Chapter: One

Signal and System :- An introduction

The concept of signals & systems arise in a wide variety of fields and the ideas and techniques associated with these concepts play an important role in such diverse areas of science and technology as communications, aeronautics and astronautics, circuit design, acoustics, seismology, biomedical engineering, energy generation and distribution systems, chemical process control etc.

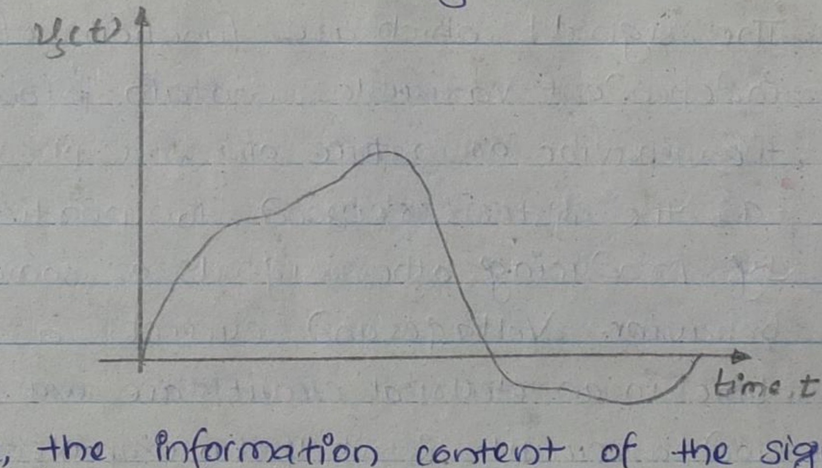
The signals which are functions of one or more independent variables, contain information about the behavior or nature of some phenomenon, whereas the systems respond to particular signals by producing other signals or some desired behavior. Voltages and currents as a function of time in an electrical circuit are examples of signals and a circuit is itself an example of a system, which in this case responds to applied voltages & currents.

Signal

Definition :-

- The variation of the dependent parameter with respect to the independent parameter is known as signal. Signals contain ~~contain~~ information about a variety of things and activities in our physical world.

Apparently, a signal is a time-varying quantity that can be represented by a graph such as that shown in figure below.



Here, the information content of the signal is represented by the changes in its magnitude as time progresses i.e; the signal waveform containing the information in the 'wiggles.' In figure the variation in magnitude of $v_s(t)$ with respect to the time is given. ie; $v_s(t)$ is dependent parameter whereas t is independent.

To extract required information from a set of signals, it needs to process the signals in some predetermined manner. Therefore signal processing is usually most conveniently performed by electronic systems. For this to be possible the signal must first be converted into an electric signal i.e. voltage or current. This process is accomplished by devices known as transducers.

Types of signal :-

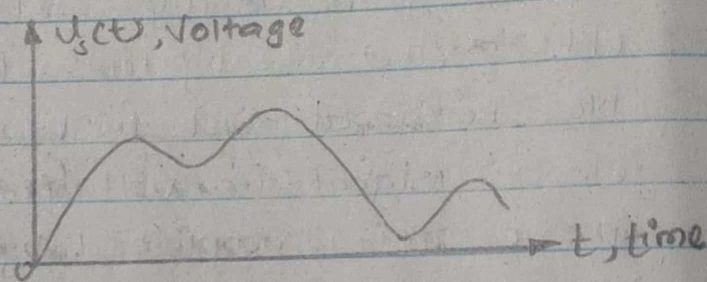
- Mainly there are two types of signal.

(1) Analog signal and
(2) Digital signal.

(1) Analog Signal :-

The magnitude of an anlog signal can take-on any value i.e. the amplitude of an analog-signal exhibits a continuous variation over its range of activity. An electronic circuit, which process such signals are known as analog circuits. An analog signal can be defined a continuous function such as a plot of current or voltage against time or displacement against force.

The voltage signal depicted in fig (a) below is called an analog signal i.e; the signal is analogous to the physical quantity that it represents.



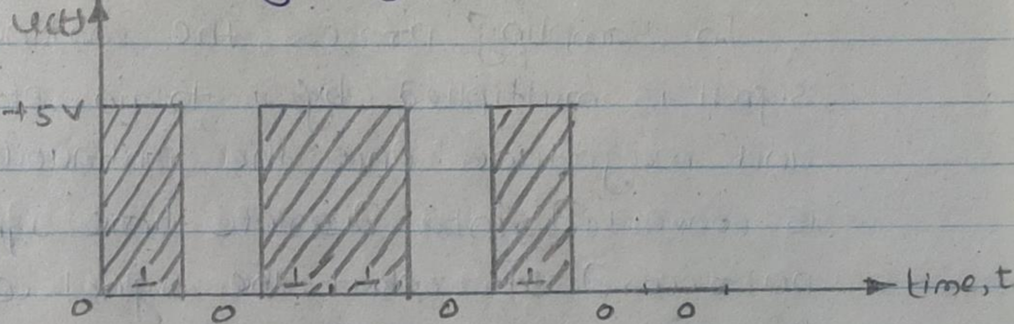
(a) fig: Analog signal.

(ii) Digital Signal:-

When a sequence of numbers is used to represents the signal magnitude at an instant of time, the resulting signal is called a digital signal. It is simply a sequence of numbers that represent the magnitude of successive signal samples. In other words, A digital signal consists of a number of discrete & discontineous pulses whose time relationship contains information regarding magnitude or the nature of quantity.

The digital signals have only two voltage levels, which can be labeled low and high.

The following fig(b) shows the time variation of such digital signal. The waveform is a pulse-train with 0V representing a 0 signal or logic 0 & +5V representing logic 1.



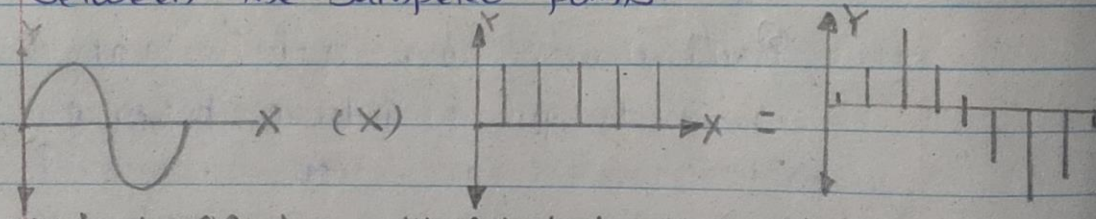
(b) fig: Digital signal.

- A digital signal is acheived through the sequence of steps of conversion of an analog signal to its equivalent digital form. by means of of Analog to digital converter. For this the maximum permissible rate of change of analog voltage and maximum permissible - frequency has to be fed to ADC.

Sampling process:-

The operation that transforms continuous time signal into discrete time signal is known as the sampling process. The main purpose of signal sampling is efficient use of data processing & data transmission unit.

In sampling process, the continuous time signal is multiplied by a train of pulse of unit magnitude. Once the continuous time signal is converted into discrete time signal, there is no record of what the signal was doing between the sampeled points.



(i) Analog signal (ii) Unit train pulse (iii) Discrete signal

For sufficiently low frequency, signal can be assumed that missing data falls on st. lines between two known sample points.
In order not to lose the identity of continuous time signal, when it is sampled, the sampling theorem states that " If the highest frequency content in the i/p signal is fin in Hz, then the i/p signal can be recovered without any distortion if it is sampled at the rate of

$2f_n$ Hz." This rate is known as Nyquiest rate ie: $f_s \geq 2f_n$. If the sampling frequency (f_s) is smaller compared to frequency of i/p signal then the reconstructed signal wave form is diff--erent than the original signal.

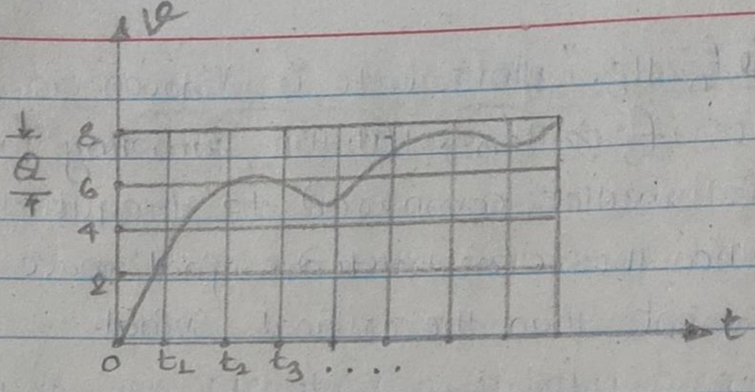
☆ Quantization Process:-

- when the value of any sampled signals falls between two permitted output states, it must be road on the permitted state nearest the actual value of the input signal.

The process of representing a continuous or analog signal by finite number of discrete state is called quantization. The standard number system used for processing the digital signal is binary number system. In this system, the code group consists of n-pulses, each indi-cating 0 or 1. The quantization level (Q) is the range between adjacent decision points and is given by

$$Q = \frac{\text{full scale Range}}{2^n}$$

where; LSB of digital signal is quantization level. as shown in fig below.



Sampled periods

The analog signal must be rounded off to a quantization level. the error varies from 0 to $\pm Q/2$, no matter how many bits are used, there is always some quantization error in the process of analog to digital conversion.

✡ Overview:

* Continuous time and Discrete time signal

In case of continuous-time signals the independent variable is continuous, and thus these signals are defined for a continuous value of the independent variable. A speech signal as a function of time and atmospheric pressure as a function of altitude are examples of continuous time signal.

Any continuous time signal can be symbollically represented as $x(t)$. where the function $x(t)$ has a continuous value with respect to the independent variable t. The illustration of a continuous time signal 't' is given below.

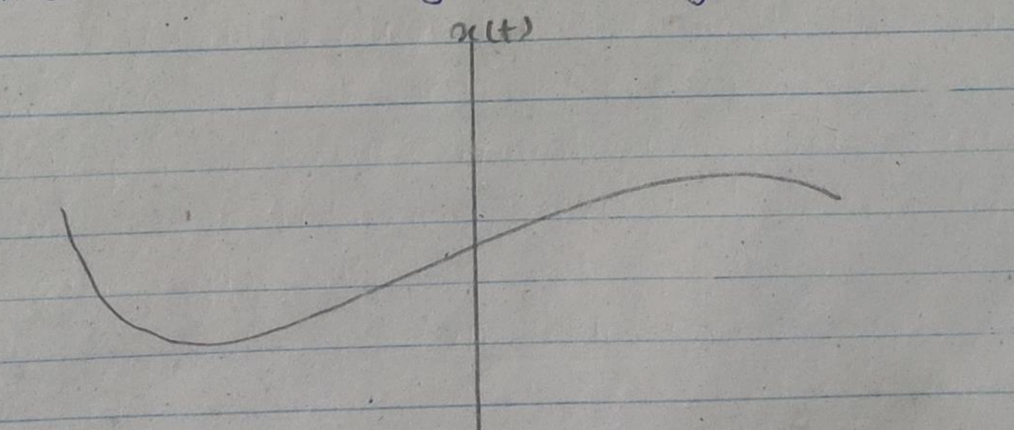


fig: continuous time signal.

On the other hand, discrete time signals are defined only at discrete times & consequently, for these signals, the independent variable takes on only a discrete set of values.

Any discrete time signals can be symbollically represented as $x[n]$, where n denotes the discrete time independent variable such that the signal $x[n]$ is defined only for integer values of the independent variable. This signal may represent a phenomenon for which 'n' is inherently discrete. Signals such as demographic data are examples of this type. The illustration of discrete time signal is given below.

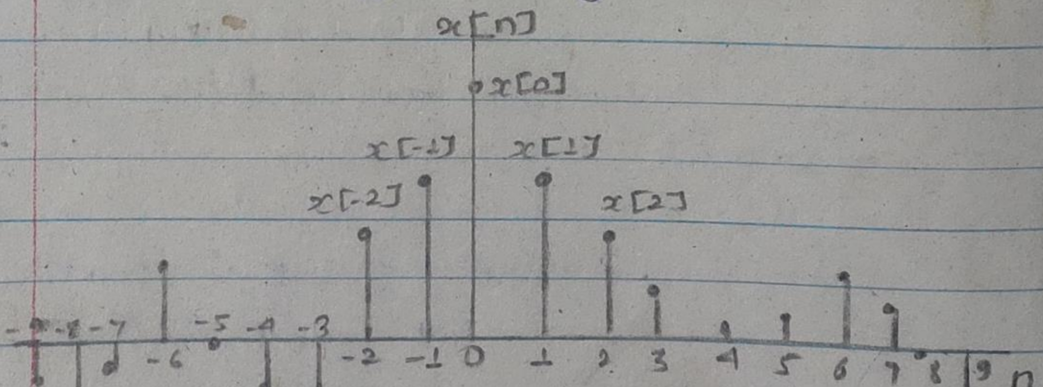
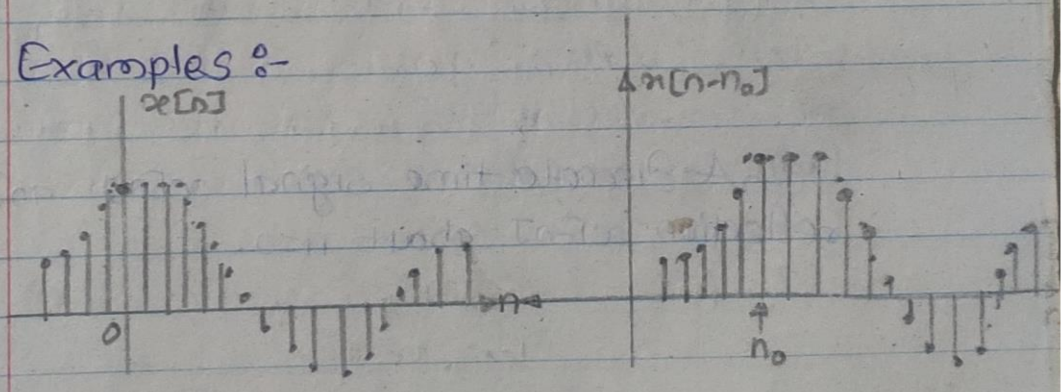


fig: Discrete time signal.

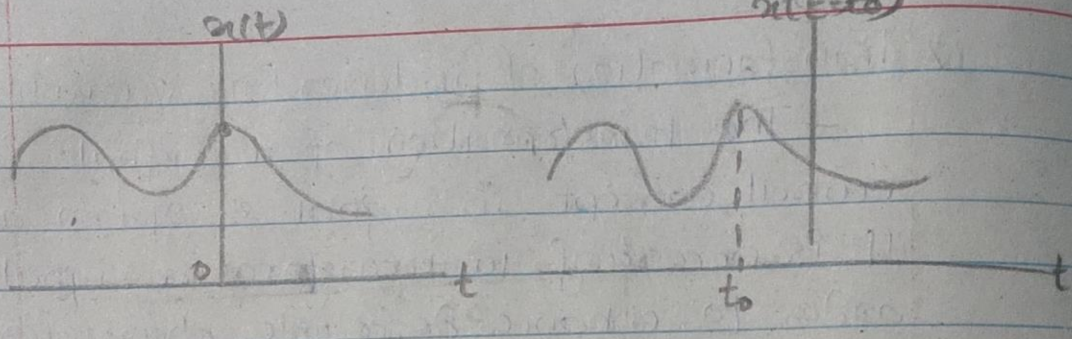
☆ Transformation of Independent Variables :-

- The transformation of a signal is a central concept in signal & system analysis. It is necessary to transform a signal in order to enhance desirable characteristics, to remove unnecessary noises or to balance the several components of the signal. Thus an elementary signal transformations involve simple modification of the independent variable i.e, the time axis in order to acheive such attempts.

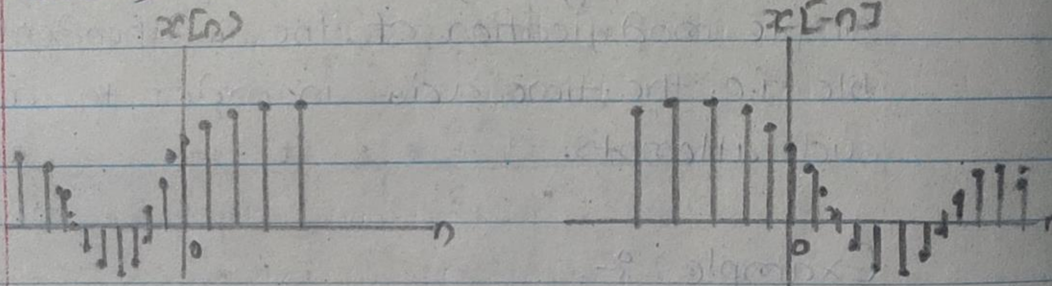
Examples :-



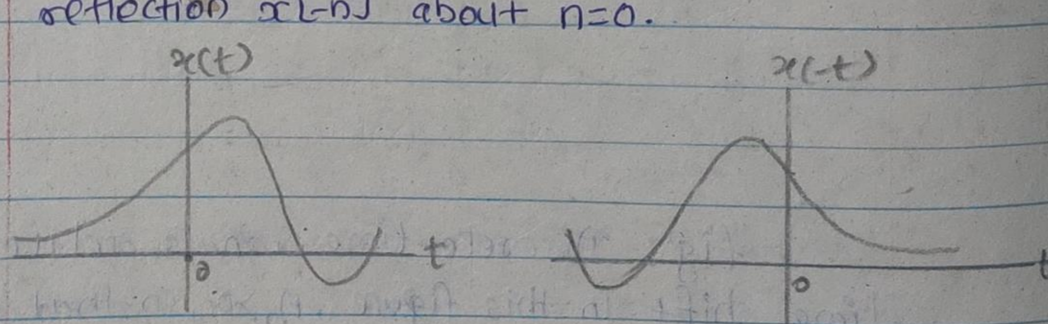
(A) fig: Discrete-time signals related by a time shift. In this figure $n_0 > 0$, so that $x[n-n_0]$ is a delayed version of $x[n]$.



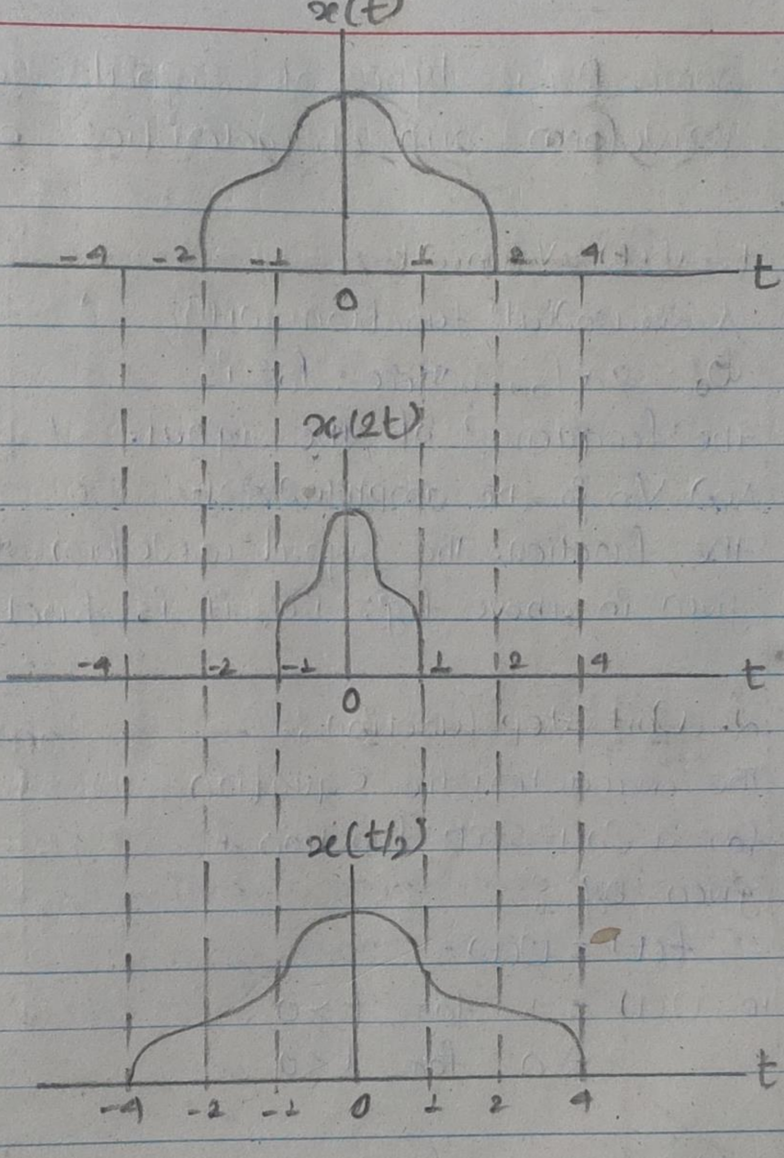
(B) fig: Continuous time signal related by a time shift. In this figure $t_0 < 0$, so that $x(t-t_0)$ is an advanced version of $x(t)$.



(C) A discrete time signal $x[n]$ and its reflection $x[-n]$ about $n=0$.



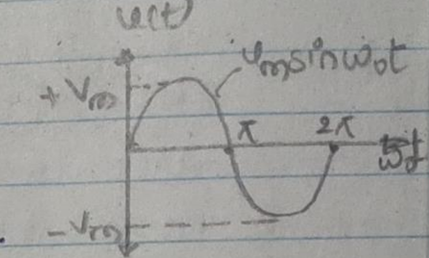
(D) A continuous time signal $x(t)$ and its reflection $x(-t)$ about $t=0$.



(E) Continuous-time signals related by time scaling.

⚝ Some Basic types of signals and their waveform with characteristics equation.

1. $v(t) = V_m \sin \omega_0 t$



A sinusodial function with $\omega_0 = 2\pi f_0$, where f_0 is the frequency of the signal. And V_m is the amplitude of the function. The signal waveform of $v(t)$ is given in above fig: i.e; it is function of time.

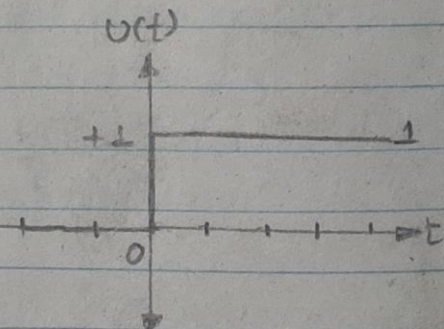
2. Unit step function :-

The characteristic equation for a unit-step function is given by :

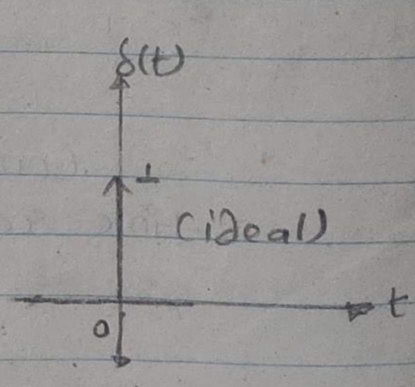
$$f(t) = U(t).$$

i.e $U(t) = 1$ for $t \geq 0$

$= 0$ for $t < 0$



3. Unit Impulse function :-



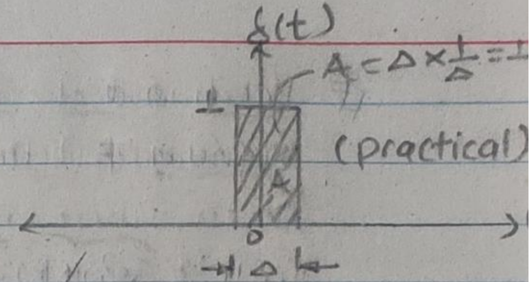
It is also called fermi dirac function or delta function of which the char^{ts} equation is given by :

e

$f(t) = \delta(t)$

ie: $\delta(t) = 1$, for $t = 0$

$= 0$, for $t \neq 0$



4. Signam function:-

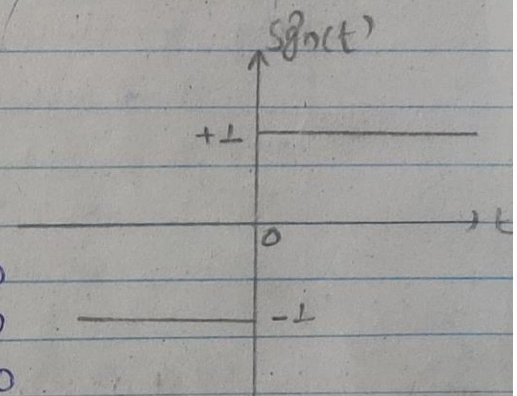
The characteristics eqⁿ of a signam function is given as:

$$f(t) = sgn(t)$$

ie; $sgn(t) = -1$, for $t < 0$

$= 1$, for $t > 0$

$= 0$, for $t = 0$

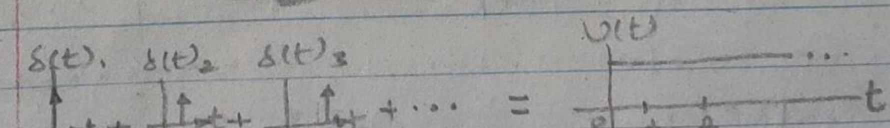


☆ Relation between Unit step fxn. and Unit impulse function.

ie; $$U(t) = \int_{0}^{\infty} \delta(t)\, dt$$

OR

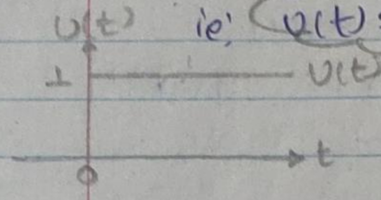
$$\frac{d}{dt}[U(t)] = \delta(t)$$

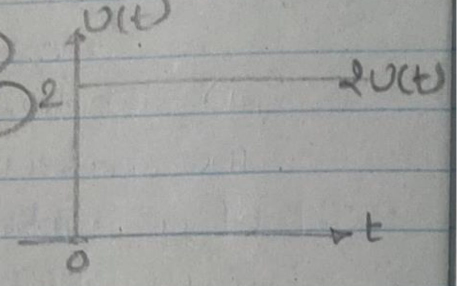


☆ Relation Between Unit step function and signum function :-

ie; $\boxed{sgn(t) = 2u(t) - 1}$

ie; $\boxed{u(t) = \frac{1}{2}\{sgn(t) + 1\}}$





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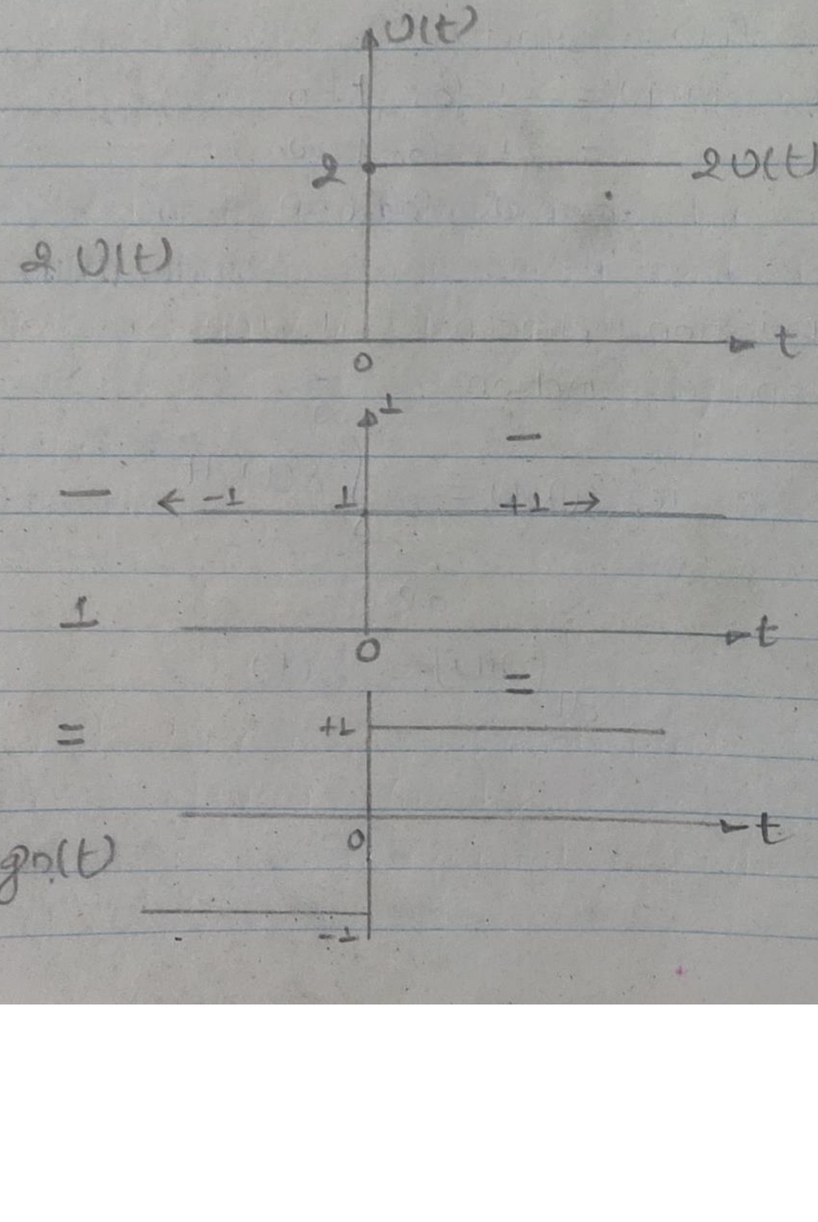


Classification of signal :-

1) Discrete time & continuous time signal
2) Deterministic & Random signal
3) Causual & Non-Causual signal
4) Energy & Power signal
5) Periodic & Non-Periodic signal
6) Even and odd signal.

Deterministic & Random Signal :-

Deterministic signals are those, which can be determine by a simple algebric - equation or graphical rebresentation.

Random signal are those, which can not be determined by a simple algebric equation or graphical representation. It is tabular form or graphical representation. It is also called non-deterministic signal.

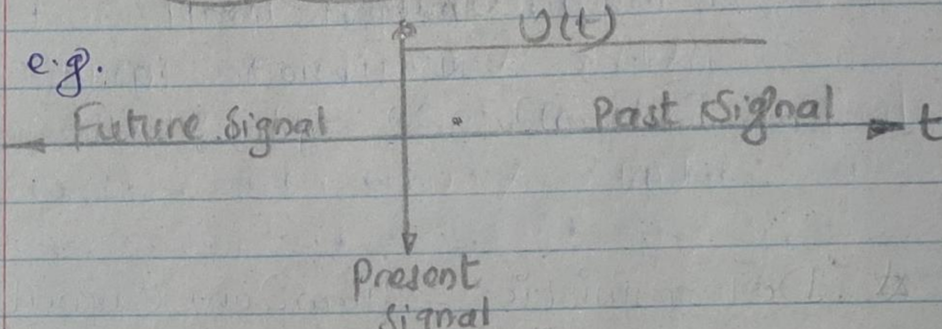
p.t.o.

Causual & Non-Causual signal :-

Causual signals are those signals in which there is no signal part for $t<0$

i.e. $x(t) = 0$ for $t<0$

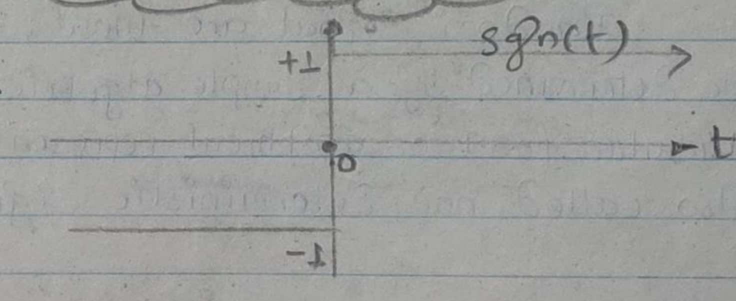
e.g.



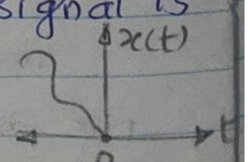
Non Causual signal is the signal in which there is some signal part for $t<0$.

i.e; $x(t) \neq 0$ for $t<0$

e.g.



Note:- If there is no signal part for $t>0$ i.e. $x(t)=0$ for $t>0$, such signal is known as Anti-causual signal.



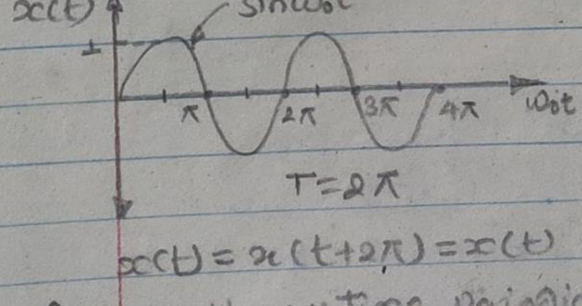
Periodic & Non-periodic Signal :

A periodic signal is that signal, which repeats itself over a continuous interval of time regularly. A signal $x(t)$ is said to be periodic with T if

$$x(t+T) = x(t)$$ for all values of 't'. In other words, a periodic signal has the property that it is unchanged by a time shift of T. In this case the signal $x(t)$ is periodic with period T.

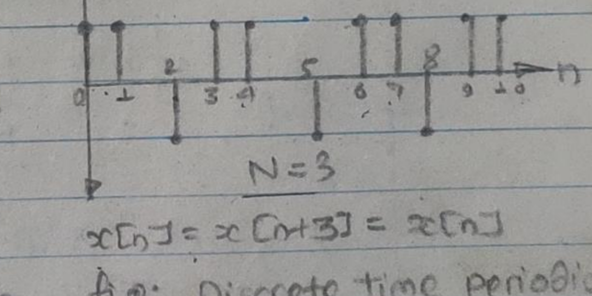
Periodic signals are defined analogously in discrete time. Specifically, a discrete time signal $x[n]$ is periodic with period N, where N is positive integer, if it is unchanged by a time shift of N, then

$$x[n] = x[n+N]$$ for all values of N.



$x(t) = x(t+2\pi) = x(t)$

fig: continuous time periodic signal



$x[n] = x[n+3] = x[n]$

fig: Discrete time periodic signal

A non periodic signal is that, signal which doesnot repeat itself over a continuous interval of time regularly. i.e, it is very hard to get the periodic time T at regular interval of time

then; $x(t+T) \neq x(t)$

Similarly for discrete time signal

$x(n+N) \neq x[n]$

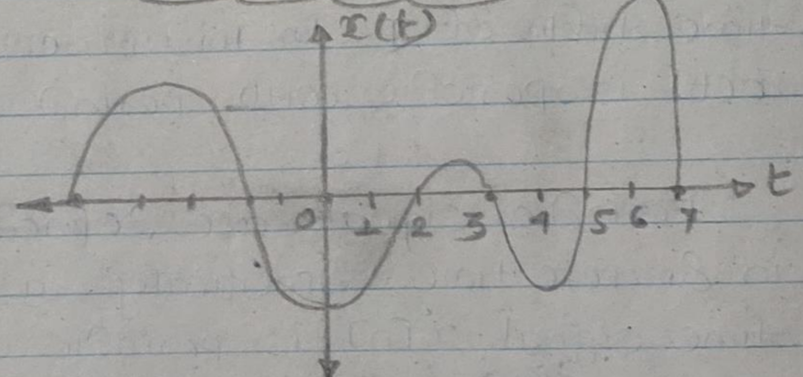


fig: Continuous Time non Periodic Signal

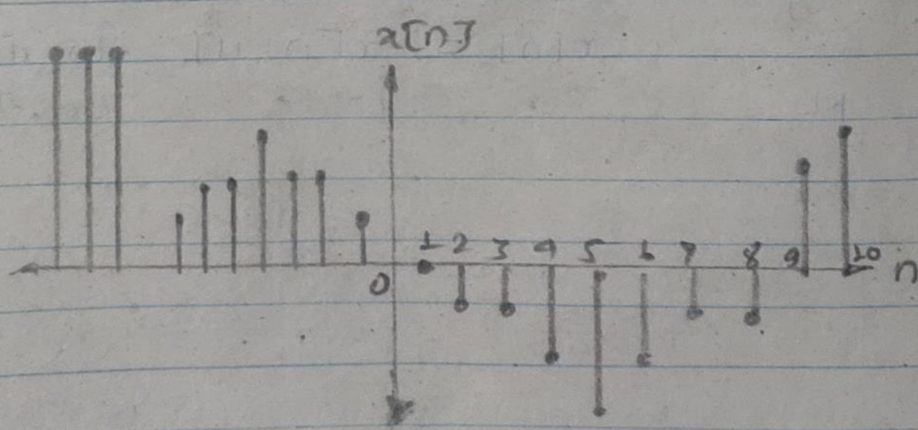


fig: Discrete Time non periodic signal

☆ Even and Odd signal :-

- A signal $x(t)$ or $x[n]$ is referred to as an even signal if it is identical to its time-reversed counterpart i.e. with its reflection about the origin.

A continuous time signal $x(t)$ is said to be even, if

$$x(-t) = x(t)$$

While a discrete time signal $x[n]$ is even, if

$$x[-n] = x[n]$$

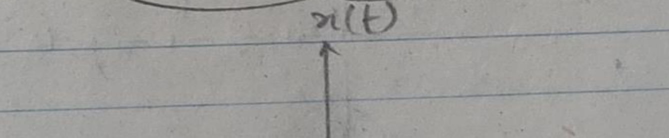


fig: An even continuous-time signal

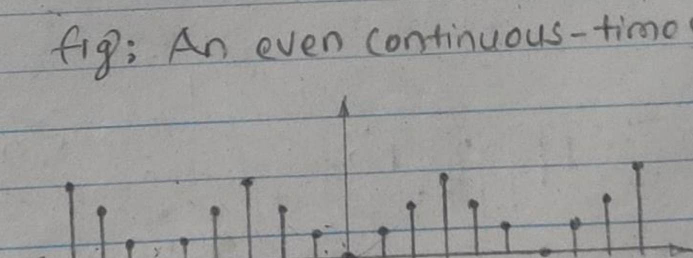


fig: An even discrete time signal.

Similarly,
A continuous time signal $x(t)$ is said to be odd, if $x(-t) = -x(t)$ — (I)
while, a discrete time signal $x[n]$ is said to be odd if $x[-n] = -x[n]$ — (II)

An odd signal must necessarily be 0 at $t=0$ or $n=0$, since equations (I) & (II) require that $x(0) = -x(0)$ and $x[0] = -x[0]$.

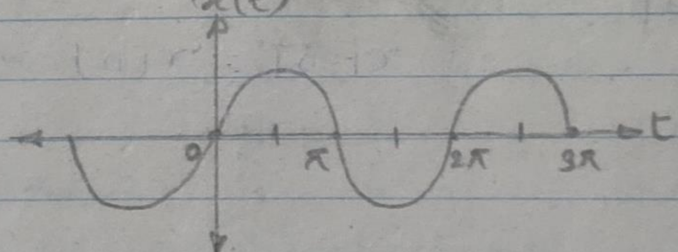


fig: An odd continuous time signal.

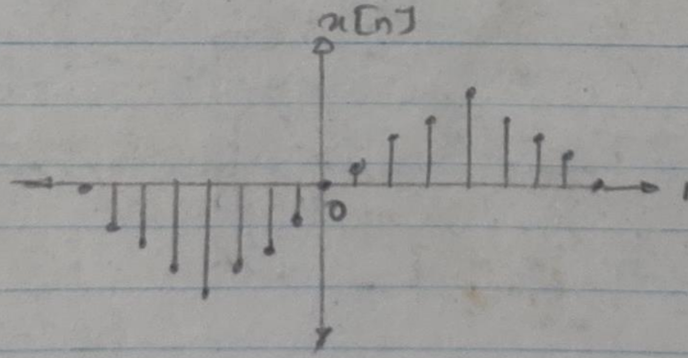


fig: An odd discrete time signal.

Any signal can be broken into a sum of even and odd signal.
ie: any signal can be expressed as

$$x(t) = E_v\{x(t)\} + O_d\{x(t)\}$$

where;

$E_v\{x(t)\} = \frac{1}{2}\{x(t) + x(-t)\}$ which is referred to as the even part of $x(t)$.

And

$O_d\{x(t)\} = \frac{1}{2}\{x(t) - x(-t)\}$ is the odd part of $x(t)$.

Exactly analogous definition hold in the case of discrete time signal.

ie; $$x[n] = E_v\{x[n]\} + O_d\{x[n]\}$$

where;

$$E_v\{x[n]\} = \frac{1}{2}\{x[n] + x[-n]\}$$

And

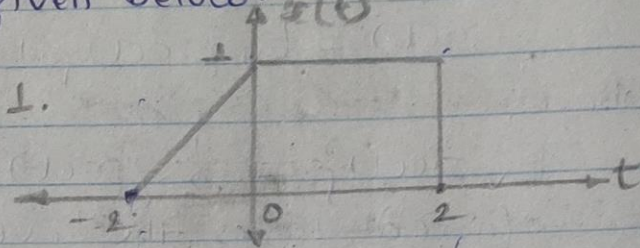
$$O_d\{x[n]\} = \frac{1}{2}\{x[n] - x[-n]\}$$

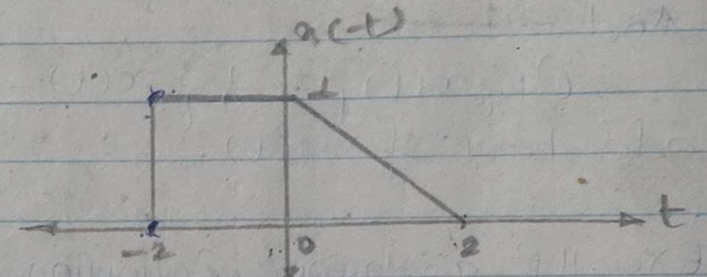
☆ Any signal, composed of even & odd part

Examples:-

* Find the even and odd part of the signal given below.

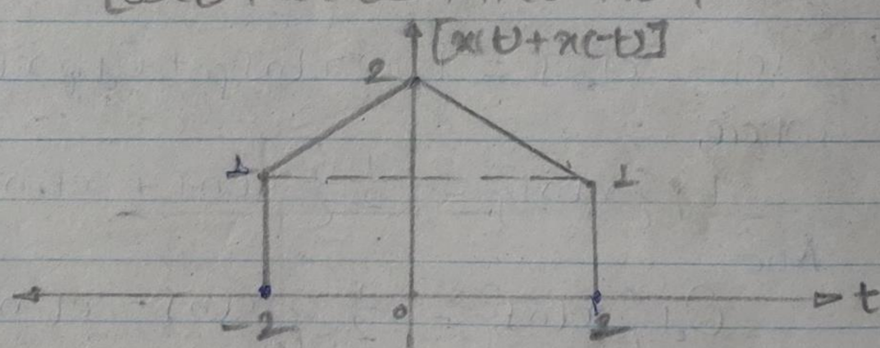
Q.1.





Now,

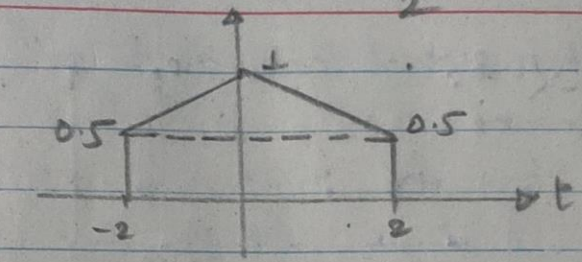
[x(t) + x(-t)] takes the form.



Finally, $E_v\{x(t)\}$ is given by

$$E_v\{x(t)\} = \frac{[x(t)+x(-t)]}{2}$$

$$\frac{[x(t)+x(-t)]}{2} = E_v\{x(t)\}$$

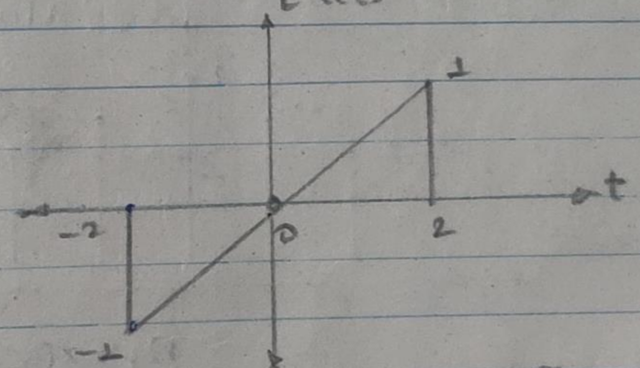


For odd part, we have

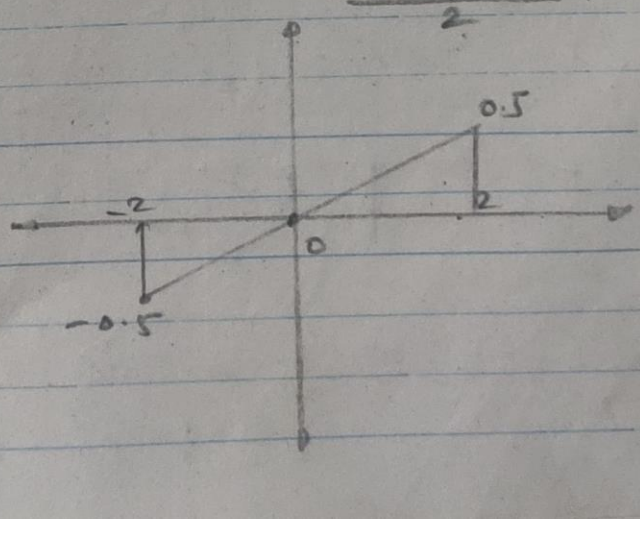
$$O_d\{x(t)\} = \frac{[x(t)-x(-t)]}{2}$$ &

this will take the form as follows:

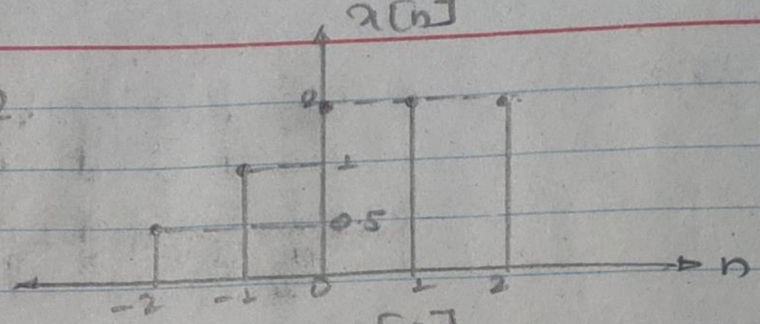
$[x(t)-x(-t)]$

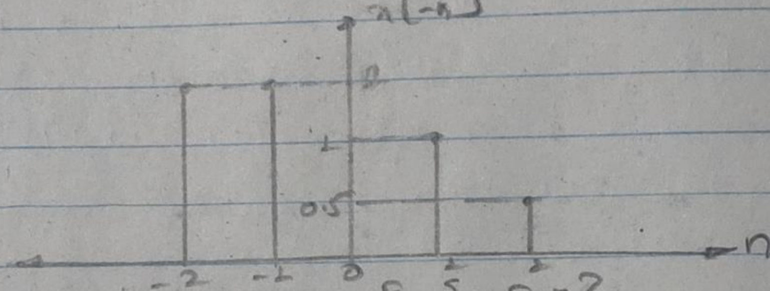


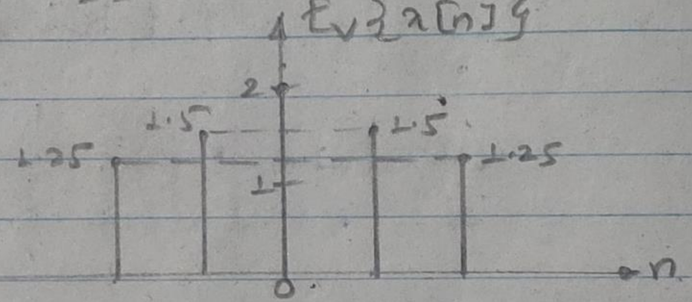
$$\frac{[x(t)-x(-t)]}{2} = O_d\{x(t)\}$$

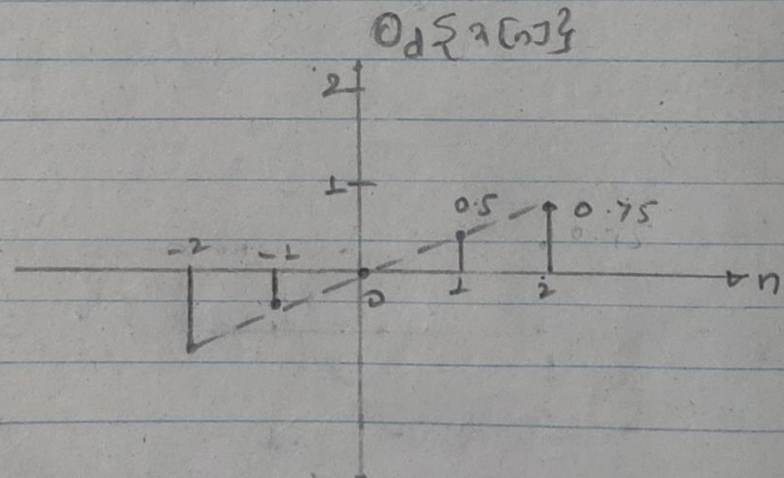


Q.2

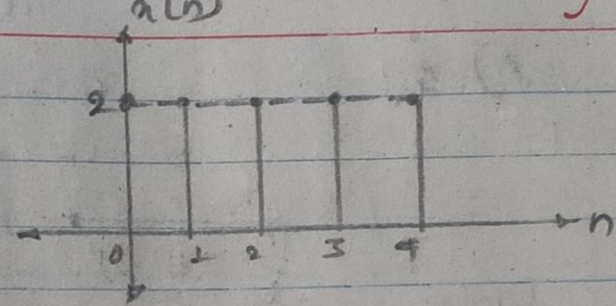

x[n]
2
1
0.5
-2
-1
0
1
2
n

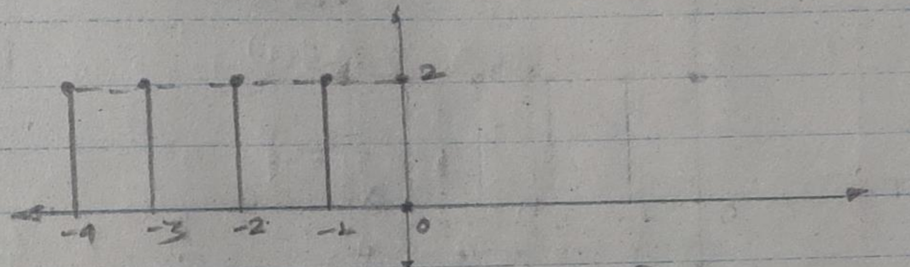

x[-n]
2
1
0.5
-2
-1
0
1
2
n

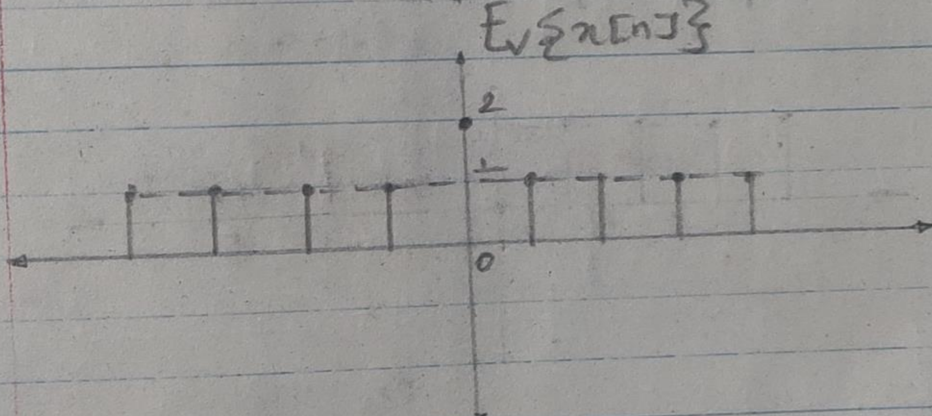

Ev{x[n]}
2
1.5
1.5
1.25
1.25
1
0
n

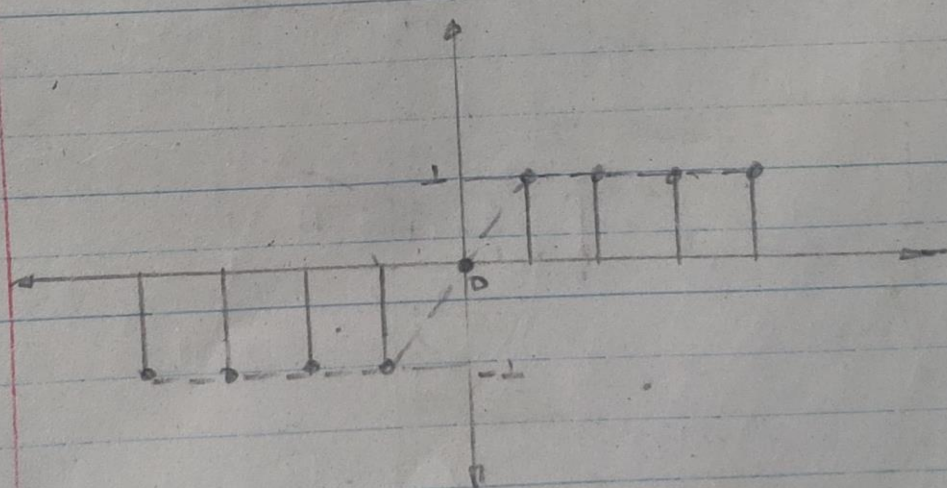

Od{x[n]}
2
1
0.5
0.75
-2
-1
0
1
2
n

Q.3.

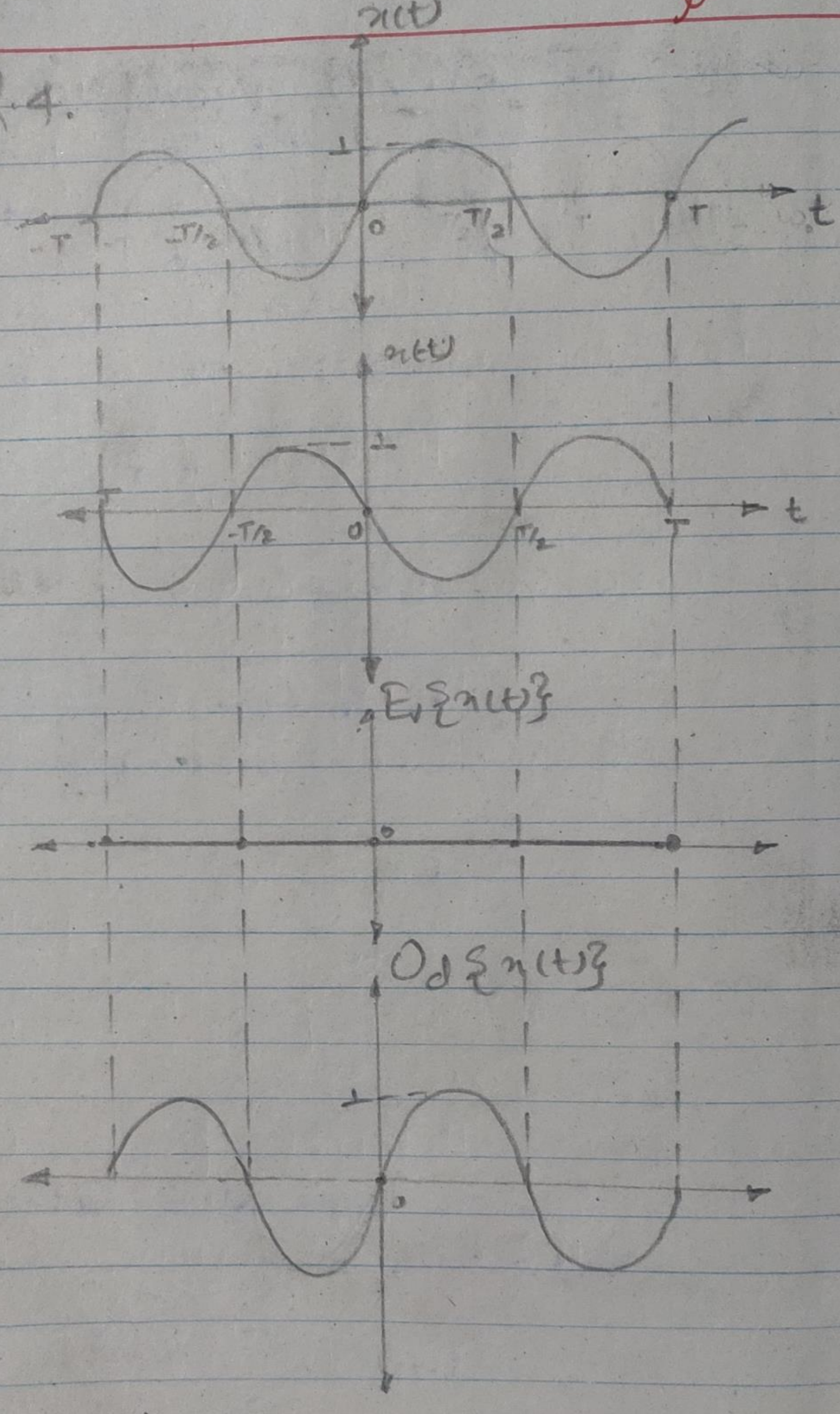








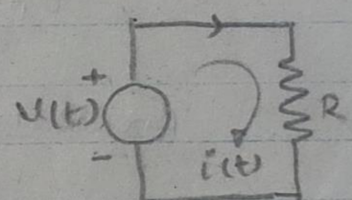
Q.4.



☆ Energy and power signal :-

- The signals, we consider can be directly related to physical quantities capturing power & energy in a physical system. For example, if $v(t)$ & $i(t)$ are, respectively, the voltage and current across a resistor 'R' then the instantaneous Power is $p(t) = v(t) \cdot i(t)$

$$\therefore p(t) = \frac{1}{R} v^2(t)$$



when, $R = 1\Omega$, then

$$p(t) = v^2(t)$$

Now,
The energy of the signal can be given by integrating the instantaneous power over infinite time.

i.e. $$E_\infty = \int_{-\infty}^{\infty} v^2(t)\, dt$$

For a signal $x(t)$, the energy can be given by

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Similarly;
For a discrete time signal $x[n]$ the energy can be given by

$$E_\infty = \sum_{-\infty}^{\infty} |x[n]|^2$$

The total energy expended over the time interval $t_1 \leq t \leq t_2$ is

$$E_T = \int_{t_1}^{t_2} p(t)\,dt = \int_{t_1}^{t_2} \frac{1}{R} v^2(t)\,dt$$

For $R = 1\,\Omega$ and continuous time signal $x(t)$

$$E_T = \int_{t_1}^{t_2} |x(t)|^2\,dt$$

For $R = 1\,\Omega$ and discrete time signal $x[n]$

$$E_N = \sum_{n=0}^{n-1} |x[n]|^2$$

Summary:-

$$(E_T)_\infty = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2\,dt$$

$$(E_T) = \int_{-T/2}^{T/2} |x(t)|^2\,dt$$

$$(E_N)_\infty = \lim_{N \to \infty} \sum_{n=0}^{N-1} |x[n]|^2$$

$$E_N = \sum_{n=0}^{N-1} |x[n]|^2$$

The average power of the signal can be given as:

$$P_{av} = \lim_{T \to \infty} \frac{E_\infty}{T}$$

i.e; $$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \quad — (i)$$

Similarly, for discrete time signal,

$$P_{av} = \lim_{N \to \infty} \frac{1}{N} \cdot \sum_{n=0}^{N-1} |x[n]|^2 \quad — (ii)$$

Energy signal :-

- A signal which has non zero & non infinite energy value is called energy signal.
  i.e; for energy signal, $0 < E < \infty$

Power signal :-

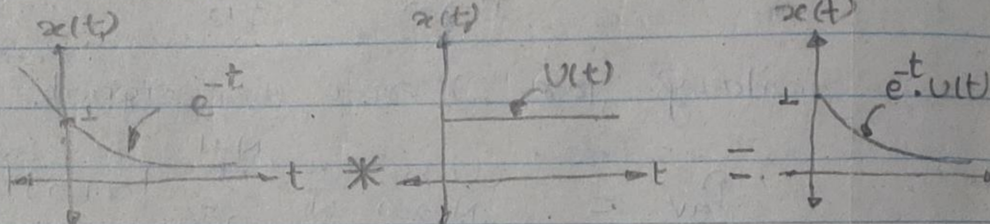
- A signal which has non zero & non infinite power value is called power signal.
  i.e; for power signal, $0 < P < \infty$

Energy and power signals are mutuall[y] exclusive. Generally periodic signals are powe[r] signal whereas aperiodic signals are energy signal.

Examples:-

plot the signals and find whether the signal is energy or power type.

Q.1 $x(t) = e^{-t} \cdot u(t)$



Solution:-

$x(t) = e^{-t} \cdot u(t)$

$\therefore |x(t)| = e^{-t}$

now;

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \lim_{T \to \infty} \int_{0}^{T/2} e^{-2t} dt$$

$$= \lim_{T \to \infty} \left(-\frac{e^{-2t}}{2} \right) \Bigg|_0^{T/2}$$

$$= \lim_{T \to \infty} \left(-\frac{e^{-T}}{2} + \frac{1}{2} \right)$$

$\therefore E = \frac{1}{2}$, which is finite

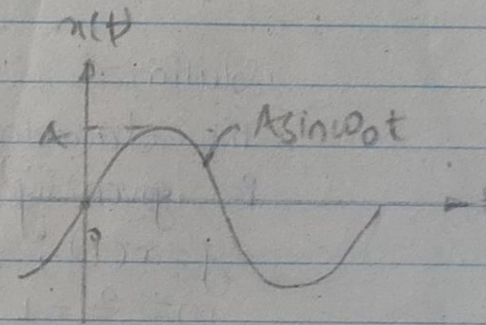
and $0 < \frac{1}{2} < \infty$

$\therefore$ $x(t)$ is an energy signal.

Also, $P = \frac{E_\infty}{T} = \lim_{T\to\infty} \frac{1}{2T} = 0$

$\therefore$ $x(t) = e^{-t} \cdot U(t)$ is not a power signal.

Q.2. $x(t) = A\sin\omega_0 t$



Since it is a periodic signal, it must be a power signal.

proof :-

we have;

$$P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} A^2\sin^2\omega_0 t\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\cdot\frac{A^2}{2}\int_{-T/2}^{T/2} (1-\cos 2\omega_0 t)\, dt$$

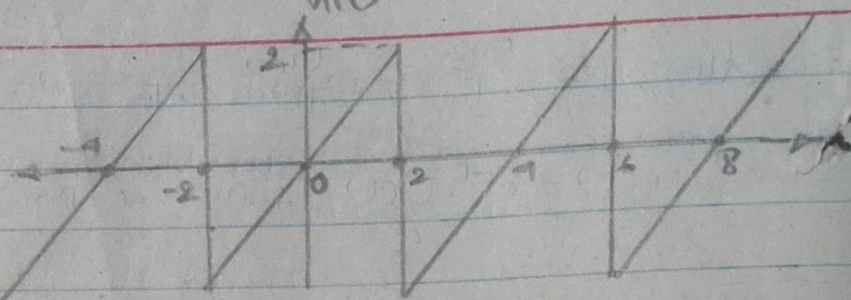
$$= \lim_{T\to\infty} \frac{A^2}{2}\cdot\frac{1}{T}\left(t - \frac{\sin 2\omega_0 t}{2\omega_0}\right)\Bigg|_{-T/2}^{T/2}$$

$$= \frac{A^2}{2}\cdot\lim_{T\to\infty}\frac{1}{T}\left(\frac{T}{2}+\frac{T}{2} - \frac{\sin\omega_0 t}{2\omega_0} - \frac{\sin\omega_0 T}{2\omega_0}\right)$$

$$= \frac{A^2}{2}\lim_{T\to\infty}\left(1 - \frac{\sin\omega_0 t}{\omega_0 T}\right)$$

$$= \frac{A^2}{2} - \lim_{T\to\infty}\frac{\sin\omega_0 T}{\omega_0 T} = \frac{A^2}{2} \quad \text{ie; } 0 < \frac{A^2}{2} < \infty$$

Q.3.



Solution:

The characteristics equation for given signal is given by, $y = mx + c$, where;

$y = x(t)$, $x = t$,

$m = \frac{2}{2} = 1$, $c = 0$

$\therefore x(t) = t$

Since, the signal is also periodic, it must be a power signal,

proof:

we have; $P = \lim_{T \to \infty} \frac{1}{T} \int_{\langle T \rangle} |x(t)|^2 dt$

where, $\langle T \rangle$: $t_1 = -2$ & $t_2 = 2$

$$\therefore P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \lim_{T \to \infty} \frac{1}{4} \int_{-2}^{2} t^2 . dt$$

$$= \lim_{T \to \infty} \frac{1}{4} . \frac{t^3}{3} \Big|_{-2}^{2}$$

$$= \frac{1}{4} . \left(\frac{8}{3} + \frac{8}{3} \right) = \frac{4}{3}$$

i.e. $0 < \frac{4}{3} < \infty$ $\therefore x(t) \Rightarrow$ power sig

Since; $P = \lim_{T\to\infty} \frac{E_\infty}{T}$

ie; $\frac{4}{3} = \lim_{T\to\infty} \frac{E_\infty}{4}$

ie: $\frac{4}{3} = \frac{E_\infty}{4}$

$\therefore$ $E_\infty = \frac{16}{3}$ for one period.

for infinite period ie: $n \to \infty$

$E_\infty = \frac{16}{3} \times \infty = \infty$

$\therefore$ it cannot be reffered as energy signal.

power of Energy signal $= 0$
Energy of power signal $= \infty$

✡ Discrete Sinusoidal Signal :-

- A discrete time sinusoidal signal can be defined as $x[n] = A\cos(\omega_0 n + \phi)$

$$= A\cos(2\pi f_0 n + \phi)$$

i.e; $x[n] = A\cos\left(\frac{2\pi}{N}.n + \phi\right)$ — ①

where; ω_0 = radian per sample

N = number of samples per second

example :-

$$x[n] = 2\cos\left(\frac{\pi}{6}.n\right)$$

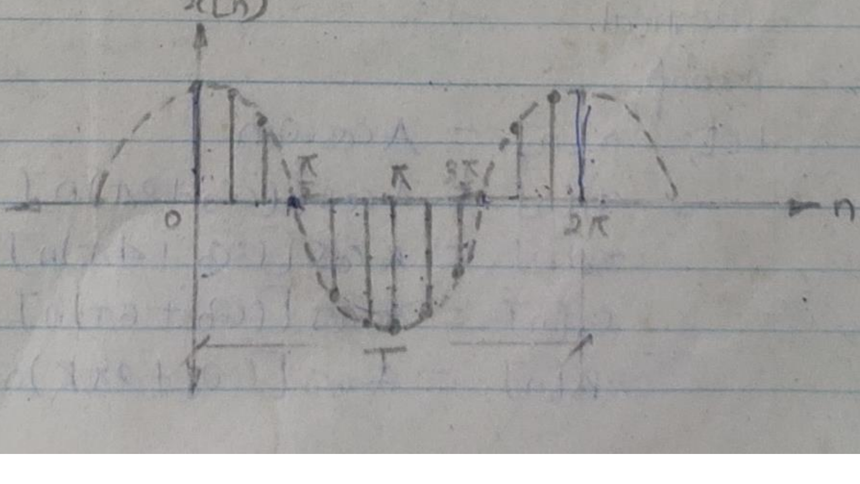
comparing this equation with ①, we get,

$\phi = 0$, $\omega_0 = \frac{\pi}{6}$ (rad/sample)

i.e; $2\pi f_0 = \frac{\pi}{6}$

$\therefore$ $f_0 = \frac{1}{12}$, i.e; $N = 12$ samples/sec

Now, plotting the given signal with calculated values, we get:



Properties of DT sinusoidial signal :-

1) DT sinusodial is periodic only if its frequency is a rational number.

proof :-

If $x[n]$ is periodic with N, then

$$x[n+N] = x[n]$$

now,

If $x[n] = A\cos\omega_0 n$, then

$$x[n+N] = A\cos\omega_0(n+N)$$

for, $A\cos\omega_0 n = A\cos\omega_0(n+N)$

$$\omega_0 N = 2\pi K$$

ie. $2\pi f_0 N = 2\pi K$

ie; $f_0 = \left(\frac{K}{N}\right)$

$\cos\theta = \cos\alpha$

$\theta = 2n\pi \pm \alpha$

$\omega_0 n = 2\pi K \pm \omega_0 n + \omega_0 N$

$\omega_0 N = 2\pi K$

2) DT sinusodial signal, whose frequenci are repeated by integer multiple of 2π are identical.

proof:

Let; $x[n] = A\cos\omega_0 n$

$$x_1[n] = A\cos[(\omega_0 + 2\pi)n]$$

$$x_2[n] = A\cos[(\omega_0 + 4\pi)n]$$

$$x_3[n] = A\cos[(\omega_0 + 6\pi)n]$$

$$x_K[n] = A\cos[(\omega_0 + 2\pi K)n]$$

$\therefore$ $x[n] = x_k[n]$

3) DT sequence exhibits highest oscillation rate when $\omega = \pi$ or $(-\pi)$

e.g. when:

(i) $\omega_0 = \frac{\pi}{8}$, $f_0 = \frac{1}{16}$ $\Rightarrow$ $N = 16$

(ii) $\omega_0 = \frac{\pi}{4}$, $f_0 = \frac{1}{8}$ $\Rightarrow$ $N = 8$

(iii) $\omega_0 = \frac{\pi}{2}$, $f_0 = \frac{1}{4}$ $\Rightarrow$ $N = 4$

(iv) $\omega_0 = \pi$, $f_0 = \frac{1}{2}$ $\Rightarrow$ $N = 2$

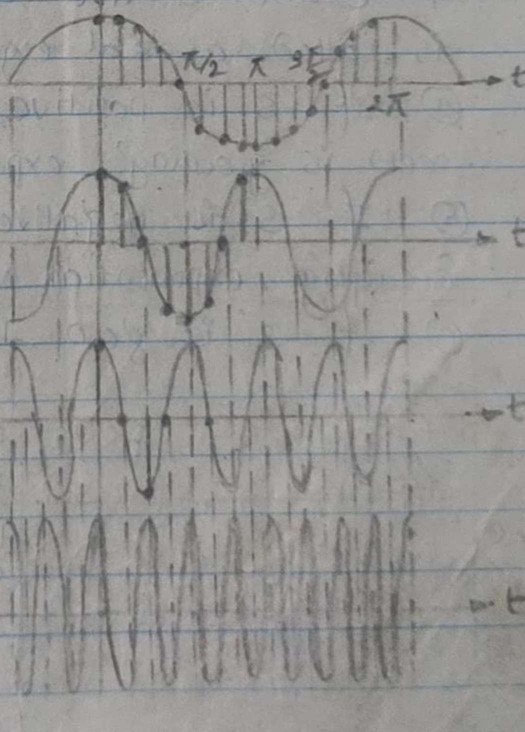
example:- $x[n] = \cos(\omega_0 n)$

(i) 16 samples/sec.

(ii) 8 samples/sec.

(iii) 4 samples/sec.

(iv) 2 samples/sec.



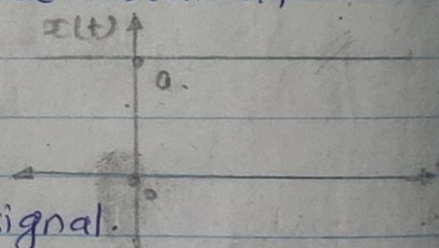
☆ Continuous Time Exponential Signal :-

- The continuous time exponential signal is of the form $x(t) = a e^{st}$, where, a and s are, in general, complex numbers. Depending upon the values of these parameters, the complex exponential can exhibit several different characteristics.

(A) Real Exponential Signal :-

- Since; $x(t) = a e^{st}$ —①

If a and s are real, then the signal $x(t)$ is said to be real exponential signal.

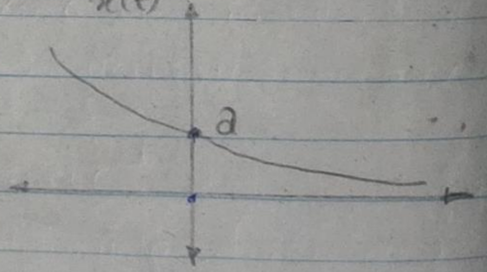


Case I :

If $s = 0$, then $x(t) = a$

ie, $x(t)$ is a constant signal.

Case II :-



If $s < 0$, then

at $t = 0$, $x(t) = a$

at $t = \infty$, $x(t) = 0$

at $t = -\infty$, $x(t) = \infty$

i.e, the signal in this case is known as decaying exponential signal.

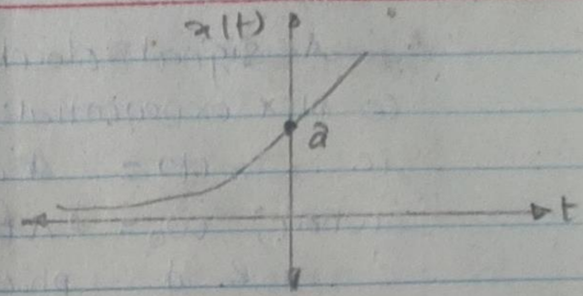
Case III

If $s>0$, then

at $t=0$, $x(t)=a$

at $t=-\infty$, $x(t)=0$

at $t=\infty$, $x(t)=\infty$



B. Periodic Complex Exponential Signal :-

– A periodic complex exponential signal is obtained by constraing 's' to be purely imaginary.

ie; $x(t) = e^{j\omega_0 t}$

Here, the property of the signal $x(t)$ is that it is periodic with period T, if

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$$

or; $e^{j\omega_0 (t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$

∴ For periodicity,

$$e^{j\omega_0 T} = 1 \quad -\circledast$$

If $\omega_0 = 0$, $x(t) = 1$, which is periodic for any value of T.

If $\omega_0 \neq 0$, then the fundamental period T_0 of $x(t)$ for which for which eq $*$ holds is

$T_0 = \dfrac{2\pi}{|\omega_0|}$, Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental Period.

A signal closely related to the periodic complex exponential is the sinusodial signal.

i.e. $x(t) = A\cos(\omega_0 t + \phi)$

where; $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

& ϕ = phase angle.

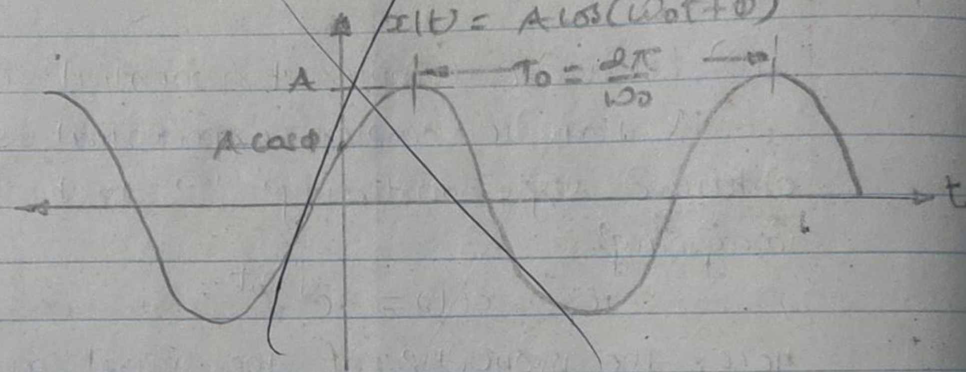


fig: continuous-time sinusodial signal.

✡ General Complex Exponential Signals:-

- The most general case of complex exponential can be expressed in terms of the real exponential and periodic complex exponential.

As we have; $x(t) = a e^{st}$

If a is expressed in polar form and 's' is in rectangular form, then;

$S = \sigma + j\omega$ and $a = |a| \angle \theta_a = |a| e^{j\theta_a}$

Now; the signal becomes;

$$x(t) = |a| e^{j\theta_a} \cdot e^{(\sigma + j\omega)t}$$
$$= |a| e^{j\theta_a} \cdot e^{\sigma t} \cdot e^{j\omega t}$$
$$= |a| e^{\sigma t} \cdot e^{j(\omega t + \theta_a)}$$

ie; $x(t) = |a| e^{\sigma t} \{\cos(\omega t + \theta_a) + j \sin(\omega t + \theta_a)\}$

$\therefore$ $Re[x(t)] = |a| e^{\sigma t} \cdot \cos(\omega t + \theta_a)$ and

$Im[x(t)] = |a| e^{\sigma t} \sin(\omega t + \theta_a)$

Case I :

when $\sigma = 0$, the real and imaginary parts of a complex exponential are sinusodial.

Case II :

when $\sigma > 0$, they corresponds to sinusodial signal multiplied by a growing exponential.

Case III :-

when $\sigma < 0$, they correspods to sinusodial signal multiplied by a decaying exponential.

The II and III cases are shown in fig. below where, the dashed lines correspond to the functions $\pm |a| e^{\sigma t}$ ie; magnitude of complex exponential.

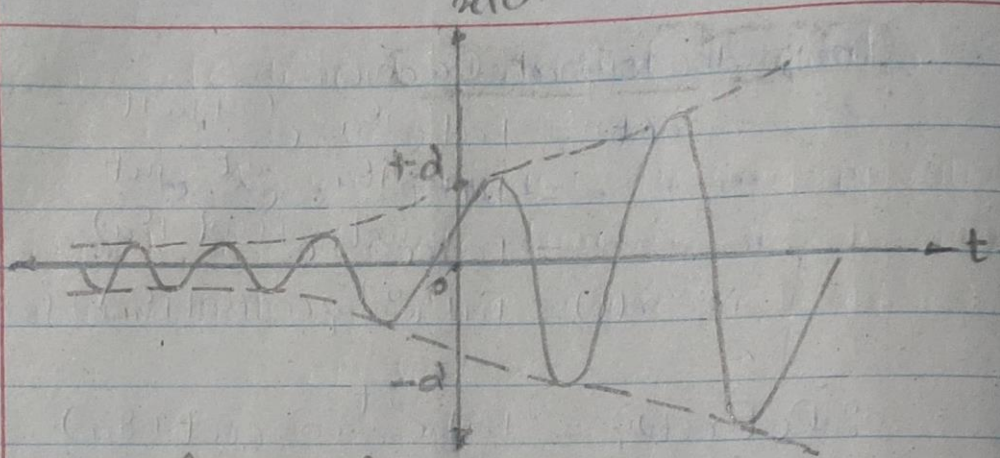


fig:- for case II
Growing Exponential Signal.

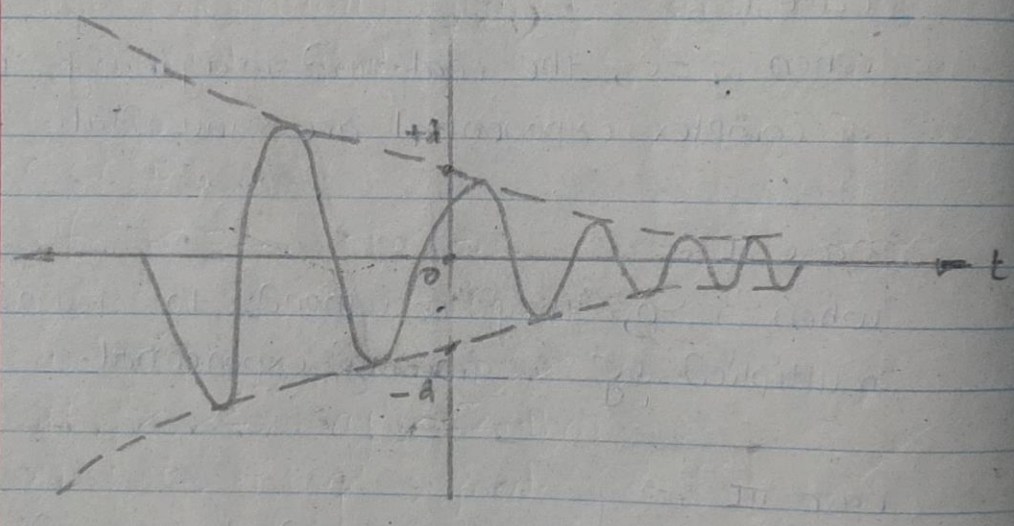


fig: for case III
Decaying Exponential Signal.

Analysis for case I

As we have;

$Re\{x(t)\} = |a| e^{\sigma t} \cdot \cos(\omega t + \theta_a)$ &

$Im\{x(t)\} = |a| e^{\sigma t} \cdot \sin(\omega t + \theta_a)$

For $\sigma = 0$

ie: $s = \sigma + j\omega_0 = j\omega_0$

Let; $\theta_a = 0$, then

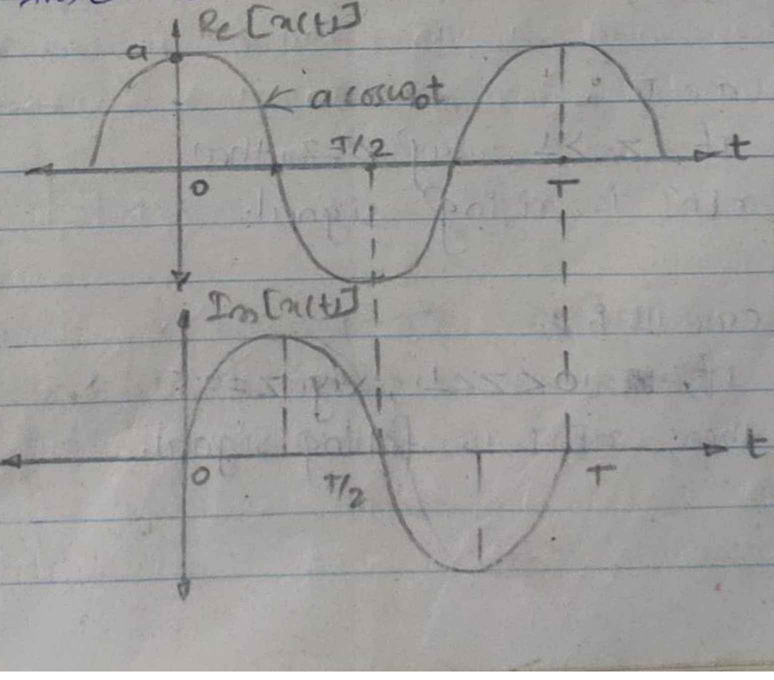
$x(t) = |a| e^{st} = |a| e^{j\omega_0 t}$

$= a e^{j\omega_0 t}$

ie: $x(t) = a[\cos\omega_0 t + j\sin\omega_0 t]$

$\therefore$ $Re[x(t)] = a\cos\omega_0 t$ and

$Im[x(t)] = a\sin\omega_0 t$



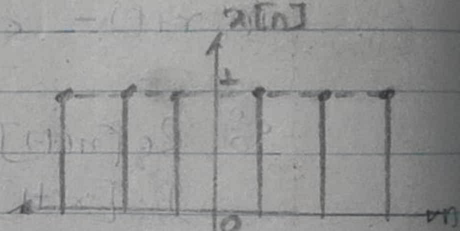
Discrete Time Exponential Signal :-

- As in continuous time, exponential signal, in discrete time is the complex exponential signal or sequence, defined by

$$x[n] = z^n$$

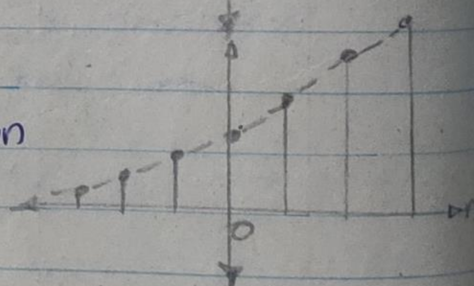
* Real Exponential Signal :-

- For all real value of z, a discrete time exponential signal is said to be real exponential signal.



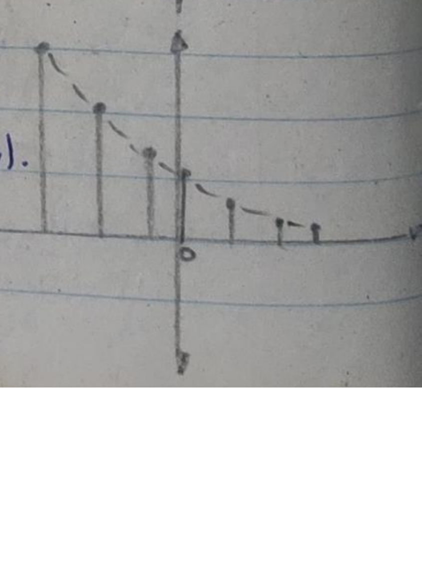
Case I :

If, $z = 1$, $x[n] = 1$



Case II :

If, $z > 1$, say $z = 2$, then $x[n]$ is rising signal.

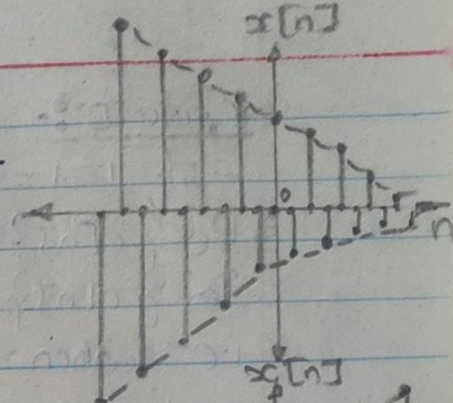


Case III :

If, $0 < z < 1$, say $z = 0.5$ then; $x[n]$ is falling signal.

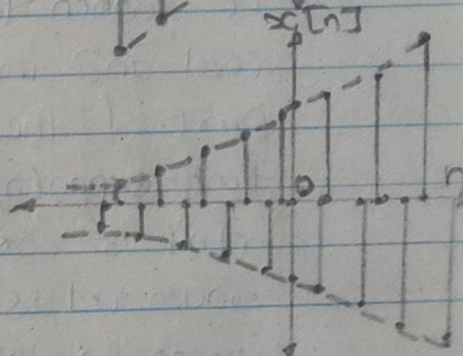
Case IV

If, $-1 < z < 0$, say $z = -0.5$
then, the signal $x[n]$ can be drawn as given in fig.



Case V

If $z < -1$, say $z = -2$
then the signal $x[n]$ can be characterized by the alongside figure.



* Complex Exponential Signal:-

- For purely complex value of z, discrete time exponential signal $x[n]$ is said to be complex exponential signal.

$$\text{i.e., } x[n] = z^n = [|z| e^{j\theta}]^n$$

$$= |z|^n \cdot e^{j\theta n}$$

$$\therefore x[n] = |z|^n [\cos\theta n + j\sin\theta n]$$

where;

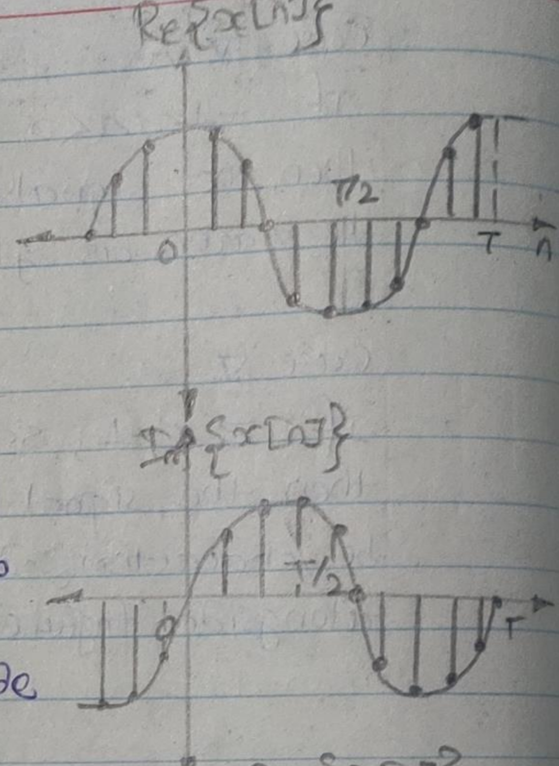
$$Re\{x[n]\} = |z|^n \cos\theta n.$$

$$Im\{x[n]\} = |z|^n \sin\theta n.$$

Case I :-

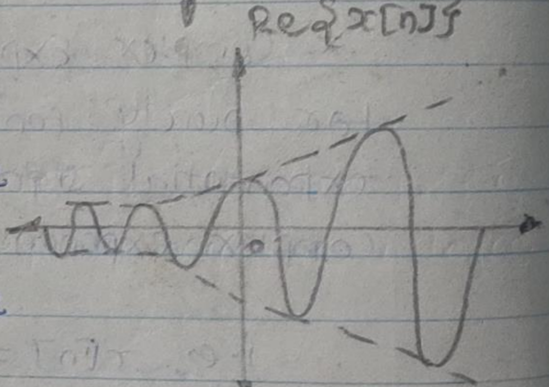
- If $|z| = 1$, then $Re\{x[n]\} = \cos\theta n$, & $Im\{x[n]\} = \sin\theta n$.

i.e., when $|z|=1$ the real and imaginary part of the discrete time exponential signal $x[n]$ is corresponds to sinusodial signal as shown in the alongside figure.



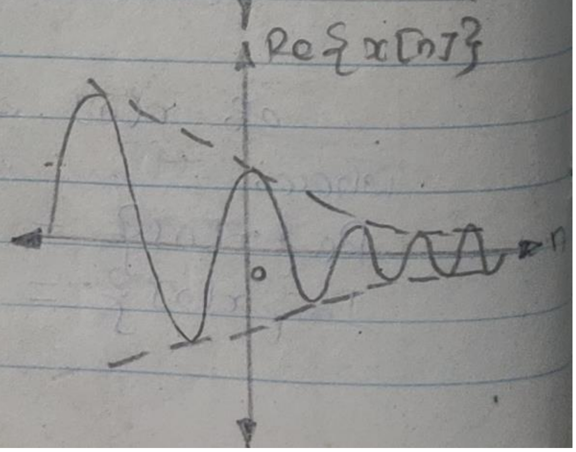
Case II :-

- If $|z| > 1$, then the signal $x[n]$ goes on, as given in the figure alongside.



Case III :-

- If $|z| < 1$, then the signal $x[n]$ will be as shown in the fig. alongside.



Imp

☆ Transformation of Independent Variable:-

- Transformation of independent - variable means the modification of independent variable i.e. continuous time (t) or discrete time [n]. This transformation

(I) allows us to introduce several basic properties of signals and system and their - impact on signal processing.

(II) allows us to define and characterised classes of system i.e. whatever they are time variant or not, linear OR non-linear, Causual or non causual, stable or non-stable etc.

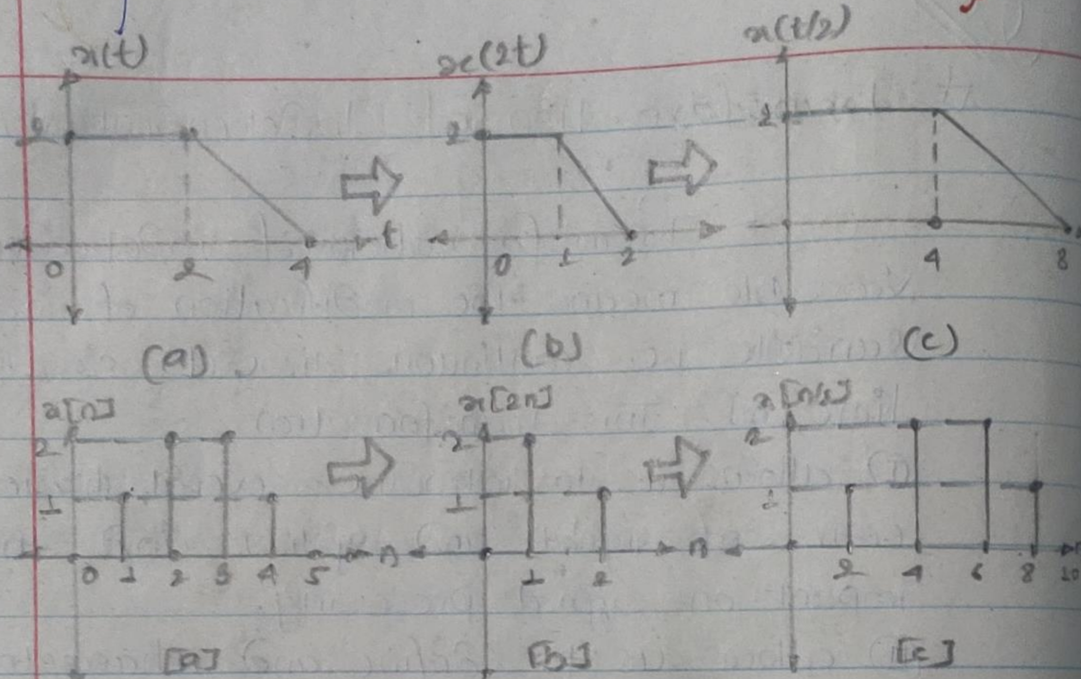
Basic types of transformations are:-

(I) Scaling
(II) Folding
(III) Shifting.

(1) Scaling:-

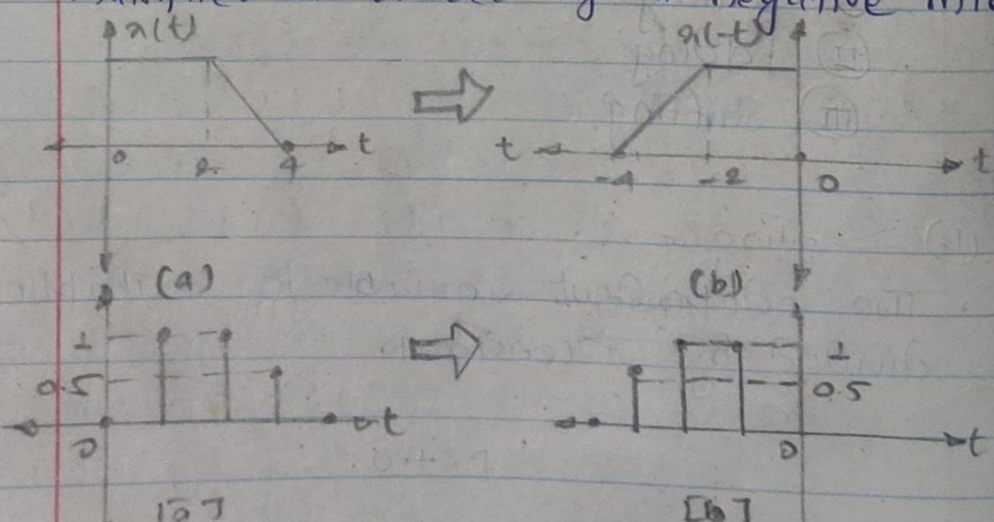
The independent variable is multiplied or divided by a constant.

P.T.O.



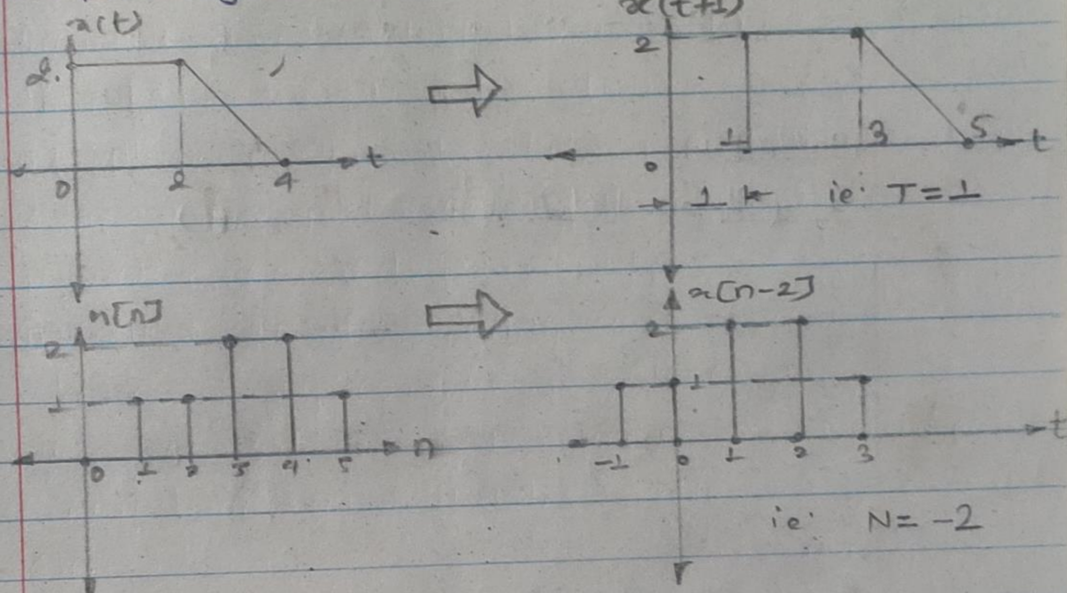
② Folding :-

The independent variable i.e. (t) or [n] is multiplied or divided by a negative integer.



(3) Shifting :-

- The independent variable i.e. either a continuous time (t) or discrete time [n] is shifted by a time period T or N respectively.

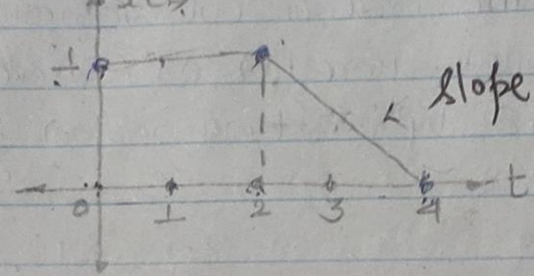


Note: ☆ Priority of Transformation :-

when, the case of two or more transformation techniques occurs at a time (e.g. folding, shifting, scaling), then for that case a priority is given for them to take an action one at a time; i.e; folding, shifting, & scaling is the order of priority.

Example :-

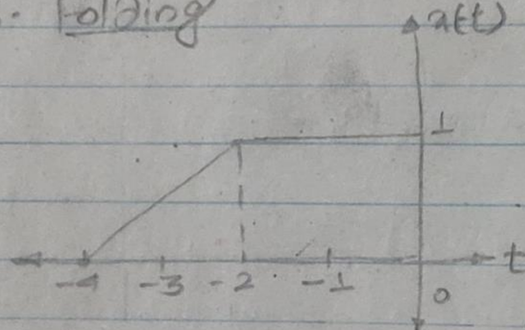
Q.1 Find $x(-t+1)$ of the following signal :-



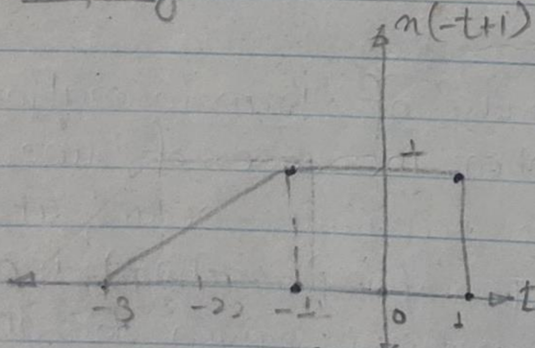
$\frac{0-1}{4-2}$ $-\frac{1}{2}t$

Solution:

Step 1 :- Folding



Step 2 :- Shifting



Note: Process:

$-t+1 = 0 \Rightarrow t = 1$

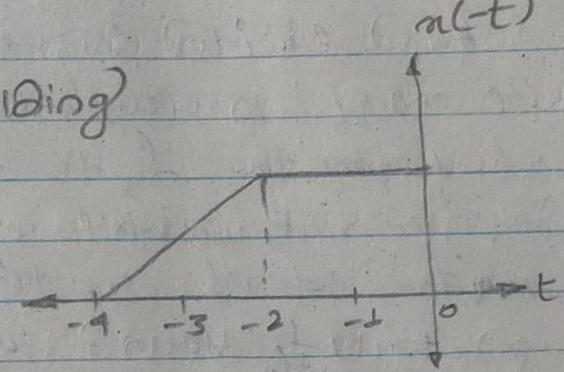
$-t+1 = 2 \Rightarrow t = -1$.

$-t+1 = 4 \Rightarrow t = -3$

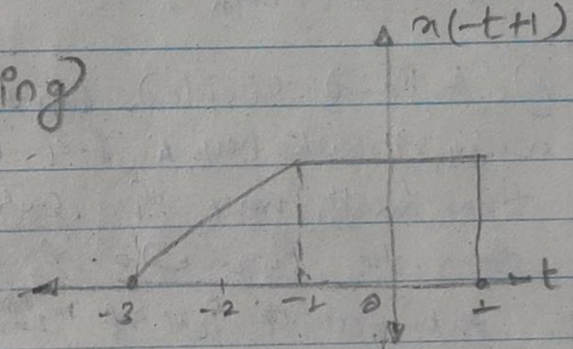
Q.2 Find $x(-2t+1)$ for the signal of Q.1

Solution:

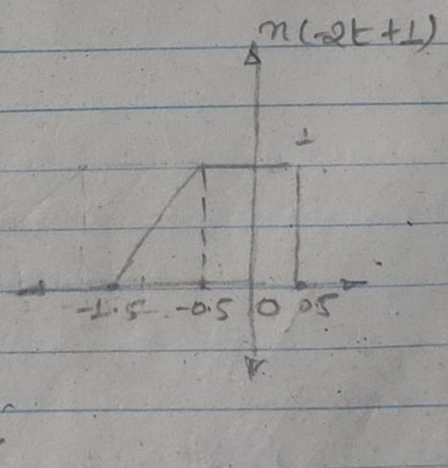
Step 1: Folding



Step 2: Shifting



Step 3: Scaling



Note: Process

t=0 $-2t+1=0 \Rightarrow t=0.5$

t=2 $-2t+1=2 \Rightarrow t=-0.5$

=4 $-2t+1=4 \Rightarrow t=-1.5$

Chapter: 4 :-

☆ Linear Time Invariant System :- (LTI)

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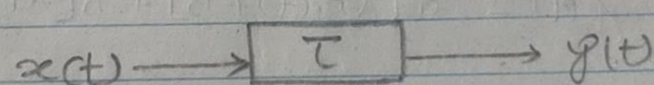
Linear System :-

- A system is said to be linear, either continuous or discrete if it holds the principle of superposition i.e. the response of a linear system to a weighted sum of inputs is equal to the same weighted sum of output signals. Each output signal being associated with a particular input acting on the system independently of all other input signals.

- Any system which violates the principle of superposition is said to be non linear system. e.g. for the following system, it is



linear, if

$$T[ax_1(t)] + T[bx_2(t)] = T[ax_1(t) + bx_2(t)]$$

Graphically this can be represented by

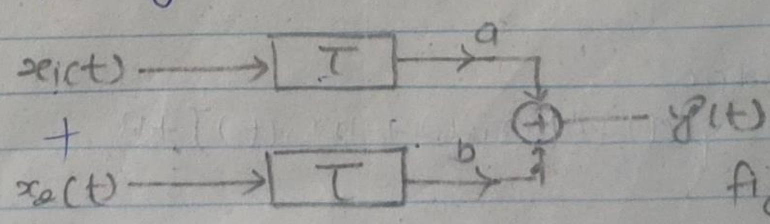


fig: ⓐ

OR

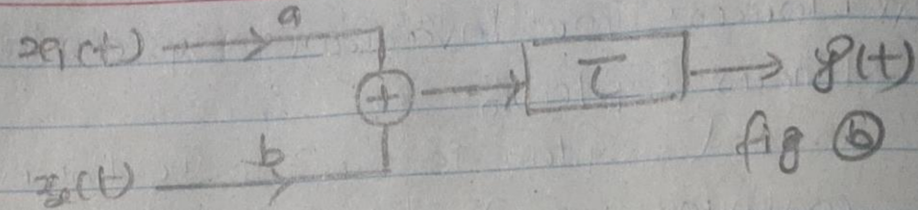


fig (b)

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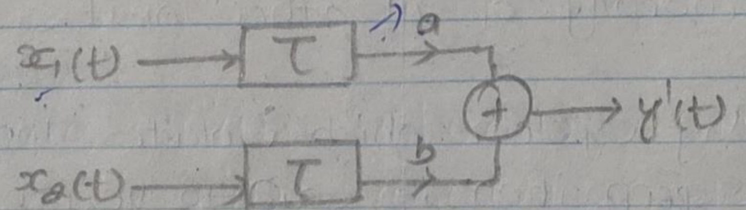
Example:-

Check whether the following system is linear or not.

(a) $y(t) = Ax(t) + B$

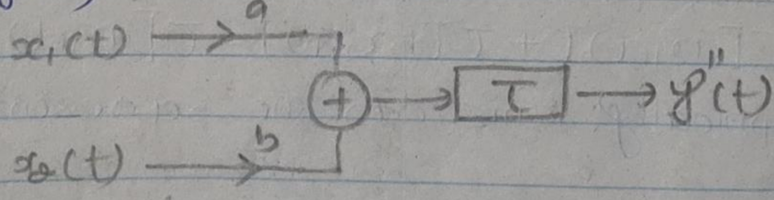
Solution:-



where;

$$y'(t) = a[Ax_1(t) + B] + b[Ax_2(t) + B]$$

Again;



where;

$$y''(t) = A[ax_1(t) + bx_2(t)] + B$$

Since; $y'(t) \neq y''(t)$

the given system is non-linear.

Note :-

$$ax_1(t) \longrightarrow \boxed{T} \longrightarrow ay_1(t)$$

$$bx_2(t) \longrightarrow \boxed{T} \longrightarrow by_2(t)$$

where; $\left.\begin{array}{l} ax_1(t) \rightarrow ay_1(t) \\ bx_2(t) \rightarrow by_2(t) \end{array}\right\}$ homogenity property.

$$ax_1(t) + bx_2(t) \longrightarrow \boxed{T} \longrightarrow ay_1(t) + by_2(t)$$

where;

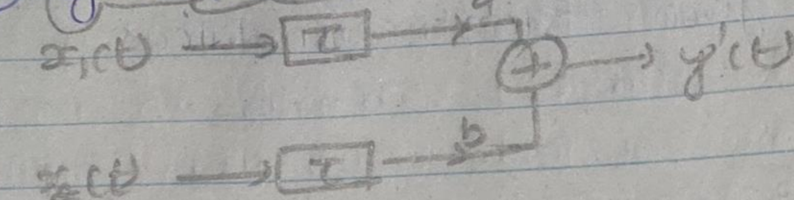
$ax_1(t) + bx_2(t) \longrightarrow ay_1(t) + by_2(t)$

ie; additive property.

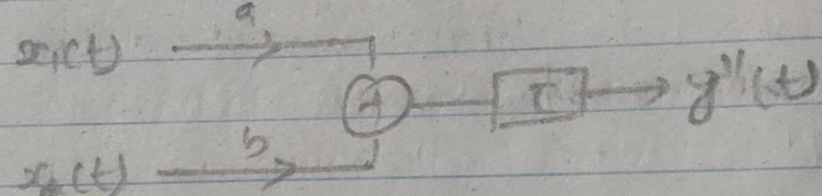
$\therefore$ Linearity = (Homogenity + Additive) Property.

$x(t)$

(b) $y(t) = x(-t)$



where; $y'(t) = a x_1(-t) + b x_2(-t)$



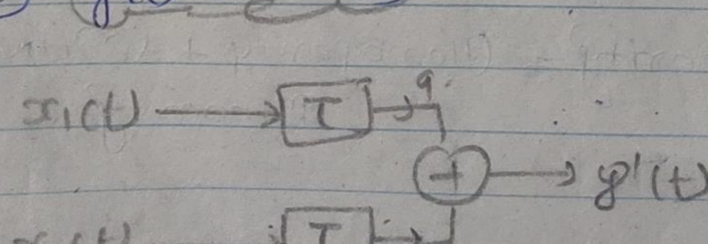
where; $y''(t) = T[a x_1(t) + b x_2(t)]$

ie; $y''(t) = a x_1(-t) + b x_2(-t)$

Since; $y'(t) = y''(t)$

so; $y(t) = x(-t)$ is a linear system.

(c) $y(t) = A x(t)$



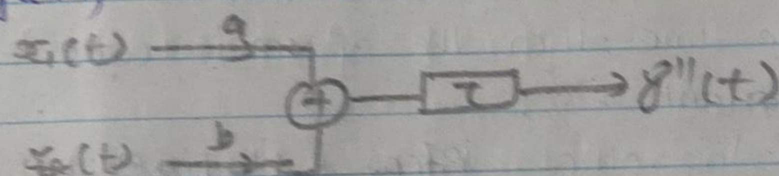
where;

$y'(t) = aT[x_1(t)] + bT[x_2(t)]$

$\quad = aAx_1(t) + bAx_2(t)$

$\therefore y'(t) = A[ax_1(t) + bx_2(t)]$

Again;



where;

$y''(t) = T[ax_1(t) + bx_2(t)]$

$\quad = A[ax_1(t) + bx_2(t)]$

since; $y'(t) = y''(t)$

so; $y(t) = Ax(t)$ is a linear system.

☆ Time Invariant & Time Variant System :-

- The property of system either continuous or discrete, is not altemed with time then the system is time invariant or shift invariant. In other words if the input & output characteristics doesn't change with the time, then the system is called time invariant.

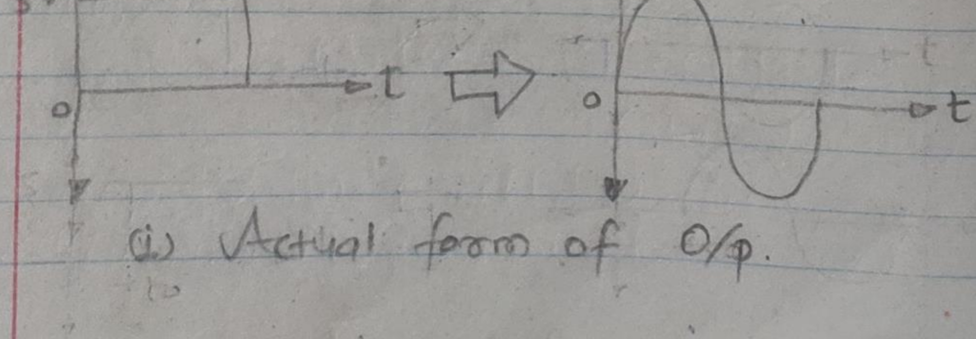
Any other system violating the above property is called time variant system.

If $x[n] \xrightarrow{T} y[n]$ is a time invariant system, then

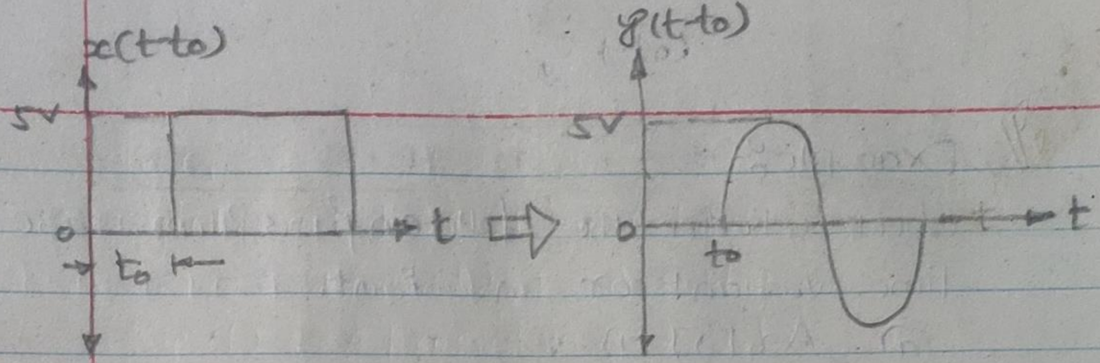
$x[n-k] \xrightarrow{T} y[n-k]$

For continuous time system :-

$x[t] \xrightarrow{T} y(t)$ is time invariant if $x(t-t_0) \xrightarrow{T} y(t-t_0)$.



(i) Actual form of O/p.



(ii) Actual shift to time domain.

ie; $x(t) \longrightarrow y(t)$

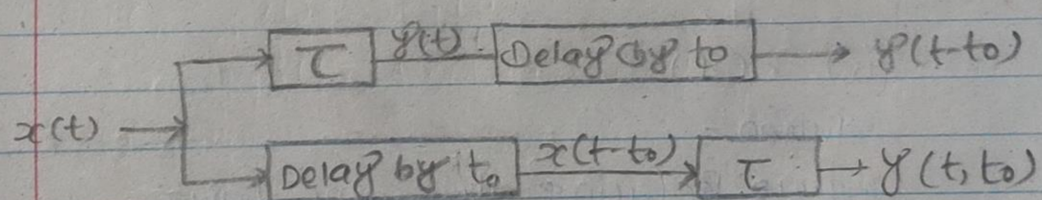
$x(t-t_0) \longrightarrow y(t-t_0)$

Similarly,

for discrete time signal, if

$x[n] \longrightarrow y[n]$

$x[n-N] \longrightarrow y[n-N]$ then the signal is said to be time invariant.



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so;

If the system is time invariant, then

$$y(t-t_0) = y(t, t_0)$$

3) Example:-

Check whether the following systems are time variant or invariant.

(a) $Ax[n]+B$

Solution:- Referring the above figure: ***

We have; $y[n] = Ax[n]+B$

$\therefore y[n-k] = Ax[n-k]+B \quad —(I)$

Again;

$y[n,k] = T[x[n-k]]$

$= Ax[n-k]+B \quad —(II)$

Since;

$y[n-k] = y[n,k]$

So, the function is time invariant.

(b) $x(-t)$

Solution:-

Again Referring the same figure:

As we have; $y(t) = x(-t)$

$\therefore y(t-t_0) = x(-t-t_0) \quad —(I)$

also,

$y(t,t_0) = T[x(t-t_0)]$

$= x(-t+t_0) \quad —(II)$

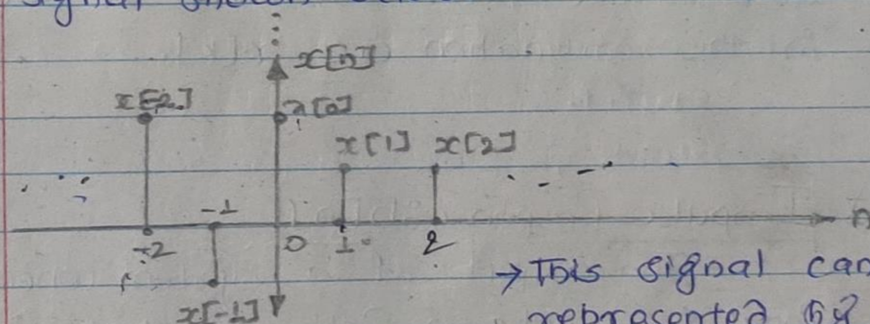
Since; $y(t-t_0) \neq y(t,t_0)$, the given function is time variant.

Linear Time Invariant System :-

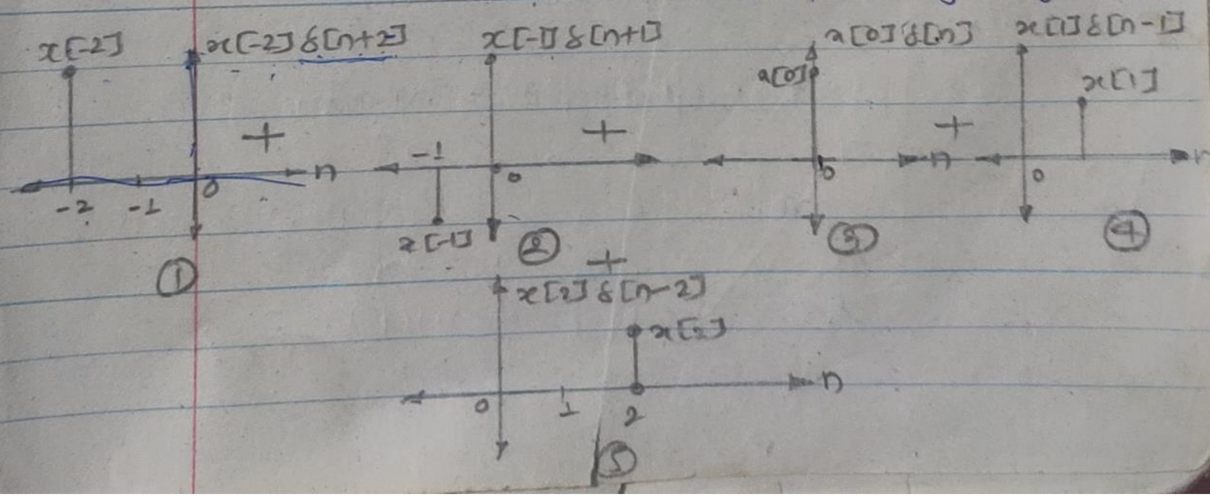
- A system, which is both linear & time invariant is called Linear time Invariant (LTI) system.

* Representation of Discrete Time Signals in terms of impulses :-

Let us consider a signal $x[n]$, which can be represented by any one type of signal shown below.



→ This signal can be represented by sum of following signals.



So, from figure, we have;

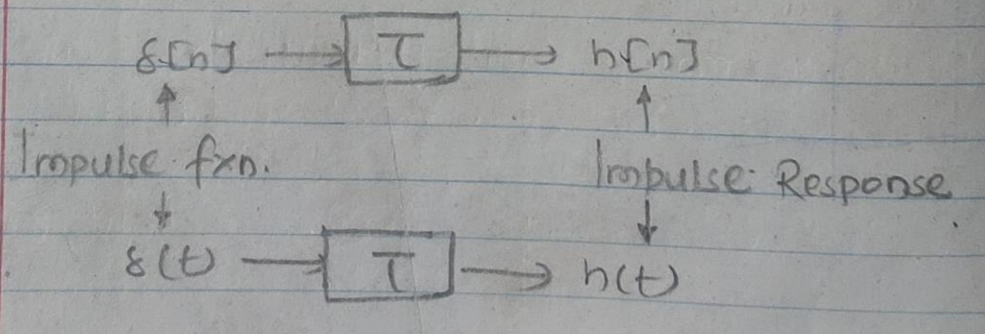
$$x[n] = \dots + \dots + x[-2]\,\delta[n+2] + x[-1]\,\delta[n+1] + x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[2]\,\delta[n-2] + \dots$$

$$\therefore \; x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Hence; any arbitrary sequence $x[n]$ can be represented by a combination of shifted unit impulses, $\delta[n-k]$, where the weights in this combination are $x[k]$.

✡ Impulse Response: $h[n]$

— The output of a system when the input is impulse or delta function is called impulse response of the system.



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☆ Convolution Sum Representation of LTI System.

- Let $h[n]$ be the response of a system to $\delta[n]$. So $h[n,k]$ or $h_k[n]$ is the response of the system to $\delta[n-k]$. Let, $x[n]$ be the input of the system and $y[n]$ be the output as shown in fig.

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

Then, the input $x[n]$ can be represented as;

$$x[n] = \sum_{k=-\infty}^{\infty} x[k].\delta[n-k] \quad —①$$

we have;

$$T\{\delta[n]\} = h[n], \text{ and}$$

$$T\{\delta[n-k]\} = h_k[n].$$

If the system is time invariant, then

$$T\{\delta[n-k]\} = h[n-k]. ✓$$

Now;

$$y[n] = T\{x[n]\}$$

$$= T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot T\{\delta[n-k]\}$$

$$\therefore \; y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

which is the convolution sum representation of Linear Time Invariant (LTI) system.

or, $y[n] = x[n] * h[n]$

where, $*$ - mathmatical operator - a representation of convolution sum of $x[n]$ and $h[n]$.

Equivalently;

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Imp// ☆ Example :-

Find the output of the system given by

$$x[n] = \{1, \underset{\uparrow}{1}, 1, 1\}$$

$$h[n] = \{6, 5, \underset{\uparrow}{4}, 3, 2, 1\}$$

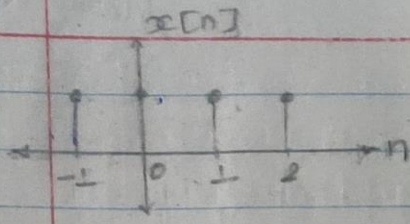


fig ①

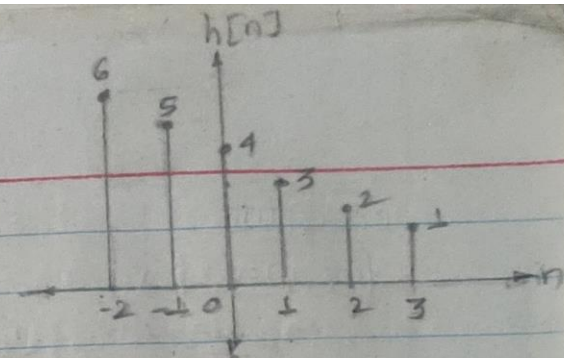


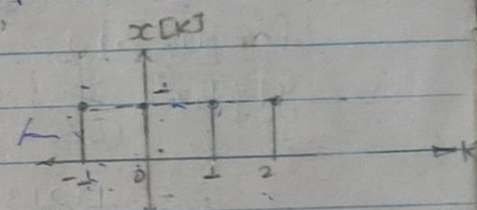
fig ⑪

Solution:

$$y[n] = x[n] * h[n] = ?$$

As we have;

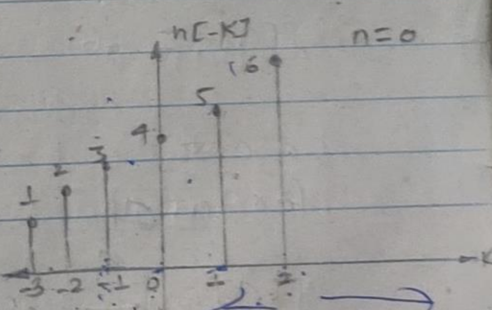
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\cdot h[n-k]$$



Case(I)

For $\boxed{n=0}$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]\,h[-k]$$



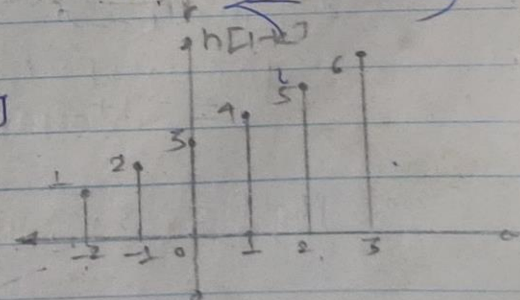
$$= 3+4+5+6$$

$$\therefore y[0] = 18$$

Case (II)

For $\boxed{n=1}$

$$y[1] = \sum_{k=-\infty}^{\infty} x[k]\,h[1-k]$$



$$= 2+3+4+5$$

$$\therefore y[1] = 14$$

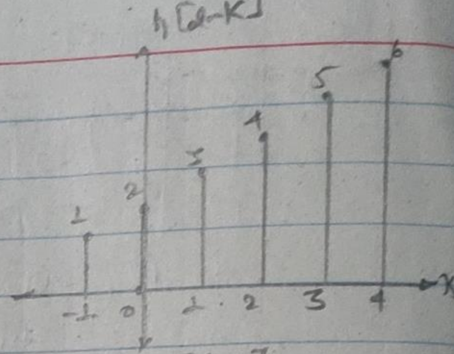
Case III

for $\boxed{n=2}$

$$y[2] = \sum_{k=-\infty}^{\infty} x[k]\,h[2-k]$$

$$= 1+2+3+4$$

$\therefore$ $y[2] = 10$.



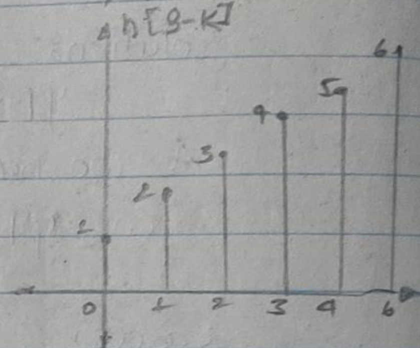
Case IV

for $\boxed{n=3}$

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]\,h[3-k]$$

$$= 1+2+3$$

$\therefore$ $y[3] = 6$



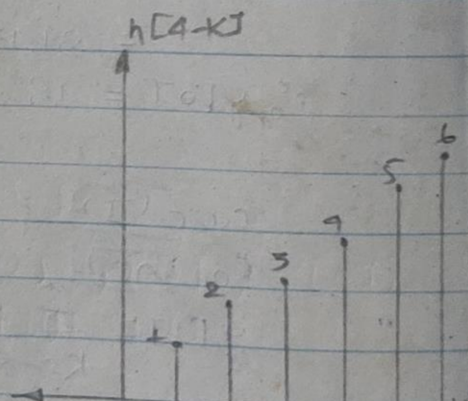
Case V

for $\boxed{n=4}$

$$y[4] = \sum_{k=-\infty}^{\infty} x[k]\,h[4-k]$$

$$= 1+2$$

$\therefore$ $y[4] = 3$.



Case VI

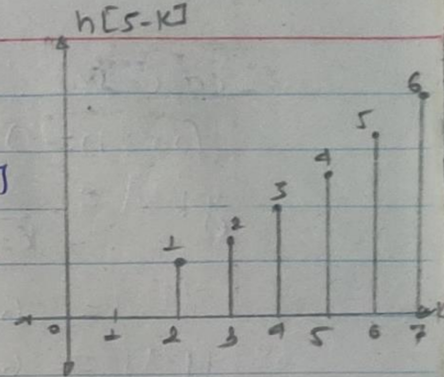
for $\boxed{n = 5}$

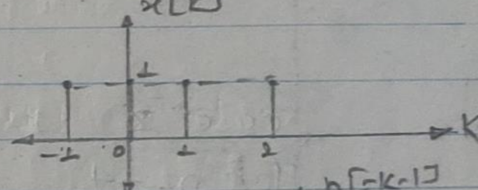
$$y[5] = \sum_{K=-\infty}^{\infty} x[K]\,h[5-K]$$

$$= 1$$

similarly, for $\boxed{n > 5}$

$$y[n] = 0$$





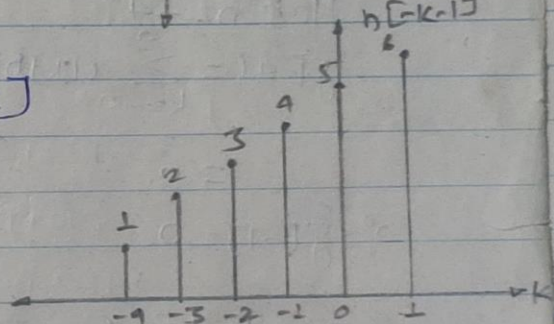
Case VII

for $\boxed{n = -1}$

$$y[-1] = \sum_{K=-\infty}^{\infty} x[K]\,h[-1-K]$$

$$= 4+5+6$$

$\therefore\ y[-1] = 15$



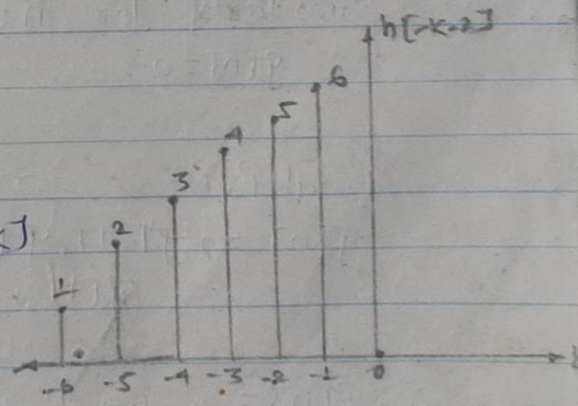
Case VIII

for $\boxed{n = -2}$

$$y[-2] = \sum_{K=-\infty}^{\infty} x[K]\,h[-2-K]$$

$$= 5+6$$

$\therefore\ y[-2] = 11$

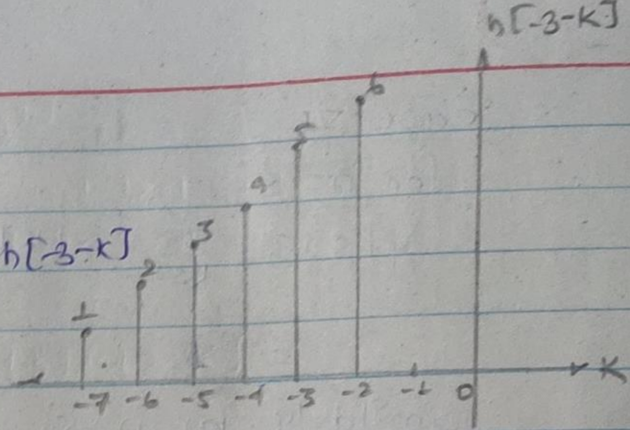


Case IX

for $n=-3$

$$y[-3] = \sum_{k=-\infty}^{\infty} x[k]h[-3-k]$$

$$= 6$$



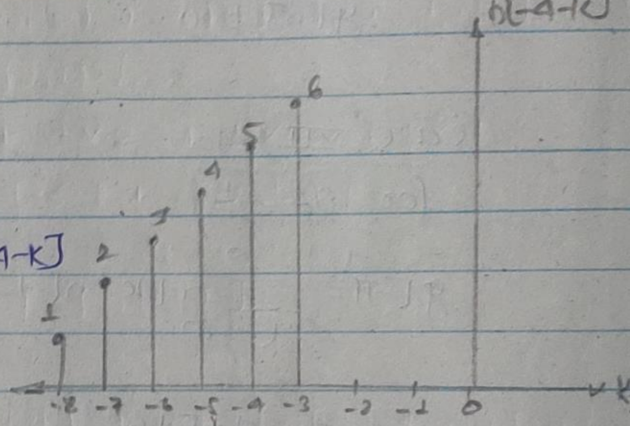
$\therefore y[-3] = 6$

Case X

for $n=-4$

$$y[-4] = \sum_{k=-\infty}^{\infty} x[k]h[-4-k]$$

$$= 0$$



$\therefore y[-4] = 0$

Similarly for $\boxed{n<-4}$

$y[n] = 0$

Finally,

$y[n] = \{y[-4], y[-3], y[-2], y[-1], y[0], y[1], y[2], y[3], y[4]\}$

$\therefore y[n] = \{6, 11, 15, \underset{\uparrow}{18}, 14, 10, 6, 3, 1\}$

Alternative Method :-

As we have,

(a) $x[n] = \{1, \underset{\uparrow}{1}, 1, 1\}$

i.e. $x[n] = \{x[-1], x[0], x[1], x[2]\}$

(b) $h[n] = \{6, 5, \underset{\uparrow}{4}, 3, 2, 1\}$

i.e. $h[n] = \{h[-2], h[-1], h[0], h[1], h[2], h[3]\}$

$$\therefore y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$

$$y[0] = \sum_{K=-1}^{2} x[K]\, h[-K]$$

$$= x[-1]h[1] + x[0]h[0] + x[1]h[-1] + x[2]h[-2]$$

$$= 1\times3 + 1\times4 + 1\times5 + 1\times6$$

$$\therefore y[0] = 18$$

$$y[1] = \sum_{K=-1}^{2} x[K]\, h[1-K]$$

$$= x[-1]h[2] + x[0]h[1] + x[1]h[0] +$$

$x[2]\,h[-1]$

$= 1\times 2 + 1\times 3 + 1\times 4 + 1\times 5$

$\therefore y[1] = 2+3+4+5 = 14$

$$y[2] = \sum_{k=-1}^{2} x[k]\,h[2-k]$$

$= x[-1]\,h[3] + x[0]\,h[2] + x[1]\,h[1] + x[2]\,h[0]$

$= 1\times 1 + 1\times 2 + 1\times 3 + 1\times 4$

$\therefore y[2] = 1+2+3+4 = 10.$

$$y[3] = \sum_{k=-1}^{2} x[k]\,h[3-k]$$

$= x[-1]\,h[4] + x[0]\,h[3] + x[1]\,h[2] + x[2]\,h[1]$

$= 1\times 0 + 1\times 1 + 1\times 2 + 1\times 3$

$\therefore y[3] = 1+2+3 = 6$

$$y[4] = \sum_{k=-1}^{2} x[k]\,h[4-k]$$

$= x[-1]\,h[5] + x[0]\,h[4] + x[1]\,h[3] + x[2]\,h[2]$

$= 0+0+1\times 1 + 1\times 2$

$\therefore y[4] = 1+2 = 3.$

$$y[5] = \sum_{k=-1}^{2} x[k]\,h[5-k]$$

$$= x[-1]h[6] + x[0]h[5] + x[1]h[4] + x[2]h[3]$$

$$= 1\times 0 + 1\times 0 + 1\times 0 + 1\times 1$$

$\therefore y[5] = 1$

Now, for $\boxed{n>5}$ $y[n] = 0$

Similarly,

$$y[-1] = \sum_{k=-1}^{2} x[k]\,h[-1-k]$$

$$= x[-1]h[0] + x[0]h[-1] + x[1]h[-2] + x[2]h[-3]$$

$$= 1\times 4 + 1\times 5 + 1\times 6 + 1\times 0$$

$\therefore y[-1] = 4+5+6 = 15$

$$y[-2] = \sum_{k=-1}^{2} x[k]\,h[-2-k]$$

$$= x[-1]h[-1] + x[0]h[-2] + x[1]h[-3] + x[2]h[-4]$$

$$= 1\times 5 + 1\times 6 + 1\times 0 + 1\times 0$$

$\therefore y[-2] = 5+6 = 11.$

$$y[-3] = \sum_{k=-1}^{2} x[k]\, h[-3-k]$$

$$= x[-1]h[-2] + x[0]h[-3] + x[1]h[-4] + x[2]h[-5]$$

$$= 1\times 6 + 1\times 0 + 1\times 0 + 1\times 0$$

$\therefore$ $y[-3] = 6$

hence,

for $\boxed{n<-3;}$ $y[n]=0$

$\therefore$ $y[n] = \{ y[-3], y[-2], y[-1], y[0], y[1], y[2], y[3], y[4], y[5] \}$

ie $y[n] = \{ 6, 11, 15, 18, 14, 10, 6, 3, 1 \}$ //

Note:-

If $x[n]$ has N samples and $h[n]$ has M samples then the number of samples in $y[n]$ is $M+N-1$

P.T.O.

Alternative Method :-

- Add and overlap Method :-

x[n] \ h[n]	6	5	4	3	2	1
1	6	5	4	3	2	1
1	6	5	4	3	2	1
1	6	5	4	3	2	1
1	6	5	4	3	2	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0

$\therefore y[n] = \{6, 11, 15, 18, 14, 10, 6, 3, 1\}$

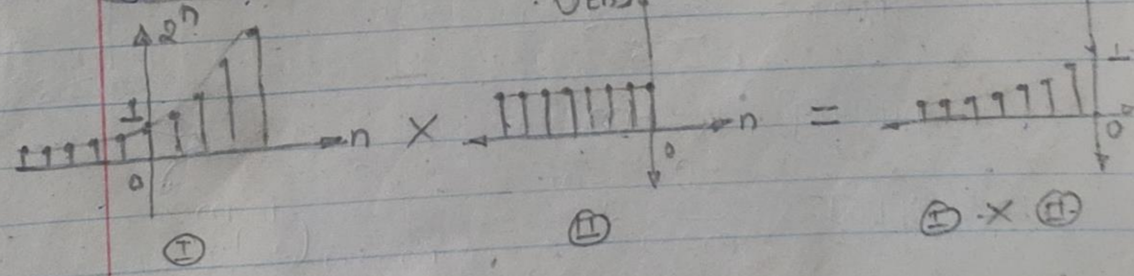
Example :- 1

Find the convolution of following signal:

$x[n] = 2^n\, u[-n]$ — (I)

$h[n] = u[n]$ — (II)

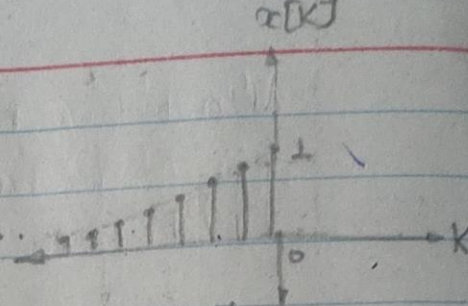
Solution :-

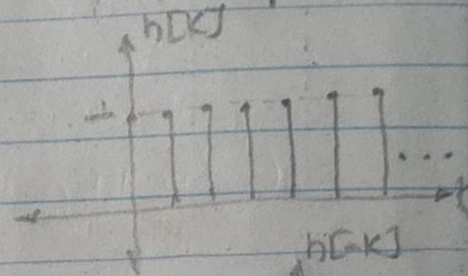


$h(1-k)$

As, we have;

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

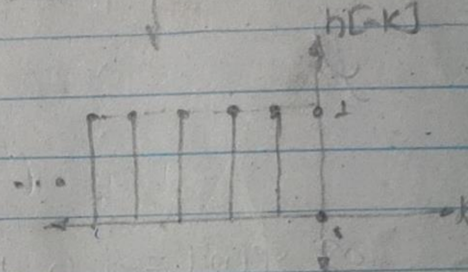




* case I :- $\boxed{\text{for } n \geq 0}$

$$y[n] = \sum_{k=-\infty}^{0} 2^k \cdot 1$$

$$= \sum_{k=-\infty}^{0} 2^k$$

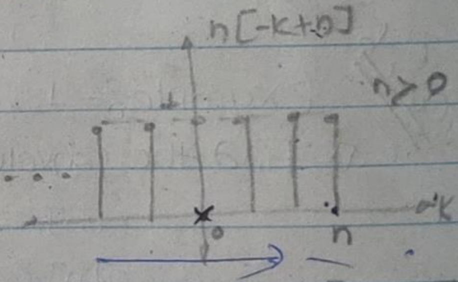


putting $k = -m$

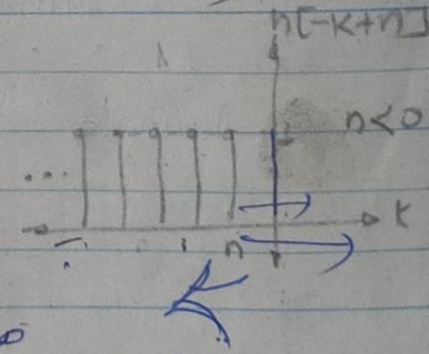
$$y[n] = \sum_{m=\infty}^{0} 2^{-m}$$

$$= \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m$$

$$\therefore y[n] = \frac{1}{1-\frac{1}{2}} = 2$$



* case II : $\boxed{\text{for } n < 0}$

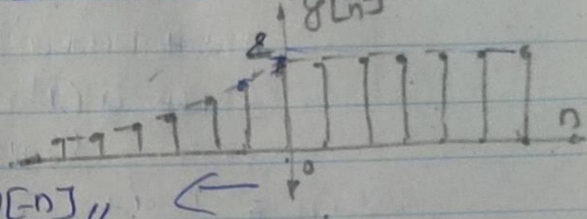


$$y[n] = \sum_{k=-\infty}^{n} 2^k \cdot 1$$

putting, $k = -m$

$$\therefore y[n] = \sum_{m=\infty}^{-n} 2^{-m} = \sum_{m=-n}^{\infty} \left(\frac{1}{2}\right)^m$$

$\therefore y[n] = \dfrac{\left(\frac{1}{2}\right)^{-n}}{\left(1-\frac{1}{2}\right)} = 2\cdot 2^{n} = 2^{n+1}$



Finally;

$y[n] = 2\,U[n] + 2^{n+1}\,U[-n]$ //

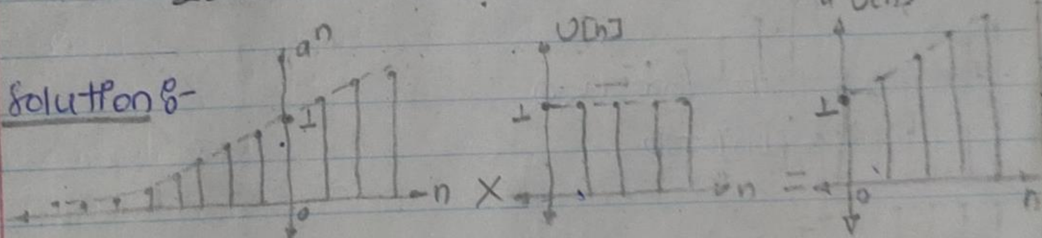
Example :- 2

Find the convolution of following two signals.

$x_1[n] = a^n\,U[n]$ — ①

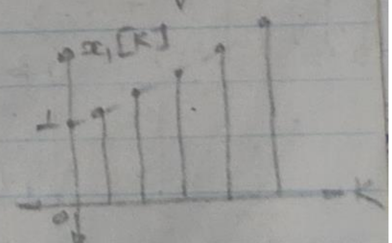
$x_2[n] = U[n]$ — ②

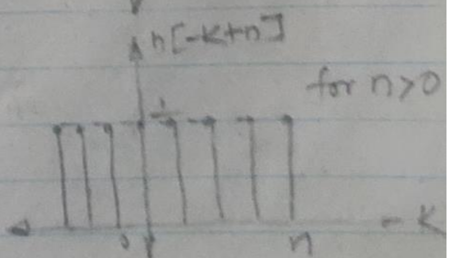
Solution :-



As we have;

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\,h[n-K]$$





Case I : for $n>0$

$$y[n] = \sum_{K=0}^{n} a^{K}\cdot 1.$$

i.e. $y[n] = 1 + a + a^2 + a^3 + \cdots + a^{n-1} + a^n$

$$= 1 + (a + a^2 + a^3 + \cdots + a^{n-1} + a^n)$$

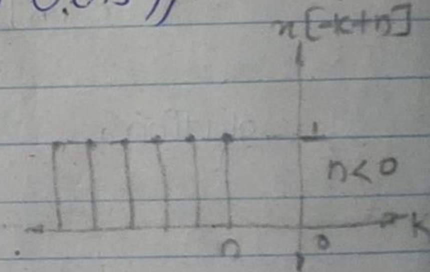
$$= 1 + \frac{a(1-a^n)}{(1-a)}$$

$$= \frac{1-a+a-a^{n+1}}{1-a}$$

$\therefore$ $y[n] = \dfrac{1-a^{n+1}}{1-a}$ // $= \left(\dfrac{1}{1-a}\right)(1-a^{n+1})$

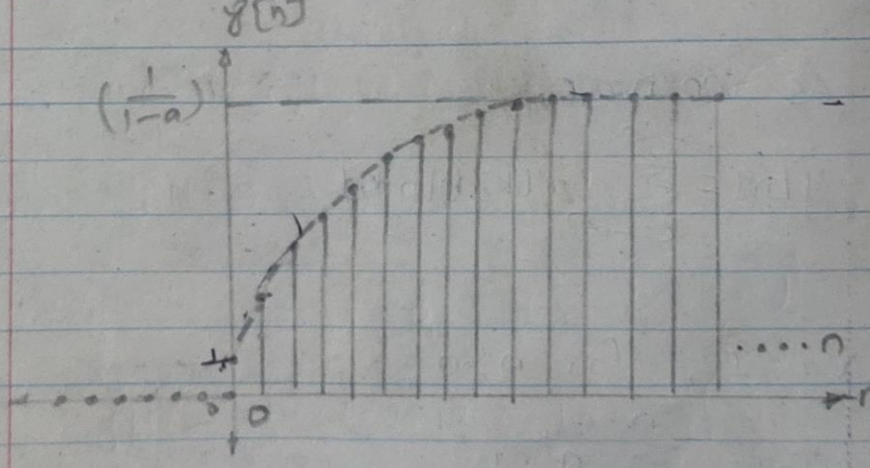
i.e., for all n

$$y[n] = \left(\frac{1}{1-a}\right)(1-a^{n+1})\,u[n] \text{ //}$$



But for $n<0$

$y[n] = 0$



☆ Representation of continuous time signals in terms of impulses:

- If a continuous time signal is denoted by $x(t)$, then;

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

ie; any continuous time signal $x(t)$ can be represented by an integration of shifted unit impulses, $\delta(t-\tau)$. In this case $x(\tau)$ is the weight of the integration whereas τ is an orbitary time period.

imp

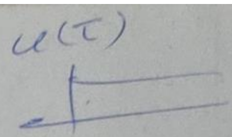
☆ Convolution Integral representation of LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

where; $y(t)$ is the convolution integral of the signals $x(t)$ and $h(t)$.

i.e; $y(t) = x(t) * h(t)$

$$\therefore\ y(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau$$



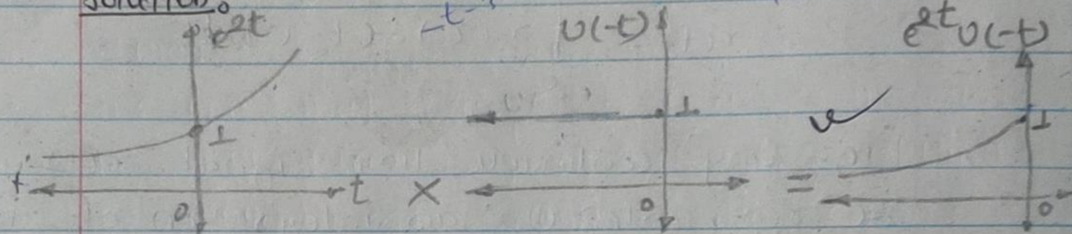
☆ Example :- 1

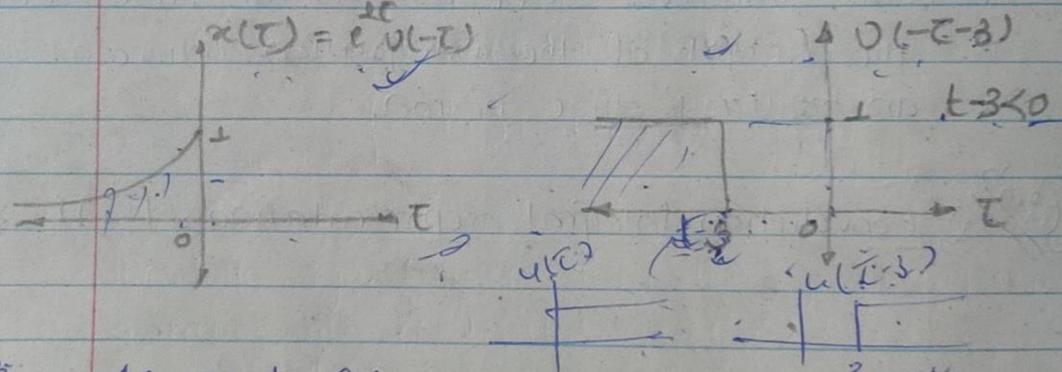
Find the output of the system given by;

$$x(t) = e^{2t}\,u(-t)$$

$$h(t) = u(t-3)$$

Solution :-





As we have;

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

Case I : for $\boxed{t<3}$ $\boxed{(\because\ t-3<0)}$

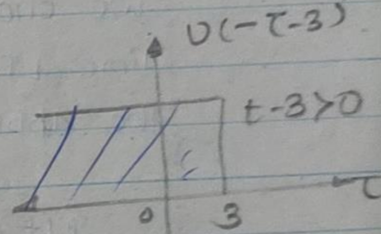
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{t-3} e^{2\tau}\cdot 1\,d\tau$$

ie $y(t) = \left. \frac{e^{2\tau}}{2} \right|_{-\infty}^{t-3} = \frac{1}{2} e^{2(t-3)}$ for $t-3<0$

ie $y(t) = \frac{e^{2t-6}}{2} \cdot u(-t+3)$ //

$\boxed{\text{Case II :}}$ for $\boxed{t-3>0}$ ie $\boxed{t>3}$

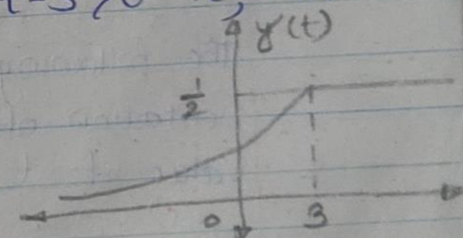
$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$



$$= \int_{-\infty}^{0} e^{2\tau} \cdot 1\, d\tau$$

$$= \left. \frac{e^{2\tau}}{2} \right|_{-\infty}^{0}$$

$\therefore$ $y(t) = \frac{1}{2}$ for $t-3>0$ ie, $t>3$.



ie $y(t) = \frac{1}{2} u(t-3)$ //

Finally; $y(t) = \frac{e^{2t-6}}{2} u(-t+3) + \frac{1}{2} u(t-3)$ //.

p.t.o.

Example :- 2

Find the convolution of following signals :-

$x(t) = e^{-at} u(t)$, for $a > 0$

$h(t) = u(t)$

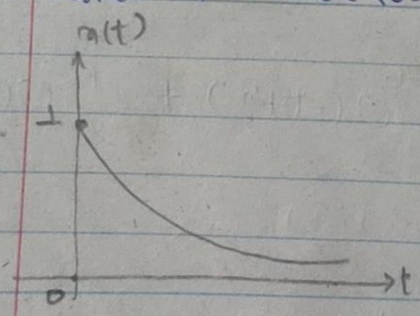
Solution :-

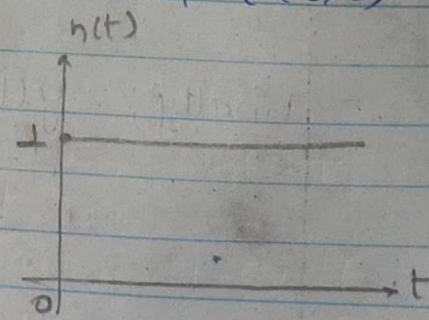
We know that o/p, $y(t)$ of a continuous time LTI system is given by the convolution of the applied i/p & unit impulse response as

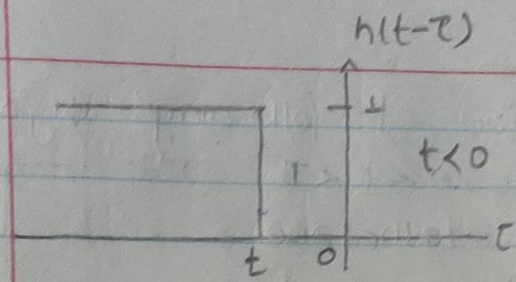
$$y(t) = x(t) * h(t)$$

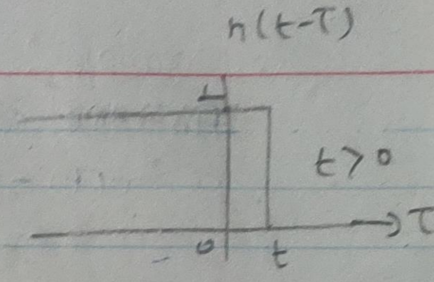
$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \quad -①$$

The following figures show the graphical representation of $x(t)$, $h(t)$ & $h(t-\tau)$ for a -ve value of t $(t<0)$ & +ve value of t $(t>0)$









It is observed that; for $t<0$, $x(t)\cdot h(t-\tau)$ is zero. ∴ From eq. (1)

$$y(t) = 0 \quad \text{for } t<0$$

Also, for $t>0$,

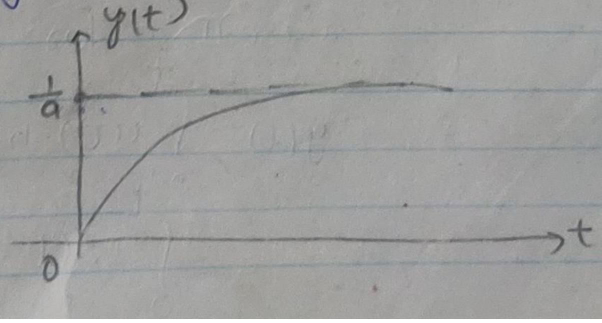
$$x(t)h(t-\tau) = e^{-a\tau}, \quad \text{for } 0<\tau<t$$
$$= 0, \quad \text{elsewhere}$$

$$\therefore y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t e^{-a\tau}d\tau$$

$$= -\frac{1}{a}\left[e^{-a\tau}\right]_0^t$$

$$= \frac{1}{a}(1-e^{-at})$$

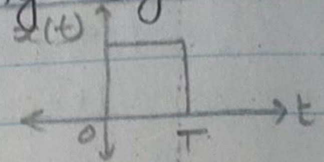
$$\therefore y(t) = \frac{1}{a}(1-e^{-at})u(t)$$



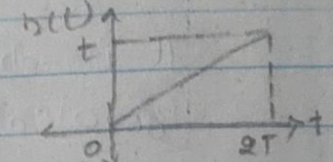
Example :- 3

Find the convolution of following signals.

$x(t) = 1 , \quad 0 < t < T$
$\quad\; = 0 , \quad \text{otherwise}$



$h(t) = t , \quad 0 \leq t < 2T$
$\quad\; = 0 , \quad \text{otherwise}$

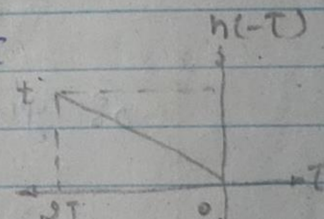


Solution:-

$y(t) = x(t) * h(t) = ?$

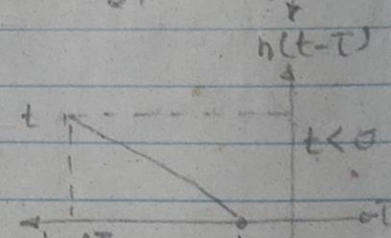
As we have;

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

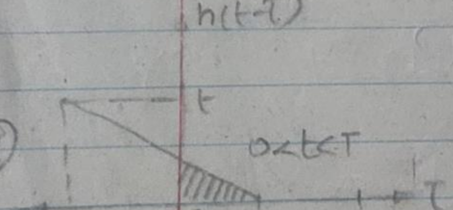


Case I: $\boxed{\text{for } t<0}$

$y(t) = 0.$



(I)

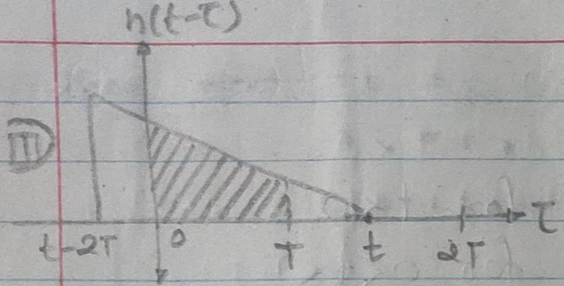


(II)

Case II: for $\boxed{0<t<T}$

$$y(t) = \int_0^t x(\tau) \cdot h(t-\tau)\, d\tau$$

$$= \int_0^t 1 \cdot (t-\tau)\, d\tau = \frac{t^2}{2}$$



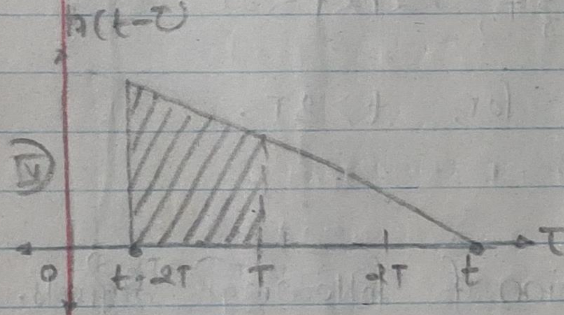
Case III : $\boxed{T < t < 2T}$

$$y(t) = \int_0^T x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_0^T 1 \cdot (t-\tau)\, d\tau$$

$$\therefore\ y(t) = \left(t\cdot\tau - \frac{\tau^2}{2}\right)\Big|_0^T$$

i.e, $y(t) = Tt - \frac{T^2}{2}$ //



Case IV : $\boxed{2T < t < 3T}$

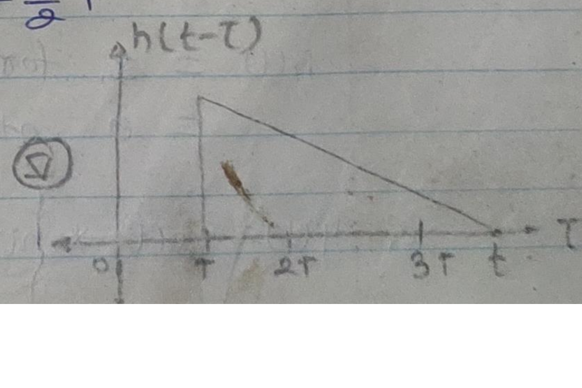
$$y(t) = \int_{t-2T}^{T} (t-\tau)\, d\tau$$

$$\therefore\ y(t) = \left(t\cdot\tau - \frac{\tau^2}{2}\right)\Big|_{t-2T}^{T}$$

$$= Tt - \frac{T^2}{2} - t(t-2T) + \frac{(t-2T)^2}{2}$$

$$= Tt - \frac{T^2}{2} - t^2 + \cancel{2Tt} + \frac{t^2}{2} - \cancel{\frac{4Tt}{2}} + \frac{4T^2}{2}$$

ie, $y(t) = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2$



Case V for; $\boxed{t > 3T}$

$y(t) = 0$

Finally;

$y(t) = 0$, for $t < 0$

$= \frac{t^2}{2}$, for $0 < t < T$

$= Tt - \frac{T^2}{2}$, for $T < t < 2T$

$= \frac{1}{2}t^2 + Tt + \frac{3}{2}T^2$, for $2T < t < 3T$

$= 0$, for $t > 3T$.

(1) ☆ Properties of LTI system:-
(property of convolution)

① Every linear time invariant system can be represented by a convolution integral for continuous time system and convolution sum for discrete time system.

② An LTI system exhibits commutative law. i.e; $x(t) * h(t) = h(t) * x(t)$

or graphically this can be represented as

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow x(t) * h(t)$$

$$|||$$

$$h(t) \longrightarrow \boxed{x(t)} \longrightarrow h(t) * x(t)$$

Proof :-

Let; $y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

putting; $t-\tau = \alpha$

$d\tau = -d\alpha$

$\therefore \quad y(t) = -\int_{\infty}^{-\infty} x(t-\alpha)\, h(\alpha)\, d\alpha$

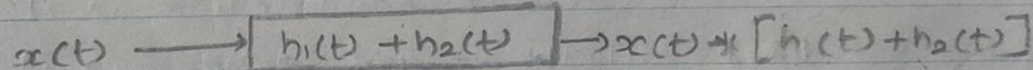
$$= \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha$$

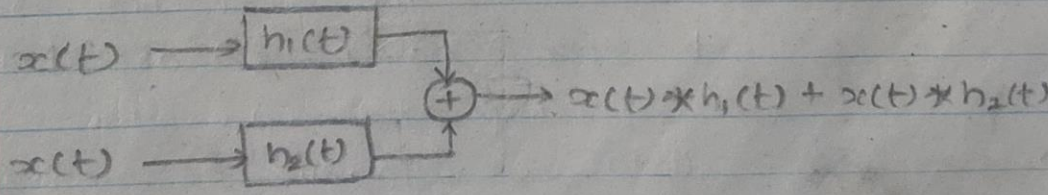
i.e; $y(t) = h(t) * x(t)$ //.

(3.) An LTI system is Distributive in addition

i.e; $x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$

or; Graphically;





Similarly; for discrete counterpart

$$x[n] * [h_1[n] + h_2[n]] = x[n] * h_1[n] + x[n] * h_2[n]$$

(4) LTI system exhibits associative law

i.e; $[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$

Graphically;

$$x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow [x(t) * h_1(t)] * h_2(t)$$

$$x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow x(t) * [h_1(t) * h_2(t)]$$

Similarly;

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * [h_1[n] * h_2[n]]$$

System :-

- A mathmatical model of a physical or logical entity or process that relates its input or excitation signals to the output or response signals is known as system. A system maps input signals into output signals.

input (excitation) $\longrightarrow$ $\boxed{\text{system } T\{\ldots\}}$ $\longrightarrow$ output (Response)

Types of system :-

① A continuous time system :- This system takes continuous time signal as an input & maps it into continuous time signal as output.

$$x(t) \longrightarrow \boxed{\text{C.T. System}} \longrightarrow y(t)$$

② A discrete time system :- This system takes discrete time signal as an input & manipulates it to give a discrete time signal as output.

$$x[n] \longrightarrow \boxed{\text{D.T. System}} \longrightarrow y[n]$$

Properties of system :-

1. Linear & Non-linear System.
2. Time Variant & time invariant System.
3. Static (Memoryless) & Dynamic (Memory) system.
4. causual and Non causual System.
5. Stable and Unstable system.
6. Invertible and Non-invertible system.
7. FIR and IIR system.
8. Recursive & Non Recursive system.

FIR → Finite Impulse Response

IIR → Infinite Impulse Response.

3. Static (Memoryless) & Dynamic (Memory) system :-

- A system is memory less, if its current ~~input~~ output at any time depends only on the input at the same time but doesn't depend on passed or future inputs. In any other case, the system is said to be dynamic or memory system.

For static system; $y(t) = f\{x(t)\}$
$y[n] = f\{x[n]\}$

For dynamic system; $y(t) = f\{x(t), x(t-t_0)\}$
$y[n] = f\{x[n], x[n-n_0]\}$

for an LTI system o/p is convolution between i/p and impulse response.
i.e; $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

This system, to be memoryless, the only way is $h(t) = 0$, for $t \neq 0$.
In this case impulse response is given by
$h(t) = K\delta(t)$.

4. Casual and Non_Casual system

$x(t) \longrightarrow \boxed{L(t)} \longrightarrow y(t)$

- A casual system is one for which the output of the system at any time depends only on present and past inputs, but doesn't depend on future inputs or mathmatically;

$y(t) = f\{x(t), x(t-1), x(t-2) \ldots\}$

i.e. $y(t) = f\{x(t-t_0)\}$, where $t_0 \geq 0$

Similarly, for discrete system

$y[n] = f\{x[n], x[n-1], x[n-2] \ldots\}$

i.e. $y[n] = f\{x[n-n_0]\}$, where, $n_0 \geq 0$.

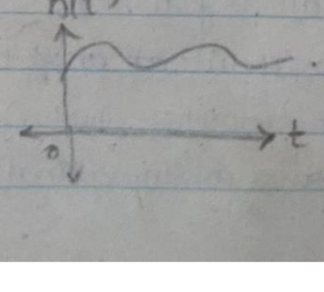
Any system violating the above condition falls under non-casual system. It is apparent that in real time signal application, we can't get future values of the signal, therefore non-casual systems are not physically realizable.

Paley-Wiener Time Domain criterian suggests that, a system is casual if its impulse response is casual.

i.e; $h(t) = 0$ for $t < 0$

$\neq 0$ otherwise

$x(t)$ —— $\boxed{h(t)}$ —— $y(t)$



similarly, for discrete system

$$h[n] = 0 \quad \text{for } n < 0$$
$$\neq 0 \quad \text{otherwise}$$

proof :-

The o/p of LTI system can be given by

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$$

OR

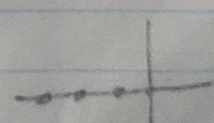
$$y[n] = \sum_{K=-\infty}^{\infty} h[K]\, x[n-K]$$

so, the o/p at $n = n_0$ is given by

$$y[n_0] = \sum_{K=-\infty}^{\infty} h[K]\, x[n_0-K]$$

$$= \cdots + h[-2]\, x[n_0+2] + h[-1]\, x[n_0+1] + h[0]\, x[n_0] + h[1]\, x[n_0-1] + \cdots$$

For the system to be casual



$$\ldots h[-2] = h[-1] = 0$$

ie; $h[n] = 0$ for $n < 0$

For casual LTI system, we can write

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

putting $n-k=m$

$$y[n] = \sum_{m=n}^{-\infty} h[n-m]\, x[m]$$

$$= \sum_{m=-\infty}^{n} x[m]\, h[n-m]$$

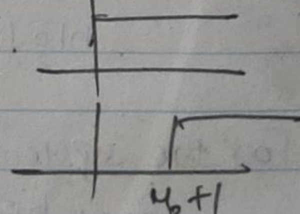
$$\therefore\ y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

proved

$$y[n-1] = \sum_{k=-\infty}^{n-1} x[k]$$

$$y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k]$$

$$= \sum_{k=-\infty}^{n-1} x[k] + x[n]$$



$n-k=p \quad n-p$

श्री

Q. ☆ Proove that response of a Casual System to a casual input signal is also casual.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Given; $x[n] = 0$; for $n<0$

$h[n] = 0$; for $n<0$

To proove; $y[n] = 0$, for $n<0$

we have;

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

since; $h[n]$ is casual, we can write

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

$k=p$
$p=n$
$p=-\infty$

we can also write;

$$y[n] = \sum_{k=-\infty}^{n} x[k]\, h[n-k]$$

since; $x[n]$ is casual

$\ldots = x[-4] = x[-3] = x[-2] = x[-1] = 0$

$$\therefore y[n] = \sum_{k=0}^{n} x[k]\, h[n-k]$$

This implies that $y[n] = 0$, for $n<0$, so $y[n]$ is casual.

✡ Stable & Unstable system :-

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

There are various definitions of stability of system.
BIBO (Bounded i/p - Bounded o/p) stability is one of them. If a system gives bounded o/p for bounded i/p then the system is called BIBO stable. i.e; for i/p $|x[n]| \leq M_x$ the system produces o/p as $|y[n]| \leq M_y$. where M_x & M_y are finite values.

& ✡ An LTI system is BIBO stable iff its impulse response is absolutely summable.

proof:-
Input is stable i.e; $|x[n]| \leq M_x$
we have;

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

i.e. $$|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[k]\, h[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |x[k]\, h[n-k]|$$

ie; $$\left|\sum_{k=-\infty}^{\infty} x[k]\, h[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |x[k]|\,|h[n-k]|$$

$$\leq \sum_{K=-\infty}^{\infty} M_x |h[n-K]|$$

$$\leq M_x \sum_{K=-\infty}^{\infty} |h[n-K]|$$

for $|y[n]|$ to be finite

$$\sum_{K=-\infty}^{\infty} |h[n-K] < \infty$$

ie. $$\sum_{K=-\infty}^{\infty} |h[n]| \leq M_x$$

ie; $h[n]$ must be absolutely summable.

Q. ☆ Find the range of α for which the given system is BIBO stable, $h[n] = \alpha^n u[n]$.

solution;
For BIBO stability, $\sum_{n=-\infty}^{\infty} h[n]$ must be finite

$$= \sum_{n=0}^{\infty} \alpha^n \quad (\text{range } |\alpha|<1)$$

Q. Determine the system is stable or not.

$$h[n] = 3^{-n} u[n].$$

Solution:

for BIBO stability $\sum_{n=-\infty}^{\infty} h[n]$ must be constant.

i.e. finite.

i.e. $$\sum_{n=0}^{\infty} 3^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$$

where; $\alpha = \frac{1}{3} < 1$ and also

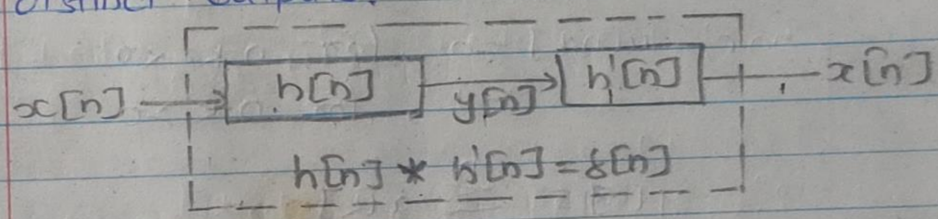
$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$$

$$= \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \Rightarrow \text{finite.}$$

Hence; the system is BIBO stable.

Invertible & Non Invertible system :-

- Any system is said to be invertible if an inverse system exists, such that when connected in series with the original system produces an o/p equal to the i/p of first system. In another way, a system is said to be invertible if distinct inputs lead to distinct outputs.



For a system to be invertible, the overall system response should be $\delta[n]$ or $\delta(t)$

i.e, $h[n] * h'[n] = \delta[n]$

$h(t) * h'(t) = \delta(t)$

Q. If $h[n] = u[n]$, prove that its inverse system is given by $h'[n] = \delta[n] - \delta[n-1]$.

* Hint: $x[n] * \delta[n-1] = x[n-1]$

Solution:-

We have; $h[n] = u[n]$

Using the convolution sum, we have;

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]u[n-k]$$

since; $u[n-k] = 0$, for $n-k < 0$

$= 1$, for $n-k \geq 0$

$$\therefore \quad y[n] = \sum_{k=-\infty}^{n} x[k]$$

Let us check;

$$h[n] * h'[n] = u[n] * \{\delta[n] - \delta[n-1]\}$$

$$= u[n] * \delta[n] - u[n] * \delta[n-1]$$

$$= u[n] - u[n-1]$$

$$= \delta[n]$$

Hence; proved//.

Imp

☆ Transmission of signals in discrete time system.

* Finite Impulse Response & Infinite impulse Response system.

- An LTI system can be also classified according to the length of the impulse response. If the impulse response is of finite duration, then the system is known as FIR system. so;

For casual FIR system:-

$h[n]=0$ for $n<0$ & $n \geq M-1$

Hence, for casual FIR system the o/p can be given as

$$y[n] = \sum_{k=0}^{M-1} h[k]\, x[n-k]$$

On the other hand if the impulse response is of infinite duration then the system is called IIR system. The convolution formula for casual IIR system becomes;

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

ॐ

Q. Prove that the impulse response of moving average system is FIR.

Solution:-

The o/p of moving average system is

$$y[n] = \frac{1}{M+1} \sum_{K=0}^{M} x[n-K]$$

for $x[n] = \delta[n]$

$$y[n] = h[n] = \frac{1}{M+1} \sum_{K=0}^{M} \delta[n-K]$$

Let; $M=3$;

$$h[n] = \frac{1}{4} \sum_{K=0}^{3} \delta[n-K]$$

so;

$$h[0] = \frac{1}{4} \sum_{K=0}^{3} \delta[-K]$$

$$= \frac{1}{4} \left\{ \delta[0] + \cancel{\delta[-1]}^{0} + \cancel{\delta[-2]}^{0} + \cancel{\delta[-3]}^{0} \right\} = \frac{1}{4}$$

$$h[1] = \frac{1}{4} \sum_{K=0}^{3} \delta[1-K]$$

$$= \frac{1}{4} \left\{ \cancel{\delta[1]}^{0} + \delta[0] + \cancel{\delta[-1]}^{0} + \cancel{\delta[-2]}^{0} \right\} = \frac{1}{4}$$

Similarly;

$h[2] = h[3] = \frac{1}{4}$ but;

$h[4] = 0$

$\vdots \quad \vdots$

$h[n] = 0$

Finally;

$$h[n] = \frac{1}{4} \quad \text{for } 0 \le n \le 3$$
$$= 0 \quad \text{otherwise}$$

so $h[n]$ is finite, hence system is FIR.

In general;

$$h[n] = \frac{1}{M+1}, \text{ for } 0 \le n \le M$$
$$= 0, \text{ otherwise.}$$

Ans

Q. Prove that the cumulative average system is IIR.

Solution:- The o/p of cumulative average system is;

$$y[n] = \frac{1}{n+1} \sum_{k=0}^{n} x[k]$$

for; $x[n] = \delta[n]$

$$y[n] = h[n] = \frac{1}{n+1} \sum_{k=0}^{n} \delta[k]$$

so;

$$y[1] = \frac{1}{2} \sum_{k=0}^{1} \delta[k] = \frac{1}{2}\{\delta[0] + \cancel{\delta[1]}^{0}\} = \frac{1}{2}$$

$$y[2] = \frac{1}{3} \sum_{k=0}^{2} \delta[k] = \frac{1}{3}\{\delta[0] + \cancel{\delta[1]}^{0} + \cancel{\delta[2]}^{0}\} = \frac{1}{3}$$

similarly; $y[3] = \frac{1}{4}$

$$\vdots$$

$$y[n] = \frac{1}{n+1}$$

$$y[\infty] = 0$$

In general;

$$y[n] = h[n] = \frac{1}{n+1} \sum_{k=0}^{n} \delta[k]$$

$\therefore$ $h[n] = \frac{1}{n+1}$. so it has non-zero value over ∞- period. The system is IIR.

* Recursive and Non Recursive System :-

$$x[n] \longrightarrow \boxed{f\{x[n], x[n-n_0]\}} \longrightarrow y[n]$$

- If output of a system is function of or dependent on present and past i/p only then the system is called non recursive system. This system is presented in block diagram form as shown above.

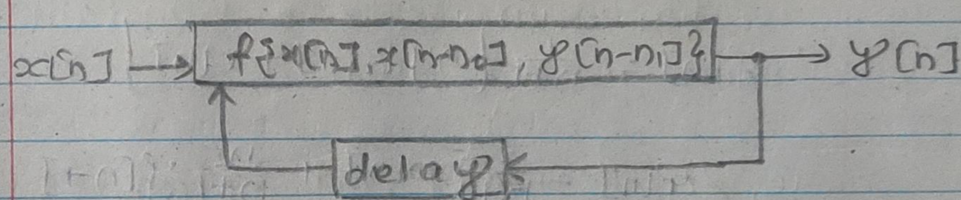
eg for casual FIR system

$$y[n] = \sum_{k=0}^{M} h[k]\, x[n-k]$$

i.e, $y[n] = h[0]x[n] + h[1]x[n-k] + \cdots + h[M]\,x[n-M]$

This shows that casual FIR system described by convolution sum are non-recursive.

Sometimes, it is necessary to express the present o/p, not only interms of past and present i/ps but also interms of past o/ps, this economizes the memory required, these systems are called recursive system and can be represented by following diagram.



Recursive realization reduces the memory needed, no. of addition and multiplication also. So recursive realization is more advantageous than non-recursive realization.

Q. Realize the following function recursively.

$$y[n] = \frac{1}{n+1} \sum_{k=0}^{n} x[k] \text{, non recursive \& IIR}$$

Solution:-

$$y[n] = \frac{1}{n+1} \sum_{k=0}^{n} x[k] \quad \text{--- (1)}$$

$$y[n-1] = \frac{1}{n-1+1} \sum_{k=0}^{n-1} x[k] = \boxed{\frac{1}{n} \sum_{k=0}^{n-1} x[k]}$$

$\therefore \sum_{k=0}^{n-1} x[k] = n\,y[n-1]$ — (ii)

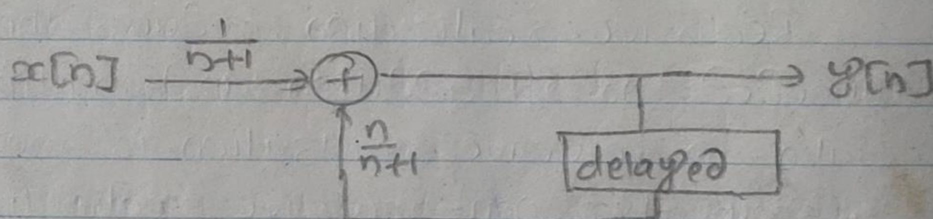
from (i)

$$y[n] = \frac{1}{n+1} \sum_{k=0}^{n} x[k] = \frac{1}{n+1}\left[\sum_{k=0}^{n-1} x[k] + x[n]\right] \quad \text{— (iii)}$$

now, from (ii) & (iii)

$$y[n] = \frac{1}{n+1}\left[n\,y[n-1] + x[n]\right]$$

ie, $y[n] = \frac{1}{n+1} x[n] + \frac{n}{n+1} y[n-1]$



Constant co-efficient difference equation :-

- An LTI system can be characterized by constant co-efficient difference equation given by;

$$\sum_{K=0}^{N} a_K y[n-K] = \sum_{K=0}^{M} b_K x[n-K]$$

for; $a_0 = 1$

$$y[n] + \sum_{K=1}^{N} a_K y[n-K] = \sum_{K=0}^{M} b_K x[n-K]$$

$$\therefore y[n] = \sum_{K=0}^{M} b_K x[n-K] - \sum_{K=1}^{N} a_K y[n-K]$$

System function / Transfer function :-

- It is the ratio of Laplace transform of o/p and Laplace transform of i/p of a system for continuous time system & ratio of z-transform of o/p and z-transform of i/p of a system for discrete time system.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\therefore H(s) = \frac{Y(s)}{X(s)}$$

where; $H(s)$ is transfer function, which is the laplace transform of impulse response $h(t)$.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$H(z) = \frac{Y(z)}{X(z)};$$ where; $H(z)$ is transfer function which is the z-transform of impulse response $h[n]$.

Transfer function of LTI system described by constant co-efficient difference equation we have;

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \quad \text{--- (1)}$$

Taking z-transform on both sides:

$$y[n] \xrightarrow{Z} Y(z)$$

$$x[n] \xrightarrow{Z} X(z)$$

$$y[n-k] \xrightarrow{Z} Y(z) z^{-k}$$

$$x[n-k] \xrightarrow{Z} X(z) z^{-k}$$

Now;

$$Y(z) = \sum_{k=0}^{M} b_k X(z) z^{-k} - \sum_{k=1}^{N} a_k Y(z) z^{-k}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k \cdot z^{-k} - \sum_{k=1}^{N} a_k \cdot \frac{Y(z)}{X(z)} \cdot z^{-k}$$

$$\text{ie, } H(z)\left(1 + \sum_{k=1}^{N} z^{-k}\right) = \sum_{k=0}^{M} b_k \cdot z^{-k}$$

$$\therefore H(z) = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} z^{-k}}$$

imp

☆ $\boxed{\text{Structure for IIR System}}$:-

- The general IIR system can be given by

$$y[n] = \sum_{k=0}^{M} b_k\, x[n-k] - \sum_{k=1}^{N} a_k\, y[n-k]$$

Its transfer fxn. is given by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=0}^{N} a_k \cdot z^{-k}}$$

Let;

$$H_1(z) = \sum_{k=0}^{M} b_k \cdot z^{-k}$$

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

So that;

$$H(z) = H_1(z) \cdot H_2(z)$$

$$\frac{Y(z)}{X(z)} = \frac{V(z)}{X(z)} \cdot \frac{Y(z)}{V(z)}$$

Hence;

$$H_1(z) = \frac{V(z)}{X(z)} = \sum_{K=0}^{M} b_K z^{-K}$$

ie; $$\frac{V(z)}{X(z)} = \sum_{K=0}^{M} b_K z^{-K}$$

ie; $$V(z) = \sum_{K=0}^{M} b_K X(z) z^{-K}$$

Tacing inverse z-transform

$$v[n] = \sum_{K=0}^{M} b_K x[n-K] \quad -①$$

Again;

$$H_2(z) = \frac{Y(z)}{V(z)} = \frac{1}{1 + \sum_{K=1}^{N} a_K z^{-K}}$$

ie; $$V(z) = Y(z)\left[1 + \sum_{K=1}^{N} a_K z^{-K}\right]$$

ie; $$V(z) = Y(z) + \sum_{K=1}^{N} a_K Y(z) z^{-K}$$

taking inverse z-transform

$$y[n] + \sum_{k=1}^{N} a_k\, y[n-k] = V[n]$$

$$\therefore y[n] = V[n] - \sum_{k=1}^{N} a_k \cdot y[n-k] \quad — (ii)$$

$$\text{ie: } y[n] = V[n] + \sum_{k=1}^{N} (-a_k)\, y[n-k]$$

From (i)

$V[n] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]$

From (ii)

$y[n] = V[n] + (-a_1) y[n-1] + \dots + (-a_N) y[n-N]$

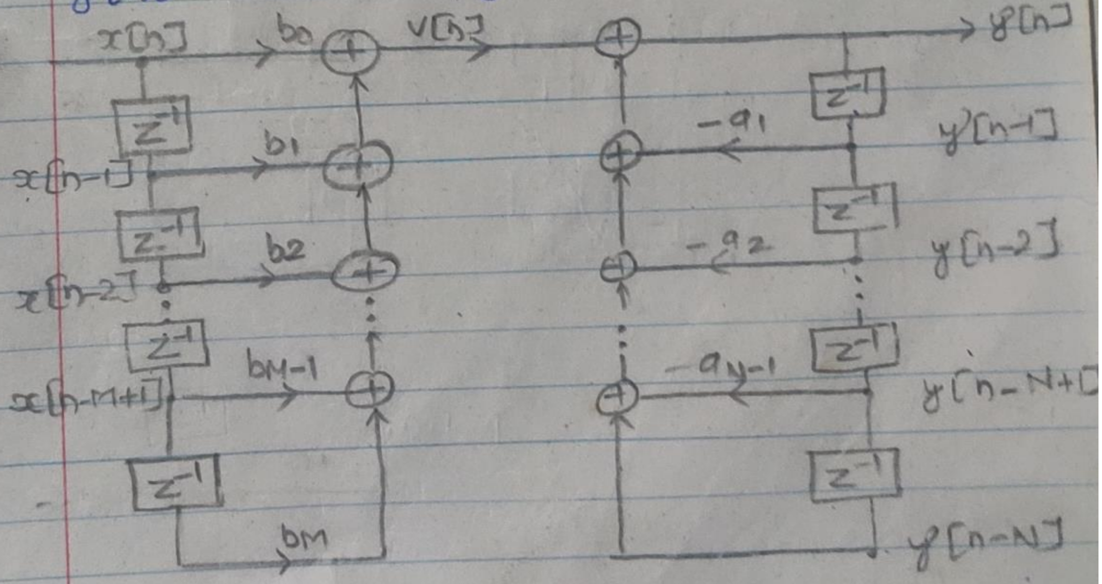


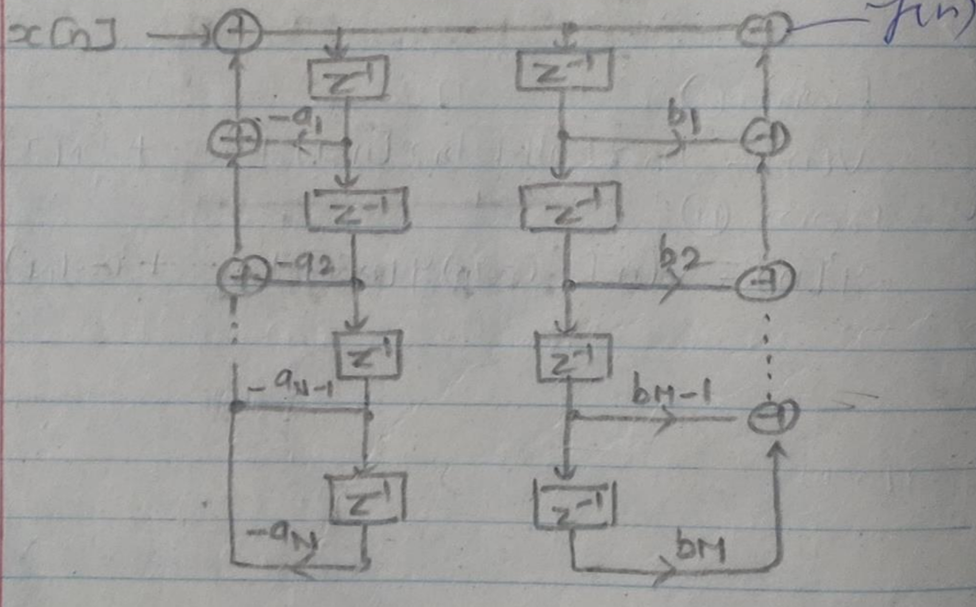
Fig: direct form I realization

This structure for IIR system is called direct form I realization. Since, this realization has high no. of memories; addition and multiplications; another compact realization is used called direct form II realization.

We know that;

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H(z) = H_2(z) \cdot H_1(z)$$



p.t.o.

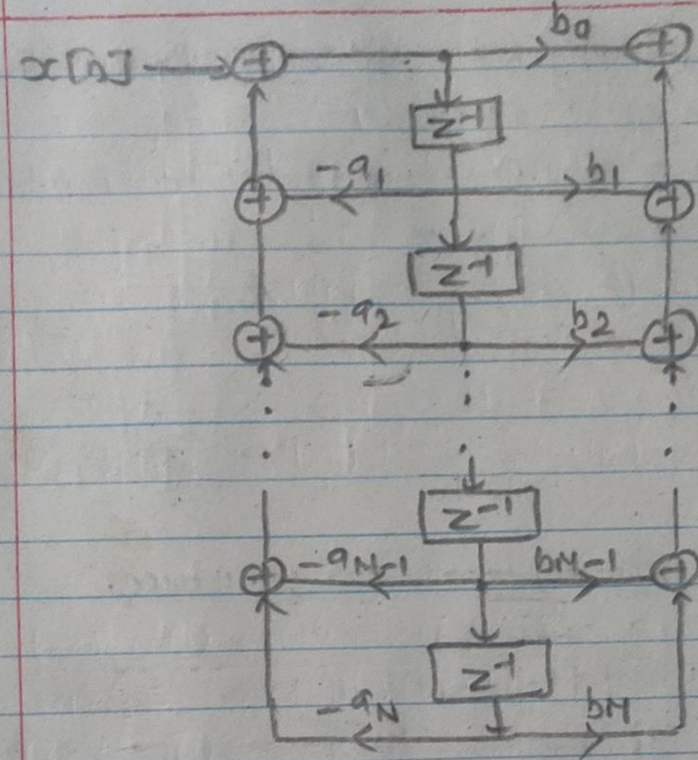


fig: direct form II realization for M=N.

☆ Example:-

① Obtain the direct form I & direct form II realization for the system characterized by

$$y[n] = \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] + x[n] + \frac{1}{3} x[n-1]$$

Solution:-

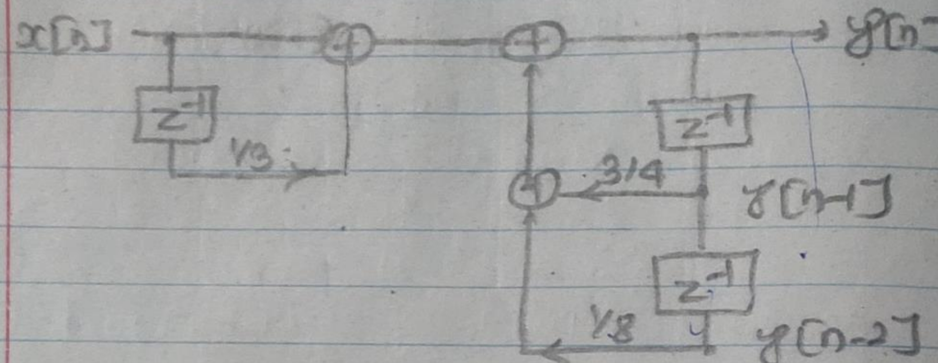
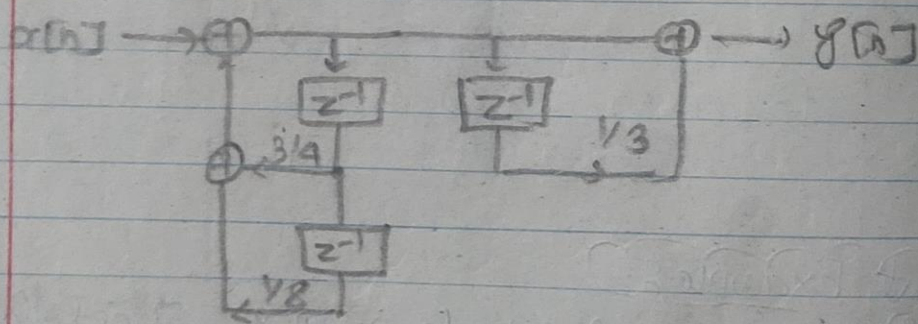


fig: Direct form I structure



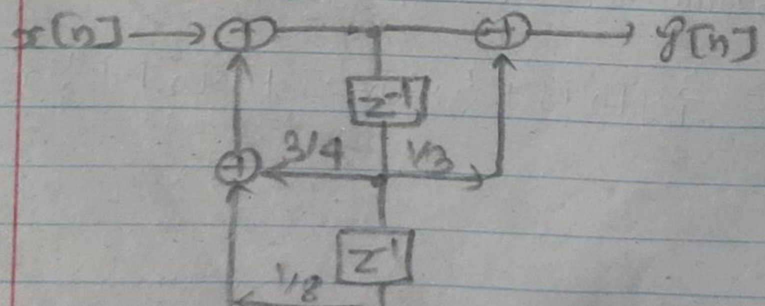


fig: Direct form II structure

h

(2) $H(z) = \dfrac{2+z^{-1}+3z^{-2}+5z^{-3}}{5+\frac{1}{3}z^{-1}+2z^{-2}+3z^{-3}}$

Solution:-

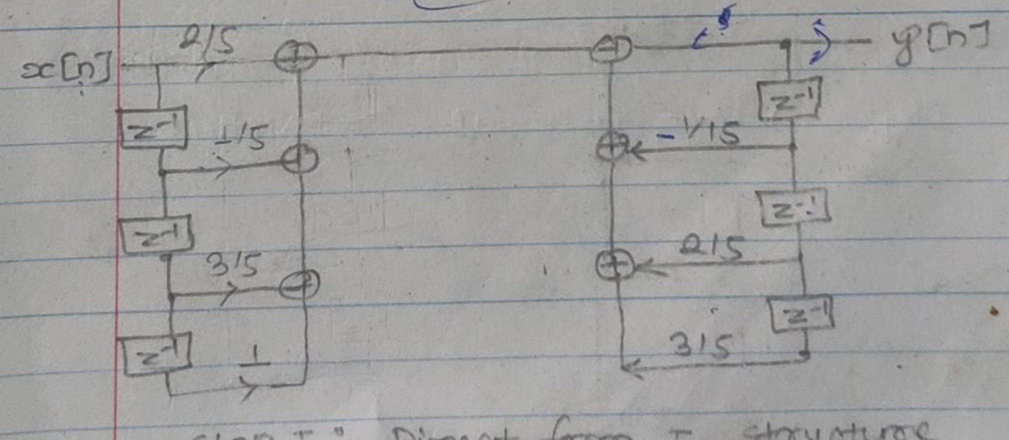
i.e, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{2+z^{-1}+3z^{-2}+5z^{-3}}{5+\frac{1}{3}z^{-1}+2z^{-2}+3z^{-3}}$

i.e, $5Y(z) + \frac{1}{3}Y(z)z^{-1} + 2Y(z)z^{-2} + 3Y(z)z^{-3} =$
$2X(z) + X(z)z^{-1} + 3X(z)z^{-2} + 5X(z)z^{-3}$

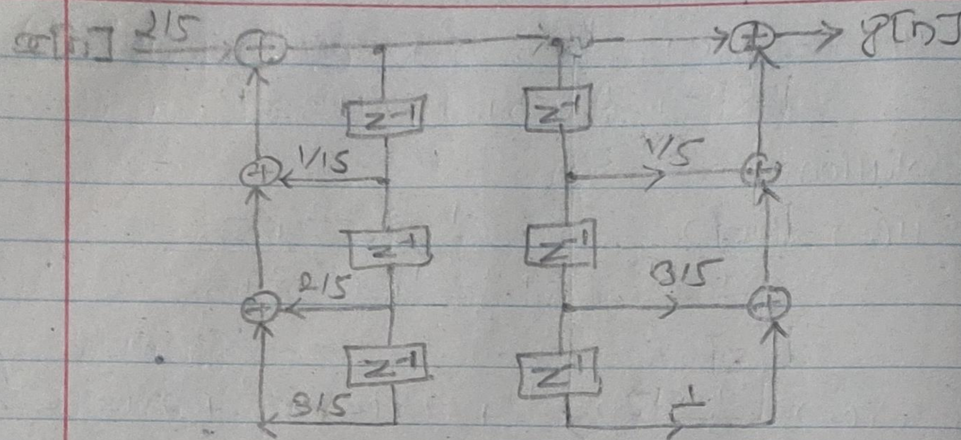
On Inverse z-transformation

$5y[n] + \frac{1}{3}y[n-1] + 2y[n-2] + 3y[n-3] =$
$2x[n] + x[n-1] + 3x[n-2] + 5x[n-3]$

$\therefore$ $y[n] = \frac{2}{5}x[n] + \frac{1}{5}x[n-1] + \frac{3}{5}x[n-2] + x[n-3] -$
$\frac{1}{15}y[n-1] + \frac{2}{5}y[n-2] + \frac{3}{5}y[n-3]$



Step I : Direct form I structure



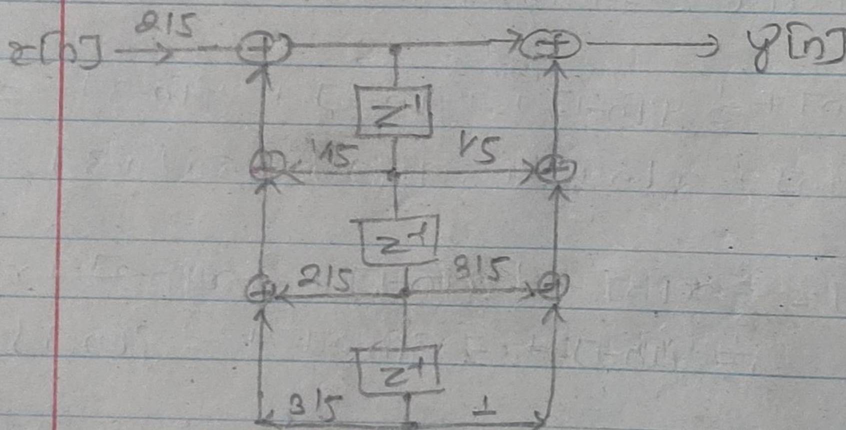


fig: Direct form II structure.

☆ Structure for FIR system :-

We have;

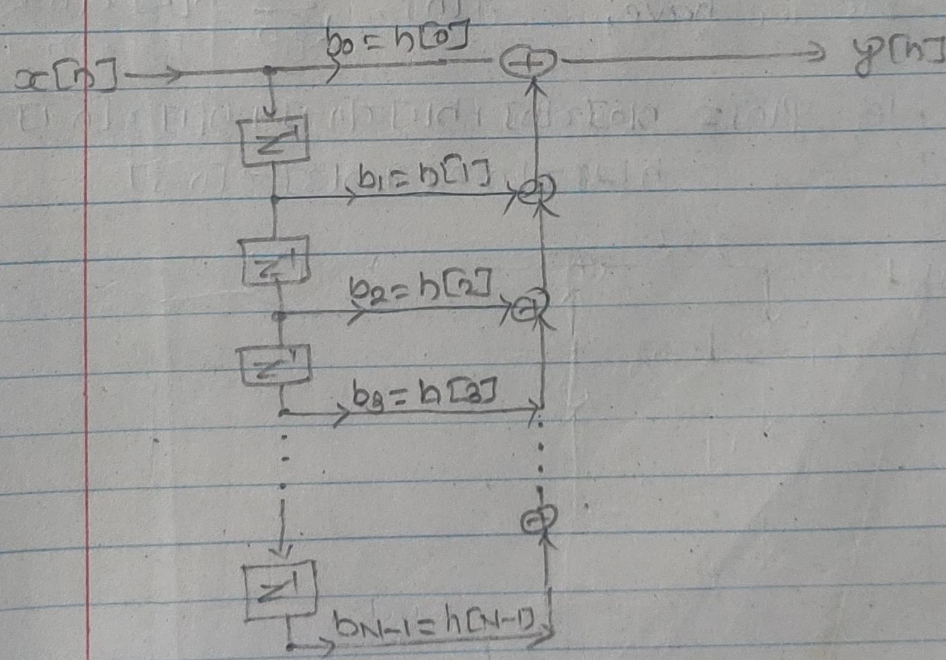
For FIR system having length of 'N' is

$$y[n] = \sum_{K=0}^{N} h[K]\, x[n-K]$$

where; $h[K]$ are constants, so we can write $h[K] = b_K$ for $K=0$ to $K=N-1$

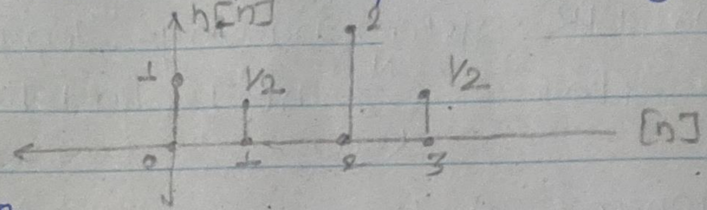
$$\therefore \; y[n] = \sum_{K=0}^{N-1} b_K\, x[n-K]$$

ie; $y[n] = b_0 x[n] + b_1 x[n-1] + \cdots b_{N-1} x[n-N+1]$



Example:-

Draw the structure for following system:-



Solution:-

From figure;

$h[0] = b_0 = 1$

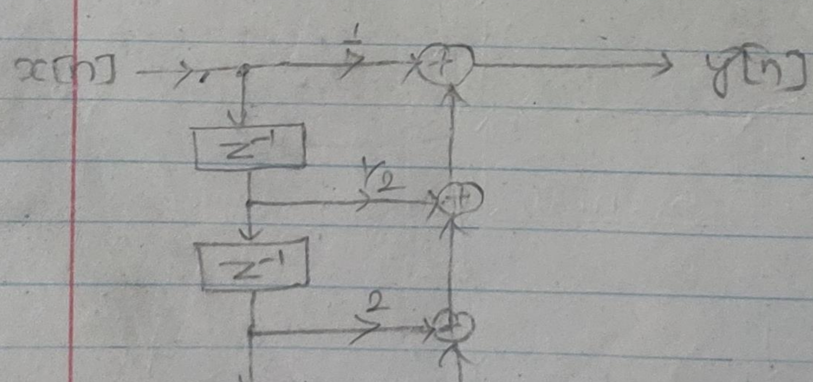
$h[1] = b_1 = \frac{1}{2}$

$h[2] = b_2 = 2$

$h[3] = b_3 = \frac{1}{2}$

As we have; $$y[n] = \sum_{k=0}^{3} h[k]\,x[n-k]$$

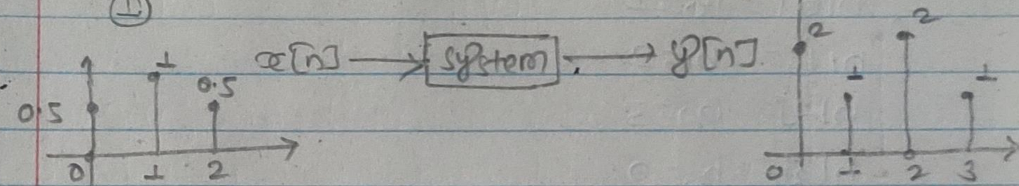
ie; $$y[n] = h[0]x[n] + h[1]x[n-1] + h[1]x[n-1] + h[2]x[n-2] + h[3]x[n-3]$$



Questions:-

(a) Draw the direct form (I) & (II) for the following system:-

(I)



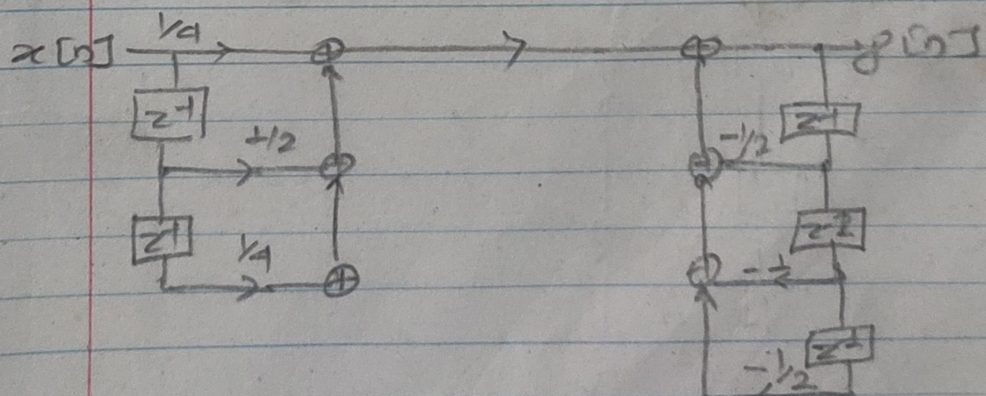
solution:

Here:

$0.5x[n] + x[n-1] + 0.5x[n-2] = 2y[n] + y[n-1] + 2y[n-2] + y[n-3]$

ie, $y[n] = \frac{1}{2}\left\{\frac{1}{2}x[n] + x[n-1] + \frac{1}{2}x[n-2] - y[n-1] - 2y[n-2] - y[n-3]\right\}$

$= \frac{1}{4}x[n] + \frac{1}{2}x[n-1] + \frac{1}{4}x[n-2] - \frac{1}{2}y[n-1] - y[n-2] - \frac{1}{2}y[n-3]$



Direct form I realization.

Q) Realize the system

$h[n] = \{ \underset{\uparrow}{1}, 2, 3, 0, 4 \}$

$h[0] = b_0 = 1$

$h[1] = b_1 = 2$

$h[2] = b_2 = 3$

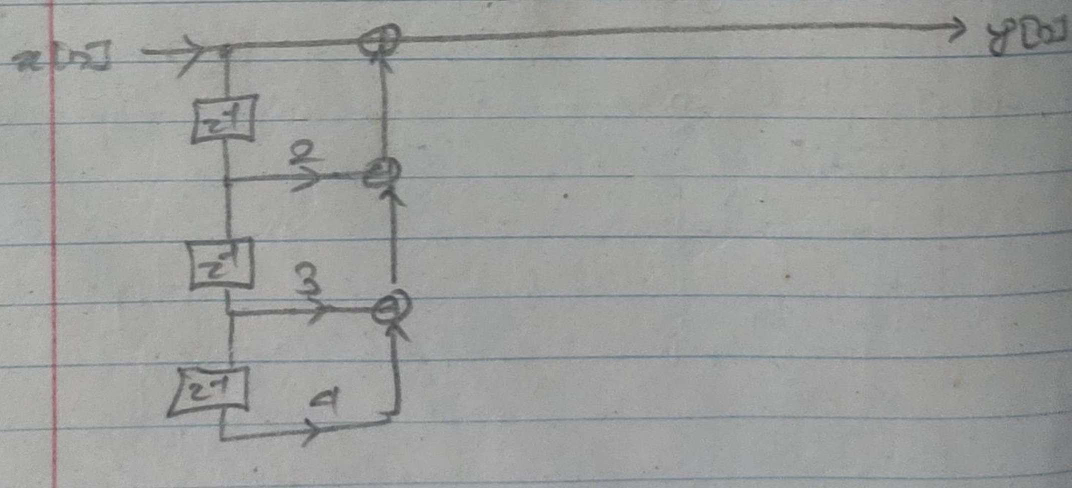
$h[3] = b_3 = 0$

$h[4] = b_4 = 4$

As we have: $y[n] = \sum_{k=0}^{4} h[k]\, x[n-k]$

$\therefore y[n] = x[n] + 2x[n-1] + 3x[n-2] + 0 + 4x[n-4]$

$= x[n] + 2x[n-1] + 3x[n-2] + 4x[n-4]$



(a) $H(z) = \dfrac{2+3z^{-2}+z^{-4}}{5+3z^{-1}+z^{-5}}$

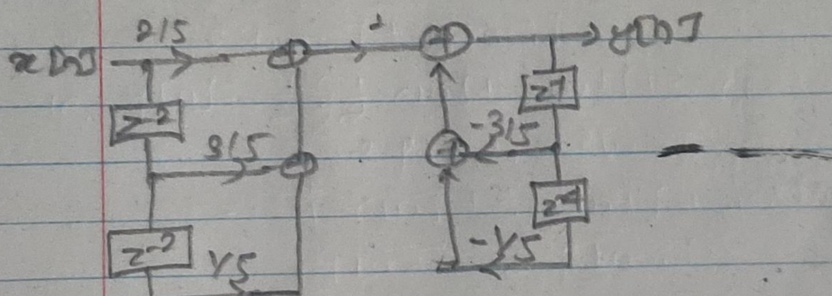
ie, $\dfrac{Y(z)}{X(z)} = \dfrac{2+3z^{-2}+z^{-4}}{5+3z^{-1}+z^{-5}}$

ie $5Y(z)+3Y(z)z^{-1}+Y(z)\cdot z^{-5} = 2X(z)+3X(z)z^{-2}$
$+ X(z)\cdot z^{-4}$

ie, On inverse z-transform

$5y[n]+3y[n-1]+y[n-5] = 2x[n]+3x[n-2] +$
$x[n-4]$

$y[n] = \frac{2}{5}x[n]+\frac{3}{5}x[n-2]+\frac{1}{5}x[n-4]-\frac{3}{5}y[n-1]-\frac{1}{5}y[n-5]$



(d) $H(z) = 1 + \frac{3}{4} z^{-1} + \frac{17}{18} z^{-2} + \frac{9}{4} z^{-3} + z^{-4}$

i.e. $Y(z) = X(z) + \frac{3}{4} X(z) z^{-1} + \frac{17}{18} X(z) z^{-2} + \frac{3}{4} X(z) z^{-3} + X(z) z^{-4}$

On inverse Z-transform

$y[n] = x[n] + \frac{3}{4} x[n-1] + \frac{17}{18} x[n-2] + \frac{3}{4} x[n-3] + x[n-4]$