Chapter 1 REVIEW OF NETWORK ANALYSIS

1.1 KIRCHHOFF'S LAW

A German physicist Gustav Kirchoff developed two laws enabling easier analysis of circuits containing interconnected impedances in 1845. The first law deals with the flow of current and is popularly known as Kirchoff's current law (KCL) while the second one deals with voltage drop in a closed circuit and is known as Kirchhoff's voltage law (KVL). These laws concern the algebraic sum of voltages around a loop and currents entering or leaving a node. The word algebraic is used to indicate that summation is carried out taking into account the polarities of voltages and direction of currents.

1.1.1 Kirchhoff's Current Law (KCL) or Kirchhoff's Point Law

It states that- 'in any electrical network, the algebraic sum of currents meeting at any node of a circuit is zero'. In other words, it simply means that the total current leaving a junction is equal to the total current entering that junction. It is obviously true because there is no accumulation of charge at the junction of the network.

Consider the case of a few conductors meeting at a point A as shown in figure 1.1 (a). Some conductors have currents leading to point A, whereas some have currents leading away from point A. Assuming the incoming currents to be positive and the outgoing currents negative.

We have,

$$I_1 + (-I_2) + (-I_3) + (+I_4) + (-I_5) = 0$$

or, $$I_1 + I_4 - I_2 - I_3 - I_5 = 0$$

or, $$I_1 + I_4 = I_2 + I_3 + I_5$$

∴ Incoming currents = Outgoing currents

We can express the above conclusion as $\Sigma I = 0$ at a junction.

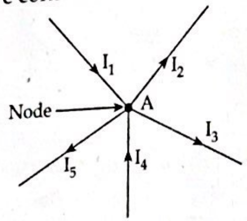


Figure 1.1: Explanation of KCL

1.1.2 Krichhoff's Voltage Law (KVL) or Mesh Law

It states that-the algebraic sum of voltages or voltage drop in any closed path in a network, traversed in a single direction is zero. In other words, the algebraic sum of the products of currents and resistances in each of the conductors in any closed path or mesh in a network plus the algebraic sum of the emfs in that path is zero.

i.e., $\Sigma IR + \Sigma emf = 0$.......... around a mesh.

In figure 1.2, if we travel clockwise in the network along the direction of the current, application of KVL yields,

$$-V_1 + IR_1 + V_2 + IR_2 + IR_3 = 0$$

$$V_1 = I\,(R_1 + R_2 + R_3) + V_2$$

or, $$V_1 - V_2 = I\,(R_1 + R_2 + R_3)$$

or, $$I = \frac{V_1 - V_2}{R_1 + R_2 + R_3}$$

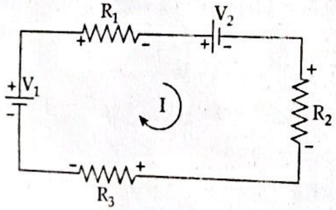
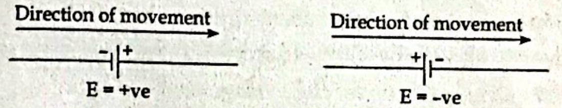


Figure 1.2: Explanation of KVL

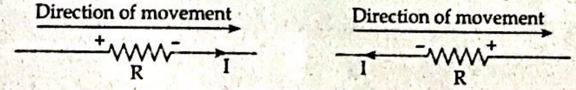
We consider the voltage drop as positive when current flows from positive to negative potential.

Sign Convention

a) Give positive sign to all rise in voltage and negative sign to voltage drops. Thus, if we move from negative (-ve) terminal of a battery to positive (+ve) terminal, a positive sign should be given, since there is a rise in voltage. On the other hand, if we go from positive (+ve) terminal to negative terminal, a negative sign should be given, since there is voltage drop.



b) For a conductor of resistance R, if the direction of current is same/opposite as the direction of movement, then voltage should be taken negative/positive, since current flows from higher voltage to lower one and hence there is a voltage drop/voltage rise while crossing the conducting element.



1.2 MESH ANALYSIS OR LOOP ANALYSIS

The mesh or loop analysis is based on Krichoff's voltage law. Here the currents in different meshes are assigned continuous paths so that they do not split at a junction into branch currents. In this method, loop voltage equations are written by KVL in terms of unknown loop currents. Circuits with voltage sources are comparatively easier to be solved by this method.

Figure 1.3 shows that two batteries having emf E_1 and E_2 are connected in a network containing five resistors. There are two loops and the respective loop currents are I_1 and I_2. Applying KVL in loop 1, we have,

$$-E_1 + I_1 R_1 + (I_1 - I_2) R_2 + I_1 R_4 = 0$$

or, $$E_1 = I_1 (R_1 + R_2 + R_4) - I_2 R_2 \quad (1)$$

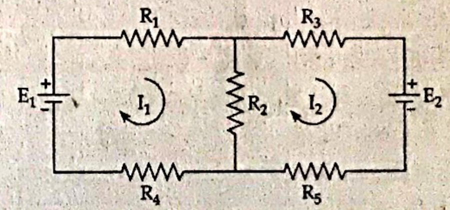


Figure 1.3: Illustration of mesh analysis

Applying KVL in loop 2, we get,

$$E_2 + I_2 R_3 + (I_2 - I_1) R_2 + I_2 R_5 = 0$$

or, $$E_2 = I_1 R_2 - (R_2 + R_3 + R_5) I_2 \quad (2)$$

Solving equations **(1)** and **(2)**, we can find the values of I_1 and I_2 and subsequently branch currents can be evaluated.

DC circuit analysis procedure using loop equations

i) Covert all current sources to voltage sources.
ii) Draw all loop currents in a clockwise direction and identify them.
iii) Indentify all resistor voltage drops as + to – in the direction of the loop current and assume these drops to be positive.
iv) Identify all voltage sources according to their correct polarity.
v) Write the equations for the voltage drops around each loop in turn by equating the sum of the voltage drops to zero.
vi) Solve the equations to find the unknown currents and/or voltage drops.

1.2.1 Mesh Analysis Using Matrix Form

Let us consider the network shown in figure 1.4. It contains three meshes. The three mesh currents are I_1, I_2 and I_3 and they are assumed to flow in a clockwise direction.

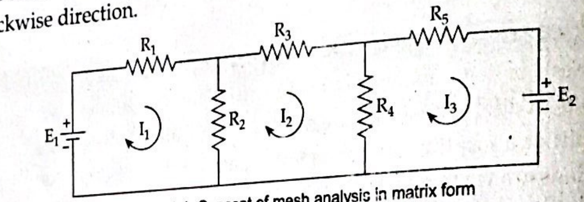


Figure 1.4: Concept of mesh analysis in matrix form

Applying KVL to mesh 1,

or $\quad -E_1 + (I_1 - I_2) R_2 + I_1 R_1 = 0$

or, $\quad I_1 R_1 + (I_1 - I_2) R_2 = E_1$

or, $\quad I_1 (R_1 + R_2) - I_2 R_2 = E_1$

or, $\quad I_1 (R_1 + R_2) + I_2 (-R_2) = E_1 \quad (1)$

Applying KVL to mesh 2,

or, $\quad (I_2 - I_1) R_2 + I_2 R_3 + (I_2 - I_3) R_4 = 0$

or, $\quad -I_1 R_2 + I_2 (R_2 + R_3 + R_4) - I_3 R_4 = 0$

or, $\quad I_1 (-R_2) + I_2 (R_2 + R_3 + R_4) + I_3 (-R_4) = 0 \quad (2)$

Applying KVL to mesh 3,

$\quad E_2 + I_3 R_5 + (I_3 - I_2) R_4 = 0$

or, $\quad -I_2 R_4 + I_3 (R_4 + R_5) = -E_2 \quad (3)$

It should be noted that the sings of resistances in the above equations have been so arranged as to make the items containing self-resistances positive. The matrix equivalent of the three equations is,

$$\begin{bmatrix} R_1 + R_2 & -R_2 & 0 \\ -R_2 & R_2 + R_3 + R_4 & -R_4 \\ 0 & -R_4 & R_4 + R_5 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} E_1 \\ 0 \\ -E_2 \end{bmatrix}$$

In general, the resistance matrix [R] can be written as

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

where, R_{11} = Self-resistance of mesh 1 = $R_1 + R_2$

R_{22} = Self resistance of mesh 2 = $R_2 + R_3 + R_4$

R_{33} = Self resistance of mesh 3 = $R_4 + R_5$

$R_{12} = R_{21}$

= –[Sum of all the resistances common to meshes 1 and 2] = $-R_2$

$R_{23} = R_{32}$

= –[Sum of all the resistances common to meshes 2 and 3] = $-R_4$

$R_{13} = R_{31}$

= – [Sum of all the resistances common to meshes 3 and 1] = 0

[R_{11}, R_{22}, R_{33} are called diagonal elements of the resistance matrix while R_{12}, R_{13}, R_{21}, R_{23} are called off-diagonal elements].

1.3 NODAL ANALYSIS

Nodal analysis is based on Krichhoff's current law. This method has the advantage that a minimum number of equations are needed to determine the unknown quantities. Moreover, it is particularly suited for networks having many parallel branches and also when there are current sources in the network. For the application of this method one of the nodes in the network is regarded as the reference or datum node or zero potential node. The number of simultaneous equations to be solved becomes (n–1), where n is the number of independent nodes.

Nodal analysis procedure

i) Convert all voltage sources to current sources and redraw the circuit diagram.
ii) Indentify all nodes and choose a reference node. (Usually, the common node is the reference node).
iii) Write the equation for the currents flowing into and out of each node, with the exception of the reference node.
iv) Solve the equation to determine the node voltage and the required branch currents.

1.3.1 Nodal Analysis with Sources

I) First Case

Consider the circuit of figure 1.5 which has three nodes. One of these *i.e.,* node 3 has been taken in as the reference node. V_A represents the potential of node 1 with reference to the datum node 3. Similarly, V_B is the potential difference between node 2 and node 3. Let the current directions which have been chosen arbitrary be as shown.

For node 1, the following current equation can be written with the help of KCL,

$$I_1 = I_4 + I_2$$

Now,

$$I_1 R_1 = E_1 - V_A$$

$$\therefore \quad I_1 = \frac{(E_1 - V_A)}{R_1} \tag{1}$$

Obviously,

$$I_4 = \frac{V_A}{R_A}$$

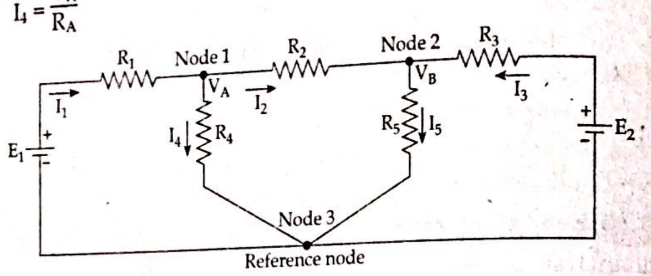


Figure 1.5:

Also, $I_2 R_2 = V_A - V_B \; (\because V_A > V_B)$

$$\therefore \quad I_2 = \frac{(V_A - V_B)}{R_2}$$

Substituting these values in equation (1), we get,

$$\frac{E_1 - V_A}{R_1} = \frac{V_A}{R_4} + \frac{V_A - V_B}{R_2}$$

On simplifying the above, we get,

$$V_A\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4}\right) - \frac{V_B}{R_2} - \frac{E_1}{R_1} = 0 \tag{2}$$

The current equation for node 2 is $I_5 = I_2 + I_3$

$$\text{or,} \quad \frac{V_B}{R_5} = \frac{V_A - V_B}{R_2} + \frac{E_2 - V_B}{R_3} \tag{3}$$

$$\text{or,} \quad V_B\left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_5}\right) - \frac{V_A}{R_2} - \frac{E_2}{R_3} = 0 \tag{4}$$

Though the above nodal equations (2) and (3) seem to be complicated, they employed a very simple and systematic arrangement of terms which can be written simply by inspection. Equation (2) at node 1 is represented by,

i) The product of node potential V_A and $\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4}\right)$ *i.e.*, the sum of the reciprocals of the branch resistance connected to this anode.

ii) Minus the ratio of adjacent potential V_B and the interconnecting resistance R_2.

iii) Minus ratio of adjacent battery or generator voltage E_1 and interconnecting resistance R_1.

iv) All the above set to zero.

Same is the case with equation (3) which applies to node 2.

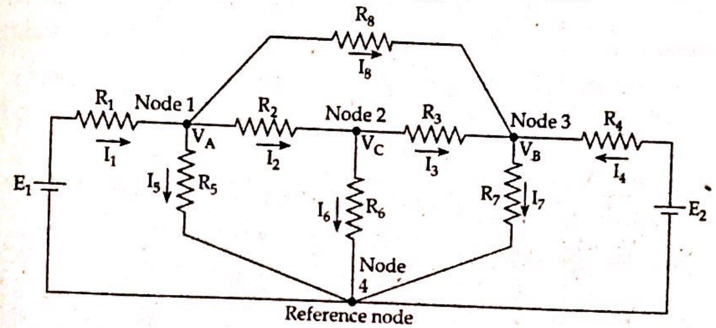


Figure 1.6:

Using conductance's instead of resistances, the above two equations may be written as,

$$V_A (G_1 + G_2 + G_4) - V_B G_2 - E_1 G_1 = 0 \tag{4}$$

$$V_B (G_2 + G_3 + G_5) - V_A G_2 - E_2 G_3 = 0 \tag{5}$$

To emphasize the procedure given above, consider the circuit of figure 1.6. The three node equation are,

$$V_A\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_5} + \frac{1}{R_8}\right) - \frac{V_C}{R_2} - \frac{V_B}{R_8} - \frac{E_1}{R_1} = 0 \quad \text{(node 1)}$$

$$V_B\left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_7} + \frac{1}{R_8}\right) - \frac{V_C}{R_3} - \frac{V_A}{R_8} - \frac{E_4}{R_4} = 0 \quad \text{(node 3)}$$

$$V_C\left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_6}\right) - \frac{V_A}{R_2} - \frac{V_B}{R_3} = 0 \quad \text{(node 2)}$$

After finding different node voltages, various currents can be calculated using ohm's law.

II) Second Case

Now, consider the case when a third battery of emf E_3 is connected between nodes 1 and 2 as shown in figure 1.7.

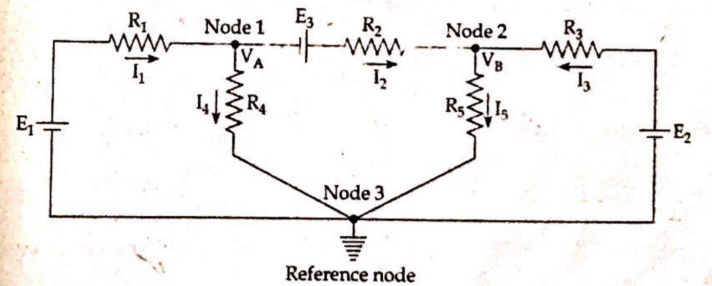


Figure 1.7:

It must be noted that as we travel from node 1 to node 2, we go from –ve terminal of E_3 to its +ve terminal. Hence, according to the sign convention, E_3 must be taken as positive. However, if we travel from node 2 to node 1, we go from the +ve to the –ve terminal of E_3. Hence when viewed from node 2, E_3 is taken negative.

For node 1

$$I_1 - I_4 - I_2 = 0$$

or, $I_1 = I_4 + I_2$ as per KCL

Now,

$$I_1 = \frac{E_1 - V_A}{R_1}$$

$$I_2 = \frac{V_A + E_3 - V_B}{R_2}$$

$$I_4 = \frac{V_A}{R_4}$$

$$\therefore \quad \frac{E_1 - V_A}{R_1} = \frac{V_A}{R_4} + \frac{V_A + E_3 - V_B}{R_2}$$

$$\text{or,} \quad V_A\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) - \frac{E_1}{R_1} + \frac{V_B}{R_2} + \frac{E_2}{R_2} = 0 \quad (1)$$

It is exactly the same expression as given under the first case discussed above except for the additional term involving E_3. This additional term is taken as $+\frac{E_3}{R_2}$ (and not as $-\frac{E_3}{R_2}$) because this third battery is so connected that when viewed from node 1, it represents a rise in voltage. Had it been connected the other way around the additional term would have been taken as $-\frac{E_3}{R_2}$.

For node 2

$$I_2 + I_3 - I_5 = 0$$

or, $I_2 + I_3 = I_5$ as per KCL

Now,

$$I_2 = \frac{V_A + E_3 - V_B}{R_2}, \; I_3 = \frac{E_2 - V_B}{R_3}, \; I_5 = \frac{V_B}{R_5}$$

$$\therefore \quad \frac{V_A + E_3 - V_B}{R_2} + \frac{E_2 - V_B}{R_3} = \frac{V_B}{R_5}$$

On simplifying, we get,

$$V_B\left(\frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{R_5}\right) - \frac{E_2}{R_3} - \frac{V_A}{R_2} - \frac{E_3}{R_2} = 0 \quad (2)$$

As seen, the additional terms is $-\frac{E_3}{R_2}$ (and not $+\frac{E_3}{R_2}$) because a viewed from this node, E_3 represents a fall in potential.

1.3.2 Nodal Analysis with Current Sources

Consider the network of figure 1.8 (a) which has two current sources and three nodes out of which 1 and 2 are independent ones whereas number 3 is the reference node. The given circuit has been redrawn for ease of understanding and is shown in figure 1.8 (b). The current directions have been taken on the assumption that,

i) Both V_1 and V_2 are positive with respective to the reference node. That is why their respective currents flow from nodes 1 and 2 to node 3.

ii) V_1 is positive with respect to V_2 because current has been shown flowing from node 1 to node 2.

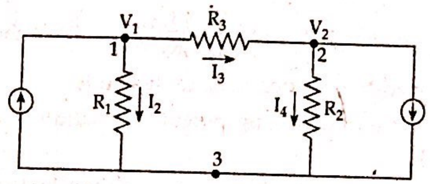


Figure 1.8 (a)

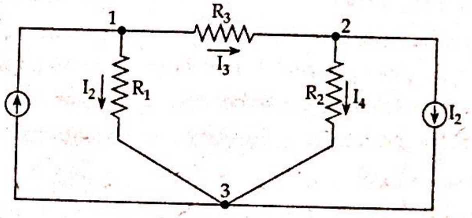


Figure 1.8 (b)

A positive result will confirm out assumption whereas a negative one will indicate that actual direction is opposite to that assumed.

Applying KCL to each node and using ohm's law to express branch currents in terms of node voltages and resistances.

Node 1:

$$I_1 - I_2 - I_3 = 0 \text{ or, } I_1 = I_2 + I_3$$

Now,

$$I_2 = \frac{V_1}{R_1} \text{ and } I_3 = \frac{V_1 - V_2}{R_3}$$

$$\therefore \quad I_1 = \frac{V_1}{R_1} + \frac{V_1 - V_2}{R_3}$$

$$\text{or,} \quad V_1\left(\frac{1}{R_1} + \frac{1}{R_3}\right) - \frac{V_2}{R_3} = I_1 \quad (1)$$

Node 2:

$$I_3 - I_2 - I_4 = 0$$

or, $I_3 = I_2 + I_4$

Now,

$$I_4 = \frac{V_2}{R_2} \text{ and } I_3 = \frac{V_1 - V_2}{-R_3} \text{ – as before}$$

$\therefore$ $\frac{V_1 - V_2}{R_3} = I_2 + \frac{V_2}{R_2}$

or, $V_2\left(\frac{1}{R_2} + \frac{1}{R_3}\right) - \frac{V_1}{R_3} = -I_1$ **(2)**

The above two equations can also be written by simple inspection. *For example*, equation (1) is represented by,

i) Product of potential V_1 and $\left(\frac{1}{R_2} + \frac{1}{R_3}\right)$ *i.e.*, sum of the reciprocals of the branch resistances connected to this node.

ii) Minus the ratio of adjoining potential V_2 and the interconnecting resistance R_3.

iii) All the above equated to the current supplied by the current source connected to this node.

This current is taken positive if flowing into the node and negative if flowing out of it (as per sign convention) same remarks apply to equation (ii) where I_2 has been taken negative because it flows away from node 2.

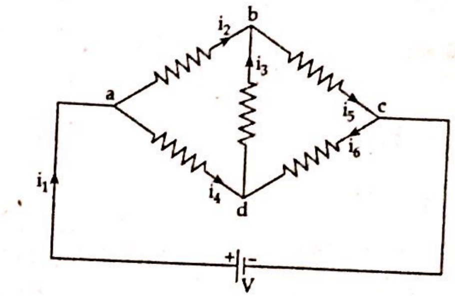
In terms of branch conductance's, the above two equations can be put as,

$$V_1 (G_1 + G_3) - V_2 G_3 = I_1$$

and, $V_2 (G_2 + G_3) - V_1 G_3 = -I_2$

SOLVED NUMERICAL EXAMPLES

1. Find the magnitude and direction of the unknown currents. Given; $i_1 = 20$ A, $i_2 = 12$ A and $i_5 = 8$ A.



Solution:

Applying KCL at node 'a',

$$-i_1 + i_2 + i_4 = 0$$

or, $i_4 = i_1 - i_2 = 20 - 12 = 8$ A

Applying KCL at node 'b',

$$-i_2 - i_3 + i_5 = 0$$

or, $i_3 = i_5 - i_2 = 8 - 12 = -4$ A

Applying KCL at node 'd',

$$-i_4 + i_3 - i_6 = 0$$

or, $i_6 = i_3 - i_4 = -4 - 8 = -12$ A

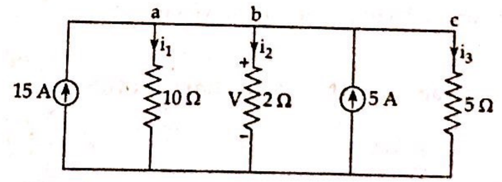
We can interpret as follows:

$\therefore$ $i_3 = -4$ A (from d to b) = 4 A (from b to d)

$\therefore$ $i_4 = 8$A (from a to d)

$\therefore$ $i_6 = -12$ A (from c to d) = 12 A (from d to c)

2. Find V. Also find the magnitudes and direction of the unknown currents through 10 Ω, 2 Ω and 5 Ω resistors.



Solution:

Applying KCL at node 'a'

$$-15 + i_1 + i_2 - 5 + i_3 = 0$$

or, $i_1 + i_2 + i_3 = 20$ **(1)**

From ohm's law,

$$i_2 = \frac{V}{2};\ i_1 = \frac{V}{10} \text{ and } i_3 = \frac{V}{5}$$

From equation (1), we have,

$$\frac{V}{10} + \frac{V}{2} + \frac{V}{5} = 20$$

or, $V + 5V + 2V = 200$

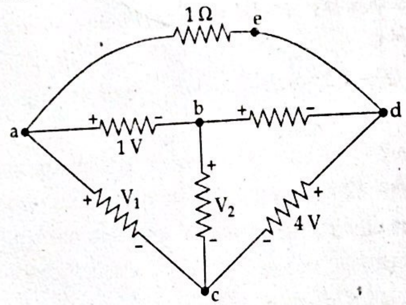
∴ $V = 25\ V$

Hence, $i_1 = \frac{V}{10} = \frac{25}{10} = 2.5\ A$

$$i_2 = \frac{V}{2} = \frac{25}{2} = 12.5\ A$$

$$i_3 = \frac{V}{5} = \frac{25}{5} = 5\ A$$

3. In the network of figure shown, find V_1 and V_2 using KVL.



Solution:

In loop 'abca' from KVL, we can write,

$1 + V_2 - V_1 = 0$

or, $V_1 - V_2 = 1$ (1)

In loop 'bcdb', using KVL,

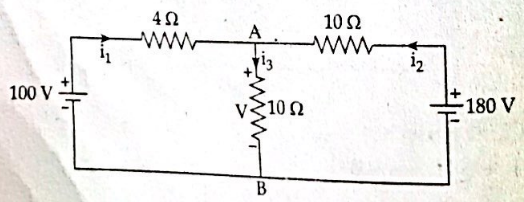
$-V_2 + 1 + 4 = 0$

or, $V_2 = 5\ V$

Replacing the value of V_2 in equation (1), we get,

$V_1 = 6\ V$

4. Find the voltage V in the circuit shown in figure.



Solution:

Considering 'B' as reference node, $V_B = 0$. Let V_A be the potential at node 'A'.

Obviously, $V_A - V_B = V$ *i.e.*, $V_A = V$

Using nodal analysis at node 'A', we get,

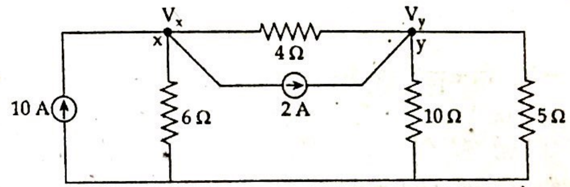
or, $\frac{V_A - 100}{4} + \frac{V_A}{10} + \frac{V_A - 180}{10} = 0$

or, $\frac{V - 100}{4} + \frac{V}{10} + \frac{V - 180}{10} = 0$

or, $V\left(\frac{1}{10} + \frac{1}{10} + \frac{1}{4}\right) = 43$

∴ $V = \frac{43 \times 10}{4.5} = 95.55\ V$

5. Find the node voltage V_x and V_y using nodal analysis.



Solution:

At node 'x', we have,

$$-10 + \frac{V_x}{6} + 2 + \frac{V_x - V_y}{4} = 0$$

or, $V_x\left(\frac{1}{4} + \frac{1}{6}\right) - \frac{V_y}{4} = 8$

or, $5\ V_x - 3\ V_y = 96$ (1)

Applying nodal analysis at 'y', we get,

or, $-2 + \frac{V_y - V_x}{4} + \frac{V_y}{10} + \frac{V_y}{5} = 0$

or, $-\frac{V_x}{4} + V_y\left(\frac{1}{4} + \frac{1}{5} + \frac{1}{10}\right) = 2$

or, $5V_x - 11\ V_y = -40$ (2)

Solving equation (1) and (2), we get,

∴ $V_x = 29.4\ V$

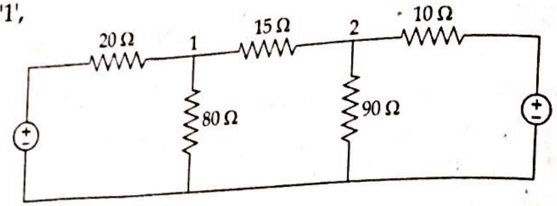
and, $V_y = 17\ V$

6. Find current in the 15 Ω resistor using nodal analysis

Solution:

Let us first designate the nodes '1' and '2' in figure and assume nodal voltages to be V_1 and V_2 respectively.

At node '1',



or, $\frac{V_1 - 400}{20} + \frac{V_1}{80} + \frac{V_1 - V_2}{15} = 0$

or, $V_1\left(\frac{1}{20} + \frac{1}{80} + \frac{1}{15}\right) - \frac{V_2}{15} = 20$

or, $\frac{31}{240} V_1 - \frac{1}{15} V_2 = 20$ **(1)**

Similarly, using nodal analysis at node '2',

$$\frac{V_2 - 200}{10} + \frac{V_2}{90} + \frac{V_2 - V_1}{15} = 0$$

or, $-\frac{V_1}{15} + V_2\left(\frac{1}{10} + \frac{1}{15} + \frac{1}{90}\right) V_2 = 20$

or, $-\frac{1}{15} V_1 + \frac{16}{90} V_2 = 20$ **(2)**

Solving equation (1) and (2), we get,

$\therefore$ $V_1 = 264.88$ V

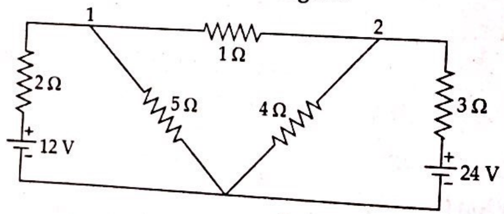
$\therefore$ $V_2 = 211.33$ V

Hence, current in the 15 Ω resistor is,

$$I_{15} = \frac{V_1 - V_2}{15} = \frac{264.88 - 211.33}{15} = 3.57 \text{ A}$$

This current is directed from node '1' to node '2'.

7. **Obtain the current through the 1 Ω resistor using node voltage method for the circuit shown in figure.**



Solution:

At node '1', we have,

$$\frac{V_1 - 12}{2} + \frac{V_1}{5} + \frac{V_1 - V_2}{1} = 0$$

or, $V_1\left(\frac{1}{5} + \frac{1}{2} + 1\right) - V_2 = 6$

or, $17 V_1 - 10 V_2 = 60$ **(1)**

At node '2', we have,

or, $\frac{V_2 - 24}{3} + \frac{V_2}{4} + \frac{V_2 - V_1}{1} = 0$

or, $-V_1 + V_2\left(\frac{1}{3} + \frac{1}{4} + 1\right) = 8$

or, $-V_1 + \frac{19}{12} V_2 = 8$

or, $-V_1 + \frac{19}{12} V_2 = 8$

or, $-12 V_1 + 19 V_2 = 96$ **(2)**

On solving equation (1) and (2), we get,

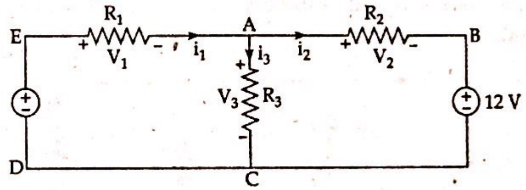
$\therefore$ $V_1 = 10.35$ V and, $V_2 = 11.6$ V

Hence the current through 1 Ω resistor is,

$$I_1 = \frac{V_2 - V_1}{1} = \frac{11.6 - 10.35}{1} = 1.25 \text{ V}$$

(Directed from node '2' and to node '1'.)

8. **Consider the circuit shown in figure. Find each branch current and voltage across each branch when $R_1 = 8$ Ω, $V_2 = -10$ volts, $i_3 = 2$ A and $R_3 = 1$ Ω. Also find R_2.**



Solution:

Applying KCL at node A, we get,

$$i_1 = i_2 + i_3$$

Using ohms law for R_3, we get,

$$V_3 = R_3 i_3 = 1 \times 2 = 2 \text{ V}$$

Applying KVL for the loop EACDE, we get,

$$-10 + V_1 + V_3 = 0$$

or, $V_1 = 10 - V_3 = 8$ V

Ohm's law for R_1 is,

$$V_1 = i_1 R_1$$

or, $i_1 = \frac{V_1}{R_1} = 1$ A

Hence, $i_2 = i_1 - i_3 = 1 - 2 = -1$ A

From the circuit,

$$V_2 = R_2 i_2$$

$\therefore$ $R_2 = \frac{V_2}{i_2} = \frac{-10}{-1} = 10$ Ω

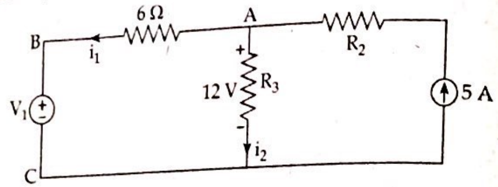
9. **For the circuit shown in figure, find i_1 and V_1 given $R_3 = 6\ \Omega$.**

Solution:

Applying KCL at node A, we get,

$-i_1 - i_2 + 5 = 0$

From Ohm's law,



$12 = i_2R_3$

$\therefore \quad i_2 = \frac{12}{R_3} = \frac{12}{6} = 2\ A$

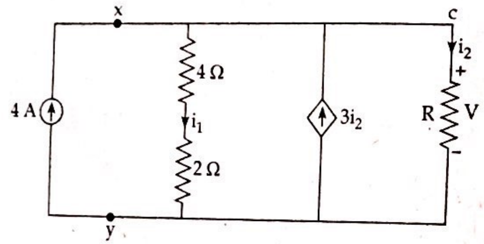
Hence, $i_1 = 5 - i_2 = 3A$

Applying KVL clockwise to the loop CBAC, we get,

$-V_1 - 6\ i_1 + 12 = 0$

$\therefore \quad V_1 = 12 - 6i_1 = 12 - 6 \times 3 = -6$ volts

10. **Find the current i_2 and voltage V for the resistor R in figure when R = 16 Ω.**



Solution:

Applying KCL at node x, we get,

$4 - i_1 + 3_{i_2} - i_2 = 0$

Also,

$i_1 = \frac{V}{4+2} = \frac{V}{6}$

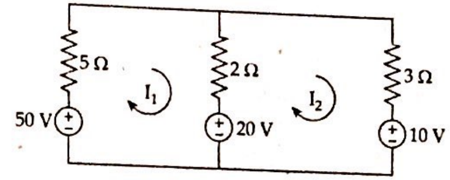
$i_2 = \frac{V}{R} = \frac{V}{16}$

Hence, $4 - \frac{V}{6} + 3 \times \frac{V}{16} - \frac{V}{16} = 0$

$\therefore \quad V = 96$ volts

and, $i_2 = \frac{V}{6} = \frac{96}{16} = 6\ A$

11. **From the mesh analysis find the current flow through a 50 V source in figure.**



Solution:

In loop 1, we have,

$-50 + 5\ i_1 + (i_1 - i_2)\ 2 + 20 = 0$

or, $\quad 7i_1 - 2i_2 - 30 = 0$ (1)

In loop 2, we have,

$3i_2 + 10 - 20 + (i_2 - i_1)_2 = 0$

or, $\quad i_1 = \frac{5}{2} i_2 - 5$ (2)

Replacing the value of i_1 from equation (2) to equation (1), we get,

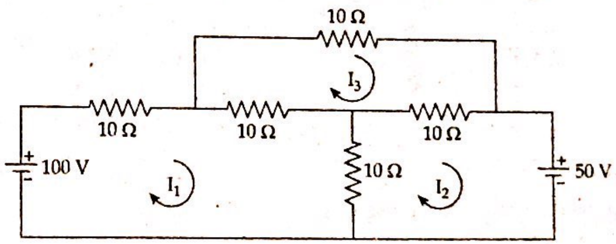
$7\left(\frac{5}{2} i_2 - 5\right) - 2i_2 - 30 = 0$

$\therefore \quad i_2 = 4.19\ A$

Thus, $i_1 = \frac{5}{2} \times 4.19 - 5 = 5.475\ A$

The current through the 50 V source is thus 5.4756 A

12. **Find the mesh currents in the figure using mesh current method.**



Solution:

Applying KVL in loop 1,

$-100 + 10\ (I_1 - I_2) + 10\ (I_1 - I_3) + 10\ I_1 = 0$

or, $\quad I_1\ (10 + 10 + 10) - 10\ I_2 - 10\ I_3 = 100$

or, $\quad 30\ I_1 - 10\ I_2 - 10\ I_3 = 100$

Applying KVL in loop 2,

$50 + 10\ (I_2 - I_3) + 10\ (I_2 - I_1) = 0$

or, $\quad -10\ I_1 + I_2\ (10 + 10) - 10\ I_3 = -50$

or, $\quad -10\ I_1 + 20\ I_2 - 10\ I_3 = -50$

Applying KVL in loop 3,

$$10 I_3 + 10 (I_3 - I_1) + 10 (I_3 - I_2) = 0$$

or, $-10 I_1 - 10 I_2 + I_3 (10 + 10 + 10) = 0$

or, $-10 I_1 - 10 I_2 + 30 I_3 = 0$

The above equations in matrix from can be written as,

$$\begin{bmatrix} 30 & -10 & -10 \\ -10 & 20 & -10 \\ -10 & -10 & 30 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 100 \\ -50 \\ 0 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 30 & -10 & -10 \\ -10 & 20 & -10 \\ -10 & -10 & 30 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ -50 \\ 0 \end{bmatrix}$$

or,

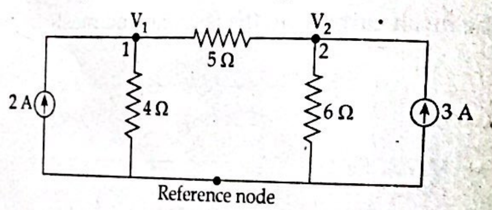
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{8000} \begin{bmatrix} 500 & 400 & 300 \\ 400 & 800 & 400 \\ 300 & 400 & 500 \end{bmatrix} \begin{bmatrix} 100 \\ -50 \\ 0 \end{bmatrix}$$

Hence, $I_1 = \dfrac{500 \times 100 - 400 \times 50}{8000} A = 3.75 A$

$$I_2 = \frac{400 \times 100 - 50 \times 800}{8000} A = 0 A$$

$$I_3 = \frac{300 \times 100 - 400 \times 50}{8000} A = 1.25 A$$

13. **In the circuit shown in figure, determine the voltage at nodes and 2 with respect to the reference point.**



Solution:

Applying nodal analysis at node 1,

$$\frac{V_1}{4} + \frac{V_1 - V_2}{5} - 2 = 0$$

or, $9 V_1 - 4 V_2 = 40$ (1)

Applying nodal analysis at node 2,

$$\frac{V_2 - V_1}{5} + \frac{V_2}{6} - 3 = 0$$

or, $-6 V_1 + 11 V_2 = 90$ (2)

Solving equations (1) and (2), we get,

$\therefore$ $V_1 = 10.667 V$

$\therefore$ $V_2 = 14 V$

BOARD EXAMINATION SOLVED QUESTIONS

1. **Write short notes on mesh and nodal analysis.**
[2013/Fall, 2018/Spring, 2019/S, 2020/Fall]

Solution: See the topic 1.2 for mesh analysis and 1.3 for nodal analysis.

2. **Write short notes on mesh analysis.** **[2016/Spring]**

Solution: See the topic 1.2.

3. **Write short notes on nodal analysis.** **[2018/Fall]**

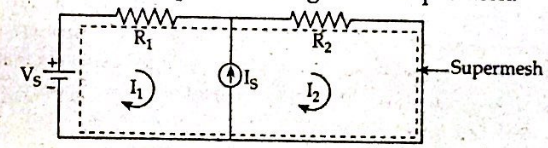
Solution: See the topic 1.3.

4. **Write short notes on supermesh analysis.** **[2019/Fall]**

Solution:

Supermesh and Supernode concept:

A supermesh occurs when a current source is contained between two essential meshes. The circuit is first treated as if the current sources is not there. This leads to one equation that incorporates two mesh currents. Once this equation is formed, an equation is needed that relates the two mesh current with the current source. This will be an equation where the current source is equal to one of the mesh currents minus the other. The following is a simple example of dealing with a supermesh.

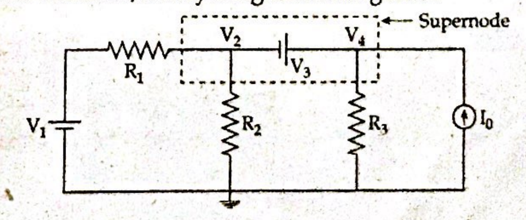


Loop 1 and loop 2 forms supermesh, applying KVL on supermesh,

$$+ V_s - I_1 R_1 - I_2 R_2 = 0 \quad (1)$$

and, Current, $I_s = I_2 - I_1$ (2)

A supernode is the combination of two extraordinary nodes (excluding the reference node) between which a voltage source exists. The voltage source may be of the independent or dependent type and the voltage source may include elements in parallel with it but not in series with it. If one of the two nodes of a supernode is a reference node, it is called a quasi supernode. If a supernode contains a resistor in parallel with the voltage source, the resistor will exercise no influence on the currents and voltages in the other parts of the circuit, and therefore, it may be ignored altogether.



Here,

$$\frac{V_2 - V_1}{R_1} + \frac{V_2}{R_2} + \frac{V_4}{R_3} = 0 \quad (1)$$

and, $V_4 - V_2 = V_3$ (2)

On solving equation (1) and (2); we can get required voltages. V_2 and V_4 and other parameters.

Supermesh analysis is a better technique instead of using mesh analysis in such a complex electric circuit, where two meshes have a current source as a common element. This is same where we use supernode circuit analysis instead nodal circuit analysis to simplify such a circuit where the assign supernode, completely enclosing the voltage source inside the supernode and reducing the number of none reference nodes by one for each voltage source. A supermesh is formed is formed when the adjacent meshes share a common current source and none of these (adjacent) meshes contains a current source in the outer loop.

☼☼☼

Chapter 2

CIRCUIT DIFFERENTIAL EQUATIONS (FORMULATIONS AND SOLUTIONS)

2.1 THE DIFFERENTIAL OPERATOR (P-OPERATOR)

Differential operator or P-operator is an operator which transforms the function into its derivative. P-operator is defined as $P \approx \frac{d}{dt}$.

If f(t) is the function, then $\frac{df}{dt}(t) \approx Pf(t)$

Similarly, second order P-operator can be written as,

$$\frac{d^2f(t)}{dt^2} \approx P^2 f(t)$$

In general, n^{th} operator P-operator

$$\frac{d^n f(t)}{dt^n} \approx P^n f(t)$$

We know,

$$v_R = i_R R \text{ (Resistor)}$$

$$v_L = i_L \frac{d_{iL}}{dt} \text{ (Inductor)}$$

$$v_C = \frac{1}{C}\int_{-\infty}^{t} i_C \, dt \text{ (Capacitor)}$$

In terms of P-operator,

$$v_R = i_R R \quad \text{(a)}$$

$$v_L = PL_{iL} \quad \text{(b)}$$

$$v_C = \frac{1}{PC} i_C \quad \text{(c)}$$

2.2 OPERATIONAL IMPEDANCE

Operational impedance is the ratio of voltage and current in terms of differential operator (P-operator).

From equation (a), (b) and (c), the operational impedance for inductor and capacitor are PL and $\frac{1}{PC}$ respectively. These are same as ohms law for the resistor except R is replaced by PL and $\frac{1}{PC}$ in case of inductor and capacitor respectively.

Consider a series R-L-C circuit as shown below,

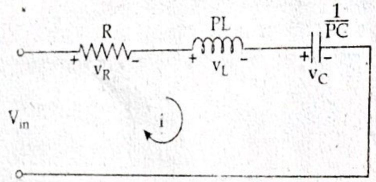


Figure 2.(r

From figure,

$$V_{in} = v_R + v_L + v_C = iR + iPL + i\frac{1}{PC} = i\left(R + PL + \frac{1}{PC}\right)$$

Hence operational impedance of above circuit is,

$$\left(R + PL + \frac{1}{PC}\right)$$

2.3 FORMULATION OF CIRCUIT DIFFERENTIAL EQUATIONS

Circuit differential equations may be obtained with or without using P-operator. For complicated network, using P-operator is a convenient method for formulating circuit differential equation.

2.4 COMPLETE RESPONSE (TRANSIENT AND STEADY STATE) OF FIRST ORDER DIFFERENTIAL EQUATIONS WITH OR WITHOUT INITIAL CONDITIONS

The response of the circuit to both an input and the initial conditions is called the complete response of the circuit. Thus the zero-input response and the zero state response are special case of the complete response.

Let us demonstrate that for the simple linear RC circuit considered, the complete response is the sum of the zero-input response and the zero-state response. Consider the circuit shown in figure 2.2, where the capacitor is initially charged; that is $v(0) = V_0 \neq 0$ and a current input is switched into the circuit at $t = 0$.

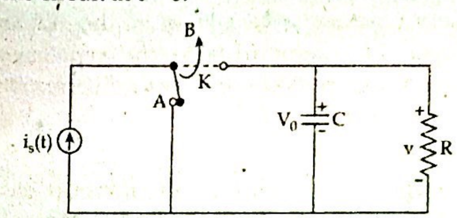


Figure 2.1: RC circuit with $v(0) = V_0$ is excited by a current source i_s (t). The switch K is flipped from A to B at t = 0.

By definition, the complete response is the waveform caused by both the input and the initial state V_0.

$$C\frac{dv}{dt} + Gv = i_s(t) \qquad t \geq 0$$

with $v(0) = V_0$

where, V_0 is the initial voltage of the capacitor. Let v_i be the zero-input response. By definition, it is the solution of,

$$C\frac{dv_i}{dt} + Gv_i = 0 \qquad t \geq 0$$

with $v_i(0) = V_0$.

Let, v_0 be the zero-state response. By definition, it is the solution of

$$C\frac{dv_0}{dt} + Gv_0 = i_s(t) \qquad t \geq 0$$

with $v_0(0) = 0$

From these four equations, we obtain, by addition,

$$C\frac{d}{dt}(v_i + v_0) + G(v_i + v_0) = i_s(t) \qquad t \geq 0$$

and, $v_i(0) + v_0(0) = V_0$

Hence complete response v is given by,

$$v(t) = v_i(t) + v_0(t) \qquad t \geq 0$$

i.e., the complete response v is the sum of the zero-input response v_i and the zero-state response v_0.

2.4.1 Various Responses

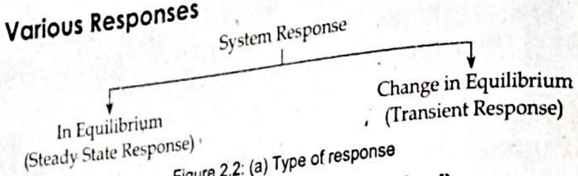


Figure 2.2: (a) Type of response

2.4.1.1 Transient Response (Transient Means Short Lived)

The value of voltage and current during the transient period are known as the transient responses. It is also defined as the part of the total time response that goes as time becomes large. It depends upon the network elements alone and independent of the forcing function (source).

The complementary function is the solution of the differential equation with forcing function set to zero and hence the complementary function represents the source free response or simply free response or natural response or transient response.

2.4.1.2 Steady State Response

The values of voltage and current after the transient has died out are known as the steady state responses. It is also defined as the part of the total time response which remains after the transient has passed. It depends on both the network elements and forcing function.

The particular integral represents the forced response or steady state response. It satisfies the differential equation but not the initial conditions.

NOTE: The total or complete response of a network is the sum of the transient response and the steady state response or the sum of the natural and forced responses.

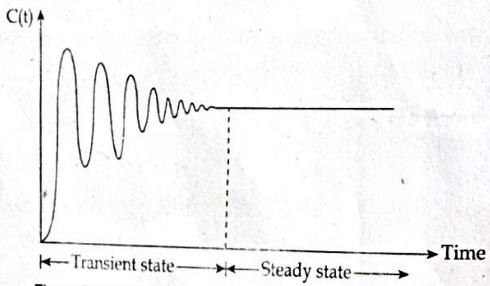


Figure 2.2: (b) Transient and steady state response analysis

Mathematically,

$$C(t) = C_{tr}(t) + C_{ss}(t)$$

where, $C_{tr}(t)$ is the transient response.

$C_{ss}(t)$ is the steady state response.

2.4.1.3 Zero Input Response

The values of voltage and current that result from initial conditions when the excitation (input) or forcing function is zero are known as zero input response.

2.4.1.4 Zero State Response

The value of voltage and current for an excitation which is applied when all initial conditions are zero are known as zero state responses. Such a network is also said to be at rest or initially relaxed.

2.5 CHARACTERISTICS OF VARIOUS NETWORK ELEMENTS

An electric circuit or electric network is an interconnection of electrical elements linked together in a closed path so that an electric current may continuously flow. Alternatively, an electrical network is essentially a pipe-line that facilitates the transfer of charge from one point to another.

a) Resistor

A resistor is a passive two-terminal component that implements electrical resistance as a circuit element. Resistance is the physical property of an element or device that resist the flow of current; it is represented by the symbol R. The resistor shown in figure 2.3 define a linear proportionality relationship between v(t) and i(t), namely,

$$v(t) = Ri(t)$$

$$i(t) = Gv(t) \qquad \left(\because \text{Conductance, } G = \frac{1}{R}\right)$$

where, R is given in ohms and G in mho.

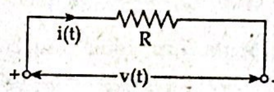


Figure 2.3: Resistor

b) Capacitor

A capacitor is a two terminal element that is a model of a device consisting of two conducting plates separated by a dielectric material. Capacitance is a measure of the ability of a device to store energy in the form of an electric field.

For the capacitor shown in figure 2.4, the v-i relationships are

$$i(t) = C\frac{dv(t)}{dt}$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t) = \frac{1}{C}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{C}\int_{0}^{0} i(t)\,dt = v_C(0^-) + \frac{1}{C}\int_{0}^{t} i(t)\,dt$$

(a) Capacitor (b) Capacitor with initial voltage $v_C(0^-)$

Figure 2.4:

where, C is given in Farads. The initial value $v_C(0^-)$ is the voltage across the capacitor just before the switching action. It can be regarded as the independent voltage source as shown in figure 2.4 (b). Also $v_C(0^-) = v_C(0^+)$ for all excitations except impulses and derivatives of impulses.

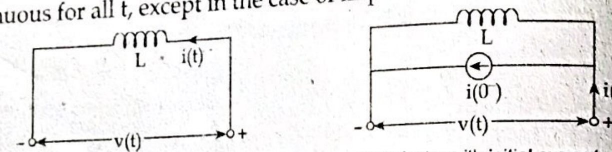
c) **Inductor**

An inductor, also called a coil, choke or reactor is a passive two terminal electrical component that stores energy in a magnetic field when electric current flows through it. The inductor in figure 2.5 (a) describes a dual relationship between voltage and current when compared to a capacitor. The v-i relationships are,

$$v(t) = L\frac{di}{dt}$$

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt = \frac{1}{L}\int_{-\infty}^{t} v(t)\,dt + \frac{1}{L}\int_{0^-}^{t} v(t)\,dt = i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} v(t)\,dt$$

where, L is given in Henrys. The initial current $i_L(0^-)$ can be regarded as an independent current source as shown in 2.5 (b). As is true for the voltage across the capacitor, the current through the inductor is similarly continuous for all t, except in the case of impulse excitations.



(a) Inductor (b) Inductor with initial current

Figure 2.5:

When the network elements are interconnected, the resulting i-v equations are integrodifferential equations relating the excitation (voltage or current sources) to the response (the voltages and currents of the elements). There are basically two ways to write these network equations. The first way is to use mesh equations and the second is node equations. Mesh equations are based upon Kirchhoff's voltage law. On the mesh basis, we establish a fictitious set of loop currents with a given reference direction, and write the equations for the sum of voltages around the loops. As the reader might recall from his previous studies, if the number of branches in the network is B, and if the number of nodes is N, the number of independent loop equations for the network is B – N + 1. We must, in addition, choose the mesh currents such that at least one mesh current passes through every element in the network.

2.6 INITIAL CONDITIONS IN NETWORKS

There are many reasons for studying initial and final conditions. The most important reason is that the initial and final conditions evaluate the arbitrary constants that appear in the general solution of a differential equation. In this section, we concentrate on finding the change in selected variables in a circuit when a switch is thrown open from closed position or vice versa. The time of throwing the switch is considered to be t = 0 and we want to determine the value of the variable at $t = 0^-$ and at $t = 0^+$, immediately before and after throwing the switch. Thus a switched circuit

is an electrical circuit with one or more switches that open or close at time t = 0. We are very much interested in the change in currents and voltages of energy storing elements after the switch is thrown since these variables along with the source will dictate the circuit behaviour for t > 0.

Initial conditions in a network depend on the past history of the circuit (before $t = 0^-$) and structure of the network at $t = 0^+$, (after switching). Past history will show up in the form of capacitor voltages and inductor currents.

2.6.1 Initial and Final Conditions for Resistor

The cause-effect relation for an ideal resistor is given by v = Ri. From this equation, we find out the current through a resistor will change instantaneously if the voltage changes instantaneously. Similarly, voltage will change instantaneously if current changes instantaneously.

2.6.2 Initial and Final Conditions for Inductor

The switch is closed at t = 0. Hence, $t = 0^-$ corresponds to the instant when the switch is just open and $t = 0^+$ corresponds to the instant when the switch is just closed.

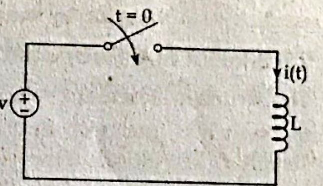


Figure 2.6: Circuit for explaining switching action of an inductor

The expression for current through the inductor is given by,

$$i = \frac{1}{L}\int_{-\infty}^{t} v\,dt = \frac{1}{L}\int_{-\infty}^{0^-} v\,dt + \frac{1}{L}\int_{0^-}^{t} v\,dt$$

$$\therefore \quad i(t) = i(0^-) + \frac{1}{L}\int_{0^-}^{t} v\,dt$$

Putting $t = 0^+$ on both sides, we get,

$$i(0^+) = i(0^-) + \frac{1}{L}\int_{0^-}^{0^+} v\,dt$$

$$\therefore \quad i(0^+) = i(0^-)$$

The above equation means that the current in an inductor cannot change instantaneously. Consequently, if $i(0^-) = 0$, we get $i(0^+) = 0$. This means that at $t = 0^+$, inductor will act as an open circuit, irrespective of the voltage across the terminals. If $i(0^-) = I_0$, then $i(0^+) = I_0$. In this case at $t = 0^+$, the inductor can be thought of as a current source of I_0 A. The equivalent circuits of an inductor at $t = 0^+$ is shown in figure 2.7.

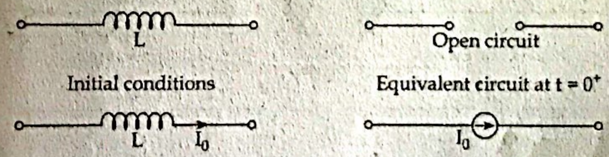


Figure 2.7: Initial-condition equivalent circuits of an inductor

The final condition equivalent circuit of an inductor is derived from the basic relationship,

$$v = L\frac{di}{dt}$$

Under steady condition, $\frac{di}{dt} = 0$. This means, $v = 0$ and hence L acts as short at $t = \infty$ (final or steady state). The final condition equivalent circuits of an inductor is shown in figure 2.8.

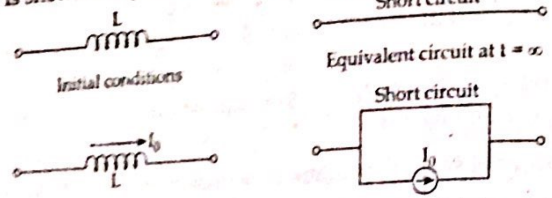


Figure 2.8: The final-condition equivalent circuits of an inductor

2.6.3 Initial and Final Conditions for Capacitor

The switch is closed at $t = 0$. Hence, $t = 0^-$ corresponds to the instant when the switch is just open and $t = 0^+$ corresponds to the instant when the switch is just closed. The expression for voltage across the capacitor is given by,

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,dt$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{0^-} i\,dt + \frac{1}{C}\int_{0^-}^{t} i\,dt$$

$$\therefore \quad v(t) = v(0^-) + \frac{1}{C}\int_{0^-}^{t} i\,dt$$

Evaluating the expression at $t = 0^+$, we get,

$$v(0^+) = v(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i\,dt$$

$$\therefore \quad v(0^+) = v(0^-)$$

Thus the voltage across a capacitor cannot change instantaneously.

If $v(0^-) = 0$, then $v(0^+) = 0$. This means that a $t = 0^+$, capacitor (acts as short circuit. Conversely, if $v(0^-) = \frac{q_0}{C}$ then $v(0^+) = \frac{q_0}{C}$.

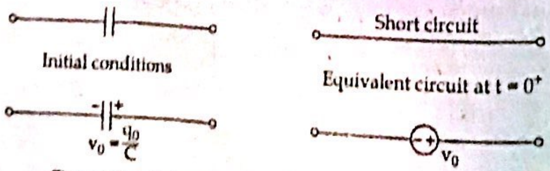


Figure 2.9: Initial-condition equivalent circuits of a capacitor

The final condition equivalent network is derived from the basic relationship, $i = C\frac{dv}{dt}$ under steady state condition, $\frac{dv}{dt} = 0$. This is, at $t = \infty$, $i = 0$. This means that at $t = \infty$ or in steady state, capacitor C acts as an open circuit. The final condition equivalent circuits of a capacitor is shown in figure 2.10.

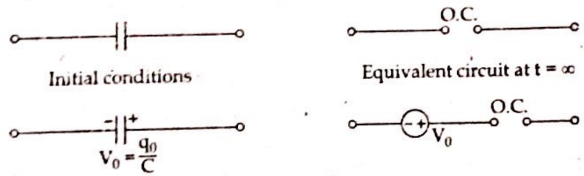


Figure 2.10: Final condition equivalent circuits of a capacitor

2.7 PROCEDURE FOR EVALUATING INITIAL CONDITIONS

There is no unique procedure that must be followed in solving for initial conditions. We usually solve for initial values of currents and voltages and then solve for the derivatives. For finding initial values of currents and voltages, an equivalent network of the original network at $t = 0^+$ is constructed according to the following rules.

i) Replace all inductors with open circuit or with current sources having the value of current flowing at $t = 0^+$.

ii) Replace all capacitor with short circuits or with a voltage source of value $v_0 = \frac{q_0}{C}$ if there is an initial charge.

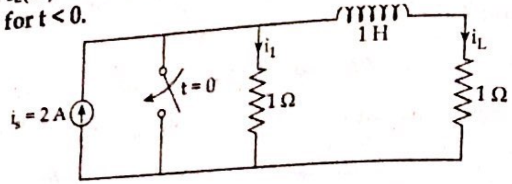
iii) Resistors are left in the network without any changes.

The equivalent circuit for the three parameters (R, L and C) at $t = 0^+$ and $t = \infty$ are shown in table below.

Elements with initial conditions	Equivalent circuit at t = 0+	Equivalent circuit at t = ∞
R	R	R
L	O.C.	S.C.
L, I_0	I_0	S.C., I_0
C	S.C.	O.C.
V_0	V_0	V_0 O.C.

SOLVED NUMERICAL EXAMPLES

1. **Find $i_L(0^+)$ and $i_1(0^+)$ of the circuit shown. The circuit is in steady state for $t < 0$.**



Solution:

The symbol for the switch implies that it is open at $t = 0^-$ and then closed at $t = 0^+$. The circuit is in steady state with the switch open. This means that $t = 0^-$, inductor L is short.

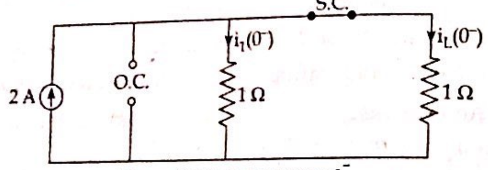


Figure: Original circuit at $t = 0^-$

Using the current division rule,

$$i_L(0^-) = \frac{2 \times 1}{1 + 1} = 1A$$

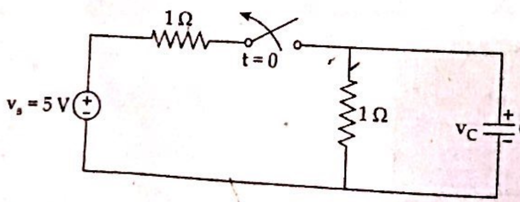
Since the current in an inductor cannot change instantaneously, we have,

$$i_L(0^+) = i_L(0^-) = 1A$$

At, $t = 0^-$, $i_1(0^-) = 2 - 1 = 1A$

But the current in a resistor can change instantaneously at $t = 0^+$, the switch is just closed, the voltage across R_1 will be equal to zero because of the switch being short circuited and hence $i_1(0^+) = 0$. Thus the current in the resistor changes abruptly from 1 A to 0 A.

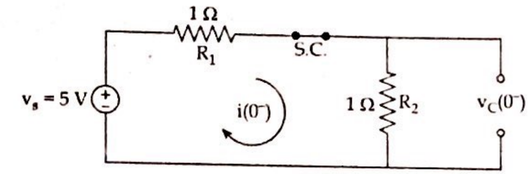
2. **Find $v_C(0^+)$. Assume that the switch was in closed state for a long time.**



Solution:

The symbol for the switch implies that it is closed at $t = 0^-$ and then opens at $t = 0^+$. Since the circuit is in steady state with the switch closed, the capacitor is represented as an open circuit at $t = 0^-$. The equivalent circuit

at $t = 0^-$ is shown in figure.



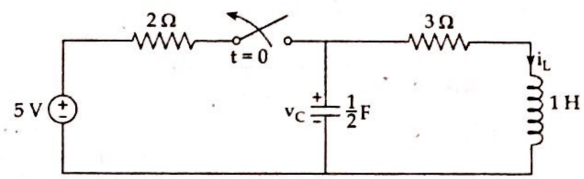
$$v_C(0^-) = i(0^-)\, R_2$$

Using the principle of voltage divider,

$$v_C(0^-) = \frac{V_S}{R_1 + R_2} R_2 = \frac{5 \times 1}{1 + 1} = 2.5 \text{ V}$$

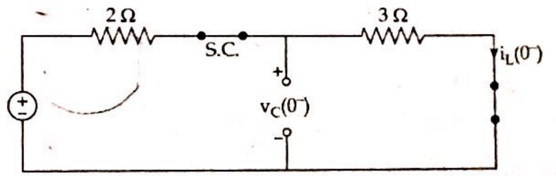
Since the voltage across a capacitor cannot change instant aneously, we have $v_C(0^+) = v_C(0^-) = 2.5$ V

3. **Find $i_L(0^+)$ and $v_C(0^+)$. The circuit is in steady state with the switch in closed condition.**



Solution:

The symbol for the switch implies it is closed at $t = 0^-$ and then opens at $t = 0^+$. In order to find $V_C(0^-)$ and $i_L(0^-)$ we replace the capacitor by an open circuit and the inductor by a short circuit, because in the steady steady state L acts as a short circuit and C as an open circuit.



$$i_L(0^-) = \frac{5}{2 + 3} = 1A.$$

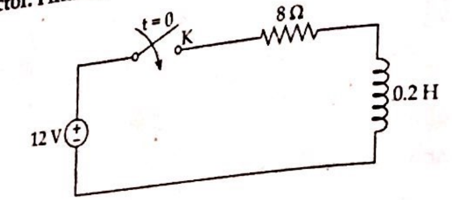
Using the voltage divider principle, we have,

$$v_C(0^-) = \frac{5 \times 3}{3 + 2} = 3 \text{ V}$$

so, $v_C(0^+) = v_C(0^-) = 3$ V

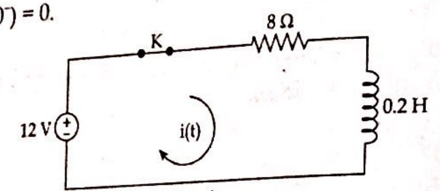
and, $i_L(0^+) = i_L(0^-) = 1$ A

4. In the given network, K is closed at $t = 0$ with zero current in the inductor. Find the values of $i, \frac{di}{dt}, \frac{d^2i}{dt}$ at $t = 0^+$ if $R = 8\ \Omega$ and $L = 0.2$ H.



Solution:

The symbol for the switch implies that it is open at $t = 0^-$ and then closes at $t = 0^+$. Since the current i through the inductor at $t = 0^-$ is zero, it implies that $i(0^+) = i(0^-) = 0$.



Applying KVL clockwise to the circuit,

$$Ri + L\frac{di}{dt} = 12$$

or, $$8i + 0.2\frac{di}{dt} = 12 \quad (1)$$

At, $t = 0^+$,

$$8i(0^+) + 0.2\frac{di(0^+)}{dt} = 12$$

or, $$8 \times 0 + 0.2\frac{di(0^+)}{dt} = 12$$

$\therefore$ $$\frac{di}{dt}(0^+) = \frac{12}{0.2} = 60\ \text{A/sec}$$

Differentiating equation (1) with respect to t, we get,

$$8\frac{di}{dt} + 0.2\frac{d^2i}{dt^2} = 0$$

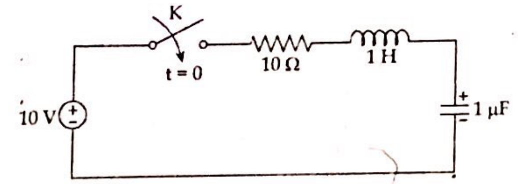
At $t = 0^+$, the above equation becomes,

$$8\frac{di(0^+)}{dt} + 0.2\frac{d^2i(0^+)}{dt^2} = 0$$

or, $$80 \times 60 + 0.2\frac{d^2i(0^+)}{dt^2} = 0$$

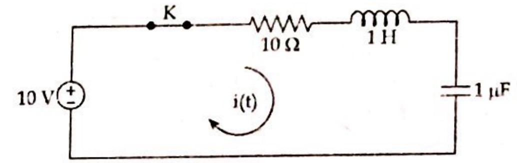
$\therefore$ $$\frac{d^2i(0^+)}{dt^2} = -2{,}400\ \text{A/sec}^2$$

5. In the network shown in figure below, the switch is closed at $t = 0$. Determine $i, \frac{di}{dt}, \frac{d^2i}{dt^2}$ at $t = 0^+$.



Solution:

The symbol for the switch implies that it is open at $t = 0^-$ and then closes at $t = 0^+$. Since there is no current through the inductor at $t = 0^-$, it implies that $i(0^+) = i(0^-) = 0$.



Applying KVL clockwise for the circuit,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(t)\,dt = 10 \quad (1)$$

or, $$Ri + L\frac{di}{dt} + v_C(t) = 10 \quad (2)$$

Putting $t = 0^+$ in equation (2), we get,

$$Ri(0^+) + L\frac{di(0^+)}{dt} + v_C(0^+) = 10$$

or, $$R \times 0 + L\frac{di(0^+)}{dt} + 0 = 10$$

$\therefore$ $$\frac{di(0^+)}{dt} = \frac{10}{L} = \frac{10}{1} = 10\ \text{A/sec}$$

Differentiating equation (1) with respect to t, we get,

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i(t)}{C} = 0$$

At $t = 0^+$,

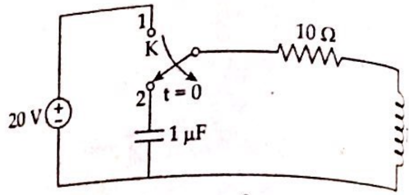
$$R\frac{di(0^+)}{dt^2} + L\frac{d^2i(0^+)}{dt^2} + \frac{i(0^+)}{C} = 0$$

or, $$R \times 10 + L\frac{d^2i(0^+)}{dt^2} + \frac{0}{C} = 0$$

or, $$10 \times 10 + 1 \times \frac{d^2i(0^+)}{dt^2} + 0 = 0$$

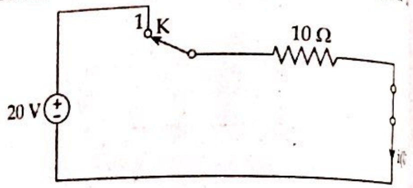
$\therefore$ $$\frac{d^2i(0^+)}{dt^2} = -100 \text{ A/sec}^2$$

6. Refer the circuit shown in figure. The switch K is changed from position 1 to position 2 at t = 0. Steady-state condition having been reached at position 1. Find the values of i, $\frac{di}{dt}$, $\frac{d^2i}{dt^2}$ at $t = 0^+$.



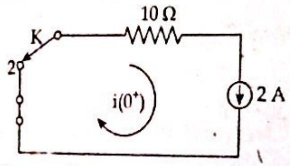
Solution:

The symbol for switch K implies that it is in position 1 at $t = 0^-$ and position 2 at $t = 0^+$. Under steady state condition, inductor acts as a short circuit. Hence at $t = 0^-$, the circuit diagram is as shown in figure.

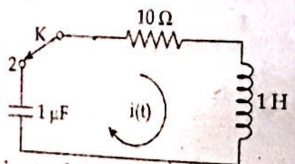


$$i(0^-) = \frac{20}{10} = 2\text{A}$$

Since the current through an inductor cannot change instantaneously, $i(0^+) = i(0^-) = 2$ A. Since there is no initial charge on the capacitor, $v_C(0^-) =$ Since the voltage across a capacitor cannot change instantaneously, $v_C(0^+) = v_C(0^-) = 0$. Hence at $t = 0^+$, the circuit diagram is,



For $t \geq 0^+$, the circuit diagram is,



Applying KVL clockwise to the circuit,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{0^+}^{t} i(t)\,dt = 0 \quad (1$$

$$Ri(t) + L\frac{di(t)}{dt} + v_C(t) = 0 \quad (2$$

At $t = 0^+$ equation (1) becomes,

$$Ri(0^+) + L\frac{di(0^+)}{dt} + v_C(0^+) = 0$$

or, $$10 \times 2 + 1\frac{di(0^+)}{dt} + 0 = 0$$

or, $$20 + \frac{di(0^+)}{dt} = 0$$

$\therefore$ $$\frac{di(0^+)}{dt} = -20 \text{ A/sec}$$

Differentiating equation (2) with respect to t, we get,

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$$

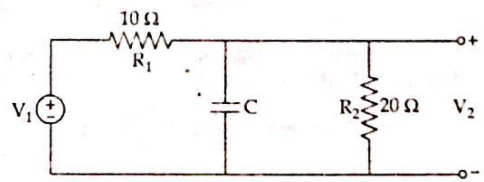
At $t = 0^+$, we get,

$$R\frac{di(0^+)}{dt} + L\frac{d^2i(0^+)}{dt^2} + \frac{i(0^+)}{C} = 0$$

or, $$10 \times (-20) + 1\frac{d^2i(0^+)}{dt^2} + \frac{2}{C} = 0$$

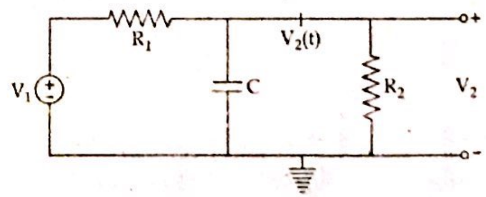
$\therefore$ $$\frac{d^2i(0^+)}{dt^2} = -2 \times 10^6 \text{ A/sec}^2$$

7. In the network shown in figure below, $V_1(t) = e^{-t}$ for $t \geq 0$ and is zero for all $t < 0$. If the capacitor is initially uncharged, determine the value of $\frac{d^2V_2}{dt^2}$ and $\frac{d^3V_2}{dt^3}$ at $t = 0^+$.



Solution:

Since the capacitor is initially uncharged, $V_2(0^+) = 0$



Applying KCL at node $V_2(t)$,

$$\frac{V_2(t) - V_1(t)}{R_1} + C\frac{dV_2(t)}{dt} + \frac{V_2(t)}{R_2} = 0$$

or, $$\left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_2(t) + C\frac{dV_2(t)}{dt} = \frac{V_1(t)}{R_1}$$

or, $$0.15\,V_2 + 0.05\frac{dV_2}{dt} = 0.1\,e^{-t} \quad (1)$$

Putting $t = 0^+$ we get

$$0.15\,V_2(0^+) + 0.05\frac{dV_2(0^+)}{dt} = 0.1$$

or, $$0.15 \times 0 + 0.05\frac{dV_2(0^+)}{dt} = 0.1$$

$\therefore$ $$\frac{dV_2}{dt} = \frac{0.1}{0.05} = 2 \text{ volts/sec}$$

Differentiating equation (1) with respect to t, we get,

$$0.15\frac{dV_2}{dt} + 0.05\frac{d^2V_2}{dt^2} = -0.1\,e^{-t} \quad (2)$$

Putting $t = 0^+$ in equation (2), we get,

$$\frac{d^2V_2}{dt^2} = \frac{-0.1 - 0.3}{0.05} = -8 \text{ volts/sec}^2$$

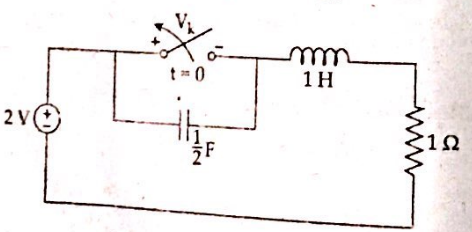
Again differentiating equation (2) with respect to t, we get,

$$0.15\frac{d^2V_2}{dt^2} + 0.05\frac{d^3V_2}{dt^3} = 0.1\,e^{-t}$$

Putting $t = 0^+$,

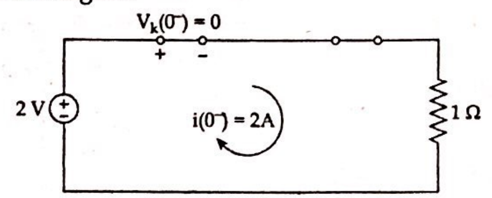
$$\frac{d^3V_2(0^+)}{dt^3} = \frac{0.1 + 1.2}{0.05} = 26 \text{ volts/sec}^3$$

8. **Refer the circuit shown in figure. The circuit is in steady state with switch K is closed. At $t = 0$, the switch is opened. Determine the voltage across the switch, V_K and $\frac{dV_k}{dt}$ at $t = 0^+$.**



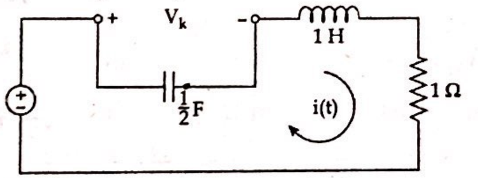
Solution:

The switch remains closed at $t = 0^-$ and open at $t = 0^+$. Under steady state condition, inductor acts as a short circuit and hence the circuit diagram at $t = 0^-$ is as shown in figure.



Hence, $V_k(0^+) = V_k(0^-) = 0$ V

For $t \geq 0^+$, the circuit diagram is,



$$i(t) = C\frac{dV_k}{dt}$$

At $t = 0^+$, we get

$$i(0^+) = C\frac{dV_k(0^+)}{dt}$$

Since the current through an inductor cannot change instantaneously, we get,

$$i(0^+) = i(0^-) = 2A$$

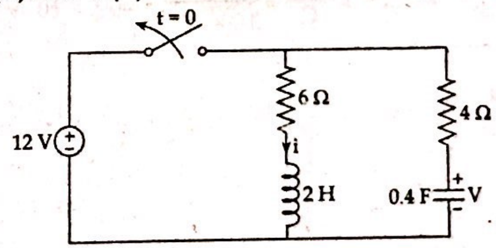
Hence, $$2 = C\frac{dV_k(0^+)}{dt}$$

$\therefore$ $$\frac{dV_k(0^+)}{dt} = \frac{2}{C} = \frac{2}{\left(\frac{1}{2}\right)} = 4 \text{ V/sec.}$$

9. **For the circuit shown, find,**

i) $i(0^+)$ **and** $v(0^+)$ **ii)** $\frac{di(0^+)}{dt}$ **and** $\frac{dv(0^+)}{dt}$

iii) $i(\infty)$ **and** $V(\infty)$

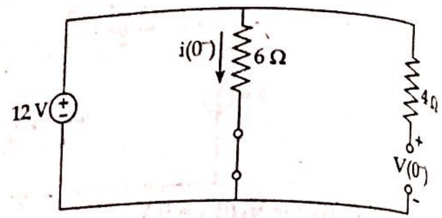


Solution:

i) From the symbol of the switch, we find that at $t = 0^-$, the switch is closed and at $t = 0^+$, it is open. At $t = 0^-$, the circuit has reached steady state so that the equivalent circuit is as shown in figure.

$i(0^-) = \frac{12}{6} = 2\text{ A}$.

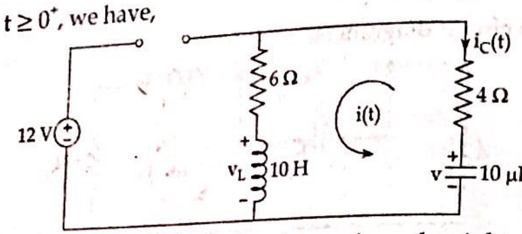
$v(0^-) = 12\text{ V}$



Hence, $i(0^+) = i(0^-) = 2\text{ A}$

$\therefore \quad v(0^+) = v(0^-) = 12\text{V}$

ii) For $t \geq 0^+$, we have,



Applying KVL anticlockwise to the mesh on the right,

$$v_L(t) - v(t) + 10i(t) = 0$$

Putting $t = 0^+$, we get,

$$v_L(0^+) - v(0^+) + 10i(0^+) = 0$$

$$v_L(0^+) - 12 + 10 \times 2 = 0$$

$$\therefore \quad v_L(0^+) = -8\text{ V}$$

The voltage across the inductor is given by,

$$v_L = L\frac{di}{dt}$$

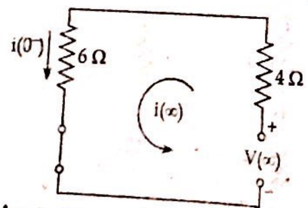
or, $$\frac{di(0^+)}{dt} = \frac{1}{L}v_L(0^+) = \frac{1}{10} \times (-8) = -0.8\text{ A/sec}$$

Similarly, the current through the capacitor is,

$$i_C = C\frac{dv}{dt}$$

$$\therefore \quad \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{i(0^+)}{C} = \frac{-2}{10 \times 10^{-6}} = -0.2 \times 10^6\text{ V/s}$$

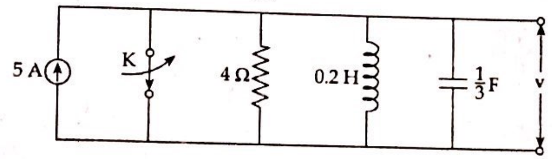
iii)



As t approaches infinity, the switch is open and the circuit has attained steady state. $i(\infty) = 0$ and $V(\infty) = 0$.

BOARD EXAMINATION SOLVED QUESTIONS

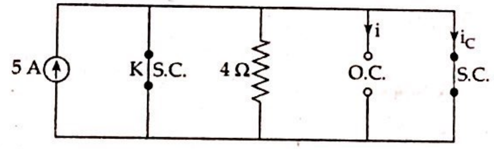
1. **For the network shown in figure below, the switch K is opened at t = 0. Find $i(0^+)$, $\frac{di(0^+)}{dt}$, $\frac{d^2i(0^+)}{dt^2}$.** **[2019/Spring]**



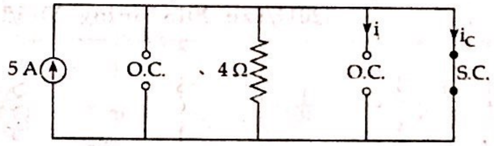
Solution:

The current (i) is not specified, so assuming it to be current through inductor.

At $t = 0^-$, the resulting circuit is,



Clearly, we see that $i(0^-) = 0$ A due to short circuit due to switch K. At $t = 0^+$, the equivalent circuit is,



We know,

For inductor, $i(0^+) = i(0^+) = 0\text{ A}$

For capacitor, $v(0^-) = 0\text{ V}$

Since this is parallel circuit, we can write,

$$v = v_{4\Omega} = v_L$$

so, $$v_L(0^+) = v(0^+) = 0\text{ V}$$

We have, $v_L(0^+) = L\frac{di(0^+)}{dt}$

or, $$0 = 0.2 \times \frac{di(0^+)}{dt}$$

$$\therefore \quad \frac{di(0^+)}{dt} = 0\text{ A/sec}$$

If i_C be current through capacitor as shown in figure, $i_C(0^+) = 5\text{ A}$

or, $$C\frac{dv(0^+)}{dt} = 5$$

$\therefore \quad \frac{dv(0^+)}{dt} = 15\ V/sec$

This also means, $\frac{dv_L(0^+)}{dt} = \frac{dv(0^+)}{dt} = 15\ V/sec$

or, $\frac{d}{dt}\left(0.2 \times \frac{di(0^+)}{dt}\right) = 15$

or, $0.2 \times \frac{d^2i(0^+)}{dt^2} = 15$

$\therefore \quad \frac{d^2i(0^+)}{dt^2} = 75\ A/sec^2$

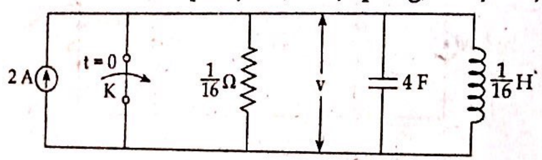
Hence, $\frac{di(0^+)}{dt} = 0\ A/sec$

$\frac{dv(0^+)}{dt} = 15\ V/sec$

$\frac{d^2i(0^+)}{dt^2} = 75\ A/sec^2$

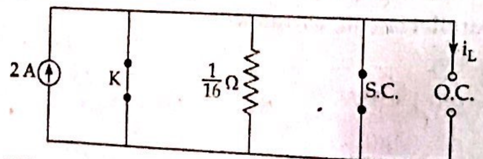
2. At t = 0, the switch K is opened. Determine $v(0^+)$, $\frac{dv(0^+)}{dt}$, $\frac{d^2v(0^+)}{dt^2}$

[2011/Fall, 2015/Spring, 2015/Fall, 2018/Fall]



Solution:

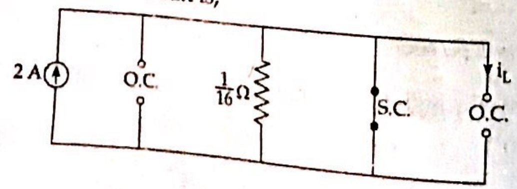
At $t = 0^-$, the equivalent circuit is,



Here, $v(0^-) = 0\ V$

If i_L is current through inductor as shown in figure, $i_L(0^-) = 0$ A.

At $t = 0^+$, the equivalent circuit is,

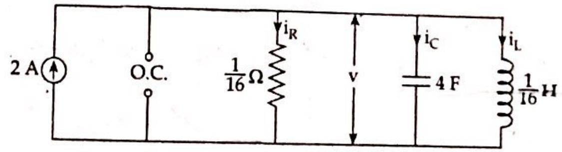


We know,

For inductor, $i_L(0^-) = i_L(0^+) = 0$ A

For capacitor, $v_C(0^-) = v_C(0^+) = 0$ V

For t > 0, the equivalent circuit is,



Applying KCL,

$2 = i_R + i_L + i_C$

or, $\frac{v(t)}{\left(\frac{1}{16}\right)} + 4 \times \frac{dv(t)}{dt} + i_L(t) = 2$

or, $4\frac{dv(t)}{dt} + 16\,v(t) + i_L(t) = 2 \qquad (1)$

At $t = 0^+$,

or, $4\frac{dv(0^+)}{dt} + 16\,v(0^+) + i_L(0^+) = 2$

or, $4\frac{dv(0^+)}{dt} = 16 \times 0 + 0 = 2$

$\therefore \quad \frac{dv(0^+)}{dt} = \frac{2}{4} = 0.5\ V/sec \qquad (2)$

As this is a parallel circuit, we have,

Voltage across inductor $(v_L) = v$

so, $v_L(0^+) = v(0^+)$

or, $L\frac{di_L(0^+)}{dt} = 0$

$\therefore \quad \frac{di_L(0^+)}{dt} = 0\ A/sec \qquad (3)$

Differentiating equation (1) with respect to t,

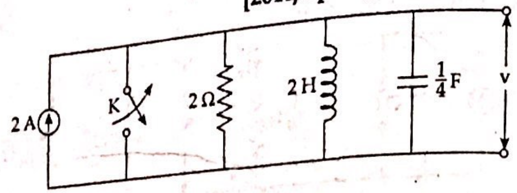
$4\frac{d^2v(t)}{dt^2} + 16\frac{dv(t)}{dt} + \frac{di_L(t)}{dt} = 0$

Now, substituting value of $\frac{dv(0^+)}{dt}$ and $\frac{di_L(0^+)}{dt}$ from (2) and (3), we get,

$\therefore \quad \frac{d^2v(0^+)}{dt^2} = \frac{-8}{4} = -2\ V/sec^2$

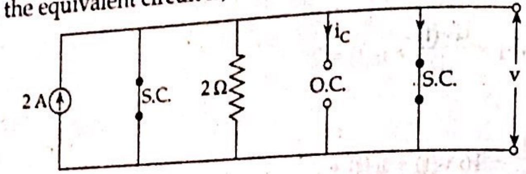
3. **In the given RLC parallel circuit, the switch K is opened at t = 0. Determine $v(0^+)$, $\frac{dv(0^+)}{dt}$, $\frac{d^2v(0^+)}{dt^2}$.**

[2011/Spring, 2018/Spring, 2018/Fall]



Solution:

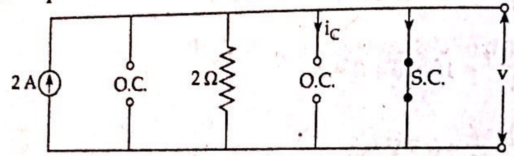
For $t = 0^-$, the equivalent circuit is,



Here; $v(0^-) = 0$ V

$i_L(0^-) = 0$ A (i_L is current through inductor)

For $t = 0^+$, the equivalent circuit is,

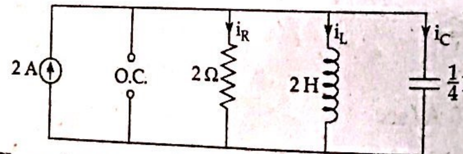


We know,

$v(0^-) = v(0^+) = 0$ V

$i(0^-) = i(0^+) = 0$ A

For $t > 0$, we have,



Applying KCL, we have,

$$2 = i_R + i_C + i_L$$

or, $$2 = \frac{v}{2} + \frac{1}{4}\frac{dv}{dt} + i_L \quad \text{(1)}$$

At $t = 0^+$,

$$\frac{v(0^+)}{2} + \frac{1}{4}\frac{dv(0^+)}{dt} + i_L(0^+) = 2$$

Substituting values of $v(0^+)$ and $i_L(0^+)$,

$$\frac{0}{2} + \frac{1}{4}\frac{dv(0^+)}{dt} + 0 = 2$$

$\therefore$ $$\frac{dv(0^+)}{dt} = 8 \text{ V/sec}$$

Since this is a parallel circuit,

Voltage across inductor, $v_L = v$

or, $v_L(0^+) = v(0^+)$

or, $$L\frac{di_L(0^+)}{dt} = v(0^+)$$

or, $$2 \times \frac{di_L(0^+)}{dt} = 0$$

$\therefore$ $$\frac{di_L(0^+)}{dt} = 0 \text{ A/sec}$$

Differentiating equation (1) with respect to t,

$$\frac{1}{2}\frac{dv(t)}{dt} + \frac{1}{4}\frac{d^2v(t)}{dt^2} + \frac{di_L(t)}{dt} = 0$$

At $t = 0^+$,

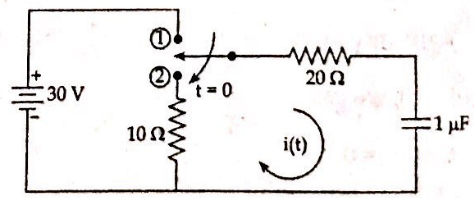
$$\frac{1}{2}\frac{dv(0^+)}{dt} + \frac{1}{4}\frac{d^2v(0^+)}{dt^2} + \frac{di_L(0^+)}{dt} = 0$$

Replacing value of $\frac{dv(0^+)}{dt}$ and $\frac{di_L(0^+)}{dt}$, we get,

$$\frac{1}{2} \times 8 + \frac{1}{4}\frac{d^2v(0^+)}{dt^2} + 0 = 0$$

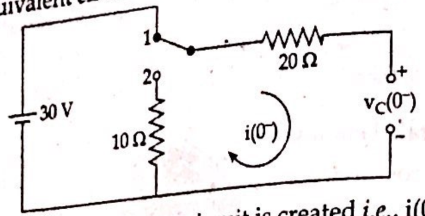
$\therefore$ $$\frac{d^2v(0^+)}{dt^2} = -4 \times 4 = -16 \text{ V/sec}^2$$

4. **In the circuit, switch K is changed from position (1) to (2) at time t = 0, steady state condition having reached before switching. Find i(t), $\frac{di(t)}{dt}$ and $\frac{d^2i(t)}{dt^2}$ at t = 0.** **[2017/Fall]**



Solution:

At t = 0^-, the equivalent circuit is,



The capacitor is energized, so open circuit is created *i.e.,* $i(0^-) = 0$ A

Applying KVL, to find $v_C(0^-)$,

or, $30 = 20 \times i(0^-) + v_C(0^-)$

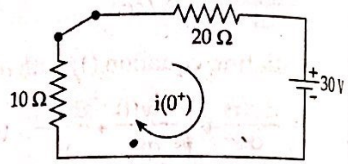
or, $30 = 20 \times i(0^-) + v_C(0^-)$

$\therefore$ $v_C(0^-) = 30$ V

We know, for capacitor,

$$v_C(0^-) = v_C(0^+) = 30 \text{ V}$$

At t = 0^+, the equivalent circuit is,



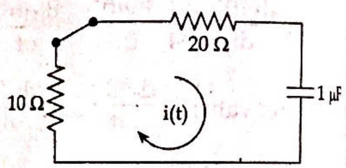
Applying KVL,

$$20 \times i(0^+) + 30 \text{ V} + 10 \times i(0^+) = 0$$

or, $30\, i(0^+) = -30$

$\therefore$ $i(0^+) = -1$ A

The negative sign indicates counterclockwise flow of current or opposite to the direction we have considered. For t > 0, the equivalent circuit is,



Applying KVL, we have,

or, $20\, i(t) + \dfrac{1}{10^{-6}} \displaystyle\int_{-\infty}^{\infty} i(t)\, dt + 10\, i(t) = 0$

or, $30\, di(t) + 10^6 \displaystyle\int_{-\infty}^{\infty} (t) = 0$ (1)

Differentiating equation (1) with respect to t, we get,

$$30 \frac{di(t)}{dt} + 10^6\, i(t) = 0 \quad (2)$$

At t = 0^+,

$$30 \times \frac{di(0^+)}{dt} + 10^6\, i(0^+) = 0$$

Replacing value of $i(0^+)$, we get,

or, $30 \dfrac{di(0^+)}{dt} + 10^6 (-1) = 0$

$\therefore$ $\dfrac{di(0^+)}{dt} = \dfrac{10^6}{30} = 3.33 \times 10^4$ A/sec

Again, differentiating equation (2) with respect to t,

$$30 \frac{d^2i(t)}{dt} + 10^6 \frac{di(t)}{dt} = 0$$

At t = 0^+,

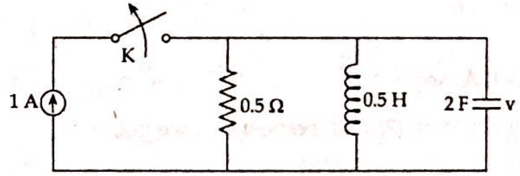
$$30 \frac{d^2i(0^+)}{dt} + 10^6 \frac{di(t)}{dt} = 0$$

Replacing value of $\dfrac{di(0^+)}{dt}$, we get,

or, $30 \dfrac{d^2i(0^+)}{dt^2} + 10^6 \times 3.33 \times 10^4 = 0$

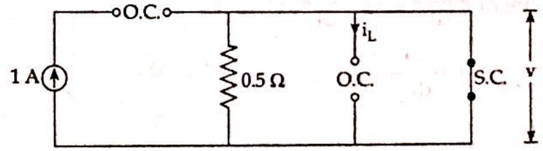
$\therefore$ $\dfrac{d^2i(0^+)}{dt^2} = -1.11 \times 10^9$ A/sec^2

5. In the given RLC parallel circuit, the switch K is closed at t = 0. Determine $v(0^+)$, $\dfrac{dv}{dt}(0^+)$, $\dfrac{d^2v(0^+)}{dt^2}$. **[2016/Fall]**



Solution:

At t = 0^-, the equivalent circuit is,



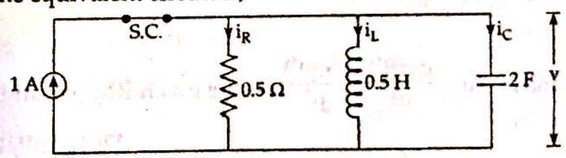
Here; $i_L(0^-) = 0$ A

$v(0^-) = 0$ V

For inductor $i_L(0^-) = i_L(0^+) = 0$ A

For capacitor, $v(0^-) = v(0^+) = 0$ V

For t > 0, the equivalent circuit is,



Applying KCL,

$$i_R + i_C + i_L = 1$$

or, $$\frac{v}{0.5} + 2\frac{dv}{dt} + i_L = 1$$

At $t = 0^+$,

$$2v(0^+) + i_L(0^+) + 2\frac{dv(0^+)}{dt} = 1$$

Replacing value of $v(0^+)$ and $i_L(0^+)$, we get,

$$2 \times 0 + 0 + 2\frac{dv(0^+)}{dt} = 1$$

$\therefore$ $$\frac{dv(0^+)}{dt} = 0.5\ \text{V/sec}$$

Since this is a parallel circuit,
Voltage across inductor, $v_L = v$

or, $$v_L(0^+) = v(0^+)$$

or, $$L\frac{di_L(0^+)}{dt} = v(0^+)$$

or, $$0.5\frac{di_L(0^+)}{dt} = 0$$

$\therefore$ $$\frac{di_L(0^+)}{dt} = 0\ \text{A/sec}$$

Differentiating equation (2) with respect to t, we get,

$$2\frac{dv(0^+)}{dt} + \frac{di_L(0^+)}{dt} + 2\frac{d^2v(0^+)}{dt^2} = 0$$

Replacing value of $\frac{dv(0^+)}{dt}$ and $\frac{di(0^+)}{dt}$, we get,

$$2 \times 0.5 + 0 + 2\frac{dv(0^+)}{dt} = 0$$

$\therefore$ $$\frac{d^2v(0^+)}{dt^2} = \frac{-1}{2} = -0.5\ \text{V/sec}^2$$

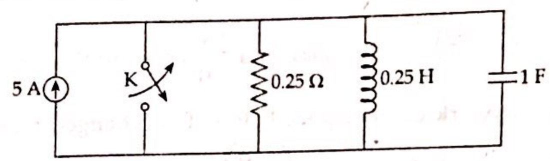
Hence, $v(0^+) = 0$ V

$$\frac{dv(0^+)}{dt} = 0.5\ \text{V/sec}$$

$$\frac{d^2v(0^+)}{dt^2} = -0.5\ \text{V/sec}^2$$

6. Determine $v(0^+)$, $\frac{dv(0^+)}{dt}$, $\frac{d^2v(0^+)}{dt^2}$ on the given RLC parallel circuit.

[2014/Fall, 2011/Fall]

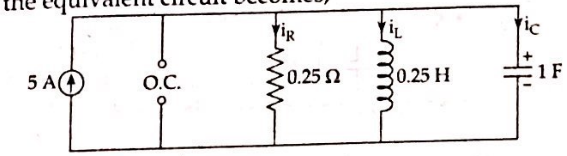


Solution:

For $t < 0$, the inductor is unsaturated and the capacitor is uncharged, so inductor acts as open circuit and capacitor acts as short circuit.

$\therefore$ $i_L(0^-) = 0$ A and $v_C(0^-) = 0$ V

For $t > 0$, the equivalent circuit becomes,



Here, $v_C(0^+) = 0$ V and $i_L(0^+) = 0$ A.

Applying KCL,

$$i_R + i_L + i_C = I_0 \quad (1)$$

or, $$\frac{v}{R} + \frac{1}{L}\int v\,dt + C\frac{dv}{dt} = 5$$

or, $$\frac{v}{1} + \frac{1}{0.25}\int v\,dt + 0.25\frac{dv}{dt} = 5$$

or, $$v + 4\int v\,dt + 0.25\frac{dv}{dt} = 5 \quad (2)$$

Differentiating equation (2) with respect t, we get,

$$\frac{dv}{dt} + 4v(t) + 0.25\frac{d^2v(t)}{dt^2}$$

or, $$0.25\frac{d^2v(t)}{dt^2} + \frac{dv(t)}{dt} + 4V(t) = 0 \quad (3)$$

At $t = 0^+$, equation (2) becomes,

$$v(0^+) + 4i_L(0^+) + 0.25\frac{dv(0^+)}{dt} = 5$$

or, $$0 + 4 \times 0 + 0.25\frac{dv(0^+)}{dt} = 5$$

$\therefore$ $$\frac{dv(0^+)}{dt} = \frac{5}{0.25} = 20\ \text{V/sec}$$

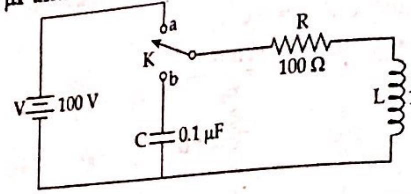
Again, at $t = 0^+$, equation (3) becomes,

$$0.25\frac{d^2v(0^+)}{dt^2} + 20 + 4 \times 0 = 0$$

$\therefore$ $$\frac{d^2v(0^+)}{dt^2} = \frac{-20}{0.25} = -80\ \text{V/sec}^2$$

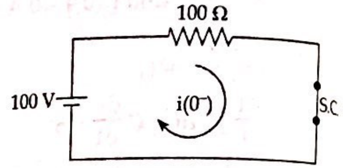
Hence, $v(0^+) = 0\ V$; $\frac{dv(0^+)}{dt} = 20\ V/sec$ and $\frac{d^2v(0^+)}{dt^2} = -80\ V/sec^2$

7. In the network of the figure below, K is changed from position 'a' to 'b' at t = 0. Solve for i, $\frac{di}{dt}$, $\frac{d^2i}{dt^2}$ at t = 0⁺ if R = 100 Ω, L = 1 H, C = 0.1 μF and V = 100 V. **[2011/Spring]**



Solution:

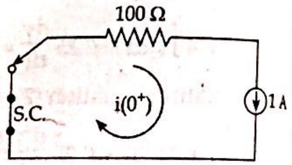
At $t = 0^-$, switch K is at position 'a'. Hence, the inductor acts as short circuit. The equivalent circuit at t = 0.



∴ $i_L(0^-) = \frac{V}{R} = \frac{100}{100} = 1\ A$

Also, $v_C(0^-) = 0\ V$

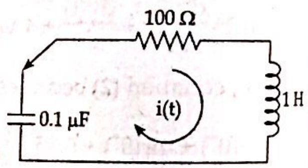
At $t = 0^+$, the switch K is at position 'b'. Hence the inductor behaves as the current source by 1 A. Also, initially capacitor is uncharged and after switching the capacitor behaves as short circuit.



Hence, $i_L(0^+) = 1\ A$

$v_C(0^+) = 0\ V$

For t ≥ 0, the circuit becomes,



Applying KVL,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = 0 \quad (1)$$

or, $Ri(0^+) + L\frac{di(0^+)}{dt} + 0 = 0$

or, $100\,i(0^+) + 1 \times \frac{di(0^+)}{dt} = 0$

∴ $\frac{di(0^+)}{dt} = -100\,i(0^+) = -100 \times 1 = -100\ A/sec$

Differentiating equation (1) with respect to, we get,

$$R\frac{di(t)}{dt} + L \times \frac{d^2i(t)}{dt^2} + \frac{i(t)}{C} = 0$$

At $t = 0^+$,

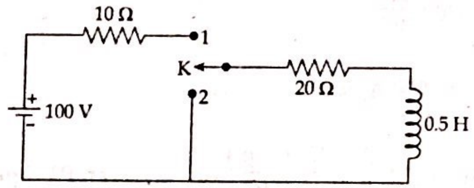
$$R\frac{di(0^+)}{dt} + L \times \frac{d^2(0^+)}{dt} + \frac{i(0^+)}{C} = 0$$

or, $100 \times (-100) + 1 \times \frac{d^2i(0^+)}{dt^2} + \frac{1}{0.1 \times 10^{-6}} = 0$

or, $\frac{d^2i(0^+)}{dt^2} = 10{,}000 - 1{,}00{,}00{,}000$

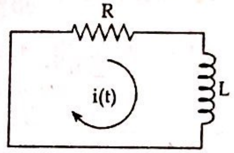
∴ $\frac{d^2i(0^+)}{dt^2} = -9.99 \times 10^6\ A/sec^2$

8. Switch K is kept at position 2 and steady state condition is reached. At t = 0, the switch is moved to position 1. Find, i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at t = 0. **[2012/Fall]**

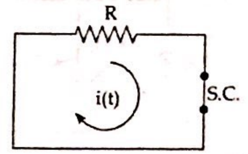


Solution:

When K is at position '2',



At steady state



Steady state value of current, $i(0^-) = \frac{V}{R} = \frac{0}{20} = 0\ A$

At position 1, $i(0^+) = 0\ A = i(0^-)$

Applying KVL,

$$R_1\,i(t) + R_2\,(t) + L\frac{di(t)}{dt} = V_0$$

or, $R = R_1 + R_2 = 10 + 20 = 30\ \Omega$

or, $R\,i(t) + L\frac{di(t)}{dt} = V_0$

or, $30\,i(t) + 0.5\frac{di(t)}{dt} = 100 \quad (1)$

Differentiating equation (1) with respect to t, we get,

$$0 = 30\frac{di(t)}{dt} + 0.5\frac{d^2i(t)}{dt^2} \quad (2)$$

At $t = 0^+$, the inductor behave as an open circuit,

Thus, $i(0^+) = 0$ A

From equation (1), we get,

or, $30 \times i(0^+) + 0.5 \dfrac{di(0^+)}{dt} = 100$

or, $30 \times 0 + 0.5 \dfrac{di(0^+)}{dt} = 100$

$\therefore \dfrac{di(0^+)}{dt} = \dfrac{100}{0.5} = 200$ A/sec

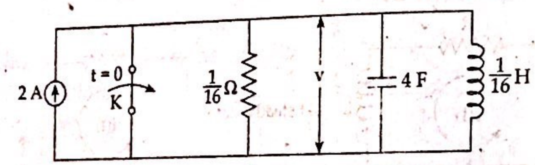
From equation (2), we have,

$0 = 30 \times 200 + 0.5 \times \dfrac{d^2i(0^+)}{dt^2}$

or, $\dfrac{d^2i(0^+)}{dt^2} = -30 \times \dfrac{200}{0.5}$

$\therefore \dfrac{d^2i(0^+)}{dt^2} = -12{,}000$ A/sec^2

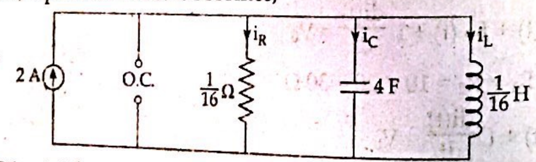
9. Determine $v(0^+)$, $\dfrac{dv(0^+)}{dt}$, $\dfrac{d^2v(0^+)}{dt^2}$ on the given RLC parallel circuit. [2012/Spring]



Solution:

For $t < 0$, the inductor is unsaturated and capacitor is uncharged. Hence $v_C(0^-) = 0$ V and $i_L(0^-) = 0$ A. This means inductor acts as open circuit and capacitor acts as short circuit.

For $t > 0$ so, equivalent circuit becomes,



Here, $i_L(0^-) = i_L(0^+) = 0$ A

and, $v_C(0^-) = v_C(0^+) = 0$ V

Hence, $v(0^+) = 0$ V

Applying KCL,

or, $I_0 = i_R + i_L + i_C$

or, $2 = \dfrac{v}{R} + \dfrac{1}{L}\int v\, dt + C\dfrac{dv}{dt}$

or, $2 = \dfrac{v}{\left(\frac{1}{16}\right)} + \dfrac{1}{\left(\frac{1}{16}\right)}\int v\, dt + 4\dfrac{dv}{dt}$

or, $2 = 16v + 16\int v\, dt + 4\dfrac{dv}{dt}$

$\therefore 1 = 8v + 8\int v\, dt + 2\dfrac{dv}{dt}$ (2)

Differentiating with respect to time 't'

or, $0 = 8\dfrac{dv}{dt} + 8v + 2\dfrac{d^2v}{dt^2}$

or, $\dfrac{d^2v}{dt^2} + \dfrac{4dv}{dt} + 4v = 0$ (3)

At $t = 0^+$ in equation (2), we get,

$1 = 8\, v(0^+) + 8i_L(0^+) + \dfrac{2dv}{dt}(0^+)$

or, $1 = 8 \times 0 + 8 \times 0 + \dfrac{2dv(0^+)}{dt}$

$\therefore \dfrac{dv(0^+)}{dt} = \dfrac{1}{2}$ V/sec = 0.5 V/sec

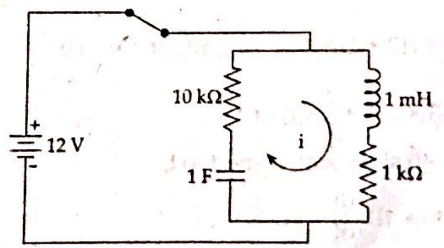
Again, at $t = 0^+$ in equation (3), we get,

$\dfrac{d^2v(0^+)}{dt^2} + 4\dfrac{dv(0^+)}{dt} + 4v(0^+) = 0$

or, $\dfrac{d^2v(0^+)}{dt^2} + 4 \times 0.5 + 4 \times 0 = 0$

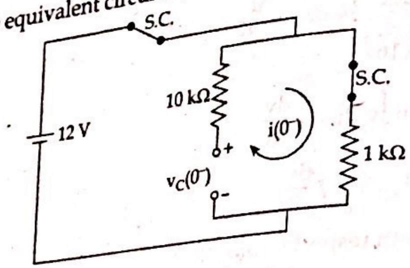
$\therefore \dfrac{d^2v(0^+)}{dt^2} = -2$ V/sec^2

10. The switch S in the given circuit is opened at time $t = 0$. Determine the current i and its derivatives at $t = 0^+$. [2013/Fall]



Solution:

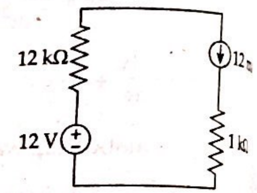
At $t = (0^-)$, the equivalent circuit is,



Here, $i(0^-) = \frac{12V}{1 \times 10^3 \Omega} = 12\ mA$

$v_C(0^+) = 12\ V$

At $t = (0^+)$, the equivalent circuit as shown in diagram.



We know,

$i(0^+) = i(0^-) = 12\ mA$

$v_C(0^+) = v_C(0^-) = 0\ V$

Applying KVL

$12 \times 10^3 \times i(0^+) - v_L(0^+) + 1 \times 10^3 \times i(0^+) = V_C(0^+)$

or, $12 \times 10^3 \times 12 \times 10^{-3} - v_L(0^+) + 1 \times 10^3 \times 12 \times 10^{-3} = 12$

or, $144 - v_L(0^+) + 12 = 12$

$\therefore$ $v_L(0^+) = 144\ V$

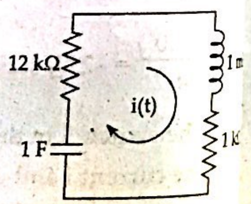
We know,

$v_L(0^+) = L\frac{di(0^+)}{dt}$

or, $144 = 1 \times 10^{-3}\frac{di(0^+)}{dt}$

$\therefore$ $\frac{di(0^+)}{dt} = 144 \times 10^3\ A/s$

For $t > 0$, the equivalent circuit as shown in diagram alongside.



Applying KVL,

$$i \times 10^{-3}\frac{di}{dt} + (12 + 1) \times 10^3 i + \frac{1}{1}\int_0^t i(t)\,dt + v_L = 0$$

or, $10^{-3}\frac{di}{dt} + 13 \times 10^3 i + \int_0^t i(t)\,dt + 12 = 0$

Differentiating both sides with respect to t,

$$10^{-3}\frac{d^2i}{dt^2} + 13 \times 10^3\frac{di}{dt} + i(t) = 0$$

At $t = 0^+$

$$10^{-3}\frac{d^2i(0^+)}{dt^2} + 13 \times 10^3\frac{di(0^+)}{dt} + i(0^+) = 0$$

or, $10^{-3}\frac{d^2i(0^+)}{dt^2} + 13 \times 10^3 \times 144 \times 10^3 + 12 \times 10^{-3} = 0$

$\therefore$ $\frac{d^2i(0^+)}{dt^2} = -1.872 \times 10^{12}\ A/sec^2$

11. Write short notes on operation impedance of passive elements. **[2019/Fall]**

Solution: See the topic 2.2.

12. Write short notes on differential operator. **[2018/Spring]**

Solution: See the topic 2.1.

13. Write short notes on transient and steady state. **[2011/Spring, 2015/Fall, 2015/Spring]**

Solution: See the topic 2.4.1.1 and 2.4.1.2.

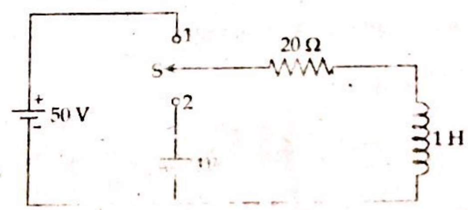
14. Write short notes on operation impedance. **[2016/Spring, 2017/Spring]**

Solution: See the topic 2.2.

15. Write short notes on complete response of the circuit. **[2012/Spring]**

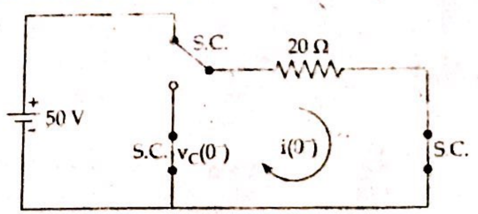
Solution: See the topic 2.4.

16. The switch S is changed from position '1' to '2' at t = 0. Find value of $i(0^+)$, $\frac{di(0^+)}{dt}$, $\frac{d^2i(0^+)}{dt^2}$. **[2020/Fall]**



Solution:

At $t = 0^-$, the equivalent circuit is,



Inductor is energized while capacitor is denergized.

$$i(0^-) = \frac{50}{20} = 2.5 \text{ A}$$

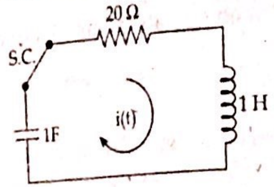
$$v_C(0^-) = 0 \text{ V}$$

We know,

$$i(0^+) = i(0^-) = 2.5 \text{ A}$$

$$v_C(0^+) = v_C(0^-) = 0 \text{ V}$$

The equivalent circuit for t > 0 is,



Applying KVL,

$$20\, i(t) + 1 \times \frac{di(t)}{dt} + v_C(t) = 0 \qquad (1)$$

At $t = 0^+$,

$$20\, i(0^+) + \frac{di(0^+)}{dt} + v_C(0^+) = 0$$

or, $$20 \times 2.5 + \frac{di(0^+)}{dt} + 0 = 0$$

$\therefore$ $$\frac{di(0^+)}{dt} = -50 \text{ A/s}$$

From equation (1), we get,

$$20\, i(t) + \frac{di(t)}{dt} + \frac{1}{1}\int_0^t i(t)dt + v_C(0^+) = 0$$

Differentiating both sides with respect to t,

$$20\frac{di(t)}{dt} + \frac{d^2i(t)}{dt^2} + i(t) = 0$$

At $t = 0^+$,

$$\frac{d^2i(0^+)}{dt^2} + 20 \times \frac{di(0^+)}{dt} + i(0^+) = 0$$

or, $$\frac{d^2i(0^+)}{dt^2} + 20 \times (-50) + 2.5 = 0$$

$\therefore$ $$\frac{d^2i(0^+)}{dt^2} = 9975.5 \text{ A/sec}^2$$

☆☆☆

Chapter 3

CIRCUIT DYNAMICS

3.1 INTRODUCTION

Whenever a circuit is switched from one condition to another, either by a change in the applied source or a change in the circuit elements, there is a transition period during which the branch currents and elements voltages change from their former values to new ones. This period is called the transient. After the transient has passed, the circuit is said to be in steady state.

Now, the linear differential equation that describes the circuit will have two parts to its solution; the complementary function corresponds to the transient and the particular solution corresponds to the steady-state.

The v-i relation for an inductor or a capacitor is a differential. A circuit containing an inductor L or a capacitor C and resistors will have current and voltage variables given by differential equations of the same form. It is a linear first order differential equation with constant. L and C are storage elements. Circuits have two storage elements like one L and one C are referred to as second order circuits.

Therefore, the series or parallel combinations of R and L or R and C are first order circuits and RLC in series or RLC in parallel are typical second order circuits. The circuit changes are assumed to occur at time $t = 0$ are represented by a switch. The switch may be supposed to closed on and open/off at $t = 0$ as shown in figure 3.1(a) or (b) respectively, for convenience, it is defined that;

$t = 0^-$, the instant prior to $t = 0$, and

$t = 0^+$, the instant immediately after switching

Figure 3.1: Switch S is (a) closed at t = 0, (b) opened at t = 0

Switching on or off an element or source in a circuit at $t = 0$ will not disturb the storage element so that $i_L(0^-) = i_L(0^+)$ and $v_C(0^-) = v_C(0^+)$. This provides a basis for constructing an equivalent circuit for a charged capacitor voltage (V_0) and current (I_0) carrying inductor.

The voltage-current relationships of the three circuit elements R, L and C are given in table below.

Parameter	Basic relationship	Voltage current relationships	Energy
$R\left(G = \frac{1}{R}\right)$	$v(t) = Ri(t)$	$v_R(t) = R\, i_R(t)$ $i_R(t) = G\, v_R(t)$	$W_R(t) = \int_{-\infty}^{t} v_R(t) i_R(t) dt$
L	$\psi(t) = Li(t)$	$v_L(t) = L\frac{di_L(t)}{dt}$ $i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)dt$ $i_L(t) = \frac{1}{L}\int_{0}^{t} v_L(t)\, dt + i_L(0^+)$	$W_L(t) = \frac{1}{2} Li_L^2(t)$
C	$q(t) = Cv(t)$	$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(t)dt$ $v_C(T) = \frac{1}{C}\int_{0}^{t} i_C(t)dt + v_C(0^+)$ $i_C(t) = C\frac{dv_C(t)}{dt}$	$W_C(t) = \frac{1}{2} Cv_C^2(t)$

3.2 FIRST ORDER DIFFERENTIAL EQUATIONS

A differential equation is an equation that relates one or more functions and their derivatives. In applications, the functions generally represent physical quantities, the derivatives represent their rate of change, and the differential equation defines a relationship between the two. The differential equation considered have the general form,

$$F[x(t), x'(t)......., x^{(n)}(t), t] = 0 \qquad (1)$$

where, t is the independent variable and x(t) is a function dependent upon t. The superscripted terms $x^{(i)}(t)$ indicate the i^{th} derivative of x(t) with respect to t, namely,

$$x^{(i)}(t) = \frac{d^{(i)}x(t)}{dt^i} \qquad (2)$$

The solution of F = 0 in equation (1) is x(t) and must be obtained as an explicit function of t. When we substitute the explicit solution x(t) into F, the equation must equal zero. If F in equation (1) is an ordinary linear differential equation, it is given by the general equation

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + ... + a_1 x^1(t) + a_0 x(t) = f(t) \qquad (3)$$

The order of the equation is n, the order of the highest derivative term. The term f(t) on the right hand side of the equation is the forcing function or driver and is independent of x(t). When f(t) is identically zero, the equation is said to be homogenous; otherwise, the equation is non-homogenous.

Definition of first order differential equation

A first order differential equation is an equation of the form $F(t, y, y') = 0$. A solution of a first order differential equation is a function f(t) that makes $F(t, f(t), f'(t)) =$ for every value of t.

Here, F is a function of three variables which we label t, y and y. The term 'first order' means that the first derivative of y appears, but no higher order derivatives do.

Ordinary differential equation

An ordinary differential equation is one in which there is only one independent variable. As a result there is no need for partial derivatives. The coefficients a_n, a_{n-1}, ..., a_2, a_1, a_0 are constant, independent of the variable t.

Linear differential equation

A differential equation is linear if it contains only terms of the first degree in x(t) and all its higher derivatives, as given by equation (3).

3.2.1 First Order Homogenous Differential Equations

A first order differential equation $\frac{dy}{dx} = f(x, y)$ is called homogenous equation, if the right side satisfies the condition $f(tx, ty) = f(x, y)$ for all t. In order words, the right side is a homogenous function (with respect to the variables x and y) of the zero order.

$$f(tx, ty) = t^0 f(x, y) = f(x, y)$$

We have,

$$\frac{dy(t)}{dt} + Py(t) = 0$$

where, P is any constant

$$\frac{dy(t)}{y(t)} = -Pdt$$

On integrating,

$$ln\, y(t) = -Pt + K'$$

Take $K' = \ln K$

$\ln y(t) = \ln e^{-Pt} + \ln K = \ln (K e^{-Pt})$

With the equation in this form, the antilogarithm may be taken to give general solution, $y(t) = K e^{-Pt}$

where, K is a constant

If the constant K is evaluated, the solution is a particular solution.

3.2.2 First Order Non-Homogenous Differential Equation

We have,

$$\frac{dy(t)}{dt} + Py(t) = Q$$

where, P is a constant and Q may be a function of independent variable or a constant. The equation is not altered if every term is multiplied by the e^{Pt}

i.e., $$e^{Pt}\frac{dy(t)}{dt} + e^{Pt}Py(t) = Q e^{Pt}$$

Since $d(x\cdot y) = x\,dy + y\,dx$, so left hand side of above equation is equal to $\frac{d}{dt}[y(t)\,e^{Pt}]$. Thus we have,

$$\frac{d}{dt}[y(t)\,e^{Pt}] = Q e^{Pt}$$

Thus, equation may be integrated to give

$$y(t)\,e^{Pt} = \int Q e^{Pt}\,dt + K$$

$$y(t) = e^{-Pt}\int Q e^{Pt} + K e^{-Pt}$$

where, K is a constant

The first term of above solution is known as the particular integral; the second is known as the complementary function. Note that the particular integral does not contain the arbitrary constant, and the complementary function does not depend on the forcing function Q.

If Q is a constant, then

$$y(t) = e^{-Pt} Q \frac{e^{Pt}}{P} + K e^{-Pt}$$

$$\therefore \quad y(t) = \frac{Q}{P} + K e^{-Pt}$$

3.3 RESPONSE OF R-L CIRCUIT WITH

3.3.1 DC Excitation

Consider a series R-L circuit as shown in figure 3.2. The switch S is closed at time $t = 0$.

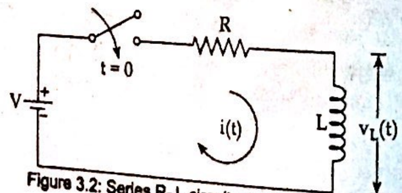


Figure 3.2: Series R-L circuit with DC excitation

Applying KVL,

$$L\frac{di(t)}{dt} + Ri(t) = V$$

or, $$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{V}{L}$$

General solution of this differential equation is,

$$i(t) = \frac{V}{R} + K e^{\frac{-Rt}{L}}$$

Since inductor behaves as an open circuit at switching $i(0^+) = 0$

or, $$0 = \frac{V}{R} + K$$

or, $$K = \frac{-V}{R}$$

Hence, $$i(t) = \frac{V}{R} - \frac{V}{R}e^{\frac{-Rt}{L}} = \frac{V}{R}\left(1 - e^{\frac{-Rt}{L}}\right)$$

and voltage across the resistor and inductor are given as,

$$v_R(t) = i(t)\,R$$

or, $$v_R(t) = V\left(1 - e^{\frac{-Rt}{L}}\right)$$

$$v_L(t) = L\frac{di(t)}{dt}\ [\text{or } V - V_R(t)] = L\frac{V}{R}\left[0 - \left(\frac{-R}{L}\right)e^{\frac{-Rt}{L}}\right]$$

or, $$v_L(t) = V e^{\frac{-Rt}{L}}$$

At $t = 0$, $i(t) = 0$ and $v_L(t) = V$, $v_R(t) = 0$

At $t = \infty$, $i(t) = \frac{V}{R}$ and $v_L(t) = 0$, $v_R(t) = V$

At $t = \frac{L}{R} = \tau$, $i(t) = \frac{V}{R}(1 - e^{-1}) = 0.632\frac{V}{R}$

and, $v_L(t) = Ve^{-1} = 0.368$ V, $v_R(t) = 0.632$ V

These i(t) and $v_R(t)$ and $v_L(t)$ are plotted in figure 3.3 (a) and 3.3 (b).

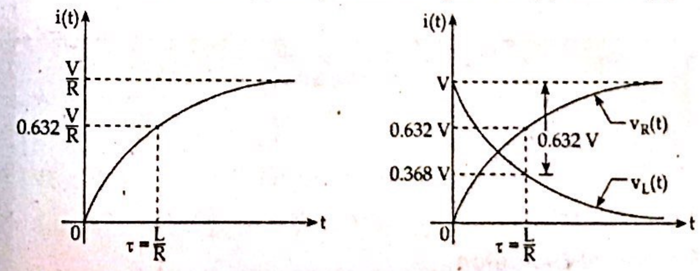


Figure 3.3: (a) for i(t) and (b) for $v_R(t)$ and $v_L(t)$

NOTE: $\tau = \frac{L}{R}$ is known as the time constant of the circuit and is defined as the interval after which current or voltage changes 63.2% of its total change.

Let us now analyze another transient condition of the R-L circuit as the circuit reaches at steady state (at $t = \infty$) and suddenly the voltage is withdrawn by opening the switch S and throwing it to S' as shown in figure 3.4 at $t = 0$

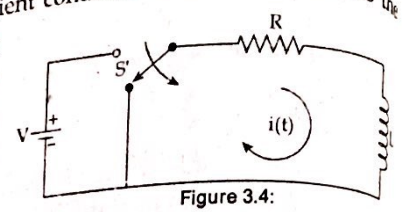


Figure 3.4:

Then, $L\dfrac{di'(t)}{dt} + Ri'(t) = 0$

General solution of this differential equation is given as $i'(t) = K e^{\frac{-Rt}{L}}$

However at $t = 0^+$, the inductor keep the steady state current

$$i'(0^+) = i(\infty) = \frac{V}{R}$$

or, $\dfrac{V}{R} = K' e^0$ or $K' = \dfrac{V}{R}$

Hence, $i'(t) = \dfrac{V}{R} e^{\frac{-Rt}{L}}$

The corresponding voltages across the resistor and inductor are given as,

$$v_R'(t) = i'(t).\ R = Ve^{\frac{-Rt}{L}}$$

and, $v_L'(t) = L\dfrac{di'(t)}{dt} = -Ve^{\frac{-Rt}{L}}$

[As $v_R(t) + v_L'(t) = 0$]

These i(t) and $v_R(t)$ and $v_L'(t)$ are plotted in figure 3.5 (a) and (b)

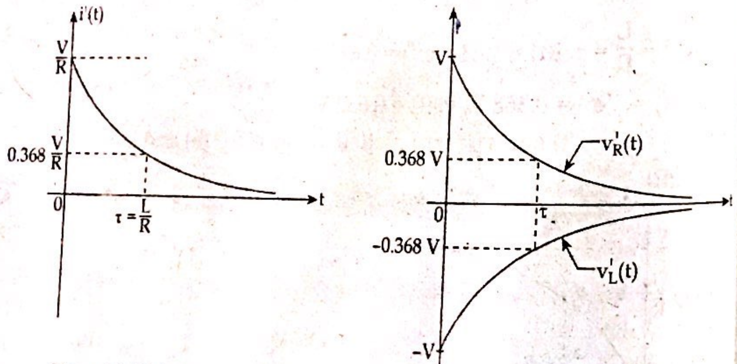


Figure 3.5: (a) for i(t) and (b) for $v_R'(t)$ and $v_L'(t)$

3.3.2 Sinusoidal Excitation

Consider a series R-L circuit excited by a sinusoidal voltage source as shown in figure 3.6. The switch is closed at time $t = 0$.

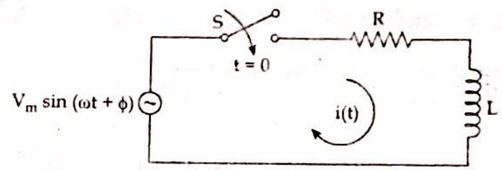


Figure 3.6:

Applying KVL,

$$L\frac{di(t)}{dt} + Ri(t) = V_m \sin(\omega t + \phi)$$

or, $\dfrac{di(t)}{dt} + \dfrac{R}{L} i(t) = \dfrac{V_m}{L} \sin(\omega t + \phi)$ (1)

This is a non-homogenous equation. The current i(t) consists of the sum of complementary function $i_C(t)$ and particular integral $i_P(t)$.

i.e., $i(t) = i_c(t) + i_P(t)$

The complementary function of equation **(1)** is

$$i_c(t) = K e^{\frac{-Rt}{L}}$$

and the particular integral of equation **(1)** is,

$$i_P(t) = e^{\frac{-Rt}{L}} \int \frac{V_m}{L} \sin(\omega t + \phi)\, e^{\frac{Rt}{L}}\, dt$$

$$= \frac{V_m e^{\frac{-Rt}{L}}}{2jL} \left[e^{j(\omega t + \phi)} - e^{-j(\omega t + \phi)}\right] e^{\frac{Rt}{L}}\, dt$$

$$= \frac{V_m e^{\frac{-Rt}{L}}}{2jL} \left[\frac{e^{j(\omega t + \phi) + \frac{R}{L}}}{j\omega + \frac{R}{L}} - \frac{e^{-j(\omega t + \phi) + \frac{R}{L}}}{-j\omega + \frac{R}{L}}\right]$$

$$= \frac{V_m}{2jL} \left[\frac{e^{j(\omega t + \phi)}\left(-j\omega + \frac{R}{L}\right) - e^{-j(\omega t + \phi)}\left(j\omega + \frac{R}{L}\right)}{\left(j\omega + \frac{R}{L}\right)\left(-j\omega + \frac{R}{L}\right)}\right]$$

$$= \frac{V_m}{L\left(\omega^2 + \frac{R^2}{L^2}\right)} \left[\frac{R}{L} \sin(\omega t + \phi) - \omega \cos(\omega t - \phi)\right]$$

$$= \frac{V_m}{R^2 + \omega^2 L^2} [R \sin(\omega t + \phi) - \omega L \cos(\omega t + \phi)]$$

This can be reduced to a single sinusoid in the form

$$i_P(t) = \frac{V_m}{R^2 + \omega^2 L^2} [C \sin(\omega t + \phi + \theta)]$$

where, C and θ can be determined as, $C \cos\theta = R$ and $C \sin\theta = -\omega L$

i.e., $C = \sqrt{R^2 + \omega^2 L^2}$ and $\theta = -\tan^{-1}\left(\dfrac{\omega L}{R}\right)$

Replacing value of C and θ,

$$i_p(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

Hence,

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)\frac{\omega L}{R} + K e^{\frac{-Rt}{L}}$$

Since inductor behaves as a open circuit at switching,

$\therefore \quad i(0^+) = 0$

or, $$0 = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\phi - \tan^{-1}\frac{\omega L}{R}\right) + K$$

or, $$K = \frac{-V_m}{\sqrt{R^2 + \omega^2 L^2}} \sin\left(\phi - \tan^{-1}\frac{\omega L}{R}\right)$$

$$\therefore \quad i(t) = \frac{V_m}{Z}\left(\sin(\omega t + \phi + \theta) - \sin(\phi + \theta)\, e^{\frac{-Rt}{L}}\right)$$

where, $\theta = -\tan^{-1}\left(\frac{\omega L}{R}\right)$

and, $Z = \sqrt{R^2 + \omega^2 L^2}$ = Impedance of R-L circuit

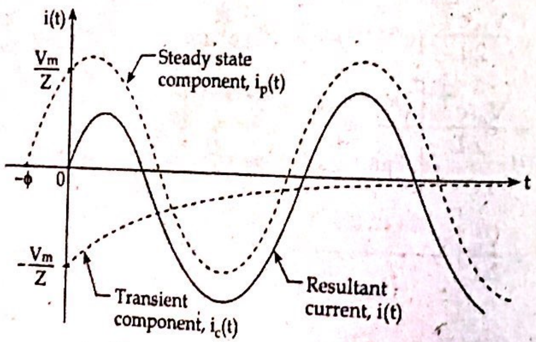


Figure 3.7: Variation of i(t) with time t

It is observed that if the angle φ which represents the angle of the sinusoid at the time switch S is closed, has the value $\phi = \theta\left(= -\tan^{-1}\frac{\omega L}{R}\right)$, the constant K will have zero value and the transient current $i_c(t)$ will vanish. In other words, if the switch is closed at the proper instant, there will be no transient. The first term of i(t) is the steady state current which lags the applied voltage by $\theta = -\tan^{-1}\left(\frac{\omega L}{R}\right)$, and the second term is the transient current with decay factor $e^{\frac{-Rt}{L}}$, which dies with time constant $\frac{L}{R}$.

3.4 RESPONSE OF R-C CIRCUIT WITH

3.4.1 DC Excitation

Consider a series R-C circuit as shown in figure 3.8. The switch S is closed at time t = 0.

By Kirchhoffs voltage law,

$$Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)\, dt = V$$

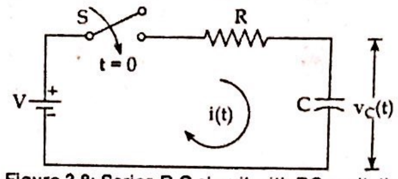


Figure 3.8: Series R-C circuit with DC excitation

or, $$Ri(t) + \frac{1}{C}\int_{0}^{t} i(t)\, dt + v_C(0^+) = V$$

Assume, initially capacitor was uncharged *i.e.,*

$$v_C(0^+) = 0$$

Differentiating, we get,

$$R\frac{di(t)}{dt} + \frac{1}{C} i(t) = 0$$

or, $$\frac{di(t)}{dt} + \frac{1}{RC} i(t) = 0$$

General solution of this differential equation is,

$$i(t) = K\, e^{\frac{-t}{RC}}$$

At $t = 0^+$, $i(0^+) = \frac{V}{R}$

(Since capacitor behaves as a short circuit at switching)

or, $$\frac{V}{R} = K e^0 = K$$

Hence, $i(t) = \frac{V}{R} e^{\frac{-1}{RC}}$ and voltage across the resistor and capacitor are,

$$v_R(t) = i(t)\, R$$

or, $$v_R(t) = V\, e^{\frac{-t}{RC}}$$

or, $$v_C(t) = \frac{1}{C}\int_0^t i(t)\, dt$$

or, $$[v - v_R(t)] = \frac{1}{C}\frac{v}{R}\left(e^{\frac{-t}{RC}}\right)\Big|_0^t (-RC) = -ve^{\frac{-t}{RC}}\Big|_0^t = -v\left(e^{\frac{-t}{RC}} - 1\right)$$

$$\therefore \quad v_C(t) = v\left(1 - e^{\frac{-t}{RC}}\right)$$

At t = 0 ; $i(t) = \frac{V}{R}$ and $v_C(t) = 0$, $v_R(t) = V$

At, t = ∞ ; (t) = 0 and $v_C(t) = V$, $v_R(t) = 0$

At, t = RC = τ ; $i(t) = \frac{V}{R} e^{-1} = 0.368 \frac{V}{R}$

and, $v_C(t) = V(1 - e^{-1}) = 0.632$ V

$v_R(t) = 0.3668$ V

These $i(t)$, $v_R(t)$ and $v_C(t)$ are plotted in figure 3.9 (a) and 3.9 (b).

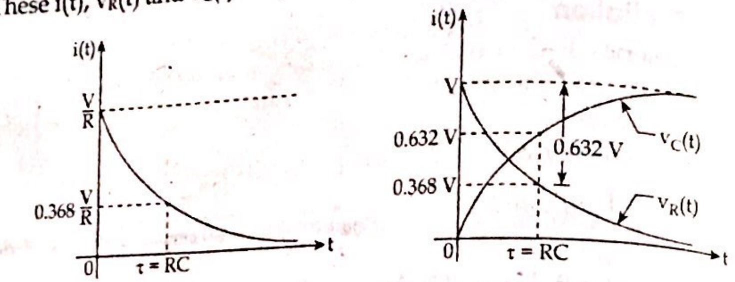


Figure 3.9: (a) for i(t) and (b) for $V_R(t)$ and $V_C(t)$

Here, $\tau = RC$ is known as time constant of the circuit.

Let us now analyse another transient condition of the R-C circuit as the circuit reaches at steady state (at $t = \infty$) and suddenly the voltage is withdrawn by opening the switch S and throwing it to S' as shown in figure 3.10 at $t = 0$.

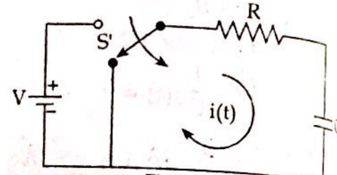


Figure 3.10:

Then by KVL,

$$Ri'(t) + \frac{1}{C}\int_0^t i(t) + v_C(0^+) = 0$$

Differentiating, we get

$$R\frac{di(t)}{dt} + \frac{1}{C}i'(t) = 0$$

Its solution is, $i'(t) = K\,e^{\frac{-1t}{RC}}$

However, at $t = 0^+$, the capacitor keeps the steady state voltage $v_C(0^+) = V$, and the direction of $i(t)$ during discharge is negative. Thus,

$$i'(0^+) = \frac{-V}{R}$$

or, $$\frac{-V}{R} = K'\,e^0 \text{ or } K' = \frac{-V}{R}$$

Hence, $$i'(t) = \frac{-V}{R}e^{\frac{-1t}{RC}}$$

The corresponding transient voltage across the resistor and capacitor are given by,

$$v_R'(t) = i'(t)\,R$$

or, $$v_R'(t) = V\,e^{\frac{-1t}{RC}}$$

and, $$v_C'(t) = \frac{1}{C}\int i'(t)\,dt$$

or, $$v_C'(t) = V\,e^{\frac{-1t}{RC}}$$

[Obviously $v_R'(t) + v_C'(t) = 0$]

These $i'(t)$, $v_R'(t)$ and $v_C'(t)$ are plotted in figure 3.11(a) and (b).

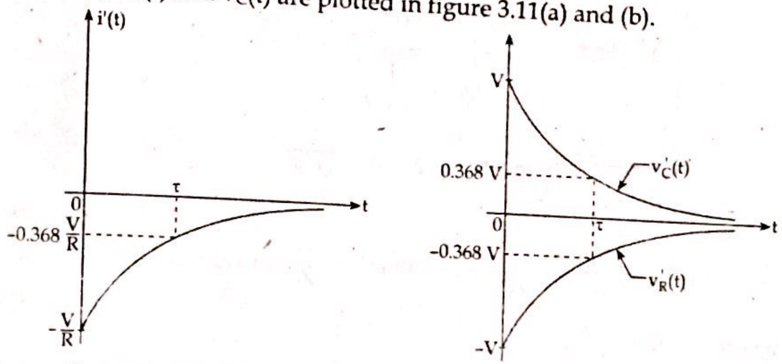


Figure 3.11: (a) for $i'(t)$ and (b) for $V_R'(t)$ and $V_C'(t)$

3.4.2 Sinusoidal Excitation

Consider a series R-C circuit excited by a sinusoidal voltage source as shown in figure 3.12. The switch S is closed at time $t = 0$.

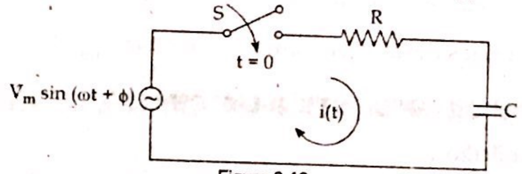


Figure 3.12:

Applying KVL,

$$Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt + v_C(0^+) = V_m \sin(\omega t + \phi)$$

On differentiating, we get,

$$\frac{di(t)}{dt} + \frac{1}{RC}i(t) = \frac{V_m}{R}\omega\cos(\omega t + \phi)$$

General solution of this differential equation is,

$$i(t) = i_C(t) + i_p(t) = K\,e^{\frac{-t}{RC}} + \left[\frac{V_m \sin \omega t + \phi}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\right] + \tan^{-1}\frac{1}{\omega CR}$$

Since capacitor behaves as a short circuit at switching,

$\therefore$ $$i(0^+) = \frac{V_m \sin\phi}{R}$$

or, $$\frac{V_m \sin\phi}{R} = K + \frac{V_m}{R^2 + \left(\frac{1}{\omega C}\right)^2}\sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

or, $$K = \frac{V_m \sin\phi}{R} - \frac{V_m}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$\therefore \quad i(t)\left[\frac{V_m \sin\phi}{R} - \frac{V_m \sin\left(\phi + \tan^{-1}\frac{1}{\omega CR}\right)}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}}\right] e^{\frac{-t}{RC}}$$

$$+ \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}\right)^2}} \sin\left(\omega t\ \phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

or, $$i(t) = \left(\frac{V_m \sin\phi}{R} - \frac{V_m}{Z}\sin(\phi+\theta).e^{\frac{-t}{RC}} + \frac{V_m}{Z}\sin(\omega t + \phi + \theta)\right)$$

where, $\theta = \tan^{-1}\frac{1}{\omega CR}$

and, $Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$ = impedance of R-C circuit

The first term of i(t) is the transient current with decay factor $e^{\frac{-t}{RC}}$, whi dies out with time constant RC and the second term is the steady sta current which leads the applied voltage by $\theta = \tan^{-1}\frac{1}{\omega CR}$.

3.5 RESPONSE OF SERIES R-L-C CIRCUIT WITH

3.5.1 DC Excitation

Consider a series R-L-C series circuit with dC excitation of V_0. volts shown in figure 3.13. The switch S is closed at time t = 0.

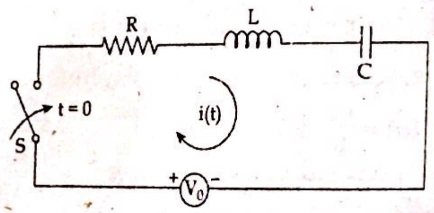


Figure 3.13

Applying KVL in the series RLC circuit,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = V_0$$

At $t = 0^+$, after the switch is closed and differentiating

$$L\frac{d^2i(t)}{dt^2} + \frac{Rdi(t)}{dt} + \frac{i(t)}{C}0$$

or, $$\left(P^2 + \frac{R}{L}P + \frac{1}{LC}\right)i(t) = 0$$

Equation **(1)** is the second order linear homogenous equation. Th characteristic equation then becomes $P^2 + \frac{R}{L}P + \frac{1}{LC} = 0$ where, the coefficie are constant. The roots of the characteristic equation then become.

$$P_1, P_2 = \frac{\frac{-R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

Let, $\alpha = \frac{-R}{2L}$

and, $\beta = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$

Hence, $P_1 = \alpha + \beta$ and, $P_2 = \alpha - \beta$

Also, the solution of differential equation **(1)** becomes

$$i(t) = K_1 e^{P_1 t} + K_2 e^{P_2 t} \quad \textbf{(2)}$$

where, K_1 and K_2 being the constants.

Case 1: when $\frac{R^2}{2L} > \frac{1}{LC}$

This time β is positive real quantity. Hence the roots P_1 and P_2 are real but unequal.

$\therefore \quad P_1 = \alpha + \beta\ ;\ P_2 = \alpha - \beta$

and, $i(t) = K_1 e^{(\alpha+\beta)t} + K_2 e^{(\alpha-\beta)t}$

$$= e^{\alpha t}(K_1 e^{\beta t} + K_2 e^{-\beta t}) \quad \textbf{(3)}$$

The current response is over-damped

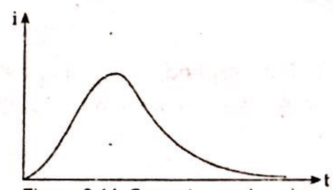


Figure 3.14: Current over-damping

Case 2: when $\frac{R^2}{2L} < \frac{1}{LC}$

This time β is imaginary and then the roots P_1 and P_2 are complex conjugate.

$\therefore \quad P_1 = \alpha + j\beta\ ;\ P_2 = \alpha - j\beta$

and, $i(t) = K_1 e^{(\alpha+j\beta)t} + K_2 e^{(\alpha-j\beta)t}$

$$= e^{\alpha t}[K_1 \cos(\beta t) + K_2 \sin(-\beta t)] \quad \textbf{(4)}$$

Thus the current is under damped or oscillatory.

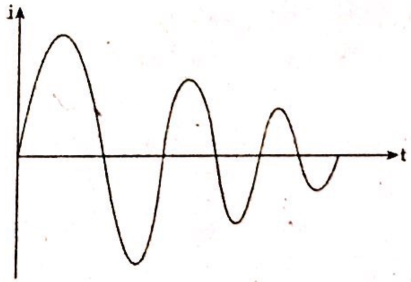


Figure 3.15: Under damped oscillation

Case 3: when $\frac{R^2}{2L} = \frac{1}{LC}$

This time β is zero. Hence the roots P_1 and P_2 are real and equal.

$$P_1 = P_2 = \alpha$$

$$\therefore \quad i(t) = K_1 e^{\alpha t} + K_2 t e^{\alpha t} = e^{\alpha t}(K_1 + K_2 t).$$

The current response is then a critically damped one.

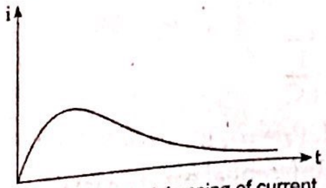


Figure 3.16: Critical damping of current

3.6.2 Sinusoidal Excitation

Consider a series R-L-C circuit with sinusoidal excitation as shown figure 3.17. The switch S is closed at time t = 0.

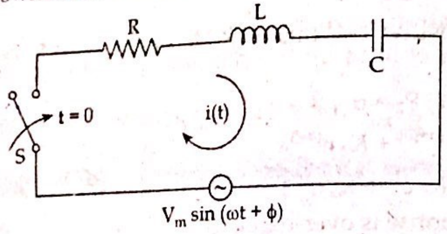


Figure 3.17:

Let the applied voltage be, $v = V_m \sin(\omega t + \phi)$,

As soon as the switch is closed, the mesh equation becomes,

$$i(t) R + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\, dt = V_m \sin(\omega t + \phi)$$

On differentiating,

$$\left(P^2 + \frac{R}{L}P + \frac{1}{LC}\right) i(t) = \frac{\omega V_m}{L}\cos(\omega t + \phi)$$

The particular solution can be obtained as follows

Let, $i_p = A\cos(\omega t + \phi) + B\sin(\omega t + \phi)$

$i_p' = -A\sin(\omega t + \phi) + B\omega\cos(\omega t + \phi)$

$i_p'' = -A\omega^2\cos(\omega t + \phi) - B\omega^2\sin(\omega t + \phi)$

Substituting the value of i_p and i_p'' in equation (2), we get,

or, $[-A\omega^2\cos(\omega t + \phi) - B\omega^2\sin(\omega t + \phi)] + \frac{R}{L}[-A\omega\sin(\omega t + \phi) + B$

$\cos(\omega t + \phi)] + \frac{1}{LC}A\cos(\omega T + \phi) + B\sin(\omega t + \phi) = \frac{\omega V_m}{L}\cos(\omega t$

Comparing both sides, we get,

or, $-B\omega^2 - A\frac{\omega R}{L} + \frac{B}{LC} = 0$

or, $A\left(\frac{\omega R}{L}\right) + B\left(\omega^2 - \frac{1}{LC}\right) = 0$ (3)

For cosine coefficients,

$$-A\omega^2 + B\frac{\omega R}{L} + A\frac{1}{LC} = \omega\frac{V_m}{L}$$

or, $A\left(-\omega^2 + \frac{1}{LC}\right) + B\frac{\omega R}{L} = \omega\frac{V_m}{L}$ (4)

From equation (3), we get,

$$A = -B\frac{\left(\omega^2 - \frac{1}{LC}\right)}{\frac{\omega R}{L}} = B\frac{\left(\frac{1}{LC} - \omega^2\right)}{\left(\frac{\omega R}{L}\right)}$$

On substituting in equation (4), we get,

$$B\left[\left(-\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right] = \frac{\omega^2 R\, V_m}{L^2}$$

$$\therefore \quad B = \frac{V_m\frac{\omega^2 R}{L^2}}{\left[\left(-\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right]} = \frac{V_m\frac{\omega^2 R}{L^2}}{\left[\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}$$

Thus, $A = \dfrac{B\left(-\omega^2 + \frac{1}{LC}\right)}{\frac{\omega R}{L}} = \dfrac{V_m\omega\left(\frac{1}{LC} - \omega^2\right)}{L\left[\left(\frac{1}{LC} - \omega^2\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}$

Hence with the values of A and B,

$$i_p = \frac{V_m\,\omega\left(-\omega^2 + \frac{1}{LC}\right)}{L\left[\left(\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}\cos(\omega t + \phi) + \frac{V_m\frac{\omega^2 R^2}{L^2}}{\left[\left(-\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}\sin(\omega t + \phi)$$

Let, $M\sin\theta = \dfrac{V_m\omega\left(-\omega^2 + \frac{1}{LC}\right)}{L\left[\left(\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}$

and, $M\cos\theta = \dfrac{V_m\frac{\omega^2 R^2}{L^2}}{\left[\left(-\omega^2 + \frac{1}{LC}\right)^2 + \frac{\omega^2 R^2}{L^2}\right]}$

Also, $\dfrac{M\sin\theta}{M\cos\theta} = \tan\theta$

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$$\text{or,}\quad \frac{\frac{V_m\omega}{L}\left(-\omega^2+\frac{1}{LC}\right)}{V_m\frac{\omega^2R^2}{L^2}} = \frac{L}{\omega R}\left(-\omega^2+\frac{1}{LC}\right) = \frac{1}{R}\left(-\omega L+\frac{1}{\omega C}\right)$$

$$\therefore\quad \theta = \tan^{-1}\left[\frac{1}{R}\left(-\omega L+\frac{1}{\omega C}\right)\right]$$

Thus, $i_p = M\sin\theta\cos(\omega t+\phi) + M\cos\theta\sin(\omega t+\phi)$

$= [\sin(\omega t+\phi)\cos\theta + \cos(\omega t+\phi)\sin\theta]\ M = M\sin(\omega t+\phi+\theta)$

$$\therefore\quad i_p = M\left[\sin\omega t+\phi+\tan^{-1}\frac{1}{R}\left(\frac{1}{\omega C}-\omega L\right)\right]$$

However, $M = \sqrt{M^2\cos^2\theta + M^2\sin^2\theta} = \dfrac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}$

$$\text{Thus,}\quad i_p = \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}\sin\left(\omega t+\phi+\text{tab}^{-1}\frac{\left(\frac{1}{\omega C}-\omega L\right)}{R}\right)$$

The complementary function ic being equal to the dC response of RL circuit.

Then, for over damped case, when $\left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$

$$i(t) = e^{\alpha t}(k_1 e^{\beta t} + k_2 e^{-\beta t}) + \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}+\omega L\right)^2}}\sin\omega t+\phi+\tan^{-1}\frac{\left(\frac{1}{\omega C}-\omega L\right)}{R}$$

For under damped case, when $\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}$

$$i(t) = e^{\alpha t}(k_1\cos\beta t + k_2\sin\beta t) + \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}+\omega L\right)^2}}$$

$$\sin(\omega t+\phi) + \tan^{-1}\frac{\left(\frac{1}{\omega C}-\omega L\right)}{R}$$

For critically damped case, when $\left(\frac{R}{2L}\right)^2 = \frac{1}{LC}$

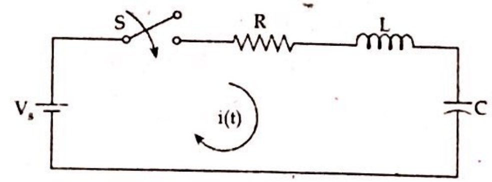
$$i(t) = e^{\alpha t}(k_1 + k_2 t) + \frac{V_m}{\sqrt{R^2+\left(\frac{1}{\omega C}+\omega L\right)^2}}\sin\left(\omega t+\phi+\tan^{-1}\frac{\left(\frac{1}{\omega C}-\omega L\right)}{R}\right)$$

SOLVED NUMERICAL EXAMPLES

1. Consider the RLC series shown in figure below.

$V_S = 2\ V;\ R = 6\ \Omega;\ L = 2\ H;\ C = 0.25\ F.$

Determine $i(0^+)$, $\frac{di}{dt}(0^+)$, $\frac{d^2i}{dt^2}(0^+)$ and $i(t)$.



Solution:

The differential equation for the current in the circuit given is given by Kirchhoff's law as,

$$L\frac{di(t)}{dt} + R_i(t) + \frac{1}{C}\int i(t)\,dt = V_S \qquad (1)$$

On differentiating,

$$\text{or,}\quad L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$$

$$\text{or,}\quad 2\frac{d^2i(t)}{dt^2} + 6\frac{di(t)}{dt} + \frac{1}{0.25}i(t) = 0$$

$$\text{or,}\quad \frac{d^2i(t)}{dt^2} + 3\frac{di(t)}{dt} + 2\,i(t) = 0 \qquad (2)$$

Substituting $P^2 = \frac{d^2i(t)}{dt}$ and $P = \frac{di}{dt}$, thus

$$P^2 + 3P + 2 = 0$$

This equation has the roots $P_1 = -1$ and $P_2 = -2$, so, that the general solution is,

$$i(t) = K_1 e^{-t} + K_2 e^{-2t} \qquad (3)$$

The constants K_1 and K_2 can be evaluated for a specific problem by a knowledge of the initial conditions. If the switch S is closed at t = 0, then,

$$i(0^+) = 0 \qquad (4)$$

(Because current cannot change instantaneously in the inductor or inductor behaves as a open circuit at $t = 0^+$).

In equation (1), the second and third voltage terms are zero at the instant of switching, $Ri(0^+)$ being zero because $i(0^+) = 0$ and $\frac{1}{C}\int_{-\infty}^{0^+} i(t)dt$ being zero because it is the initial voltage across the capacitor.

$$\text{Hence,}\quad \frac{di(0^+)}{dt} = \frac{V_S}{L} = \frac{2}{2} = 1\ \text{A/sec} \qquad (5)$$

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From equation (2), we get,

$$\frac{d^2i(0^+)}{dt^2} + 3\frac{di(0^+)}{dt} + 2\,i(0^+) = 0$$

$$\therefore \quad \frac{d^2i(0^+)}{dt^2} = -3 \times 1 - 2 \times 0 = -3\ A/sec^2 \qquad (6)$$

The above two initial conditions (4) and (5), substituted into the general equation (3) give the equations,

$$K_1 + K_2 = 0 \text{ and } -K - 2K_2 = 1$$

The solution of these equations is $K_1 = 1$ and $K_2 = -1$.

Hence the particular solution is,

$$i(t) = e^{-t} - e^{-2t}\ A$$

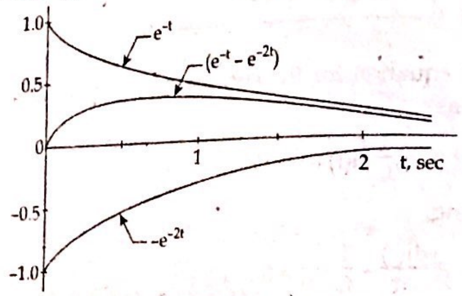
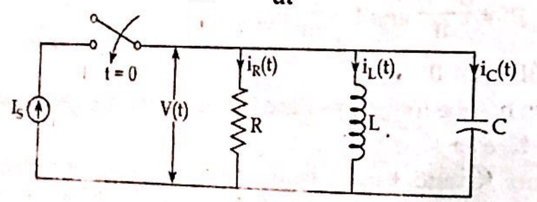


Figure: Response including the two component parts that sum to give the response.

2. **Consider the RLC parallel circuit shown in figure below;**

$$I_S = 2A,\ R = \frac{1}{16}\,\Omega,\ L = \frac{1}{16}\,H,\ C = 4\ F.$$

Determine the $V(0^+)$, $\frac{dV(0^+)}{dt}$, $\frac{d^2V(0^+)}{dt^2}$ and $V(t)$.



Solution:

The circuit equation by KVL is,

$$I_S = i_R(t) + i_L(t) + i_C(t)$$

or, $$I_S = \frac{V(t)}{R} + \frac{1}{L}\int V(t)\,dt + C\frac{dV(t)}{dt} \qquad (1)$$

Differentiating and using numerical values for R, L and C gives,

or, $$0 = 16\frac{dV(t)}{dt} + 16\,V(t) + 4\frac{d^2V(t)}{dt^2}$$

or, $$\frac{d^2V(t)}{dt^2} + 4\frac{dV(t)}{dt} + 4\,V(t) = 0 \qquad (2)$$

Substituting P^2 for $\frac{d^2V(t)}{dt^2}$ and P for $\frac{dV(t)}{dt}$, thus $P^2 + 4P + 4 = 0$.

This equation has the repeated roots $P_{1,2} = -2$. Thus the general solution to our problem with repeated roots $V(t) = K_1 e^{-2t} + K_2 t\, e^{-2t}$ (3)

From the circuit, $V(0^+)$ must equal zero since the capacitor acts as a short circuit at the initial instant, *i.e.*, $V(0^+) = 0$ (4)

In equation (1), the first and second current terms are zero at the instant of switching, because $V(0^+) = 0$ initial instant. Hence,

$$\frac{dV(0^+)}{dt} = \frac{I_S}{C} = \frac{2}{4} = \frac{1}{2}\ V/sec \qquad (5)$$

From equation (2)

$$\frac{d^2V(0^+)}{dt^2} + 4 \times \frac{1}{2} + 4 \times 0 = 0$$

$$\therefore \quad \frac{d^2V(0^+)}{dt^2} = -2\ V/sec^2 \qquad (6)$$

Replacing value of equation (4) and (5) in equation (3), give the equations

$$K_1 = 0 \text{ and } -2K + K_2 = \frac{1}{2} \text{ or } K_2 = \frac{1}{2}$$

Hence the particular solution of our problem is,

$$V(t) = \frac{1}{2}\,t\,e^{-2t}\ V \qquad (7)$$

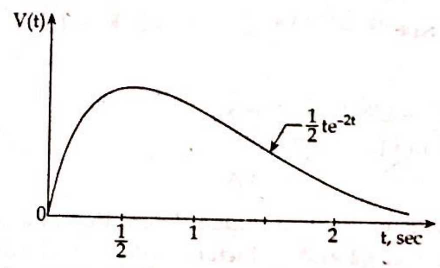


Figure: Voltage response of network of given circuit as given by equation (7).

3. **In a series R-L-C circuit, R = 5 Ω, L = 1H and C = 1F. A dc voltage of 20 V is applied at t = 0. Obtain i(t).**

Solution:

Applying KVL yields,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = V \qquad (1)$$

On differentiation,

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{i(t)}{C} = 0$$

or, $$\left(LP^2 + RP + \frac{1}{C}\right) i = 0 \qquad (2)$$

Thus the characteristic equation is,

$$LP^2 + RP + \frac{1}{LC} = 0$$

or, $$P^2 + 5P + 1 = 0$$

∴ $$P_1, P_2 = \frac{-5 \pm \sqrt{25-4}}{2} = \frac{-5 \pm 4.58}{2} = -0.21, -4.79$$

∴ The solution becomes

$$i(t) = K_1 e^{-0.21t} + K_2 e^{-4.79t}$$

As current in inductor cannot change instantaneously and also voltage across the capacitor does not change instantaneously.

Hence, $i(0^+) = 0$ and $\frac{1}{C}\int i\, dt = 0$

Thus, from equation (1)

$$L\frac{di(t)}{dt} = V \text{ at } t = 0^+$$

∴ $$\frac{di(0^+)}{dt} = \frac{V}{L} = \frac{20}{1} = 20 \text{ A/sec}$$

Again, substituting, $i = 0$ at $t = 0^+$ in equation (4), we get,

$$0 = K_1 e^{-0.21\times0} + K_2 e^{-4.79\times0} = K_1 + K_2$$

Again differentiating equation (4), we get,

$$\frac{di}{dt} = -0.21\, K_1 e^{-0.21t} - 4.79\, K_2 e^{-4.79\times t} = 0.21\, K_1 - 4.79\, K_2$$

At $t = 0^+$,

or, $20 = 0.21\, K_2 - 4.79\, K_2 = -4.58\, K_2$ [∵ $K_1 + K_2 = ...$

∴ $K_2 = -4.37$ and $K_1 = 4.37$

∴ $i(t) = (4.37\, e^{-0.21t} - 4.37\, e^{-4.79t})$ A

4. **A dC voltage of 200 V is suddenly applied to a series L-R circuit having R = 20 Ω and inductance 0.2 H. Determine the voltage drop across the inductor at the instant of switching on and 0.02 seconds later.**

Solution:

Since inductor initially behaves as a open circuit at $t = 0^+$, then $i(0^+) = ...$ hence, at instant of switching, the voltage drop across inductor is 200 ... After the instant of switching the current is given by,

$$i(t) = \frac{V}{R}\left(1 - e^{\frac{-Rt}{L}}\right) = \frac{200}{20}\left(1 - e^{\frac{-20t}{0.2}}\right) = 10\,(1 - e^{-100t})$$

At $t = 0.02$ sec,

$$i(t) = 10(1 - e^{-100\times0.02}) = 8.646 \text{ A}$$

or, $$Ri(t) + L\frac{di(t)}{dt} = 200$$

or, $$L\frac{di(t)}{dt} = 200 - R\, i(t)$$

The voltage across the inductor after lapse of 0.02 second from switching is given as $L\frac{di(0.02)}{dt} = 200 - 20 \times 8.664 = 27$ V.

5. **A resistance R and 5 μF capacitor are connected in series across a 100 V dC supply. Calculate the value of R such that the voltage across the capacitor becomes 50 V in 5 sec after the circuit is switched on.**

Solution:

In case of charging, the voltage at any time across the capacitor is given as,

$$v_C(t) = V\left(1 - e^{\frac{-t}{RC}}\right)$$

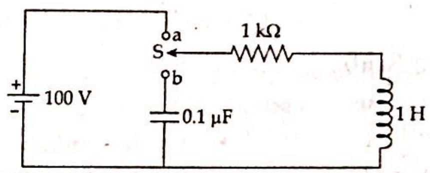
or, $$50 = 100\left(1 - e^{\frac{-5}{R\times5\times10^{-6}}}\right)$$

or, $$0.5 = \left(1 - e^{\frac{-10^6}{R}}\right)$$

or, $$e^{\frac{-10^6}{R}} = 0.5$$

∴ $$R = 1.45 \times 10^6\ \Omega$$

6. **In the given circuit as shown in figure below, switch S is changed from position 'a' to 'b' at t = 0. Find values of i, $\frac{di}{dt}$, $\frac{d^2i}{dt^2}$ at $t = 0^+$.**



Solution:

At position 'a';

The steady state value of current $i(0^-) = \frac{100}{1{,}000} = 0.1$ A

At position 'b';

$$i(0^+) = i(0^-) = 0.1 \text{ A}$$

Applying KVL, we have

$$1{,}000\, i(t) + 1 \cdot \frac{di(t)}{dt} + \frac{1}{0.1 \times 10^{-6}}\int_{-\infty}^{t} i(t)\, dt = 0 \qquad (1)$$

Since initially capacitor is uncharged and at switching instant capacitor behaves as a short circuit *i.e.*, last term of left hand side of above equation is zero. Then from equation (1), we get,

∴ $$1000\, i(t) + \frac{di(t)}{dt} = 0$$

$$\therefore \quad \frac{di(0^+)}{dt} = -1000\, i(0^+) - 1{,}000 \times 0.1 = -1000 \text{ A/sec}$$

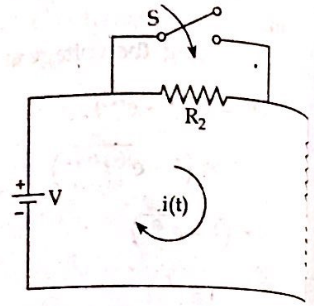
On differentiating equation (1) becomes,

$$\frac{d^2i(t)}{dt^2} = -1{,}000 \frac{di(t)}{dt} \frac{-1}{0.12 \times 10^{-6}} i(t)$$

$$\therefore \quad \frac{d^2i(0^+)}{dt^2} = -1{,}000 \times (-100) - 10^7 \times (0.1) = -(-10^5 + 10^6)$$

$$= -9 \times 10^5 \text{ A/sec}^2$$

7. In the network shown in figure, switch S is closed at t = 0, a steady state current having previously been attained. Solve for the current as a function of time.



Solution:

Steady state current (before the switching action takes place),

$$i(0^-) = \frac{V}{R_1 + R_2}$$

(Since inductor behaves as a short circuit at $t = \infty$).

When switch is closed, R_2 is short-circuited

Applying KVL,

$$V = L\frac{di(t)}{dt} + R_1\, i(t)$$

or, $$\frac{di(t)}{dt} + \frac{R_1}{L} i(t) = \frac{V}{L}$$

The general solution of the above V equation is given as differential,

$$i(t) = \frac{V}{R_1} + K e^{\frac{-R_1 t}{L}}$$

and, $$i(0^+) = i(0^-) = \frac{V}{R_1 + R_2}$$

or, $$\frac{V}{R_1 + R_2} = \frac{V}{R_1} + K$$

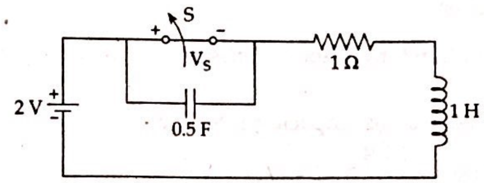
Hence, $$K = \frac{-V}{R_1}\left(\frac{R_2}{R_1 + R_2}\right)$$

Thus, the solution of this problem is,

$$i(t) = \frac{V}{R_1} - \frac{VR_2}{R_1(R_1 + R_2)} \cdot e^{\frac{-R_1 t}{L}}$$

$$\therefore \quad i(t) = \frac{V}{R_1}\left(1 - \frac{R_2}{R_1 + R_2} \cdot e^{\frac{-R_1 t}{L}}\right) \text{A}$$

8. The circuit shown in figure is in the steady state with the switch S is closed. The switch is opened at t = 0. Determine voltage across the switch, V_S and $\frac{dV_S}{dt}$ at $t = 0^+$.



Solution:

When circuit is in the steady state with the switch S closed, capacitor is short circuited *i.e.*, voltage across capacitor is zero.

or, $V_S(0^-) = 0$ and steady state current $i(0^-) = \frac{2}{1} = 2$ A

and when switch is opened at t = 0, the capacitor behaves as a short circuit, so

$$V_S \text{ at } t = 0^+ \text{ or } V_S(0^+) = 0$$

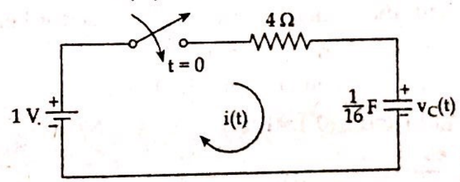
$$i(0^+) = i(0^-) = 2 \text{ A}$$

or, $$C\frac{dV_S}{dt} = i(t)$$

or, $$\frac{dV_S}{dt} = \frac{1}{C} i(t)$$

Hence, $$\frac{dV_S}{dt}(0^+) = \frac{1}{C} i(0^+) = \frac{1}{0.5} \times 2 = 4 \text{ V/sec}$$

9. Using classical method of solution of differential equations, find the value of $v_C(t)$ for $t > 0$ in the circuit shown in figure. Assume initial condition $v_C(0^-) = 9$ V.



Solution:

Applying KVL in the circuits, we have,

$$1 = 4i(t) + v_C(t)$$

and, $$v_C(t) = 9 + \frac{1}{C}\int_0^t i(t)\, dt$$

Hence, $$1 = 4i(t) + 9 + 16\int_0^t i(t)\, dt$$

or, $$4i(t) + 16\int_0^t i(t)\, dt = -8$$

or, $i(t) + 4\int_0^t i(t)\,dt = -2$

On differentiating,

$$\frac{di(t)}{dt} + 4i(t) = 0$$

The general solution for the above differential equation is,

$$i(t) = K e^{-4t}$$

Since initial voltage across capacitor is 9 V, therefore,

Initial current $i(0^+) = \frac{1-9}{4} = -2 = K e^0$

$\therefore$ $K = -2$

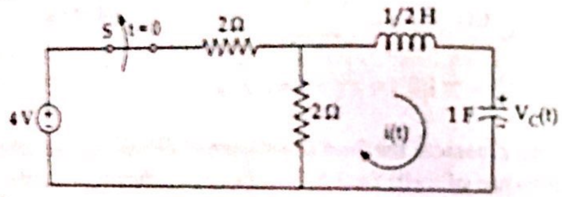
So the value of the current, $i(t) = -2e^{-4t}$

Then, $v_C(t) = 9 + \frac{1}{C}\int_0^t i(t)\,dt$

or, $v_C(t) = 9 + 16\left(\frac{-2e^{-4t}}{-4}\right)\Big|_0^t = 9 + 8e^{-4t}\Big|_0^t$

$\therefore$ $v_C(t) = 9 + (8e^{-4t} - 8) = 1 + 8e^{-4t}$ V

10. The circuit shown in figure is in steady state with the switch closed. The switch is opened at t = 0. Determine $i_L(t)$ in the circuit.



Solution:

At steady state with the switch S closed. The capacitor behaves as a open circuit. Hence $v_C(0^-) = 2$ V and $i_L(0^-) = 0$.

Now, switch S is opened, then applying KVL with

$$v_C(0^+) = v_C(0^-) \text{ and } i_L(0^+) = i_L(0^-)$$

or, $2i_L(t) + \frac{1}{2}\frac{di_L(t)}{dt} + \frac{1}{1}\int_0^t i_L(t) + v_C(0^-) = 0$

On differentiating, we get,

$$\frac{2di_L}{dt} + \frac{1}{2}\frac{d^2i_L(t)}{dt^2} + i_L(t) = 0$$

or, $\frac{d^2i_L(t)}{dt^2} + 4\frac{di_L(t)}{dt} + 2i_L(t) = 0$

Characteristic equation is $P^2 + 4P + 2 = 0$ having the roots $P_1 = -3.414$, $P_2 = -0.586$

Thus, $i_L(t) = K_1 e^{-3.414t} + K_2 e^{-0.586t}$

$i_L(0^+) = 0$ requires that $K_1 + K_2 = 0$

Now, $v_L(0^+) = -v_C(0^+) = -2 = \frac{L\,di_L(0^+)}{dt}$

or, $\frac{2di_L(0^+)}{dt} = \frac{-2}{L} = \frac{-2}{\left(\frac{1}{2}\right)} = -4$

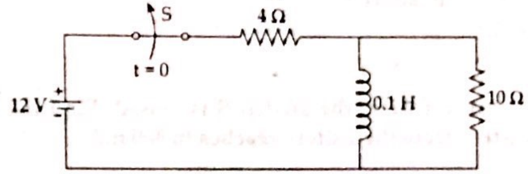
so, $-3.414K_1 - 0.586K_2 = -4$

Solving for K_1 and K_2 yields

$\therefore$ $K_1 = 1.414$ and $K_2 = -1.414$

Thus, $i_L(t) = 1.414\left(e^{-3.14t} - e^{-0.586t}\right)$ A.

11. The 12 V battery in figure is disconnected (opened) at t = 0. Find the inductor current and voltage as a function of time.



Solution:

Assume the switch S has been closed for a long time before t = 0. The inductor behaves as a short circuit,

i.e., $v_L(0^-) = 0$ and $i_L(0^-) = \frac{12}{4} = 3$ A

After the battery is disconnected at t > 0, applying KVL,

or, $0.1\frac{di_L}{dt} + 10\,i_L(t) = 0$

or, $\frac{di_L(t)}{dt} + 100\,i_L(t) = 0$

This gives $i_L(t) = K e^{-100t}$

Since $i_L(0^+) = i_L(0^-) = 3 = Ke^0$

$\therefore$ $K = 3$

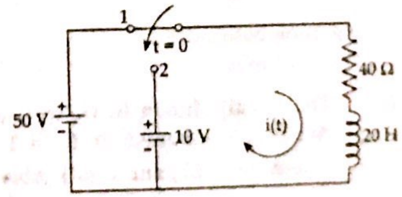
Hence, $i_L(t) = 3e^{-100t}$

$\therefore$ $v_L(t) = L\frac{di_L(t)}{dt}$

$= 0.1 \times 3 \times (-100)\,e^{-100t}$

$= -30\,e^{-100t}$ V

12. The switch in figure has been in position 1 for a long time; it is moved to 2 at t = 0. Obtain the expression for i, for t > 0.



Solution:

When the switch is at 1,

$$i(0^-) = \frac{50}{40} = 1.25\ A$$

When the switch is at 2,

$$i(0^+) = i(0^-) = 1.25\ A$$

Applying KVL,

or, $10 = 40\, i(t) + 20 \frac{di(t)}{dt}$

or, $\frac{di(t)}{dt} + 2i(t) = 0.5$

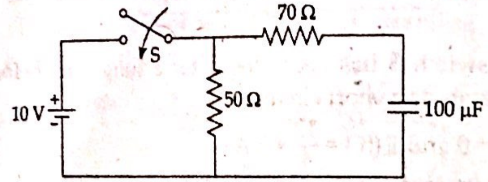
$\therefore$ $i(t) = \frac{0.5}{2} + K e^{-2t}$

Putting $i(0^+) = 1.25$ in above equation,

or, $1.25 = 0.25 + K$

Hence, $i(t) = 0.25 + e^{-2t}$ A.

13. **In the given figure, the switch S is closed. Find the time whe[n] the current from the battery reaches to 500 mA.**



Solution:

Let the current through 50 Ω be I_1 and through 70 Ω or 100 μF be I_2 afte[r] the switch S is closed.

$$I_1 = \frac{10}{50} = 0.2\ A = 200\ mA$$

However, $I = I_1 + I_2$

or, $500 = 200 + I_2$

$\therefore$ $I_2 = 300$ mA

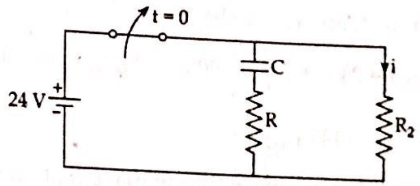
This I_2 equals to $\frac{70}{10} e^{\frac{-t}{70 \times 100 \times 10^{-6}}}$

or, $300 = 7 \times e^{\frac{-t}{7000 \times 10^{-6}}}$ $\left[\because i_C(t) = \frac{V}{R} e^{\frac{-t}{RC}}\right]$

Taking ln-on both sides,

$\therefore$ $t = 5.2$ m sec

14. **The circuit shown in figure was under steady state before the switch was opened. If $R_1 = 1\ \Omega$, $R_2 = 2\ \Omega$ and $C = 0.167$ F, determine $v_C(0^-)$ and $v_C(0^+)$. Also find $i(0^+)$.**



Solution:

Since at steady state, capacitor behaves as a open circuit or the voltage cannot change instantaneously in the capacitor, we have,

$$v_C(0^-) = v_C(0^+) = 24\ V$$

After the switch is opened at $t = 0^+$,

$$v_C = v_{R1} + v_{R2}$$

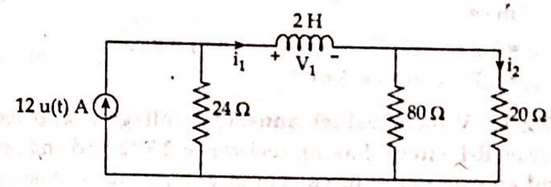
or, $24 = i(1 + 2)$

$\therefore$ $i(0^+) = 8A$

15. **For the network shown in figure below, find i_1, i_2 and V_1 at**

a) $t = 0^-$ b) $t = 0^+$

c) $t = \infty$ d) $t = 50$ ms

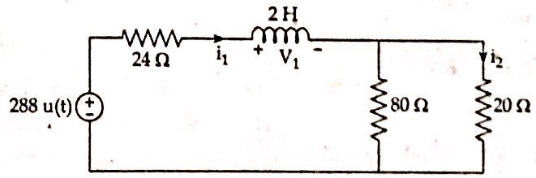


Solution:

a) At $t = 0^-$: By the definition of step signal at $t = 0^-$. There is no energy source.

Hence, $i(0^-) = 0$ A; $i_2(0^-) =$ A; $V_1(0^-) = 0$ V

b) At $t = 0^+$. First we convert current source of 12 u(t) in an equivalent voltage source = 12 u(t) × 24 = 288 u(t) as shown in figure.



Applying KVL,

$$288\, u(t) = 24\, i_1 + 2\frac{di_1}{dt} + 80\,(i_1 - i_2) \quad (1)$$

and, $i_2 = i_1 \times \frac{80}{20 + 80}$ (By current division rule)

$\therefore$ $i_2 = 0.8\, i_1$ (2)

From equation (1) and (2), we have,

$$288\,u(t) = 24\,i_1 + 2\frac{di_1}{dt} + 80\,(i_1 - 0.8\,i_1) = 40\,i_1 + 2\frac{di_1}{dt}$$

or, $\frac{di_1}{dt} + 20\,i_1 = 144\,u(t)$

The general solution of above differential equation is given as,

$$i(t) = \frac{144}{20} + K\,e^{-20t}$$

At $t = 0^+$, inductor behaves as a open circuit, *i.e.*, $i(0^+) = 0$. This gi...

$$K = \frac{-144}{20}$$

Hence, $i(t) = \frac{144}{20}(1 - e^{-20t}) = 7.2\,(1 - e^{-20t})$.

and, $i_2(t) = 0.8\,i_1(t) = 5.76\,(1 - e^{-20t})$

so, $i(0^+) = 0$ A; $i_2(0^+) = 0$ A; $V_1(0^+) = 288$ V

c) At $t = \infty$. Since inductor behaves as a short circuit at $t \to \infty$,

so, $V_1(\infty) = 0$.

d) At t = 50 m sec

∴ $i_1 = 7.2\,(1 - e^{20\times 50\times 10^{-3}}) = 7.2\,(1 - e^{-1}) = 4.55$ A

∴ $i_2 = 5.76\,(1 - e^{-1}) = 3.64$ A

16. **A 50 Hz 300 V (peak value) sinusoidal voltage is applied at t... to a series R-L circuit having resistance 2.5 Ω and inductance... H. Find an expression of current at any instant t. Also, calcul... the value of transient current, steady current and result... current 0.01 sec after switching on.**

Solution:

Impedance of R-L circuit,

$$Z = \sqrt{R^2 + (2\pi fL)^2}$$

∴ $Z = \sqrt{(2.5)^2 + (2\pi \times 50 \times 0.1)^2} = 31.50\ \Omega$

Transient current,

$$i_C(t) = K\,e^{\frac{-Rt}{L}} = K\,e^{-25t}$$

Steady state current,

$$i_p(t) = \frac{V_m}{Z}\sin\left(\omega t + \phi - \tan^{-1}\frac{\omega L}{R}\right)$$

$$= \frac{300}{31.5}(314t - 85.45°)$$

$\left(\text{Since voltage is applied at } t = 0, \textit{ i.e., } \phi = 0 \text{ and } \tan^{-1}\frac{314 \times 0.1}{2.5} = 85.45°\right)$

$$= 9.52 \sin(314\,t - 1.49)$$

$$= 9.52 \sin(314\,t - 1.49)$$

Hence, $i(t) = i_C(t) + i_p(t) = K\,e^{-25t} + 9.52 \sin(314\,t - 1.49)$

Since $i(0^+) = 0$ A,

∴ $0 = K + 9.52 \sin(-1.49)$

∴ $K = 9.49$

Hence required expression of current is given by,

$$i(t) = 9.49\,e^{-25t} + 9.52 \sin(314\,t - 1.49)$$

At t = 0.01 sec,

∴ $i(t) = i_C(t) + i_p(t) = 16.88$ A

17. **A voltage V = 300 sin 314 t is applied at t = 2.14 m sec to a series R-C circuit having resistance 10Ω and capacitance 200 μF. Find an expression for current. Also, find the value of current 1 msec after switching on.**

Solution:

Since the voltage is not applied at t = 0 but at ϕ

where, $\phi = 2.14$ msec $= 2.14 \times 10^{-3} \times 314 = 0.672$ radian

Impedance of R-C circuit,

$$Z = \sqrt{10^2 + \left(\frac{1}{314 \times 200 \times 10^{-6}}\right)^2} = 18.8\ \Omega$$

Transient current,

$$i_C(t) = K\,e^{\frac{-t}{RC}} = K\,e^{\frac{-t}{10 \times 200 \times 10^{-6}}} = K\,e^{-500t}$$

Steady state current,

$$i_p(t) = \frac{V_m}{Z}\sin\left(\omega t + \phi + \tan^{-1}\frac{1}{\omega CR}\right)$$

$$= \frac{300}{18.8}\sin\left[314t + 0.672 + \tan^{-1}\left(\frac{1}{314 \times 200 \times 10^{-6} \times 10}\right)\right]$$

$$= 15.96 \sin(314\,t + 0.672 + 1.59)$$

Hence, $i(t) = i_C(t) + i_p(t) = K\,e^{-500t} + 15.96 \sin(314\,t + 0.672 + 1.59)$

Since capacitor behaves as a short circuit at switching.

∴ $i(2.14 \text{ m sec}) = 300 \sin\frac{(314 \times 2.14 \times 10^{-3})}{10} = 18.67$ A

Hence, $18.67 = K(1) + 15.96 \sin(0.672 + 1.59)$

∴ $K = 18.67 - 12.29 = 6.38$

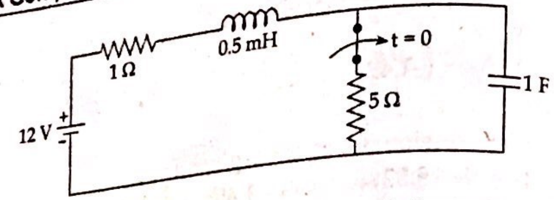
Therefore, the required expression of current given by,

$$i(t) = 6.38\,e^{-500\times 10^{-3}} + 15.96 \sin(314 \times 10^{-3} + 2.262)$$

$$= 3.87 + 8.55$$

$$= 12.42 \text{ A}$$

18. **The switch in current of figure has been closed for a very long time. It opens at t = 0. Find $v_C(t)$ for t > 0 using differential equation approach.**



Solution:

At steady state (with switch is closed), inductor behaves as a short circuit while capacitor behaves as a open circuit, circuit is shown in figure (a).

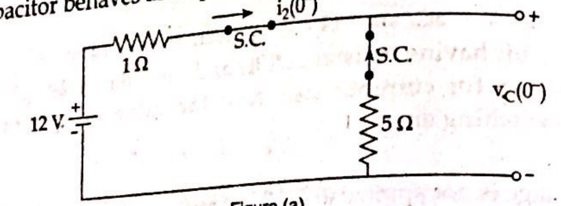
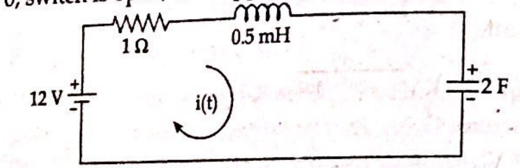


Figure (a)

$\therefore \quad i_L(0^-) = \dfrac{12}{1+5} = 2$ A

$v_C(0^-) = 2 \times 5 = 10$ V

Now at t = 0, switch is open, then applying KVL,



or, $\quad 12 = 1 \times i(t) + 0.5 \times \dfrac{di(t)}{dt} + \dfrac{1}{2}\int_0^t i(t)\,dt + 10$

or, $\quad 2 = i(t) + 0.5\dfrac{di(t)}{dt} + \dfrac{1}{2}\int_0^t i(t)\,dt$

On differentiating,

$$0 = \frac{di(t)}{dt} + 0.5\frac{d^2i(t)}{dt} + \frac{1}{2}i(t)$$

or, $\quad \dfrac{d^2i(t)}{dt^2} + 2\dfrac{di(t)}{dt} + i(t) = 0$

Characteristic equation is $P^2 + 2P + 1 = 0$

or, $\quad (P+1)^2 = 0$

Having the roots, $P_{1,2} = -1$

Thus, the general solution,

$i(t) = (K_1 + K_2t)\,e^{-t}$

At $t = 0^+$,

i) $\quad i(0^+) = i_L(0^-) = 2 = K_1 e^0$

$\therefore \quad K_1 = 2$

ii) $\quad v_C(0^+) = v_C(0^-) = 10 = \dfrac{1}{2}\int_0^t i(t)\,dt\Big|_{t=0} + 10$

Gives $K_2 = 2$

Hence, $i(t) = (2 + 2t)\,e^{-t}$

so, $\quad v_C(t) = \dfrac{1}{2}\int_0^t i(t)\,dt + 10 = \int_0^t (1+t)\,e^{-t}\,dt + 10$

$= (1+t)\,-e^{-t}\Big|_0^t - \int_0^t 1 - e^{-t}\Big|_0^t\,dt + 10$

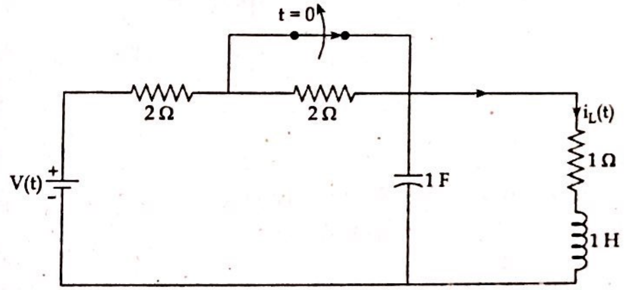
$= (1+t)\,(1+e^{-t}) - \int_0^t (1 - e^{-t})\,dt + 10$

$= 1 - te^{-t} - e^{-t} + t - (t + e^{-t})\Big|_0^t + 10$

$= 1 - te^{-t} - e^{-t} + t - (t + e^{-t} - 1) + 10$

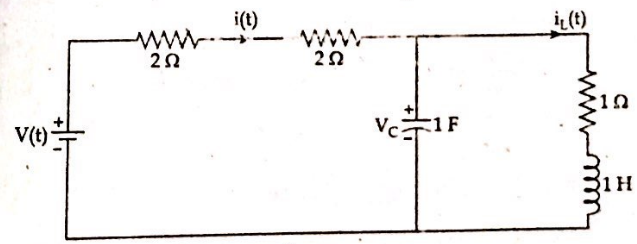
$= 12 - te^{-t}$ V

19. **Consider the network shown in figure. The switch is initially closed for long time. The switch is opened at t = 0. Find differential equation relating $i_L(t)$ with v(t) and also evaluate initial conditions.**



Solution:

When switch is initially closed for a long time, the capacitor behaves as an open circuit and the inductor behaves as an open circuit and the inductor behaves as a short circuit *i.e.*, $v_C(0^-) = \dfrac{V(t)}{3}$ and $i_L(0^-) = \dfrac{V(t)}{3}$. After switch is opened, the circuit is reduced as shown in figure.



Applying KVL, in outer loop, we have,

$$V(t) = 4\,i(t) + 1\,i_L(t) + 1\frac{di_L(t)}{dt}$$

or, $V(t) = 4\,i(t) + i_L(t) + \frac{di_L(t)}{dt}$

But, $i(t) = C\frac{dv_C(t)}{dt} + i_L(t)$

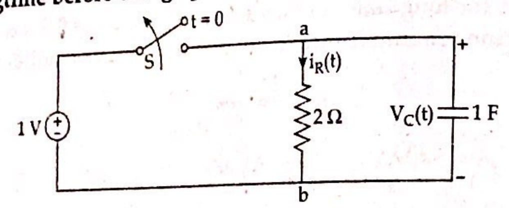
or, $i(t) = 1\frac{d}{dt}\left[1 \cdot i_L(t) + 1 \cdot \frac{di_L(t)}{dt}\right] + i_L(t) = \frac{di_L(t)}{dt} + \frac{d^2i_L(t)}{dt^2} + i_L(t)$

From equation (1) and (2), we have,

$$V(t) = 4\left[\frac{di_L(t)}{dt} + \frac{d^2i(t)}{dt^2} + i_L(t)\right] + i_L(t) + \frac{di_L(t)}{dt}$$

$$\therefore \quad V(t) = \frac{4d^2i_L(t)}{dt^2} + 5\frac{di_L(t)}{dt} + 5\,i_L(t)$$

20. **Calculate the voltage, $V_C(t)$ and current $i_R(t)$ for ≥ 0 for the circuit shown in figure below. Assume that switch S was closed for longtime before being opened at t = 0.**



Solution:

At steady state with switch S closed, $v_C(0^-) = 1$ V

When switch is opened at t = 0, applying KVL,

$$1 = 2\,i_R(t) + \frac{1}{1}\int_0^t i_R(t)$$

On differentiating, we get,

$$2\frac{di_R(t)}{dt} + i_R(t) = 0$$

Its solution is,

$$i_R(t) = K\,e^{\frac{-1t}{2}}$$

At $t = 0^+$; $i_R(0^+) = \frac{1}{2} = K$

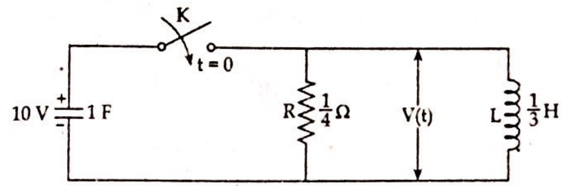
Hence, $i_R(t) = \frac{1}{2}e^{\frac{-1t}{2}}$

and, $-v_C(t) = 1 - \frac{1}{1}\int_0^t i_R(t) = 1 - \int_0^t \frac{1}{2}e^{\frac{-1t}{2}}\,dt$

$$= 1 - \frac{1}{2}\left.\frac{e^{\frac{-1t}{2}}}{\left(\frac{-1}{2}\right)}\right|_0^t = 1 - \left(1 - e^{\frac{-1t}{2}}\right) = e^{\frac{-1t}{2}}$$

$\therefore \quad v_C(t) = e^{\frac{-1t}{2}}$ V

21. **In the circuit shown below, capacitor C has an initial voltage v_C = 10 volts and at the same instant, current through inductor L is zero. The switch K is closed at time t = 0. Find the expression for the voltage V(t) across the inductor L using differential equation formulation.**



Solution:

$$v_C = \frac{1}{C}\int_0^t i(t)\,dt + V(t)$$

or, $10 = \frac{1}{1}\int_0^t i(t)\,dt + V(t)$

where, i(t) be the current through the capacitor. On differentiating,

$$0 = i(t) + \frac{dv(t)}{dt} \qquad (1)$$

Applying KVL,

$$i(t) = \frac{V(t)}{\left(\frac{1}{4}\right)} + \frac{1}{\left(\frac{1}{3}\right)}\int_0^t V(t)\,dt$$

or, $i(t) = 4\,V(t) + 3\int_0^t V(t)\,dt \qquad (2)$

On differentiating,

$$0 = 4\frac{dV(t)}{dt} + 3\,V(t) + \frac{d^2V(t)}{dt^2}$$

The general solution of the above differential equation is,

$$V(t) = K_1\,e^{-t} + K_2\,e^{-3t}$$

But at $t = 0^+$; $V(0^+) = 10 = K_1 + K_2 \qquad (3)$

$i_L(0^+) = 3\int V(0^+)'dt = 0 \qquad (4)$

(Since inductor behaves as an open circuit at $t = 0^-$)

or, $3\left[-K_1 - \frac{K_2}{3}\right] = 0$

or, $K_1 = \frac{-K_2}{3} \qquad (5)$

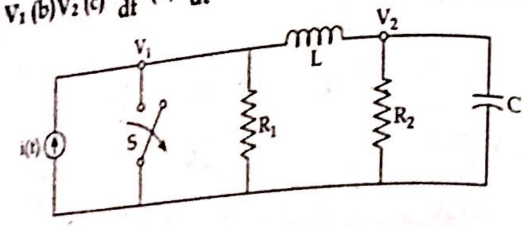
From equation (4) and (5), we have,

$\therefore \quad K_1 = -5, K_2 = 15$

Now, from equations (3) and (6), we have,

$$V(t) = -5e^{-t} + 15e^{-3t}\ V$$

22. The circuit shown below has two independent node pairs. If switch S is opened at t = 0. Find the following quantities at t = 0⁺: (a) V_1 (b) V_2 (c) $\frac{dV_1}{dt}$ (d) $\frac{dV_2}{dt}$.



Solution:

At $t = 0^+$, inductor and capacitor behaves as open circuit and short circuit respectively. Hence,

$$V_1(0^+) = R_1 i_1(0^+)$$

$$V_2(0^+) = 0$$

At any time t,

$$V_1(t) = R_1\left[i(t)\frac{-1}{L}\int\{V_1(t) - V_2(t)\}dt\right]$$

On differentiating, we have,

$$\frac{dV_1(t)}{dt} = R_1\left[\frac{di(t)}{dt}\frac{-1}{L}\{V_1(t) - V_2(t)\}\right]$$

At $t = 0^+$,

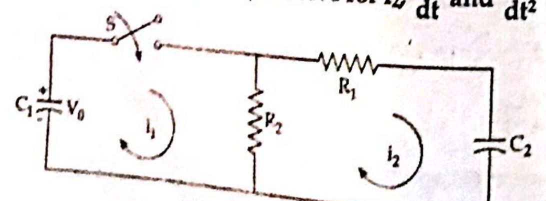
$$\frac{dV_1(0^+)}{dt} = R_1\left[\frac{di(0^+)}{dt} - \frac{R_1}{L}i(0^+)\right]$$

Also at $t = 0^+$, applying KCL,

$$0 = \frac{V_2(0^+)}{R_2} + C\frac{dV_2(0^+)}{dt}$$

$$\therefore \quad \frac{dV_2(0^+)}{dt} = 0$$

23. In the circuit shown, the capacitor C_1 is charged to voltage V_0 and the switch S is closed at t = 0. When $R_1 = 2\ M\Omega$, $V_0 = 1000$ V, $R_2 = 1\ M\Omega$, $C_1 = 10\ \mu F$ and $C_2 = 20\ \mu F$. Solve for i_2, $\frac{di_2}{dt}$ and $\frac{d^2i_2}{dt^2}$ at $t = 0^+$.



Solution:

At $t = 0^+$, both the capacitors behave as the short circuit. Hence V_0 is the voltage across R_2 as well as R_1.

Hence, $i_2(0^+) = \frac{V_0}{R_1} = \frac{1000}{2 \times 10^6} = 500 \times 10^{-6}$ A

Now applying KVL in both the loops, we have,

$$V_0 = \frac{1}{C_1}\int i_1(t)\,dt + R_2\,[i_1(t - i_2(t)] \quad (1)$$

and, $$R_2\,[i_2(t) - i_1(t) + R_2 i_2 + \frac{1}{C_2}\int i_2(t)\,dt = 0$$

or, $$(R_1 + R_2)\,i_2(t) + \frac{1}{C_2}\int i_2(t)\,dt = R_2\, i_2\,(t) \quad (2)$$

On differentiating equation (1) and (2), we have,

$$0 = \frac{1}{C_1}i_1(t) + R_2\left[\frac{di_1(t)}{dt} - \frac{di_2}{dt}\right] \quad (3)$$

and, $$(R_1 + R_2)\frac{di_2(t)}{dt} + \frac{1}{C_2}i_2(t) = R_2\frac{di_1(t)}{dt} \quad (4)$$

Putting the values of $i_1(t)$ and $R_2\frac{di_1(t)}{dt}$ from equations (2) and (4) in equation (3), we have,

$$0 = \frac{1}{C_1}\left[\frac{R_1 + R_2}{R_2}i_2(t) + \frac{1}{R_2C_2}\int i_2(t)\,dt\right] + (R_1 + R_2)\frac{di_2(t)}{dt} + \frac{1}{C_2}i_2(t) - \frac{R_2 di_2(t)}{dt}$$

or, $$\left[\frac{R_1 + R_2}{C_1R_2} + \frac{1}{C_2}\right]i_2(t) + \frac{1}{C_1R_2C_2}\int i_2(t)\frac{-1}{R_1R_2C_1C_2}\int i_2(t)\,dt \quad (5)$$

At $t = 0^+$, $\frac{1}{C_2}\int i_2(t)\,dt = 0$ (Since initially capacitor C_2 behaves as a short circuit).

$$\frac{di_2(0^+)}{dt} = -\left(\frac{C_2R_1 + C_2R_2 + C_1R_2}{R_1R_2C_1C_2}\right)i_2(0^+)$$

$$= -\left(\frac{40 + 20 + 10}{400}\right) \times (500 \times 10^{-6})$$

$$= -87.5 \times 10^{-6}\ \text{A/sec}$$

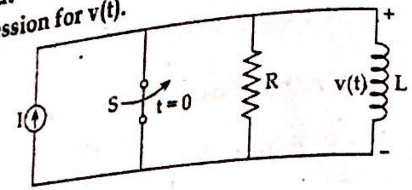
On differentiating equation (5) becomes,

$$\frac{d^2i_2(t)}{dt^2} = -\left(\frac{C_1R_1 + C_2R_2 + C_1R_2}{R_1R_2C_1C_2}\right)\frac{di_2(t)}{dt} = \frac{-1}{R_1R_2C_1C_2}i_2(t)$$

At $t = 0^+$,

$$\frac{d^2i_2(t)}{dt^2}(0^+) = \frac{-70}{400}(-87.5 \times 10^{-6})\frac{-1}{400}(500 \times 10^{-6}) = 14.06 \times 10^{-6}\ \text{A/sec}^2$$

24. **The circuit shown has the switch S opened at t = 0. Solve $\frac{dV}{dt}$ and $\frac{d^2V}{dt^2}$ at $t = 0^+$, if I = 1 A, R = 100 Ω and L = 1 H. Also find the expression for v(t).**



Solution:

Applying KCL,

$$I = \frac{v(t)}{R} + \frac{1}{L}\int V(t)\,dt$$

Differentiating and using numerical values gives,

$$0 = \frac{1}{100}\frac{dv(t)}{dt} + V(t)$$

On differentiating equation (2) becomes,

$$0 = \frac{1}{100}\frac{d^2v(t)}{dt^2} + \frac{dv(t)}{dt}$$

At $t = 0^+$, inductor behaves as an open circuit, hence $v(0^+) = IR = 100\,V$

From equation (2), we have,

$$\frac{dv(0^+)}{dt^2} = -100\frac{dv}{dt}(0^+) = 10^6\ V/sec^2$$

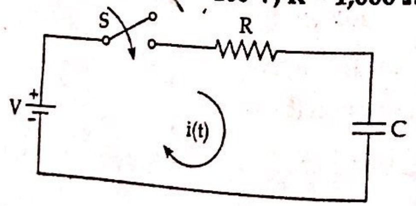
Now, the general solution of the equation (2) is,

$$v(t) = K\,e^{-100t}$$

But at $t = 0^+$, $v(0^+) = 100$ gives K = 100

Hence, $v(t) = 100\,e^{-100t}$ V.

25. **In the circuit shown below, the switch S is closed at t = 0 with capacitor uncharged. Find the values of i, $\frac{di}{dt}$ and, $\frac{d^2i}{dt^2}$ at $t = 0^+$, for element values as follows: V = 100 V, R = 1,000 Ω and C = 1 μF.**



Solution:

Applying KVL,

$$V = Ri(t) + \frac{1}{C}\int i(t)\,dt$$

Differentiating and using element value gives,

$$0 = 1000\frac{di(t)}{dt} + \frac{1}{1 \times 10^{-6}}\,i(t) \qquad (2)$$

Again differentiating equation (2) becomes,

$$0 = 1000\,\frac{d^2i(t)}{dt^2} + \frac{1}{1 \times 10^{-6}}\frac{di(t)}{dt} \qquad (3)$$

At $t = 0^+$, capacitor behaves as a short circuit, hence the second voltage term of equation (1) is zero. Therefore, from equation (1), we have,

$$i(0^+) = \frac{V}{R} = \frac{100}{1000} = 0.1\ A$$

From equation (2), we have,

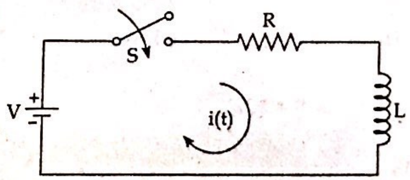
$$1000\frac{di(t)}{dt} = \frac{-1}{1 \times 10^{-6}}\,i(0^+)$$

or,
$$\frac{di(0^+)}{dt} = \frac{-1}{1000}\left[\frac{1}{1 \times 10^{-6}} \times 0.1\right] = -100\ A/sec$$

From equation (2), we have,

$$\therefore \quad \frac{d^2i(0^+)}{dt^2} = \frac{-1}{1000}\left[\frac{1}{1 \times 10^{-6}} \times (-100)\right] = 1 \times 10^5\ A/sec^2$$

26. **In the given circuit, S is closed at t = 0 with zero current in the inductor. Find the values of i, $\frac{di}{dt}$ and L = 1 H and V = 100 V.**



Solution:

Applying KVL,

$$V = Ri(t) + \frac{L\,di(t)}{dt} \qquad (1)$$

On differentiating equation (1) becomes,

$$0 = \frac{Rdi(t)}{dt} + \frac{Ld^2i(t)}{dt^2} \qquad (2)$$

At $t = 0^+$, inductor behaves as an open circuit hence $i(0^+) = 0$.

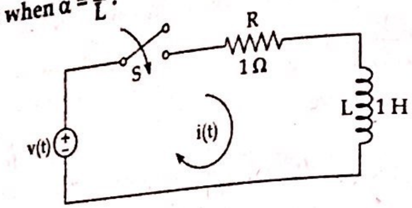
From equation (1), we have,

$$\frac{di(0^+)}{dt} = \frac{V}{L} = \frac{100}{1} = 100\ A/sec$$

From equation (2), we have,

$$\frac{d^2i(0^+)}{dt^2} = \frac{-R}{L}\frac{di(0^+)}{dt} = \frac{-10}{1} \times 100 = -1{,}000\ A/sec^2.$$

27. In the circuit shown below, the voltage source follows the l... $v(t) = V e^{-\alpha t}$, where α is a constant. The switch is closed at t = ... (a) solve for the current assuming that $\alpha \neq \frac{R}{L}$ (b) solve for the current when $\alpha = \frac{R}{L}$.



Solution:

At $t = 0^+$, applying KVL,

$$Ve^{-\alpha t} = Ri(t) + L\frac{di(t)}{dt}$$

or, $$\frac{di(t)}{dt} + \frac{R_i(t)}{L} = \frac{V}{L}e^{-\alpha t}$$

General solution of the above differential equation is,

$$i(t) = e^{\frac{Rt}{L}}\int \frac{V}{L}e^{-\alpha t}\cdot e^{\frac{Rt}{L}} + Ke^{\frac{Rt}{L}} = \frac{V}{L}e^{\frac{-Rt}{L}}\int e^{\left(\frac{R}{L}-\alpha\right)} dt + K e^{\frac{-Rt}{L}}$$

a) When $\alpha \neq \frac{R}{L}$, from equation (1), we have,

$$i(t) = \frac{V}{R-\alpha L}e^{-\alpha t} + K e^{\frac{-Rt}{L}}$$

Since $i(0^+) = 0$,

or, $$K = \frac{-V}{R-\alpha L}$$

Hence, $$i(t) = \frac{V}{R-\alpha L}\left(e^{-\alpha t} - e^{\frac{-Rt}{L}}\right) A.$$

b) When $\alpha = \frac{R}{L}$, from equation (1), we have,

$$i(t) = \frac{V}{L}e^{\frac{-Rt}{L}}\int e^{o}\, dt + K e^{\frac{-Rt}{L}} = \frac{V}{L}e^{\frac{-Rt}{L}}\cdot t + K e^{\frac{-Rt}{L}}$$

Again, $i(0^+) = 0$ gives $K = 0$.

Hence, $$i(t) = \frac{Vt}{L}e^{\frac{-Rt}{L}} A.$$

BOARD EXAMINATION SOLVED QUESTIONS

1. What are second order circuits? For the series circuit having R = 100 Ω, C = 1 μF with initial charge 50 micro coulombs. A voltage V = 100 volts is applied at t = 0. Find the expression for the resulting current in the circuit for t ≥ 0. [2019/Spring]

Solution:

Definition part:

Circuit that include an inductor, capacitor and resistor connected in series or parallel are called second order circuits. Analysis of second order circuit yields a second-order differential equation. Since second order circuits have two energy-storage types, the circuits can have the following forms;

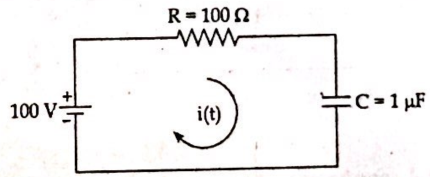
i) Two capacitors
ii) Two inductors
iii) One capacitor and one inductor
  a) Series RLC circuit
  b) Parallel RLC circuit
  c) Others

Numerical part:

Given that;

$R = 100\ \Omega$
$C = 1\ \mu F$
$q_0 = 50\ \mu C$ (q_0 is initial charge)
$V = 100\ V$

The equivalent circuit is,



For t > 0, applying KVL,

$$R\,i(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)\, dt = V$$

or, $$100\, i(t) + \frac{1}{10^{-6}}\int_{-\infty}^{\infty} i(t)\, dt = 100$$

or, $$100\, i(t) + 10^{6}\int_{0}^{\infty} i(t)\, dt + \frac{50 \times 10^{-6}}{10^{-6}} = 100$$

$$\left[\because \int_{-\infty}^{\infty} i(t)\, dt = \int_{-\infty}^{0} i(t)\, dt + \int_{0}^{\infty} i(t)\, dt \text{ and, given} \int_{-\infty}^{0} i(t)\, dt = q_0 = 50\ \mu C\right]$$

or, $100\, i(t) + 10^6 \int_0^\infty i(t)\, dt = 50$

Differentiating with respect to t and dividing by 100, we get,

or, $\frac{100}{100}\frac{di(t)}{dt} + \frac{10^6}{100} i(t) = 0$

or, $\frac{di(t)}{dt} + 10^4 i(t) = 0$

Let the transient response be $i_t(t) = Ke^{st}$

then, $\frac{di_t(t)}{dt} = Kse^{st} = si_t(t)$

Thus, equation (1) becomes,

$si_t(t) + 10^4 i_t(t) = 0$

$\therefore$ $s = -10^4$

so, transient response, $i_t(t) = Ke^{-10^4 t}$ A

Now, for $i(0^+)$, the initial voltage on capacitor is,

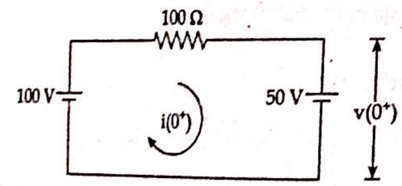
$v(0^+) = \frac{q_0}{C} = 50$ V

so, $i(0^+) = \frac{100 - 50}{100} = 0.5$ A

Applying this condition in equation (2), we get,

$i(0^+) = Ke^{-10^4 \times 0}$

$\therefore$ $K = 0.5$

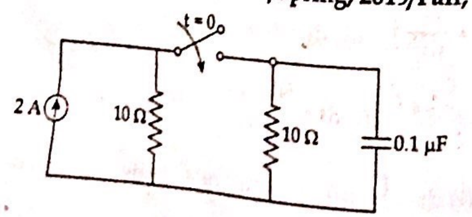


Hence, required expression of current,

$i(t) = 0.5\, e^{-10^4 t}$ A.

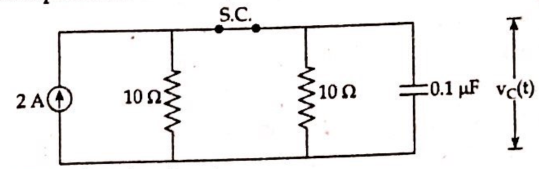
2. **Find the total response $v_C(t)$ for the voltage across the capacitor in circuit given below using classical method. The switch S is closed at t = 0.**

[2013/Spring, 2015/Fall, 2018/Spring, 2019/Fall, 2019/Spring]



Solution:

For t > 0, the equivalent circuit is,



Applying KCL in the circuit,

$\frac{v_C(t)}{10} + \frac{v_C(t)}{10} + 0.1 \times 10^{-6} \frac{dv_C(t)}{dt} = 2$

or, $\frac{dv_C(t)}{dt} + 10^6 v_C(t) + 10^6 v_C(t) = 2 \times 10^7$

or, $\frac{dv_C(t)}{dt} + 2 \times 10^6 v_C(t) = 2 \times 10^7$ (1)

Since source is constant, let forced response, $v_{CF}(t)$ be also constant. Such that $\frac{dv_{CF}(t)}{dt} = 0$.

So equation (1) becomes,

$0 + 2 \times 10^{-6} v_{CF}(t) = 2 \times 10^7$

$\therefore$ $v_{CF}(t) = 20$ V

For transient response $v_{Ct}(t)$ of equation (1), we make the equation homogenous,

$\frac{dv_C(t)}{dt} + 2 \times 10^6 v_{ct}(t) = 0$

Let $v_C(t) = Ke^{st}$, then,

or, $\frac{dv_C(t)}{dt} = sK e^{st} = sv_{Ct}(t)$,

Thus equation (1) becomes,

$sv_C(t) + 2 \times 10^6 v_C(t) = 0$

$\therefore$ $s = -2 \times 10^6$

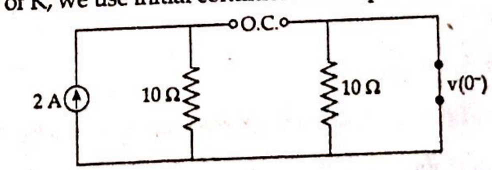
Hence, $v_{Ct}(t) = Ke^{-2 \times 10^{-6} t}$

Thus, total response is,

$v_C(t) = v_{CF}(t) + v_{CE}(t)$

or, $v_C(t) = 20 + Ke^{-2 \times 10^6 t}$ (2)

To find value of K, we use initial condition. The equivalent circuit at $t = 0^-$ is,



$v_C(0^-) = 0$ V

We know, for capacitor, $v_C(0^-) = v_C(0^+) = 0$ V

Applying this condition in equation (2), we get,

$v_C(0^+) = 20 + Ke^{-2\times 10^6\times 0}$

or, $0 = 20 + K \times 1$

$\therefore$ $K = -20$

Thus voltage across the capacitor in the given circuit is,

$v_C(t) = 20 \times 20\, e^{-2\times 10^6 t}$

$\therefore$ $v_C(t) = 20\,(1 - e^{-2\times 10^6 t})$ V

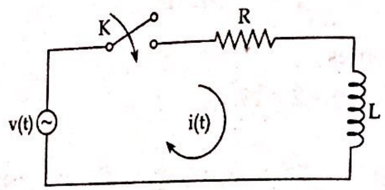
3. **An R-L circuit is connected to an average voltage v(t) = 80 sin (400t + 20) volts at t = 10, R = 10 Ω and L = 0.5 H. Find the equation for the current using classical method.** **[2018/Spring]**

Solution:

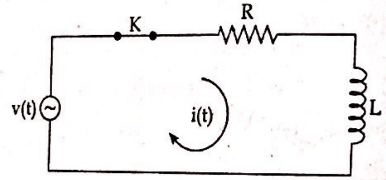
$v(t) = 80 \sin(400t + 20)$ V

$R = 10\,\Omega$

$L = 0.5$ H



For t > 0, the equivalent circuit is,



Applying KVL, we get,

$$Ri(t) + L\frac{di(t)}{dt} = v(t)$$

or, $$10\, i(t) + 0.5\frac{di(t)}{dt} = 80 \sin(400t + 20) \quad (1)$$

or, $$\frac{di(t)}{dt} + 20\, i(t) = 160 \sin(400t + 20) \quad (2)$$

Expressing equation (1) in operator form,

$i(0.5P + 10) = 80 \sin(400t + 20)$

or, $$i = \frac{80 \sin(400t + 20)}{0.5P + 10}$$

We have a sinusoidal source, so its force response is given by,

$$i_f(t) = \frac{80}{|Z|} \sin(400t + 20 - \phi)$$

where, $|Z| = \sqrt{R^2 + L^2\omega^2}$ and $\phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$

and, $\omega = 400$ rad/sec

$$|Z| = \sqrt{(10)^2 + (0.5 \times 400)^2} = 200.25\,\Omega$$

$$\phi = \tan^{-1}\left(\frac{400 \times 0.5}{10}\right) = 87.14°$$

so, $$i_f(t) = \frac{80}{200.25} \sin(400t + 20 - 87.14°)$$

$\therefore$ $i_f(t) = 0.4 \sin(400t - 67.14°)$ A

To find transient response, $i_t(t)$, we make equation (2) homogenous and find its characteristic equation which is,

$s + 20 = 0$

$\therefore$ $s = -20$

Thus, transient response is, $i_t(t) = Ke^{-20t}$

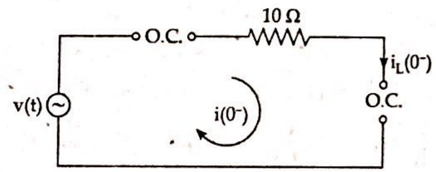
Thus, overall response is,

$i(t) = i_f(t) + i_t(t)$

$\therefore$ $i(t) = 0.4 \sin(400t - 67.14°) + Ke^{-20t}$ (3)

Using initial condition to find value of K,

The circuit at $t = 0^-$ is,



Here; $i_L(0^-) = 0$ A

Since this is a series circuit, $i(0^-) = i_L(0^+) = 0$ A

For inductor, $i_L(0^+) = i_L(0^-) = 0$ A

so, $i(0^+) = i_L(0^+) = 0$ A

Applying this condition in equation (3), we get,

$i(0^+) = 0.4 \sin(400 \times 0 - 6.47°) + Ke^{-20\times 0}$

or, $0 = 0.4 \sin(-67.47°) + K$

$\therefore$ $K = 0.37$

Hence, equation for current, $i(t) = 0.37\, e^{-20t} + 0.4 \sin(400t - 67.14°)$ A

4. **Find the complete response i(t) for the RL series circuit with R = 4.8 ohm, L = 4 H, supply voltage is 2 sin (10t). Use classical approach.** **[2017/Spring]**

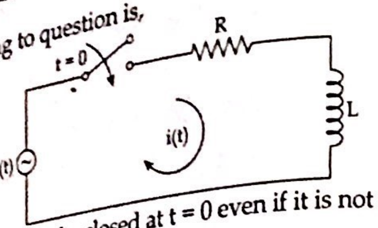
Solution:

$R = 4.8\,\Omega$

$L = 4$ H

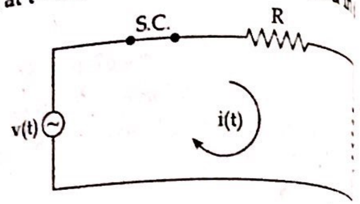
$v(t) = 2 \sin(10t)$ V

The circuit according to question is,



Here, we assume switch to be closed at t = 0 even if it is not mentioned in question. This helps to find $i(0^+)$.

For t > 0, the circuit will be,



Applying KVL, we get,

$$Ri(t) + L\frac{di(t)}{dt} = v(t)$$

or, $$4.8\, i(t) + 4\frac{di(t)}{dt} = 2 \sin 10t$$

or, $$\frac{di(t)}{dt} + 1.2\, i(t) = 0.5 \sin 10t$$

The forced response, $i_f(t)$ due to sinusoidal source will be,

$$i_f(t) = \frac{V_m}{|Z|} \sin(\omega t - \phi)$$

where, $V_m = 2$ (Peak source voltage)

ω = Angular frequency = 2 rad/sec

$$|Z| = \sqrt{R^2 + \omega^2 L^2} = \sqrt{(4.8)^2 + (10)^2 \times (4)^2} = 40.29\ \Omega$$

and, $$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right) = \tan^{-1}\left(\frac{40}{4.8}\right) = 83.16°$$

Thus, $i_f(t) = \frac{2}{40.29} \sin(10t - 83.16°) = 0.05 \sin(10t - 83.16°)$ A

For transient response, we make equation (1) homogenous and find characteristic equation which is,

$$s + 1.2 = 0$$

$\therefore$ $s = -1.2$

Transient response, $i_t(t) = Ke^{-1.2t}$ A

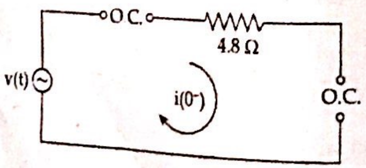
The overall response is,

$$i(t) = i_f(t) + i_t(t)$$

$\therefore$ $i(t) = 0.05 \sin(10t - 83.16°) + Ke^{-1.2t}$

Applying initial condition to find value of K,

The circuit at $t = 0^-$ is,



This is an open circuit so $i(0^-) = 0$ A. As $i(0^-)$ is current through inductor as well, we can write,

$$i(0^+) = i(0^-) = 0 \text{ A}$$

Using equation (2), we get,

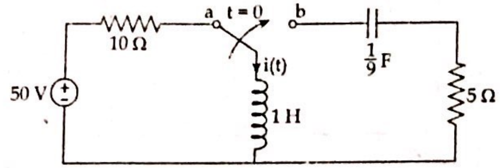
$$i(0^+) = 0.05 \sin(10 \times 0 - 83.16°) + K e^{-1.2\times 0}$$

or, $$0 = 0.05 \times (-0.99) + K$$

$\therefore$ $K = 0.05$

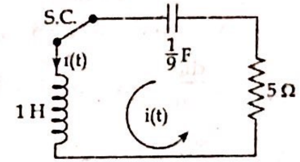
Hence, complete response, $i(t) = 0.05 \sin(10t - 83.16°) + 0.05\, e^{-1.2t}$ A

5. Find the through inductor for t > 0. **[2017/Spring]**



Solution:

For t > 0, the equivalent circuit is,



Applying KVL, we have,

$$1 \times \frac{di(t)}{dt} + 5\, i(t) + \frac{1}{\left(\frac{1}{9}\right)}\int_{-\infty}^{\infty} i(t)dt = 0$$

Different with respect to t,

$$\frac{d^2i(t)}{dt^2} + 5\frac{di(t)}{dt} + 9\, i(t) = 0 \qquad (1)$$

For forced response, $i_f(t)$, Let $i_f(t)$ be constant as source is constant, so

$$\frac{d^2i_f(t)}{dt} = \frac{di_f(t)}{dt} = 0$$

Then equation (1) becomes,

$$0 + 5 \times 0 + 9 \times i_f(t) = 0$$

$\therefore$ $i_f(t) = 0$ A

For transient response, we make equation (1) homogenous and find its characteristic equation which is,

$$s + 5s + 9 = 0$$

$\therefore$ $s_1, s_2 = -2.5 + 1.66j, -2.5 - 1.66j$

Since roots are complex conjugates, the transient response is,

$$i_t(t) = e^{-2.5t}(K_1 \cos 1.66jt + K_2 \sin 1.66t)$$

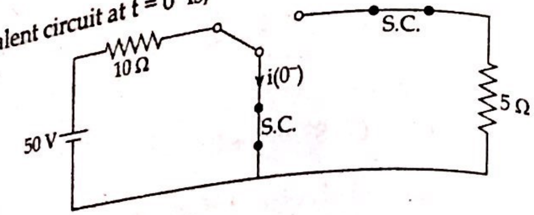
Thus, overall response is,

$$i(t) = i_f(t) + i_t(t)$$

or, $$i(t) = e^{-2.5}(K_1 \cos 1.66t + K_2 \sin 1.66t) \qquad (2)$$

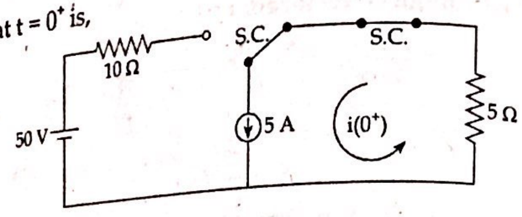
To find the values of K_1 and K_2, we initial conditions.

The equivalent circuit at $t = 0^-$ is,



Here, $i(0^-) = \frac{50}{10} = 5$ A

The circuit at $t = 0^+$ is,



Here; $i(0^+) = i(0^-) = 5$ A

Applying KVL, we have,

$$v_L(0^+) = v_C(0^+) + 5 \times i(0^+) = 0$$

or, $L\frac{di(0^+)}{dt} + 0 + 5 \times 5 = 0$

or, $1\frac{di(0^+)}{dt} + 25 = 0$

$\therefore$ $\frac{di(0^+)}{dt} = -25$ A/s

Applying equation (3) in (2), we have,

$$i(0^+) = e^{-2.5\times0}(K_1 \cos 1.66 \times 0 + K_2 \sin 1.66 x_0)$$

or, $5 = K_1$

$\therefore$ $K_1 = 5$

Different equation (2) with respect to t,

$$\frac{di(t)}{dt} = -2.5\, e^{-2.5t}(K_1 \cos 1.66t + K_2 \sin 1.66t) + e^{-2.5t}(-1.66\, K_1 \sin 1.66t + 1.66\, K_2 \cos 1.66t)$$

Applying equation (4), we get,

At $t = (0^+)$,

$$\frac{di(0^+)}{dt} = -2.5\, e^{-2.5\times0}(K_1 \cos 1.66 \times 0 + K_2 \sin 1.66 \times 0) + e^{-2.5\times0}(-1.66\, K_1 \sin 1.66 \times 0 + 1.66\, K_2 \cos 1.66 \times 0)$$

or, $-25 = -2.5 \times 5 + 1.66\, K_2$

$\therefore$ $K_2 = -7.53$

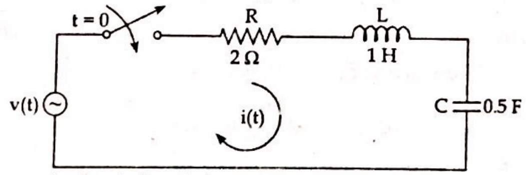
Hence, current through inductor, $i(t) = e^{-2.5t}(5 \cos 1.66t - 7.53 \sin 1.66t)$

6. **A series RLC circuit with R = 2 Ω, C = 0.5 F and L = 1 H is excited by a voltage of v(t) = sin t for t ≥ 0. For the elements value specified find the current i(t) through the circuit if the switch is closed at t = 0.** **[2017/Spring]**

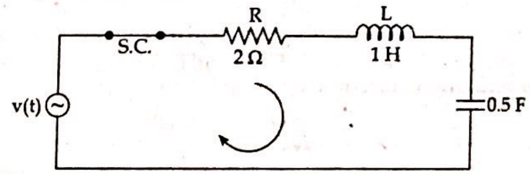
Solution:

R = 2 Ω   C = 0.5 F

L = 1 H   v(t) = sin V



For t > 0, the equivalent circuit is,



Applying KVL, we get,

$$Ri(t) + \frac{L\,di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{\infty} i(t)\,dt = v(t)$$

or, $2\,i(t) + \frac{di(t)}{dt} + 2\int_{-\infty}^{\infty} i(t)\,dt = \sin t$

Differentiating both sides with respect to we get,

$$\frac{d^2i(t)}{dt^2} + 2\frac{di(t)}{dt} + 2\,i(t) = \cos t \qquad (1)$$

Now, the steady state response, $i_f(t)$ is given by,

$$i_f(t) = \frac{V_m}{|Z|}\sin(\omega t - \phi)$$

where, $V_m = 1;\ \omega = 1$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{(2)^2 + \left(1 \times 1\ \frac{-1}{1 \times 0.5}\right)^2} = 2.24$$

$$\phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) = \tan^{-1}\left(\frac{-1}{2}\right) = 296.56$$

$$= \left[\tan^{-1}\left(\frac{y}{x}\right);\ y \text{ is negative, x is positive so we add } + 270°\right]$$

so, $i_f(t) = \frac{1}{2.24}\sin(t - 296.56°)$ A $= 0.45 \sin(t - 296.56°)$ A

To find transient response, we make equation (1) homogenous and calculate its characteristic equation which is,

$$s^2 + 2s + 2 = 0.$$

$\therefore \quad s_1, s_2 = -1 + i; -1 - i$

Since roots are complex conjugates, the transient response is,

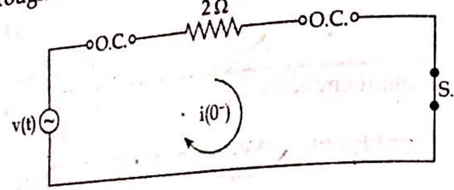
$$i_t(t) = e^{-t}(K_1 \cos t + K_2 \sin t)$$

Thus, overall response is,

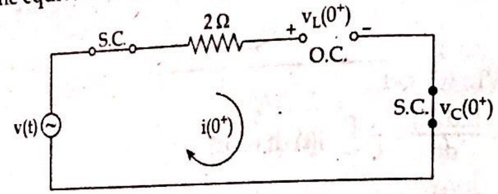
$$i(t) = i_f(t) + i_t(t)$$

$\therefore \quad i(t) = 0.45 \sin(t - 296.56°) + e^{-t}(K_1 \cos t + K_2 \sin t)$

For values of K_1 and K_2, applying initial conditions, $i(0^-) = i_L(0^-) = 0$ A is current through inductor. For $t = 0^-$, the equivalent circuit is,



For $t = 0^+$, the equivalent circuit is,



The circuit is still open due to inductor, so,

$$i(0^+) = i_L(0^+) = 0 \text{ A}$$

Applying KVL,

$$\cos 0 = 2 \times i(0^+) + v_L(0^+) + v_C(0^+) \text{ ; } v_C(0^+) \text{ is short circuit, so } v_C(0^+) = 0 \text{ V}$$

or, $\quad 1 = 2 \times 0 + v_L(0^+) + 0$

$\therefore \quad v_L(0^+) = 0$ V

We know,

$$v_L(0^+) = L\frac{di(0^+)}{dt}$$

or, $\quad \dfrac{0}{1} = \dfrac{di(0^+)}{dt}$

$\therefore \quad \dfrac{di(0^+)}{dt} = 0$ A/sec

Applying equation (3) in equation (2), we get,

$$i(0^+) = 0.45 \sin(0 - 296.56°) + e^{-0}(K_1 \cos 0° + K_2 \sin 0°)$$

or, $\quad 0 = 0.45 \times (0.89) + K_1$

$\therefore \quad K_1 = -0.40$

Different equation (2) with respect to t, we get,

$$\frac{di(t)}{dt} = 0.45 \cos(t - 296.56°) + (-1)\, e^{-t}(K_1 \cos t + K_2 \sin t) + e^{-t}(K_1 \sin t + K_2 \cos t)$$

Applying equation (4), we get,

$$\frac{di(0^+)}{dt} = 0.45 \sin(0 - 296.56°) - e^{-0}(K_1 \cos 0t + K_2 \sin 0) + e^{-0}(K_1 \sin 0 + K_2 \cos 0)$$

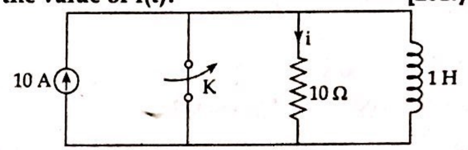
or, $\quad 0 = 0.45 \times (-0.45) - (-0.4) + K_2$

$\therefore \quad K_2 = -0.2$

Thus, the current through the circuit,

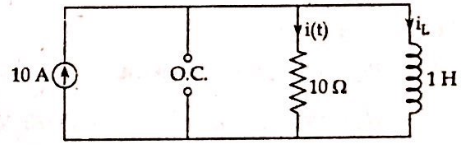
$$i(t) = 0.45 \sin(t - 296.56°) + e^{-t}(-0.2 \cos t - 0.2 \sin t) \text{ A}$$

7. **In the given circuit, the switch K is opened at time t = 0. Find the solution for current i(t) using classical method. At t = 5 second, what is the value of i(t)?** **[2017/Fall, 2013/Fall]**



Solution:

The equivalent circuit for t > 0 is,



Applying KCL, we get,

$$i(t) + i_L(t) = 10$$

or, $\quad \dfrac{V}{10} + \dfrac{1}{1}\displaystyle\int_{-\infty}^{\infty} V\, dt = 10$

Differentiating with respect to t, we get,

or, $\quad \dfrac{1}{10}\dfrac{dV}{dt} + V = 0$

or, $\quad \dfrac{dV}{dt} + 10\,V = 0 \qquad (1)$

The forced response, v_f is constant as source is constant.

so, $\quad \dfrac{dv_f}{dt} = 0$, then equation (1) becomes,

$$0 + 10\, v_f = 0$$

$\therefore \quad v_f = 0$

For transient response, we find characteristic equation of equation (1),

$$s + 10 = 0$$

$\therefore \quad s = -10$

so, transient response, $v_t(t) = Ke^{-10t}$

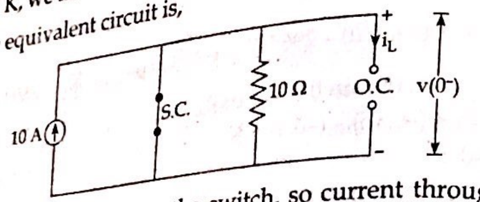
Hence, overall response is,

$$v(t) = v_f(t) + v_t(t)$$

or, $$v(t) = Ke^{-10t}$$

For value of K, we find initial condition, $v(0^+)$,

At $t = 0^-$, the equivalent circuit is,

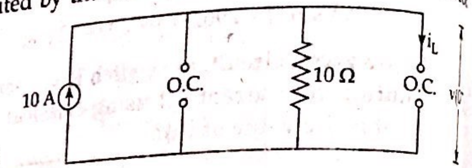


The circuit is short circuited by the switch, so current through inductor

$i_L(0^-) = 0$ A.

For $t = 0^+$, the circuit is,

$$i_L(0^+) = i_L(0^-) = 0 \text{ A}$$



All current flows through resistor, so,

$$v(0^+) = 10 \times 10 = 100 \text{ V}$$

Using this condition in equation (2), we get,

$$v(0^+) = Ke^{-100}$$

$\therefore$ $K = 100$

Hence, $v(t) = 100\, e^{-10t}$

Now, current i(t) as shown in figure is,

$$i(t) = \frac{v(t)}{10} = \frac{100\, e^{-10t}}{10}$$

$\therefore$ $i(t) = 10\, e^{-10t}$ A

At t = 5 seconds

$\therefore$ $i(5) = 10\, e^{-10 \times 5} = 1.93 \times 10^{-21}$ A

At 5 seconds, value of i(t) is 1.93×10^{-21} A

8. In the series RLC network, V = 100 V, L = 0.2, H, R = 30 Ω, C = 40 μF. Obtain the expression for i(t). Using classical approach. Assume there is no initial charge on capacitor or current in the inductor. **[2014/Fall, 2017/Fall]**

Solution:

Given that;

$R = 30\,\Omega$ $\quad$ $L = 0.2$ H

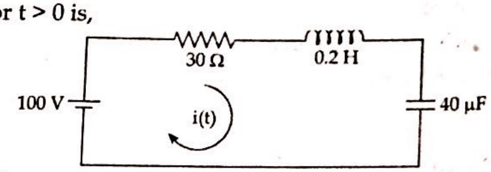
$C = 40\,\mu F$ $\quad$ $V = 100$ V

$i(t) = ?$

$q_0 = 0$ C in capacitor

$I_0 = 0$ A in inductor.

The circuit for t > 0 is,



Applying KVL,

$$30\, i(t) + 0.2\frac{di(t)}{dt} + \frac{1}{40 \times 10^{-6}}\int_{-\infty}^{t} i(t)\, dt = 100$$

Differentiating both sides with respect to t, we get,

$$0.2\frac{d^2i(t)}{dt^2} + 30\frac{di(t)}{dt} + 25000\, i(t) = 0$$

or, $$\frac{d^2i(t)}{dt^2} + 150\frac{di(t)}{dt} + 125000\, i(t) = 0 \quad (1)$$

The forced response, $i_f(t)$ will be constant as source is constant

so, $$\frac{d^2i_f(t)}{dt^2} = \frac{di_f(t)}{dt^2} = 0$$

Hence, equation (1) becomes,

$$125000\, i_f(t) = 0$$

$\therefore$ $i_f(t) = 0$

For transient response, the characteristic equation of homogenous equation (1) is,

$$s^2 + 150\, s + 125000 = 0$$

or, $s_1, s_2 = -75 + 345.51\, j\,; -75 - 345.51\, j$

The roots are complex conjugates, so the transient response will be,

$$i_t(t) = e^{-75t}\,(K_1 \cos 345.51\, t) + K_2 \sin 345.51\, t) \text{ A}$$

Overall response is;

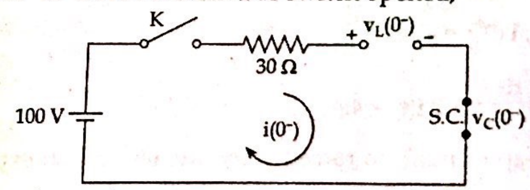
$$i(t) = i_f(t) + i_t(t)$$

$$i(t) = e^{-75t}\,(K_1 \cos 345.51\, t) + K_2 \sin 345.51\, t) \quad (2)$$

For K_1 and K_2, we use initial condition given to us; *i.e.*, $v_C(0^-) = 0$ V

and, $i(0^-) = 0$ A.

The circuit at $t = 0^-$ can be assumed as switch opened,



For inductor and capacitor,

$$i(0^+) = i(0^-) = 0 \text{ A}$$

$$v_C(0^+) = v_C(0^-) = 0 \text{ V}$$

Applying KVL, we get,

$$100 = 30 \times i(0^+) + V_L(0^+) + V_C(0^+)$$

or, $100 = 30 \times 0 + V_L + 0$

$\therefore$ $v_L(0^+) = 100$

Also, $0.2\dfrac{di(0^+)}{dt} = 100$

$\therefore$ $\dfrac{di(0^+)}{dt} = 500$ A/sec

Applying equation (3) on (2), we get,

$$i(0^+) = e^{-75\times 0}(K_1 \cos(345.51 \times 0) + K_2 \sin(345.51 \times 0)$$

or, $0 = K_1 \times 1 + K_2 \times 0$

$\therefore$ $K_1 = 0$

Different equation (2) with respect to t,

$$\frac{di(t)}{dt} = -75\, e^{-75t}(K_1 \cos 345.51t + K_2 \sin 345.51t) + e^{-75t}(345.51 \times (-1) K_1 \sin 345.51\, t + 345.51 K_1 \cos 345.51t)$$

Applying equation (4), we get,

or, $\dfrac{di(0^+)}{dt} = -75\, e^{-75\times 0}(K_1 \times \cos 0 + K_2 \times \sin 0) + e^{-75\times 0}(-345.51 K_1 \sin 0 + 345.51 K_2 \cos 0)$

or, $500 = 0 + 1 \times (0 + 345.51 K_2)$

$\therefore$ $K_2 = 1.45$

Hence, expression for current is,

$$i(t) = 1.45\, e^{-75t} \sin(345.51\, t) \text{ A}$$

9. **Determine the solution of current differential equation $i''(t) + 3i'(t) + 2i(t) = 4e^t$ with initial conditions $i(0^+) = 1$, $i'(0^+) = -1$ using classic method.** **[2012/Spring, 2014/Spring, 2016/Spring]**

Solution:

Given that;

$$i''(t) + 3i'(t) + 2i(t) = 4e^t$$

Initial conditions:

$$i(0^+) = 1;\ i'(0^+) = -1 \text{ or } \frac{di(0^+)}{dt} = -1$$

Now, $\dfrac{d^2i(t)}{dt^2} + 3\dfrac{di(t)}{dt} + 2\, i(t) = 4e^t$

The source is exponential, so forced response, $i_f(t)$ is also exponential

$i_f(t) = Ke^t$

so, $\dfrac{di_f(t)}{dt} = Ke^t$ and $\dfrac{d^2i_f(t)}{dt^2} = Ke^t$

Thus, equation (1) becomes

$$Ke^t + 3Ke^t + 2Ke^t = 4e^t$$

or, $6K = 4$

$\therefore$ $K = \dfrac{2}{3} = 0.67$

Hence, $i_f(t) = 0.67\, e^t$

For transient response, we make equation (1) homogenous and find its characteristic equation which is,

$$s^2 + 3s + 2 = 0$$

$\therefore$ $s_1, s_2 = -1, -2$

The roots are real and distinct, so the response, $i_t(t)$ is,

$$i_t = K_1 e^{-t} + K_2 e^{-2t}$$

Overall response is,

$$i(t) = i_f(t) + i_t(t)$$

$\therefore$ $i(t) = 0.67\, e^t + K_1 e^{-t} + K_2 e^{-2t}$ (2)

Using initial conditions,

$$i(0^+) = 0.67\, e^0 + K_1 e^{-0} + K_2 e^{-0}$$

or, $K_1 + K_2 = 1 - 0.67 = 0.33$ (3)

Different equation (2) with respect to t, we get,

$$\frac{di(t)}{dt} = 0.67\, e^t - K_1 e^{-t} - 2K\, e^{-2t}$$

Applying $\dfrac{di(0^+)}{dt} = -1$

$$\frac{di(0^+)}{dt} = 0.67\, e^0 - K_1 e^{-0} - 2K_2 e^{-0}$$

or, $-1 - 0.67 = -K_1 - 2K_2$

or, $K_1 + 2K_2 = 1.67$ (4)

Solving equation (3) and (4), we get,

$\therefore$ $K_1 = -1.01$

$\therefore$ $K_2 = 1.34$

Hence, $i(t) = 0.67\, e^t - 1.01\, e^{-t} + 1.34\, e^{-2t}$ A

10. **A dc voltage of 100 V is applied to a series RC circuit at t = 0 having R = 10 Ω and C = 4 μF. Use classical method to find the current through the capacitor $i_C(t)$ for t > 0. Assume initial charge on the capacitor $q_C(0) = 800$ μC.** **[2016/Spring]**

Solution:

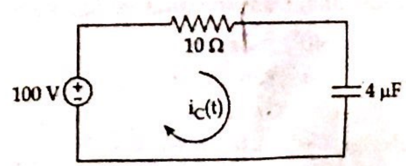
Given that;

V = 100 V

R = 10 Ω

C = 4 μF

$q_0(C) = 800$ μC



The required circuit for t > 0 is,

Since this is a series circuit, current through the circuit is equal to current through capacitor, so we use $i_C(t)$.

Applying KVL,

$$R\, i_C(t) + \frac{1}{C}\int_{-\infty}^{\infty} i_C(t)\, dt = V$$

or, $$10\, i_C(t) + \frac{1}{4\times 10^{-6}}\int_{-\infty}^{t} i_C(t)\, dt = 100$$

Differentiating with respect to t, we get,

or, $$10\frac{di_C(t)}{dt} + 250000\, i_C(t) = 0$$

or, $$\frac{di_C(t)}{dt} + 25000\, i_C(t) = 0$$

Since source is constant, the forced response, $i_{Cf}(t)$ is also constant. Th...

$$\frac{di_{Cf}(t)}{dt} = 0$$

Then, equation (1) becomes,

$$25000\, i_{Cf}(t) = 0$$

$\therefore$ $i_{Cf}(t) = 0$ A

For transient response, we find characteristic equation of homogeneous equation (1),

$$s + 25000 = 0$$

$\therefore$ $s = -25000$

The transient response is,

$$i_{Ct}(t) = Ke^{-25000t}$$

Thus, overall response is,

$$i_C(t)\, i_{Cf}(t) = i_{Ct}(t)$$

or, $i_C(t) = Ke^{-25000t}$

Using initial condition to find value of K,

$$q(0^+) = q(0^-) = 800\ \mu C$$

Voltage on capacitor, $v_C(0^+) = v_C(0^-) = \dfrac{800\ \mu C}{4\mu F} = 200$ V.

Current at $t = 0^+$ is,

$$i_C(0^+) = \frac{100 - 200}{10} = -10 \text{ A}$$

(-Ve sign indicates that current flows from capacitor to voltage source)

Thus, using this in equation (2), we have,

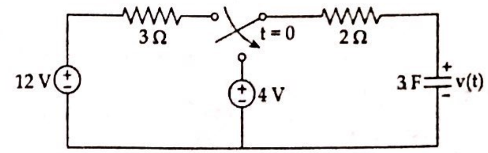
$$i_C(0^+) = Ke^{-25000\times 0}$$

or, $-10 = K \times 1$

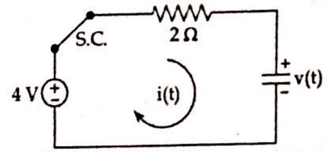
$\therefore$ $K = -10$

Hence, $i_C(t) = -10\, e^{-25000t}$ A

11. **Initially the switch is connected to 12 V and at t = 0 switch connects to 4 V source. Find the current through capacitor and voltage across capacitor at t > 0. [2016/Spring]**



Solution:



For t > 0, the equivalent circuit is,

Applying KVL,

$$2 \times i(t) + v(t) = 4$$

Since i(t) is current flowing through capacitor too, we can write,

$$i(t) = C\frac{dv(t)}{dt}$$

or, $$i(t) = 3\frac{dv(t)}{dt} \quad (1)$$

Replacing value of i(t) in KVL equation

$$2 \times 3\frac{dv(t)}{dt} + v(t) = 4$$

or, $$\frac{dv(t)}{dt} + \frac{1}{6}v(t) = \frac{2}{3} \quad (2)$$

Since source is constant, we take forced response, $v_f(t)$ as constant so that $\dfrac{dv_f(t)}{dt}$ 0.

Then equation (2) becomes,

$$\frac{1}{6}v_f(t) = \frac{2}{3}$$

$\therefore$ $v_f(t) = 4$ V

For transient response, we make equation (2) homogenous and find characteristic equation, *i.e.,*

$$s + \frac{1}{6} = 0$$

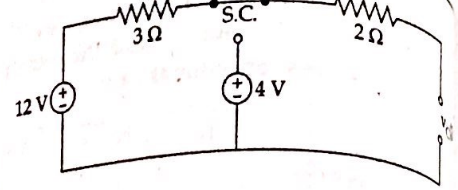
$\therefore$ $s = \dfrac{-1}{6}$

Hence, transient response, $v_t(t) = K\, e^{-1/6t}$

Overall response is, $v(t) = v_f(t) + v_t(t)$

or, $$v(t) = 4 + K\, e^{-1/6t} \quad (3)$$

To find value of K, we use initial condition, $v(0^+)$



The circuit at $t = 0^-$ is,
The circuit is open due to energized capacitor. To find $v(0^-)$, we use KVL,

$$3 \times i(0^-) + 2 \times i(0^-) + v(0^-) = 12$$

Since the circuit is open, current, $i(0^-) = 0$ A

or, $3 \times 0 + 2 \times 0 + v(0^-) = 12$

∴ $v(0^-) = 12$ V

For capacitor

$$v(0^-) = v(0^+) = 12 \text{ V}$$

Applying this in equation (3), we get,

$$v(0^+) = 4 + K e^{-\frac{1}{6} \times 0}$$

or, $12 = 4 + K$

∴ $K = 8$

Hence, equation of voltage across capacitor is

$$v(t) = 4 + 8 e^{-\frac{1}{6}t}$$

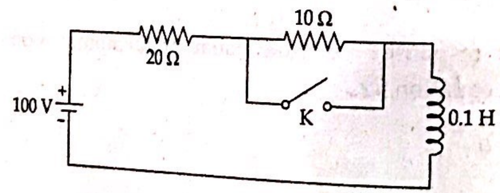
Now, to find current across capacitor, we use equation (1) and substitu... value of v(t) from equation (4), so we get,

$$i(t) = 3 \frac{d}{dt}(4 + 8 e^{-\frac{1}{6}t}) = 3 \times \left(\frac{-1}{6}\right) \times 8 e^{-\frac{1}{6}t}$$

∴ $i(t) = -4 e^{-\frac{1}{6}t}$ A

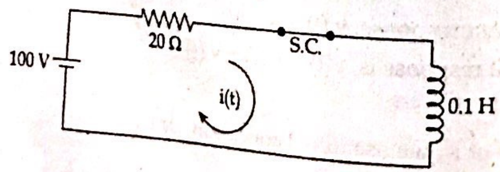
The negative sign shows that flow of current is opposite to what we ha... initially assumed.

12. **A dc voltage of 100 V is applied in the adjoining circuit and th... switch K is open. The switch K is closed at time t = 0. Find th... complete expression for the current. [2016/Fall, 2012/Fa...**



Solution:

The circuit at $t = 0^-$ is,



Applying KVL, we get,

or, $0.1 \frac{di(t)}{dt} + 20\, i(t) = 100$

or, $\frac{di(t)}{dt} + 200\, i(t) = 1000$ (1)

As source is constant, the forced response, $i_f(t)$ is also constant, so, $\frac{di_f(t)}{dt} = 0$, then equation (1) becomes,

or, $200\, i_f(t) = 1000$

∴ $i_f(t) = 5$ A

For transient response, we make equation (1) homogenous and find its characteristic equation which is,

$$s + 200 = 0$$

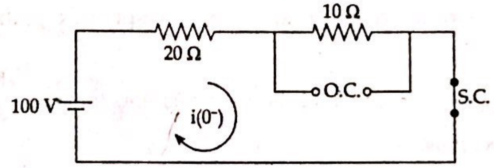
∴ $s = -200$

Hence transient response, $i_t(t) = Ke^{-200t}$ A

Overall response, $i(t) = i_f(t) + i_t(t) = 5 + Ke^{-200t}$ (2)

Using initial condition to find the value of K,

The equivalent circuit at $t = 0^-$ is,



$$i(0^-) = i(0^+) = 3.33 \text{ A}$$

Using this in equation (2), we get,

$$i(0^+) = 5 + K e^{-200 \times 0}$$

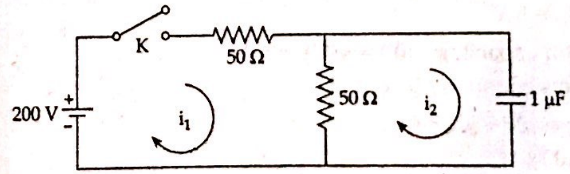
or, $3.33 = 5 + K$

∴ $K = -1.73$

Thus, complete expression of current,

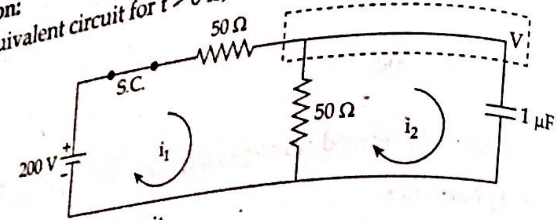
$$i(t) = 5 - 1.73\, e^{-200t} \text{ A}$$

13. **A DC voltage of 200 V is suddenly applied in the network shown in figure below. Find the transient currents in both the loops and obtain the transient voltage across the capacitor. [2016/Fall]**



Solution:

The equivalent circuit for $t > 0$ is,



Applying KCL in the circuit,

$$1 \times 10^{-6}\frac{dv}{dt} + \frac{v}{50} + \frac{v-200}{50} = 0$$

or, $$10^{-6}\frac{dv}{dt} + \frac{v}{25} = 4$$

or, $$\frac{dv}{dt} + 40000\,V = 4 \times 10^{6}$$

The transient response of equation (1) is found by making equation homogenous and finding its characteristic equation.

$$s + 40000 = 0$$

$\therefore$ $$s = -40000$$

Hence, transient response, $v_t(t) = Ke^{-40000t}$

To find the valve of K, we first find complete response v(t) and use initial condition.

The forced response is constant as source is constant, so, $\frac{dv_f}{dt} = 0$, the equation (1) becomes,

$$40000\,v_f(t) = 4 \times 10^{6}$$

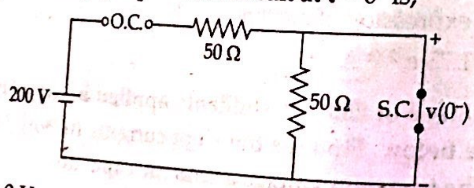
$\therefore$ $$v_f(t) = 100\text{ V}$$

Hence, overall response is,

$$v(t) = v_f(t) + v_t(t)$$

or, $$v(t) = 100 + Ke^{-40000t}$$

For initial condition $v(0^+)$, equivalent circuit at $t = 0^-$ is,



Clearly $v(0^-) = 0$ V

We know, for capacitor, $v(0^-) = v(0^+) = 0$ V

Applying this equation (2), we get,

$$v(0^+) = 100 + K e^{-40000 \times 0}$$

or, $$0 = 100 + K$$

$\therefore$ $K = -100$

so, $v(t) = 100 - 100\,e^{-40000t}$

Thus, transient response $v_t(t)$,

$$v_t(t) = -100\,e^{-40000t}\text{ V}$$

Since v is voltage across capacitor as well, this expression gives transient voltage across capacitor. Now, current through capacitor,

$$i_2(t) = 1 \times 10^{-6}\frac{dV_t(t)}{dt}$$

$$= 10^{-6}\frac{d}{dt}(-100\,e^{-40000t})$$

$$= 10^{-6}(-40000)(-100\,e^{-40000t})$$

$$= 4e^{-40000t}\text{ A}$$

For transient current i_1, we have,

$$i_1(t) = \frac{200 - v_t(t)}{50} = \frac{4-(-100\,e^{-40000t})}{50}$$

$\therefore$ $i_1(t) = 4 + 2\,e^{-40000t}$ A

Here, transient response of $i_1(t)$ is,

$\therefore$ $i_1(t) = 2\,e^{-40000t}$ A

$\therefore$ $i_2(t) = 4\,e^{-40000t}$ A

$\therefore$ $v_f(t) = -100\,e^{-4000t}$ V

14. A dc voltage of 1 V is applied to a series RC circuit at t = 0, having R = 4 Ω and C = 1/16 F. Use classical method to find the voltage across the capacitor $v_C(t)$ for t > 0. Assume initial voltage across capacitor, $v_C(0^-) = 9$ V **[2016/Fall]**

Solution:

Given that;

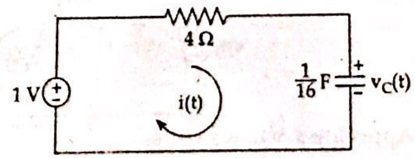
$V = 1$ V

$R = 4\ \Omega$

$C = \frac{1}{16}\ \Omega$

$v_C(0^-) = 9$ V

The circuit for $t > 0$ is,



Applying KVL,

$$4i(t) + v_C(t) = 1$$

Since i(t) is current through capacitor as well, we can write,

$$i(t) = \frac{1}{16}\frac{dv_C}{dt}, \text{ so,}$$

or, $$4 \times \frac{1}{16}\frac{dv_C}{dt} + v_C(t) = 1$$

or, $$\frac{dv_C(t)}{dt} + 4\,v_C(t) = 4 \quad (1)$$

As source is constant, the forced response, $v_{Cf}(t)$ is also constant, thus,

$$\frac{dv_{Cf}(t)}{dt} = 0$$

Then equation (1) becomes,

$$4\, v_{Cf}(t) = 4$$

$\therefore$ $v_{Cf}(t) = 1$

To find transient response, $v_{Ct}(t)$, we make equation (1) homogenous, find its characteristic equation, which is, $s + 4 = 0$

$\therefore$ $s = -4$

so, $v_{Ct}(t) = Ke^{-4t}$

Overall response, $V_C(t) = v_{Cf}(t) + v_{Ct}(t)$

or, $v_C(t) = 4 + Ke^{-4t}$

To find value of K, using initial condition *i.e.*, $v_C(0^-) = 9$ V

So, equation (2) becomes,

$v_C(0^-) = 4 + Ke^{-4\times 0}$

or, $9 = K + 4 + K$

or, $9 - 4 = K$

$\therefore$ $K = 5$

Hence, voltage across capacitor, $v_C(t) = (4 + 5\,e^{-4t})$ V.

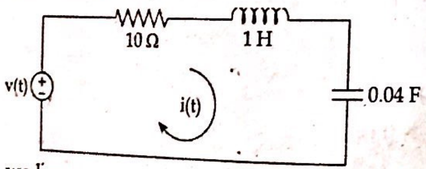
15. An exponential voltage, $v = 4e^{-3t}$ is applied to a series RLC circuit consisting of resistor R = 10 Ω, inductor, L = 1 H and capacitor C = 0.04 F at time t = 0. The inductor and capacitor are initially uncharged. Obtain the complete solution for the current i(t) in the circuit. **[2015/Spring]**

Solution:

$v(t) = 4\,e^{-3t}$ V ; L = 1 H

R = 10 Ω ; C = 0.04 F

The equivalent circuit for t > 0 is,



Applying KVL, we have,

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{\infty} i(t)\,dt = v(t)$$

or, $$10\,i(t) + 1\frac{di(t)}{dt} + \frac{1}{0.04}\int_{-\infty}^{t} i(t)\,dt = 4\,e^{-3t}$$

Different with respect to t,

$$\frac{d^2i(t)}{dt^2} + 10\frac{di(t)}{dt} + 25\,i(t) = -12\,e^{-3t}$$

Since source is exponential, the forced response, $i_f(t)$ will be exponential as well.

Let, $i_f(t) = Ke^{-3t}$, then,

$$\frac{di_f(t)}{dt} = -3\,Ke^{-3t} \text{ and } \frac{d^2i_f(t)}{dt^2} = 9\,Ke^{-3t}$$

Thus equation (1) becomes,

$9\,K\,e^{-3t} + 10 \times (-3\,Ke^{-3t}) + 25\,(K\,e^{-3t}) = -12\,e^{-3t}$

or, $4\,Ke^{-3t} = -12\,e^{-3t}$

$\therefore$ $K = -3$

Hence, $i_f(t) = -3\,e^{-3t}$ A

For transient response, we make equation (1) homogenous and find its characteristic equation which is,

$s^2 + 10s + 25 = 0$

$\therefore$ $s_1, s_2 = -5, -5$

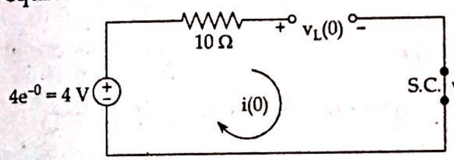
The roots are real and repeated, so the transient response is

$i_t(t) = (K_1 + K_2)\,e^{-5t}$

Thus, overall response, $i(t) = i_f(t) + i_t(t)$

or, $i(t) = -3\,e^{-3t} + (K_1 + K_2t)\,e^{-5t}$ (2)

For values of K_1 and K_2, we use initial conditions. Given that the inductor and capacitor are initially uncharged, so it means inductor is replaced by open circuit and capacitor is replaced by short circuit at t = 0. The equivalent circuit at t = 0 is,



This circuit is open, so i(0) = 0 A (3)

Applying KVL,

$4 = 10 \times i(0) + v_L(0^-) + v_C(0)$

or, $4 = 10 \times 0 + v_L(0) + 0$

$\therefore$ $v_L = 4$ V

Since i is current through inductor as well, we have, $1 \times \frac{di(0)}{dt} = 4$

$\therefore$ $\frac{di(0)}{dt} = 4$ (4)

Applying equation (3) in equation (2), we get,

$i(0) = -3\,e^{-0} + (K_1 + K_2 \times 0)\,e^{-5\times 0}$

or, $0 = -3 + (K_1 + 0) \times 1$

$\therefore$ $K_1 = 3$

Different equation (2) with respect to t,

$$\frac{di(t)}{dt} = 9e^{-3t} - 5e^{-5t}(K_1 + K_2 t) + K_2 e^{-5t}$$

Applying equation (4), we get,

$$\frac{di(0)}{dt} = 9e^{-0} - 5e^{-0}(K_1 + K_2 t) + K_2 e^{-5t}$$

Applying equation (4), we get,

$$\frac{di(0)}{dt} = 9e^{-0} - 5e^{-0}(K_1 + K_2 \times 0) + K_2 e^{-0}$$

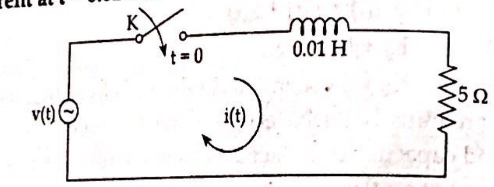
or, $4 = 9 - 5 \times K_1 + K_2$

∴ $K_2 = 4 - 9 + 5 \times 3 = 10$

Hence, complete solution for current,

$$i(t) = -3e^{-3t} + (3 + 10t)\, e^{-5t} \text{ A}$$

16. **An R-L circuit is connected to an ac voltage v(t) = 100 sin (50... 30) volts at t = 0; R = 5 Ω and L = 0.01 H. Find the equation for ... current using classical method. Also find the value of cir... current at t = 0.01 sec.** **[2015/Spri...]**



Solution:

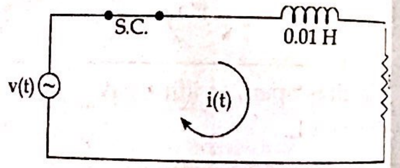
Given that;

$v(t) = 100 \sin(500t + 30)$ V at t = 0

$R = 5\Omega$

$L = 0.01$ H

The circuit for t > 0 is,



Applying KVL,

or, $L\frac{di(t)}{dt} + R\, i(t) = v(t)$

or, $0.01\frac{di(t)}{dt} + 5\, i(t) = 100 \sin(500t + 30)$

or, $\frac{di(t)}{dt} + 500\, i(t) = 10^4 \sin(500t + 30)$

The forced response, $i_f(t)$ due to sinusoidal source is,

$$i_f(t) = \frac{V_m}{|Z|}\sin(\omega t - \phi)$$

where, V_m = Maximum source voltage = 100 V

ω = Angular frequency = 500 rad/sec

$$|Z| = \sqrt{R^2 + (\omega L)^2} = \sqrt{(5)^2 (500 \times 0.01)^2} = 7.07$$

$$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right) = \tan^{-1}\left(\frac{500 \times 0.01}{5}\right) = 45°$$

so, $i_f(t) = \frac{100}{7.07}\sin(500t + 30° - 45°) = 14.14 \sin(500t - 15°)$

For transient response, we make equation (1) homogenous and find its characteristic equation, which is,

$s + 500 = 0$

∴ $s = -500$

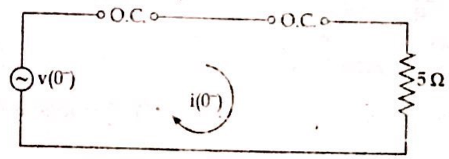
so, $i_t(t) = K\, e^{-500t}$

Overall response is, $i(t) = i_f(t) + i_t(t)$

$$i(t) = 14.14 \sin(500t - 15°) + K\, e^{-500t} \quad (2)$$

For value of K, we use initial condition at $t = 0^+$,

The equivalent circuit at $t = 0^-$ is,



The circuit is open so, $i(0^-) = 0$ A. Since $i(0^-)$ is current through inductor,

$i(0^+) = i(0^-) = 0$ A

Applying this in equation (2), we get,

$i(0^+) = 14.14 \sin(500 \times 0 - 30°) + K\, e^{-500 \times 0}$

or, $0 = 14.14 \times (-0.5) + K$

∴ $K = 7.07$

Hence, $i(t) = 14.14 \sin(500t - 15°) + 7.07\, e^{-500t}$

At t = 0.01 sec

$i(0.01) = 14.14 \sin(500 \times 0.1 - 15) + 7.07\, e^{-500 \times 0.01}$

∴ $i(0.01) = -2.41$ A

Hence the value of circuit current at t = 0.01 sec is −2.41 A

17. **An RC circuit is connected to an ac voltage e = 100 sin 100 t volts at t = 0. If R = 100 Ω and C = 100 μF, find the equations for the current using classical method.** **[2015/Fall]**

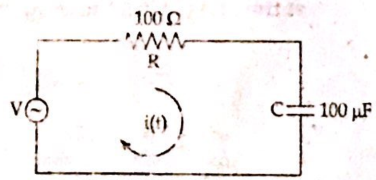
Solution:

Given that;

$e = V = 100 \sin(100t)$ volts

$R = 100\ \Omega$

$C = 100\ \mu F$



The circuit for t > 0 is,

Applying KVL,

$$100\, i(t) + \frac{1}{100 \times 10^{-6}}\int_{-\infty}^{t} i(t)\, dt = 100 \sin(100t)$$

Different with respect to t, we get,

$$100\frac{di(t)}{dt} + 10^4\, i(t) = 100 \times 100 \cos(100t)$$

or, $$\frac{di(t)}{dt} + 100\, i(t) = 100 \cos(100t)$$

The forced response, $i_f(t)$ to a sinusoidal source is, $i_f(t) = \frac{V_m}{|Z|}\sin(\omega t - \phi)$

where, V_m = Maximum source voltage = 100 V

ω = Angular source frequency = 100 rad/sec

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} = \sqrt{(100)^2 + \left(\frac{1}{100 \times 100 \times 10^{-6}}\right)^2} = 141.12\,\Omega$$

$$\phi = \tan^{-1}\left(\frac{\frac{1}{\omega C}}{R}\right) = \tan^{-1}\left(\frac{\frac{1}{100} \times 100 \times 10^{-6}}{100}\right) = 45°$$

so, $$i_f(t) = \frac{100}{141.42}\sin(100t - 45°)$$

For transient response, we make equation (1) homogenous and find characteristic equation which is,

$s + 100 = 0$

$\therefore$ $s = -100$

Hence, $i_t(t) = Ke^{-100t}$ A

Overall response, $i(t) = i_f(t) + i_t(t)$

or, $i(t) = 0.71 \sin(100t - 45°) + K e^{-100t}$

Using initial condition to find value of K,

Value of v(t) at t = 0, $v(0) = 100 \sin(100 \times 0) = 0$ V

Assuming capacitor to be initially uncharged,

$$i(0) = \frac{V}{Z} = 0 \text{ A}$$

Applying this in equation (2), we get,

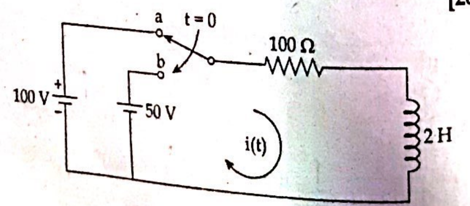
$\therefore$ $i(0) = 0.71 \sin(100 \times 0 - 45°) + K e^{-100 \times 0}$

or, $0 = 0.71 \times (-0.71) + K$

$\therefore$ $K = 0.5$

Hence, equation for current $i(t) = 0.71 \sin(100t - 45°) + 0.5\, e^{-100t}$ A

18. in the given circuit, switch K is moved position 'a' to position ... at time t = 0. Find the complete response of current i(t) for t > ... [2014/Spr...]



Solution:

Applying KVL for $t = 0^+$,

$$v = R\, i(t) + L\frac{di(t)}{dt}$$

or, $$50 = 100\, i(t) + 2\frac{di(t)}{dt} \quad (1)$$

For steady state response, i(t) is constant because voltage source is constant.

or, $50 = 100\, i_f + 0$ $\left[\because i(t) \text{ is constant so } \frac{di_f(t)}{dt} = 0\right]$

or, $50 = 100\, i_f + 0$

$\therefore$ $i_f(t) = 0.5$ A

For transient response, required homogenous equation is,

or, $$100\, i(t) + 2\frac{di(t)}{dt} = 0$$

or, $100\, i(t) + 2P\, i(t) = 0$

$\therefore$ $P = -50$

General equation for transient response is,

$i_t(t) = Ke^{Pt}$

$\therefore$ $i_t(t) = Ke^{-50t}$ (2)

Hence, complete response,

$i(t) = i_f(t) + i_t(t)$

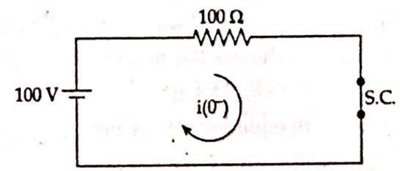
$= 0.5 + Ke^{-50t}$ (3)

At t = 0,

$i(0^+) = 0.5 + K$

$\therefore$ $K = i(0^+) - 0.5$

Since $i(0^-) = i(0^+) = \frac{V}{R} = \frac{100}{100} = 1$ A



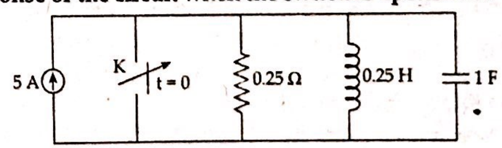
The equivalent circuit at $t = 0^-$ is,

Thus, $K = 1 - 0.5 = 0.5$

Hence, complete response of current i(t),

$$i(t) = 0.5 + 0.5\, e^{-50t} = 0.5\,(1 + e^{-50t}) \text{ A}$$

19. **For the network shown in figure below, determine the voltage response of the circuit when the switch is opened at t = 0.[2011/Fall]**



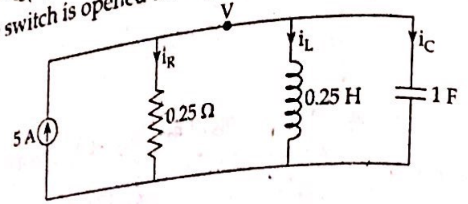
Solution:

For t < 0, the switch is closed and no current flows through the circuit, inductor acts as open circuit and capacitor acts like short circuit.

$\therefore \quad v_C(0^-) = v_C(0^+) = 0$ V

$\therefore \quad i_L(0^-) = i_L(0^+) = 0$ A

For $t > 0$, the switch is opened and applying KCL at a node,



or, $\quad I = i_C + i_R + i_L$

or, $\quad 5 = i_C + i_R + i_L$

or, $\quad 5 = C\dfrac{dv}{dt} + \dfrac{v}{R} + \dfrac{1}{L}\int v\,dt$

or, $\quad 5 = 1\dfrac{dv}{dt} + \dfrac{v}{0.25} + \dfrac{1}{0.25}\int v\,dt$

or, $\quad 5 = \dfrac{dv}{dt} + 4v + 4\int v\,dt$

Different equation (1) with respect time 't',

or, $\quad 0 = \dfrac{d^2v}{dt^2} + 4\dfrac{dv}{dt} + 4$

$\therefore \quad \dfrac{d^2v}{dt^2} + 4\dfrac{dv}{dt} + 4 = 0$

The auxiliary equation or characteristic equation is,

$$s^2 + 4s + 4 = 0$$

where, roots are $s_1 = -2$, $s_2 = -2$.

Since the roots are real and equal, the response is critically dampe responses which is defined as,

$$v(t) = (K_1 t + K_2)\, e^{-2t}$$

At $t = 0^+$ in equation (3), we get,

$$v(0^+) = (0 + K_2)\, e^{-2\times 0}$$

or, $\quad 0 = K_2 \times 1$

$\therefore \quad K_2 = 0$

Differentiating equation (3) with respect to t, we get,

$$\frac{dv(t)}{dt} = K_1 e^{-2t} + (-2K_1 t)\, e^{-2t} + (-2t K_2 e^{-2t}).$$

Putting $K_2 = 0$, we get,

or, $\quad \dfrac{dv(t)}{dt} = K_1 e^{-2t} - 2K_1 t\, e^{-2t} - 0$

At $t = 0^+$,

$$\frac{dv(0^+)}{dt} = K_1 e^{-2\times 0} - 2K_1 \times 0 \times e^{-2\times 0}$$

$\therefore \quad \dfrac{dv(0^+)}{dt} = K_1$

From equation (1), we have,

At $t = 0^+$,

$$5 = \frac{dv(0^+)}{dt} + 4v(0^+) + i_L(0^+)$$

or, $\quad 5 = \dfrac{dv(0^+)}{dt} + 4 \times 0 + 0$

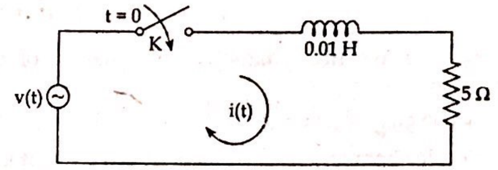
$\therefore \quad \dfrac{dv(0^+)}{dt} = 5$ V/sec

Hence, $K_1 = 5$

Replacing value of K_1 and K_2 in equation (3), we get,

$\therefore \quad v(t) = (5t + 0)\, e^{-2t}$ V $= 5t\, e^{-2t}$ V

20. An RL circuit is connected to an ac voltage V(t) = 100 sin (500t + 30) volts at t = 0, R = 5 Ω and L = 0.01 H. Find the equation for the current using classical method. Also find the value of circuit current at t = 0.01 sec. **[2011/Fall]**



Solution: Proceed as question number (3).

21. Find the total response i(t) for an R-L series circuit with components R = 10 Ω and L = 0.1 H for sinusoidal input voltage v(t) = 100 sin 200t. **[2011/Spring]**

Solution: Proceed as question number (4).

22. In the series RLC circuit the switch K is closed at t = 0, R = 7 Ω, L = 1 H, C = 0.1 F, V_0 = 10 V. Obtain the general and particular solution for current i(t). Also obtain the value of current at time t = 0.1 second. **[2011/Spring]**

Solution: Proceed as question number (8).

23. A step dc voltage of 5 volts is applied at time t = 0 to a series R-L-C circuit consisting of a resistor R = 10 Ω, inductor L = 1 H and capacitor C = 1 μF. Obtain the total solution for the current i(t). Assume zero current through L and zero voltage across C before switching. **[2012/Spring]**

Solution:

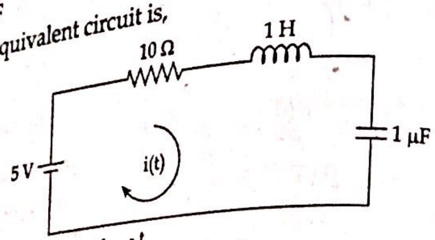
Given that;

$V = 5$ V

$R = 10\,\Omega$

$C = 1\mu F$

For $t > 0$, the equivalent circuit is,



$$10\,i(t) + 1 \times \frac{di(t)}{dt} + \frac{1}{10^{-6}}\int_{-\infty}^{t} i(t)\,dt = 5$$

Differentiating with respect to t, we get,

$$\frac{d^2i(t)}{dt^2} + 10\frac{di(t)}{dt} + 10\,i(t) = 0$$

The forced response, $i_f(t)$ is constant as source voltage is constant, so,

$$\frac{di_f(t)}{dt} = \frac{d^2i_f(t)}{dt^2} = 0, \text{ then equation (1) becomes,}$$

$$10^6\, i_f(t) = 0$$

$\therefore \quad i_f(t) = 0\text{ A}$

For transient response, we find characteristic equation of equation which is $s^2 + 10s + 10^6 = 0$

$\therefore \quad s_1, s_2 = -5 + 999.99\,j, -5 - 999.99\,j$

Since roots are complex conjugates, the transient response, it is

$$i_t(t) = e^{-5t}\,(A \cos 999.99t + B \sin 999.99t)$$

Hence overall response is,

$$i(t) = i_f(t) + i_t(t)$$

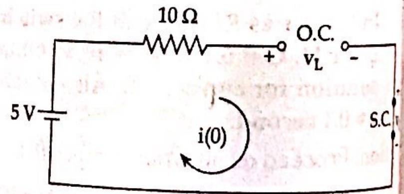
or, $\quad i(t) = e^{-5t}\,(A \cos 999.99t + B \sin 999.99t)$

Given, no initial current through an inductor and capacitor is se energized hence,

$$i(0^-) = i_L(0^-) = 0\text{ A}$$

$$v_C(0^-) = v_C(0^+) = 0\text{ A}$$

The equivalent circuit at $t = 0$ is,



Applying KVL,

or, $\quad 10 \times i(0) + v_L(0) + v_C(0) = 5$

or, $\quad 10 \times 0 + v_L(0) + 0 = 5$

$\therefore \quad v_L(0) = 5$

or, $\quad 1 \times \frac{di(t)}{dt} = 5$

Also, $\frac{di(0)}{dt} = \frac{di_L(0)}{dt} = 5\text{ A/sec}$

At $t = 0^+$, the equation (2) becomes,

$$i(0) = e^{-5(0)}\,(A \cos 999.99 \times 0 + B \sin 999.99 \times 0)$$

or, $\quad 0 = A \times 1 + B \times 0$

$\therefore \quad A = 0$

Different equation (2) with respect to t,

$$\frac{di(t)}{dt} = -5\,e^{-5t}\,(A \cos 999.99t + B \sin 999.99t)$$
$$+ e^{-5t}\,(-999.99 \sin 999.99t + 999.99\,B \cos 999.99t)$$

At $t = 0$,

$$\frac{di(t)}{dt} = -5\,e^{-5\times 0}\,(A \times 1 + B \times 0) + e^{-5\times 0}\,(-999.99\,A \times 0 + 999.99\,B)$$

or, $\quad 5 = -5 \times 1\,(0 + 0) + 1\,(0 + 999.98)$

$\therefore \quad B = 0.005$

Hence, $i(t) = 0.005\,e^{-5t} \sin (999.99t)$ A.

24. A series circuit containing $R = 7\,\Omega$, $L = 1$ H and $C = 0.1$ F is connected to 20 V dc supply at $t = 0$ sec. Find the expression of voltage across capacitor at any time using classical method considering inductor has zero current and capacitor has zero voltage initially. **[2013/Spring]**

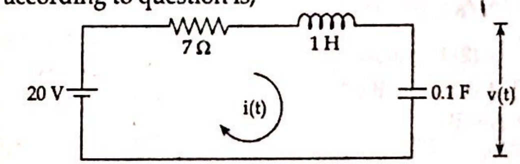
Solution:

Given that;

$R = 7\,\Omega \qquad L = 1\text{ H}$

$C = 0.1\text{ F} \qquad v = 20\text{ V}$

The circuit according to question is,



For $t > 0$, applying KVL,

$$7\,i(t) + 1 \times \frac{di(t)}{dt} + V(t) = 20$$

Since i(t) is current through capacitor as well, we can write,

$$i(t) = 0.1\frac{dv(t)}{dt} \text{ so,}$$

or, $\quad 7 \times 0.1\frac{dv(t)}{dt^2} + 1 \times 0.1\frac{d^2v(t)}{dt^2} + v(t) = 20$

or, $\quad 0.1\frac{d^2v(t)}{dt^2} + 0.7\frac{dv(t)}{dt} + 10\,\dot{v}(t) = 20$

or, $\quad \frac{d^2v(t)}{dt^2} + 7\frac{dv(t)}{dt} + 10\,v(t) = 200 \qquad (1)$

The forced response, $v_f(t)$ is constant as source is constant,

so, $\frac{d^2v_f(t)}{dt^2} = \frac{dv_f(t)}{dt} = 0$, then equation (1) becomes,

or, $0 + 7 \times 10\, v_f(t) = 200$

$\therefore$ $v_f(t) = 20$ V

For transient response, $v_t(t)$, we make equation (1). Homogenous and find its characteristic equation which is,

$s^2 + 7s + 10 = 0$

or, $s_1\, s_2 = -2, -5$

Since roots are real and distinct, the transient response is,

$v_t(t) = A\, e^{-2t} + B\, e^{-5t}$

The complete response is,

$v(t) = v_f(t) + v_t(t)$

$\therefore$ $v(t) = 20 + A\, e^{-2t} + B\, e^{-5t}$

Given, initial current is zero for inductor and capacitor has zero volt initially, i.e.,

$v_C(0^-) = 0\text{ V} = v(0)$

$i_L(0^-) = 0$ A

Since it is a series circuit, $i_C(0^-) = i_L(0^-) = 0$ A

or, $i_C(0^-) = 0$

or, $0.1 \times \frac{dv(0)}{dt} = 0$

$\therefore$ $\frac{dv(0)}{dt} = 0$ V/s

At t = 0, equation (2) becomes,

$v(0) = 20 + A\, e^{-2\times 0} + B\, e^{-5\times 0}$

or, $0 = 20 + A + B$

$\therefore$ $A + B = -20$

Different equation (2) with respect to t,

$\frac{dv(t)}{dt} = -2Ae^{-2t} - 5Be^{-5t}$

At t = 0,

$\frac{dv(0)}{dt} = -2A\, e^{-2\times 0} - 5 \times B\, e^{-5\times 0}$

or, $0 = -2A - 5B$

or, $2A + 5B = 0$

Solving equation (3) and (4), we get,

$A = -33.33$

$\therefore$ $B = -13.33$

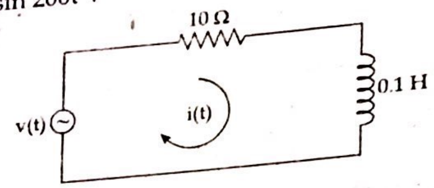
Hence, $v(t) = 20 - 33.33\, e^{-2t} + 13.33\, e^{-5t}$ V

25. Find the total response i(t) for an RL series circuit with components R = 10 Ω and L = 0.1 H for sinusoidal input voltage v(t) = 100 sin 200 t. [2014/Spring]

Solution:

$R = 10\Omega$ and $L = 0.1$ H

$V(t) = 100 \sin 200t$ V



Applying KVL,

$10\, i(t) + 0.1 \frac{di(t)}{dt} = 100 \sin 200t$

or, $\frac{di(t)}{dt} + 100\, i(t) = 100 \sin 200t$ (1)

Since source is sinusoidal, the forced response, $i_f(t)$ is given by,

$i_f(t) = \frac{V_{max}}{|Z|} = \sqrt{R^2 + (\omega L)^2} = \sqrt{(10)^2 + (200 \times 0.1)^2} = 22.36\ \Omega$

$\phi = \tan^{-1}\left(\frac{\omega L}{R}\right) = \tan^{-1}\left(\frac{200 \times 0.1}{10}\right) = 63.43°$

so, $i_f(t) = \frac{100}{22.36} \sin(200t - 6.43°)$

For transient response, we make equation (1) homogenous and find its characteristic equation which is,

$s + 100 = 0$

$\therefore$ $s = -100$

Hence, transient response, $i_t(t)$ is,

$i_t(t) = K\, e^{-100t}$

Overall response is, $i(t) = i_f(t) + i_t(t)$

$\therefore$ $i(t) = 4.47 \sin(200t - 63.43) + K\, e^{-100t}$ (2)

Assuming no current flows through inductor initially,

i.e., $i(0) = i_L(0) = 0$ A

At t = 0, equation (2) becomes

$i(0) = 4.47 (\sin 200 \times 0 - 63.43) + K\, e^{-100\times 0}$

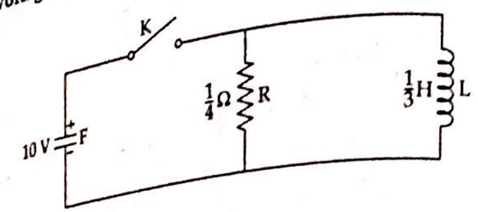
or, $0 = 4.47 \times (-0.89) + K$

$\therefore$ $K = 3.99$

Hence, $i(t) = 4.47 \sin(200t - 63.43°) + 3.99\, e^{-100t}$ A.

26. In the given circuit below, capacitor C has an initial voltage $v_C(0^-)$ = 10 volts and at the same instant current in the inductor L is

zero. Switch K is closed at time t = 0. Obtain the expression the voltage v(t) across the inductor L using classical approach. [2020]

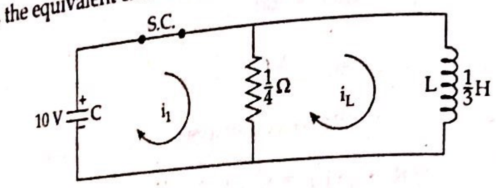


Solution:

$v_C(0^-) = 10$ V

$i_L(0^-) = 0$ A

For t > 0, the equivalent circuit is,



Applying KVL in left loop,

$$v_C(t) + \frac{1}{4}(i_1 - i_2) = 0$$

or, $$\frac{1}{1}\int_0^t i_1(t)\,dt - \frac{10}{4}(i_1(t) - i_2(t)) = 0$$

or, $$4\int_0^t i_1\,dt + (i_1 - i_2) = 40$$

Differentiating both sides with respect to t, we get,

$$4i_1(t) + \frac{di_1(t)}{dt} - \frac{di_2(t)}{dt} = 0$$

Applying KVL in right loop, we get,

$$\frac{1}{3}\frac{di_2(t)}{dt} + \frac{1}{4}(i_2 - i_1) = 0$$

$\therefore$ $$i_1 = \frac{4}{3}\frac{di_2}{dt} + i_2$$

Replacing value of i_1 from equation (2) in equation (1), we get,

or, $$4\left(\frac{4}{3}\frac{di_2}{dt} + i_2\right) + \frac{d}{dt}\left(\frac{4}{3}\frac{di_2}{dt} + i_2\right) - \frac{di_2}{dt} = 0$$

or, $$\frac{16}{3}\frac{di_2}{dt} + 4i_2 + \frac{4}{3}\frac{d^2 i_2}{dt^2} + \frac{di_2}{dt} - \frac{di_2}{dt} = 0$$

or, $$\frac{d^2 i_2}{dt^2} + \frac{1}{4}\frac{di_2}{dt} + 3i_2 = 0$$

Since source is constant, the forced response $i_2(t)$, is also constant, so,

$$\frac{d^2 i_{2f}}{dt^2} = \frac{di_{2f}}{dt} = 0$$

Now, equation (3) becomes,

or, $$0 + \frac{1}{4} \times 0 + 3i_{2f} = 0$$

$\therefore$ $$i_{2f}(t) = 0$$

For transient response, the characteristic equation of equation (3) is given by,

$$s^2 + \frac{1}{4}s + 3 = 0$$

$\therefore$ $$s = -0.125 + j\,1.73,\ -0.125 - j1.73$$

The roots are complex conjugates, so transient response is,

$$i_{2t}(t)\ e^{-0.125t}(K_1 \cos 1.73t + K_2 \sin 1.73t)$$

The total response is,

$$i_2(t) = i_{2f}(t) + i_{2t}(t)$$

or, $$i_2(t) = e^{-0.125t}(K_1 \cos 1.73t + K_2 \sin 1.73t) \quad (4)$$

We are given initial conditions;

$$v_C(0^-) = 10 \text{ V}$$

$$i_2(0^-) = i_L(0^-) = 0\text{A}$$

so, $$v_C(0^+) = 10 \text{ V} \quad (5)$$

$$i_2(0^+) = 0 \text{ A}$$

Since the capacitor, resistor and capacitor are in parallel, all elements have same potential difference, so, if v_L be voltage across inductor.

$$v_L(0^+) = v_C(0^+) = 10 \text{ V}$$

or, $$\frac{1}{3} \times \frac{di(0^+)}{dt} = 10 \text{ V}$$

or, $$\frac{di(0^+)}{dt} = 30 \text{ A/sec} \quad (6)$$

Put $t = 0^+$ in equation (4), we get,

$$i_2(0^+) = 1 \times (K_1 \times 1 + 0)$$

Replacing value of $i_2(0^+)$ from equation (5), we get,

$\therefore$ $$K_1 = 0$$

Different equation (4) with respect to t,

$$\frac{di_2(t)}{dt} = -0.125\,e^{-0.125t}(K_1 \cos 1.73t + K_2 \sin 1.73t)$$
$$+ e^{-0.125t}(-K_1\,1.73 \sin 1.73t + K_2 \times 1.73 \times \cos 1.73t)$$

At $t = 0^+$,

$$\frac{di(0^+)}{dt} = -0.125(K_1 + 0) + 1 \times (-0 + 1.73\,K_2)$$

Replacing value of $\frac{di_2(0^+)}{dt}$ from equation (6), we get,

$$30 = -0.125 \times 0 + 1.73\ K_2$$

$\therefore$ $K_2 = 17.34$

Hence, current inductor is given by,

$\therefore$ $i_2(t) = 17.34\ e^{-0.125} \sin (1.73)t$ A.

27. **Obtain the step response of series RLC circuit using cla... method. Assume that there is no current flowing through inductor and zero voltage across capacitor before switching.** [2019...]

Solution: See the topic 3.5.1.

28. **Obtain the sinusoidal response of RLC series circuit with nece... assumption.** [2018/Sp...]

Solution: See the topic 3.5.2.

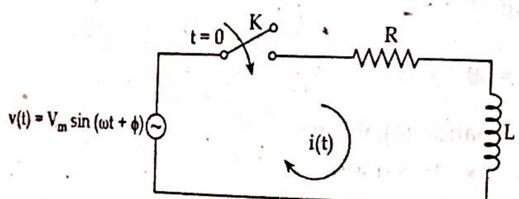
29. **Find the response of R-L circuit connected to DC voltage so... when switch is closed at t = 0 and define the time constant for...** [2018/F...]

Solution: See the topic 3.3.1.

30. **Write short notes on step response of R-C series circuit.** [2012/Spring, 2013/Fall, 2014/Fall, 2014/Spri...]

Solution: See the topic 3.4.1.

31. **An R-L circuit is connected to an AC voltage $v(t) = V_m \sin (\omega t$... The switch is closed at t = 0. Find the response of current us... classical method.** [2014/F...]



Solution: See the topic 3.3.2 "Sinusoidal excitation".

32. **Find the current response i(t) for a series R-C circuit excited by step voltage source of V volts with resistor R ohms and capaci... C farad at time $t \geq 0$. Assume that the capacitor C has an ini... voltage of V_0 volts.** [2013/Fa...]

Solution: See the topic 3.4.1.

☆☆☆

Chapter 4 REVIEW OF LAPLACE TRANSFORM

4.1 INTRODUCTION

The laplace transform is one of the mathematical tool used for the solution of linear ordinary integro-differential equations. (Mostly continuous-time systems are described by integro-differential equations). In comparison with the classical method of solving linear integro-differential equations, the laplace transform method has the following two attractive properties.

a) The homogenous equation and the particular integral o solution are obtained in one operation.
b) The laplace transform converts the integro-differential eq into an algebraic equation in s (laplace operator). It is then po to manipulate the algebraic equation by simple algebraic ru obtain the expression in suitable forms. The final solut obtained by taking the inverse laplace transform.

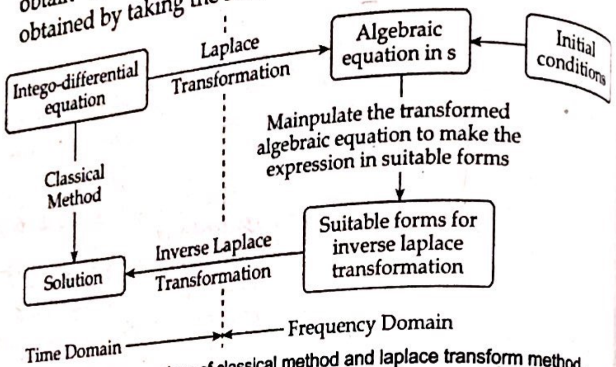


Figure 4.1: Comparison of classical method and laplace transform method.

From the second feature of laplace transform over classical method, la transformation is somewhat similar to logarithmic operation. To fin product or quotient of two number, we find,

i) Logarithm of two numbers.
ii) Add or subtract.
iii) Take antilogarithm to get product or quotient.

4.1.1 Advantages of Laplace Transform Method over Classic Method

Today, laplace transform is popularly used. The advantages of la transform over classical method are;

a) The solution of the differential equations gets greatly simplifie
b) This method yields in a single operation the complete solutio both the particular and complementary function.
c) Initial condition gets automatically specified in the transfor equation whereas in the classical method, these initial condi have to be separately specified.
d) Finally the initial conditions get included in the problem in the step in the analysis rather than in the last step.

The process of using laplace transformation is similar to the proce using logarithms which is also a type of a transformation. La transform gives systematic and routine solution for differential equati

4.2 THE LAPLACE TRANSFORMATION

The laplace transform method is a powerful technique for solving di problems. We define a laplace transform as follows;

For the time function f(t) which is zero for $t < 0$ and that satisfy the condition,

$$\int_0^\infty |f(t)\, e^{-\sigma t}|\, dt < \infty$$

For some real and positive σ, the laplace transform of f(t) is defined as,

$$F(s) = £[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

The variable s is referred to as the laplace operator, which is complex variable, *i.e.*, $s = \sigma + j\omega$. And the functions f(t) and F(s) are known as laplace transform pair.

Inverse Laplace Transformation

Given the laplace transform F(s), the operation of obtaining f(t) is termed the termed the inverse laplace transformation and is denoted by,

$$f(t) = £^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)\, e^{st}\, ds$$

Normally the transform table is used to obtain the inverse transformation.

4.3 IMPORTANT PROPERTIES OF LAPLACE TRANSFORMATION

a) Multiplication by a constant

Let K be a constant and F(s) be the laplace transform of f(t). Then,

$$£[Kf(t)] = K\,F(s).$$

b) Sum and difference

Let $F_1(s)$ and $F_2(s)$ be the laplace transform of $f_1(t)$ and $f_2(t)$ respectively. Then, $£[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$

c) Differentiation with respect to 't' (time-differentiation).

Let F(s) be the laplace transform of f(t) and let $f(0^+)$ be the value of f(t) as t approaches 0. Then,

$$£\left(\frac{df(t)}{dt}\right) = s\,F(s) - \lim_{t\to 0} f(t) = s\,F(s) - f(0^+)$$

Proof:

$$F(s) = £[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

Let, $f(t) = u$; then $\left(\frac{df(t)}{dt}\right) dt = du$

and, $e^{-st}\, dt = du$; then $v = \frac{-1}{s} e^{-st}$

On integrating,

$$F(s) = \int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= f(t)\left(\frac{-1}{s}e^{-st}\right)\Big|_0^\infty - \int_0^\infty \left(\frac{-1}{s}e^{-st}\right)\left(\frac{df(t)}{dt}\right) dt$$

$$\therefore \quad F(s) = \frac{1}{s}f(0^+) + \frac{1}{s}\int_0^\infty e^{-st}\left(\frac{df(t)}{dt}\right)dt = \frac{1}{s}f(0^+) + \frac{1}{s}£\left(\frac{df(t)}{dt}\right)$$

Hence, $£\left(\frac{df(t)}{dt}\right) = s\,F(s) - f(0^+)$

Thus the laplace transform of the second derivatives of f(t) as,

$$£\left(\frac{d^2f(t)}{dt^2}\right) = £\left[\frac{df(t)}{dt}\right] = s\,£\left(\frac{df(t)}{dt}\right) - \left.\frac{df(t)}{dt}\right|_{t=0}$$

$$= s[sF(s) - f(0^+)] - f^1(0^+) = s^2F(s) - sf(0^+) - f^1(0^+)$$

where, $f^1(0^+)$ is the value of the first derivative of f(t) as t approaches (0)

In general,

$$£\left(\frac{d^nf(t)}{dt^n}\right) = s^n F(s) - s^{n-1}f(0^+) - s^{n-2}f^1(0^+) - \ldots\ldots\ldots sf^{n-1}(0^+) - f^{n-1}(0^+)$$

d) Integration by 't' (time-integration)

If $£[f(t)] = F(s)$

Then the laplace transform of the first integral of f(t) is given by,

$$£\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Proof:

$$£\left[\int_0^t f(t)\,dt\right] = \int_0^\infty\left[\int_0^t f(t)\,dt\right]e^{-st}\,dt$$

Let, $u = \int_0^t f(t)\,dt$; then $du = f(t)\,dt$

and, $dv = e^{-st}\,dt$; then $v = \frac{-1}{s}e^{-st}$

On integrating,

$$£\left[\int_0^t f(t)\,dt\right] = \int_0^\infty v\,du$$

$$= uv\Big|_0^\infty - \int_0^\infty v\,du$$

$$= \frac{-1}{s}e^{-st}\int_0^t f(t)\,dt\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t)\,e^{-st}\,dt$$

$$= 0 - 0 + \frac{1}{s}£[f(t)] = \frac{F(s)}{s}$$

In general,

$$£\left[\int_0^{t_1}\int_0^{t_2}\ldots\ldots\ldots\int_0^{t_n} f(t)\,dt_1\cdot dt_2\ldots\ldots dt_n\right] = \frac{F(s)}{s^n}$$

The laplace transform of the indefinite integral is given as,

$$£\left(\int f(t)\,dt\right) = £\left[\int_0^t f(t)\,dt + f^{-1}(0^+)\right] = \frac{F(s)}{s} + \frac{f^{-1}(0^+)}{s}$$

where, $f^{-1}(0^+)$ is the value of the integral f(t) as t approaches zero.

e) Differentiation with respect to 's' (Frequency differentiation)

The differentiation in the s-domain corresponds to the multiplication by 't' in the time domain

i.e., $£[t\,f(t)] = \frac{-dF(s)}{ds}$

Proof:

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\infty f(t)\left(\frac{d}{ds}e^{-st}\right)dt$$

$$= \int_0^\infty f(t)\,e^{-st}(-t)\,dt = -\int_0^\infty t\cdot f(t)\,e^{-st}\,dt = -£[t\cdot f(t)]$$

f) Integration by "s" (Frequency-Integration)

The integration of F(s) in s-domain corresponds to division by "t" in the time domain.

i.e., $£\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\cdot ds$

Proof:
$$\int_s^\infty F(s)\cdot ds = \int_s^\infty\left[\int_0^\infty f(t)\,e^{-st}\,dt\right]ds = \int_0^\infty f(t)\int_s^\infty f(t)\int_s^\infty e^{-st}\,ds\,dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]\Big|_0^\infty dt = \int_0^\infty f(t)\left[0 - \frac{e^{-st}}{-t}\right]dt$$

$$= \int_0^\infty f(t)\frac{e^{-st}}{t}\,dt = £\left[\frac{f(t)}{t}\right]$$

g) Shifting theorem

i) Shifting in time (Time shifting):

The laplace transform of a shifted or delayed function is given as,

$$£[f(t-a)\cdot u(t-a) = e^{-as}F(s)$$

Proof:

Let, $t - a = y$, then $dt = dy$

$$£[f(t-a)\cdot u(t-a) = \int_{-a}^\infty f(y)\cdot u(y)\,e^{-s(y+a)}\,dy = \int_0^\infty f(y)\,e^{-s(y+a)}\,dy$$

$$= e^{-as}\int_0^\infty f(y)\,e^{-sy}\,dy = e^{-as}F(s)$$

ii) Shifting in frequency (frequency shifting)

The laplace transform of e^{-at} times a function is equal to the laplace transform of that function, with s is replaced by (s + a).

Proof:

$$£[e^{-at}f(t)] = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^\infty f(t)\,e^{-(s+a)t}\,dt = F(s+a)$$

NOTE: $£[e^{at}f(t)] = F(s-a)$

h) Initial value theorem

If the function f(t) and its first derivative $\frac{df(t)}{dt}$ are both laplace transformable,

then, $f(0^+) = \lim_{t\to 0} f(t) = \lim_{t\to 0}[s\cdot F(s)]$

Proof:

Using time-differentiation property,

$$s \cdot F(s) - f(0^+) = \int_0^\infty \frac{df(t)}{dt} e^{-st}\,dt$$

or, $$s \cdot F(s) = f(0^+) + \int_0^\infty \frac{df(t)}{dt} e^{-st}\,dt$$

or, $$\lim_{s\to\infty} [s \cdot F(s)] = f(0^+) + \lim_{s\to\infty} \int_0^\infty \frac{df(t)}{dt} e^{-st}\,dt$$

$\therefore$ $$\lim_{s\to\infty} [s \cdot F(s)] = f(0^+) + \int_0^\infty \frac{df(t)}{dt} \left(\lim_{s\to\infty} e^{-st}\right) dt = f(0^+)$$

i) Final value theorem

The final value of a function f(t) is given as,

$$f(\infty) = \lim_{s\to\infty} f(t) = \lim_{s\to 0} [s \cdot F(s)]$$

Proof:

Using time differentiation property, and we let, $s \to 0$

$$\lim_{s\to\infty} [s \cdot F(s) - f(0^+)] = \lim_{s\to 0} \int_0^\infty \frac{df(t)}{dt} e^{-st}\,dt = \int_0^\infty \frac{df(t)}{dt} \left(\lim_{s\to 0} e^{-st}\right) dt$$

$$= \lim_{s\to 0} \int_0^\infty \frac{df(t)}{dt}\,dt = f(t)\Big|_0^\infty$$

$$= \lim_{s\to\infty} [f(t) - f(0)]$$

$$\lim_{s\to 0} [s \cdot F(s)] = \lim_{s\to\infty} f(t)\ [\sin f(0^+) = f(0)]$$

j) Theorem for periodic functions

The laplace transform of a periodic function (wave) with period T is eq[illegible] to $\frac{1}{1-e^{-Ts}}$ times the laplace transform of the first cycle of that func[illegible] (wave).

Proof:

Let, $f_1(t), f_2(t), f_3(t), \ldots\ldots$ be the functions describing the first, second, th[illegible] …….. cycles of a periodic function f(t) whose time period is T. Then,

$$f(t) = f_1(t) + f_2(t) + f_3(t) + \ldots\ldots$$

$$= f_1(t) + f_1(t-T)\,u(t-T) + f_1(t-2T)\,u(t-2T) + \ldots\ldots$$

Now, Let, $\mathcal{L}[f_1(t)] = F_1(s)$

Therefore, by shifing theorem, we get,

$$\mathcal{L}[F(t)] = F_1(s) + e^{-Ts} F_1(s) + e^{-2Ts} F_1(s) + \ldots\ldots$$

$$= F_1(s)\,[1 + e^{-Ts} + e^{-2Ts} + \ldots\ldots]$$

$$= \frac{1}{1-e^{-Ts}} F_1(s)$$

k) Convolution theorem

Given two functions $f_1(t)$ and $f_2(t)$ which are zero for $t < 0$.

If $\mathcal{L}[f_1(t)] = F_1(s)$ and $\mathcal{L}[F_2(t)] = F_2(s)$, then

$\mathcal{L}^{-1}[F_1(s) \cdot F_2(s)] = f_1(t) * f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$ and is equal to

$$\int_0^t f_1(t-\tau)\, f_2(\tau)\, d\tau \text{ or } \int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau$$

l) Time scaling

If laplace transform of f(t) is F(s), then

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Table: Laplace transform pairs

S.N.	f(t)*	$F(s) = \int_0^\infty f(t)e^{-st}dt$
a)	1 or u(t), K	$\frac{1}{s}, \frac{K}{s}$
b)	t, t^n	$\frac{1}{s^2}, \frac{n!}{s^{n+1}}$
c)	$\delta(t)$	1
d)	$e^{\pm at}$	$\frac{1}{s \mp a}$
e)	$t\,e^{\pm at}$	$\frac{1}{(s \mp a)^2}$
f)	$\sin \omega t$	$\frac{\omega}{s^2 + w^2}$
g)	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
h)	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
i)	$e^{-at} \cos \omega t$	$\frac{(s+a)}{(s+a)^2 + \omega^2}$
j)	sin hat	$\frac{a}{s^2 - a^2}$
k)	cosh at	$\frac{s}{s^2 - a^2}$
l)	$e^{\pm at} f(t)$	$F(s \mp a)$
m)	$f(t \pm t_0)$	$e^{\pm t_0 s} F(s)$

All f(t) should be thought of as being multiplied by u(t), *i.e.*, f(t) = 0 for t < 0.

4.4 INITIAL AND FINAL VALUE THEOREM

4.4.1 Initial Value Theorem

The initial value theorem allows us to find the initial value $x(0)$ di... from its laplace transform $X(s)$.

If $x(t)$ is a causal signal

then, $x(0) = \lim_{s\to\infty} sX(s)$

Proof: To prove this theorem, we use the time differentiation property

$$\pounds\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0) = \int_0^\infty \frac{dx}{dt} e^{-st}\, dt$$

If we let $s \to \infty$, then integral on the right side of equation **(1)** vanishes ... to damping factor, e^{-st}

Thus,

$$\lim_{s\to\infty} [sX(s) - x(0)] = 0$$

$$\therefore \quad x(0) = \lim_{s\to\infty} s\,X(s)$$

4.4.2 Final Value Theorem

The final value theorem allows us to find the final value $x(\infty)$ direc... from its laplace transform $X(s)$. If $x(t)$ is a causal signal,

Then, $\lim_{s\to\infty} x(t) = \lim_{s\to 0} s\,X(s)$

Proof: The laplace transform of $\frac{dx(t)}{dt}$ is given by,

$$s\,X(s) - x(0) = \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt$$

Taking the limit $s \to 0$ on both sides, we get,

$$\lim_{s\to\infty} [s\,X(s) - x(0)] = \lim_{s\to\infty} \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt = \int_0^\infty \frac{dx(t)}{dt} \lim_{s\to\infty} e^{-st}\, dt$$

$$= \int_0^\infty \frac{dx(t)}{dt}\, dt = x(t)|_0^\infty = x(\infty) - x(0)$$

Since, $\lim_{s\to 0} [s\,X(s) - x(0)] = \lim_{s\to 0} [sX(s) - x(0)]$

We get, $x(\infty) - x(0) = \lim_{s\to 0} [sX(s) - x(0)]$

Hence, $x(\infty) = \lim_{s\to 0} [sX(s)]$

This proves the final value theorem. The final value theorem is ... useful since we can find $x(\infty)$ from $X(s)$. However, one must be careful ... using final value theorem since the function $x(t)$ may not have a fi... value as $t \to \infty$.

For example; Consider $x(t) = \sin at$ having $X(s) = \frac{a}{s^2 + a^2}$. Now we know ... $\sin at$ does not exist.

However, if we carefully use the final value theorem in this case, we would obtain.

$$\lim_{s\to 0} s\,X(s) = \lim_{s\to 0} s\left(\frac{a}{s^2 + a^2}\right) = 0.$$

Note that the actual function $x(t)$ does not have a limiting value as $t \to \infty$. The final value theorem has failed because the poles of $X(s)$ lie on the $j\omega$ axis. Hence, we conclude that for final value theorem to give a valid result, poles of $X(s)$ should not lie to the right side of the s-plane or on the $j\omega$ axis.

4.5 SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

The laplace transformation is used to determine the solution of integro-differential equation. A differential equation of the general form,

$$a_0 \frac{d^n i}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \ldots\ldots + a_{n-1} \frac{di}{dt} + a_{ni} = v(t)$$

becomes, as a result of the laplace transformation, an algebraic equation which may be solved for the unknown as,

$$I(s) = \frac{\pounds[v(t)] + \text{initial condition terms}}{a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1}s + a_n}$$

where, $I(s) = \pounds[i(t)]$

$I(s)$ is the ratio of two polynomials in s. Let the numerator and denominator polynomials be designated $P(s)$ and $Q(s)$ respectively, as

$$I(s) = \frac{P(s)}{Q(s)}$$

Note that $Q(s) = 0$ in the characteristic equation. If the transform term $\frac{P(s)}{Q(s)}$ can now be found in a table of transform pairs, the solution $i(t)$ can be written directly. In general however, the transform expression for $I(s)$ must be broken into simpler terms before any practical transform table can be used.

Next, we factor the denominator polynomial,

$$Q(s) = a_0 s^n + a_1 s^{n-1} + \ldots\ldots + a_{n-1} s + a_n$$
$$\equiv a_0 (s + s_1) \ldots\ldots (s + s_n)$$

or, Very compactly, $Q(s) = a_0 \prod_{j=1}^{n} (s + s_j)$

where, π indicates a product of factors and $s_1, s_2, \ldots\ldots, s_n$ are the n roots of the characteristic equation $Q(s) = 0$. Now the possible forms of these roots are discussed as;

a) Partial fraction expansion when all the roots of $Q(s)$ are simple.

If all the roots of $Q(s) = 0$, are simple, then

$$I(s) = \frac{P(s)}{(s + s_1)(s + s_2) \ldots\ldots (s + s_n)}$$
$$= \frac{K_1}{s + s_1} + \frac{K_2}{s + s_2} + \ldots\ldots + \frac{K_n}{s + s_n}$$

where, the K's are real constants called residues. Any of residues $K_1, K_2, \ldots\ldots, K_n$ can be found by multiplying I(s) by corresponding denominator factor and setting $(s + s_j)$ equal to ze

i.e., $s = -s_j \cdot A_s$

$$K_j = \left[(s + s_j)\frac{P(s)}{Q(s)}\right]_{s=-s_j}$$

b) Partial fraction expansion when some roots of Q(s) are of multiple or
If a root of Q(s) = 0, is if multiplicity r, then

$$I(s) = \frac{P(s)}{(s + s_1)^r Q_1(s)} = \frac{K_{11}}{s + s_1} + \frac{K_{12}}{(s + s_1)^2} + \ldots\ldots + \frac{K_{1r}}{(s + s_1)^r} + \ldots$$

The following equations may be used for the evaluation coefficients of repeated roots.

$$K_{1r} = (s + s_1)^r I(s)\big|_{s=-s_1}$$

$$K_{1(r-1)} = \frac{d}{ds}[(s + s_1)^r \cdot I(s)]\big|_{s=-s_1}$$

$$K_{1(r-2)} = \frac{1}{2!}\frac{d^2}{ds^2}[(s + s_1)^r \cdot I(s)]\big|_{s=-s_1}$$

$$K_{11} = \frac{1}{(r-1)!}\frac{d^{r-1}}{ds^{r-1}}[(s + s_1)^r \cdot I(s)]\big|_{s=-s_1}$$

c) Practical fraction expansion when two roots of Q(s) are of com conjugate pair.
If two roots of Q(s) = 0, which form a complex conjugate pair, th

$$I(s) = \frac{P(s)}{(s + \alpha + j\omega)(s + \alpha - j\omega) \cdot Q_1(s)}$$

$$= \frac{K_1}{(s + \alpha + j\omega)} + \frac{K_1^*}{(s + \alpha - j\omega)} + \ldots\ldots\ldots$$

where, $K_1 = (s + \alpha + j\omega) \cdot I(s)\big|_{s=-(\alpha+j\omega)}$

and, K_1^* is the complex conjugate of K_1. An expression of the shown above is necessary for each pair of compiles conjugate r

4.6 TRANSFORMED CIRCUIT COMPONENTS REPRESENTATION

4.6.1 Independent Sources

The sources v(t) and i(t) may be represented by their transformat namely V(s) and I(s) respectively as shown in figure 4.2.

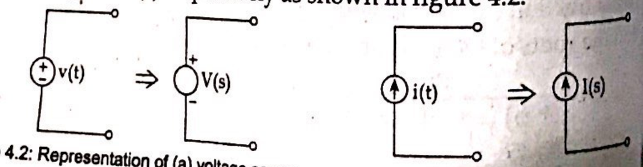


Figure 4.2: Representation of (a) voltage source (b) current source

4.6.2 Resistance Parameter

By ohm's law, the v-i relationship for a resistor in t-domain is, $v_R(t) = R\, i_R(t)$.
In the complex-frequency domain (s-domain), above equation becomes,

$$V_R(s) = R\, I_R(s)$$

From the above two equations, we observe that the representation of a resistor in t-domain and s-domain are one and the same.

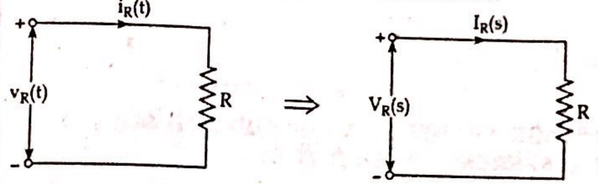


Figure 4.3: Representation of a resistor

4.6.3 Inductor Parameter

The v-i relationship for an inductor is,

$$v_L(t)\, L\frac{di_L(t)}{dt}$$

$$i_L(t) = \frac{1}{L}\int_0^t v_L(t)\, d(t) + i_L(0^+)$$

The corresponding laplace transforms are

$$V_L(s) = s_L\, I_L(s) - L\, i_L(0^+)$$

$$I_L(s) = \frac{1}{sL}V(s) + \frac{i_L(0^+)}{s}$$

From the above equations, we get the transformed circuit representation as shown in figure 4.4.

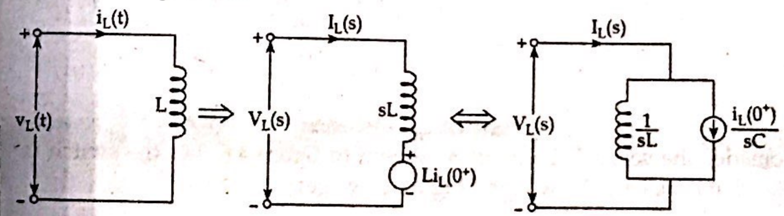


Figure 4.4: Representation of an inductor

4.6.4 Capacitor Parameter

For a capacitor, the v-i relationship is,

$$i_c(t) = C\frac{dv_c(t)}{dt} \text{ or } v_C(t) = \frac{1}{C}\int_{0^+}^t i_c(t)\, dt + v_C(0^+)$$

The corresponding laplace transform are,

$$I_C(s) = sC\, V_C(s) - C\, v_C(0^+)$$

$$\therefore \quad V_C(s) = \frac{1}{sC} I_C(s) + \frac{v_C(0^+)}{s}$$

From above equations, we get the transformed circuit representation for the capacitor as shown in figure 4.5.

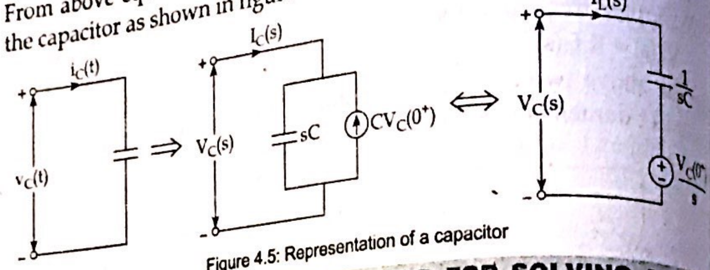


Figure 4.5: Representation of a capacitor

4.7 LAPLACE TRANSFORM METHOD FOR SOLVING A SET OF DIFFERENTIAL EQUATIONS

a) Identify the circuit variables such as inductor currents and capacitor voltages.

b) Obtain the differential equations describing the circuit and keep a watch on the initial conditions of the circuit variables.

c) Obtain the laplace transform of the various differential equations.

d) Using Cramer's rule or a similar technique, solve for one or more of the unknown variables, obtaining the solution in s domain.

e) Find the inverse transform of the unknown variable and this obtain the solution in the time domain.

4.8 RESPONSE OF R-L CIRCUIT WITH

A. DC Excitation

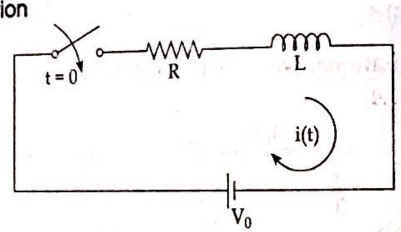


Figure 4.6: Series R-L circuit

Consider the series R-L circuit as shown in figure 4.6. Let the switch is closed at time t = 0. Now, applying KVL, we get,

$$V_0 = iR + L\frac{di}{dt}$$

Taking laplace transform

$$\frac{V_0}{s} = RI(s) + L\,[sI(s) - i(0^+)]$$

or, $$\frac{V_0}{s} = RI(s) + Ls\,I(s)$$

Since at $t = 0^+$, $i(0^+) = 0$ A

or, $$I(s)\,[R + Ls] = \frac{V_0}{s}$$

or, $$I(s) = \frac{V_0}{s(R + Ls)}$$

or, $$I(s) = \frac{\left(\frac{V_0}{L}\right)}{s\left(s + \frac{R}{L}\right)}$$

Let, $$I(s) = \frac{\left(\frac{V_0}{L}\right)}{s\left(s + \frac{R}{L}\right)} = \frac{A}{s} + \frac{B}{\left(s + \frac{R}{L}\right)} \qquad (2)$$

Solving for A and B, we get,

$$A = \frac{\left(\frac{V_0}{L}\right)}{s\left(s + \frac{R}{L}\right)} \times s\,\Bigg|_{s=0} = \frac{\left(\frac{V_0}{L}\right)}{s\left(s + \frac{R}{L}\right)}\Bigg|_{s=0}$$

∴ $$A = \frac{V_0}{R}$$

and, $$B = \frac{\left(\frac{V_0}{L}\right)}{s\left(s + \frac{R}{L}\right)} \times \left(s + \frac{R}{L}\right)\Bigg|_{s=-\frac{R}{L}} = \frac{\left(\frac{V_0}{L}\right)}{\left(\frac{-R}{L}\right)} = \frac{-V_0}{R}$$

Replacing value of A and B in equation (1), we get,

$$I(s) = \frac{V_0}{R}\left[\frac{1}{s} - \frac{1}{\left(s + \frac{R}{L}\right)}\right]$$

Taking inverse laplace transform,

∴ $$i(t) = \frac{V_0}{R}\left[1 - e^{-\frac{Rt}{L}}\right]$$

Here; $\frac{V_0}{R}$ = Steady state response

$\frac{V_0}{R}e^{-\frac{Rt}{L}}$ = Transient response

B. Sinusoidal Excitation

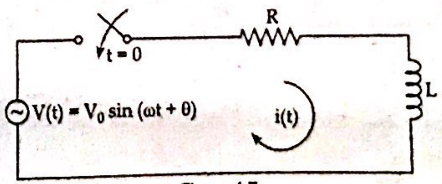


Figure 4.7:

Let us consider a series R-L circuit excited by sinusoidal driving voltage as shown in figure 4.7. For this type of excitation,

$$i(t) = -\frac{V_0}{\sqrt{R^2+\omega^2 L^2}} \sin\left(\theta - \tan^{-1}\frac{\omega L}{R}\right) e^{-\frac{Rt}{L}} + \frac{V_0}{\sqrt{R^2+\omega^2L^2}}$$

$$\sin\left(\omega t + \theta - \tan^{-1}\frac{\omega L}{R}\right) \text{ A}$$

C. **Exponential excitation**

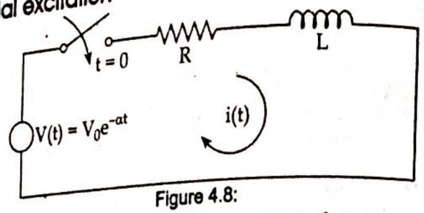


Figure 4.8:

Let us consider a series R-L circuit connected to exponential driving voltage as shown in figure 4.8. Let the switch is closed at time $t = 0^-$. Then the equivalent circuit at $t > 0$ will be,

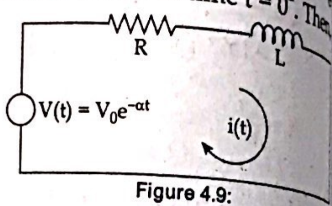


Figure 4.9:

Applying KVL, we get,

$$V_0 e^{-\alpha t} = R_i + L\frac{di}{dt}$$

Taking laplace transform on both sides,

$$\frac{V_0}{s+\alpha} = RI(s) + [sI(s) - i(0^+)]$$

or, $$\frac{V_0}{s+\alpha} = RI(s) + L[sI(s) - 0]$$

Here, $i(0^+) = 0$ because initially inductor was uncharged. So it has zero current and after switching instant it acts as open circuit.

or, $$\frac{V_0}{s+\alpha} = RI(s) + LsI(s)$$

or, $$\frac{V_0}{s+\alpha} I(s)\,[R + Ls]$$

or, $$I(s) = \frac{V_0}{(s+\alpha)(R+Ls)}$$

or, $$I(s) = \frac{V_0}{L(s+\alpha)\left(s+\frac{R}{L}\right)}$$

Let, $$I(s) = \frac{V_0}{(s+\alpha)\left(s+\frac{R}{L}\right)} = \frac{A}{s+\alpha} + \frac{B}{\left(s+\frac{R}{L}\right)} \qquad (1)$$

Case-I: If $\alpha \neq \frac{R}{L}$

$$A = \frac{V_0}{(s+\alpha)\left(s+\frac{R}{L}\right)} \times (s+\alpha)\Big|_{s=-\alpha} = \frac{V_0}{L\left(s+\frac{R}{L}\right)}\Bigg|_{s=-\alpha}$$

$$= \frac{V_0}{L\left(-\alpha+\frac{R}{L}\right)} = \frac{V_0}{L\left(\frac{-\alpha L + R}{L}\right)}$$

∴ $$A = \frac{V_0}{(R-\alpha L)}$$

Similarly,

$$B = \frac{V_0}{L(s+\alpha)\left(s+\frac{R}{L}\right)} \times \left(s+\frac{R}{L}\right)\Big|_{s=-\frac{R}{L}} = \frac{V_0}{L(s+\alpha)}\Big|_{s=-\frac{R}{L}}$$

$$= \frac{V_0}{L\left(\frac{-R}{L}+\alpha\right)} = \frac{V_0}{L\left(\frac{-R+\alpha L}{\alpha}\right)} = \frac{V_0}{(-R+\alpha L)} = \frac{-V_0}{R-\alpha L}$$

Replacing values of A and B in equation (1), we get,

$$I(s) = \frac{V_0}{(R-\alpha L)(s+\alpha)} - \frac{V_0}{(R-\alpha L)\left(s+\frac{R}{L}\right)}$$

$$V = \frac{V_0}{(R-\alpha L)}\left[\frac{1}{s+\alpha} - \frac{1}{s+\frac{R}{L}}\right]$$

Taking inverse laplace transform,

∴ $$i(t) = \frac{V_0}{(R-\alpha L)}(e^{-\alpha t} - e^{-\beta t}) \text{ A}$$

Case-II: If $\alpha = \frac{R}{L}$

$$I(s) = \frac{V_0}{L(s+\alpha)^2}$$

Taking inverse laplace transform,

∴ $$i(t) = \frac{V_0}{t} te^{-\alpha t} \text{ A}$$

4.9 RESPONSE OF R-C CIRCUIT WITH

A. DC excitation

Consider a series RC circuit as shown in figure 4.10. Let the switch is closed at time $t = 0$.

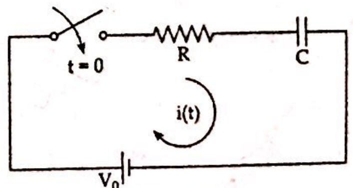


Figure 4.10:

Applying KVL, for $t > 0$,

$$V_0 = iR + \frac{1}{C}\int_{-\infty}^{t} i\,dt$$

or, $$V_0 = iR + \frac{1}{C}\int_{-\infty}^{t} i\,dt + \frac{1}{C}\int_{0}^{t} i\,dt$$

or, $V_0 = iR + 0 + \frac{1}{C}\int_0^t i\,dt$

or, $V_0 = iR + \frac{1}{C}\int_0^t i\,dt$

Taking laplace transform on both sides,

$$\frac{V_0}{s} = RI(s) + \frac{1}{C}\frac{I(s)}{s}$$

or, $\frac{V_0}{s} = I(s)\left(R + \frac{1}{Cs}\right)$

or, $I(s) = \dfrac{V_0}{Rs + \frac{1}{C}} = \dfrac{\left(\frac{V_0}{R}\right)}{s + \frac{1}{RC}}$

Taking inverse laplace transform,

$\therefore$ $i(t) = \frac{V_0}{R} e^{\frac{-t}{RC}}$ A

B. Sinusoidal excitation

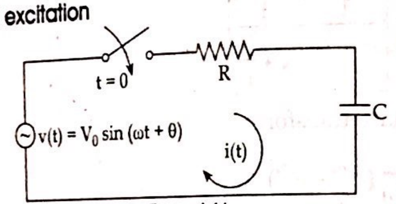


Figure 4.11:

Consider a series R-C circuit excited by sinusoidal driving voltage, V(t) as shown in figure 4.11.

For such type of response,

$$i(t) = \left[\frac{V_0}{R}\sin\theta - \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}\cos(\theta - \tan^{-1}\omega CR)e^{\frac{-C}{Rt}} + \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}\cos(\omega t + \theta - \tan^{-1}\omega RC)\right] A$$

C. Exponential excitation

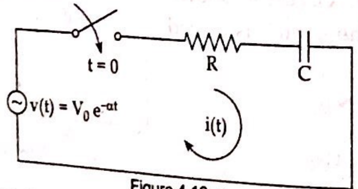


Figure 4.12:

Consider a series R-C circuit excited by an exponential driving force. Let switch is closed at time, t = 0.

Equivalent circuit at t > 0 is shown in figure 4.13.

Applying KVL,

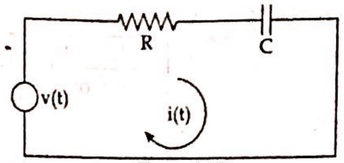


Figure 4.13:

or, $V_0 e^{-\alpha t} = Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt$

or, $V_0 e^{-\alpha t} = Ri + \frac{1}{C}\int_{-\infty}^{0} i\,dt + \frac{1}{C}\int_0^t i\,dt$

or, $V_0 e^{-\alpha t} = Ri + v_C(0^+) + \frac{1}{C}\int_0^t i\,dt$

or, $V_0 e^{-\alpha t} = Ri + 0 + \frac{1}{C}\int_0^t i\,dt$

or, $V_0 e^{-\alpha t} = Ri + \frac{1}{C}\int_0^t i\,dt$

By taking inverse laplace transform, we get,

or, $\frac{V_0}{s+\alpha} = R\,I(s) + \frac{1}{C}\frac{I(s)}{s}$

or, $\frac{V_0}{s+\alpha} = I(s)\left(R + \frac{1}{Cs}\right)$

or, $I(s) = \frac{V_0}{R}\left[\dfrac{1}{(s+\alpha)\left(s + \frac{1}{RC}\right)}\right]$ (1)

Case-I: If $\alpha \neq \frac{1}{RC}$

Let, $I(s) = \frac{V_0}{R}\left[\dfrac{1}{(s+\alpha)\left(s + \frac{1}{RC}\right)}\right] = \dfrac{A}{s+\alpha} + \dfrac{B}{\left(s + \frac{1}{RC}\right)}$ (2)

Solving for A and B,

$$A = \frac{V_0}{R} \times \frac{s}{\left(s + \frac{1}{RC}\right)}\Bigg|_{s=-\alpha} = \frac{V_0}{R} \times \frac{-\alpha}{\left(-\alpha + \frac{1}{RC}\right)} = \frac{V_0}{R}\frac{-\alpha}{\left(\frac{-\alpha RC}{RC} + 1\right)}$$

$\therefore$ $A = V_0\left(\frac{-\alpha C}{-\alpha RC + 1}\right) = V_0\left(\frac{\alpha C}{1 - \alpha RC}\right)$

Similarly,

$$B = \frac{V_0}{R}\frac{s}{(s+\alpha)\left(s + \frac{1}{RC}\right)} \times \left(s + \frac{1}{RC}\right)\Bigg|_{s=-\frac{1}{RC}}$$

$$= \frac{V_0}{R}\frac{-\frac{1}{RC}}{\left(\frac{-1}{RC} + \alpha\right)} = \frac{V_0}{R}\left(\frac{\frac{-1}{RC}}{\frac{-1 + R\alpha C}{RC}}\right)$$

$\therefore$ $B = \frac{V_0}{R}\left(\frac{-1}{RC} \times \frac{RC}{-1 + R\alpha C}\right) = \frac{V_0}{R}\left(\frac{1}{1 - \alpha RC}\right)$

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Replacing value of A and B in equation (2), we get,

$$I(s) = \frac{V_0}{R}\left[\frac{\alpha RC}{\left(\frac{\alpha RC-1}{s+\alpha}\right)} + \frac{\frac{1}{(1-\alpha RC)}}{\left(s+\frac{1}{RC}\right)}\right]$$

or, $$I(s) = \frac{V_0}{R(\alpha RC-1)} \times \left(\frac{\alpha RC}{s+\alpha} - \frac{1}{s+\frac{1}{RC}}\right)$$

Taking laplace transform,

$$i(t) = \frac{V_0}{R(\alpha RC-1)} = \left(\alpha RC\, e^{-\alpha t} - e^{\frac{-t}{RC}}\right) A$$

Case-II: If $\alpha = \frac{1}{RC}$

$$I(s) = \frac{V_0}{R} = \left[\frac{s}{(s+\alpha)\left(s+\frac{1}{RC}\right)}\right] = \frac{V_0}{R}\left[\frac{s}{\left(s+\frac{1}{RC}\right)}\right]^2$$

By taking inverse laplace,

$\therefore$ $$i(t) = \frac{V_0}{R} e^{\frac{-t}{RC}}\left(1 - \frac{1}{RC}t\right) A$$

4.10 RESPONSE OF SERIES R-L-C CIRCUIT WITH

A. DC excitation

Let us consider a series RLC circuit with DC excitation as shown in fig 4.14. Let the switch is closed at time, t = 0.

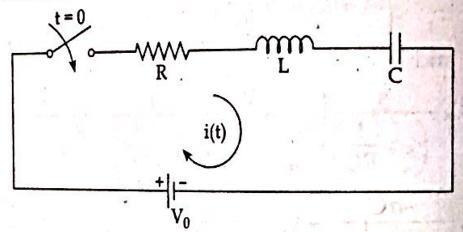


Figure 4.14:

Applying KVL to the circuit

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{0} i\, dt = V_0\, u(t)$$

or, $$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{0} i\, dt + \frac{1}{C}\int_{0}^{t} i\, dt = V_0\, u(t)$$

or, $$L\frac{di}{dt} + Ri + 0\frac{1}{C} i\, dt = V_0\, u(t)$$

Taking laplace transform on both sides,

or, $$L[s\,I(s) - i(0^+)] + RI(s) + 0 + \frac{1}{C}\frac{I(s)}{s} = \frac{V_0}{s}$$

or, $$Ls\,I(s) + RI(s) + \frac{ICs}{Cs} = \frac{V_0}{s}$$

or, $$I(s)\left(Ls + R + \frac{1}{Cs}\right) = \frac{V_0}{s}$$

or, $$I(s) = \frac{V_0}{Ls^2 + Rs + \frac{1}{C}}$$

$\therefore$ $$I(s) = \frac{V_0}{L(s-s_1)(s-s_2)}$$

where, s_1, s_2 are roots of the equations,

so, $$Ls^2 + Rs + \frac{1}{C} = 0$$

Thus, $$s_1, s_2 = \frac{-R}{2L} \pm \frac{1}{2L}\sqrt{R^2 - 4\frac{L}{C}}$$

Taking inverse laplace transform

$\therefore$ $$i(t) = \frac{\left(\frac{V_0}{L}\right)}{(s_1 - s_2)}\left(e^{s_1 t} - e^{s_2 t}\right) A$$

B. Sinusoidal excitation

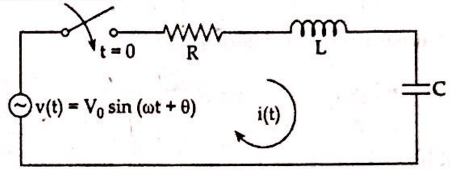


Figure 4.15:

Consider a series R-L-C circuit excited by the sinusoidal driving voltage as shown in figure 4.15. For this type of response,

$$i(t) = \left[\frac{V_0}{R}\sin\theta - \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}\cos(\theta - \tan^{-1}\omega CR)e^{\frac{-t}{Rt}} + \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}\cos(\omega t + \theta - \tan^{-1}\omega RC)\right] A$$

C. Exponential excitation

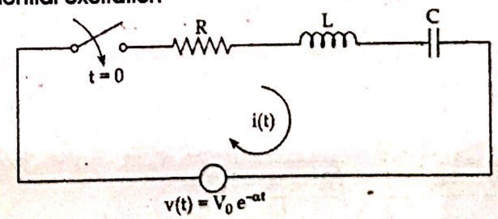


Figure 4.16:

It's equivalent circuit at time $t > 0$,

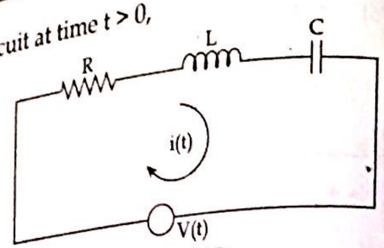


Figure 4.17:

Applying KVL,

$$V_0 e^{-\alpha t} = Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{0} i\,dt$$

or, $$V_0 e^{-\alpha t} = Ri + L\frac{di}{dt}\frac{1}{C}\int_{-\infty}^{0} i\,dt + \frac{1}{C}\int_0^t i\,dt$$

or, $$V_0 e^{-\alpha t} = Ri + L\frac{di}{dt} + V_C C_0 + \frac{1}{C}\int_0^t i\,dt$$

or, $$V_0 e^{-\alpha t} = Ri + L\frac{di}{dt} + 0 + \frac{1}{C}\int_0^t i\,dt$$

or, $$V_0 e^{-\alpha t} = Ri + L\frac{di}{dt} + 0 + \frac{1}{C}\int_0^t i\,dt$$

Taking laplace transform on both sides,

$$\frac{V_0}{s+\alpha} = RI(s) + L[s\,I(s) - i(0^+)] + \frac{1}{C}\frac{I(s)}{s}$$

or, $$\frac{V_0}{s+\alpha} = R\,I(s) + L\,[s\,I(s) - 0] + \frac{1}{C}\frac{I(s)}{s}$$

or, $$\frac{V_0}{s+\alpha} = R\,I(s) + Ls\,I(s) + \frac{1}{C}\frac{I(s)}{s}$$

or, $$\frac{V_0}{s+\alpha} = I(s)\left(R + Ls + \frac{1}{Cs}\right)$$

or, $$I(s) = \frac{V_0}{(s+\alpha)\left(Ls + R + \frac{1}{Cs}\right)} = \frac{V_0}{(s+\alpha)\left(Ls^2 + Rs + \frac{1}{C}\right)}$$

$$= \frac{V_0}{L}\left\{\frac{s}{(s+\alpha)\left(s^2 + \frac{R}{L}\right)s + \frac{1}{LC}}\right\} = \left(\frac{V_0}{L}\frac{s}{(s-s_1)(s-s_2)(s+\alpha)}\right)$$

where, s_1 and s_2 are roots of the quadratic equation,

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

Here, $$s_1 = s_2 = \frac{-\left(\frac{R}{L}\right)}{2} \pm \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}$$

Hence, $$I(s) = \frac{V_0}{L}\left[\frac{K_1}{s+\alpha} + \frac{K_2}{s-s_1} + \frac{K_3}{s-s_2}\right], \alpha \neq s_1 \neq s_2$$

Taking inverse laplace,

$\therefore$ $$i(t) = \frac{V_0}{L}(K_1 e^{-\alpha t} + K_2 e^{s_1 t} + K_3 e^{s_2 t})\text{ A}$$

4.11 RESPONSE OF PARALLEL R-L-C CIRCUIT WITH DC EXCITATION

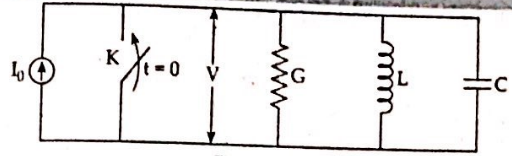


Figure 4.18:

Let us consider a parallel R-L-C circuit as shown in figure 4.18. Let the switch K is opened at time t = 0, thus connecting the DC current source I_0 to the circuit.

Now applying KCL,

$$C\frac{dV}{dt} + GV + \frac{1}{L}\int_{-\infty}^{0} v\,dt = I_0\,u(t) \qquad \left[\because \text{Conductance } G = \frac{1}{R}\right]$$

or, $$C\frac{dV}{dt} + GV + \frac{1}{L}\int_{-\infty}^{0} v\,dt + \frac{1}{L}\int_0^t v\,dt = I_0\,u(t)$$

Taking laplace transform on both sides,

$$C[s\,V(s) - v(0^+)] + G\,V(s) + \frac{1}{L}L\,\psi(0^+) + \frac{1}{L}\frac{V(s)}{s} = \frac{I_0}{s}$$

where, $\psi(0^+)$ = flux linkages and equals $Li(0^+)$

Using initial condition,

$$v(0^+) = 0\text{ V}$$
$$\psi(0) = 0$$

or, $$s\,V(s) + G\,V(s) + \frac{1}{Ls} = \frac{1}{s}I_0$$

or, $$V(s)\left[Cs + G + \frac{1}{Ls}\right] = \frac{I_0}{s}$$

Hence, $$V(s) = \frac{I_0}{s^2 + Gs + \frac{1}{L}} = \frac{I_0}{C\left(s^2 + \frac{Gs}{C} + \frac{1}{LC}\right)} = \frac{I_0}{C(s-s_1)(s-s_2)}$$

where, s_1 s_2 are the roots of the equations

$$\left(s^2 + Gs + \frac{1}{L}\right) = 0$$

Then, $$s_1, s_2 = \frac{-G}{2C} \pm \frac{1}{2C}\sqrt{G^2 - 4\frac{C}{L}}$$

General solution, $V(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$

The final form of the solution depends upon whether;

Case-I: $G^2 > \frac{4C}{L}$

In this case, quantity $\left(G^2 - \frac{4C}{L}\right)$ is positive. Hence,

$$V(t) = \frac{\left(\frac{I_0}{C}\right)}{s_1 - s_2}(e^{s_1 t} - e^{s_2 t})$$

Case-II: $G^2 = \frac{4C}{L}$

In this case, quantity $\left(G^2 - \frac{4C}{L}\right)$ is zero

$$V(t) = \frac{I_0}{C} t\, e^{st} = \frac{I_0}{C} t\, e^{\frac{-Gt}{2C}}$$

Case-III: $G^2 < \frac{4C}{L}$

In this case, quantity $\left(G^2 - \frac{4C}{L}\right)$ is negative. Hence the roots of characteristics equation are complex and are,

$$s_1 = A + jB \text{ and } s_2 = A - jB$$

where, $A = \frac{-G}{2C}$

$$B = \sqrt{\frac{\frac{4C}{L} - G^2}{2C}}$$

Thus, $V(t) = \frac{I_0 e^{\frac{-Gt}{2C}}}{\sqrt{\frac{C}{L} - \frac{G^2}{4}}} \sin\left[\sqrt{\frac{1}{LC} - \frac{G^2}{4C^2}}\right] t \;\; V$

4.12 LAPLACE TRANSFORM OF COMMON FORCING FUNCTIONS

I) Unit Step Function

The unit step function is defined as,

$$f(t) = \begin{cases} 1 & \text{for } t \ge 0 \\ 0 & \text{for } t < 0 \end{cases}$$

It is denoted by u(t)

$$\mathcal{E}[u(t)] = \int_0^\infty e^{-st}\, dt = -\left|\frac{1}{s}e^{-st}\right|_0^\infty = \frac{1}{s}$$

Hence, $\mathcal{E}[u(t)] = \frac{1}{s}$

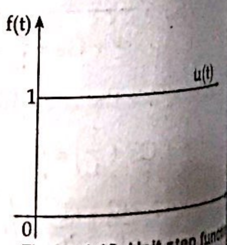


Figure 4.19: Unit step func

and, $\mathcal{E}[u(t-a)] = \int_0^\infty 1 \cdot e^{-st}\, dt = \left|\frac{e^{-st}}{s}\right|_0^\infty = e^{-as}\left(\frac{1}{s}\right)$

II) Ramp Function

If a time variant current or voltage increases linearly with time, it is known as a ramp function or a linear ramp.

A unit ramp function occurring at t = 0 denoted by r(t) equals

$$\int_0^t u(t)\, dt$$

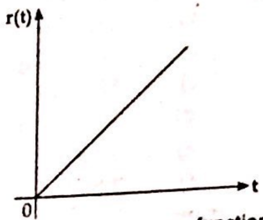


Figure 4.20: Unit ramp function

Hence, $\mathcal{E}[r(t)] = \mathcal{E}[\int_0^t u(t)\, dt] = \frac{1}{s}[\mathcal{E}(u(t))] = \frac{1}{s^2}$

III) Impulse Function

If a function has only one non-zero value and if this value is infinite, this function is known as an impulse function.

$$\mathcal{E}[\delta(t - t_1)] = \lim \frac{e^{-t_1 s} - e^{-t(t_1 + a)s}}{as}$$

where, $a = t_2 - t_1$

Hence, $t\, \delta(t - t_1) = e^{-t_1 s}$

Here, t_1 is instant at which the unit impulse appears.

When, $t_1 = 0$ *i.e.,* when unit impulse occurs at t = 0

$$\mathcal{E}(\delta t) = 1$$

so, $\mathcal{E}\delta(t) = \mathcal{E}\left[\frac{di}{dt}u(t)\right] = s\mathcal{E}[u(t)] = s \cdot \frac{1}{s}$

∴ $\mathcal{E}\delta(t) = 1$

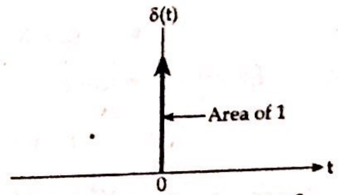


Figure 4.21: Ideal Impulse at t = 0

Hence, the laplace transform of impulse function is 1.

and, $\mathcal{E}\delta(t - t_1) = \mathcal{E}\left[\frac{d}{dt}u(t - t_1)\right] = s \cdot \mathcal{E}[u(t - t_1)] = S \cdot \frac{e^{-t_1 s}}{s} = e^{-t_1 s}$

The delta function is frequently called the unit impulse.

SOLVED NUMERICAL EXAMPLES

1. Find the laplace transform of the following standard signals.
   a) The unit step function v(t)
   b) The delayed step function Ku(t – a)
   c) The ramp function Kr(t) or Kt u(t)
   d) The delayed ramp function r(t – a)
   e) The unit impulse function δ(t)
   f) The unit doublet function δ'(t)

Solution:

a) $f(t) = u(t)$

$$F(s) = \int_0^\infty u(t) \cdot e^{-st}\, dt$$

By the definition of u(t), we have,

$$= \int_0^\infty e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$$

b) $f(t) = Ku(t-a)$

$$F(s) = \int_0^\infty Ku(t-a)\, e^{-st}\, dt$$

By the definition of u(t – a), we have,

$$= \int_0^\infty K e^{-st}\, dt = K\left.\frac{e^{-st}}{-s}\right|_0^\infty = K\frac{e^{-as}}{s}$$

c) $f(t) = Kr(t) = Kt\, u(t)$

$$F(s) = \int_0^\infty Kt\, u(t)\, e^{-st}\, dt = \int_0^\infty Kt\, e^{-st}\, dt$$

$$= K\left[t \cdot \left.\frac{e^{-st}}{-s}\right|_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-st}}{-s}\, dt\right]$$

$$= K[0-0] + \frac{K}{s}\int_0^\infty e^{-st}\, dt$$

$$= \left.\frac{Ke^{-st}}{-s}\right|_0^\infty = \frac{K}{s^2}$$

d) $f(t) = r(t-a) = (t-a)\, u(t-a)$.

$$F(s) = \int_0^\infty (t-a)\, u(t-a)\, e^{-st}\, dt$$

$$= \int_a^\infty (t-a)\, e^{-st}\, dt = (t-a)\left.\frac{e^{-st}}{-s}\right|_a^\infty - \int_a^\infty 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$= 0 - 0 + \frac{1}{s}\int_a^\infty e^{-st}\, dt$$

$$= \frac{1}{s} \cdot \left.\frac{e^{-st}}{-s}\right|_a^\infty = \frac{e^{-as}}{s^2}$$

e) $f(t) = \delta(t)$

$$F(s) = \int_0^\infty \delta(t)\, e^{-st}\, dt$$

By the definition of δ(t), we have,

$$F(s) = e^{-st}|_{t=0} = 1 \text{ (since } \delta(t) = 1 \text{ only at } t = 0)$$

f) $f(t) = \delta'(t)$

$$F(s) = s \cdot £[\delta(t)] = s$$

2. **Find the laplace transform of the following functions.**
   a) The exponential decay function Ke^{-at}.
   b) The sinusoidal function sin ωt.
   c) The cosine function cos ωt.
   d) e^{-at} sin ωt.
   e) e^{-at} t u(t).
   f) e^{-at} t u(t).
   g) sin hat.
   h) cosh at.

Solution:

a) $f(t) = K e^{-at}$

$$F(s) = \int_0^\infty K e^{-at} e^{-st}\, dt = K\int_0^\infty e^{-(s+a)t}\, dt$$

$$= K \cdot \left.\frac{e^{-(sa+a)t}}{-(s+a)}\right|_a^\infty = K\left[0 - \left\{\frac{-e^0}{s+a}\right\}\right] = \frac{K}{s+a}$$

b) $f(t) = \sin \omega t$

$$F(s) = \int_0^\infty \sin \omega t\, e^{-st}\, dt \qquad \left(\because \sin \omega t = \frac{e^{j\omega t} - e^{j\omega t}}{2j}\right)$$

$$= \int_0^\infty \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t}) \cdot e^{-st}\, dt$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{1}{2j}\left[\frac{s+j\omega - s + j\omega}{(s+j\omega)(s-j\omega)}\right]$$

$$= \frac{\omega}{s^2+\omega^2}$$

c) As similar to above case

$$f(t) = \cos \omega t, \text{ then, } F(s) = \frac{s}{s^2+\omega^2}$$

d) $f(t) = e^{-at} \sin \omega t$

$$F(s) = \frac{\omega}{(s+a)^2+\omega^2}$$

e) $f(t) = e^{-at} \cos \omega t$

$$F(s) = \frac{\omega}{(s+a)^2+\omega^2}$$

f) $f(t) = e^{-at}\, t\, u(t)$

$F(s) = \dfrac{1}{(s+a)^2}$

g) We know that,

$\sin hat = \dfrac{1}{2}(e^{at} - e^{-at})$

$$\pounds[\sin hat] = \frac{1}{2}\left[\int_0^\infty e^{at} \cdot e^{-st}\, dt - \int_0^\infty e^{-at} e^{-st}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2 - a^2}$$

h) As similar to above case,

$$\cosh at = \frac{1}{2}(e^{at} + e^{-at})$$

$$\pounds[\cosh at] = \frac{s}{s^2 - a^2}$$

3. **Find the laplace transform of the given functions.**

a) **$f(t) = \sin \omega t$.**
b) **$f(t) \cdot u(t) = \sin \omega t\, u(t)$.**
c) **$f(t - t_0) = \sin \omega(t - t_0)$.**
d) **$f(t - t_0)\, u(t) = \sin \omega\, (t - t_0) \cdot u(t)$.**
e) **$f(t)\, u(t - t_0) = \sin \omega t \cdot u(t - t_0)$.**
f) **$f(t - t_0)\, u(t - t_0) = \sin \omega\, (t - t_0) \cdot u(t - t_0)$.**

Solution:

a) $\pounds[\sin \omega t] = \dfrac{\omega}{s^2 + \omega^2}$

b) $\pounds[\sin \omega t \cdot u(t)] = \pounds[\sin \omega t) = \dfrac{\omega}{s^2 + \omega^2}$

Since the laplace transform is for $0 < t < \infty$, which is the same functions (i) and (ii).

c) $\pounds[\sin \omega(t - t_0)] = \pounds\, [\sin \omega t \cos \omega t_0 - \cos \omega t \sin \omega t_0]$

$$= \frac{\omega}{s^2 \omega^2} \cdot \cos \omega t_0 - \frac{s}{s^2 + \omega^2} \cdot \sin \omega t_0$$

$$= \frac{\omega \cos \omega t_0 - s \sin \omega t_0}{s^2 + \omega^2}$$

d) $\pounds[\sin \omega(t - t_0)\, u(t)] = \pounds \sin \omega(t - t_0)] = \dfrac{\omega \cos \omega t_0 - s \sin \omega t_0}{s^2 + \omega^2}$

e) $\pounds[\sin \omega t \cdot u(t - t_0)] = \displaystyle\int_0^\infty \sin \omega t \cdot u(t - t_0)\, e^{-st}\, dt$

$$= \int_0^\infty \sin \omega t \cdot e^{-st}\, dt$$

$$= \frac{1}{2j}\int_{t_0}^\infty [e^{(-s+j\omega)t} - e^{-(s+j\omega)t}]\, dt$$

$$= \frac{1}{2j}\left[0 - \frac{e^{(-s+j\omega)t_0}}{s - j\omega} - \frac{e^{-(s+j\omega)t_0}}{(s + j\omega)}\right]$$

$$= \frac{1}{2j} e^{-t_0 s}\left[\frac{e^{j\omega t_0}}{s - j\omega} - \frac{e^{-j\omega t_0}}{s + j\omega}\right]$$

$$= \frac{e^{-t_0 s}}{2j}\left[\frac{e^{j\omega t_0}(s + j\omega) - e^{-j\omega t_0}(s - j\omega)}{(s - j\omega)(s + j\omega)}\right]$$

$$= \frac{e^{-t_0 s}}{2j}\left[\frac{j\omega(e^{j\omega t_0} + e^{-j\omega t_0}) + s\,(e^{j\omega t_0} - e^{-j\omega t_0})}{s^2 + \omega^2}\right]$$

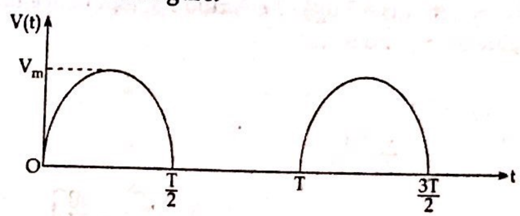
$$= e^{-t_0 s}\left[\frac{\omega \cos \omega\, t_0 + s \sin \omega t_0}{s^2 + \omega^2}\right]$$

f) $\pounds[\sin \omega\, (t - t_0) \cdot u(t - t_0)]$

Using property, shifting in time,

$$= e^{-t_0 s}\left(\frac{\omega}{s^2 + \omega^2}\right)$$

4. **Obtain the laplace transform of the veriodic, rectified half-sine wave as shown in figure.**



Solution:

We know, laplace transform of the single half-since wave is,

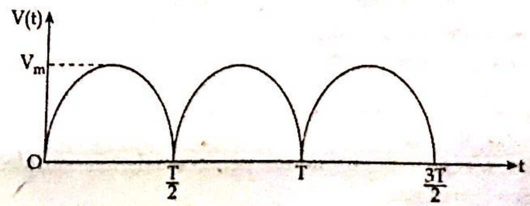
$$V_1(s) = \frac{V_m\, \omega}{s^2 + \omega^2}(1 + e^{\frac{-Ts}{2}})$$

Using theorem for periodic function

Then, $V(s) = \pounds\, V(t) = \left(\dfrac{1}{1 - e^{-Ts}}\right) \cdot V_1(s)$

$$= \left(\frac{1}{1 - e^{-Ts}}\right) \cdot V_m \cdot \frac{\omega}{s^2 + \omega^2}(1 + e^{\frac{-Ts}{2}}) = (1 + e^{\frac{-Ts}{2}}) \cdot V_m \cdot \frac{\omega}{(s^2 + \omega^2)}$$

5. **Determine the laplace transform of the periodic, rectified full-wave sine wave.**



Solution:

The laplace transform of the first cycle of the wave is given by,

$$V_1(s) = \frac{V_m \cdot \omega}{s^2+\omega^2}\left(1 + e^{\frac{-Ts}{2}}\right)$$

Therefore, the laplace transform of the periodic waveform V(t) of per is given by,

$$V(s) = \left(\frac{1}{1-e^{\frac{-Ts}{2}}}\right)\cdot\frac{V_m\omega}{s^2+\omega^2}\cdot\left(1+e^{\frac{-Ts}{2}}\right)$$

$$= \frac{V_m\cdot\omega}{s^2+\omega^2}\cdot\left[\frac{1+e^{\frac{-Ts}{2}}}{1-e^{\frac{-Ts}{2}}}\right]$$

$$= \frac{V_m\cdot\omega}{s^2+\omega^2}\cdot\left(\frac{e^{\frac{-Ts}{4}}+e^{\frac{-Ts}{4}}}{e^{\frac{-Ts}{4}}-e^{\frac{-Ts}{4}}}\right) = \frac{V_m\cdot\omega}{s^2+\omega^2}\cosh\frac{Ts}{4}$$

6. **For the current $i(t) = 5\,u(t) - 3\,e^{-2t}$, find I(s) and hence to determ the value of i(0) and i(∞).**

Solution:

$$i(t) = 5\,u(t) - 3\,e^{-2t}$$

$$I(s) = \frac{5}{s} - \frac{3}{s+2} = \frac{2s+10}{s(s+2)}$$

$$\text{Hence, } i(0^+) = \lim_{s\to\infty}[s\cdot I(s)] = \lim_{s\to\infty}\left[\frac{2s+10}{s+2}\right] = \lim_{s\to\infty}\left[\frac{2+\frac{10}{s}}{1+\frac{2}{s}}\right] = 2$$

$$\text{and, } i(\infty) = \lim_{s\to 0}[s\cdot I(s)] = \lim_{s\to 0}\left[\frac{2s+10}{s+2}\right] = 5$$

7. **Solve the differential equation $x'' + 3x' + 2x = 0,\ x(0^+) = 2,\ x'(0^+)$**

Solution:

Taking the laplace transform,

or, $s^2 X(s) - sx(0^+) - x'(0^+) + 3s\,X(s) - 3x(0^+) + 2\,X(s) = 0$

or, $(s^2+3s+2)\,X(s) = sx(0^+) + x'(0^+) + 3x(0^+)$

or, $(s^2+3s+2)\,X(s) = sx(0^+) + x'(0^+) + 3x(0^+)$

By putting the initial conditions,

or, $(s^2+3s+2)\,X(s) = 2s+3$

or, $X(s) = \dfrac{2s+3}{s^2+3s+2} = \dfrac{2s+3}{(s+1)(s+2)}$

Hence all the roots of denominator polynomials are simple. He partial fraction expansion,

$$X(s) = \frac{2s+3}{(s+1)(s+2)} = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

where, $K_1 = (s+1)\cdot X(s)|_{s=-1} = \left.\dfrac{2s+3}{s+2}\right|_{s=-1} = \dfrac{-2+3}{1+2} = 1$

and, $K_2 = (s+2)\cdot X(s)|_{s=-2} = \left.\dfrac{2s+3}{-s+1}\right|_{s=-1} = 1$

The result of the partial fraction expansion is thus,

$$X(s) = \frac{2s+3}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{1}{s+2}$$

Therefore, the solution of given differential expansion is,

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{1}{s+1} + \frac{1}{s+2}\right]$$

$\therefore\quad x(t) = e^{-t} + e^{-2t}$

8. **Find i(t), if $I(s) = \dfrac{1}{s(s+1)^2(s+2)}$.**

Solution:

$$I(s) = \frac{1}{s(s+1)^2(s+2)} = \frac{K_1}{s} + \frac{K_{21}}{s+1} + \frac{K_{22}}{(s+1)^2} + \frac{K_3}{s+2}$$

where, $K_1 = sI(s)|_{s=0} = \dfrac{1}{2}$

$K_{22} = (s+1)^2\, I(s)|_{s=-1} = -1$

$K_3 = (s+2)\, I_s|_{s=-2} = \dfrac{-1}{2}$

$K_{21} = \dfrac{d}{ds}[(s+1)^2 I(s)]|_{s=-2} = \left.\dfrac{(2s+2)}{s^2(s+2)^2}\right|_{s=-1} = 0$

The complete expansion is,

$$I(s) = \frac{1}{2s}\frac{-1}{(s+1)^2} - \frac{1}{2(s+2)}$$

Hence, $i(t) = \dfrac{1}{2} - te^{-t} - \dfrac{1}{2}e^{-2t}$ A

9. **If $I(s) = \dfrac{s^2+5s+9}{s^3+5s^2+12s+8}$; find i(t).**

Solution:

$$I(s) = \frac{s^2+5s+9}{s^3+5s^2+12s+8} = \frac{s^2+5s+9}{(s+1)(s^2+4s+8)}$$

or, $\dfrac{s^2+5s+9}{(s+1)[(s+2)^2-(j2)]} = \dfrac{[s^2+5s+9]}{(s+1)(s+2+j2)(s+2-j2)}$

or, $I(s) = \dfrac{K_1}{s+1} + \dfrac{K_2}{(s+2+j2)} + \dfrac{K_2^*}{(s+2-j2)}$

$$K_1 = (s+1)\,I(s)|_{s=-1} = \left.\frac{s^2+5s+9}{s^2+4s+8}\right|_{s=-1} = 1$$

$$K_2 = (s+2+j2)\,I(s)|_{-(2+j2)} = \frac{-1}{j4}$$

$$\therefore \quad K_2^{*} = \frac{1}{j4}$$

Therefore, the complete expansion is,

$$I(s) = \frac{1}{s+1} + \frac{\frac{-1}{j4}}{(s+2+j2)} + \frac{\frac{1}{j4}}{(s+2-j2)}$$

Hence, $i(t) = e^{-t} - \left(\frac{1}{j4}\right) e^{-(2+j2)t} + \left(\frac{1}{j4}\right) e^{-(2-j2)t}$

$$= e^{-t} - \frac{1}{j4} e^{-2t}\,[e^{-j2t} - e^{j2t}]$$

$$\therefore \quad i(t) = e^{-t} + \frac{1}{2} e^{-2t} \sin 2t \ \text{ A}$$

10. **Consider the differential equation $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 5u(t)$ where, u(t) is unit-step function. The initial conditions are $y(0^+) = -1$ and $\frac{dy}{dt}(0^+) = 2$. Determine y(t) for $t \geq 0$.**

Solution:

Taking laplace transform on both sides of given differential equation.

$$s^2 Y(s) - sy(0^+) - y'(0^+) + 3s\,Y(s) + 3y(0^+) + 2Y(s) = \frac{5}{s}$$

Substituting the values of initial conditions and solving for Y(s), we get,

$$s^2 Y(s) + s - 2 + 3s\,Y(s) + 3 + 2\,Y(s) = \frac{5}{s}$$

or, $Y(s) = \frac{-s^2 - s + 5}{s(s^2+3s+2)} = \frac{-s^2 - s + 5}{s(s+1)(s+2)}$

Explained by partial fraction expansion,

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

$$K_1 = s\,Y(s)|_{s=0} = \frac{5}{2}$$

$$K_2 = (s+1)\,Y(s)|_{s=-1} = \frac{-1+1+5}{(-1)(1)} = -5$$

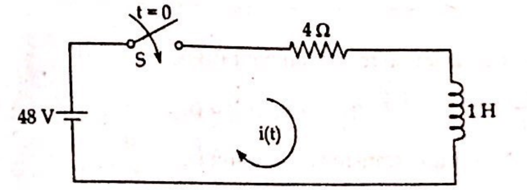
$$K_3 = (s+2)\,Y(s)|_{s=-2} = \frac{-4+2+5}{(-2)(-1)} = \frac{3}{2}$$

Hence, $Y(s) = \frac{5}{2s} - \frac{5}{s+1} + \frac{3}{2(s+1)}$

Taking the inverse laplace transform, we get the complete solution as,

$$y(t) = \frac{5}{2} - 5\,e^{-t} + \frac{3}{2} e^{-2t}\,;\ t \geq 0$$

11. **Consider the R-L circuit with R = 4 Ω, L = 1 H excited by a 48 V dc source as shown in figure. Assume the initial current through the inductor is 3 A. Using the laplace transformation, determine the current i(t); $t \geq 0$. Also draw the s-domain representation of the circuit.**



Solution:

Applying KVL,

$$Ri + L\frac{di(t)}{dt} = 48$$

Taking laplace transform

$$RI(s) + L[s\,I(s) - i_L(0^+)] = \frac{48}{s}$$

or, $4I(s) + L[s\,I(s) - 3] = \frac{48}{s}$

or, $I(s) = \frac{3s+48}{s(s+4)}$

Applying the partial fraction expansion, we get,

$$I(s) = \frac{3s+48}{s(s+4)} = \frac{K_1}{s} + \frac{K_2}{s+4}$$

where, $K_1 = s.\,I(s)|_{s=0} = \left.\frac{3s+48}{s+4}\right|_{s=0} = 12$

and, $K_2 = (s+4)\cdot I(s)|_{s=-4} = \left.\frac{3s+48}{s}\right|_{s=-4} = -9$

Then, $I(s) = \frac{12}{s} - \frac{9}{s+4}$

$$\therefore \quad i(t) = \mathcal{L}^{-1}[I(s)] = 12 - 9\,e^{-4t} \text{ A}$$

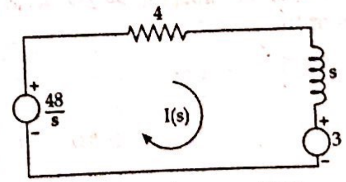
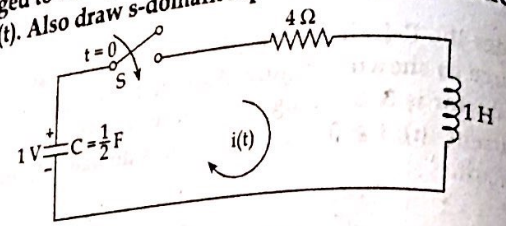


Figure: s-domain representation

12. **Consider a series R-L-C circuit with the capacitor initially charged to voltage of 1V as indicated in figure. Find the expression for i(t). Also draw s-domain representation of the circuit.**



Solution:

The differential equation for the current i(t) is,

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt + v_C(0^+) = 0$$

and the corresponding transform equation is,

$$L[s\,I(s) - i(0^+)] + R\,I(s) + \frac{1}{Cs}I(s) + \frac{v_C(0^+)}{s} = 0$$

Given, $C = \frac{1}{2}$ F, R = 2 Ω, L = 1H and $v_C(0^+) = -1$V,

Initial current $i(0^+) = 0$ because initially inductor behaves as a open circuit.
The transform equation I(s) then becomes,

$$sI(s) + 2\,I(s) + \frac{2}{s}I(s)\frac{-1}{s} = 0$$

or, $$I(s) = \frac{1}{2s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1}$$

∴ $$i(t) = \mathcal{L}^{-1}[I(s)] = e^{-t}\sin t \text{ A}$$

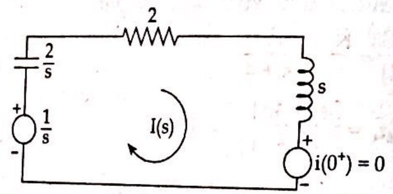
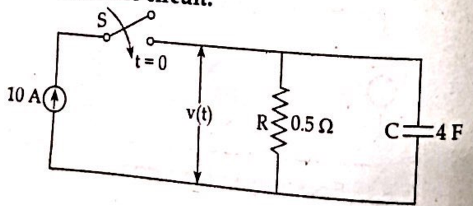


Figure: s-domain representation

13. **Consider the R-C parallel circuit with R = 0.5 Ω and C = 4 F excited by dc current source of 10 A. Determine the voltage across the capacitor by applying laplace transformation. Assume the initial voltage across the capacitor as 2 V. Also draw the s-domain representation of the circuit.**



Solution:

Applying KCL,

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} = 10$$

Taking laplace transform

$$\frac{V(s)}{R} + C[sV(s) - v(0^+)] = \frac{10}{s}$$

or, $$2\,V(s) + 4\,[s\,V(s) - 32] = \frac{10}{s}$$

∴ $$V(s) = \frac{8\,s + 10}{s(4s+2)}$$

Applying the partial fraction expansion,

$$V(s) = \frac{2s + 2.5}{s(s+0.5)} = \frac{K_1}{s} + \frac{K_2}{s+0.5}$$

$$K_1 = s\,V(s)|_{s=0} = 5$$

$$K_2 = (s + 0.5)\,V(s)|_{s=-0.5} = -3$$

∴ $$V(s) = \frac{5}{s} + \frac{-3}{s+0.5}$$

Hence, $v(t) = \mathcal{L}^{-1}[V(s)] = 5 - 3\,e^{-0.5t}$ V

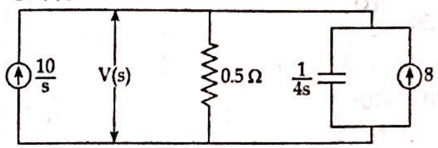
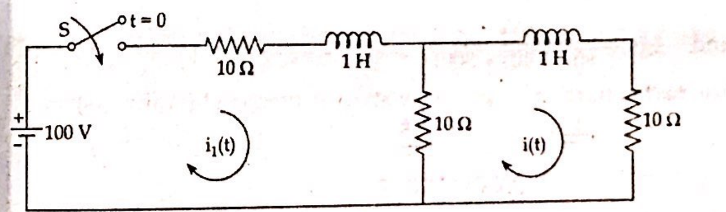


Figure: s-domain representation of the circuit

14. **In the network shown in figure, the switch S is closed at t = 0. With the network parameter values shown, find the expression for $i_1(t)$ and $i_2(t)$, if the network is unenergized before the switch is closed.**



Solution:

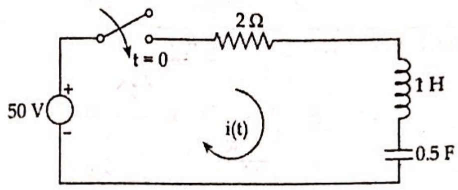
Applying KVL, loop equations are,

$$\frac{di_1(t)}{dt} + 10\,i_1(t) + 10\,[i_1(t) - i_2(t)] = 100$$

or, $$\frac{di_1(t)}{dt} + 20\,i_1(t) - 10\,i_2(t) = 100 \qquad (1)$$

∴ $i_2(t) = \frac{10}{3} - 5\,e^{-10t} + \frac{5}{3}e^{-30t}$ A

15. In the series R-L-C circuit shown. There is no initial charge on the capacitor. If the switch S is closed at t = 0, determine the resulting current.



Solution:

The time domain equation of the given circuit is,

$$2i(t) + \frac{1 \cdot di(t)}{dt} + \frac{1}{0.5}\int i(t)\,dt = 50$$

or, $$2i(t) + \frac{di(t)}{dt} + 2\int i(t)\,dt = 50 \qquad [\because v_C(0^-) = 0]$$

Taking laplace transform,

$$2I(s) + s\,I(s) - i(0^+) + 2\frac{I(s)}{s} = \frac{50}{5}$$

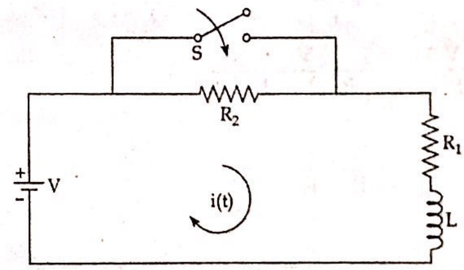
Because $i(0^+) = 0$, therefore,

$$I(s) = \frac{50}{s^2 + 2s + 2} = \frac{50}{(s+1)^2 + 1}$$

Hence, $i(t) = £^{-1}[I(s)]$

∴ $i(t) = 50\,e^{-t}\sin t$ A

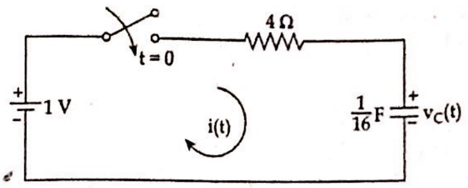
16. In the network shown in figure below, switch S is closed at t = 0, a steady state current having previously been attained. Solve for the current as a function of time using laplace transform.



Solution:

Before the switching action takes place,

$$v = i'(t)\,(R_1 + R_2) + L\frac{di'(t)}{dt}$$



Solution:

Applying KVL,

$$1 = R(i) + \frac{1}{C}\int_0^t i(t)\,dt + v_C(0^+)$$

Taking laplace transform and putting $R = 4$, $C = \frac{1}{16}$ and $v_C(0^+) = 9$ V, we have,

$$\frac{1}{s} = 4\,I(s) + \frac{16}{s}I(s) + \frac{9}{s}$$

or, $$\frac{-8}{s} = \left(4 + \frac{16}{s}\right) I(s)$$

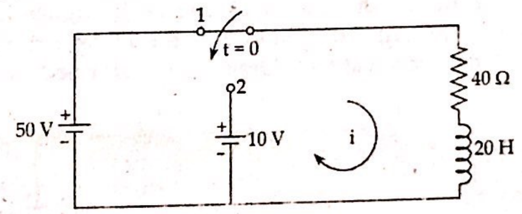
$\therefore$ $$I(s) = \frac{8}{4s + 16} = \frac{-2}{s + 4}$$

$\therefore$ $$i(t) = \mathcal{L}^{-1}[I(s)] = -2\,e^{-4t}$$

Hence, $v_C(t) = 16\int_0^t i(t)\,dt + 9 = 16\,(-2)\int_0^t e^{-4t}\,dt + 9 = 8\,[e^{-4t}]\,\big|_0^t + 9$

$\therefore$ $$v_C(t) = 1 + 8\,e^{-4t}\ \text{V}$$

18. The switch in figure below has been in position 1 for a long time; it is moved to 2 at t = 0. Obtain the expression for i, for t > 0 using laplace transform.



Solution:

With the switch on 1,

$$50 = 40\,i'(t) + 20\frac{di'(t)}{dt}$$

Taking laplace transform,

$$\frac{50}{s} = 40\,I'(s) + 20\,[s\,I'(s) - i'(0^+)]$$

Since $i'(0^+) = 0$,

$$I'(s) = \frac{50}{s(40 + 20s)} = \frac{2.5}{s(s + 2)}$$

Using partial fraction expansion,

$$I'(s) = \left(\frac{2.5}{2}\right)\frac{1}{s} + \left(\frac{2.5}{2}\right)\frac{1}{s+2} = 1.25\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

Hence, $i'(t) = \mathcal{L}^{-1}[I'(s)] = 1.25\,[1 - e^{-2t}]$

as $t \to \infty$, $i'(\infty) = 1.25$ A

When the switch on 2,

$$10 = 40\, i(t) + 20\frac{di(t)}{dt}$$

Taking laplace transform,

$$\frac{10}{s} = 40\, I(s) + 20\,[s\, I(s) - i(0^+)]$$

As $i(0^+) = i'(\infty) = 1.25$

Hence, $\frac{10}{s} = (40 + 20s)\, I(s) - 20 \times 1.25$

or, $I(s) = \dfrac{\left(\frac{10}{s} + 25\right)}{(40 + 20s)} = \dfrac{10 + 25s}{s(40 + 20s)}$

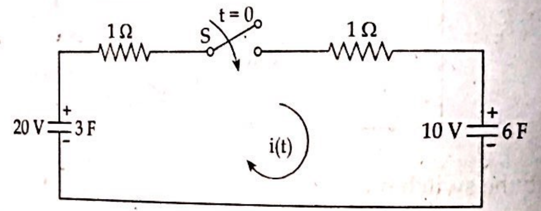
or, $I(s) = \dfrac{25(s + 0.4)}{20s(s + 2)} = 1.25\left[\dfrac{s + 0.4}{s(s + 2)}\right]$

or, $I(s) = 1.25\left[\dfrac{\left(\frac{0.4}{2}\right)}{s} + \dfrac{\left(\frac{-1.6}{-2}\right)}{s + 2}\right] = \dfrac{0.25}{s} + \dfrac{1}{s + 2}$

Hence, $(i) = \mathcal{L}^{-1}\, I(s)$

$\therefore$ $i(t)\,(0.25 + e^{-2t})$ A

19. Solve for i(t) in the circuit as shown in figure in which 3 F capacitor is initially charged to 20 V, the 6 F capacitor to 10 V, and the switch is closed at t = 0. Draw the transformed circuit.



Solution:

$$\frac{1}{3}\int_{-\infty}^{t} i(t)\,dt + 1\, i(t) + 1\, i(t) + \frac{1}{6}\int_{-\infty}^{t} i(t)\,dt = 0$$

or, $\frac{1}{3}\int_{0}^{t} i(t)\,dt - 20 + 2i\,(t) + \frac{1}{6}\int_{0}^{t} i(t)\,dt + 10 = 0$

Taking laplace transform,

$$\frac{1}{3s} I(s) \frac{-20}{s} + 2\, I(s) + \frac{1}{6s} I(s) + \frac{10}{s} = 0$$

or, $\left(\frac{1}{3s} + \frac{1}{6s}\right) I(s) + 2\, I(s) = \frac{10}{s}$

or, $\frac{1}{2s} I(s) + 2\, I(s) = \frac{10}{s}$

or, $I(s) = \frac{20}{4s + 1} = \frac{5}{s + 0.25}$

$\therefore$ $i(t) = 5\, e^{-0.25t}$ A

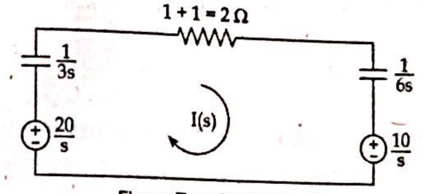
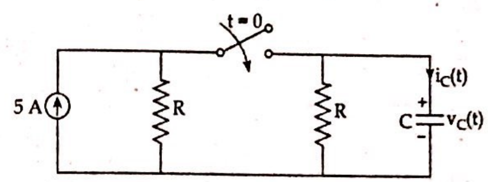


Figure: Transformed circuit

20. At t = 0, s is closed in the circuit, find $v_C(t)$ and $i_C(t)$. All initial conditions are zero.



Applying KCL,

$$I = \frac{v_C(t)}{R} + \frac{v_C(t)}{R} + C\frac{dv_C(t)}{dt}$$

Taking laplace transform,

$$\frac{I}{s} = \frac{V_C(s)}{R} + \frac{V_C(s)}{R} + C\,[s\, V_C(s) - v_C(0^+)]$$

As $v_C(0^+) = 0$,

$$V_C(s) = \frac{I}{Cs}\left[\frac{1}{s + \frac{2}{RC}}\right] = \frac{I}{C}\left[\frac{1}{s\left(s + \frac{2}{RC}\right)}\right]$$

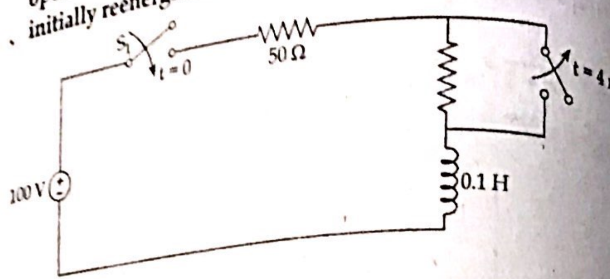
$$v_C(t) = \mathcal{L}^{-1}\,[V_C(s)] = \frac{I}{C}\mathcal{L}^{-1}\left[\frac{RC}{2}\frac{1}{s} - \frac{RC}{2}\cdot\frac{1}{s + \frac{2}{RC}}\right]$$

$$v_C(t) = \frac{IR}{2}\left(1 - e^{\frac{-2t}{RC}}\right)$$

and, $i_C(t) = C\dfrac{d}{dt}[v_C(t)] = C\dfrac{IR}{2}\left[0 - \left(\dfrac{-2}{RC}\right) e^{\frac{-2t}{RC}}\right]$

$\therefore$ $i_C(t) = I\, e^{\frac{-2t}{RC}}$ A

21. **In the circuit of figure shown, S_1 is closed at $t = 0$, and S_2 opened at $t = 4$ msec. Determine $i(t)$ for $t > 0$. Assume inductor initially reenergized.**



Solution:

For $0 \le t \le 4$ m sec; S_1 is closed and S_2 is closed loop equation becomes,

$$100 = 50\, i(t) + 0.1 \frac{di(t)}{dt}$$

Taking laplace transform,

$$\frac{100}{s} = 50\, I(s) + 0.1\, [sI(s) - i(0^+)]$$

Since, $i(0^+) = 0$ (given)

Then, $I(s) = \dfrac{100}{s(50 + 0.1s)} = \dfrac{1000}{s(s + 500)}$

Hence, $i(t) = \mathcal{L}^{-1}[I(s)] = \mathcal{L}^{-1}\left[\dfrac{2}{s} - \dfrac{2}{s + 500}\right] = 2(1 - e^{-500t})$ A

Thus, $i(4 \times 10^{-3}) = 1.729$ A

For 4 m sec $\le t \le \infty$: If $t' = t - 4 \times 10^{-3}$ then, $0 \le t' < \infty$

Loop equation becomes,

$$100 = 150\, i(t) + 0.1 \frac{di(t')}{dt}$$

Taking laplace transform,

$$\frac{100}{s} = 150\, I(s)\, 0.1\, [s\, I(s) - i(4 \times 10^{-3})]$$

or, $\dfrac{100}{s} = (150 + 0.1s)\, I(s) - 0.1 \times 1.729$

or, $I(s) = \dfrac{100 + 0.1729s}{s(0.1s + 150)} = \dfrac{1.729s + 1000}{s(s + 1500)}$

By partial fraction expansion,

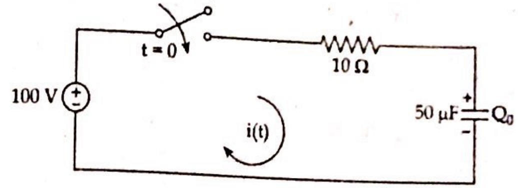
$$I(s) = \frac{0.667}{s} + \frac{1.062}{s + 1500}$$

Hence, $i(t') = 0.667 + 1.062 \times e^{-1500t'}$

or, $i(t) = 0.667 + 1.062 \times e^{-1500(t - 4 \times 10^{-3})} = 0.667 + 1.062\, e \cdot e^{-1500t}$

$\therefore$ $i(t) = 0.667 + 428.4\, e^{-1500t}$ A

22. **In the series R-C circuit, the capacitor has initial charge 2.5 mC. At $t = 0$, the switch is closed and a constant voltage source $V = 100$ V is applied, find $i(t)$.**



Solution:

Applying KVL,

$$100 = 10\, i(t) + \frac{1}{50 \times 10^{-6}} \int_0^t i(t)\, dt + v_C(0^+)$$

Taking laplace transform

with $v_C(0^+) = \dfrac{-Q_C}{C} = \dfrac{-2.5 \times 10^{-3}}{50 \times 10^{-6}} = -50$ V

or, $\dfrac{100}{s} = 10\, I(s) + \dfrac{1}{50 \times 10^{-6}} \dfrac{I(s)}{s} \dfrac{-50}{s}$

or, $I(s)\left[10 + \dfrac{1}{(50 \times 10^{-6})s}\right] = \dfrac{150}{s}$

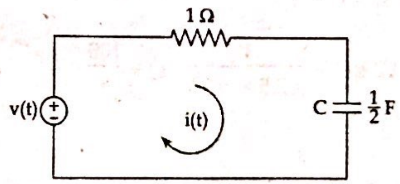
or, $I(s)\left[\dfrac{10s + (2 \times 10^4)}{s}\right] = \dfrac{150}{s}$

or, $I(s) = \dfrac{150}{10s + (2 \times 10^4)} = \dfrac{15}{s + (2 \times 10^3)}$

Hence, $i(t) = \mathcal{L}^{-1}[1(s)] = 15 \times e^{-2 \times 10^3 t}$

$\therefore$ $i(t) = 15\, e^{-2t}$ A

23. **Find the current $i(t)$ for the network shown in figure below, if the voltage source $v(t) = 2\, e^{-0.5t}\, u(t)$ and $v_C(0^+) = 0$.**



Solution:

Applying KVL,

$$2\, e^{-0.5t}\, u(t) = Ri(t) + \frac{1}{C}\int_{-\infty}^{t} i(t)\, dt$$

or, $2\, e^{-0.5t}\, u(t) = Ri(t) + \dfrac{1}{C}\displaystyle\int_0^t i(t)\, dt + v_C(0^+)$

Taking laplace transform, with $R = 1\,\Omega$, $C = \frac{1}{2}$ F and $v_C(0^+) = 0$

or, $\frac{2}{s+0.5} = I(s) + \frac{2}{s} I(s) + 0$

or, $I(s) = \frac{2s}{(s+2)(s+0.5)}$

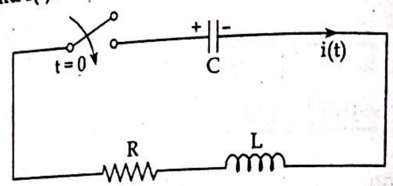
Using partial fraction expansion,

$$I(s) = \frac{\left(\frac{8}{3}\right)}{s+2} - \frac{\left(\frac{2}{3}\right)}{s+0.5}$$

Hence, $i(t) = \mathcal{L}^{-1}[I(s)] = \frac{8}{3}e^{-2t} - \frac{2}{3}e^{-0.5t}$

∴ $i(t) = \frac{2}{3}(4e^{-2t} - e^{-0.5t})\,u(t)$ A

24. In the given circuit, L = 2 H, R = 12 Ω C = 62.5 mF. The initial conditions are $v_C(0^+) = 100$ V and $i_L(0^+) = 1$ A. The switch is closed at t = 0. Find i(t).



Solution:

Applying KVL,

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(t)\,dt + v_C(0^+) = 0$$

Taking laplace transform,

$$L[s\,I(s) - i_L(0^+)] + R\,I(s) + \frac{1}{Cs}I(s) + \frac{V_C(0^+)}{s} = 0$$

or, $2[sI(s) - 1] + 12\,I(s) + \frac{1}{(62.5\times10^{-3})}I(s) + \frac{100}{s} = 0$

or, $I(s)\left[2s + 12 + \frac{16}{s}\right] = \frac{-100}{s} + 2$

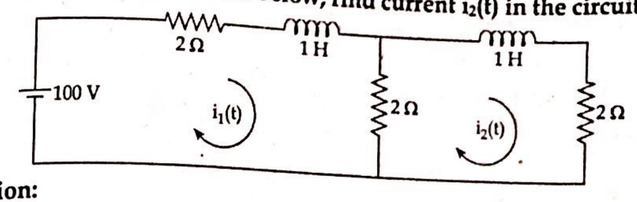
or, $I(s) = \frac{-100+2s}{2s^2+12s+16} = \frac{s-50}{s^2+6s+8} = \frac{s-50}{(s+4)(s+2)}$

Using partial fraction expansion,

$$I(s) = \frac{27}{s+4} - \frac{26}{s+2}$$

∴ $i(t) = \mathcal{L}^{-1}[I(s)] = [27e^{-4t} - 26e^{-2t}]\,u(t)$

25. In the network shown below, find current $i_2(t)$ in the circuits.



Solution:

Applying KVL, on loop 1

$$100 = 2\,i_1(t) + \frac{1\cdot di(t)}{dt} + 2\,[i_1(t) - i_2(t)]$$

Taking laplace transform with $i_1(0^+) = 0$

$$\frac{100}{s} = 2\,I_1(s) + s\,I_1(s) + 2[I_1(s) - I_2(s)]$$

or, $\frac{100}{s} = (4+s)\,I_1(s) - 2\,I_2(s)$ (1)

Applying KVL on loop 2,

$$2[i_2(t) - i_1(t)] + 1\frac{di_2(t)}{dt} + 2\,i_2(t) = 0$$

Taking laplace transform, with $i_2(0^+) = 0$

$$2[I_2(s) - I_1(s)] + s\,I_2(s) + 2\,I_2(s) = 0$$

$$(4+s)\,I_2(s) = 2\,I_1(s)$$

From equation (2), putting the value of $I_1(s)$ in equation (1), we have,

$$\frac{100}{s} = \left[(4+s)\cdot\frac{(4+s)}{2} - 2\right]I_2(s)$$

$$I_2(s) = \frac{200}{s(s^2+8s+12)} = \frac{200}{s(s+2)(s+6)}$$

Using partial fraction expansion,

$$I_2(s) = \frac{\left(\frac{50}{3}\right)}{s} - \frac{25}{s+2} + \frac{\left(\frac{25}{3}\right)}{s+6}$$

Therefore, $i_2(t) = \mathcal{L}^{-1}[I_2(s)] = \left[\frac{50}{3} - 25e^{-2t} + \frac{25}{3}e^{-6t}\right]u(t)$ A

26. Find the inverse laplace transform of $X(s) = \frac{2s+4}{s^2+4s+3}$.

Solution:

$$X(s) = \frac{2s+4}{s^2+4s+3} = \frac{2(s+2)}{(s+1)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where, $K_1 = (s+1)\,X(s)|_{s=-1} = \left.\frac{2(s+2)}{(s+3)}\right|_{s=-1} = 1$

$$K_2 = (s+3)\,X(s)\big|_{s=-3} = \frac{2(s+2)}{(s+1)}\bigg|_{s=-3}$$

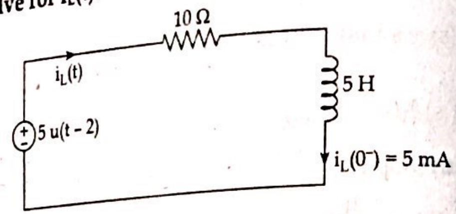
Hence, $X(s) = \frac{1}{s+1} + \frac{1}{s+3}$

We know,

$$\mathcal{L}\{e^{-at}\,u(t)\} = \frac{1}{s+a}$$

$\therefore$ $x(t) = [e^{-t} + e^{-3t}]\,u(t)$

27. **Referring to the RL circuit, write a differential equation for the inductor current $i_L(t)$ and find $I_L(s)$, the laplace transform of i_L and solve for $i_L(t)$.**



Solution:

Applying KVL clockwise, we get,

$$10\,i_L(t) + 5\frac{di_L(t)}{dt} - 5\,u(t-2) = 0$$

Taking laplace transform, we get,

$$10\,I_L(s) + 5\,[s\,I_L(s) - i_L(0^-)] = \frac{5}{s}e^{-2s}$$

or, $$I_L(s) = \frac{\frac{5}{s}e^{-2s} + 5\,i_L(0^-)}{5s+10} = \frac{e^{-2s} + 5\times10^{-3}s}{s(s+2)}$$

$$= e^{-2s}\left[\frac{K_1}{s} + \frac{K_2}{s+2}\right] + \frac{5\times10^{-3}}{s(s+2)}$$

where, $K_1 = \frac{1}{s+2}\bigg|_{s=0} = \frac{-1}{2}$

$$K_2 = \frac{1}{s}\bigg|_{s=-2} = \frac{-1}{2}$$

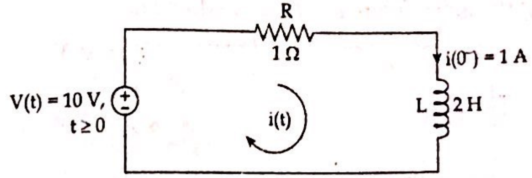
Hence, $I_L = \frac{1}{2}e^{-2s}\left[\frac{1}{s} - \frac{1}{s+2}\right] + \frac{5\times10^{-3}}{(s+2)}$

Taking inverse laplace transform,

$\therefore$ $$i_L(t) = \frac{1}{2}[u(t) - e^{-2t}\,u(t)]_{t\to t-2} + 5\times10^{-3}\,e^{-2t}\,u(t)$$

$$= \frac{1}{2}[u(t-2) - e^{-2t}\,u(t-2)] + 5\times10^{-3}\,e^{-2t}\,u(t)\ \text{A}$$

28. **Find i(0) and i(∞) using initial and final value theorems.**



Solution:

Applying KVL, we get,

or, $$i + 2\frac{di}{dt} = 10$$

Taking laplace transform,

$$I(s) + 2[s\,I(s) - i(0^-)] = \frac{10}{s}$$

or, $$I(s) + 2\,[s\,I(s) - 1] = \frac{10}{s}.$$

or, $$I(s)\,[1 + 2s] = \frac{10}{s} + 2$$

or, $$I(s) = \frac{10}{s(1+2s)} + \frac{2}{1+2s} = \frac{10+2s}{s(1+2s)} = \frac{5+s}{s\left(s+\frac{1}{2}\right)}$$

According to initial value theorem,

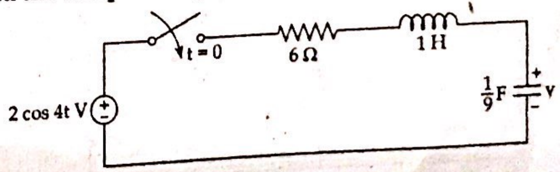
$$i(0) = \lim_{s\to\infty} s\,I(s) = \lim_{s\to\infty} s\cdot\frac{5+s}{s\left(s+\frac{1}{2}\right)} = \lim_{s\to 0} = \frac{\left(1+\frac{5}{s}\right)}{\left(1+\frac{1}{2s}\right)} = 1$$

We know from fundamentals for an inductor, $i(0^+) = i(0^-) = i(0)$. Hence, i(0) found using initial value theorem verifies the initial value of i(t) given in the problem.

From final value theorem,

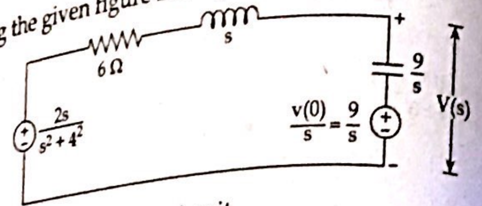
$$i(\infty) = \lim_{s\to\infty} sI(s) = \lim_{s\to 0} \frac{s(s+5)}{s\left(s+\frac{1}{2}\right)} = \frac{5}{\left(\frac{1}{2}\right)} = 10\ \text{A}$$

29. **Find the complete response for v(t) if $t \geq 0^+$. Take $v(0) = 2$ V.**



Solution:

Representing the given figure in frequency domain,



Applying KVL clockwise to the circuit,

$$\frac{-2s}{s^2+16}\left(6+s+\frac{9}{s}\right)I(s)+\frac{2}{s}=0$$

or, $$I(s)=\frac{-32}{(s^2+6s+9)(s^2+16)}$$

Hence, $$V(s)=I(s)\left[\frac{9}{s}\right]+\frac{2}{s}=\frac{2}{s}+\frac{-288}{s(s+3)^2(s^2+16)}$$

Using partial fraction, we get,

$$V(s)=\frac{2}{s}+\left[\frac{K_1}{s}+\frac{K_2}{s+3}+\frac{K_3}{(s+3)^2}+\frac{K_4}{s-j4}+\frac{K_4^*}{s+j4}\right]$$

Solving for K_1, K_2, K_3 and K_4,

$$K_1=\left.\frac{-288}{(s+3)^2(s^2+16)}\right|_{s=0}=-2$$

$$K_2=\frac{d}{ds}\left[\frac{-288}{s(s^2+16)}\right]\Bigg|_{s=-3}=-2.2$$

$$K_3=\left.\frac{-288}{s(s^2+16)}\right|_{s=-3}$$

$$K_4=\left.\frac{-288}{s(s+3)(s+j4)}\right|_{s=j4}=0.36\angle-106.2°$$

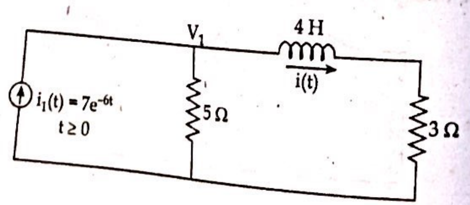
Hence,

$$V(s)=\frac{2}{s}-\frac{2}{s}+\frac{2.2}{s+3}+\frac{3.84}{(s+3)^2}+\frac{0.36\angle-106.2}{s-j4}+\frac{0.36\angle106.2}{s+j4}$$

Taking inverse laplace transform,

$$v(t)=2.2\,e^{-3t}+3.84\,t\,e^{-3t}+0.72\cos(4t-106.2°)\text{ V}$$

30. **Find i(t) for the circuit when $i_1(t)=7\,e^{-6t}$ A for $t\geq 0$ and i(0) = 0. Also find $i(\infty)$.**



Solution:

KCL at node V_1

$$\frac{V_1}{5}+i=7\,e^{-6t}$$

Also, $$V_1=3i+4\frac{di}{dt}$$

Hence, $$\frac{1}{5}\left[3i+4\frac{di}{dt}\right]+i=7\,e^{-6t}$$

or, $$\frac{4}{5}\frac{di}{dt}+\frac{8}{5}i=7\,e^{-6t}$$

or, $$\frac{di}{dt}+2i=\frac{35}{4}e^{-6t}$$

Taking laplace transform of the differential equation, we get

$$[s\,I(s)-i(0)]+2I(s)=\frac{35}{4}\left(\frac{1}{s+6}\right)$$

or, $$I(s)=\frac{35}{4}\left[\frac{1}{(s+2)(s+6)}\right]$$

Using partial fraction expansion, we get,

$$I(s)=\frac{K_1}{s+2}+\frac{K_2}{s+6}$$

∴ $$K_1=\frac{35}{16}\text{ and }K_2=\frac{-35}{16}$$

Hence,

$$I(s)=\frac{35}{16}\left[\frac{1}{s+2}\right]-\frac{35}{16}\left[\frac{1}{s+2}\right]$$

∴ $$i(t)=\frac{35}{16}[e^{-2t}-e^{-6t}]\,u(t)$$

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BOARD EXAMINATION SOLVED QUESTIONS

1. **The series circuit of RLC network consists, of R = 5 Ω, L = 1 H and C = 0.25 F with supply voltage of 10 V. Find i(t) using laplace transform. Assume zero initial current through inductor and zero initial voltage across capacitor. [2012/Fall, 2014/Fall, 2019/Spring]**

Solution:

Given that;

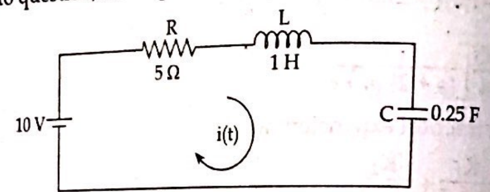
$R = 5\,\Omega$, $L = 1\,H$, $C = 0.25\,F$

Supply voltage, $v = 10\,V$

Initial condition; $i(0^-) = 0\,A$

and, $\int_{-\infty}^{0} i(t)\,dt = v_C(0^-) = 0\,V$

According to question, the required figure is,



For t > 0, applying KVL,

$$R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{\infty} i(t)\,dt = v$$

or, $5\,i(t) + 1\dfrac{di(t)}{dt} + \dfrac{1}{0.25}\int_{-\infty}^{t} i(t)\,dt = 10$

Taking laplace transform, we get,

$$5\,I(s) + [sI(s) - i(0)] + 4\frac{I(s)}{s} + 4\int_{-\infty}^{0} i(t)\,dt = \frac{10}{s}$$

Applying initial condition,

or, $5\,I(s) + [sI(s) - 0] + 4\dfrac{I(s)}{s} + 4 \times 0 = \dfrac{10}{s}$

or, $5\,I(s) + sI(s) + 4\dfrac{I(s)}{s} = \dfrac{10}{s}$

or, $(s^2 + 5s + 4)\,I(s) = 10$

or, $I(s) = \dfrac{10}{(s^2 + 5s + 4)} = \dfrac{10}{(s+1)(s+4)}$

Using partial fraction expansion,

$$I(s) = \frac{10}{(s+1)(s+4)} = \frac{A}{(s+1)} + \frac{B}{(s+4)}$$

or, $10 = A(s+4) + B(-4+1)$

Put s = –4,

so, $10 = A(-4+4) + B(-4+1)$

∴ $B = -3.3$

Put s = –1,

$10 = A(-1+4) + B(-1+1)$

∴ $A = 3.3$

Hence, $I(s) = \dfrac{3.3}{(s+1)} - \dfrac{3.3}{(s+4)}$

Taking inverse laplace transform,

∴ $i(t) = 3.3\,e^{-t} - 3.3\,e^{-4t} = 3.3\,(e^{-t} - e^{-4t})\,A$

2. **A voltage source of $v = 8\,e^{-t}$ is supplied to series RC circuits. Find the expression for i(t). Given that R = 2 ohm and C = 0.2 F when switch K is closed at t = 0. Assume zero charge in capacitor initially using Laplace transform method. [2019/Fall]**

Solution:

Given that;

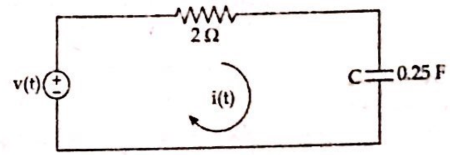
$v(t) = 8\,e^{-t}\,V$

$R = 2\,\Omega$

$C = 0.25\,F$

$v_C(0) = 0\,V$

The equivalent circuit is,



Applying KVL, we get,

$$R\,i(t) + \frac{1}{C}\int_{-\infty}^{\infty} i(t)\,dt = v(t)$$

or, $2\,i(t) + \dfrac{1}{0.25}\int_{-\infty}^{\infty} i(t)\,dt = 8\,e^{-t}$

or, $2\,i(t) + 4\int_{-\infty}^{t} i(t)\,dt + v_C(0^-) = 8\,e^{-t}$

Applying initial condition, $v_C(0^-) = 0\,V$

or, $2\,i(t) + 4\int_{0}^{t} i(t)\,dt + 0 = 8\,e^{-t}$

Taking laplace transform, we get,

or, $2\,I(s) + 4\dfrac{I(s)}{s} = \dfrac{8}{(s+1)}$

or, $(s+2)\,I(s) = \dfrac{4s}{(s+1)}$

or, $I(s) = \dfrac{4s}{(s+1)(s+2)}$

Let, $\dfrac{4s}{(s+1)(s+2)} = \dfrac{A}{(s+1)} + \dfrac{B}{(s+2)}$

or, $4s = A(s+2) + B(s+1)$

Put $s = -1$,

$4 \times (-1) = A(-1+2) + B(-1+1)$

$\therefore \quad A = -4$

Put $s = -2$,

$4 \times (-2) = A(-2+2) + B(-2+1)$

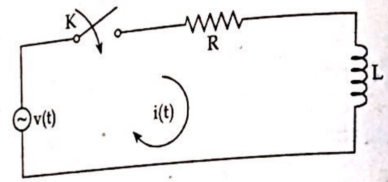
$\therefore \quad B = 8$

so, $I(s) = \frac{-4}{s+1} + \frac{8}{s+2}$

Taking inverse laplace transform, we get,

$i(t) = -4\,e^{-t} + 8\,e^{-2t}$ A

3. **An R-L series circuit is connected to an AC voltage v(t) = 80 (400t + 20), R = 10 Ω and L = 0.05 H. At t = 0, the switch K closed. Find the equation for the current using laplace transfo** **[2018/F**



Solution:

Given that;

$v(t) = 80\cos(400t + 20)$ V

$R = 10\ \Omega$

$L = 0.05$ H

For $t > 0$, applying KVL,

$$R\,i(t) + L\frac{di(t)}{dt} = v(t)$$

or, $$10\,i(t) + 0.05\frac{di(t)}{dt} = 80\cos(400t + 20)$$

Taking laplace transform, we get,

$$10\,I(s) + 0.05\,[sI(s) - i(0)] = 80\left[\frac{\cos 20 - 400\sin 20}{s^2 + (400)^2}\right]$$

From figure, $i(0) = 0$ A, so,

$$10\,I(s) + 0.05\,sI(s) = 80\left[\frac{0.94\,s - 136.81}{s^2 + (400)^2}\right]$$

or, $$(s + 200)\,I(s) = 1600\frac{(0.94\,s - 136.81)}{s^2 + (400)^2}$$

or, $$I(s) = \frac{1702.13\,(s - 145.54)}{(s + 200)\,(s^2 + 1{,}60{,}000)}$$

Let, $$\frac{1702.13\,(s - 145.54)}{(s + 200)\,(s^2 + 1{,}60{,}000)} = \frac{A}{s + 200} + \frac{Bs + C}{s^2 + 1{,}60{,}000}$$

or, $1702.13\,(s - 145.54) = A\,(s^2 + 1{,}60{,}000) + (Bs + C)\,(s + 200)$

Put $s = -200$,

or, $1702.13 = (-200 - 145.54)$

$= A[(-200)^2 + 1{,}60{,}000) + B\,(-200 + C)\,(-200 + 200)]$

$\therefore \quad A = \frac{-588154}{2{,}00{,}000} = -2.94$

Equating coefficient of s^2,

$0 = A + B$

$\therefore \quad B = 2.94$

Equating coefficient of constants,

$-1702.13 \times 145.54 = 1{,}60{,}000\,A + 200\,C$

or, $-2.47728 = 1{,}60{,}000 \times (-2.94) + 200\,C$

$\therefore \quad C = 1{,}113.36$

Hence,

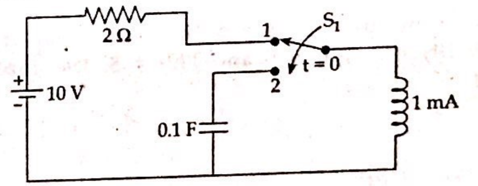
$$I(s) = \frac{-2.94}{s + 200} + \frac{2.94s + 1113.36}{s^2 + 160000}$$

$$= \frac{-2.94}{s + 200} + 2.94\frac{s}{s^2 + (400)^2} + \frac{1113.36}{s^2 + (400)^2} \times \frac{1113.36}{400}$$

Taking inverse laplace, we get,

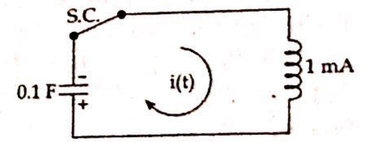
$i(t) = (-2.94\,e^{-200t} + 2.94\cos 400t + 2.78\sin 400t)$ A

4. **The voltage is applied across the RL circuit when switch S_1 is in position 1 and steady state condition is reached. Later on, at time t = 0, switch K is moved from position 1 to 2. Using laplace transformation, obtain the expression for the current i(t) in the LC circuit and solve for i(t).** **[2017/Spring]**



Solution:

For $t > 0$, the equivalent circuit is,



Applying KVL, we get,

or, $$L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{\infty} i(t)\,dt = 0$$

or, $$0.1 \times 10^{-3}\frac{di(t)}{dt} + \frac{1}{0.1}\int_{-\infty}^{\infty} i(t)\,dt = 0$$

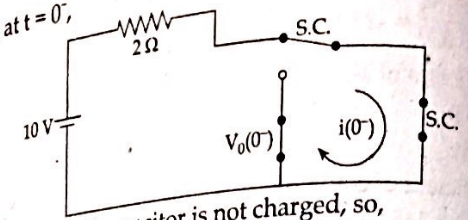
or, $$10^{-3}\frac{di(t)}{dt} + 10\int_{0}^{\infty} i(t)\,dt + v_0 = 0$$

Here, v_0 is the initial voltage on capacitor

Taking laplace transform, we get,

$$10^{-3}[sI(s) - i(0)] + 10\frac{I(s)}{s} + \frac{V_0}{s} = 0$$

From circuit at $t = 0^-$,



Inductor is charged and capacitor is not charged, so,

$$v_0(0^-) = 0\ V$$
$$i(0^-) = \frac{10}{2} = 5\ A$$

Then, $v(0^+) = 0\ V$

$$i(0^+) = i(0^-) = 5\ A$$

Hence, $10^{-3}[sI(s) - 5] + 10\frac{I(s)}{s} + 0 = 0$

or, $sI(s) - 5 + 10^4\frac{I(s)}{s} = 0$

or, $(s^2 + 10^4)\,I(s) = 5s$

or, $I(s) = 5 \times \frac{s}{s^2 + 100^2}$

Taking inverse laplace transform, we get,

$\therefore$ $i(t) = 5\cos(100\,t)$ A

5. Determine the current i(t) from the differential equation $i''(t) + 5 i'(t) + 6\, i(t) = 0$, $i(0) = 0$ and $i'(0) = 5$. Use laplace transform method. **[2017/Fa...]**

Solution:

Given that;

$$i''(t) + 5\,i'(t) + 6\,i(t) = 0$$
$$i(0) = 0\ A \text{ and } i'(0) = 5\ A$$

Taking laplace transform of equation (1),

$$[s^2 I(s) - s\,i(0) - i'(0) + 5\,[s\,I(s) - i(0)] + 6\,I(s) = 0$$

Applying initial conditions,

or, $[s^2 I(s) - 0 - 5] + 5\,(sI(s) - 0) + 6\,I(s) = 0$

or, $[s^2 + 5s + 6)\,I(s) = 5$

or, $I(s) = \frac{5}{s^2 + 5s + 6} = \frac{5}{(s+2)(s+3)}$

Let, $\frac{5}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)}$

or, $5 = A(s + 3) + B(s + 2)$

Put $s = -2$,

$$5 = A(-2 + 3) + B(-2 + 2)$$

$\therefore$ $A = 5$

Put $s = -3$,

$$5 = A(-3 + 3) + B(-3 + 2)$$

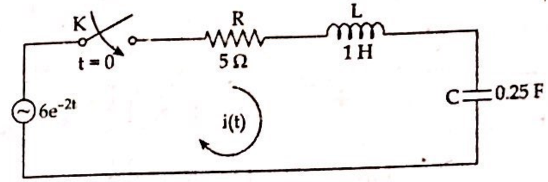
$\therefore$ $B = -5$

Hence, $I(s) = \frac{5}{(s+2)} + \frac{(-5)}{(s+3)}$

Taking inverse laplace transform, we get,

$\therefore$ $i(t) = 5\,e^{-2t} - 5\,e^{-3t} = 5\,(e^{-2t} - e^{-3t})$ A

6. Switch K is closed at t = 0. Find the time domain circuit current using laplace transform for the following circuit. **[2011/Fall, 2014/Spring]**

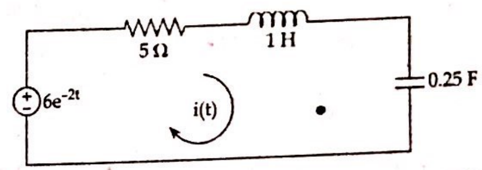


Solution:

Given that;

$$v(t) = 6\,e^{-2t}\ V$$
$$R = 5\ \Omega$$
$$L = 1\ H$$
$$C = 0.25\ F$$
$$i(0^-) = 0\ A$$
$$v_C(0^-) = 0 \text{ volt}$$

Equivalent circuit at $t > 0$ is,



Applying KVL, we get,

$$v(t) = v_R + v_L + v_C$$

or, $6e^{-2t} = i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$

or, $6e^{-2t} = 5\,i(t) + 1\frac{di(t)}{dt} + \frac{1}{0.25}\int_{-\infty}^{0} i(t)\,dt + \frac{1}{0.25}\int_0^t i(t)\,dt$

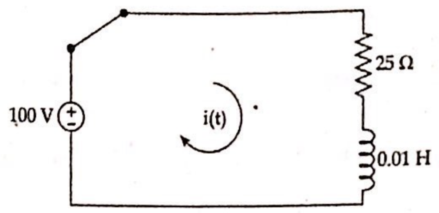
or, $6e^{-2t} = 5\,i(t) + \frac{di(t)}{dt} + v_C(0^-) + 4\int_0^t i(t)\,dt$

or, $6e^{-2t} = 5\,i(t) + \frac{di(t)}{dt} + 0 + 4\int_0^t i(t)\,dt$

or, $6e^{-2t} = 5\,i(t) + \frac{di(t)}{dt} + 4\int_0^t i(t)\,dt$

Solution:

Equivalent circuit at t > 0 is,



Applying KVL,

$$v = v_R + v_L$$

or, $100\ 25\ i(t) + 0.01 \frac{di(t)}{dt}$

or, $25\ i(t) + 0.01 \frac{di(t)}{dt} = 100$

Taking laplace transform, we get,

$$25\ I(s) + 0.01\ [sI(s) - i(0^+)] = \frac{100}{s}$$

or, $25\ I(s) + s0.01\ I(s) - 0.01 \times 2 = \frac{100}{s}$

Since $i(0^-) = \frac{50}{25} = 2$ A

$\therefore$ $i(0^-) = i(0^+) = 2$ A

or, $25\ I(s) + 0.01\ sI(s) - 0.02 = \frac{100}{s}$

or, $25\ I(s) + 0.01\ sI(s) = \frac{100}{s} + 0.02$

or, $I(s)\ (25 + 0.01s) = \frac{100 + 0.02s}{s}$

or, $I(s) = \frac{100 + 0.02s}{s(25 + 0.1s)} = \frac{A}{s} + \frac{B}{25 + 0.01s}$ (1)

Solving for A and B

$$A = \left.\frac{100 + 0.02s}{s(25 + 0.015)}\right|_{s=0} = 4$$

$$B = \left.\frac{100 + 0.02s}{s(25 + 0.01s)} \times s\right|_{s=-\frac{25}{0.01}} = 4$$

$$B = -0.03$$

Hence, $I(s) = \frac{4}{s} - \frac{0.03}{25 + 0.01s} = \frac{4}{s} - \frac{0.03}{s + 2500}$

Taking inverse laplace transform, we get,

$\therefore$ $i(t) = 4 - 0.03\ e^{-2500t}$ A

8. **A sinusoidal voltage 25 sin 10t is applied at time t = 0 to a series R-L circuit comprising R = 5 Ω and L = 1 H. By the method of laplace transformation, find current i(t). Assuming zero current through inductor before application of voltage. [2013/Spring]**

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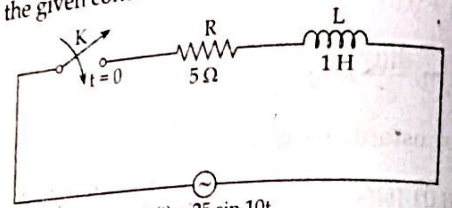
Solution:

Given that;

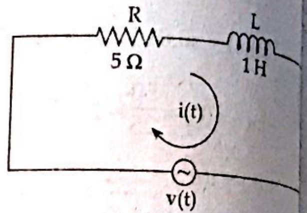
$v(t) = 25 \sin 10t$ $\quad R = 5\,\Omega$

$L = 1\,H$ $\quad i_L(0^-) = 0\text{ A} = i(0^+)$

The circuit for the given condition is,



v(t) = 25 sin 10t



$v(t) = 25 \sin 10t$

Equivalent circuit at $t > 0$ is,

Applying KVL,

$$25 \sin 10t = 25\, i(t) + \frac{di(t)}{dt}$$

Taking laplace transform, we get,

$$25\, I(s) + [sI(s) - i(0^+)] = 25\left(\frac{10}{s^2+100}\right)$$

or, $$25\, I(s) + I(s) - 0 = \frac{250}{s^2+100}$$

or, $$I(s)\,(25+s) = \frac{250}{s^2+100}$$

$$I(s) = \frac{250}{(s^2+100)\,(s+25)}$$

or, $$I(s) = \frac{250}{(s+25)\,(s^2+10^2)}$$

Let, $$I(s) = \frac{25}{(s+25)\,(s+10j)\,(s-10j)} = \frac{A}{s+25} + \frac{B}{s+10j} + \frac{B^*}{s-j10}$$

Solving for A,B and B^*, we get,

$\therefore$ $$A = \frac{25}{(s+25)\,(s+10j)\,(s-10j)} \times (s+25)\Big|_{s=-25} = \frac{1}{29}$$

$\therefore$ $$B = \frac{25}{(s+25)\,(s+10j)\,(s-10j)} \times (s+10j)\Big|_{s=-10j} = -200 - 500j$$

and, $B^* = -200 + 500j$

Hence, $$I(s) = \frac{1}{29(s+25)} + \frac{(-200-500j)}{(s+10j)} + \frac{(-200+500j)}{(s-10j)}$$

$$= \frac{1}{29}\left(\frac{1}{s+25}\right) + (-200-500j)\,\frac{1}{(s+10j)} + (-200+500j)\left(\frac{1}{s-10j}\right)$$

Taking laplace inverse, we get,

$$i(t) = \frac{1}{29}e^{-25t} + e^{-10jt}(-20-500j) + e^{10jt}(-200+500j)$$

$$= \frac{1}{29}e^{-25t} - 200\,e^{-10jt} - 500j\,e^{-10jt} - 200\,e^{10jt} + j\,500\,e^{10jt}$$

$$= \frac{1}{29}e^{-25t} - 200\,(e^{10jt} + e^{-10jt}) + 500\,(e^{-10jt} + e^{10jt})$$

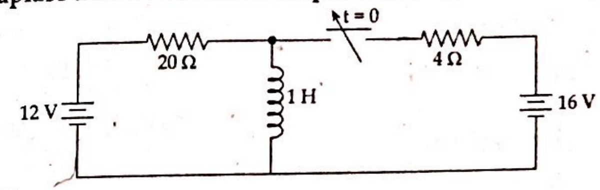
$$= 0.03\,e^{-25t} - 200 \times 2\left(\frac{e^{10jt}+e^{-10jt}}{2}\right) + 500j \times 2j\left(\frac{e^{10jt}-e^{-10jt}}{2j}\right)$$

$$= 0.03\,e^{-25t} - 400\left(\frac{e^{10jt}+e^{-10jt}}{2}\right) - 1000\left(\frac{e^{10jt}-e^{-10jt}}{2j}\right)$$

$\therefore$ $$i(t) = (0.03\,e^{-25t} - 400 \cos 10t - 1000 \sin 10t)\text{ A}$$

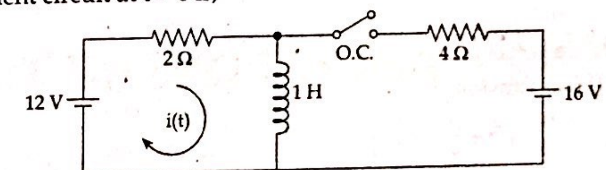
NOTE: $\sin\theta = \left(\frac{e^{j\theta}-e^{-j\theta}}{2j}\right)$ and, $\cos\theta = \left(\frac{e^{j\theta}+e^{-j\theta}}{2}\right)$

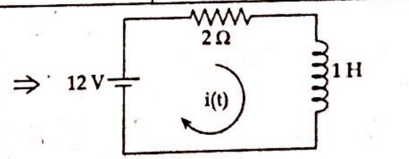
9. Find i(t) for t > 0 in the circuit shown in figure below using laplace transform. Switch is opened at t = 0. **[2014/Fall]**



Solution:

Equivalent circuit at $t > 0$ is,





Applying KVL, we get,

$$v(t) = v_R + v_L$$

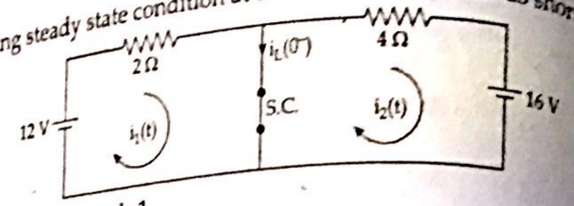
or, $$12 = 2i(t) + 1\,\frac{di(t)}{dt}$$

or, $$12 = 2i(t) + \frac{di(t)}{dt}$$

Taking laplace transform, we get,

$$\frac{12}{s} = 2\,I(s) + sI(s) - i(0^+) \qquad (1)$$

Assuming steady state condition at $t = 0^-$, so inductor acts as short c...



Applying KVL in mesh-1,

$$-2\, i_1(t) + 12 = 0$$

or, $2\, i_1(t) = 12$

∴ $i_1(t) = 6\text{ A}$

Applying KVL in mesh

or, $-4\, i_2(t) - 16 = 0$

or, $-4\, i_2(t) = 16$

∴ $i_2(t) = -4\text{ A}$

∴ $i_L(0^-) = i_1(t) - i_2(t) = 6 + 4 = 10\text{ A}$

Also, $i_L(0^-) = i_L(0^+) = 10\text{ A}$

Equation (1) becomes,

$$\frac{12}{s} = 2I(s) + s\, I(s) - 10$$

or, $\dfrac{12}{s} + 10 = I(s)\,(2 + s)$

or, $\dfrac{12 + 10s}{s} = I(s)\,(s + 2)$

or, $I(s) = \dfrac{(10s + 12)}{s(s + 2)}$

Using partial expansion,

$$I(s) = \frac{10s + 12}{s(s + 2)} = \frac{A}{s} + \frac{B}{s + 2}$$

Solving for A and B,

$$A = \frac{10s + 12}{s(s + 2)} \times s\Big|_{s=0} = \frac{10s + 12}{s + 2}\Big|_{s=0} = 6$$

and, $B = \dfrac{10s + 12}{s(s + 2)} \times (s + 2)\Big|_{s=-2} = \dfrac{10s + 12}{s + 2}\Big|_{s=-2} = 4$

or, $I(s) = \dfrac{6}{s} + \dfrac{4}{s + 2}$

Taking inverse laplace transform, we get,

∴ $i(t) = 6 + 4\,e^{-2t}\text{ A}$

10. State initial and final value theorems of laplace transform. Fi... initial value and I(t) for the function. [2014/Spri...

$$I(s) = \frac{2s + 5}{(s + 1)\,(s + 2)}$$

Solution:

See the topic 4.4 for first part question.

Given that;

$$I(s) = \frac{2s + 5}{(s + 1)\,(s + 2)}$$

Applying initial value theorem

$$f(0^+) \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} [sF(s)]$$

Now, $i(0^+) = \lim_{s\to\infty} s\,\dfrac{(2s + 5)}{(s + 1)\,(s + 2)}$

$$= \lim_{s\to\infty} \frac{2s^2 + 5}{(s + 1)\,(s + 2)} = \lim_{s\to\infty} \frac{2s^2 + 5s}{s^2 + 3s + 2}$$

$$= \lim_{s\to\infty} \frac{s^2\left(2 + \frac{5}{s}\right)}{s^2\left(1 + \frac{3}{s} + \frac{2}{s^2}\right)} = \frac{2 + 0}{1 + 0 + 0}$$

∴ $i(0^+) = 2\text{ A}$

The correction of this result can be verified by using laplace for i(t) at t = 0.

Thus, $\dfrac{2s + 5}{(s + 1)\,(s + 2)} = \dfrac{A}{s + 1} + \dfrac{B}{s + 2}$

or, $2s + 5 = A\,(s + 2) + B\,(s + 1)$

Put $s = -1$

∴ $A = 3$

Put $s = -2$,

$$2\,(-2) + 5 = A\,(-2 + 2) + B\,(-2 + 1)$$

∴ $B = -1$

Now, $I(s) = \dfrac{3}{s + 1} - \dfrac{1}{s + 2}$

Taking inverse laplace transform, we get,

$$i(t) = 3\,e^{-t} - 1\,e^{-2t}$$

At $t = 0^+$,

∴ $i(0^+) = 3 - 1 = 2\text{A}$

This verifies value obtained by using initial value theorem.

11. Find i(0) and steady state value i(t) for the function,

$$I(s) = \frac{(s + 1)}{s(s + 2)}$$ [2011/Spring]

Solution:

Given that;

$$I(s) = \frac{(s + 1)}{s(s + 2)}$$

$$s \cdot I(s) = \frac{(s + 1)}{(s + 2)}$$

Using final value theorem, we have,

$$\lim_{t\to\infty} i(t) = \lim_{s\to 0} s \cdot I(s) = \lim_{s\to 0} \frac{s+1}{s+2} = \frac{0+1}{0+2} = \frac{1}{2}$$

Hence steady state value $i(t) = \frac{1}{2}$ A

By using partial fraction expansion, we get,

Let, $I(s) = \frac{(s+1)}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$

Solving for A and B,

$$A = \frac{(s+1)}{s(s+2)} \times s\bigg|_{s=0} = \frac{s+1}{s+2}\bigg|_{s=0} = \frac{1}{2}$$

and, $B = \frac{(s+1)}{s(s+2)} \times (s+2)\bigg|_{s=-2} = \frac{s+1}{s}\bigg|_{s=-2} = \frac{1}{2}$

Hence, $I(s) = \frac{(s+1)}{s(s+2)} = \frac{1}{2s} + \frac{1}{2(s+2)}$

Taking inverse laplace transform, we get,

$$i(t) = \frac{1}{2} \times 1 + \frac{1}{2} e^{-2t}$$

or, $i(t) = \frac{1}{2} + \frac{1}{2} e^{-2t}$

or, $\lim_{t\to\infty} i(t) = \frac{1}{2} + 0$

$\therefore$ $i(t) = \frac{1}{2}$

This verifies result obtained from final value theorem for i(0),

$$i(t) = i(0) = \frac{1}{2} + \frac{1}{2} e^{o} = \frac{1}{2} + \frac{1}{2} \times 1 = 1 \text{ A}$$

12. The response v(t) in a series RLC circuit is described by the differential equation. **[2015/Spring]**

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 8v = 8\,u(t)$$

Solution:

Given that;

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 8v = 8\,u(t)$$

Taking laplace transform, we get,

$$s^2 V(s) - s\,i(0^+) - i'(0^+) + 4\,[s\,V(s) - i(0^+)] + 8\,V(s) = \frac{8}{s}$$

or, $s^2 V(s) - 3s - (-4) + 4[sV(s) - 3] - 8\,V(s) = \frac{8}{s}$

or, $s^2 V(s) - 3s + 4s\,V(s) + 8\,V(s) = \frac{8}{s}$

or, $[s^2 V(s) + 4s\,V(s) + 8\,V(s)] - 3s - 8 = \frac{8}{s}$

or, $V(s)\,(s^2 + 4s + 8) = \frac{8}{s} + 3s + 8$

or, $V(s) = \frac{8 + 3s^2 + 8s}{s(s^2 + 4s + 8)} = \frac{3s^2 + 8s + 8}{s(s^2 + 4s + 8)}$

or, $V(s) = \frac{3s^2 + 8s + 8}{s(s + 2 + 2j)(s + 2 - 2j)}$

By using partial fraction expansion,

$$V(s) = \frac{3s^2 + 8s + 8}{s(s^2 + 2s + 2j)(s + 2 - 2j)} = \frac{A}{s} + \frac{B}{s + 2 + 2j} + \frac{B^*}{s + 2 - 2j}$$

Solving for A, B and B^*,

$\therefore$ $A = \frac{3s^2 + 8s + 8}{s(s^2 + 4s + 8) \times s}\bigg|_{s=0} = 1$

$\therefore$ $B = \frac{3s^2 + 8s + 8}{s(s + 2 + 2j)(s + 2 - 2j)} \times (s + 2 + 2j)\bigg|_{s=-2-2j} = 1$

$\therefore$ $B^* = \frac{3s^2 + 8s + 8}{s(s + 2 + 2j)(s + 2 - 2j)} \times (s + 2 - 2j)\bigg|_{s=-2+2j} = 1$

Hence, $V(s) = \frac{1}{s} + \frac{1}{s + 2 + 2j} + \frac{1}{s + 2 - 2j}$

Taking inverse laplace transform, we get,

$$v(t) = 1 + e^{-(2+2j)t} + e^{-(2-2j)t}$$

$$= 1 + e^{-2t}(e^{-2jt} + e^{2jt}) = 1 + \left(\frac{e^{2jt} + e^{-2jt}}{2} \times 2\right) e^{-2t}$$

$\therefore$ $v(t) = 1 + 2\,e^{-2t} \cos 2t$ V

13. An exponential voltage $v(t) = 6\,e^{-2t}$ is suddenly at time t = 0 to a series RL circuit consisting of R = 2 Ω and L = 1 H. Using laplace transform method, find the particular solution for i(t) through the circuit. Assume no initial current through the inductor before switching. **[2016/Fall]**

Solution:

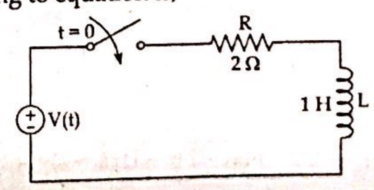
Given that;

R = 2 Ω

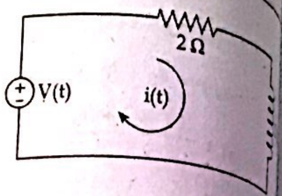
L = 1 H

$v(t) = 6\,e^{-2t}$

The circuit according to equation is,



Equivalent circuit at $t > 0$ is,



Applying KVL,

$$v(t) = v_R + v_L$$

or, $6e^{-2t} = Ri(t) + L\frac{di(t)}{dt}$

or, $6e^{-2t} = 2i(t) + \frac{di(t)}{dt}$

Taking laplace transform, we get,

or, $\frac{6}{s+2} = 2I(s) + sI(s) - i(0^+)$

Since initially zero current flows through the inductor,

i.e., $i(0^-) = i(0^+) = 0$ A

or, $\frac{6}{s+2} = 2I(s) + sI(s) - 0$

or, $\frac{6}{s+2} = I(s)(2+s)$

or, $I(s) = \frac{6}{(s+2)^2}$

Taking inverse laplace transform, we get,

$\therefore$ $i(t) = 6te^{-2t}$ A

Hence particular solution for i(t) is $6te^{-2t}$ A

14. The response i(t) in a circuit is described by the differential equation

$$\frac{d^2i(t)}{dt^2} - i = 25 + e^{2t}$$

All the initial conditions are zero. Find the response i(t) using laplace transform method. **[2016/Fall]**

Solution:

Given that; differential equation is,

$$\frac{d^2i(t)}{dt^2} - i(t) = 25 + e^{2t}$$

All initial conditions are zero, *i.e.,* $i(0^-) = 0$ and $i'(0) = 0$

Taking laplace transform of equation (1),

$$[s^2I(s) - s\,i(0) - i'(0)] - I(s) = \frac{25}{s} + \frac{1}{(s-2)}$$

or, $[s^2I(s) - 0 - 0] - I(s) = \frac{25(s-2)+s}{s(s-2)}$

or, $(s^2-1)I(s) = \frac{26s-50}{s(s-2)}$

or, $I(s) = \frac{26s-50}{s(s-2)(s^2-1^2)} = \frac{26s-50}{s(s-2)(s-1)(s+1)}$

Using partial fraction expansion, we have,

Let, $I(s) = \frac{26s-50}{s(s-2)(s^2-1^2)} = \frac{26s-50}{s(s-2)(s-1)(s+1)}$

Solving for A, B, C and D,

$\therefore$ $A = \left.\frac{26s-50}{(s+1)(s-1)(s-2)}\right|_{s=0} = -25$

$\therefore$ $B = \left.\frac{26s-50}{(s-1)(s-2)}\right|_{s=-1} = \frac{38}{3}$

$\therefore$ $C = \left.\frac{26s-50}{(s+1)(s-2)}\right|_{s=1} = 12$

$\therefore$ $D = \left.\frac{26s-50}{s(s+1)(s-1)}\right|_{s=2} = \frac{1}{3}$

Hence, $I(s) = \frac{-25}{s} + \frac{38}{3(s+1)} + \frac{12}{s-1} + \frac{1}{3(s-2)}$

Taking inverse laplace transform, we get,

$\therefore$ $i(t) = \left(-25 + \frac{38}{3}e^{-t} + 12e^{t} + \frac{1}{3}e^{2t}\right)$ A

15. A 50 Hz, 300 V sinusoidal voltage is applied at t = 0 to a series R-L circuit having resistance of 2.5 Ω and inductance 0.1 H. Find an expression of current at any instant t; Also calculate the value of the transient current, steady state current and total current at 0.01 sec after switching on using laplace transform, method. **[2016/Spring]**

Solution:

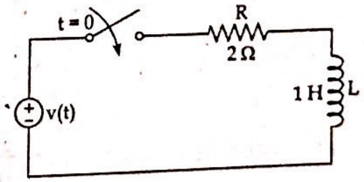
Given that;

$f = 50$ Hz $V = 300$ V

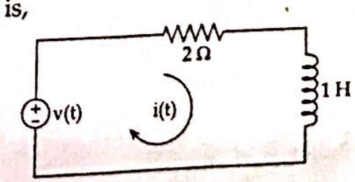
$R = 2.5\ \Omega$ $L = 0.1$ H

$\omega = 2\pi f$

$v(t) = 300 \sin(2\pi \times 50t) = 300 \sin(100\pi t)$ V



The circuit for $t > 0$ is,



Applying KVL, we get,

$$300 \sin(100\,\pi t) = 2.5\, i(t) + 0.1\,\frac{di(t)}{dt}$$

Taking laplace transform on both sides,

or, $$300\left(\frac{\omega}{s^2+\omega^2}\right) = 2.5\, I(s) + 0.1\,[s\, I(s) - i(0^+)]$$

or, $$300\left[\frac{100\pi}{s^2+(100\pi)^2}\right] = 2.5\, I(s) + 0.1\,[s\, I(s) - i(0^+)]$$

Since, initially before switch is closed, the current through the inductor is zero, i.e., $i(0^-) = i(0^+) = 0$ A

or, $$300\left[\frac{100\pi}{s^2+(100\pi)^2}\right] = 2.5\, I(s) + 0.1\, s\, I(s)$$

or, $$\frac{3\times10^4}{s^2+(100\pi)^2} = I(s)\,(s+2.5)$$

or, $$I(s) = \frac{3\times10^4\,\pi}{(s+2.5)\,[s^2+(100\pi)^2]}$$

or, $$I(s) = \frac{3\times10^4\,\pi}{(s+2.5)\,(s+j100\pi)\,(s-j100\pi)}$$

Using partial fraction expansion,

Let, $$I(s) = \frac{3\times10^4\,\pi}{(s+2.5)\,(s+j100\pi)\,(s-j100\pi)} = \frac{A}{s+2.5} + \frac{B}{s+j\,100\pi} + \frac{B^*}{s-j100\pi}$$

Solving for A, B and B^*,

$$A = \left.\frac{3\times10^4\,\pi}{s^2+(100\,\pi)^2}\right|_{s=-2.5} = 0.954$$

$\therefore$ $$B = \left.\frac{3\times10^4\,\pi}{(s+2.5)\,(s+j100\,\pi)^2}\right|_{s=-j100\pi}$$

$$= -0.477 + 3.79\times10^{-3}j$$

$$B^* = \left.\frac{3\times10^4}{(s+2.5)\,(s+j100\,\pi)}\right|_{s=j100\pi}$$

$$= -0.477 - 3.79\times10^{-3}j$$

Hence, $$I(s) = 0.95\,\frac{1}{(s+2.5)} + (-0.477 + 3.79\times10^{-3}j)\,\frac{1}{(s+j100\pi)}$$

$$+ (-0.477 - 3.79\times10^{-3}j)\,\frac{1}{(s-j100\pi)}$$

Taking inverse laplace transform, we get,

$$i(t) = 0.95\, e^{-2.5t} + (-0.477 + 3.79\times10^{-3}j)\, e^{-100\pi jt}$$

$$+ (-0.477 - 3.8\times10^{-3}j)\, e^{100\pi jt}$$

$$= 0.95\, e^{-2.5t} - 0.477\times 2\left(\frac{e^{-j100\pi t}+e^{100\pi jt}}{2}\right) - 3.79\times10^{-3}j$$

$$\times\, 2j\left(\frac{e^{100j\pi t}-e^{-100j\pi t}}{2j}\right)$$

$\therefore$ $$i(t) = 0.95\, e^{-2.5t} - 0.954\cos(100\,\pi t) + 7.58\times10^{-3}\sin(100\pi t)\ \text{A}$$

At t = 0.01 sec,

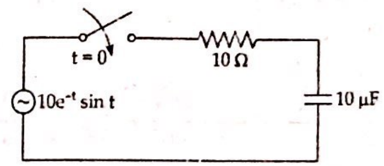
or, $$i(0.01) = 0.95\, e^{-2.5\times0.01} - 0.954\cos(100\pi\times0.01)$$

$$+ 7.58\times10^{-3}\sin(100\pi\times0.01)$$

$$= 0.9265 - 0.9526 + 7.58\times10^{-3}\times0.05480$$

$\therefore$ $$i(0.01) = -0.02568\ \text{A}$$

16. In the circuit shown below, the switch is closed at t = 0 with capacitor initially unergized. For the values given, find the value of current passing through the 10 Ω resistor using laplace transform method. **[2017/Fall]**



Solution:

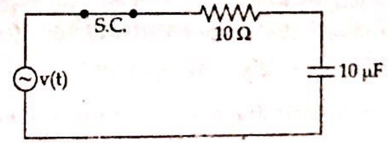
Given that;

$R = 10\,\Omega$

$C = 10\,\mu F = 10\times10^{-6}$ F

$v(t) = 10\, e^{-t}\sin t$

For $t > 0$, the equivalent circuit is,



Since this circuit is series connection, current through 10 Ω resistor is equal to current in the circuit i(t).

Applying KVL for $t > 0$,

or, $$10\, i(t) + \frac{1}{10\times10^{-6}}\int_{-\infty}^{\infty} i(t)\, dt = 10\, e^{-t}\sin t$$

Taking laplace transform, we get,

$$10\, I(s) + 10^5\,\frac{I(s)}{s} + \frac{v_0(0)}{s} = 10\times\frac{1}{(s+1)^2+1^2}$$

V_0 is initial charge on capacitor which is 0 as given that capacitor initially discharged.

or, $(s + 10^4)\, I(s) = \dfrac{s}{s^2 + 2s + 1}$

or, $I(s) = \dfrac{s}{(s + 10{,}000)(s^2 + 2s + 2)}$

$I(s) = \dfrac{s}{(s + 10{,}000)(s + 1 - i)(s + 1 + i)}$

Let, $\dfrac{s}{(s + 10{,}000)(s^2 + 2s + 2)} = \dfrac{A}{(s + 10{,}000)} + \dfrac{B}{(s + 1 - i)} + \dfrac{C}{(s + 1 + i)}$

or, $s = A(s^2 + 2s + 2) + B(s + 10{,}000)(s + 1 + i) + C(s + 10{,}000)(s + 1 - i)$

Put $s = -10{,}000$,

or, $-10{,}000 = A[(-10{,}000)^2 + 2 \times (-10{,}000) + 2]$

$\therefore$ $A = -1 \times 10^{-4}$

Put $s = -1 + i$,

$\therefore$ $B = 5 \times 10^{-5} + 4.99 \times 10^{-5} j$

Put $s = -1 - i$,

$\therefore$ $C = 5 \times 10^{-5} - 4.99 \times 10^{-5} j$

Thus, $I(s) = \dfrac{-10^4}{(s + 10{,}000)} + \dfrac{5 \times 10^{-5} + 4.99 \times 10^{-5} j}{(s + 1 - i)} + \dfrac{5 \times 10^5 - 4.99 \times 10^{-5}}{(s + 1 + i)}$

Taking inverse laplace transform, we get,

$$i(t) = -10^{-4} e^{-10{,}000t} + (5 \times 10^{-5} + 4.99 \times 10^{-5} j)\, e^{-(1-j)t} + (5 \times 10^{-5} - 4.99 \times 10^{-5} j)\, e^{-(1+j)t}$$

$$= -10^{-4} e^{-10{,}000t} + 5 \times 10^{-5} e^{-1t} j\, e^{jt} + 4.99 \times 10^{-5} j\, e^{-1t} e^{jt} + 5 \times 10^{-5} e^{-t} e^{-jt} - 4.99 \times 10^{-5} j\, e^{-t} e^{-jt}$$

$$= -10^{-4} e^{-10{,}000t} + 2 \times 5 \times 10^{-5} e^{-t} \left(\frac{e^{jt} + e^{-jt}}{2}\right) - 5 \times 10^{-5} j\, e^{-t}\, 2j \times \left(\frac{e^{jt} + e^{-jt}}{2}\right)$$

$$= -10^{-4} e^{-10{,}000t} + 10 \times 10^{-5} e^{-t} \cos t + 10 \times 10^{-5} e^{-t} (j^2) \sin t$$

$\therefore$ $i(t) = -10^{-4} e^{-10{,}000t} + 10^{-4} e^{-t} (\cos t - \sin t)$ A

17. Write short notes on initial and final value theorem.
[2014/Spring, 2016/Fall, 2018/Spring, 2019/Spring, 2020/Fall]

Solution: See the topic 4.4.

19. Obtain the voltage-current relationship for passive elements in time domain and frequency domain. Also, draw their equivalent circuits in both domain showing initial values. [2020/Fall]

Solution: See the topic 4.6.2, 4.6.3 and 4.6.4.

20. Write short notes on exponential response of R-C series circuit. [2018/Fall]

Solution: See the topic 4.9. 'C'.

21. What are the advantages of laplace method over classical method? [2013/Spring, 2015/Spring, 2016/Fall]

Solution: See the topic 4.1.1.

22. Write short notes on laplace transform and its advantage. [2014/Fall]

Solution: See the topic 4.1 and 4.1.1.

23. Define inverse transform. Find the laplace transform of common forcing functions unit step function, delta function and ramp function. [2013/Fall]

Solution: See the topic 4.2 and 4.12.

24. Write short notes on laplace transform of common forcing function. [2019/Fall]

Solution: See the topic 4.12.

25. What are the conditions necessary for the existence of laplace transform? Prove that the laplace transform of delta function is unity. [2012/Spring]

Solution:

Let, f(t) be defined for $t \geq 0$. The laplace transform of f(t) denoted by F(s)

or $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$

Provides that this (improper) integral exists, *i.e.*, that the integral convergent.

For functions of t continues $[0, \infty]$, the above transformation to the frequency domain is one to one. That is, different contirlues functions will have different transforms.

A function f(t) is called piecewise continuous if it only has finitely many (or none whatever -a continues function is considered to be "piecewise continuous"!) discontinuous on any interval [a, b], and that both one sides limits exist as approaches each of those discontinuity from within the interval. The last part of the definition means that f could have removable and/or jump discontinuities only it cannot have any infinity discontinuity.

Suppose that;

i) f is piecewise continuous on the interval $0 \leq t \leq A$ for any $A > 0$. For any $A > 0$.

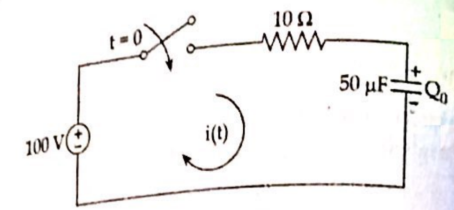
ii) $|f(t)| \leq K e^{at}$ when $t \geq M$ for any real constant a and some positive constants K and M. (This means that f is "of exponential order" *i.e.*, its rate of growth is no faster than that of exponential functions).

Then the laplace transform, $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$.

The above theorem gives a sufficient condition for the existence of laplace transform. It is not a necessary condition. A function does not need to satisfy the two conditions in order to have a laplace transform.

Solution: See the topic 4.12 for answer of 2^{nd} part question.

26. **In the series R-C circuit, the capacitor has initial charge 2.5 M... At t = 0, the switch is closed and a constant voltage so... V = 100 V is applied, find i(t) using laplace transform method.** [2020/...]



Solution:

Given that;

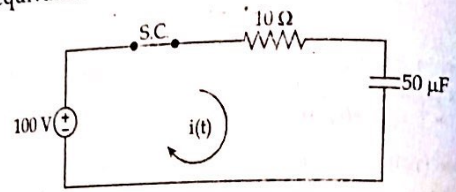
$V = 100$ V

$R = 10\ \Omega$

$C = 50\ \mu F$

$Q_0 = 2.5$ MC

For $t > 0$, the equivalent circuit is,



Using KVL,

$$100 = 10\, i(t) + \frac{1}{50 \times 10^{-6}} \int_{-\infty}^{t} i(t)\, dt$$

Taking laplace transform, we get,

$$\frac{100}{s} = 10\, I(s) + \frac{1}{50 \times 10^{-6}} \left(\frac{I(s)}{s} - \frac{Q_0}{s} \right)$$

or, $$\frac{100}{s} = 10\, I(s) + \frac{20000}{s} I(s) - \frac{2.5 \times 10^{6}}{50 \times 10^{-6} \times s}$$

or, $$(10\, s + 2000)\, I(s) = 100 + 5 \times 10^{10}$$

or, $$I(s) = \frac{5 \times 10^{9}}{s + 2000} = 5 \times 10^{9} \times \frac{1}{s + 2000}$$

Taking inverse laplace, we get,

$\therefore$ $$i(t) = 5 \times 10^{9}\, e^{-2000t}\ \text{A}.$$

❖❖❖

Chapter 5 | **TRANSFER FUNCTION**

5.1 TRANSFER FUNCTIONS OF NETWORK SYSTEM

A function relating currents or voltage at different parts of the network is called a transfer function. It is found to be mathematically similar to the transform impedance function. These functions are called network functions.

5.2 CONCEPT OF COMPLEX FREQUENCY

The solution of differential equations for the network is of the form,

$$x(t) = ke^{s_n t} \qquad (1)$$

This x(t) may be a voltage v(t) or a current i(t). Generally, x(t) is a function of time. And s_n is a complex number, which may be expressed as,

$$s_n = \sigma_n + j\omega_n \qquad (2)$$

Where, ω_n, the imaginary part of s_n is called as angular frequency (radian frequency) and it uses in time-domain equations in the forms of cos ω_{nt} or sin ω_{nt}. The dimension of the radian frequency is radian per second. The radian frequency may be expressed as,

$$\omega_n = 2\pi f_n = \frac{2\pi}{T}$$

where, f_n is the frequency in Hertz (Hz) and T is the time period in seconds. From equation (2), we can easily see that σ_n and ω_n must have identical dimensions. The dimensions of ω_n is $(\text{time})^{-1}$, since the radian is a dimensionless quantity (being length of arc per length of radius). The dimension of σ_n can be determined as σ_n appears as an exponential factor

i.e., $V = V_0 e^{\sigma_n t}$

or, $\sigma_n = \frac{1}{t} ln\left(\frac{V}{V_0}\right)$

Since the usual unit for the natural logarithm is the neper, hence, the dimension of σ_n is neper per second.

Now, the complex sum,

$$s_n = \sigma_n + j\omega_n$$

is defined as the complex frequency. The real part of the complex frequency is neper frequency, corresponds to exponential decay or exponential increase (depending on sign) and the imaginary part of the complex frequency is the radian frequency (or real frequency), corresponds to oscillations.

5.2.1 Terminal Pairs or Ports

In figure 5.1 (a), one port network is shown. The pair of terminals is connected to an energy source which is the driving force for the network, so that the pair of terminals is known as the driving point of the network.

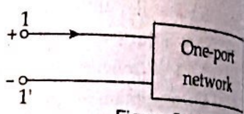


Figure 5.1 (a)

Figure 5.1 (b) shows a two-port network. The port 1-1' is assumed to be connected to the driving force (as an input), and port 2-2' is connected to a load (as an output).

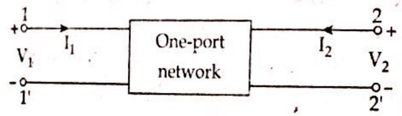


Figure 5.1: (b)

In figure 5.1 (c), N-port network is shown for the general case

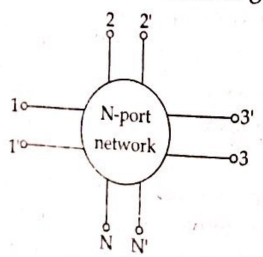


Figure 5.1: (c)

5.3 NETWORK FUNCTIONS

A. The transform impedance at a port has been defined as the ratio of voltage transform to current transform. Thus we write,

$$Z(s) = \frac{V(s)}{I(s)}$$

Similarly, the transform admittance is defined as the ratio of current transform to voltage transform,

i.e., $Y(s) = \frac{I(s)}{V(s)} = \frac{1}{Z(s)}$

The transform impedance and transform admittance must relate to the source port 1-1' or 2-2' in figure 5.1 (a) and (b). The impedance or admittance found at a given port is called a driving point impedance or admittance. i.e., transform impedances or admittances of ports 1-1' and 2-2' are also called as input driving point impedance or admittances respectively. Because of the similarity of impedance and admittance these two quantities are assigned one name "immittance" (a combination of impedance and admittance). An immittance is thus an impedance or an admittance. Table below show the immittance of circuit elements and the immittance function for some networks are given in table.

Table: Immittance function for circuit elements

Elements	Impedance function Z(s)	Admittance function Y(s)
Resistance (R) in Ω	R	$\frac{1}{R} = G$
Inductance (L) in H	sL	$\frac{1}{sL}$
Capacitance (C) in F	$\frac{1}{sC}$	sC

Hence, $Z(s) \equiv R \equiv sL = \frac{1}{sC}$

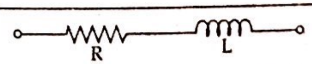
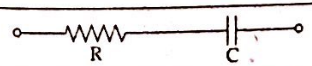
and, $Y(s) \equiv \frac{1}{R} \equiv \frac{1}{sL} \equiv sC$

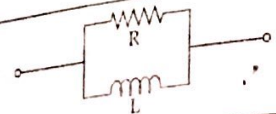
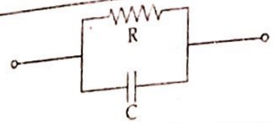
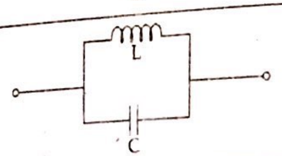
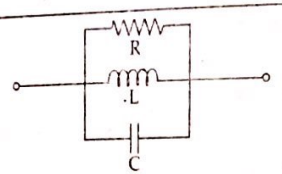
NOTE: The impedance function Z(s) and admittance function Y(s) are easily determined for series and parallel circuit respectively.

i) For series circuit; $Z(s) = Z_1(s) + Z_2(s) + Z_3(s)$

ii) For parallel circuit; $Y(s) = Y_1(s) + Y_2(s) + Y_3(s)$

Table: Immittance functions for some simple networks

Network	Impedance function Z(s)	Admittance function Y(s)
 R L	R + sL	$\frac{1}{R+sL}$
 R C	$\frac{sRC+1}{sC}$	$\frac{sC}{sRC+1}$

	$\frac{s^2 LC + 1}{sC}$	$\frac{sC}{s^2 LC + 1}$
	$\frac{sRC + s^2 LC + 1}{sC}$	$\frac{sC}{sRC + s^2 LC + 1}$
	$\frac{sRL}{RC + R}$	$\frac{sL + R}{sRL}$
	$\frac{sRL}{sRC + 1}$	$\frac{sRC + 1}{R}$
	$\frac{sL}{s^2 LC + 1}$	$\frac{s^2 LC + 1}{sL}$
	$\frac{sRL}{s^2 RLC + sL + R}$	$\frac{s^2 RLC + sL + R}{sRL}$

B. The transfer function is used to describe networks which have at least two ports. In general, the transfer function relates the transform of a quantity at one port to the transform of another quantity at another port. Thus transform functions have the following possible forms.

i) The ratio of one voltage to another current, or one current to another voltage; Z(s) or Y(s).

ii) The ratio of one voltage to another voltage, or the voltage transfer function; G(s).

iii) The ratio of one current to another current, or the current transfer function: α(s).

It is conventional, to define transfer functions as the ratio of an output quantity to an input quantity. In terms of the two port networks of figure 5.1 (b), the output quantities are $V_2(s)$ and $I_2(s)$ and the input quantities are $V_1(s)$ and $I_1(s)$. Using this scheme, there are only four basic transfer functions for the two port network and these are given as;

i) Transfer impedance function; $Z_{21}(s) = \frac{V_2(s)}{I_1(s)}$

ii) Transfer admittance function; $Y_{21}(s) = \frac{I_2(s)}{V_1(s)}$

iii) Voltage transfer function; $G_{21}(s) = \frac{V_2(s)}{V_1(s)}$

iv) Current transfer function; $\alpha_{21}(s) = \frac{I_2(s)}{I_1(s)}$

NOTE:
The ratio of an input quantity to an output quantity is termed as the inverse transfer function.

i.e., $\frac{V_1(s)}{I_2(s)} = Z_{12}(s)$; the inverse transfer impedance function.

$\frac{I_1(s)}{V_2(s)} = Y_{12}(s)$; the inverse transfer admittance function.

$\frac{V_1(s)}{V_2(s)} = G_{12}(s)$; the inverse voltage transfer function

$\frac{I_1(s)}{I_2(s)} = \alpha_{12}(s)$; the inverse current transfer function.

5.4 POLES AND ZEROS OF NETWORK FUNCTIONS

In linear RLC networks, all network functions T(s) are the rational functions of s and may be expressed as the ratio of two polynomials namely, N(s), the numerator polynomial and D(s), the denominator polynomial as,

$$T(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \dots\dots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \dots\dots + b_1 s + b_0} \quad (1)$$

or, $$T(s) = \frac{N(s)}{D(s)} = K\frac{s^n + c_{n-1} s^{n-1} + \dots\dots + c_1 s + c_0}{s^m + d_{m-1} s^{m-1} + \dots\dots + d_1 s + d_0} \quad (2)$$

where, $K = \frac{a_n}{b_m}$ is a positive constant known as scalar factor, the coefficients a, b, c and d are real and positive for passive network and no dependent sources.

The polynomial N(s) = 0 has n roots, they are called as zeros of the network function T(s) and the polynomial D(s) = 0 has m roots, they are called as poles of the T(s).

Writing equation **(2)** in terms of the roots in a factored form, we obtain,

$$T(s) = \frac{K(s - z_1)(s - z_2) \dots\dots (s - z_n)}{(s - p_1)(s - p_2) \dots\dots (s - p_m)}$$

where, z_i, the zeros of T(s) and p_i, the poles of T(s).

The value of the poles and zeros of T(s) and hence their location in s-plane completely specify the network function except for the scalar factor K.

NOTE: In the complex s-plane, a pole is denoted by a small cross (X) and a zero by a small circle (0).

Case I: where r poles (r < m) or r zeros (r < n) in equation **(3)** have the same value, the pole or zero is said to be of multiplicity r.

Case II: When n>m, we have poles at infinity of multiplicity or degree of (n – m).

Case III: When n<m, we then have zeros at infinity of multiplicity or degree of (m–n).

For any rational network function,

The number of finite poles + The number of poles at infinity

= The number of finite zeros + The number of zeros at infinity

i.e., The total number of poles = The total number of zeros

5.4.1 Necessary Conditions for Driving point Immittance Functions (with common factors in N(s) and D(s) Cancelled)

a) The coefficients in the polynomials N(s) and D(s) must be real and positive.

b) Poles and zeros must be conjugate if imaginary or complex.

c) The real part of all poles and zeros must be negative or zero, if the real part is zero, then that pole or zero must be simple *i.e.*, all the roots of N(s) = 0 and D(s) = 0 lies on the left half of s-plane and simple roots may lie on the imaginary or $j\omega$ – axis.

d) The polynomials N(s) and D(s) may not have missing terms between those of highest and lowest degrees, unless all even or all odd terms are missing.

e) The highest degree of N(s) and D(s) may differ by either zero or one only.

f) The lowest degree of N(s) and D(s) may differ by either zero or one only.

5.4.2 Necessary Conditions for Transfer Functions (with Common Factors in N(s) and D(s) Cancelled)

1. The coefficients in the polynomials N(s) and D(s) of $T = \frac{N}{D}$ must be real and those for D(s) must be positive.
2. Poles and zeroes must be conjugate if imaginary or complex.
3. The real part of poles must be negative or zero if the real part is zero, then that pole must be simple. This includes the origin.
4. The polynomial D(s) may not have missing term between that of highest and lowest degrees, unless all even or all odd terms are missing.
5. The polynomial N(s) may have terms missing and some of the coefficients may be negative.
6. The degree of N(s) may be as small as zero independent of the degree D(s).
7. i) For G and α: The maximum degree of N(s) is equal to the degree of D(s).

   ii) For Z and Y: The maximum degree of N(s) is equal to the degree of D(s) plus one.

5.5 STABILITY

Among many forms of performance specifications used in design, the most important requirement is that the system be stable. An unstable system is generally considered to be useless.

For analysis and design purposes, we can classify stability as absolute stability and relative stability. Absolute stability refers to the condition of whether the system is stable or unstable; it is a yes or no answer. Once the system is found to be stable, it is of interest to determine how stable it is and this degree of stability is a measure of relative stability.

A system is said to be stable if its output (response) cannot be made to increase indefinitely by the application of a bounded input excitation.

5.5.1 Relationship between Impulse Response and Stability

The stability of a network function T(s) can be conveniently determined by considering its response to an impulse function which is obtained by taking inverse laplace transform of the partial fraction expansion of the function.

a) Stable system

A system is said to be stable if the impulse response approaches zero for sufficiently large time.

b) Unstable system

A system is said to be unstable if the inpulse response grow without bound *i.e.*, approaches infinity for sufficiently large time.

c) Marginally stable system

A system is said to be marginally stable if the inpulse response approaches a constant non-zero value or a constant amplitude oscillation for sufficiently large time.

5.5.2 Relationship between Pole Positions and Stability

The necessary and sufficient condition for the system to be stable is that all roots of characteristic equation of the system lie in the negative half of the s-plane. In other words, we can say the poles of the transfer function $T(s) = \frac{N(s)}{D(s)}$ (*i.e.*, roots of D(s) = 0) lie in left half of s-plane. On the other hand, a system having any pole in right half (or positive half) of the s-plane will be unstable.

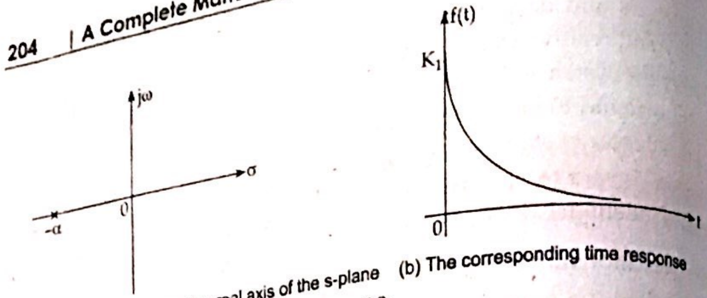
Let us consider the relationship between pole positions and the corresponding impulse responses and hence stability.

a) Poles on the negative real axis

If the network has a simple pole on the negative real axis, *i.e.*, $F(s) = \frac{K_1}{s+\alpha}$ then the corresponding impulse response for $t \geq 0$ is given by

$$f(t) = \mathcal{L}^{-1}[F(s)] = K_1 e^{-\alpha t}$$

When t is sufficiently large, f(t) approaches zero and the system is stable. The pole on the negative real axis and the corresponding time response is shown in figure 5.2 (a) and (b).



(a) Pole on the negative real axis of the s-plane (b) The corresponding time response

Figure 5.2:

If the network function F(s) has a multiple order pole *i.e.,* $F(s) = \frac{K_1}{(s+\alpha)^r}$

then the time response would be of the form of $f(t) = K_1 t^r e^{-\alpha t}$.

This response is also stable.

b) Complex poles in the left half of the s-plane

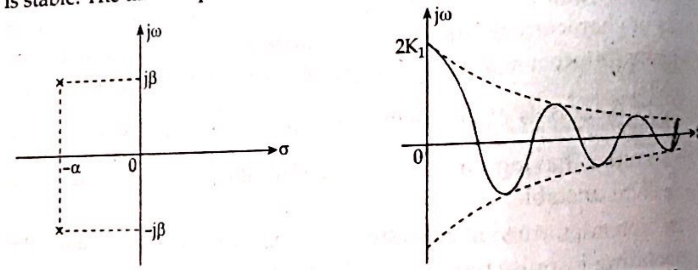
Suppose that F(s) has a pair of first order complex conjugate poles at $s = -\alpha \pm j\beta$ as shown in figure 5.3 (a).

i.e., $F(s) = \frac{K_1}{s+\alpha-j\beta} + \frac{K_1}{s+\alpha+j\beta}$

The time response due to this pair of poles is given by,

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{2K_1(s+\alpha)}{(s+\alpha)^2+\beta^2}\right] = 2K_1 e^{-\alpha t}\cos\beta t$$

When t is sufficiently large, f(t) approaches zero and therefore the system is stable. The time response is shown in figure 5.3 (b).



(a) Complex poles in the left half of s-plane (b) The corresponding time response

Figure 5.3:

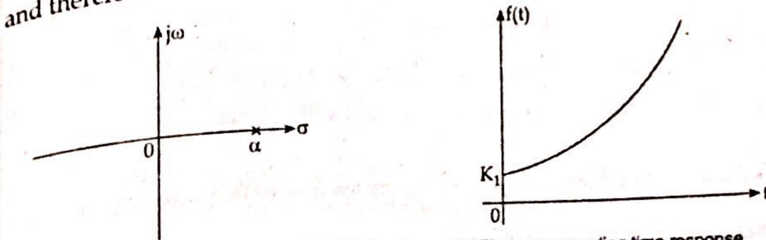
If the pair of poles were multiple-order poles in the left half of the s-plane the time response would be of the form of $f(t) = 2K_1 t^r e^{-\alpha t}\cos\beta t$

Here, also f(t) approaches zero for large t and hence the system is stable.

Thus, we have seen that both simple and complex conjugate pair of poles (either simple-order or multiple order) of a network function in the left half of the s-plane represent stable response.

c) Poles on the positive real axis

If the network function has a simple pole on the positive real axis as shown in figure 5.4 (a), the time response is of the form $f(t) = K_1 e^{\alpha t}$. This response increases exponentially with t as shown in figure 5.4 (b) and therefore the response is unstable.



(a) Pole on the positive real axis of the s-plane (b) The corresponding time response

Figure 5.4:

It can be shown that multiple-order poles on the positive real axis of the s-plane also represent the unstable state.

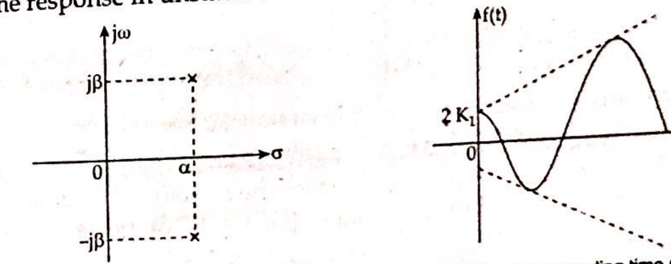
d) Complex-poles in the right half of the s-plane

Let us consider a pair of first order complex poles in the right half of s-plane at $s = \alpha \pm j\beta$ as shown in figure 5.5 (a). The time response due to this pair of poles is given by,

$$f(t) = \mathcal{L}^{-1}[F(s)] = \left[\frac{K_1}{s-\alpha_1-j\beta} + \frac{K_1}{s-\alpha+j\beta}\right]$$

or, $$f(t) = \mathcal{L}^{-1}\left[\frac{2K_1(s-\alpha)}{(s-\alpha)^2+\beta^2}\right] = 2K_1 e^{\alpha t}\cos\beta t$$

It is seen that f(t) is an exponentially increases sinusoidal and therefore the response in unstable as shown in figure 5.5 (b).



(a) Complex poles in the right half of the s-plane (b) The corresponding time response

Figure 5.5:

It can be shown that multiple order poles in the right half of the s-plane also represent unstable responses. Thus, we have seen that poles in the right half of the s-plane always represent instability (or unstable responses).

e) Poles at the origin

Consider a pole at the origin as shown in figure 5.6 (a). The impulse response is of the form

$$f(t) = \pounds^{-1}[F(s)] = \pounds^{-1}\left[\frac{K_1}{s}\right] = K$$

Since f(t) is a constant for all values of t, the impulse response is marginally stable. The time response is shown in figure 5.6 (b).

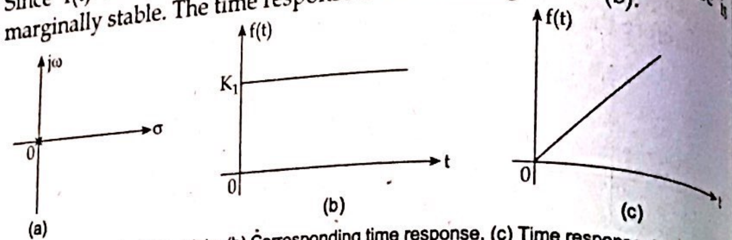


Figure 5.6: (a) Pole at the origin, (b) Corresponding time response, (c) Time response due to second order poles at the origin.

If there is a multiple-order poles at the origin, the time response would be of the form of $f(t) = K_1 t^r$. This relation shows that response is unstable. The response due to second order poles at origin is shown in figure 5.6 (c).

f) **Poles on jω-axis**

Consider a pair of first order conjugate complex poles on the imaging axis as shown in figure 5.7 (a). The time response is given by,

$$f(t) = \pounds^{-1}[F(s)]\ \pounds^{-1}\left[\frac{K_1}{s+j\beta} + \frac{K_1}{s-j\beta}\right] = \pounds^{-1}\left[\frac{2K_1 s}{s^2+\beta^2}\right] = 2K_1\cos\beta t$$

This function is shown in figure 5.6 (b), since the function f(t) oscillates for all time, it is said to be marginally stable.

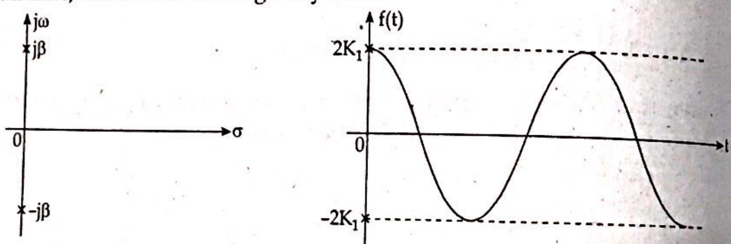


Figure 5.7: (a) poles on imaginary axis and (b) the corresponding time responses.

5.6 ROUTH HURWITZ STABILITY CRITERION

As discussed earlier, the necessary condition for the system to be stable is that the roots of the characteristic equation [D(s) = 0] or poles of the transfer function $T(s) = \frac{N(s)}{D(s)}$ have negative real parts. This ensures that the impulse response will decay exponentially with time. And if the system has some simple poles with real parts equal to zero, but none with positive real parts, the system is said to be marginally stable. On the other hand, a system having any pole with positive real parts is said to be unstable.

NOTE: If the system has multiple-order poles with real parts equal to zero (or on the jω-axis), the system is said to be unstable.

Routh-Hurwitz stability criterion is method for determining number of roots of characteristic equation *i.e.*, poles of transfer function with negative real parts, zero real parts and positive real parts and hence system stability.

Consider that the characteristic equation of a system is of the form

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + \dots\dots\dots a_1 s + a_0 = 0 \qquad (1)$$

Where all the coefficients a_i; i = 0, 1, 2, ..., n are real. In order that equation (1) not have roots with positive real parts, it is necessary that the following conditions hold:

a) All the coefficients of the equation have the same sign.
b) None of the coefficients vanishes *i.e.*, no missing term between that of highest and lowest degrees.

The two necessary conditions for equation (1) to have no roots with positive real parts can easily be checked by inspection of the equation. However, these conditions are not sufficient, for it is quite possible that an equation with all its coefficient of the same sign and non zero still may not have all the roots with negative real roots.

The necessary and sufficient condition that all roots of equation (1) have negative real parts if all the elements of the first column of the Routh's tabulation are of the same sign. The number of changes of signs in the elements of the first column equals the number of roots with positive real parts.

Routh's Tabulation

The first step in the Routh-Horwitz criterion is to arrange the coefficients of the equation (1) into two rows. The first row consists of the first, third, fifth, ..., coefficients and the second row consists of the second, fourth sixth, ..., coefficients, all counting from the highest-order term, as shown in the following tabulation:

$$\begin{matrix} a_n & a_{n-2} & a_{n-4} & \dots\dots\dots \\ a_{n-1} & a_{n-3} & a_{n-5} & \dots\dots\dots \end{matrix}$$

The next step is to form the following array of numbers by the indicated operations, illustrated here for a sixth order equation.

$$a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

s^6	a_6	a_4	a_2	a_0
s^5	a_5	a_3	a_1	0
s^4	$\frac{a_5a_4 - a_6a_3}{a_5} = A$	$\frac{a_5a_2 - a_6a_1}{a_5} = B$	$\frac{a_5a_0 - a_6.0}{a_5} = a_0$	0
s^3	$\frac{Aa_3 - a_5B}{A} = C$	$\frac{Aa_1 - a_5a_0}{A} = D$	$\frac{A\cdot0 - a_5\cdot0}{A} = 0$	0
s^2	$\frac{BC - AD}{C} = E$	$\frac{Ca_0 - A\cdot0}{C} = a_0$	$\frac{C\cdot0 - A\cdot0}{C} = 0$	0
s	$\frac{ED - Ca_0}{E} = F$	0	0	0
s^0	$\frac{Fa_0 - E\cdot0}{F} = a_0$	0	0	0

The array above is called the Routh's tabulation or Routh's array. The last row of the Routh's tabulation should always be the s^0 row.

Once the Routh's tabulation has been completed, the last step in the application of the criterion is to investigate the signs of the coefficients in the first column of the tabulation, which contains the information on the roots of the equation as discussed earlier.

Special case when Routh's tabulation terminates prematurely

Depending on the coefficients of the equation, sometimes following difficulties may occur that prevent Routh's tabulation from completing properly;

a) The first element in any row of Routh's tabulation is zero, but the others are not.

b) All elements in one row of Routh's tabulation is zero.

Case 1:

If a zero appears in the first element of a row, the elements in the next row will all become infinite and Routh's tabulation cannot be continue. To remedy the situation, we replace the zero element in the first column by an arbitrary small positive number ϵ and then proceed with Routh's tabulation.

Case 2:

If all the elements in one row of Routh's tabulation are zero before the tabulation is properly terminated, it indicates that one or more of the following conditions may exist.

i) The equation has at least one pair of real roots with equal magnitude but opposite sings (*Example*; $s = \pm 3$).

ii) The equation has one or more pairs of imaginary roots (*Example*; $s = \pm j1, s = \pm j2, \pm j5$)

iii) The equation has pairs of complex conjugate roots forming symmetry about the origin of s-plane (*Example*; $s = -1 \pm j1, s = 1 \pm j1$)

The situation with the entire row of zeros can be remedied by using the auxiliary equation $A(s) = 0$, which is formed from the coefficients of the row just above the row of zeros in Routh's tabulation. The auxiliary equation is always even polynomials. The roots of the auxiliary equation also satisfy the original equation. Thus, by solving the auxiliary equation, we also get some of the roots of the original equation. To continue with Routh's tabulation when a row of zero appears, we conduct the following steps;

i) From the auxiliary equation $A(s) = 0$ by use of the coefficients from the row just preceding the row of zeros.

ii) Take the derivatives of the auxiliary equation with respect to s; this gives $\frac{dA(s)}{ds} = 0$.

iii) Replace the row of zeros with the coefficients of $\frac{dA(s)}{ds} = 0$.

iv) Continue with Routh's tabulation in the usual manner with the newly formed row of coefficients replacing the row of zeros.

v) Interpret the changes of signs, if any, of the coefficients in the first column of Routh's tabulation in the usual manner.

5.6.1 Deficiencies of Routh Hurwitz Criterion

a) It cannot be applied to discrete-time system.

b) It gives the information about roots of characteristic equation only with respect to the left half or right half of the s-plane *i.e.*, it does not give the information about the roots on the jω-axis (stability boundary).

c) It is valid only if the characteristic equation is algebraic with real coefficients. If any of the coefficients is complex, or if the equation is not algebraic, such as containing exponential or sinusoidal function of s, the Routh's Hurwitz criterion is simply cannot be applied.

5.7 TIME-DOMAIN RESPONSE FROM POLE-ZERO PLOT

The time-domain response can be obtained from the pole zero plot of a network function. Consider a network function given by,

$$H(s) = \frac{N(s)}{D(s)} = K\frac{(s - z_1)(s - z_2) \ldots\ldots (s - z_n)}{(s - p_1)(s - p_2) \ldots\ldots (s - p_m)}$$

where, $z_1, z_2 \ldots\ldots, z_n$ are the zeros and $p_1, p_2, \ldots\ldots, p_m$ are the poles of the function H(s).

Assume that the poles zeros are distinct. Using partial fractional expansion, H(s) can be written as,

$$H(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \ldots\ldots + \frac{K_n}{s - p_m}$$

A particular coefficient K_i can be found as follows.

$$K_1 = [(s - p_i) H(s)]_{s=p_i} ; i = 1, 2, 3, \ldots\ldots, m$$

or, $$K_1 = \frac{K(p_i - z_1)(p_i - z_2) \ldots\ldots (p_i - z_n)}{(p_i - p_1)(p_i - p_2) \ldots\ldots (p_i - p_{i-1})(p_i - p_{i+1}) \ldots\ldots (p_i - p_m)}$$

Here p_i, z_n and p_m are complex numbers. Above equation contain factors lime $(p_i - z_n)$ and $(p_i - p_m)$. The difference of two complex numbers is also a complex number.

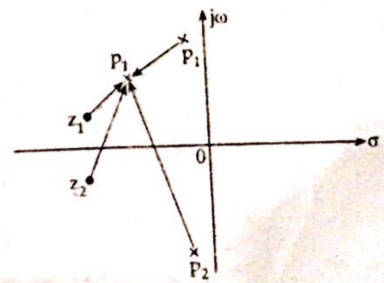


Figure 5.8

Hence the complex number $(p_i - z_i)$ is the directed line from z_1 to p_i and the complex number $(p_i - z_i)$ is the directed line from z_2 to p_i etc. Similarly, the complex number $(p_i - p_1)$ will be directed line from p_1 to p_i

etc as shown in figure 5.8. Each line is expressed in its polar from magnitude and phase.

Thus, let

$p_i - z_1 = r_{1i} \angle\alpha_{1i}$

$p_i - z_2 = r_{2i} \angle\alpha_{2i}$

$\vdots$

$\vdots$

$p_i - z_n = r_{ni} \angle\alpha_{ni}$

and, $p_i - p_1 = R_{1i} \angle\beta_{1i}$

$p_i - p_2 = R_{2i} \angle\beta_{2i}$

$\vdots$

$\vdots$

$p_i - p_m = R_{mi} \angle\beta_{mi}$

$$\therefore \quad K_i = K \frac{r_{1i}\, r_{2i} \,........\, r_{ni}}{R_{1i}\, R_{2i} \,.......\, R_{mi}} \angle(\alpha_{1i} + \alpha_{2i} + ... + \alpha_{ni}) - (\beta_{1i} + \beta_{2i} + ... + \beta_{mi})$$

This equations shows that a particular coefficient K_i can obtained from,

$$K_i = K = \frac{\text{Product of all the directed lines from all zeros to } p_i}{\text{Product of all directed lines from other remaining poles to } p_i}$$

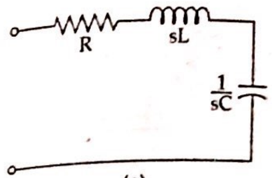
Similarly, the other constants $K_1, K_2, ..., K_{i-1}, K_{i+1}, ..., K_m$ can be calculated. The time-domain response can obtained by taking the inverse laplace transform.

$$h(t) = \pounds^{-1}[H(s)] = \pounds^{-1}\left[\frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + ... + \frac{K_m}{s - p_m}\right]$$

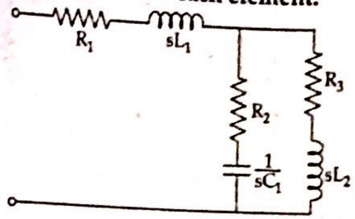
It is instead that the above graphical method is applicable only for all non-repeated poles.

SOLVED NUMERICAL EXAMPLES

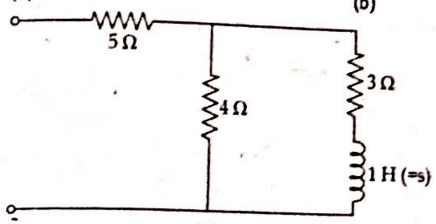
1. Obtain the driving point impedance function or transform impedance Z(s) for the networks shown in figure (a) to (c), in which transformed impedance marked for each element.



(a)



(b)



(c)

Solution:

The driving point impedance function Z(s) for the series R-L-C network of figure (a) is,

$$Z(s) = R + sL + \frac{1}{sC} = \frac{RCs + LCs^2 + 1}{sC}$$

Network shown in figure (b) contains series parallel branches only, hence,

$$Z(s) = (R_1 + sL_1) + \left[R_2 + \frac{1}{sC_1}\right] [(R_3 + sL_2)]$$

$$= (R_1 + sL_1) + \left[\frac{\left(R_2 + \frac{1}{sC_1}\right) \cdot (R_3 + sL_2)}{R_2 + \frac{1}{sC_1} + R_3 + sL_2}\right]$$

$$= \frac{a_3 s^3 + a_2 s^2 + a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}$$

where, $a_3 = L_1L_2C_1$; $b_2 = L_2C_1$

$a_2 = [(R_2 + R_3) L_1 + (R_1 + R_2)L_2] C_1$

$b_1 = (R_2 + R_3) C_1$

$a_1 = (L_1 + L_2) + (R_1R_2 + R_2R_3 + R_3R_1) C_1$

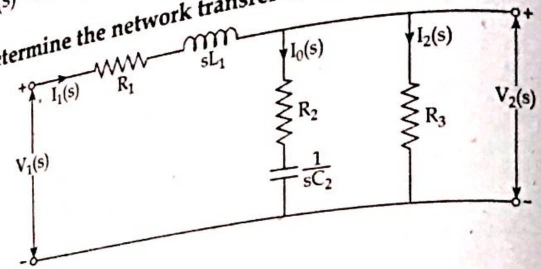
$b_0 = 1$

$a_0 = R_1R_3$

From figure (c), the impedance function for 3 Ω, 1 H series circuit is $Z_1(s) = 3 + s$. The branch $Z_1(s)$ is in parallel with branch $Z_2(s) = 4\Omega$.

Hence, $Z(s) = 5 + [(3+s) \parallel 4] = 5 + \dfrac{(3+s)\times 4}{(3+s)+4} = \dfrac{9s+47}{s+7} = 9\left(\dfrac{s+\frac{47}{9}}{s+7}\right)$

2. **Determine the network transfer function.**



Solution:

The network equations are;

$$V_2(s) = R_3 I_2(s) = \left(R_2 + \frac{1}{sC_2}\right) I_0(s) \quad (1)$$

or, $$I_0(s) = \frac{V_2(s)}{R_2 + \frac{1}{sC_2}} \quad (2)$$

$$I_1(s) = I_2(s) + I_0(s) \quad (3)$$

$$V_1(s) = V_2(s) + (R_1 + sL_1)\, I_1(s) \quad (4)$$

Transfer impedance function,

$$Z_{21}(s) = \frac{V_2(s)}{I_1(s)}$$

$$V_2(s) = \left(R_2 + \frac{1}{sC_2}\right) I_0(s)$$

$$I_1(s) = I_2(s) + I_0(s) = I_0(s)\left[\frac{I_2(s)}{I_0(s)} + 1\right]$$

$$= I_0(s)\left[\frac{R_2}{R_3} + \frac{1}{sC_2R_3} + 1\right] \quad \text{[From equation (1)]}$$

$$\therefore \quad Z_{21}(s) = \frac{R_3\,(sC_2R_2 + 1)}{C_2(R_2 + R_3)s + 1}$$

Now, Transfer admittance function, $Y_{21}(s) = \dfrac{I_2(s)}{V_1(s)}$

$$V_1(s) = V_2(s) + (R_1 + sL_1)\, I_1(s)$$

$$= R_3 I_2(s) + (R_1 + sL_1)\,[I_2(s) + I_0(s)] \quad \text{[From equation (1) and (3)]}$$

$$= I_2(s)\left[R_3 + (R_1 + sL_1) + (R_1 + sL_1)\left(\frac{R_3}{R_2 + \frac{1}{sC_2}}\right)\right]$$

$$= I_2(s)\left[\frac{(R_3 + R_1 + sL_1)(sC_2R_2 + 1) + (R_1 + sL_1)\,R_3\,sC_2}{sC_2R_2 + 1}\right]$$

$$\therefore \quad Y_{21}(s) = \frac{sC_2R_2 + 1}{L_1C_2(R_2 + R_3)\,s^2 + [C_2(R_1R_2 + R_2R_3 + R_3R_1 + L_1]s + (R_1 + R_3))}$$

and, Voltage transform function,

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{V_2(s)}{I_1(s)} \cdot \frac{I_2(s)}{V_1(s)} \cdot \frac{I_1(s)}{I_2(s)}$$

$$= Z_{21}(s) \cdot Y_{21}(s) \cdot \frac{I_1(s)}{I_2(s)}$$

or, $$I_1(s) = I_2(s) + I_0(s) = I_2(s)\left[1 + \frac{I_0(s)}{I_2(s)}\right]$$

$$= I_2(s)\left[1 + \frac{R_3}{R_2 + \frac{1}{sC_2}}\right]$$

$$= I_2(s)\left[\frac{[(sR_2C_2 + 1)] + sC_2R_3}{(sR_2C_2 + 1)}\right]$$

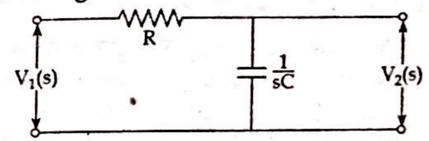
$$\therefore \quad \frac{I_1(s)}{I_2(s)} = \frac{C_2(R_2 + R_3)s + 1}{sR_2C_2 + 1}$$

$$\therefore \quad G_{21}(s) = \frac{R_3(sC_2R_2 + 1)}{L_1C_2(R_2 + R_3)\,s^2 + [C_2(R_1R_2 + R_2R_3 + R_3R_1) + L_1]\,s + (R_1 + R_3)}$$

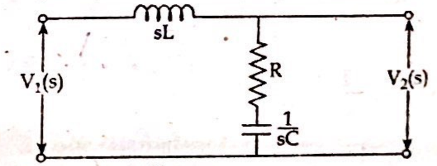
and, Current transfer function, $\alpha_{21}(s) = \dfrac{I_2(s)}{I_1(s)} = \dfrac{sR_2C_2 + 1}{C_2(R_2 + R_3)_s + 1}$

3. **Find the voltage transfer function of**

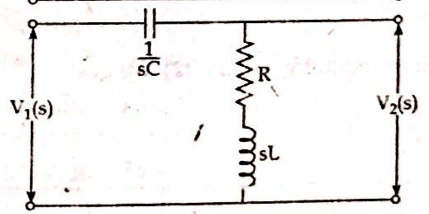
(a)



(b)



(c)



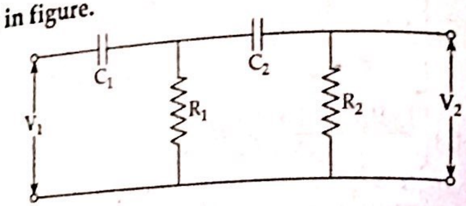
Solution:

a) $$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{sRC + 1}$$

b) $G_{21}(s) = \dfrac{R + \frac{1}{sC}}{R + \frac{1}{sC} + sL} = \dfrac{sRC + 1}{s^2 LC + sRC + 1}$

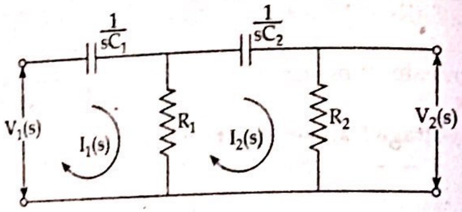
c) $G_{21}(s) = \dfrac{R + sL}{R + sL + \frac{1}{sC}} = \dfrac{sC(R + sL)}{s^2LC + sRC + 1}$

4. **Find the expression of voltage transfer functions for the network shown in figure.**



Solution:

Redrawing the circuit in s-domain,



From first (leftmost)loop,

$$V_1(s) = I_1(s)\left[R_1 + \frac{1}{sC_1}\right] - R_1 I_2(s) \quad (1)$$

From second loop,

$$-R_1 I_1(s) + \left(R_1 + R_2 + \frac{1}{sC_2}\right) I_2(s) = 0 \quad (2)$$

From third loop,

$$V_2(s) = R_2 I_2(s)$$

Using cramer's rule, from equation (1) and (2), we get,

$$I_2(s) = \frac{\begin{vmatrix} R_1 + \frac{1}{sC} & V_1(s) \\ -R_1 & 0 \end{vmatrix}}{\begin{vmatrix} R_1 + \frac{1}{sC_1} & -R_1 \\ -R_1 & R_1 + R_2 + \frac{1}{sC_2} \end{vmatrix}} = \frac{R_1 V_1(s)}{\left(R_1 + \frac{1}{sC_1}\right)\left(R_1 + R_2 + \frac{1}{sC_2}\right) - R_1^2}$$

$$= \frac{R_1 C_1 C_2 s^2 V_1(s)}{(R_1 + R_2)C_2 s + 1 + R_1 C_1 s(R_2 C_2 s + 1)}$$

From equation (3), we have,

$$V_2(s) = R_2\left[\frac{R_1 C_1 C_2 s^2 \cdot V_1(s)}{(R_1 + R_2) C_2 s + 1 + R_1 C_1 s(R_2 C_2 s + 1)}\right]$$

$$\text{Hence, } G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \left[\frac{R_1 R_2 C_1 C_2 s^2}{(R_1 + R_2) C_2 s + 1 + R_1 C_1 s (R_2 C_2 s + 1)}\right]$$

5. **Obtain the pole-zero location for the function**

$$T(s) = \frac{(2s + 4)(s + 4)}{s(s + 1)(s + 3)}$$

Solution:

The poles are at $s = 0, s = -1, s = -3$

The zeros are at $s = -2, s = -4, s = \infty$

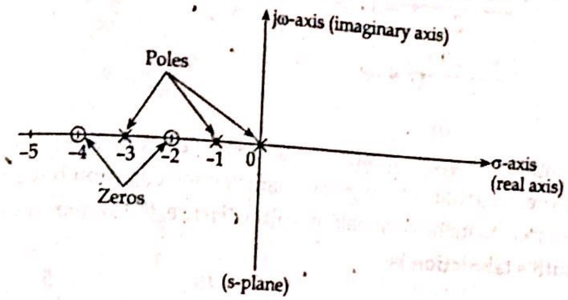


Figure: Pole zero diagram

6. **Check whether given functions are suitable in representing the driving point immittance functions or not.**

a) $Z(s) = \dfrac{4s^4 + s^2 - 3s + 1}{s^3 + 2s^2 + 2s + 10}$ b) $Y(s) = \dfrac{15(s^3 + 2s^2 + 3s + 4)}{s^4 + 8s^3 + 6s^2}$

c) $Z(s) = \dfrac{s^2 + s + 2}{2s^2 + s + 1}$

Solution:

a) No; one coefficient is missing and one is negative.

b) No; the lowest degrees of N(s) and D(s) differ by two.

c) Yes; all conditions are satisfied.

7. **Check whether given functions are suitable in representing the transfer functions or not.**

a) $G_{21}(s) = \dfrac{3s + 2}{5s^3 + 4s^2 + 1}$ b) $\alpha_{21}(s) = \dfrac{2s^2 + 5 + 1}{s + 7}$

c) $Z_{21}(s) = \dfrac{1}{s^3 2s}$ d) $G_{21}(s) = \dfrac{2s^2 + 5}{3s^2 + 9s + 1}$

Solution:

a) No; coefficient is missing in polynomial D(s).

b) No; the degree of N(s) is greater than D(s).

c) Yes; all conditions are satisfied.

d) Yes; all conditions are satisfied.

8. Determine the stability of the systems whose characteristics equation are;

a) $s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$

b) $s^4 + 8s^3 + 18s^2 + 16s + 5 = 0$

Solution:

a) Routh's tabulation is,

s^4	1	3	5
s^3	2	4	0
s^2	$\frac{2\times3-4\times1}{2}=1$	$\frac{2\times5-0\times1}{2}=5$	
s^1	$\frac{1\times4-2\times5}{1}=-6$		
s^0	5		

As the number of change of sign in the first column of the tabulation is two, therefore, two roots of the given characteristic equation have positive real parts or located in the right half of s-plane. Hence, the system is unstable.

b) Routh's tabulation is

s^4	1	18	5
s^3	8	16	0
s^2	$\frac{8\times18-16\times1}{8}=16$	$\frac{8\times5-1\times0}{8}=5$	
s^1	$\frac{16\times16-8\times5}{16}=27$	0	
s^0	5		

As the elements of the first column are all positive hence all the roots of the characteristic equation located in the left half s-plane or having negative real parts. Hence, the system is stable.

9. Calculate the range of K for which the systems is given by characteristic equation.

$s^3 + 7s^2 + 10s + 10K = 0$ is stable

Solution:

Routh's tabulation is,

s^3	1	10
s^1	7	10K
s^1	$\frac{70-10K}{7}$ 10	
s^0	10K	

For stability, there should be no sign changes in the first column *i.e.,*

$\frac{70-10K}{7} > 0$ or $K < 7$ and $10K > 0$ or $K > 0$

Hence the range of k is $0 < K > 7$.

10. Check the stability of the system with characteristic equation.

$2s^2 + s^4 + 6s^3 + 3s^2 + s + 1 = 0$

Solution:

The Routh's tabulation is,

s^5	2		
s^4	1	6	1
s^3	$\frac{1\times6-2\times3}{1}0=\epsilon$	3	1
s^2	$\frac{3\epsilon+1}{\epsilon}$	$\frac{1-2}{1}=-1$	
s^1	$\frac{\frac{-3\epsilon-1}{\epsilon}-\epsilon}{\frac{3\epsilon+1}{\epsilon}}$		
s^0	1		

As $\epsilon \to 0$, the elements of s^1 row become $\frac{-3\epsilon-1-\epsilon^2}{3\epsilon+1}$ (or negative); for ϵ is small and positive.

Hence, there are two sign changes in the first column elements. This indicates that there are two roots in the right half s-plane. Hence, the system is unstable.

11. Check the stability of the system with characteristic equation.

$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$

Solution:

The Routh's tabulation is,

s^5	1	8	7
s^4	4	8	4
s^3	$\frac{4\times8-8\times1}{4}=6$	$\frac{4\times7-4\times1}{4}=6$	0
s^2	$\frac{6\times8-6\times4}{6}=4$	$\frac{6\times4-0\times4}{6}=4$	
s^1	$\frac{4\times6-4\times6}{4}=0$	0	

Since a row of zeros appear prematurely, we form the auxiliarly equation using the coefficients of the s^2 row.

$$A(s) = 4s^2 + 4 = 0$$

The derivative of A(s) with respect to s *i.e.,* $\frac{dA(s)}{ds} = 8s + 0 = 0$

From which the coefficients 8 and 0 replace the zeros in the s^1 row of the original tabulation. The remaining portion of the Routh's tabulation is,

s^1	8	$0 \rightarrow$ Coefficient of $\frac{dA(s)}{ds}$
s^0	4	

Since there are no sign changes in the first column in the entire Routh's tabulation, the given characteristic equation does not have any root in the right half s-plane. Solving the auxiliary equation, $4s^2 + 4 = 0$ gives two roots at $s = \pm j$ which are also two of the roots of given characteristic equation. Thus, the characteristic equation has two roots on the jω-axis. Hence, the system is marginally stable.

12. **If $H(s) = \frac{s(s+1)}{(s+4)(s^2+6s+18)}$; find h(t) using the pole-zero diagram of the function.**

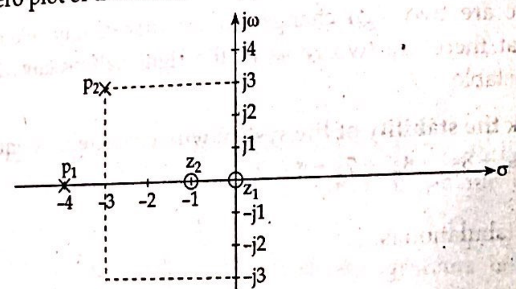
Solution:

In factored form,

$$H(s) = \frac{8(s+1)}{(s+4)(s+3+j3)(s+3-j3)}$$

There are two zeros, one at z = 0 and the other at z = –1 and three poles, $p_1 = -4$, $p_2 = -3 - j3$ and $p_3 = -3 + j3$.

The pole-zero plot of the function H(s) is shown in figure.



$$H(s) = \frac{K_1}{s+4} + \frac{K_2}{s+3+j3} + \frac{K_3}{s+3-j3}$$

For the pole $p_1 = -4$

$$K_1 = \frac{K\text{ (Product of all the directed line from all zeros to } p_1)}{\text{Product of all the directed lines from other remaining poles to } p_1}$$

Here, K = 1,

Hence,

$$K_1 = \frac{(p_1 - z_1)(p_1 - z_2)}{(p_1 - p_2)(p_1 - p_3)} = \frac{(-4-0)(-4+1)}{(-4+3+j3)(-4+3-j3)} = \frac{12}{(-1)^2 + (3)^2} = 1.2$$

For the pole $p_2 = -3 - j3$

$$K_2 = \frac{(p_2 - z_1)(p_2 - z_2)}{(p_2 - p_1)(p_2 - p_3)} = \frac{(-3-j3-0)(-3-j3+1)}{(-3-j3+4)(-3-j3+3-j3)}$$

$$= \frac{(-3-j3)(-2-j3)}{(1-j3)(-j6)}$$

$$= \frac{(6-9)+j(6+9)}{-18-j6}$$

$$= \frac{-3+j15}{-18-j6} = \frac{1-j5}{6+j2}\left[\frac{6-j2}{6-j2}\right]$$

$$= \frac{-4-j32}{36+4} = \frac{1}{10}(-1-j8)$$

For the pole $p_3 = -3 + j3$

Since the poles p_2 and p_3 are conjugate pairs, K_3 is the complex conjugate of K_2.

i.e., $K_3 = K_2^* = \frac{1}{10}(-1+j8)$

$$H(s) = \frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \frac{K_3}{s-p_3}$$

Now, taking laplace inverse of H(s), we get,

$$h(t) = \mathcal{L}^{-1}[H(s)]$$

$$= K_1 e^{P_1 t} + K_2 e^{P_2 t} + K_3 e^{P_3 t}$$

$$= 1.2\, e^{-4t} + \frac{1}{10}(-1-j8)\, e^{(-3-j3)t} + \frac{1}{10}(-1+j8)\, e^{(-3+j3)t}$$

$$= 1.2e^{-4t} + e^{-3t}\left[\frac{-1}{10}(e^{j3t} + e^{-j3t}) + \frac{8}{10} j\,(e^{j3t} - e^{-j3t})\right]$$

$$= 1.2e^{-4t} + e^{-3t}\left[\frac{-1}{10}(2\cos 3t) - \frac{8}{10}(2\sin 3t)\right]$$

$$\left[\because \cos at = \frac{e^{jat} + e^{-jat}}{2} \text{ and } \sin at = \frac{e^{jat} - e^{-jat}}{2}\right]$$

$$\therefore \quad h(t) = 1.2\, e^{-4t} \frac{-1}{5} e^{-3t}[\cos 3t + 8\sin 3t]$$

BOARD EXAMINATION SOLVED QUESTIONS

1. **From pole zero plot, obtain the time domain response for i(t) using pole zero method.** **[2019/Spring, 2018/Spring]**

$$I(s) = \frac{3s}{(s+2)(s+3)}$$

Solution:

Given that;

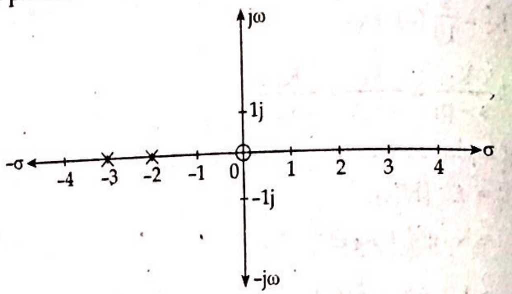
$$I(s) = \frac{3s}{(s+2)(s+3)}$$

The zeros lies at $s = 0$ and $s = \infty$ and the poles lies at

$s + 2 = 0$ and $s + 3 = 0$

$\therefore$ $s = -2$ $\therefore$ $x = -3$

The pole-zero plot is;



Let, $\frac{3s}{(s+2)(s+3)} = \frac{K_1}{s+2} + \frac{K_2}{s+3}$

We have,

$$K_1 = [I(s)(s+2)]|_{s=-2} = \frac{3s}{(s+2)(s+3)} \times (s+2)\Big|_{s=-2} = \frac{3\times(-2)}{(-2)+3} = -6$$

Similarly,

$$K_2 = [I(s)(s+)]|_{s=-3} = \frac{3\times(-3)}{(-3+2)} = -9$$

so, $I(s) = \frac{-6}{s+2} + \frac{(-9)}{s+3}$

Taking inverse laplace transform of I(s), we get,

$i(t) = -6\,e^{-2t} - 9\,e^{-3t}$ A

Hence, time domain response for $i(t) = -6\,e^{-2t} - 9\,e^{-3t}$ A.

2. **Using Routh Hurwitz criteria, check the stability of the system and how many roots of the following polynomial are in the right half of s-plane and left hand of s-plane.** **[2019/Fall]**

$$Q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$$

Solution:

Given that;

$Q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63$

The Routh-Hurwitz table is,

s^5	1	4	3
s^4	1	24	63
s^3	-20	-60	
s^2	17.5	63	
s^1	12	0	
s^0	63		

Since first column of R-H table consists the element with change in sign, the system is unstable. There are two sign changes (*i.e.,* 1 to -20 and -20 to + 17.5), so,

Number of roots on right half of s-plane = 2

Number of roots on left half of s-plane = 5 − 2 = 3

[Hint: Total number of roots = Maximum power of s. Here, maximum power of s = 5]

3. **Form routh array for the following characteristic equation and state whether the system is stable or not.**

$$Q(s) = 5s^5 + 3s^4 + 2s^3 + 2s^2 + s + 1$$ **[2018/Fall]**

Solution:

Given that;

$Q(s) = 5s^5 + 3s^4 + 2s^3 + 2s^2 + s + 1$

The routh-Hurwitz table is,

s^5	5	2	1
s^4	3	2	1
s^3	$\frac{-4}{3}$	$\frac{-2}{3}$	
s^2	$\frac{1}{2}$	1	
s^1	2	0	
s^0	1		

Since there is sign change in the 3rd row of 1st column, $\left(i.e., 3 \text{ to } -\frac{4}{3}\right)$ and $\left(\frac{-4}{3} \text{ to } \frac{1}{2}\right)$, we can say that the given system is unstable. (Two roots in right half s-plane)

4. **Find the number of poles in the given characteristic equation and determine the range of K for the system to be stable, where, K is the adjustable loop gain.**

$$q(s) = s^3 + 2s^2 + 4s + K$$ **[2017/Spring, 2011/Fall, 2015/Fall]**

Solution:

Given that;

$$q(s) = s^3 + 2s^2 + 4s + K$$

R-H table:

s^3	1	4
s^2	2	K
s^1	$\left(\frac{8-K}{2}\right)$	0
s^0	K	

For system to be stable,

$$\frac{8-K}{2} > 0 \text{ and } K > 0$$

or, $4 - \frac{K}{2} > 0$

or, $4 > \frac{K}{2}$

or, $8 > K$

Hence range of K should be $0 < K < 8$ to be system stable.

5. Find the value K for which $s^3 + 7s^2 + 10s + 10K = 0$ is stable. **[2017/Fall]**

Solution:

Given that;

$$s^3 + 7s^2 + 10s + 10K = 0$$

The R-H table is

s^3	1	10
s^2	7	10K
s^1	$\left(\frac{70-10K}{7}\right)$	0
s^0	10K	

For system to be stable, there should be no sign change in 1st column, *i.e.,*

$$\left(\frac{70-10K}{7}\right) > 0 \quad \text{and} \quad 10K > 0$$

or, $70 - 10K > 0$ $\quad K > 0$

or, $70 > 10K$

$\therefore$ $K < 7$

So, for system to be stable K should be greater than 0 and less than 7, *i.e.,* $0 < K < 7$.

6. Plot the poles and zeros in s-plane for the network function,

$$V(s) = \frac{10s\,(s+1)}{(s+3)\,(s+2)\,(s+5)}$$

Also obtain V(t) using pole-zero diagram and determine value of V(t) for t = 1 sec. **[2016/Spring]**

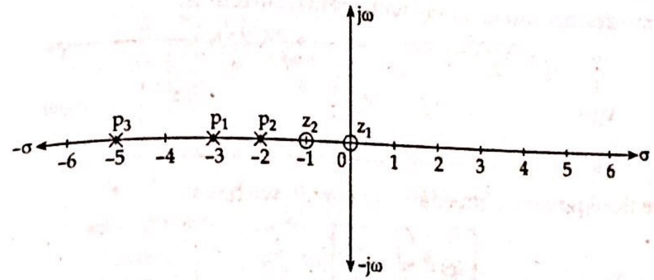
Solution:

Given that;

$$V(s) = \frac{10\,s(s+1)}{(s+2)\,(s+3)\,(s+5)}$$

The poles are at $p_1 = -3$, $p_2 = -2$ and $p_3 = -5$.

The zeros are at $z_1 = 0$ and $z_2 = -1$



Now, For the pole, $p_1 = -3$

$$\therefore \quad A = 10 \times \frac{(p_1 - z_1)\,(p_1 - z_2)}{(p_1 - p_2)\,(p_1 - p_3)} = -30$$

For the pole, $p_2 = -2$

$$B = 10 \times \frac{(p_2 - z_1)\,(p_2 - z_2)}{(p_2 - p_3)\,(p_2 - p_1)}$$

$$= 10 \times \frac{(-2-0)\,(-2+1)}{(-2+5)\,(-2+3)}$$

$$= 10 \times \left(\frac{-2}{3}\right) \times \left(\frac{-1}{1}\right) = \left(\frac{20}{3}\right)$$

For the pole, $p_3 = -5$,

$$C = \frac{10 \times (p_3 - z_1)\,(p_3 - z_2)}{(p_3 - p_1)\,(p_3 - p_2)}$$

$$= \frac{10 \times (-5-0)\,(-3+1)}{(-5+2)\,(-5+3)}$$

$$= 10 \times \frac{(-5) \times (-2)}{(-3) \times (-2)} = \frac{50}{3}$$

i.e., $$V(s) = \frac{A}{(s+3)} + \frac{B}{(s+2)} + \frac{C}{(s+5)} = \frac{-30}{(s+3)} + \frac{20}{3(s+2)} + \frac{50}{3(s+5)}$$

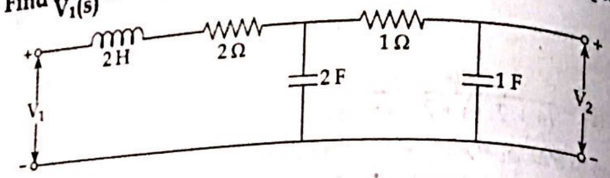
Taking inverse laplace transform, we get,

$$V(t) = \left(-30\,e^{-3t} + \frac{20}{3}\,e^{-2t} + \frac{50}{3}\,e^{-5t}\right) V$$

At t = 1 sec,

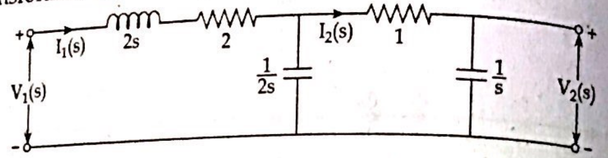
$$\therefore \quad V(1) -30\,e^{-3} + \frac{20}{3}\,e^{-2} + \frac{50}{3}\,e^{-5} = -1.493 + 0.9022 + 0.1122 = -0.478 \text{ Volts}$$

7. **Find $\frac{V_2(s)}{V_1(s)}$ for the following network.** **[2016/Fall]**



Solution:

The transformed circuit along with branch current is,



If Z be the equivalent impedance of circuit, we have,

$$Z(s) = 2s + 2 + \left[\frac{\frac{1}{2s} \times \left(1 + \frac{1}{s}\right)}{\frac{1}{2s} + 1 + \frac{1}{s}}\right]$$

$$= 2 + 2s + \frac{\left(1 + \frac{s}{2s^2}\right)}{\left(\frac{1 + 2s + 2}{2s^2}\right)}$$

$$= 2 + 2s + \left(\frac{1 + s}{2s + 3}\right)$$

$$= \frac{(2 + 2s)(2s + 3) + (1 + s)}{(2s + 3)}$$

$$= \frac{(4s + 6 + 4s^2 + 6s) + (1 + s)}{(2s + 3)}$$

$$= \frac{4s^2 + 11s + 7}{2s + 3}$$

We have, from figure

$$V_2(s) = I_2(s) \times \frac{1}{s} \quad (1)$$

Using current divider rule,

$$I_2(s) = \frac{\frac{1}{2s}}{\left(\frac{1}{2s} + 1 + \frac{1}{s}\right)} \times I_1(s)$$

or, $$I_2(s) = \frac{s}{(2s + 3)} I_1(s)$$

Then equation (1) becomes,

$$V_2(s) = \frac{s}{2s + 3} \times I_1(s) \times \frac{1}{s} = \frac{1}{(2s + 3)} I_1(s) \quad (2)$$

We know,

$$V_1(s) = Z(s) \times I_1(s)$$

or, $$V_1(s) = \frac{4s^2 + 11s + 7}{2s + 3} \times I_1(s) \quad (3)$$

Using equation (2) and (3), we have,

$$\frac{V_2(s)}{V_1(s)} = \frac{\left(\frac{1}{2s + 3}\right) \times I_1(s)}{\left(\frac{4s^2 + 11s + 7}{2s + 3}\right) \times I_1(s)}$$

$\therefore$ $$\frac{V_2(s)}{V_1(s)} = \frac{1}{4s^2 + 11s + 7}$$

8. **The characteristics equation of the system is given by $s^3 + 3s^2 + 3s + A = 0$, where A is adjustable. Determine the stability of the system using Routh-Hurwitz criteria.** **[2015/Spring]**

Solution:

Let, $P(s) = s^3 + 3s^2 + 3s + A$

The R-H table

s^3	1	3
s^2	3	A
s^1	$\frac{(9 - A)}{3}$	0
s^0	A	

For system to be stable,

$$\frac{9 - A}{3} > 0 \quad \text{and,} \quad A > 0$$

or, $$3 - \frac{A}{3} > 0$$

$\therefore$ $A < 9$

Hence the range of A is $0 < A < 9$ to be system stable, *i.e.*, the system will be stable if the value of A lies between 0 to 9.

9. **From pole zero plot, obtain the time response for,**

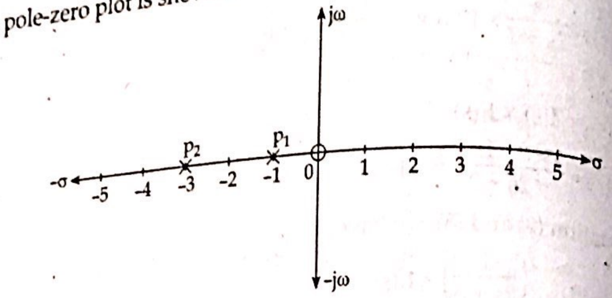
$$I(s) = \frac{4s}{(s + 1)(s + 3)}$$ **[2014/Fall, 2015/Fall]**

Solution:

$$I(s) = \frac{4s}{(s + 1)(s + 3)}$$

There is one zero at $z = 0$ and two poles at $p_1 = -1$ and $p_2 = -3$

The pole-zero plot is shown below



Let, $I(s) = \frac{4s}{(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$

For the pole $p_1 = -1$,

$$A = 4 \times \frac{(p_1 - z)}{(p_1 - p_2)} = 4 \times \frac{(-1-0)}{[-1-(-3)]} = \frac{-4}{2} = -2$$

For the pole, $p_2 = -3$,

$$B = 4 \times \frac{(p_1 - z)}{(p_1 - p_2)} = 4 \times \frac{(-3-0)}{[-3-(-1)]} = \frac{-12}{-3+1} = 6$$

Thus, $I(s) = \frac{-2}{(s+1)} + \frac{6}{(s+3)}$

Taking inverse laplace transform, we get,

$$i(t) = -2\,e^{-t} + 6\,e^{-3t}\ A$$

Hence time response of I(s) is $i(t) = -2\,e^{-t} + 6\,e^{-3t}$ A

10. Plot the poles and zero in s-plane for the network function.

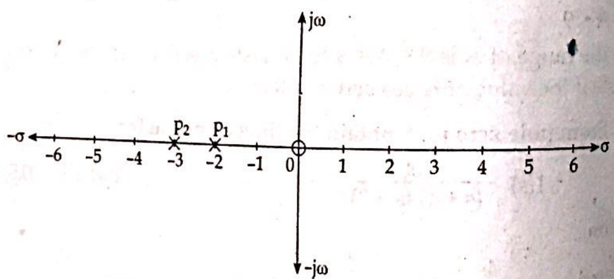
$$V(s) = \frac{10s}{(s+2)(s+3)}$$ **[2011/Spring, 2013/Spring, 2014/Spring]**

Solution:

$$V(s) = \frac{10s}{(s+2)(s+3)}$$

There is one zero at z = 0 and two poles at $p_1 = -2$ and $p_2 = -3$

The pole-zero plot on s-plane is:



Here; $V(s) = \frac{10s}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)}$

For the pole, $p_1 = -2$,

$$A = 10 \times \frac{(p_1 - z)}{(p_1 - p_2)} = 10 \times \frac{(-2-0)}{[-2-(-3)]} = \frac{-20}{1} = -20$$

For the pole, $p_2 = -3$,

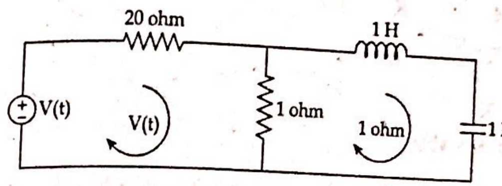
$$B = 10 \times \frac{(p_2 - z)}{(p_2 - p_1)} = 10 \times \frac{(-3-0)}{[-3-(-2)]} = \frac{-30}{-1} = 30$$

Thus, $V(s) = \frac{-20}{(s+2)} + \frac{30}{(s+3)}$

Taking inverse laplace transform, we get,

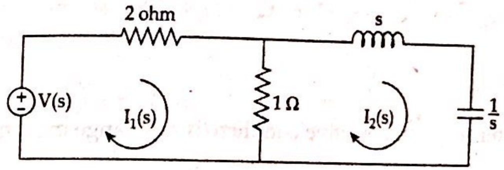
$$V(t) = -20\,e^{-2t} + 30\,e^{-3t}\ V$$

11. For the network shown below, determine the driving point impedance. **[2015/Fall]**



Solution:

The transformed circuit is,



Diving point impedance, $Z_{11}(s) = \frac{V(s)}{I_1(s)}$

$Z_{11}(s)$ is also the equivalent impedance of circuit

$$\therefore\ Z_{11}(s) = 2 + 1 \times \frac{\left(s + \frac{1}{s}\right)}{1 + s + \frac{1}{s}} = 2 + \frac{\frac{(s^2+1)}{s}}{\frac{(s^2+s+1)}{s}}$$

$$= 2 + \frac{s^2+1}{s^2+s+1} = \frac{2s^2+2s+2+s^2+1}{s^2+s+1}$$

$$= \frac{3s^2+2s+3}{s^2+s+1}$$

12. Determine the range of K for which the system is stable using R-H criteria, **[2012/Spring, 2013/Spring, 2014/Spring]**

$$Q(s) = s^4 + s^3 + s^2 + s + K$$

Solution:

Given that;

$$Q(s) = s^4 + s^3 + s^2 s + K$$

The Routh-Hurwitz table is;

		1	K
s^4	1	1	0
s^3	1	K	
s^2	0		
s^1			
s^0			

Since the element of first column, third row is 0, replacing 0 with ϵ,

		1	K
s^4	1	1	0
s^3	1	K	0
s^2	ϵ		
s^1	$\frac{\epsilon - K}{\epsilon}$	0	
s^0	K		

For system to be stable,

$$\frac{\epsilon - K}{\epsilon} > 0 \text{ and } K > 0$$

or, $1 - \frac{K}{\epsilon} > 0$

or, $1 > \frac{K}{\epsilon}$

$\therefore$ $K < \epsilon$

If $K < 1$, the term $1 - \frac{K}{\epsilon}$ is positive and there is no change in signs

If $K > 1$, the term $1 - \frac{K}{\epsilon}$ is negative, so number of roots in right hand side is 2. Due to change in sign, system is unstable.

13. **Check the stability of the system whose characteristic equation is given by, $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5 = 0$ using routh Hurwitz criteria.** **[2013/Fall]**

Solution:

$$P(s) = s^5 + s^4 + 2s^3 + 2s^2 + 3s + 5$$

The R-H table is

s^5	1	2	3
s^4	1	2	5
s^3	0	-2	0
s^2			
s^1			
s			

Since element in the first column, third row is 0, replacing it with ϵ. Then,

s^5	1	2	3
s^4	1	2	5
s^3	ϵ	-2	0
s^2	$\left(\frac{2\epsilon + 2}{\epsilon}\right)$	5	0
s^1	$\frac{-2(2\epsilon + 2) - 5\epsilon^2}{2\epsilon + 2}$	0	
s^0	5		

For stability,

$$\frac{2\epsilon + 2}{\epsilon} > 0 \text{ and } \left(\frac{-4\epsilon - 4 - 5\epsilon^2}{2\epsilon + 2}\right) > 0$$

or, $2 + \frac{2}{\epsilon} > 0$

or, $2 > \frac{-2}{\epsilon}$

$\therefore$ $\epsilon > -1$

As $\epsilon \to 0$, first term of fifth row has value -2, thus there are sign changes in the first column making system unstable.

14. **Check the stability of the system given below expressed in polynomial as $Q(s) = s^3 + 2s^2 + 2s + 40$ using routh Hurwitz criteria.** **[2012/Fall]**

Solution:

$$Q(s) = s^3 + 2s^2 + 2s + 40$$

The R-H table is

s^3	1	2
s^2	2	40
s^1	-18	0
s	40	

Since first column of R-H table consists the element with change in sign (2 to -18 and -18 to 40) the system is unstable.

15. **Plot the poles and zeros in s-plane and obtain T(t) for the transfer function of a network given by,**

$$T(s) = \frac{3s}{(s + 2)(s^2 + 2s + 2)}$$ **[2012/Fall, 2013/Fall]**

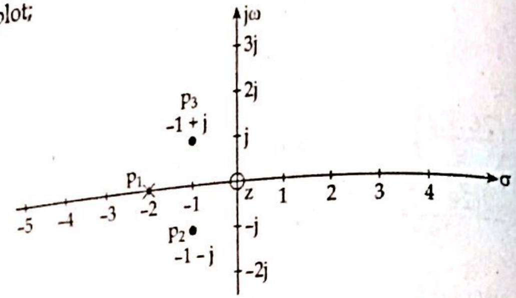
Solution:

$$T(s) = \frac{3s}{(s + 2)(s^2 + 2s + 2)} = \frac{3s}{(s + 2)(s + 1 + j)(s + 1 - j)}$$

Here; one zero is at $z = 0$

Poles are at; $p_1 = -2$, $p_2 = -1 - j$ and, $p_3 = -1 + j$

Pole-zero plot;



Here;

$$T(s) = \frac{3s}{(s+2)(s+1+j)(s+1-j)} = \frac{A}{s+2} + \frac{B}{s+1+j} + \frac{C}{s-1-j}$$

For the pole, $p_1 = -2$

$$\therefore \quad A = 3\frac{(p_1 - z)}{(p_1 - p_2)(p_1 - p_3)}$$

$$= 3\frac{(-2-0)}{(-2+1+j)(-2+1-j)} = -3 \times \frac{-2}{(-1+j)(-1-j)}$$

$$= -3 \times \frac{-2}{(-1)^2 - (j)^2} = \frac{-6}{-1-(-1)}$$

For the pole, $p_2 = -1 - j$

$$\therefore \quad B = 3 \times \frac{(-1-j-0)}{(-1-j+2)(-1-j+1-j)} = 3\frac{(-1-j)}{(1-j)(-2j)} = -3 \times \frac{(1+j)}{-2(1+j)} = \frac{3}{2}$$

$$\therefore \quad C = B^* = \frac{3}{2}$$

Hence, $T(s) = \frac{A}{s+2} + \frac{B}{s+1+j} + \frac{C}{s+1-j}$

or, $T(s) = \frac{-3}{s+2} + \frac{3}{2(s+1+j)} + \frac{3}{2(s+1-j)}$

Taking laplace inverse,

$$T(t) = -3e^{-2t} + \frac{3}{2}e^{-(1+j)t} + \frac{3}{2}e^{-(1-j)t}$$

or, $T(t) = -3e^{-2t} + \frac{3}{2}e^{-t}(e^{-jt} + e^{jt})$

$$= -3e^{-2t} + \frac{3}{2}e^{-t}(2\cos t) \qquad \left[\because \cos t = \frac{e^{jt} + e^{-jt}}{2}\right]$$

$\therefore$ $T(t) = -3e^{-2t} + 3e^{-t}\cos t = -3e^{-t}(e^{-t} - \cos t)$

16. **Form the routh array for the following characteristic equation and find whether the system is stable or not.** **[2011/Spring]**
$s^6 + 3s^5 + 5s^4 + 9s^3 + 8s^2 + 6s + 4 = 0$

Solution:

Routh array is:

s^6	1	3	5	8	4
s^5	3	9	6		
s^4	2	6	4		
s^3	0	0			
s^2					
s^1					
s^0					

The elements of fourth row are zero, so required auxiliary equation,

$$A(s) = 2s^4 + 6s^2 + 4$$

or, $\frac{dA(s)}{ds} = 8s^3 + 12s$

Now,

s^6	1	5	8	4
s^5	3	9	6	
s^4	2	6	4	
s^3	8	12		
s^2	3	4		
s^1	$\frac{4}{3}$			
s^0	4			

Since there is no sign change in the first column. Hence, system is stable.

17. **Define transfer function.** **[2011/Fall, 2015/Fall, 2020/Fall]**
Solution: See the topic 5.1.

18. **What does the Routh criteria state?** **[2011/Spring, 2018/Fall]**
Solution: See the topic 5.6.

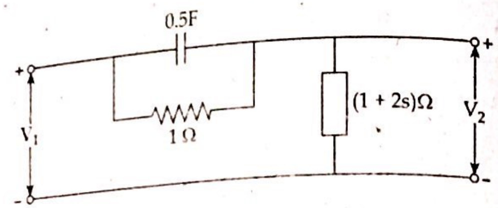
19. **Write short notes on poles and zero of network function.** **[2017/Fall]**
Solution: See the topic 5.4.

20. **Write short notes on time domain behaviour from pole zero location.** **[2017/Spring]**
Solution: See the topic 5.7.

21. **Write short notes on transfer function.** **[2018/Fall]**
Solution: See the topic 5.1.

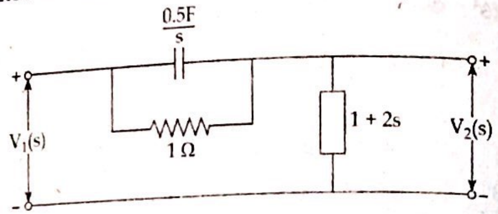
22. **From the pole zero plot, obtain the time domain response of given function, $I(s) = \frac{s(s+1)}{(s+4)(s^2+6s+8)}$** **[2020/Fall]**
Solution: See the 'solved example' question number 12.

23. **Find the driving point impedance $Z_{11}(s)$ and voltage transfer function $G_{21}(s)$ in laplace domain in the circuit given below.** [2020/Fall]



Solution:

The transformed circuit is,



$$Z_1(s) = \frac{\frac{0.5}{s} \times 1}{\frac{0.5}{s} + 1} = \frac{0.5}{s + 0.5} = \frac{1}{1 + 2s}$$

$$Z_2(s) = 1 + 2s$$

Now,

$$V_1(s) = I_1(s) \times [Z_1(s) + Z_2(s)]$$

$$= \left(\frac{1}{1 + 2s} + 1 + 2s\right) I_1(s)$$

$$= \frac{(4s^2 + 4s + 2)}{1 + 2s} I_1(s) = \frac{2\,(2s^2 + 2s + 1)}{1 + 2s} I_1(s)$$

$$V_2(s) = Z_2(s) \times I_1(s) = (1 + 2s)\, I_1(s)$$

So, driving point impedance,

$$Z_{11}(s) = \frac{V_1(s)}{I_1(s)} = \frac{\frac{2(2s^2 + 2s + 1)}{(1 + 2s)}}{I_1(s)} \times I_1(s) = \frac{2(2s^2 + 2s + 1)}{(1 + 2s)}$$

and, Voltage transfer function,

$$G_{21}(s) = \frac{V_2(s)}{V_1(s)} = \frac{(1 + 2s)}{\frac{2(2s^2 + 2s + 1)}{(1 + 2s)}} I_1(s) = \frac{4s^2 + 4s + 1}{4s^2 + 4s + 2}$$

✧✧✧

Chapter 6

FOURIER SERIES TRANSFORM

6.1 BASIC CONCEPT OF FOURIER SERIES AND ANALYSIS INTRODUCTION

A signal is periodic if, for some positive value of T. $x(t) = x(t + T)$ for all t.

Fundamental or time period (T)

Minimum positive, non-zero value of T, for which above equation is satisfied. Fundamental frequency $(\omega_0) = \frac{2\pi}{T}$

As we know, there are two basic periodic signals:

a) The sinusoidal signal; $x(t) = \cos \omega_0 t$ or $\sin \omega_0 t$

b) The complex exponential; $x(t) = e^{j\omega_0 t}$

Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $(T) = \frac{2\pi}{\omega_0}$.

Thus the linear combination of harmonically related compels exponentials of the form,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{-2\pi t}{T}} \qquad (A)$$

$k = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$ is also periodic with period T.

The terms for $k = \pm 1 \rightarrow$ both have fundamental frequency equal to ω_0 and referred to as the fundamental components or first harmonic components. The terms $k = \pm 2$ are periodic with half the period or (equivalent, twice the frequency) of the fundamental components and are referred to as the second harmonic components. Therefore generally, the components for $k = \pm N$ are referred to as the N^{th} harmonic components.

6.2 FOURIER SERIES

6.2.1 Fourier Series Representation

The representation of a periodic signal in the form of equation (A) is referred to as the Fourier series representation. There are some alternative forms of Fourier series of real periodic signals. Suppose that x(t) is real and can be represented in the form of equation (A). Then,

$$x(t) = x^*(t) \qquad \text{[Since x(t) is real]}$$

$$= \sum_{k=-\infty}^{\infty} a_k^* e^{jk\omega_0 t}$$

Replacing k by −k in the summation, we have,

$$x(t) = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

Which by comparison with equation (A), we get,

$$a_k = a_{-k}^*$$

or, $$a_k^* = a_{-k} \qquad (1)$$

From equation (A), we get,

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[e_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

Replacing a_k^* for a_{-k} from equation (1), we obtain,

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[e_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right]$$

Since the two terms inside the summation are complex conjugates of each other. Hence,

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left\{a_k e^{jk\omega_0 t}\right\}$$

If a_k is expressed in polar form as,

$$a_k = A_k' e^{j\theta_k}$$

Then, $$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\left(A_k' e^{j\theta_k} \cdot e^{jk\omega_0 t}\right)$$

i.e., $$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k' \cos(k\omega_0 t + \theta k) \qquad (2)$$

Another form is obtained by writing a_k in rectangular form as,

$$a_k = B_k' + jC_k' \cos \text{ (where } B_k' \text{ and } C_k' \text{ are both real).}$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\mathrm{Re}\,(B_k' + jC_k')\, e^{jk\omega_0 t})$$

i.e., $$x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k' \cos k\,\omega_0 t - C_k' \sin k\,\omega_0 t] \qquad (3)$$

Thus, for real periodic functions, the Fourier series in terms of complex exponentials as given in equation (1), is mathematically equivalent to either of the two forms in equation (2) and (3) that are trigonometric functions.

6.2.2 Determination of the Fourier Coefficients

a) For complex exponential functions

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T} x(t)\, e^{-jk\omega_0 t}\, dt$$

The coefficient a_0 is the dc or constant component of x(t) and is given by above equation with k = 0 *i.e.,*

$$a_0 = \frac{1}{T}\int_T x(t)\,(t)\, dt$$

(Which is simply the average value of x(t) over one period)

b) For trigonometric functions

Equation (3) is identical to the equation,

$$x(t) = a_0 + \sum_{k=1}^{\infty} [A_k \cos k\,\omega_0 t + B_k \sin k\,\omega_0 t]$$

(where, $A_k = B_k'$ and $B_k = -C_k'$; another constant)

$$A_k = \frac{2}{T}\int_T x(t) \cos k\,\omega_0 t$$

$$B_k = \frac{2}{T}\int_T x(t) \sin k\,\omega_0 t$$

and, $a_0 = \frac{1}{T}\int_T x(t)\, dt$; same as in previous case

6.2.3 Magnitude and Phase Spectra of a Periodic Signal

Let the complex Fourier coefficients a_k be expressed as,

$$a_k = |a_k|\, e^{j\theta_k}$$

where, $|a_k|$ is the magnitude and θ_k is the phase of k^{th} harmonic component. A plot of $|a_k|$ versus the angular frequency ω(or k) is called the magnitude spectrum of the periodic signal x(t), and a plot of θ_k of $\angle a_k$ versus ω or k is called the phase spectrum of x(t). Since here k assumes only integers, the magnitude and phase spectra are not continuous curves but discrete in nature, *i.e.*, appear only at the discrete in nature *i.e.*, appear only at the discrete frequencies. They are therefore referred to as discrete frequency spectra.

For a real valued periodic signal x(t)

$a_{-k} = a_k^*$ Thus,

$|a_{-k}| = |a_k|$ and $\angle a_{-k} = -\angle a_k$

∴

i) Magnitude spectrum of a real valued period signal x(t) is an even function of ω or k.

ii) Phase spectrum of a real valued periodic signal x(t) is an odd function of ω or k.

Combining the above two statements, it can be stated that spectra of a real valued periodic signal x(t) is conjugate symmetric.

6.3 EFFECTS OF SYMMETRY

The Fourier series of only even periodic function f(t) consists of cosine terms only and the Fourier series for any odd periodic function f(t) consists of sine terms only.

$$a_0 = \frac{1}{T}\int_T f(t)\, dt$$

$$A_k = \frac{2}{T}\int_T (t) \cos k\omega_0 t\, dt$$

$$B_k = \frac{2}{T}\int_T f(t) \sin k\omega_0 t\, dt$$

Recall cos $k\omega_0$t is an even function while sin $k\omega_0$t is an odd function.

a) If f(t) is an even function of t_1 then f(t). cos $k\omega_0$t is also an even function and f(t). sin $k\omega_0$t is an odd function of t.

Hence, $B_k = 0$

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(t)\, dt$$

$$A_0 = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\, dt \sin k\omega_0 t\, dt$$

6.4 FOURIER TRANSFORM

A non-periodic signal may be assumed as a limiting case of a periodic signal where the period of the signal approaches infinity. We can use this approach to develop the frequency domain representation of a non periodic signal over an entire interval.

For a continuous-time periodic signal

$$x_p(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t} \quad (A)$$

where, $\omega_0 = \frac{2\pi}{T} \rightarrow$ Angular frequency (fundamental)

$T \rightarrow$ Time period (fundamental)

and, $a_k = \frac{1}{T}\int_T x_p(t) \cdot e^{-jk\omega_0 t}\, dt$

Let us define $X(jk\,\omega_0) = \frac{2\pi}{\omega_0} \cdot a_k = \int_T x_p(t)\, e^{-jk\omega_0 t}\, dt$ (B)

or, $a_k = \frac{\omega_0}{\pi} X(jk\omega_0 t)$

Hence, from equation (A), we get,

$$x_p(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)\, e^{jk\omega_0 t}\, \omega_0 \quad (C)$$

A non-periodic continuous time signal x(t) can be viewed as periodic signal $x_p(t)$ with time period $T \rightarrow \infty$ and frequency $\omega_0 \rightarrow 0$.

As $T \rightarrow \infty$, $x_p(t) \rightarrow x(t)$

Also, $k\omega_0 \rightarrow \omega$ (continuous variable)

$\omega_0 \rightarrow d\omega$ (differential variable)

From equation (B), we get,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad (1)$$

and, from equation (C),

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \quad (2)$$

Equation (1) is known as Fourier transform or Fourier integral of non-periodic signal x(t) and equation (2) is known as inverse Fourier transform of X(jω). Mathematically,

$$X(j\omega) = F[x(t)] = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

and, $x(t) = F^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

Therefore, X(jω) is the frequency domain representation of time domain function x(t). This means that we are converting a time domain function into its frequency domain representation with the help of Fourier transform, and vice-versa.

Hence, (t) and X(jω) are a Fourier transform pair. Symbolically, this maybe expressed as $x(t) \leftrightarrow X(j\omega)$.

Here, double ended arrow means that Fourier transform of x(t) is X(jω) and inverse Fourier transform of X(jω) is x(t).

In general, Fourier transform X(jω) is complex function of ω and may be expressed as,

$$X(j\omega) = |X(j\omega)| \cdot e^{j\theta(\omega)}$$

where, $|X(j\omega)|$ = Amplitude spectrum (plot of $|X(j\omega)|$ versus ω)

$\text{Arg}[X(j\omega)] = \theta(\omega)$ = Phase spectrum (plot of Arg (X(jω)] versus ω.

The signal is uniquely defined in time domain as well as in frequency domain. The spectrum $X(j\omega)$ is continuous in frequency unlike Fourier series representation of periodic signal where the spectrum is discrete.

For a real valued function $x(t)$, its conjugate $x^*(t) = x(t)$

$$x^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt \qquad [\text{since } x^*(t) = x(t)]$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(-\omega t)}\, dt$$

$$= X(-j\omega)$$

Thus it follow that for real valued function $x(t)$.

$$|X(-j\omega)| = |X(j\omega)|$$

and, $\text{Arg}\,[X(-j\omega)] = -\text{Arg}[X(j\omega)]$

Therefore,

i) Amplitude spectrum of a real valued signal $x(t)$ is an even function.

ii) Phase spectrum of a real valued signal $x(t)$ is an odd function.

Combining the above two statements, it can be stated that spectrum of a real valued signal is conjugate symmetric.

6.5 EXISTENCE OF FOURIER TRANSFORM

A function $x(t)$ is said to be Fourier transformable, if it is satisfied following Dirichlet conditions.

a) The function $x(t)$ is absolutely integrable *i.e.,* $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$

b) The function has finite number of maxima and minima, if any, within a finite interval of time.

c) The function is single valued and has finite number of discontinuities, if any, within a finite interval of time.

6.6 FOURIER TRANSFORM OF PERIODIC SIGNALS

We can also develop Fourier transform representations for periodic signals, directly from its Fourier series representation. The resulting transform consists of a train of impulses in the frequency domain.

To suggest the general result, let us consider a signal $x(t)$ with fourier transform $X(j\omega)$ that it is a single impulse of area 2π at $\omega = \omega_0$ *i.e.,*

$$X(j\omega) = 2\pi\, \delta(\omega - \omega_0)$$

To determine the signal $x(t)$ for which this is the fourier transform, we can apply the inverse transform relation to obtain,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, \delta(\omega - \omega_0)\, e^{j\omega t}\, d\omega = e^{j\omega_0 t}$$

More generally if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency *i.e.,*

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\omega_0)$$

Hence, $x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{-jk\omega_0 t}$

We see that the above equation corresponds exactly to the Fourier series representation of a periodic signal with Fourier series coefficients a_k.

6.7 TABLE OF FOURIER TRANSFORM PAIRS

S.N.	x(t)	X(jω)
1.	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
2.	$e^{-at}u(-t)$	$\frac{1}{a-j\omega}$
3.	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$
4.	$t\,e^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$
5.	$\delta(t)$	1
6.	$\delta(t \pm t_0)$	$e^{\pm j\omega t_0}$
7.	1	$2\pi\delta(\omega)$
8.	$e^{j\omega_0 t}$	$2\pi\delta(\omega_0)$
9.	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
10.	$\sin\omega_0 t$	$j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
11.	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
12.	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$
13.	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
14.	$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sin c}\left(\frac{\omega T}{2}\right)$
15.	$\Delta\left(\frac{t}{T}\right)$	$\frac{T}{2}\,\text{sin c}^2\left(\frac{\omega T}{4}\right)$
16.	$\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\omega_0\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_0)$

Here all $a > 0$ and $\omega_0 \neq \frac{2\pi}{T}$, where T-fundamental period of the x(t).