systems are also called fixed systems.

Similarly if the input/output characteristics of the discrete time system do not change with shift of time origin, such systems are called shift invariant or time invariant systems. Let the system has input $x(n)$ and corresponding output $y(n)$, i.e. $y(n) = f[x(n)]$. Then the system is shift invariant or time invariant if and only if,

$$f[x(n-k)] = y(n-k)$$

3 Linear and Non-linear Systems

Definition : *A system is said to be linear if it satisfies the superposition principle.*

Consider the two systems defined as follows :

$y_1(t) = f[x_1(t)]$ i.e. $x_1(t)$ is input and $y_1(t)$ is output.

and $y_2(t) = f[x_2(t)]$ i.e. $x_2(t)$ is input and $y_2(t)$ is output.

Then the continuous time system is linear if,

$$f[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

Here a_1 and a_2 are arbitrary constants. This condition states that combined response due to $x_1(t)$ and $x_2(t)$ together is same as the sum of individual responses for a linear system.

Similarly, the discrete time system is said to be linear if it satisfies superposition principle. Consider the two systems defined as follows :

$y_1(n) = f[x_1(n)]$ i.e. $x_1(n)$ is input and $y_1(n)$ is output.

$y_2(n) = f[x_2(n)]$ i.e. $x_2(n)$ is input and $y_2(n)$ is output.

Then the discrete time system is linear if,

$$f[a_1 x_1(n) + a_2 x_2(n)] = a_1 y_1(n) + a_2 y_2(n)$$

4 Causal and Non-causal Systems

> **Definition** : *The system is said to be causal if its output at any time depends upon present and past inputs only.*

i.e.,

$$y(t_0) = f\left[x(t); t \le t_0\right]$$

Thus the output at time t_0, depends on inputs before t_0. The causal system is not anticipatory. Similarly, a discrete time system is said to be causal if its output at any instant depends upon present and past input samples only. i.e.,

$$y(n) = f\left[x(k); k \le n\right]$$

Thus the output is the function of $x(n)$, $x(n-1), x(n-2), x(n-3)$ ... etc. For causal system. The system is non-causal if its output depends upon future inputs also, i.e. $x(n+1)$, $x(n+2)$, $x(n+3)$ ... etc.

Normally all causal systems are physically realizable. There is no system which can generate the output for inputs which will be available in future. Such systems are non-causal, and they are not physically realizable.

.5 Stable and Unstable Systems

> **Definition** : *When every bounded input produces bounded output, then the system is called Bounded Input Bounded Output (BIBO) stable.*

This criteria is applicable for both the continuous time and discrete time systems. The input is said to be bounded if there exists some finite number M_x such that,

$$\left.\begin{array}{ll} \text{CT input :} & |x(t)| \le M_x < \infty \\ \text{DT input :} & |x(n)| \le M_x < \infty \end{array}\right\}$$

Similarly the output is said to be bounded if there exists some finite number M_y such that,

$$\left.\begin{array}{ll} \text{CT output :} & |y(t)| \le M_y < \infty \\ \text{DT output :} & |y(n)| \le M_y < \infty \end{array}\right\}$$

If the system produces unbounded output for bounded input, then it is unstable.

6 Invertability and Inverse Systems

Definition : ***A system is said to be invertible if there is unique output for every unique input.***

Fig. 1.6.3 shows this concept.

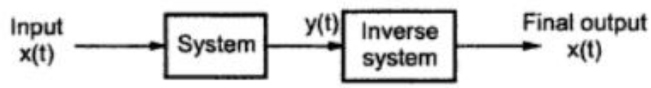


Fig. 1.6.3 Invertible system

If the system is invertible, there exists an inverse system. If these two systems are cascaded as shown in figure, then final output is same as input.

If the system is denoted by H, then its inverse system is denoted by H^{-1}. Then cascading the two systems gives,

$$H\,H^{-1} = 1$$

Frequency Response of LTI Systems

The LTI systems form an important class in communication. The amplitude and phase response, realizability, bandwidth, distortion during transmission of signal are all very important concepts related to design and implementation of systems.

Frequency Response

The frequency response of the system gives the variation of magnitude and phase of the system output with respect to frequency on application of input. We know that the output y (t) of the system is given as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

This equation gives time response of the LTI system.

The RHS of the above equation represents convolution of input signal x (t) and impulse response h (t). By applying fourier transform to above equation,

$$F[y(t)] = F\left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right]$$

We know that convolution of two functions is transformed into multiplication of their fourier transforms. By applying this to above equation,

$$y(t) = K\, x(t-t_0)$$

Here, K = constant represents change in amplitude.

& t_0 = time delay in transmission of signal through a system.

By taking fourier transform of both sides of above equation

$$Y(f) = F[y(t)] = F\left\{K\, x(t-t_0)\right\}$$

$\therefore$ From the time shifting property of FT,

$$Y(f) = K X(f)\, e^{-j2\pi f t_0}$$

Transfer function H (f) is given from equation 2.13.2

$$H(f) = \frac{Y(f)}{X(f)}$$

Putting for RHS from equation 2.13.4 in above equation,

$$H(f) = K\, e^{-j2\pi f t_0}$$

This equation gives the transfer function for a distortionless system. It is clear from above equation that, the magnitude of the transfer function is 'K', which is independent of frequency. That is the transfer function has constant amplitude at all frequencies. The phase shift of above equation is,

$$\theta(f) = -2\pi f t_0$$
$$= (-2\pi t_0) f$$

That is the phase shift is linearly proportional to frequency. Here the phase shift is linear at all frequencies. This can be expressed with the example.

Let there be a signal in time domain as

$$x(t) = \cos(2\pi f t)$$

Now let the output signal be same in amplitude but shifted in time by t_0 seconds. i.e.

$$y(t) = \cos[2\pi f(t - t_0)]$$

This equation can also be written as,

$$y(t) = \cos(2\pi f t - 2\pi f t_0) = \cos(2\pi f t - \theta(f))]$$

Thus phase shift of y (t) is,

$$\theta(f) = -2\pi f t_0$$

which is proportional to frequency `f'.

$$F\left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right] = X(f)\, H(f)$$

Here, H(f) is called transfer function of the system,

$$\therefore \qquad Y(f) = H(f)\, X(f)$$

Thus for a Linear Time Invariant system fourier transform of the output is equal to product of the transfer function of the system and fourier transform of the input. The above equation gives frequency response of the system.

Distortionless Transmission Through System

A distortion less transmission means output of the system is an exact replica of the input signal. The difference between input and output of such system is that,

1. Amplitude of the output signal may increase or decrease by some factor with respect to input and
2. The output signal may be delayed in time with respect to input signal because of system delay.

Therefore output signal y (t) can be written interms of input x (t) as,

Response of a Linear System

Impulse Response

Convolution relates input and output of LTI sytem.

It is given as,

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

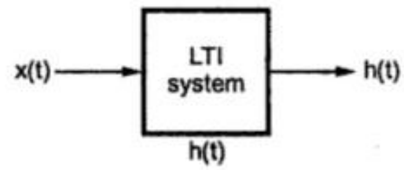


Fig. 1 Input and output of LTI system

Here h (t) is called impulse response of the system.

It is characteristic of a particular system. Impulse response h (t) of the system is obtained at the output by applying unit impulse $\delta(t)$ at the input. i.e.,

when $\quad x(t) = \delta(t),\ y(t) = h(t)$

Frequency Response

Frequency response analysis and differential equations etc. can be analyzed with the help of Fourier representations. For example, the fourier transform $X(\omega)$ gives frequency spectrum of the signal. We know that output of the system is,

$$y(t) = x(t) * h(t)$$

By convolution theorem above equation becomes

$$Y(\omega) = X(\omega) \cdot H(\omega) \quad \text{or} \quad Y(f) = X(f) \cdot H(f)$$

and
$$y(t) = IFT\{X(\omega) \cdot H(\omega)\}$$

Thus output $y(t)$ can be obtained by taking the inverse fourier transform of the product $X(\omega)$. Let us now study these aspects.

The convolution is given as,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Let the input be $e^{j\omega t}$, i.e. sinusoid. Then above equation becomes,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

In the above equation the integral represents fourier transform of $h(\tau)$. i.e.,

In the above equation the integral represents fourier transform of $h(\tau)$. i.e.,

$$y(t) = e^{j\omega t}\, H(\omega)$$

Here $H(\omega)$ is the Fourier transform of $h(t)$. The above equation shows that output $y(t)$ contains the same signal as input $e^{j\omega t}$ multiplied by $H(\omega)$. This $H(\omega)$ is called frequency response of the system.

Again consider the convolution,

$$y(t) = x(t) * h(t)$$

By convolution property of Fourier transform we can write above equation as,

$$Y(\omega) = X(\omega)\, H(\omega) \quad \text{or} \quad Y(f) = X(f) \cdot H(f)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \quad \text{or} \quad H(f) = \frac{Y(f)}{X(f)}$$

Here $H(\omega)$ represent the frequency response of the LTI-CT system. These functions are also called as system transfer functions.

Example .1 : *The impulse response of the continuous time system is given as,*

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Determine the frequency response and plot the magnitude phase plots.

Solution : Take Fourier transform of the given impulse response. i.e.,

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{RC} e^{-t/RC} u(t) e^{-j\omega t} dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t/RC} e^{-j\omega t} dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t\left(j\omega + \frac{1}{RC}\right)} dt$$

$$= \frac{1}{RC} \left(-\frac{1}{j\omega + \frac{1}{RC}}\right) \left[e^{-t\left(j\omega + \frac{1}{RC}\right)}\right]_0^{\infty}$$

$$= \frac{1/RC}{j\omega + 1/RC} = \frac{1}{1 + j\omega RC}$$

Now let us determine the magnitude and phase of $H(\omega)$. Let us rearrange above equation as,

$$H(\omega) = \frac{1}{1+j\omega RC} \times \frac{1-j\omega RC}{1-j\omega RC} = \frac{1-j\omega RC}{1+(\omega RC)^2}$$

$$= \frac{1}{1+(\omega RC)^2} + j\frac{-\omega RC}{1+(\omega RC)^2}$$

Thus $H(\omega)$ is expressed into its real and imaginary parts. Now magnitude can be obtained as,

$$|H(\omega)| = \left\{\frac{1}{[1+(\omega RC)^2]^2} + \frac{(\omega RC)^2}{[1+(\omega RC)^2]^2}\right\}^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}}$$

This is the magnitude response of the given system. And the phase response will be,

$$\angle H(\omega) = tan^{-1}\left\{\frac{(-\omega RC)\,/\,[1+(\omega RC)^2]}{1\,/\,[1+(\omega RC)^2]}\right\}$$

$$= -\,tan^{-1}\,(\omega RC)$$

Let $RC = 1$, then magnitude and phase response will be,

$$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

Fig. .2 shows the magnitude and phase response as given by above equations.

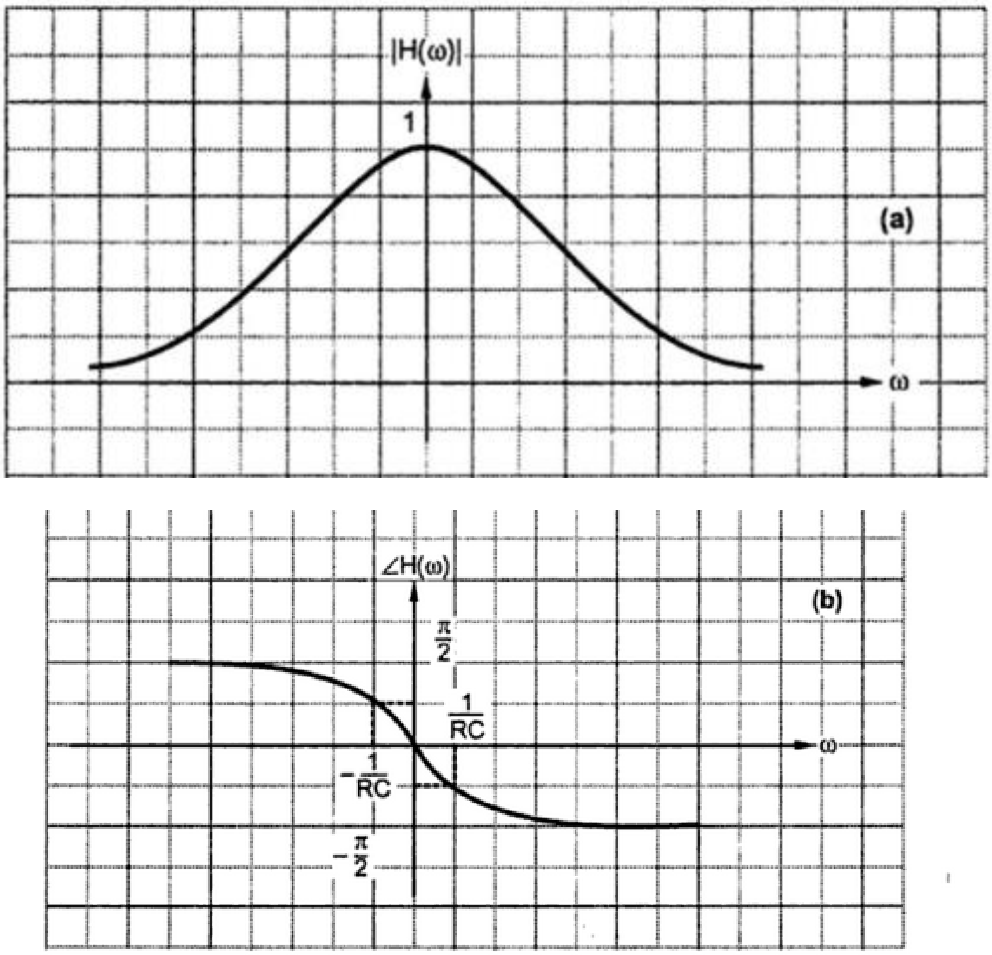


Fig. 4.4.2 (a) Magnitude response (b) Phase response

In this figure observe that the magnitude response is symmetric but phase response is antisymmetric. Magnitude response is monotonically decreasing. Hence this is a lowpass filter.

➡ **Example .2 :** *The system produces the output of* $y(t) = e^{-t}u(t)$ *for an input of* $x(t) = e^{-2t}\,u(t)$. *Determine the impulse response and frequency response of the system.*

Solution : Here $\quad y(t) = e^{-t}\,u(t)$

and
$$x(t) = e^{-2t}\,u(t)$$

Consider the standard Fourier transform pair,

$$e^{-at}\,u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

Hence fourier transforms of $y(t)$ and $x(t)$ will be,

$$Y(\omega) = \frac{1}{1+j\omega}$$

and
$$X(\omega) = \frac{1}{2+j\omega}$$

From equation 4.5.3 we can obtain the transfer function as,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Putting the values of $X(\omega)$ and $Y(\omega)$, Putting the values of $X(\omega)$ and $Y(\omega)$,

$$H(\omega) = \frac{1/(1+j\omega)}{1/(2+j\,\omega)} = \frac{2+j\omega}{1+j\omega}$$

Let us multiply the numerator and denominator by $1-j\omega$ i.e.,

$$H(\omega) = \frac{2+j\omega}{1+j\omega} \times \frac{1-j\omega}{1-j\omega}$$

$$= \frac{2+(\omega)^2}{1+(\omega)^2} + j\,\frac{-\omega}{1+(\omega)^2}$$

Hence magnitude of $H(\omega)$ will be,

$$|H(\omega)| = \left\{\left[\frac{2+(\omega)^2}{1+(\omega)^2}\right]^2 + \left[\frac{(\omega)}{1+(\omega)^2}\right]^2\right\}^{\frac{1}{2}}$$

Simplifying the above equation we get,

$$|H(\omega)| = \sqrt{\frac{4+(\omega)^2}{1+(\omega)^2}}$$

This is the magnitude response of the system. And the phase response will be,

$$\angle H(\omega) = \tan^{-1} \frac{(-\omega)\,/\,[1+(\omega)^2]}{[2+(\omega)^2]\,/\,[1+(\omega)^2]} = -\tan^{-1}\left\{\frac{\omega}{2+(\omega)^2}\right\}$$

Now consider the transfer function of equation 4.4.8. i.e.,

$$H(\omega) = \frac{2+j\omega}{1+j\omega}$$

Let us rearrange the above equation as,

$$H(\omega) = \frac{1+j\omega+1}{1+j\omega} = 1+\frac{1}{1+j\omega}$$

Inverse Fourier transform of above equation becomes,

$$h(t) = IFT\{H(\omega)\} = \delta(t)+e^{-t}\,u(t)$$

This is the impulse response of the given system.

.1 Ideal Low Pass Filters

An ideal low-pass filter transmits (passes to output) all of the signals below certain frequency 'B' Hz without any distortion. The range of frequencies from '0' Hz to 'B' Hz is called passband of the lowpass filter. It rejects all the signals which lie outside of the passband. The frequency 'B' Hz is called cut-off frequency of the ideal lowpass filter.

Since the filter is ideal and distortionless the phase change should follow equation

Therefore transfer function of ideal lowpass filter can be written as,

$$H(f) = K\, e^{-j2\pi f t_0}\ ; \qquad -B \le f < B$$

$$= 0 \qquad ; \qquad |f| > B \qquad \dots (\cdot\ 1)$$

This equation is obtained from equation 4.5.3, we earlier derived for distortionless transmission. Here the amplitude 'K' can be assumed to be unity for convenience. ∴ By K = 1 in above equation, the transfer function will be,

$$H(f) = e^{-j2\pi f t_0}\ ; \qquad -B \le f < B$$

$$= 0 \qquad ; \qquad |f| > B \qquad \dots (\cdot.2)$$

We know that transfer function H (f) is the Fourier transform of impulse response h (t). Therefore h (t) can be obtained for ideal lowpass filter by taking IFT of equation .2. Therefore by definition of IFT,

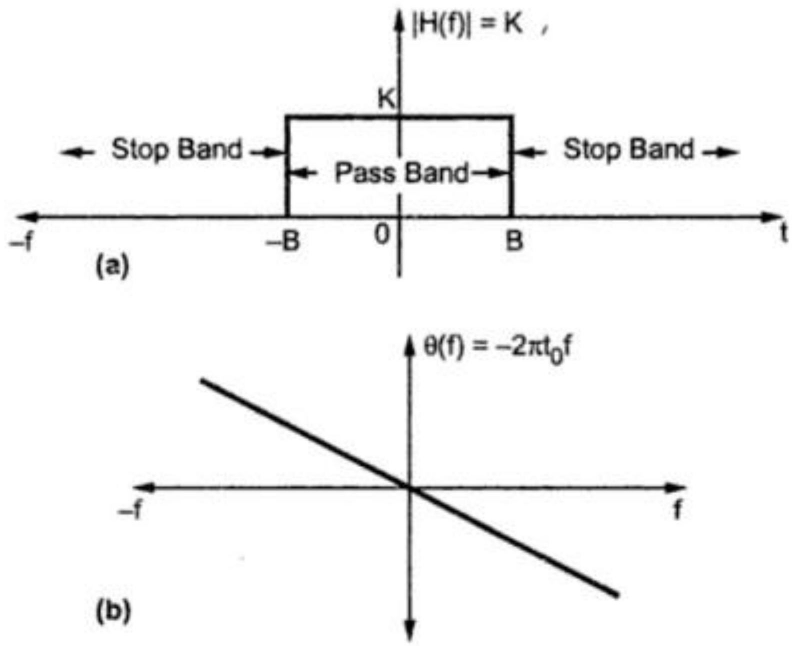


Fig. 1 (a) Magnitude response of ideal lowpass filter
(b) Phase response of ideal lowpass filter

$$h(t) = \int_{-B}^{B} e^{-j2\pi f t_0} \; e^{j2\pi f t} \; df$$

$$= \int_{-\infty}^{\infty} e^{j2\pi f(t-t_0)} \, df$$

$$= \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi f(t-t_0)}\right]_{-B}^{B}$$

$$= \frac{1}{j2\pi(t-t_0)} \left[e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)}\right]$$

$$= \frac{1}{\pi(t-t_0)} \left[\frac{e^{j2\pi B(t-t_0)} - e^{-j2\pi B(t-t_0)}}{2j}\right]$$

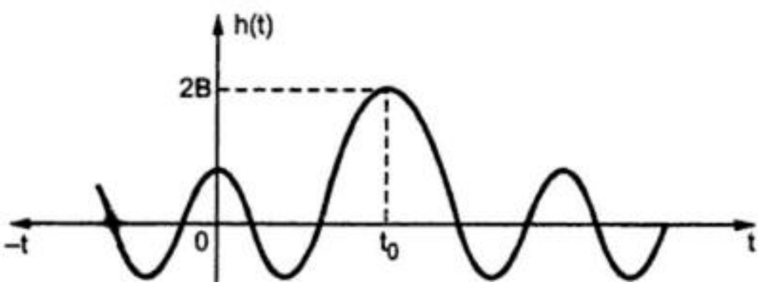
$$= \frac{1}{\pi(t-t_0)} \sin\left[2\pi B(t-t_0)\right]$$

$$= 2B\left(\frac{\sin\left[2\pi B(t-t_0)\right]}{2\pi B(t-t_0)}\right)$$

$$= 2B \, sinc\left[2B(t-t_0)\right]$$

The above equation gives impulse response of an ideal lowpass filter.

Fig. .2 shows the impulse response h (t) of above equation of ideal lowpass filter.



The above figure shows that impulse response exists for negative values of 't'. But actually unit impulse is applied at t = 0 always. Thus the response appears before the unit impulse is applied. Practically it is impossible to implement such a system. And h (t) = 0 for $t < 0$ for causal system.

Therefore it is clear that although ideal lowpass filter is very desirable it cannot be physically realizable. Practically unit impulse to the filter is applied at t = 0 [i.e. $\delta(t) = 1$ applied at t = 0], then impulse response of the filter should start at the most at $t \geq 0$, and not at t < 0, i.e. negative values of 't'.

.2 Other Ideal Filters such as HPF, BPF etc.

In previous section we studied realizability of ideal lowpass filter. But its response begins before input is applied. This means ideal lowpass filter is anticipatory. Hence it is not physically realizable. Similarly other filters like ideal highpass filters and ideal bandpass filters have frequency response as shown in Fig. 4.6.3. These filters have sharp transition at cut-off frequencies. These sharp transition in frequency response results in non-casual impulse response (i.e. it begins before input is applied.) This means all ideal filters are physically not realizable since their impulse response is non-casual.

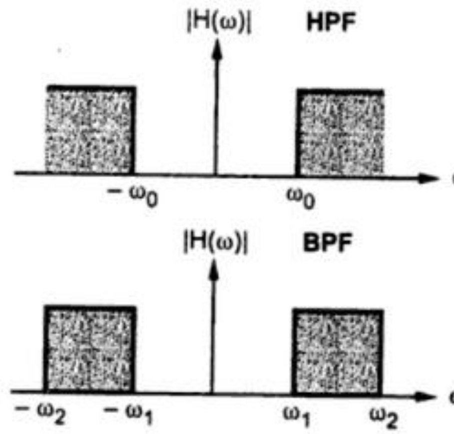


Fig. .3 Ideal filters

UNIT IV:

CONVOLUTION AND CORRELATION OF SIGNALS

.1 INTRODUCTION

Convolution is a mathematical way of combining two signals to form a third signal. Convolution is important because it relates the input signal and impulse response of the system to the output of the system. Correlation is again a mathematical operation that is similar to convolution. Correlation also uses two signals to form a third signal. It is very widely used in practice, particularly in communication engineering. Basically, it compares two signals in order to determine the degree of similarity between them. Radar, sonar, and digital communications use correlation of signals very extensively. Correlation may be cross correlation or autocorrelation. When one signal is correlated with another signal to form a third signal, it is called cross correlation. When a signal is correlated with itself to form another signal, it is called autocorrelation.

In this chapter, the properties of convolution, convolution theorems associated with Fourier transforms and graphical convolution of two signals are discussed. Also correlation, energy density spectra, power density spectra, Rayleigh's theorem and Parseval's theorem are discussed.

.2 CONCEPT OF CONVOLUTION

Convolution is a mathematical operation which is used to express the input–output relationship of an LTI system. It is a most important operation in LTI continuous-time systems. It relates input and impulse response of the system to output.

An arbitrary driving function $x(t)$ can be expressed as a continuous sum of impulse functions. The response $y(t)$ is then given by the continuous sum of responses to various impulse components. In fact, the convolution integral precisely expresses the response as a continuous sum of responses to individual impulse components.

Consider an LTI system which is initially relaxed at $t = 0$. If the input to the system is an impulse, then the output of the system is denoted by $h(t)$ and is called the impulse response of the system.

The impulse response is denoted as:

$$h(t) = T[\delta(t)]$$

We know that any arbitrary signal $x(t)$ can be represented as:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

The system output is given by

$$y(t) = T[x(t)] \quad = \quad y(t) = T\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\, d\tau\right]$$

$\therefore$ For a linear system,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, T[\delta(t-\tau)]\, d\tau$$

If the response of the system due to impulse $\delta(t)$ is $h(t)$, then the response of the system due to delayed impulse is:

$$h(t, \tau) = T[\delta(t - \tau)]$$

Substituting this value of $T[\delta(t - \tau)]$ in the expression for $y(t)$, we have

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

This is called convolution integral, or simply convolution. The convolution of two signals $x(t)$ and $h(t)$ can be represented as:

$$y(t) = x(t) * h(t)$$

In general, the lower and upper limits of integration in the convolution integral depend on whether the signal $x(t)$ and the impulse response $h(t)$ are causal or not. If $h(t)$ is causal, then $h(t - \tau) = 0$ for $\tau > t$. Therefore, the upper limit of integration is t for a causal $h(t)$. If $x(t)$ is causal, then $x(t) = 0$ for $t < 0$. Therefore, the lower limit of integration is 0 for a causal $x(t)$. Thus,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{if both } x(t) \text{ and } h(t) \text{ are non-causal}$$

$$= \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau \quad \text{if } x(t) \text{ is non-causal and } h(t) \text{ is causal}$$

$$= \int_{0}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \quad \text{if } x(t) \text{ is causal and } h(t) \text{ is non-causal}$$

$$= \int_{0}^{t} x(\tau)\, h(t-\tau)\, d\tau \quad \text{if both } x(t) \text{ and } h(t) \text{ are causal}$$

.3 PROPERTIES OF CONVOLUTION

Let us consider two signals $x_1(t)$ and $x_2(t)$. The convolution of two signals $x_1(t)$ and $x_2(t)$ is given by

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x_2(\tau)\, x_1(t-\tau)\, d\tau$$

The properties of convolution are as follows:

Commutative property The commutative property of convolution states that

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

Distributive property The distributive property of convolution states that

$$x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$

Associative property The associative property of convolution states that

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

Shift property The shift property of convolution states that if

$$x_1(t) * x_2(t) = z(t)$$

Then $\quad x_1(t) * x_2(t-T) = z(t-T)$

Similarly, $\quad x_1(t-T) * x_2(t) = z(t-T)$

and $\quad x_1(t-T_1) * x_2(t-T_2) = z(t-T_1-T_2)$

Convolution with an impulse: Convolution of a signal $x(t)$ with a unit impulse is the signal itself. That is,

$$x(t) * \delta(t) = x(t)$$

Width property Let the duration of $x_1(t)$ and $x_2(t)$ be T_1 and T_2 respectively. Then the duration of the signal obtained by convolving $x_1(t)$ and $x_2(t)$ is $T_1 + T_2$.

EXAMPLE 1 Find the convolution of the following signals:

(i) $x_1(t) = e^{-2t}u(t)$; $x_2(t) = e^{-4t}u(t)$

(ii) $x_1(t) = t\,u(t)$; $x_2(t) = t\,u(t)$

(iii) $x_1(t) = \cos t\,u(t)$; $x_2(t) = u(t)$

(iv) $x_1(t) = e^{-3t}u(t)$; $x_2(t) = u(t+3)$

(v) $x_1(t) = r(t)$; $x_2(t) = e^{-2t}u(t)$

Solution:

(i) Given $$x_1(t) = e^{-2t}u(t);\ x_2(t) = e^{-4t}u(t)$$

We know that $$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau$$

$$\therefore \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} e^{-2\tau}u(\tau)\,e^{-4(t-\tau)}u(t-\tau)\,d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t-\tau) = 1$ for $(t-\tau) \geq 0$ or for $\tau < t$.
Hence $u(\tau)\,u(t-\tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\,u(t-\tau) = 0$.

$$\therefore \quad x_1(t) * x_2(t) = \int_0^t e^{-2\tau}\,e^{-4(t-\tau)}\,d\tau$$

$$= e^{-4t}\int_0^t e^{2\tau}\,d\tau = e^{-4t}\left[\frac{e^{2\tau}}{2}\right]_0^t = e^{-4t}\left(\frac{e^{2t}-1}{2}\right) = \frac{e^{-2t}-e^{-4t}}{2} \quad (\text{for } t \geq 0) \quad = \frac{e^{-2t}-e^{-4t}}{2}u(t)$$

(ii) Given $$x_1(t) = t\,u(t);\ x_2(t) = t\,u(t)$$

We know that $$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau$$

$$\therefore \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \tau u(\tau)\,(t-\tau)\,u(t-\tau)\,d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t-\tau) = 1$ for $(t-\tau) \geq 0$ or for $\tau < t$.
Hence $u(\tau)\,u(t-\tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\,u(t-\tau) = 0$.

$$\therefore\ x_1(t) * x_2(t) = \int_0^t \tau(t-\tau)\,d\tau = \int_0^t t\tau\,d\tau - \int_0^t \tau^2 d\tau = t\left[\frac{\tau^2}{2}\right]_0^t - \left[\frac{\tau^3}{3}\right]_0^t$$

$$= t\left(\frac{t^2}{2} - 0\right) - \left(\frac{t^3}{3} - 0\right) = \frac{t^3}{2} - \frac{t^3}{3} = \frac{t^3}{6} \quad (\text{for } t \geq 0) \quad \therefore \quad x_1(t) * x_2(t) = \frac{t^3}{6} u(t)$$

(iii) Given $x_1(t) = \cos t\, u(t);\ x_2(t) = u(t)$

We know that $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \ \therefore \quad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \cos\tau\, u(\tau)\, u(t-\tau)\, d\tau$

$u(\tau) = 1$ for $\tau > 0$ and $u(t - \tau) = 1$ for $(t - \tau) \geq 0$ or for $\tau < t$.

Hence $u(\tau)\ u(t - \tau) = 1$ only for $0 < \tau \leq t$. For all other values of τ, $u(\tau)\ u(t - \tau) = 0$.

$$\therefore \qquad x_1(t) * x_2(t) = \int_0^t \cos\tau\, d\tau = [\sin\tau]_0^t = \sin t \quad \text{for } t \geq 0$$

$$\therefore \qquad x_1(t) * x_2(t) = \sin t\, u(t)$$

(iv) Given $x_1(t) = e^{-3t} u(t);\ x_2(t) = u(t + 3)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \qquad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} e^{-3\tau}\, u(\tau)\, u(t+3-\tau)$$

In this case, $u(\tau) = 0$ for $\tau < 0$ and $u(t + 3 - \tau) = 0$ for $\tau > t + 3$.

$u(\tau)\ u(t + 3 - \tau) = 1$ only for $0 < \tau < t + 3$. For all other values of τ, $u(\tau)\ u(t + 3 - \tau) = 0$.

$$\therefore \qquad x_1(t) * x_2(t) = \int_0^{t+3} e^{-3\tau}\, d\tau = \left[\frac{e^{-3\tau}}{-3}\right]_0^{t+3} = \frac{e^{-3(t+3)} - 1}{-3} = \frac{1 - e^{-3(t+3)}}{3}$$

$$\therefore \qquad y(t) = 0 \qquad (\text{for } t < -3) \quad = \frac{1 - e^{-3(t+3)}}{3} \quad (\text{for } t > -3)$$

(v) Given $x_1(t) = r(t) = tu(t);\ x_2(t) = e^{-2t}\, u(t)$

We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

$$\therefore \qquad x_1(t) * x_2(t) = \int_{-\infty}^{\infty} \tau u(\tau)\, e^{-2(t-\tau)}\, u(t-\tau)\, d\tau$$

$u(\tau) = 1$ for $\tau > 0$ and $u(t - \tau) = 1$ for $(t - \tau) \geq 0$ or for $\tau < t$.
Hence $u(\tau)\ u(t - \tau) = 1$ only for $0 < \tau < t$. For all other values of τ, $u(\tau)\ u(t - \tau) = 0$.

$$\therefore \quad x_1(t) * x_2(t) = \int_0^t \tau e^{-2(t-\tau)} d\tau \quad = e^{-2t} \int_0^t \tau e^{2\tau}\, d\tau = e^{-2t} \left\{ \left[\frac{\tau e^{2\tau}}{2} \right]_0^t - \int_0^t \frac{e^{2\tau}}{2} d\tau \right\}$$

$$= e^{-2t} \left\{ \left[\frac{te^{2t}}{2} - \left[\frac{e^{2\tau}}{4} \right]_0^t \right] \right\} = e^{-2t} \left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4} \right) \quad = \frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4} \quad (\text{for } t \geq 0)$$

$$\therefore \quad x_1(t) * x_2(t) = \left(\frac{t}{2} - \frac{1}{4} + \frac{e^{-2t}}{4} \right) u(t)$$

.4 CONVOLUTION THEOREMS

Convolution of signals may be done either in time domain or in frequency domain. So there are following two theorems of convolution associated with Fourier transforms:

1. Time convolution theorem
2. Frequency convolution theorem

.4.1 Time Convolution Theorem

The time convolution theorem states that convolution in time domain is equivalent to multiplication of their spectra in frequency domain. Mathematically, if

$$x_1(t) \longleftrightarrow X_1(\omega)$$

and $$x_2(t) \longleftrightarrow X_2(\omega)$$

Then $$x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)\, X_2(\omega)$$

Proof: $$\mathrm{F}[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\omega t}\, dt$$

We have $$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t - \tau)\, d\tau$$

$$\therefore \quad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} [x_1(\tau)\, x_2(t-\tau)\, d\tau] \right\} e^{-j\omega t}\, dt$$

Interchanging the order of integration, we have

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

Letting $t - \tau = p$, in the second integration, we have

$$t = p + \tau \text{ and } dt = dp$$

$$\therefore \quad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega(p+\tau)}\, dp \right] d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(p)\, e^{-j\omega p}\, dp \right] e^{-j\omega \tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, X_2(\omega)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau\, X_2(\omega)$$

$$= X_1(\omega)\, X_2(\omega)$$

$$\therefore \quad x_1(t) * x_2(t) \longleftrightarrow X_1(\omega)\, X_2(\omega)$$

This is time convolution theorem.

4.2 Frequency Convolution Theorem

The frequency convolution theorem states that the multiplication of two functions in time domain is equivalent to convolution of their spectra in frequency domain. Mathematically, if

$$x_1(t) \longleftrightarrow X_1(\omega)$$

and

$$x_2(t) \longleftrightarrow X_2(\omega)$$

Then

$$x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)]$$

Proof:

$$F[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} [x_1(t)\, x_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) e^{j\lambda t}\, d\lambda \right] x_2(t)\, e^{-j\omega t}\, dt$$

Interchanging the order of integration, we get

$$F[x_1(t)\, x_2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) \left[\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, e^{j\lambda t}\, dt \right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) \left[\int_{-\infty}^{\infty} x_2(t)\, e^{-j(\omega-\lambda)t}\, dt \right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\lambda) X_2(\omega - \lambda) d\lambda$$

$$= \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)]$$

$\therefore$
$$x_1(t)\, x_2(t) \longleftrightarrow \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

or
$$2\pi x_1(t)\, x_2(t) \longleftrightarrow X_1(\omega) * X_2(\omega)$$

This is frequency convolution theorem in radian frequency.
In terms of frequency, we get

$$\mathrm{F}[x_1(t)\, x_2(t)] = X_1(f) * X_2(f)$$

EXAMPLE .2 Find the convolution of the signals $x_1(t) = e^{-at} u(t)$; $x_2(t) = e^{-bt} u(t)$ using Fourier transform.

Solution: Given
$$x_1(t) = e^{-at} u(t) \qquad X_1(\omega) = \frac{1}{a + j\omega}$$

$\therefore$
$$x_2(t) = e^{-bt} u(t) \qquad X_2(\omega) = \frac{1}{b + j\omega}$$

We know that $\quad \mathrm{F}[x_1(t) * x_2(t)] = X_1(\omega)\, X_2(\omega) \qquad x_1(t) * x_2(t) = \mathrm{F}^{-1}[X_1(\omega)\, X_2(\omega)]$

$\therefore$
$$x_1(t) * x_2(t) = \mathrm{F}^{-1}\left[\frac{1}{(a + j\omega)(b + j\omega)}\right] = \mathrm{F}^{-1}\left[\frac{1}{(b-a)}\left(\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right)\right]$$

$\therefore$
$$= \frac{1}{b-a}\left[\mathrm{F}^{-1}\left(\frac{1}{a + j\omega}\right) - \mathrm{F}^{-1}\left(\frac{1}{b + j\omega}\right)\right]$$

$\therefore$
$$= \frac{1}{b-a}\left[e^{-at} u(t) - e^{-bt} u(t)\right]$$

EXAMPLE .3 Find the convolution of the signals $x_1(t) = 2e^{-2t} u(t)$ and $x_2(t) = u(t)$ using Fourier transform.

Solution: Given
$$x_1(t) = 2e^{-2t} u(t) \qquad X_1(\omega) = \frac{2}{j\omega + 2}$$

$\therefore$
$$x_2(t) = u(t) \qquad X_2(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$\therefore X_1(\omega)\, X_2(\omega) = \frac{2}{j\omega + 2}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \frac{2}{j\omega(j\omega + 2)} + \frac{2\pi\delta(\omega)}{j\omega + 2}$$

Since $x_1(t) * x_2(t) = \mathrm{F}^{-1}[X_1(\omega)\, X_2(\omega)]$, we have

$$x_1(t) * x_2(t) = \mathrm{F}^{-1}\left[\frac{2}{j\omega(j\omega + 2)} + \frac{2\pi\delta(\omega)}{j\omega + 2}\right] = \mathrm{F}^{-1}\left[\frac{1}{j\omega} - \frac{1}{j\omega + 2} + \frac{2\pi\delta(\omega)}{j\omega + 2}\right]$$

Since $\delta(\omega) = 1$ for $\omega = 0$ and $\delta(\omega) = 0$ for $\omega \neq 0$, we have $\dfrac{2\pi\delta(\omega)}{j\omega + 2} = \pi\delta(\omega)$.

.6 SIGNAL COMPARISON: CORRELATION OF FUNCTIONS

Concept of correlation

The signals may be compared on the basis of similarity of waveforms. Quantitatively, a comparison may be based upon the amount of the component of one waveform contained in the other waveform. If $x_1(t)$ and $x_2(t)$ are two waveforms, then the waveform $x_1(t)$ contains an amount $C_{12}x_2(t)$ of that particular waveform $x_2(t)$ in the interval (t_1, t_2), where

$$C_{12} = \frac{\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt}{\int_{t_1}^{t_2} x_2^2(t)\, dt}$$

The magnitude of the integral in the numerator might be taken as an indication of similarity.

If this integral vanishes, i.e

$$\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt = 0$$

then the two signals have no similarity over the interval (t_1, t_2). Such signals are said to be orthogonal over the specified interval.

The integral $\int_{t_1}^{t_2} x_1(t)\, x_2(t)\, dt$ forms the basis of comparison of the two signals $x_1(t)$ and $x_2(t)$ over the interval (t_1, t_2).

In general we are interested in comparing the two signals over the interval $(-\infty, \infty)$. So the test integral becomes

$$\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt$$

However, there is a difficulty with this test integral which can be illustrated with the example of radar pulse. Figure 7.36 shows a transmitted pulse and a received pulse which is delayed w.r.t. transmitted pulse by T s. Obviously, the two waveforms are identical except that one is delayed w.r.t. the other. Yet the test integral $\int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt$ yields zero because the product $x_1(t)\, x_2(t)$ is zero everywhere. This indicates that the two waveforms have no measure of similarity which is obviously a wrong conclusion. Hence in order to search for a

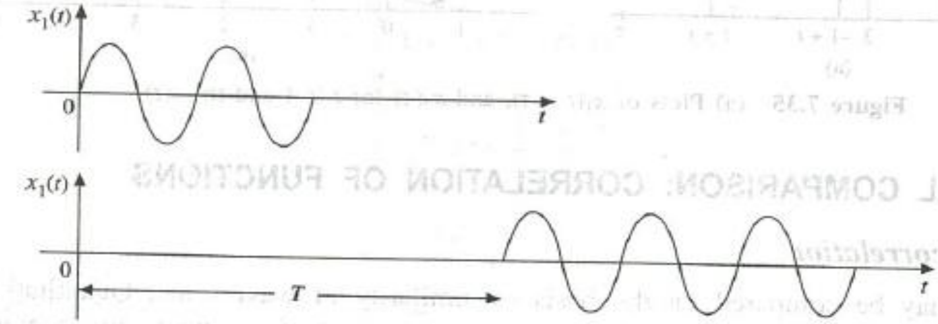


Figure 6 Signal comparison.

similarity between the two waveforms, we must shift one waveform w.r.t. the other by various amounts and see whether a similarity exists for some amount of shift of one function w.r.t. the other.

Therefore, the test integral is modified as $\int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$

where τ is the searching or scanning parameter. This integral is a function of τ. This integral is known as the cross correlation function between $x_1(t)$ and $x_2(t)$ and is denoted by $R_{12}(\tau)$.

It is immaterial whether we shift the function $x_1(t)$ by an amount of τ in the negative direction or shift the function $x_2(t)$ by the same amount in the positive direction. Thus

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2(t)\, dt$$

Thus the correlation of two functions or signals or waveforms is a measure of similarity between those signals. The correlation is of two types: cross correlation and autocorrelation. The autocorrelation and cross correlation are defined separately for energy (or aperiodic) signals and power (or periodic) signals.

.6.1 Cross Correlation

The cross correlation between two different waveforms or signals is a measure of similarity or match or relatedness or coherence between one signal and the time delayed version of another signal. That means the cross correlation between two signals indicates how much one signal is related to the time delayed version of another signal.

Cross correlation of energy signals

Consider two general complex signals $x_1(t)$ and $x_2(t)$ of finite energy. The cross correlation of these two energy signals denoted by $R_{12}(\tau)$ is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2^*(t)\, dt$$

If the two signals $x_1(t)$ and $x_2(t)$ are real, then $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt = \int_{-\infty}^{\infty} x_1(t+\tau)\, x_2(t)\, dt$

If $x_1(t)$ and $x_2(t)$ have some similarity, then the cross correlation $R_{12}(\tau)$ will have some finite value over the range of τ. Also if

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = 0 \quad \text{i.e. if} \quad R_{12}(0) = 0$$

then the two signals $x_1(t)$ and $x_2(t)$ are called *orthogonal signals*. That is the cross correlation for orthogonal signals is zero.

Another form of cross correlation between $x_2(t)$ and $x_1(t)$ is defined as: $R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$

In the above equations, the cross correlation function $R_{12}(\tau)$ is a function of the variable τ. The variable τ is called the *delay parameter* or the *scanning parameter* or the *searching parameter*. It is time delay or time shift of one of the two signals. The delay parameter τ determines the correlation between two signals. Two signals with no cross correlation at $\tau = 0$ can have significant cross correlation by adjusting the parameter τ. Two signals for which the cross correlation is zero for all values of τ are called *uncorrelated* or *incoherent signals*.

Properties of cross correlation function for energy signals

Following are the properties of cross correlation for energy signals:

1. The cross correlation functions exhibit conjugate symmetry, i.e.

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

   That is unlike convolution, cross correlation is not in general commutative, i.e.

$$R_{12}(\tau) \neq R_{21}(\tau)$$

2. If $R_{12}(0) = 0$

   i.e. if $$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = 0$$

   then the two signals are said to be orthogonal over the entire time interval.

3. The cross correlation of two energy signals corresponds to the multiplication of the Fourier transform of one signal by the complex conjugate of Fourier transform of second signal.

   i.e. $$R_{12}(\tau) \longleftrightarrow X_1(\omega) X_2^*(\omega)$$

   This is known as *correlation theorem*.

Cross correlation of power (periodic) signals

The cross correlation function $R_{12}(\tau)$ for two periodic signals $x_1(t)$ and $x_2(t)$ may be defined with the help of average form of correlation. If the two periodic signals $x_1(t)$ and $x_2(t)$ have the same time period T, then cross correlation is defined as:

$$R_{12}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_1(t) x_2^*(t - \tau)\, dt$$

then the two signals $x_1(t)$ and $x_2(t)$ are called *orthogonal signals*. That is the cross correlation for orthogonal signals is zero.

Another form of cross correlation between $x_2(t)$ and $x_1(t)$ is defined as:

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t - \tau)\, dt$$

In the above equations, the cross correlation function $R_{12}(\tau)$ is a function of the variable τ. The variable τ is called the *delay parameter* or the *scanning parameter* or the *searching parameter*. It is time delay or time shift of one of the two signals. The delay parameter τ determines the correlation between two signals. Two signals with no cross correlation at $\tau = 0$ can have significant cross correlation by adjusting the parameter τ. Two signals for which the cross correlation is zero for all values of τ are called *uncorrelated* or *incoherent signals*.

Properties of cross correlation function for energy signals

Following are the properties of cross correlation for energy signals:

1. The cross correlation functions exhibit conjugate symmetry, i.e.

$$R_{12}(\tau) = R_{21}^{*}(-\tau)$$

That is unlike convolution, cross correlation is not in general commutative, i.e.

$$R_{12}(\tau) \neq R_{21}(\tau)$$

2. If $R_{12}(0) = 0$

i.e. if

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^{*}(t)\, dt = 0$$

then the two signals are said to be orthogonal over the entire time interval.

3. The cross correlation of two energy signals corresponds to the multiplication of the Fourier transform of one signal by the complex conjugate of Fourier transform of second signal.

i.e.

$$R_{12}(\tau) \longleftrightarrow X_1(\omega) X_2^{*}(\omega)$$

This is known as *correlation theorem.*

Cross correlation of power (periodic) signals

The cross correlation function $R_{12}(\tau)$ for two periodic signals $x_1(t)$ and $x_2(t)$ may be defined with the help of average form of correlation. If the two periodic signals $x_1(t)$ and $x_2(t)$ have the same time period T, then cross correlation is defined as:

$$R_{12}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_1(t) x_2^{*}(t-\tau)\, dt$$

The cross correlation of two periodic functions is defined in another form as:

$$R_{21}(\tau) = \frac{1}{T} \int_{-T/2}^{T/2} x_2(t) x_1^{*}(t-\tau)\, dt$$

Properties of cross correlation function for power (periodic) signals

Following are the properties of cross correlation for power signals:

1. The Fourier transform of the cross correlation of two signals is equal to the multiplication of Fourier transform of one signal and complex conjugate of Fourier transform of other signal.

$$R_{12}(\tau) \longleftrightarrow X_1(\omega) X_2^*(\omega)$$

2. If $R_{12}(0) = 0$,

i.e. if
$$\underset{T\to\infty}{\text{Lt}} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t)\, x_2^*(t)\, dt = 0$$

then the signals are said to be orthogonal over the entire time interval.

3. The cross correlation exhibits conjugate symmetry, i.e.

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

4. Unlike convolution, the cross correlation is not commutative, i.e.

$$R_{12}(\tau) \neq R_{21}(\tau)$$

EXAMPLE ..12 Prove that $R_{12}(\tau) = R_{21}^*(-\tau)$ i.e. the cross correlation exhibits conjugate symmetry.

Solution: The cross correlation of two signals $x_1(t)$ and $x_2(t)$ is given as:

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t-\tau)\, dt$$

Let $t - \tau = n$ in the above equation for $R_{12}(\tau)$,

$$\therefore \quad R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(n+\tau) x_2^*(n)\, dn$$

Also we know that

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\, x_1^*(t-\tau)\, dt$$

Let $t = n$ in the above equation for $R_{21}(\tau)$.

$$\therefore \quad R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(n)\, x_1^*(n-\tau)\, dn$$

$$\therefore \quad R_{21}^*(\tau) = \int_{-\infty}^{\infty} x_2^*(n)\, x_1(n-\tau)\, dn$$

$$\therefore \quad R_{21}^*(-\tau) = \int_{-\infty}^{\infty} x_2^*(n)\, x_1(n+\tau)\, dn$$

Comparing the above two equations for $R_{12}(\tau)$ and $R_{21}^*(-\tau)$, we can write

$$R_{12}(\tau) = R_{21}^*(-\tau)$$

6.2 Autocorrelation

The autocorrelation function gives the measure of match or similarity or relatedness or coherence between a signal and its time delayed version. This means that the autocorrelation function is a special form of cross correlation function. It is defined as the correlation of a signal with itself.

The autocorrelation is defined separately for energy signals and power signals.

Autocorrelation for energy signals

The autocorrelation of an energy signal $x(t)$ is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

where τ is called the delay parameter and the signal $x(t)$ is shifted by τ in positive direction.

If $x(t)$ is shifted by τ in negative direction, then

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau)x^*(t)\, dt$$

Properties of autocorrelation function of energy signals

Following are the properties of autocorrelation for energy signals:

1. The autocorrelation function exhibits conjugate symmetry, i.e.

$$\boxed{R(\tau) = R^*(-\tau)}$$

Thus, it states that the real part of $R(\tau)$ is an even function of τ and the imaginary part of $R(\tau)$ is an odd function of τ.

Proof: The autocorrelation of an energy signal $x(t)$ is given by

$$R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\, dt$$

Taking the complex conjugate, we have

$$R^*(\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t-\tau)\, dt$$

$$\therefore \quad R^*(-\tau) = \int_{-\infty}^{\infty} x^*(t)\, x(t+\tau)\, dt = R(\tau)$$

$$\therefore \quad R(\tau) = R^*(-\tau)$$

2. The value of autocorrelation function of an energy signal at origin (i.e. at $\tau = 0$) is equal to the total energy of that signal. i.e.

$$R(0) = E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

Proof: We have

Proof: We have

$$R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)\,dt$$

Putting $\tau = 0$ gives

$$R(0) = \int_{-\infty}^{\infty} x(t)x^*(t)\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt = E$$

3. If τ is increased in either direction, the autocorrelation $R(\tau)$ reduces. As τ reduces autocorrelation, $R(\tau)$ increases and it is maximum at $\tau = 0$, i.e. at the origin. Therefore,

$$\boxed{|R(\tau)| \le R(0) \quad (\text{for all } \tau)}$$

Proof: Consider the functions $x(t)$ and $x(t+\tau)$. $[x(t) \pm x(t+\tau)]^2$ is always greater than or equal to zero since it is squared, i.e.

$$x^2(t) + x^2(t+\tau) \pm 2x(t)\,x(t+\tau) \ge 0$$

or
$$x^2(t) + x^2(t+\tau) \ge \pm 2x(t)\,x(t+\tau)$$

Integrating both the sides, we get

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt + \int_{-\infty}^{\infty} |x(t+\tau)|^2\,dt \ge 2\int_{-\infty}^{\infty} x(t)\,x(t+\tau)\,dt$$

$\therefore$ $\quad E + E \ge 2R(\tau) \quad$ [If $x(t)$ is real valued function]

$\therefore$ $\quad E \ge R(\tau)$

or $\quad R(0) \ge |R(\tau)| \quad$ (Since $R(0) = E$)

4. The autocorrelation function $R(\tau)$ and energy spectral density function $\psi(\omega)$ of energy signal form a Fourier transform pair.

$$R(\tau) \xleftrightarrow{FT} \psi(\omega)$$

Autocorrelation theorem

The autocorrelation theorem states that the Fourier transform of autocorrelation function $R(\tau)$ yields the energy density function of signal $x(t)$, i.e.

$$\mathrm{F}[R(\tau)] = |X(\omega)|^2 = \psi(\omega)$$

Proof: The Fourier transform of autocorrelation function $R(\tau)$ is:

$$\mathrm{F}[R(\tau)] = \int_{-\infty}^{\infty} R(\tau)\,e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\,x(t-\tau)\,e^{-j\omega\tau}\,dt\,d\tau$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt \int_{-\infty}^{\infty} x(t-\tau)\,e^{j\omega(t-\tau)}\,d\tau$$

$$= X(\omega)\int_{-\infty}^{\infty} x(t-\tau)\,e^{j\omega(t-\tau)}\,d\tau$$

Letting $t - \tau = n$ in the second integral, we have

$$F[R(\tau)] = X(\omega) \int_{-\infty}^{\infty} x(n) e^{j\omega n}\, dn$$

$$= X(\omega)\, X(-\omega) = |X(\omega)|^2$$

$$= \psi(\omega)$$

Autocorrelation function for power (periodic) signals

The autocorrelation function of a periodic signal with any period T is given by

$$R(\tau) = \operatorname*{Lt}_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

Properties of autocorrelation function for power signals

Following are the properties of autocorrelation function for power signals:

1. The autocorrelation function exhibits conjugate symmetry, i.e.

$$R(\tau) = R^*(-\tau)$$

Proof: We have

$$R(\tau) = \operatorname*{Lt}_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

$$\therefore \quad R^*(\tau) = \operatorname*{Lt}_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t-\tau)\, dt$$

$$\therefore \quad R^*(-\tau) = \operatorname*{Lt}_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^*(t)\, x(t+\tau)\, dt = R(\tau)$$

$$\therefore \quad R(\tau) = R^*(-\tau)$$

2. The autocorrelation function at origin is equal to the average power of that signal, i.e.

$$R(0) = P = \operatorname*{Lt}_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |x(t)|^2 dt$$

3. The autocorrelation function $R(\tau)$ has maximum value at the origin, i.e.

$$|R(\tau)| \le R(0)$$

The value of autocorrelation reduces as τ increases from origin.

4. The autocorrelation function $R(\tau)$ and power spectral density $S(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \xleftrightarrow{FT} S(\omega)$$

5. The autocorrelation function is periodic with the same period as the periodic signal itself, i.e.

$$R(\tau) = R(\tau \pm nT), \quad n = 1, 2, 3, \ldots$$

.7 ENERGY DENSITY SPECTRUM

Spectral density It is the distribution of energy or power of a signal per unit bandwidth as a function of frequency.

Energy signals Signals with finite energy and zero average power, i.e. $0 < E < \infty$ and $P = 0$ are called energy signals, e.g. aperiodic signals like pulse.

Normalized energy The normalized energy, or simply energy of a signal $x(t)$ is defined as the energy dissipated by a voltage signal applied across 1-Ω resistor (or by a current signal flowing through 1-Ω resistor). Mathematically,

$$\boxed{E = \int_{-\infty}^{\infty} |x(t)|^2\, dt}$$

The energy of a signal exists only if E is finite, i.e. only if $0 < E < \infty$.

Parseval's theorem for energy signals (Rayleigh's energy theorem) Parseval's theorem defines the energy of a signal in terms of its Fourier transform. Using Parseval's theorem, the energy of a signal $x(t)$ can be evaluated directly from its frequency spectrum $X(\omega)$ without the knowledge of its time domain version, i.e. $x(t)$.

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

or

$$E = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

Proof: Consider a function $x(t)$ such that

$$x(t) \longleftrightarrow X(\omega)$$

Let $x^*(t)$ be the conjugate of $x(t)$ such that

$$x^*(t) \longleftrightarrow X^*(-\omega)$$

The energy of a signal $x(t)$ is given by

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{\infty} x^*(t)\, x(t)\, dt$$

Replacing $x(t)$ by its inverse Fourier transform, we have

$$E = \int_{-\infty}^{\infty} x^*(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\right] dt$$

Interchanging the order of integration,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) X^*(-\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Let

$$\omega = 2\pi f$$

$\therefore$

$$d\omega = 2\pi\, df$$

Normally $X(2\pi f)$ is written as $X(f)$, then we have

$$\boxed{E = \int_{-\infty}^{\infty} |X(f)|^2\, df}$$

This is called Parseval's theorem for energy signals (also called Rayleigh's energy theorem).

Energy spectral density

The ESD function gives the distribution of energy of a signal in the frequency domain. For an energy signal, the total area under the spectral density curve plotted as a function of frequency is equal to the total energy of the signal. It is also called energy density spectrum (ESD or ED). It is designated by $\psi(\omega)$ and given by

$$\psi(\omega) = |X(\omega)|^2$$

Let $x(t)$ and $y(t)$ be the input and output respectively of a linear system.
Let $x(t) \longleftrightarrow X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$ and $H(\omega)$ be the system transfer function.
Then, we have

$$Y(\omega) = H(\omega)\ X(\omega)$$

The ESD of the input $x(t)$ is:

$$\psi_x(\omega) = |X(\omega)|^2$$

The ESD of the output $y(t)$ is:

$$\psi_y(\omega) = |Y(\omega)|^2$$

$$\therefore \quad \psi_y(\omega) = |Y(\omega)|^2 = |H(\omega)\,X(\omega)|^2 = |H(\omega)|^2\,|X(\omega)|^2 = |H(\omega)|^2\,\psi_x(\omega)$$

$$\therefore \quad \psi_y(\omega) = |H(\omega)|^2\,\psi_x(\omega)$$

Thus, the ESD of the output (response) of a linear system is the product of ESD of input (excitation) and square of the magnitude of the transfer function.

Energy of the output signal $E_y = \int_{-\infty}^{\infty} \psi_y(f)\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_y(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2\,\psi_x(\omega)\,d\omega$

$$= \frac{1}{2\pi}2\int_{0}^{\infty} |H(\omega)|^2\,\psi_x(\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{\infty} |H(\omega)|^2\,\psi_x(\omega)\,d\omega$$

If the LTI system is an ideal LPF with lower and upper cutoff frequencies f_L and f_H respectively, then $|H(\omega)| = 1$ for $f_L < f < f_H$.

$$\therefore \quad E_y = \frac{1}{\pi}\int_{f_L}^{f_H} \psi_x(\omega)\,d\omega$$

or

$$E_y = \frac{1}{\pi}\int_{f_L}^{f_H} \psi_x(2\pi f)\,2\pi df = 2\int_{f_L}^{f_H} \psi_x(f)\,df$$

Properties of ESD: The following are the properties of ESD.

1. The total area under the energy density spectrum is equal to the total energy of the signal.

i.e.
$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df$$

2. If $x(t)$ is the input to an LTI system with impulse response $h(t)$, then the input and output ESD functions are related as:

$$\psi_y(\omega) = |H(\omega)|^2\, \psi_x(\omega)$$

or
$$\psi_y(f) = |H(f)|^2\, \psi_x(f)$$

3. The autocorrelation function $R(\tau)$ and ESD $\psi(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

or
$$R(\tau) \longleftrightarrow \psi(f)$$

.8 POWER DENSITY SPECTRUM

Power signals Signals with finite average power and infinite energy, i.e. $0 < P < \infty$ and $E = \infty$ are called power signals, e.g. periodic signals.

Average power It is defined as the average power dissipated by a voltage $x(t)$ applied across 1-Ω resistor (or by a current signal flowing through 1-Ω resistor). Mathematically,

$$\boxed{P = \operatorname*{Lt}_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt}$$

The power P defined above is actually the mean square value or the time average of the squared signal. Thus, we may write

$$P = \overline{x^2(t)} = \operatorname*{Lt}_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} |x(t)|^2\, dt$$

Parseval's power theorem Parseval's power theorem defines the power of a signal in terms of its Fourier series coefficients, i.e. in terms of the harmonic components present in the signal. Mathematically, it is given by

$$P = \sum_{n=-\infty}^{\infty} |C(n)|^2$$

Proof: Consider a function $x(t)$. We know that

$$|x(t)|^2 = x(t)\, x^*(t)$$

where $x^*(t)$ is the conjugate of $x(t)$.

The average power of $x(t)$ for one cycle is:

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt$$

But, we have the exponential Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$\therefore \quad P = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t} x^*(t)\, dt$$

Interchanging the order of summation and integration, we get

$$P = \sum_{n=-\infty}^{\infty} C_n \frac{1}{T}\int_{-T/2}^{T/2} x^*(t)\, e^{jn\omega t}\, dt$$

$$= \sum_{n=-\infty}^{\infty} C_n C_n^* = \sum_{n=-\infty}^{\infty} |C_n|^2$$

$$\therefore \quad \boxed{P = \sum_{n=-\infty}^{\infty} |C_n|^2}$$

This is called Parseval's power theorem. It states that the power of a signal is equal to the sum of square of the magnitudes of various harmonics present in the discrete spectrum.

Power spectral density (PSD)

The distribution of average power of the signal in the frequency domain is called power spectral density or power density or power density spectrum (PSD or PD).

To derive the PSD, assume the power signal as a limiting case of the energy signal. Consider a power signal $x(t)$, extending to infinity as shown in Figure 7.37.

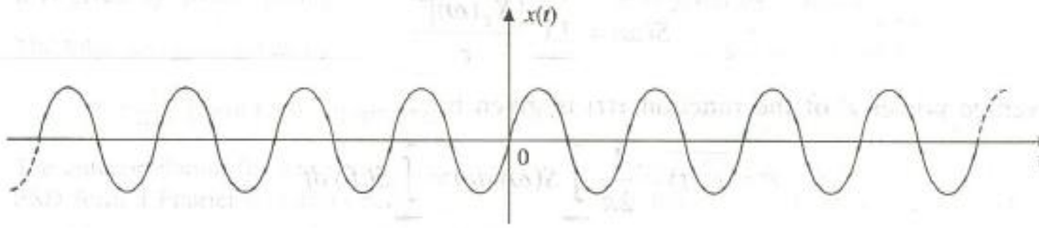


Figure 7.37 Power signal.

Let us truncate this signal so that it is zero outside the interval $|\tau/2|$ as shown in Figure 7.38. Let this truncated signal be $x_\tau(t)$.

$$\therefore \qquad x_\tau(t) = \begin{cases} x(t), & |t| < \dfrac{\tau}{2} \\ 0, & \text{elsewhere} \end{cases}$$

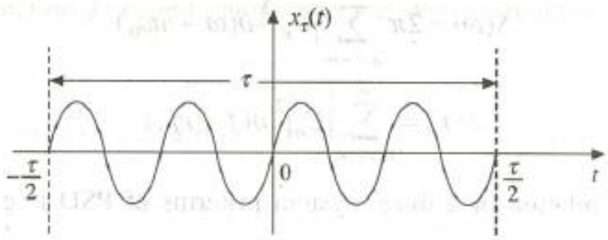


Figure 7.38 Truncated power signal.

The signal $x_\tau(t)$ is of finite duration τ and hence it is an energy signal with energy E given by

$$E = \int_{-\infty}^{\infty} |x_\tau(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_\tau(\omega)|^2 \, d\omega$$

where $\qquad x_\tau(t) \longleftrightarrow X_\tau(\omega)$

As $x(t)$ over the interval, $\left(-\dfrac{\tau}{2} \text{ to } \dfrac{\tau}{2}\right)$ is same as $x_\tau(t)$ over the interval $-\infty$ to ∞, we have

$$\int_{-\infty}^{\infty} |x_\tau(t)|^2 \, dt = \int_{-\tau/2}^{\tau/2} |x(t)|^2 \, dt$$

$$\therefore \qquad \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} |x(t)|^2 \, dt = \frac{1}{2\pi} \frac{1}{\tau} \int_{-\infty}^{\infty} |X_\tau(\omega)|^2 \, d\omega$$

$$\therefore \quad P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \operatorname*{Lt}_{\tau\to\infty} \frac{|X_\tau(\omega)|^2}{\tau}\, d\omega$$

If $\tau \to \infty$, $|X_\tau(\omega)|^2/\tau$ approaches a finite value.

Let this finite value is denoted by $S(\omega)$, i.e.

$$S(\omega) = \operatorname*{Lt}_{\tau\to\infty} \frac{|X_\tau(\omega)|^2}{\tau}$$

The average power P of the function $x(t)$ is given by

$$P = \overline{x^2(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df$$

where $\overline{x^2(t)}$ is the mean square value of $x(t)$.

The average power is, therefore, given by

$$P = 2\frac{1}{2\pi}\int_{0}^{\infty} S(\omega)\, d\omega = \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\, d\omega = 2\int_{0}^{\infty} S(f)\, df$$

The PSD of a periodic function is given by

$$S(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(\omega - n\omega_0)$$

or alternately

$$S(f) = \sum_{n=-\infty}^{\infty} |C_n|^2\, \delta(f - nf_0)$$

The input and output relation of a linear system in terms of PSD is given by

$$S_y(\omega) = |H(\omega)|^2\, S_x(\omega)$$

or

$$S_y(f) = |H(f)|^2\, S_x(f)$$

Properties of PSD The following are the properties of PSD:

1. The area under the PSD function is equal to the average power of that signal, i.e.

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df$$

2. The input and output PSDs of an LTI system are related as:

$$S_y(\omega) = |H(\omega)|^2\, S_x(\omega)$$

3. The autocorrelation function $R(\tau)$ and PSD $S(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow S(\omega)$$

The comparison of ESD and PSD is given in Table 7.1.

Table .1 Comparision of ESD and PSD

S.No.	ESD	PSD
1.	It gives the distribution of energy of a signal in frequency domain.	It gives the distribution of power of a signal in frequency domain.
2.	It is given by $\psi(\omega) = \lvert X(\omega)\rvert^2$	It is given by $S(\omega) = \underset{\tau\to\infty}{\text{Lt}} \frac{\lvert X(\omega)\rvert^2}{\tau}$
3.	The total energy is given by $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, d\omega = \int_{-\infty}^{\infty} \psi(f)\, df$	The total power is given by $P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, d\omega = \int_{-\infty}^{\infty} S(f)\, df$
4.	The autocorrelation for an energy signal and its ESD form a Fourier transform pair. $R(\tau) \longleftrightarrow \psi(\omega)$ or $R(\tau) \longleftrightarrow \psi(f)$	The autocorrelation for a power signal and its PSD form a Fourier transform pair $R(\tau) \longleftrightarrow S(\omega)$ or $R(\tau) \longleftrightarrow S(f)$

9 RELATION BETWEEN AUTOCORRELATION FUNCTION AND ENERGY/POWER SPECTRAL DENSITY FUNCTION

.9.1 Relation between ESD and Autocorrelation Function $R(\tau)$

The autocorrelation function $R(\tau)$ and energy spectral density function $\psi(\omega)$ form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

Proof: The autocorrelation of a function $x(t)$ is given as:

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\, x^*(t-\tau)\, dt$$

Replacing $x^*(t-\tau)$ by its inverse transform, we have

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega(t-\tau)}\, d\omega\right]^* dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega(t-\tau)}\, d\omega\right] dt$$

Interchanging the order of integration, we have

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right] e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)\, X(\omega)\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\omega)\, e^{j\omega\tau}\, d\omega \qquad [\text{since } \lvert X(\omega)\rvert^2 = \psi(\omega)]$$

$$= F^{-1}[\psi(\omega)]$$

$\therefore \quad \psi(\omega) = F[R(\tau)]$

This proves that $R(\tau)$ and $\psi(\omega)$ form a Fourier transform pair.

$$R(\tau) \longleftrightarrow \psi(\omega)$$

9.2 Relation between Autocorrelation Function $R(\tau)$ and Power Spectral Density (PSD)

The autocorrelation function $R(\tau)$ and the power spectral density (PSD), $S(\omega)$ of a power signal form a Fourier transform pair, i.e.

$$R(\tau) \longleftrightarrow S(\omega)$$

Proof: The autocorrelation function of a power (periodic) signal $x(t)$ in terms of Fourier series coefficients is given as:

$$R(\tau) = \sum_{n=-\infty}^{\infty} C_n C_{-n} e^{jn\omega_0 \tau}$$

where C_n and C_{-n} are the exponential Fourier series coefficients.

$$\therefore \quad R(\tau) = \sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0 \tau}$$

Taking Fourier transform on both sides, we have

$$\mathrm{F}[R(\tau)] = \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} |C_n|^2 e^{jn\omega_0 \tau} \right) e^{-j\omega\tau}\, d\tau$$

Interchanging the order of integration and summation, we get

$$\mathrm{F}[R(\tau)] = \sum_{n=-\infty}^{\infty} |C_n|^2 \int_{-\infty}^{\infty} e^{-j\tau(\omega - n\omega_0)}\, d\tau$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{\infty} |C_n|^2 \delta(f - nf_0)$$

The RHS is the PSD $S(\omega)$ or $S(f)$ of the periodic function $x(t)$.

$$\therefore \quad \mathrm{F}[R(\tau)] = S(\omega) \quad [\text{or } S(f)]$$

and $$\mathrm{F}^{-1}[S(\omega)] \quad \{\text{or } \mathrm{F}^{-1}[S(f)]\} = R(\tau)$$

i.e. $$R(\tau) \longleftrightarrow S(\omega) \quad [\text{or } S(f)]$$

10 RELATION BETWEEN CONVOLUTION AND CORRELATION

There is a striking resemblance between the operation of convolution and correlation. Indeed the two integrals are closely related. To obtain the cross correlation of $x_1(t)$ and $x_2(t)$ according to the equation $R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t-\tau)\, dt$, we multiply $x_1(t)$ with function $x_2(t)$

displaced by τ sec. The area under the product curve is the cross correlation between $x_1(t)$ and $x_2(t)$ at τ. On the other hand, the convolution of $x_1(t)$ and $x_2(t)$ at $t = \tau$ is obtained by folding $x_2(t)$ backward about the vertical axis at the origin and taking the area under the product curve of $x_1(t)$ and the folded function $x_2(-t)$ displaced by τ. It, therefore, follows that the cross correlation of $x_1(t)$ and $x_2(t)$ is the same as the convolution of $x_1(t)$ and $x_2(-t)$.

The same conclusion can be arrived at analytically as follows:

The convolution of $x_1(t)$ and $x_2(-t)$ is given by

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(\tau - t)\, d\tau$$

Replacing the dummy variable τ in the above integral by another variable n, we have

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n - t)\, dn$$

Changing the variable from t to τ, we get

$$x_1(\tau) * x_2(-\tau) = \int_{-\infty}^{\infty} x_1(n)\, x_2(n - \tau)\, dn = R_{12}(\tau)$$

Hence $$R_{12}(\tau) = x_1(t) * x_2(-t)\big|_{t=\tau}$$

Similarly, $$R_{21}(\tau) = x_2(t) * x_1(-t)\big|_{t=\tau}$$

All of the techniques used to evaluate the convolution of two functions can be directly applied in order to find the correlation of two functions. Similarly, all of the results derived for convolution also apply to correlation.

If one of the function is an even function of t, let us say $x_2(t)$ is an even function of t, i.e.

$$x_2(t) = x_2(-t)$$

then the cross correlation and convolution are equivalent.

UNIT V:

LAPLACE TRANSFORMS:

Laplace Transforms

The Laplace transform is another mathematical tool used for analysis of signals and systems. Laplace transform provides broader characterization of the signals and systems compared to Fourier transform. Laplace transform can be used for the analysis of unstable systems, where Fourier transform has limitations. There are two types of Laplace transforms.

i) Bilateral or two-sided Laplace transform

ii) Unilateral or single-sided Laplace transform

.1 Definition of Laplace Transform

The bilateral Laplace transform is defined as,

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt}$$

Here the integration is taken from $-\infty$ to $+\infty$. Hence it is called bilateral or two sided laplace transform. Here $f(t)$ is the time domain signal and $F(s)$ is its Laplace transform.

.1 Definition of Laplace Transform

The bilateral Laplace transform is defined as,

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt}$$

Here the integration is taken from $-\infty$ to $+\infty$. Hence it is called bilateral or two sided laplace transform. Here $f(t)$ is the time domain signal and $F(s)$ is its Laplace transform.

The unilateral Laplace transform is given as,

$$\boxed{F(s) = \int_{0-}^{\infty} f(t)\, e^{-st}\, dt}$$

Here observe that integration is taken from 0 to ∞. The unilateral Laplace transform is mainly useful for analysis of causal signals. The lower limit is taken as 0 –. This is to include the time just before zero. It can be taken as 0+ also. For continuous function $f(0-) = f(0+)$. Thus for a continuous function, integration is effectively taken from 0 to ∞, since value of $f(t)$ just before and after zero is same. The $f(t)$ and $F(s)$ is called Laplace transform pair. It is written as,

$$f(t) \leftrightarrow F(s)$$

The variable 's' is the complex frequency. It is given as,

$$s = \sigma + j\omega$$

Here σ is the attenuation constant or damping factor and ω is the angular frequency. With above value of 's' we can write equation 2.5.1 as,

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-\sigma t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[f(t)\, e^{-\sigma t}\right] e^{-j\omega t}\, dt$$

The above equation shows that $F(s)$ is basically the Fourier transform of $f(t)\, e^{-\sigma t}$. This is the relationship between Laplace and Fourier transforms. The Fourier transform given by above equation must exist, which is actually Laplace transform. Hence sufficient condition for $f(t)$ to be Laplace transformable is that

$$\int_0^{\infty} |f(t)|\, e^{-\sigma t}\, dt < \infty$$

for real and positive values of σ.

The inverse Laplace transform is given as,

$$\boxed{\textbf{Inverse Laplace Transform : } f(t) = \frac{1}{2\pi j} \int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)\, e^{st}\, ds}$$

.2 Properties of Laplace Transform

.1 Linearity

Let $f_1(t) \leftrightarrow F_1(s)$ and $f_2(t) \leftrightarrow F_2(s)$ be the two Laplace transform pairs. Then linearity property states that,

$$\mathcal{L}\left[a_1 f_1(t) + a_2 f_2(t)\right] = a_1 F_1(s) + a_2 F_2(s)$$

Here a_1 and a_2 are constants.

Proof : Let us find the Laplace transform of $a_1 f_1(t) + a_2 f_2(t)$ by applying definition. i.e.,

$$\begin{aligned}\mathcal{L}\left[a_1 f_1(t) + a_2 f_2(t)\right] &= \int_{0-}^{\infty} \left[a_1 f_1(t) + a_2 f_2(t)\right] e^{-st}\, dt \\ &= a_1 \int_{0-}^{\infty} f_1(t)\, e^{-st}\, dt + a_2 \int_{0-}^{\infty} f_2(t)\, e^{-st}\, dt \\ &= a_1 F_1(s) + a_2 F_2(s)\end{aligned}$$

.2 Shifting Theorem (Translation in Time Domain)

Let $f(t) \leftrightarrow F(s)$ be a Laplace transform pair. If $f(t)$ is delayed by time t_0, then its Laplace transform is multiplied by e^{-st_0}. i.e.,

$$\mathcal{L}\left[f(t-t_0)\right] = e^{-st_0} F(s)$$

Here t_0 is a constant.

Proof : Consider the Laplace transform of $f(t-t_0)$ by definition,

$$\mathcal{L}\left[f(t-t_0)\right] = \int_{t_0}^{\infty} f(t-t_0)\, e^{-st}\, dt$$

Here lower limit of integration is taken as t_0. This is because the function $f(t-t_0)$ is shifted at $t = t_0$.

Let $\tau = t - t_0$

$\therefore$ $d\tau = dt$

when $t = t_0$, $\tau = 0$ and

at $t = \infty$, $\tau = \infty$

With this substitution equation 2.5.6 (a) becomes,

$$\begin{aligned}\mathcal{L}\left[f(t-t_0)\right] &= \int_{0-}^{\infty} f(\tau)\, e^{-s(\tau+t_0)} d\tau \\ &= e^{-st_0} \int_{0-}^{\infty} f(\tau)\, e^{-s\tau}\, d\tau\end{aligned}$$

$$= e^{-st_0} F(s)$$

Since $\int_{0-}^{\infty} f(\tau)\, e^{-s\tau}\, d\tau = F(s)$ By definition of LT.

.3 Complex Translation or Translation in Frequency Domain

Let $f(t) \leftrightarrow F(s)$ be a Laplace transform pair, then

$$\boxed{F(s-a) = \mathcal{L}\left[e^{at} f(t)\right]}$$

Here a is a complex number.

Proof : Let us find the Laplace transform of $e^{at} f(t)$. By definition of Laplace transform, we have,

$$\mathcal{L}\left[e^{at} f(t)\right] = \int_{0-}^{\infty} e^{at} f(t)\, e^{-st}\, dt$$

$$= \int_{0-}^{\infty} f(t)\, e^{-(s-a)t}\, dt$$

$$= F(s-a)$$

.4 Differentiation Theorem (Differentiation in Time Domain)

Let $f(t) \leftrightarrow F(s)$ be a laplace transform pair. Then,

$$\boxed{\mathcal{L}\left[\frac{d}{dt} f(t)\right] = s\,F(s) - f(0-)}$$

Here $f(0-)$ is the value of $f(t)$ at $t = 0-$.

'$t = 0-$' indicates the time just before t = 0. In other words, t = 0 – is equivalent to $\lim\limits_{\varepsilon \to 0} t = -\varepsilon$.

Proof :

Let us write $\frac{d}{dt} f(t) = f'(t)$. Consider Laplace transform of $f'(t)$ by definition. i.e.,

$$\mathcal{L}\left[f'(t)\right] = \int_{0-}^{\infty} e^{-st} f'(t)\, dt$$

Integrating RHS of above equation by parts we get, Integrating RHS of above equation by parts we get,

$$\begin{aligned}
\mathcal{L}\,[f'(t)] &= \left[e^{-st}\,f(t)\right]_{0-}^{\infty} - \int_{0-}^{\infty} (-s)\,e^{-st} f(t)\;dt \\
&= \left[e^{-st} f(t)\right]_{0-}^{\infty} + s\int_{0-}^{\infty} f(t)\,e^{-st}\;dt \\
&= e^{-\infty}\,f(\infty) - e^{0}\,f(0-) + s\int_{0-}^{\infty} f(t)\,e^{-st}\;dt \\
&= -f(0-) + s\,F(s) \quad \text{since } e^{-\infty} = 0 \text{ and } e^{0} = 1 \\
&= s\,F(s) - f(0-) \quad \text{which is proved.}
\end{aligned}$$

This theorem can be further expanded as follows :

$$\begin{aligned}
\mathcal{L}\left[\frac{d^2}{dt^2} f(t)\right] &= s\,[s\,F(s) - f(0-)] - f'(0-) \\
&= s^2\,F(s) - s\,f(0-) - f'(0-) \\
\mathcal{L}\left[\frac{d^3}{dt^3} f(t)\right] &= s^3\,F(s) - s^2\,f(0-) - s\,f'(0-) - f''(0-)
\end{aligned}$$

.5 Integration Theorem

Let $\mathcal{L}\,[f(t)] = F(s)$, then Laplace transform of integral of $f(t)$ is given as,

$$\boxed{\mathcal{L}\left[\int_{0-}^{t} f(t)\,dt\right] = \frac{F(s)}{s}}$$

Proof : By definition of Laplace transform we have,

$$\mathcal{L}\left[\int_{0-}^{t} f(t)\,dt\right] = \int_{0-}^{\infty} e^{-st}\left[\int_{0-}^{t} f(t)\,dt\right] dt$$

Integrating the above relation by parts we get,

$$\mathcal{L}\left[\int_{0-}^{t} f(t)\,dt\right] = \left[\frac{e^{-st}}{-s}\int_{0-}^{t} f(t)\,dt\right]_{0-}^{\infty} - \int_{0-}^{\infty} \frac{e^{-st}}{-s}\cdot f(t)\,dt$$

In the above expression,

$e^{-st} \to 0$ as $t \to \infty$ and $\int_{0-}^{t} f(t)\;dt\big|_{t=0-} = 0$

$$\therefore \quad \mathcal{L}\left[\int_{0-}^{t} f(t)\,dt\right] = 0 + \frac{1}{s}\int_{0-}^{\infty} e^{-st} f(t)\,dt$$

$$= \frac{1}{s}\cdot F(s) \text{ or } \frac{F(s)}{s}$$

This relation can be generalized for multiple integrals. i.e.,

$$\mathcal{L}\left[\int_{0}^{t_1}\int_{0}^{t_2} \dots \int_{0}^{t_n} f(t)\,dt_1, dt_2, \dots dt_n\right] = \frac{F(s)}{s^n}$$

.6 Differentiation by 's'

Let $\mathcal{L}\,[f(t)] = F(s)$. Then differentiation in complex frequency domain corresponds to multiplication by t in the time domain. i.e.,

$$\boxed{\mathcal{L}\,[t\,f(t)] = -\frac{d}{ds}F(s)}$$

Proof : By definition of Laplace transform,

$$F(s) = \int_{0-}^{\infty} f(t)\,e^{-st}\,dt$$

Differentiate this equation with respect to s *i.e.*,

$$\frac{d}{ds}F(s) = \int_{0-}^{\infty} f(t)\cdot\frac{d}{ds}e^{-st}\,dt$$

$$= -\int_{0-}^{\infty} t\, f(t) \cdot e^{-st}\, dt$$

$$= -\mathcal{L}\, [t\, f(t)] \qquad \text{By definition.}$$

7 Initial Value Theorem

If $\mathcal{L}\, [f(t)] = F(s)$, then initial value of $f(t)$ is given as,

$$\boxed{f(0+) \;=\; \lim_{t \to 0+} f(t) \;=\; \lim_{s \to \infty} [s\, F(s)]}$$

provided that first derivative of $f(t)$ should be Laplace transformable.

Proof : We know that,

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = s\, F(s) - f(0-) \quad \text{By equation 2.5.8}$$

Let us take limit of the above equation as s tends to ∞ i.e.,

$$\lim_{s \to \infty} \mathcal{L}\left[\frac{d}{dt} f(t)\right] = \lim_{s \to \infty} \{[s\ F(s)] - f(0-)\}$$

Consider LHS of above equation. i.e.,

$$\lim_{s \to \infty} \mathcal{L}\left[\frac{d}{dt} f(t)\right] = \lim_{s \to \infty} \int_{0-}^{\infty} \frac{d}{dt} f(t)\, e^{-st}\ dt \qquad \text{By definition of LT.}$$

$$= 0 \quad \text{since } \lim_{s \to \infty} e^{-st}\ dt = 0$$

Putting this value of $\lim_{s \to \infty} \mathcal{L}\left[\frac{d}{dt} f(t)\right]$ in equation 2.5.11 (a) we get,

$$0 = \lim_{s \to \infty} \{s\, F(s) - f(0-)\}$$

$$\therefore \qquad f(0-) = \lim_{s \to \infty} [s\, F(s)]$$

$f(0-)$ indicates the value of $f(t)$ just before $t = 0$ and $f(0+)$ indicates the value of $f(t)$ just after $t = 0$. If function $f(t)$ is continuous at $t = 0$, then its value just before and just after $t = 0$ will be same. i.e.,

$$f(0+) = f(0-) \quad \text{for } f(t) \text{ continuous at } t = 0.$$

Putting this value of $f(0-)$ in equation 2.5.11 (b) we get,

$$f(0+) = \lim_{s \to \infty} [s\,F(s)]$$

This equation is used to determine the initial value of $f(t)$ and its derivative.

.8 Final Value Theorem

If $\mathcal{L}\,[f(t)] = F(s)$, then final value of $f(t)$ is given as,

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} [s\,F(s)]}$$

Proof : We know that,

$$\mathcal{L}\left[\frac{d}{dt} f(t)\right] = s\,F(s) - f(0-) \quad \text{By equation 2.5.8.}$$

Let us take limit of the above equation as s tends to zero i.e.,

$$\lim_{t \to \infty} \mathcal{L}\left[\frac{d}{dt} f(t)\right] = \lim_{s \to 0} \{s\,F(s) - f(0-)\}$$

$$= \lim_{s \to 0} [s\,F(s)] - f(0-)]$$

Consider LHS of above equation i.e.,

$$\lim_{t \to \infty} \mathcal{L}\left[\frac{d}{dt} f(t)\right] = \lim_{s \to 0} \int_{0-}^{\infty} \frac{d}{dt} f(t)\, e^{-st}\, dt \quad \text{By definition of LT.}$$

$$= \int_{0-}^{\infty} \frac{d}{dt} f(t)\, dt \quad \text{since} \quad \lim_{s \to 0} e^{-st} = 1$$

$$= [f(t)]_{0-}^{\infty}$$

$$= \lim_{t \to \infty} f(t) - f(0-)$$

Putting this expression of $\lim_{s \to 0} \mathcal{L}\left[\frac{d}{dt} f(t)\right]$ in equation 2.5.12 (a) we get,

$$\lim_{t \to \infty} f(t) - f(0-) = \lim_{s \to 0} \{s\,F(s)\} - f(0-)$$

$$\therefore \quad \lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s) \quad \text{which is proved.}$$

The final value theorem is useful in analysis and design of feedback control systems.

.10 Convolution Theorem

The convolution theorem of Laplace transforms states that,

If $F_1(s)$ is Laplace transform of $f_1(t)$ and $F_2(s)$ is Laplace transform of $f_2(t)$ then,

$$\boxed{\mathcal{L}[f_1(t) * f_2(t)] = F_1(s) \cdot F_2(s)}$$

That is, the Laplace transform of convolution of two functions is equivalent to multiplication of their Laplace transforms.

Proof :

The convolution of two functions is represented as,

$$\begin{aligned} u(t-\tau) &= 1 \qquad \text{for } t \geq \tau \\ &= 0 \qquad \text{for } t < \tau \end{aligned}$$

Using this step function we can write,

$$f_1(t) * f_2(t) = \int_0^{\infty} f_1(t-\tau)\, u(t-\tau)\, f_2(\tau)\, d\tau$$

Here we used step function to change integration limits from 0 to t to 0 to ∞. Taking Laplace transform of above equation we get,

$$\begin{aligned} \mathcal{L}[f_1(t) * f_2(t)] &= \int_0^{\infty}\left[\int_0^{\infty} f_1(t-\tau)\, u(t-\tau)\, f_2(\tau)\, d\tau\right] e^{-st}\, dt \\ &= \int_0^{\infty} e^{-st} \int_0^{\infty} f_1(t-\tau)\, u(t-\tau)\, f_2(\tau)\, d\tau\, dt \end{aligned}$$

Put $x = t - \tau$, then $t = x + \tau$

$$\therefore \qquad e^{-st} = e^{-s(x+\tau)} = e^{-sx} \cdot e^{-s\tau}$$

$$\text{and} \qquad dx = dt$$

$$\therefore \quad \mathcal{L}[f_1(t) * f_2(t)] = \int_0^{\infty}\int_0^{\infty} f_1(x)\, u(x)\, f_2(\tau)\, e^{-s\tau} \cdot e^{-sx}\, d\tau \cdot dx$$

$$= \int_{0}^{\infty} f_1(x)\, u(x)\, e^{-sx}\, dx \int_{0}^{\infty} f_2(\tau)\, e^{-s\tau}\, d\tau$$

In the above equation $U(x) = 1$ for $x \geq 0$ hence it can be dropped. Then we have,

$$\therefore \quad \mathcal{L}[f_1(t) * f_2(t)] = \int_{0}^{\infty} f_1(x)\, e^{-sx}\, dx \cdot \int_{0}^{\infty} f_2(\tau)\, e^{-s\tau}\, d\tau$$

$$= F_1(s) \cdot F_2(s) \qquad \text{By definition of LT}$$

This is the proof of convolution theorem of equation

Example .1 : *Find out the Laplace transform of an exponential function which is given as,*

$$f(t) = e^{at}$$

[The above function can also be written as $e^{at}\, u(t)$ to indicate that e^{at} exists only for $t \geq 0$ since $u(t) = 1$ for $t \geq 0$].

Solution : By definition of Laplace transform we have

$$\mathcal{L}\left[e^{at}\right] = \int_{0-}^{\infty} e^{at}\; e^{-st}\, dt$$

Here $u(t)$ is dropped since integration is for $t \geq 0$ and $u(t) = 1$ for $t \geq 0$.

$$= \int_{0-}^{\infty} e^{-(s-a)t}\, dt$$

$$= -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_{0-}^{\infty}$$

$$= \frac{1}{s-a}$$

thus, $$\boxed{\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}}$$

For $s < a$, the laplace transform cannot be calculated since the integral is unbounded. Therefore the region of convergence is $s > a$. This is shown in Fig. 1. The shaded area shows the ROC. Thus the laplace transform pair is,

$$\boxed{e^{at}\, u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a}, \quad ROC \quad s > a}$$

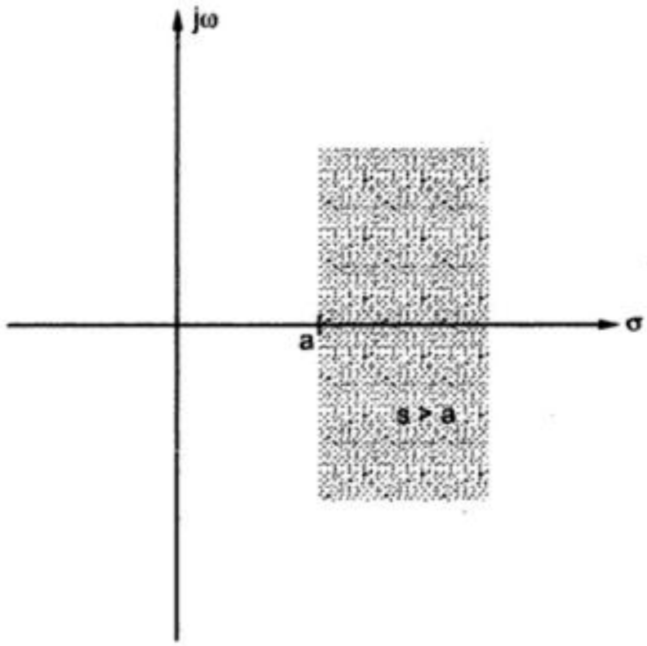


Fig. .1 ROC for $e^{at}\, u(t)$

Example .2 : *Find out the Laplace transform of unit step function. The unit step function is given as,*

$$u(t) = 1 \quad \text{for } t \geq 0 \quad \text{and}$$
$$= 0 \quad \text{otherwise.}$$

Solution : By definition of Laplace transform we have,

$$\mathcal{L}[u(t)] = \int_{0-}^{\infty} 1 \cdot e^{-st}\, dt$$

$$= -\frac{1}{s}\left[e^{-st}\right]_0^{\infty}$$

$$= \frac{1}{s}$$

Thus, $\mathcal{L}[u(t)] = \frac{1}{s}$

Example 3 : *Find out the Laplace transform of ramp function. The ramp function is given as,*

$$r(t) = t \qquad \text{for } t \geq 0$$
$$= 0 \qquad \text{otherwise.}$$

or $$r(t) = t\,u(t)$$

Solution : By definition of Laplace transform we have,

$$\mathcal{L}\,[r(t)] = \int_{0-}^{\infty} t\, e^{-st}\, dt$$

Integrating the above equation by parts we get,

$$\mathcal{L}\,[r(t)] = \left[t \cdot \frac{e^{-st}}{-s}\right]_{0-}^{\infty} - \int_{0-}^{\infty} 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$= \left[\frac{t\, e^{-st}}{-s}\right]_{0-}^{\infty} - \left[\frac{e^{-st}}{-s^2}\right]_{0-}^{\infty}$$

$$= \frac{1}{s^2}$$

Thus $$\boxed{\mathcal{L}\; t\,u(t) = \frac{1}{s^2}}$$

If the slope of the ramp is 'K', then it is given as,

$$f(t) = K\,r(t) \qquad K \text{ is slope.}$$
$$= K\,t$$

The Laplace transform of this function will be,

$$\mathcal{L}\,[K\,r(t)] = \frac{K}{s^2}$$

If the unit ramp function is delayed by time t_0, it is given as,

$$r(t-t_0) = t-t_0 \quad \text{for} \quad t \geq t_0$$
$$= 0 \quad \text{otherwise.}$$

By shifting property of Laplace transform

$$\mathcal{L}\,[r(t-t_0)] = \frac{e^{-st_0}}{s^2}$$

Similarly,

$$\mathcal{L}\,[K r(t-t_0)] = \frac{K e^{-st_0}}{s^2}$$

Example .5 : *Find out the Laplace transform of sine wave. A sine wave is given as,*

$$f(t) = A \sin \omega_0 t$$

Solution : We know that $\sin \omega_0 t$ can be represented using Euler's identity as,

$$\sin \omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$\therefore$ Equation 2.5.19 (a) becomes,

$$\therefore \quad f(t) = \frac{A}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

Taking Laplace transform of both sides,

$$\mathcal{L}\,[f(t)] = \frac{A}{2j}\left\{\mathcal{L}\left[e^{j\omega_0 t}\right] - \mathcal{L}\left[e^{-j\omega_0 t}\right]\right\}$$

By equation 2.5.15 the Laplace transform of e^{at} is given as,

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

$$\therefore \qquad \mathcal{L}\left[e^{j\omega_0 t}\right] = \frac{1}{s-j\omega_0}$$

and $$\mathcal{L}\left[e^{-j\omega_0 t}\right] = \frac{1}{s+j\omega_0}$$

Putting these values in equation 2.5.19 (b) we get,

$$\mathcal{L}\,[f(t)] = \frac{A}{2j}\left\{\frac{1}{s-j\omega_0}-\frac{1}{s+j\omega_0}\right\}$$

$$= \frac{A}{2j}\cdot\frac{2j\omega_0}{s^2+\omega_0^2}$$

$$= \frac{A\omega_0}{s^2+\omega_0^2}$$

Thus, $$\boxed{\mathcal{L}\,[A\sin\omega_0 t] = \frac{A\omega_0}{s^2+\omega_0^2}}$$

Example 8 : *A damped cosine wave is given as,*

$$f(t) = e^{-at}\cos\omega t$$

find out Laplace transform of this signal.

Solution : With the help of Euler's identity,

$$f(t) = e^{-at}\left[\frac{e^{j\omega t}+e^{-j\omega t}}{2}\right]$$

$$= \frac{1}{2}\cdot\left[e^{-(a-j\omega)t}+e^{-(a+j\omega)t}\right]$$

By taking Laplace transform of both sides,

$$\mathcal{L}\,f(t) = \frac{1}{2}\,L\left\{e^{-(a-j\omega)t}+e^{-(a+j\omega)t}\right\}$$

$$= \frac{1}{2} \cdot \left\{ \frac{1}{s+(a-j\,\omega)} + \frac{1}{s+(a+j\,\omega)} \right\}$$

$$= \frac{1}{2} \cdot \frac{2(s+a)}{(s+a)^2+\omega^2}$$

$$= \frac{s+a}{(s+a)^2+\omega^2}$$

Thus, $\boxed{\mathcal{L}\left[e^{-at}\cos\omega t\right] = \frac{s+a}{(s+a)^2+\omega^2}}$

Example .2 : *Determine the laplace transform and ROC for the signal*

$$x(t) = -e^{at}\,u(-t)$$

Solution : Laplace transform is given as,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} -e^{at}\, u(-t)\, e^{-st}\, dt$$

We know that,

$$u(-t) = \begin{cases} 0 & for \quad t \geq 0 \\ 1 & for \quad t < 0 \end{cases}$$

Hence the integration limits of laplace transform will be changed as follows :

$$X(s) = \int_{-\infty}^{0} -e^{at}\, e^{-st}\, dt$$

$$= -\int_{-\infty}^{0} e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{s-a}\right]_{-\infty}^{0}$$

$$= \lim_{t\to 0}\left[\frac{e^{-(s-a)t}}{s-a}\right] - \lim_{t\to -\infty}\left[\frac{e^{-(s-a)t}}{s-a}\right]$$

The second term will converge if power of exponent is negative. Note that 't' tends to $-\infty$. Hence $(s-a)$ must be negative to make overall exponent negative. Therefore we can write,

$$X(s) = \frac{e^{-(s-a)0}}{s-a} - \frac{e^{-(s-a)(-\infty)}}{s-a}$$

$$= \frac{1}{s-a} - \frac{0}{s-a} \ \textit{for} \ (s-a)<0$$

$$= \frac{1}{s-a} \ \textit{for} \ s-a<0 \ \textit{or} \ s<a$$

Thus the laplace transform will converge if $s<a$. For $s>a$, the integration will be unbounded. Fig. 2 shows the ROC of $X(s)$.

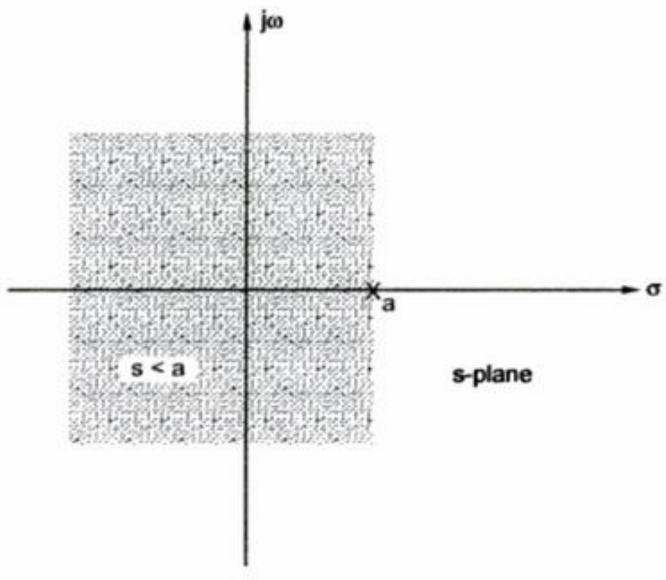


Fig. 2 ROC for $-e^{at}\,u(-t)$

The laplace transform pair can be written as,

$$\boxed{-e^{at}\,u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a}, \ ROC \ s<a}$$

Relationship between Fourier Transform and Laplace Transform

We know that fourier transform is given as,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Fourier transform can be calculated only if $x(t)$ is absolutely integrable. i.e.,

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

We know that $s = \sigma + j\omega$. Hence equation 2.3.1 can be written as,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-\sigma t}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left\{x(t) e^{-\sigma t}\right\} e^{-j\omega t}\, dt$$

Comparing above equation with equation 2.3.6 we find that, laplace transform of $x(t)$ is basically the fourier transform of $x(t)\, e^{-\sigma t}$. If $s = j\omega$, i.e. $\sigma = 0$, then above equation becomes,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= X(j\omega)$$

Thus $$X(s) = X(j\omega) \quad \textit{when} \quad s = j\omega$$

This means laplace transform is same as fourier transform when $s = j\omega$. Above equation shows that fourier transform is special case of laplace transform. Thus laplace transform provides broader characterization compared to fourier transform. $s = j\omega$ indicates imaginary axis in complex s-plane. Thus laplace transform is basically fourier transform on imaginary $(j\omega)$ axis in the s-plane.

Convergence of laplace transform

From equation 2.3.8 we know that laplace transform is basically the fourier transform of $x(t)e^{-\sigma t}$. Hence if fourier transform of $x(t)e^{-\sigma t}$ exists, then laplace transform of $x(t)$ exists. For fourier transform to exist, $x(t)e^{-\sigma t}$ must be absolutely integrable. i.e.,

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|\, dt < \infty$$

Z–TRANSFORMS:

The z-transform is a generalization of the discrete-time Fourier transform we learned in Chapter 5. As we will see, z-transform allows us to study some system properties that DTFT cannot do.

1 The z-Transform

Definition 21. *The z-transform of a discrete-time signal $x[n]$ is:*

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}. \qquad (.1)$$

We denote the z-transform operation as

$$x[n] \longleftrightarrow X(z).$$

In general, the number z in (.1) is a complex number. Therefore, we may write z as

$$z = re^{jw},$$

where $r \in \mathbb{R}$ and $w \in \mathbb{R}$. When $r = 1$, (.1) becomes

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn},$$

which is the discrete-time Fourier transform of $x[n]$. Therefore, DTFT is a special case of the z-transform! Pictorially, we can view DTFT as the z-transform evaluated on the unit circle:

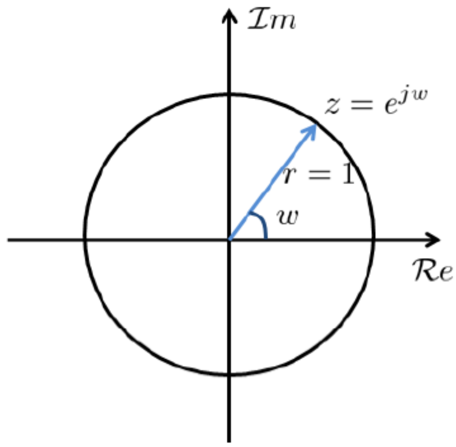


Figure .1: Complex z-plane. The z-transform reduces to DTFT for values of z on the unit circle.

When $r \neq 1$, the z-transform is equivalent to

$$\begin{aligned} X(re^{jw}) &= \sum_{-\infty}^{\infty} x[n]\left(re^{jw}\right)^{-n} \\ &= \sum_{-\infty}^{\infty} \left(r^{-n}x[n]\right)e^{-jwn} \\ &= \mathcal{F}\left[r^{-n}x[n]\right], \end{aligned}$$

Example 1. Consider the signal $x[n] = a^n u[n]$, with $0 < a < 1$. The z-transform of $x[n]$ is

$$
\begin{aligned}
X(z) &= \sum_{-\infty}^{\infty} a^n u[n] z^{-n} \\
&= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n .
\end{aligned}
$$

Therefore, $X(z)$ converges if $\sum_{n=0}^{\infty} \left(az^{-1}\right)^n < \infty$. From geometric series, we know that

$$
\sum_{n=0}^{\infty} \left(rz^{-1}\right)^n = \frac{1}{1 - az^{-1}},
$$

with ROC being the set of z such that $|z| > |a|$.

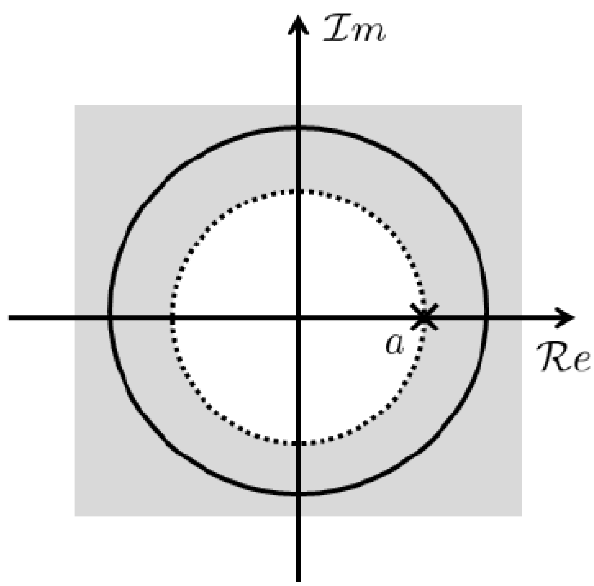


Figure .2: Pole-zero plot and ROC of Example 1.

Example 2. Consider the signal $x[n] = -a^n u[-n-1]$ with $0 < a < 1$. The z-transform of $x[n]$ is

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} \\ &= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n \\ &= 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n . \end{aligned}$$

Therefore, $X(z)$ converges when $|a^{-1}z| < 1$, or equivalently $|z| < |a|$. In this case,

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}},$$

with ROC being the set of z such that $|z| < |a|$. Note that the z-transform is the same as that of Example 1. The only difference is the ROC. In fact, Example 2 is just the left-sided version of Example 1!

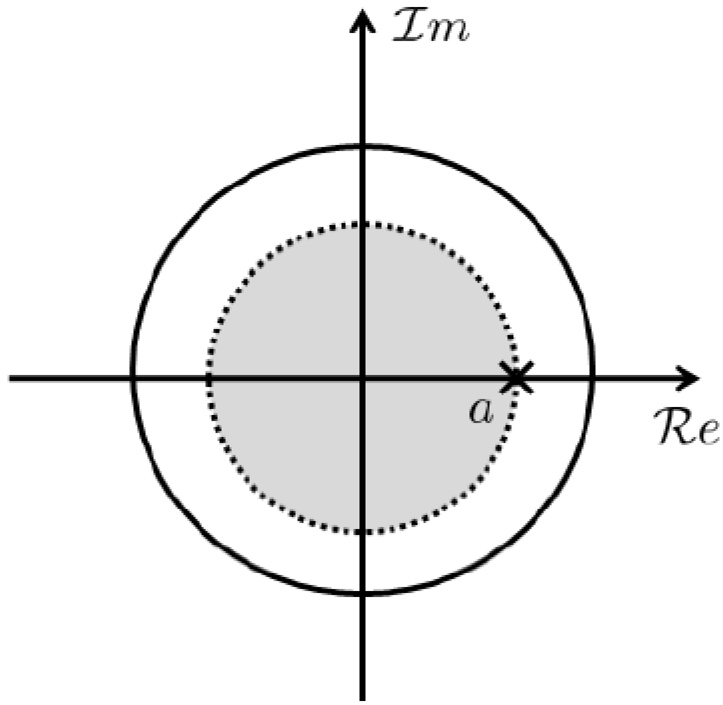


Figure 3: Pole-zero plot and ROC of Example 2.

Example 3. Consider the signal

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n].$$

The z-transform is

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty}\left[7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right] u[n]z^{-n} \\
&= 7\sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^n u[n]z^{-n} - 6\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^n u[n]z^{-n} \\
&= 7\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right) - 6\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)
\end{aligned}$$

For $X(z)$ to converge, both sums in $X(z)$ must converge. So we need both $|z| > |\frac{1}{3}|$ and $|z| > |\frac{1}{2}|$. Thus, the ROC is the set of z such that $|z| > |\frac{1}{2}|$.

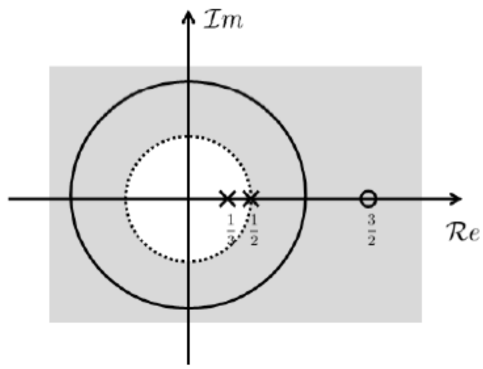


Figure .4: Pole-zero plot and ROC of Example 3.

Properties of ROC

Property 1. *The ROC is a ring or disk in the z-plane center at origin.*

Property 2. *DTFT of $x[n]$ exists if and only if ROC includes the unit circle.*

Proof. By definition, ROC is the set of z such that $X(z)$ converges. DTFT is the z-transform evaluated on the unit circle. Therefore, if ROC includes the unit circle, then $X(z)$ converges for any value of z on the unit circle. That is, DTFT converges. □

Property 3. *The ROC contains no poles.*

Property 4. *If $x[n]$ is a finite impulse response (FIR), then the ROC is the entire z-plane.*

Property 5. *If $x[n]$ is a right-sided sequence, then ROC extends outward from the outermost pole.*

Property 6. *If $x[n]$ is a left-sided sequence, then ROC extends inward from the innermost pole.*

Proof. Let's consider the right-sided case. Note that it is sufficient to show that if a complex number z with magnitude $|z| = r_0$ is inside the ROC, then any other complex number z' with magnitude $|z'| = r_1 > r_0$ will also be in the ROC.
Now, suppose $x[n]$ is a right-sided sequence. So, $x[n]$ is zero prior to some values of n, say N_1. That is

$$x[n] = 0, \qquad n \leq N_1.$$

Consider a point z with $|z| = r_0$, and $r_0 < 1$. Then

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=N_1}^{\infty} x[n]r_0^{-n} < \infty, \end{aligned}$$

because $r_0 < 1$ guarantees that the sum is finite.
Now, if there is another point z' with $|z'| = r_1 > r_0$, we may write $r_1 = ar_0$ for some $a > 1$. Then the series

$$\begin{aligned} \sum_{n=N_1}^{\infty} x[n]r_1^{-n} &= \sum_{n=N_1}^{\infty} x[n]a^{-n}r_0^{-n} \\ &\leq a^{N_1} \sum_{n=N_1}^{\infty} x[n]r_0^{-n} < \infty. \end{aligned}$$

So, z' is also in the ROC. □

Property 7. *If $X(z)$ is rational, i.e., $X(z) = \frac{A(z)}{B(z)}$ where $A(z)$ and $B(z)$ are polynomials, and if $x[n]$ is right-sided, then the ROC is the region outside the outermost pole.*

Proof. If $X(z)$ is rational, then by (Appendix, A.57) of the textbook

$$X(z) = \frac{A(z)}{B(z)} = \frac{\sum_{k=0}^{n-1} a_k z^k}{\prod_{k=1}^{r}(1 - p_k^{-1}z)^{\sigma_k}},$$

where p_k is the k-th pole of the system. Using partial fraction, we have

$$X(z) = \sum_{i=1}^{r}\sum_{k=1}^{\sigma_i} \frac{C_{ik}}{(1 - p_i^{-1}z)^k}.$$

Each of the term in the partial fraction has an ROC being the set of z such that $|z| > |p_i|$ (because $x[n]$ is right-sided). In order to have $X(z)$ convergent, the ROC must be the intersection of all individual ROCs. Therefore, the ROC is the region outside the outermost pole. $\square$

For example, if

$$X(z) = \frac{1}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{2}z^{-1})},$$

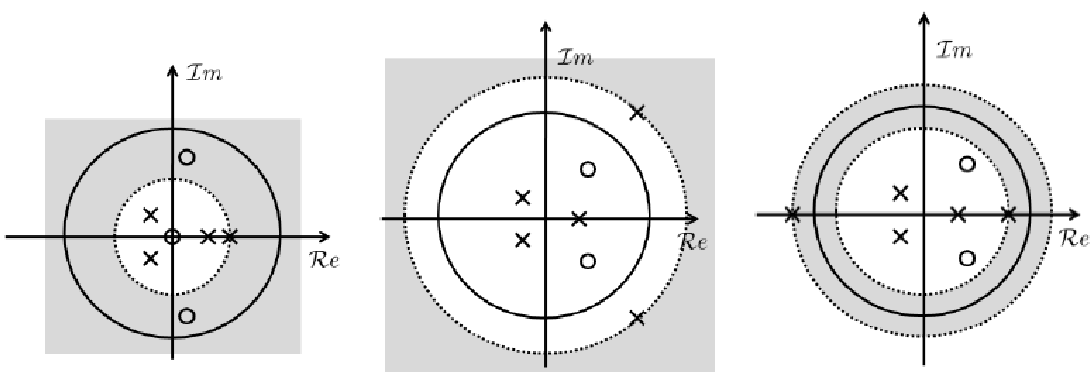
then the ROC is the region $|z| > \frac{1}{2}$.

Property 9. *A discrete-time LTI system is stable if and only if ROC of $H(z)$ includes the unit circle.*

Proof. A system is stable if and only if $h[n]$ is absolutely summable, if and only if DTFT of $h[n]$ exists. Consequently by Property 2, ROC of $H(z)$ must include the unit circle. $\square$

Property 10. *A causal discrete-time LTI system is stable if and only if all of its poles are inside the unit circle.*

Examples.



Causal, Stable   Causal, Unstable   Not causal, Stable