Click on wanted Notes/solutions to Download from website:

1)Engineering Chemistry
2)C programming 2015 Fall solution
3)OOPS
4)Electronic Device
5)Network theory
6)Electromagnetic Field and Theory
7)Logic circuit
8)Electronic circuits
9)Electrical Machine
10)Instrumentation
11)Advance Electrical Machine
12)Control system
13)Signals and systems
14)Probability and statistics
15)Transmission and distribution system
16)Microprocessor Board Questions solutions
17)Switch gear and protection system
18)Power electronics 6th semester Note
19)Research methodology
20)Power plant Equipment
21)Engineering Economics
22)Communication system Engineering
23)Power system analysis
24)C programming lecture Note
25)OOPS lecture Note

And other many Notes / solutions/Projects are available :)

Scanned with CamScanner

Introduction

1 chapter = 1

1.A. Signals : 2016/s

A detectable physical quantity or impulse (as a voltage, current, or magnetic field strength) by which messages or information can be transmitted is known as signal. A signal is a function of independent variables that carry some information. A signal is a physical quantity that varies with the time, space or any other independent variable by which information can be conveyed.

For example: a speech signal can be represented mathematically by acoustic pressure as a function of time. and a picture can be represented by brightness as a function of two spatial variable.

Systems : 2016/s

A system is any physical set of components that takes a signal and produces a signal.

Classification of one-Dimensional signal:

i Continuous-time signals
II Discrete-time signals.

I. Continuous-time signals :

A continuous time signal is defined as a mathematical continuous function that is defined continuously in the time domain. For continuous time signals, the independent variable is time, t. A continuous time signal has a value defined at each point in time. speech signal and the temperature of the room are examples of

Scanned with CamScanner

continuous time signal.
E.g: velocity, voltage.
Denote by $x(t)$, where the time interval may be bounded, (finite) or infinite.

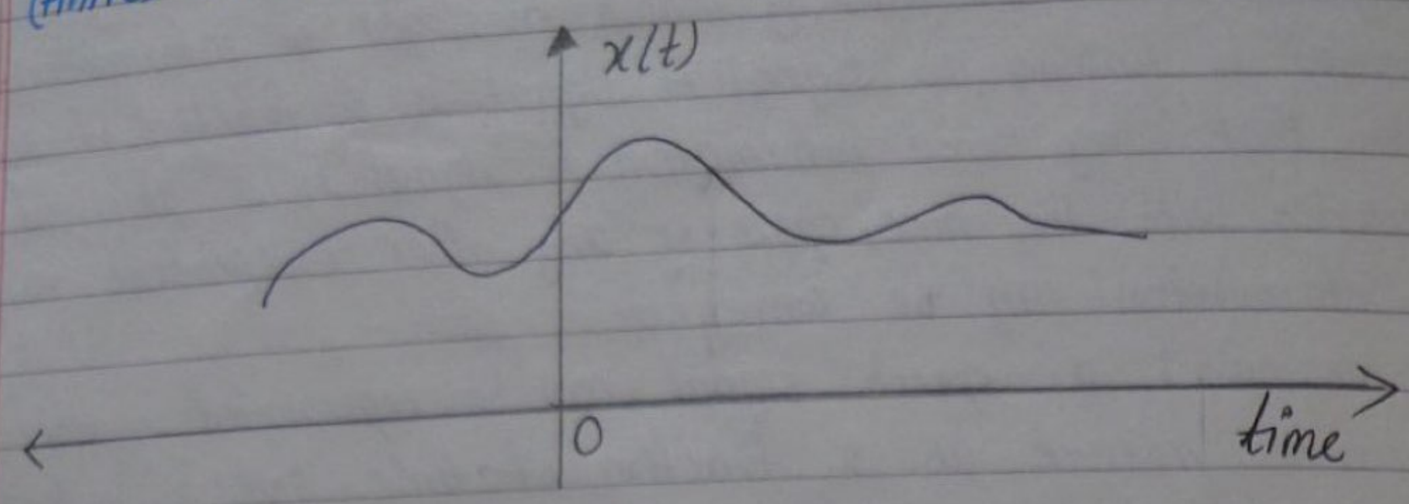


Fig: Continuous time signal

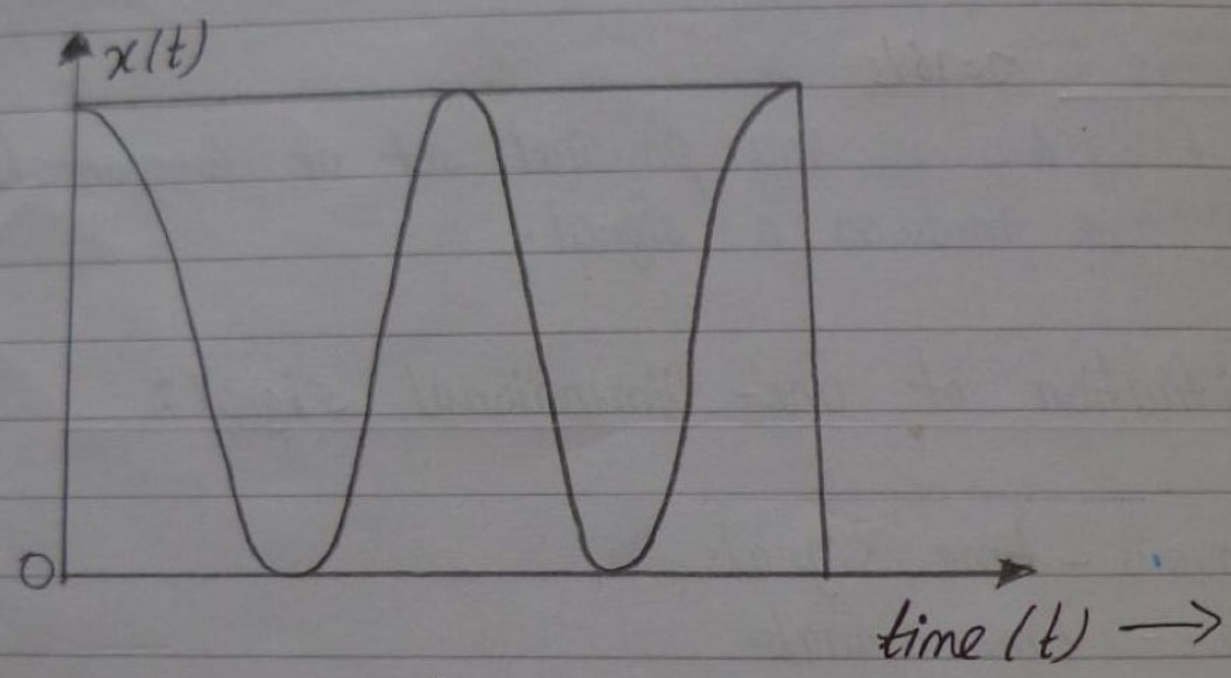


Fig: Continuous time signal.

This type of signal shows continuity both in amplitude and time. These will have values at each instant of time. Sine and cosine functions are best example of continuous time signal.

Scanned with CamScanner

3

II Discrete time signals :

A discrete time signal is defined only at certain time instant. and the amplitude between two time instants is just not defined. It can be obtained by periodic sampling of continuous-time signals. For eg: the crime rate, budget of the country is represented in discrete form.

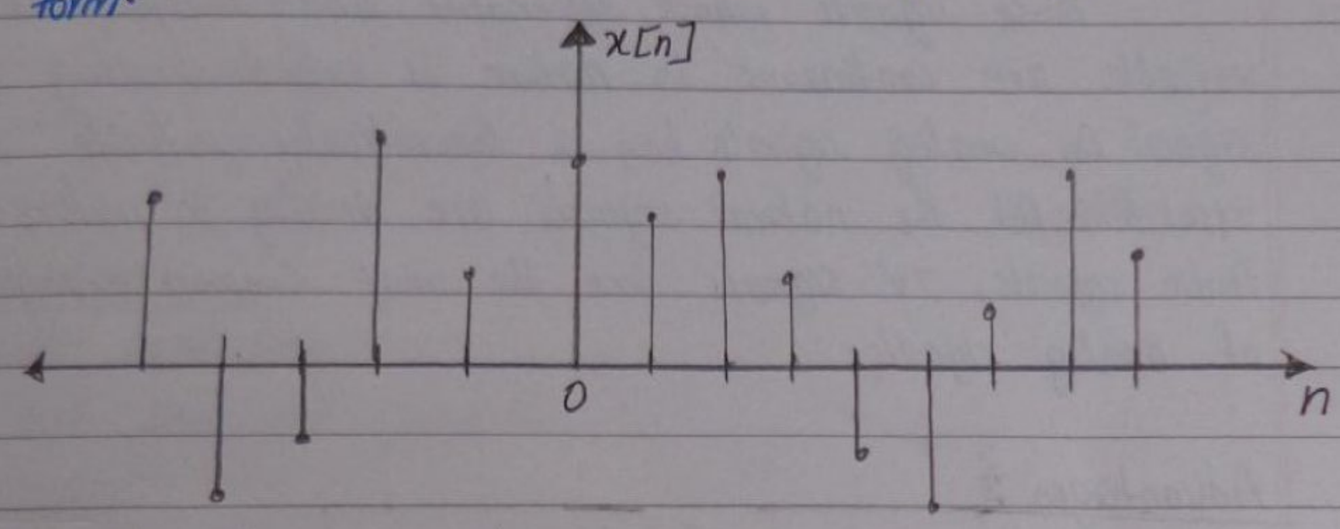


Fig : Discrete - time signal.

Representation of discrete - time signals :

a) sequence Representation :

$$x[n] = \{ \ldots -1, 1, \underset{\uparrow}{1}, 1, -1, 1, \ldots \}$$

where the functional value indicated by the arrow (↑) represents the value of function at the origin.

b. Tabular representation :

n	= ---	-1	0	1	2	3	---
x[n]	= ----	0	1	2	4	-6	----

c. Functional Representation

Scanned with CamScanner

d. Graphical Representation.

Classification of signals based on the conditions of dependent and independent variable :

I Analog signal :

Those signals whose dependent and independent variable are continuous in nature is called analog signal. An analog signals has a theoretically infinite resolution. All the natural signals are analog in nature. Audio signals, TV signals are the most common examples of analog signals.

Advantages :

- Has infinite amount of signal resolution.
- Higher density
- processing is simpler
- No quantization error
- requires less Bandwidth.

Disadvantages :

- quality often degraded due to noise.
- Requires high quality processing
- high power requirements

II. Digital signals :

Those signals whose both dependent & independent variable are discrete in nature are called digital signals. The amplitude of digital signals are constant. A digital

Scanned with CamScanner

5

signal is a signal that is constructed from a discrete set of wave forms of a physical quantity so as to represent a sequence of discrete values. A logic signal is a digital signal with only two possible values.

Advantages :

- easy to implement
- cheaper than analog
- security, Flexibity

Disadvantages :

- not wide bandwidth
- High cost and not portable.
- less accuracy.

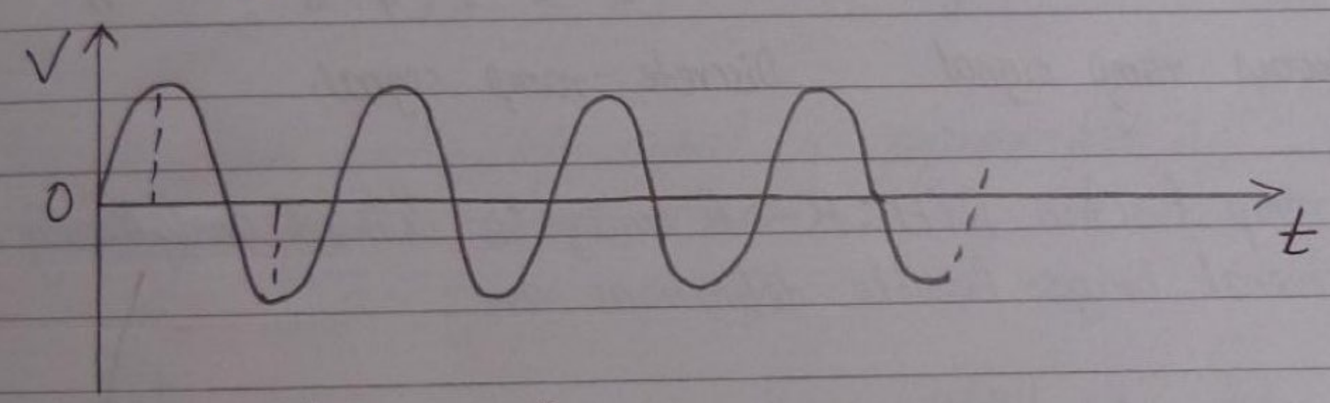


Fig : analog signal.

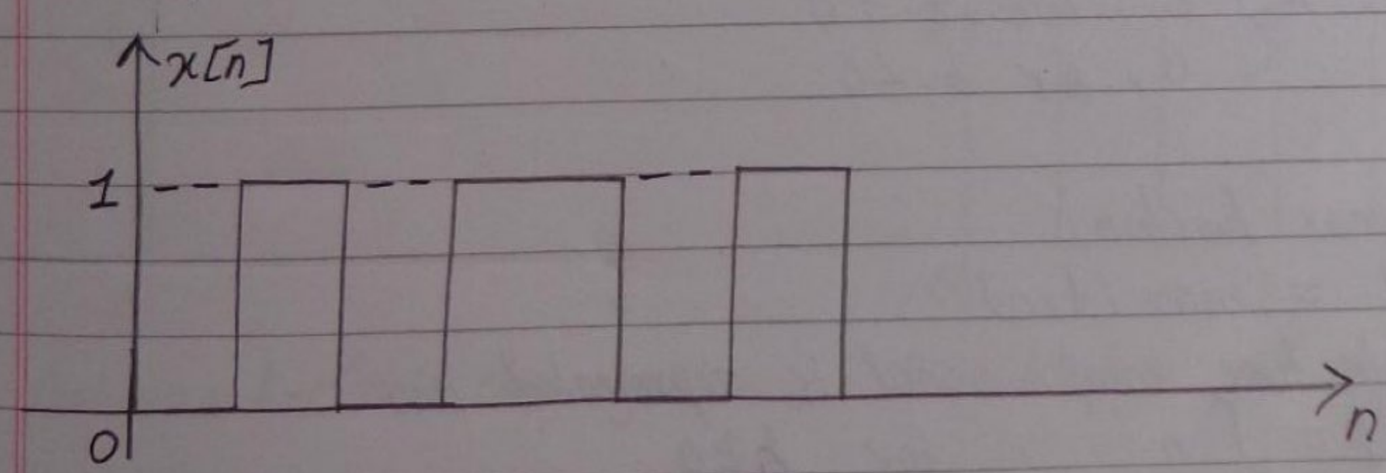


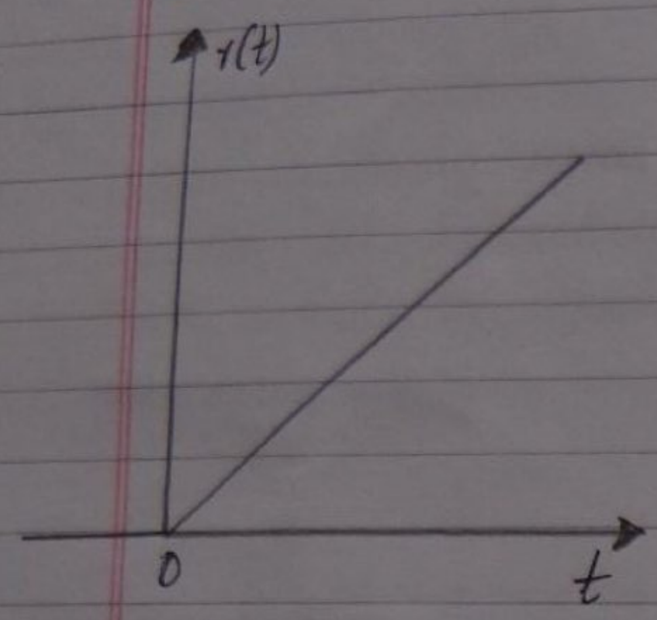
Fig : digital signal.

Scanned with CamScanner

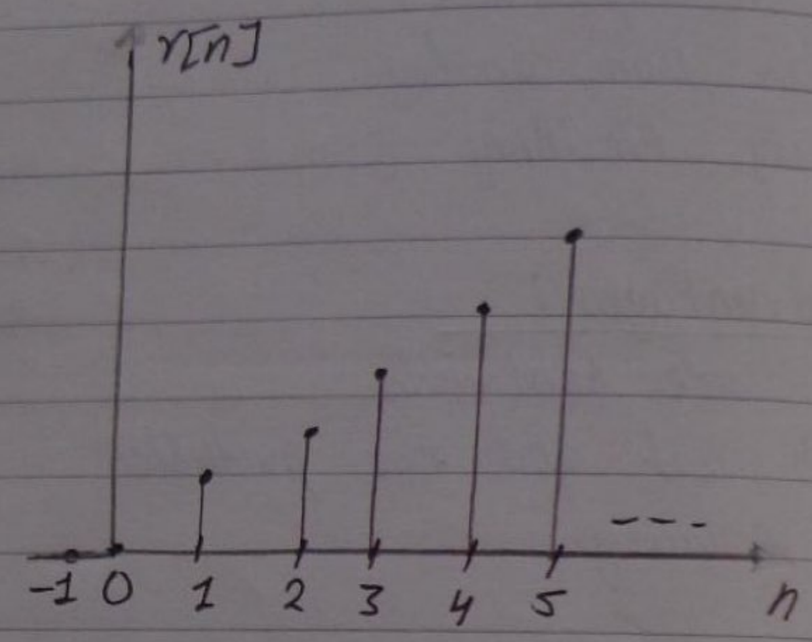
Primary signals :

I. Ramp signals :

Signal increasing linearly with time is called as ramp signal. Continuous time ramp signal is represented by $r(t)$ and discrete time ramp signal by $r[n]$.



Continuous ramp signal



Discrete ramp signal.

The ramp function $R(t): R \rightarrow R$ may be defined analytically in several ways. Possible definitions are;

- A system of equations:

$$R(t) = \begin{cases} t, & \text{for } t \geq 0 \\ 0, & \text{for } t < 0 \end{cases}$$

- The max function,

$$r(t) = \max(t, 0)$$

Discrete time ramp signal is represented as,

$$r[n] = \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

Scanned with CamScanner

7

Area under unit ramp is unity.

$$\int u(t) = \int 1 = t = r(t)$$

$$\therefore \; u(t) = \frac{dr}{dt}$$

II. Unit – step signal : 2017/5

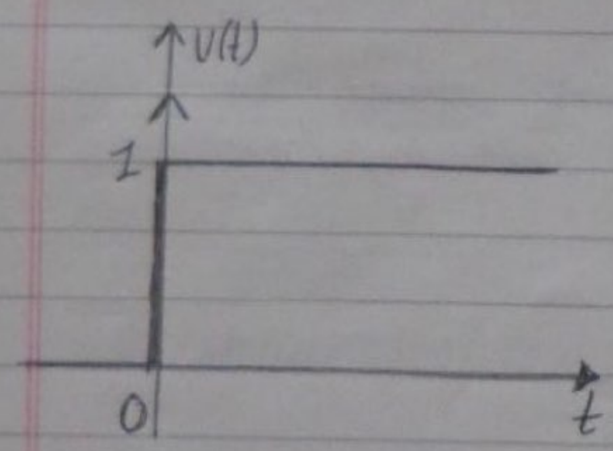
Signals that begins from the time instant '0' and goes up to time constant '$+\infty$' with the constant amplitude of unity is called unit-step signal. It is discontinuous function whose value is zero for negative argument and one for positive argument.

mathematically, continuous time unit step signal,

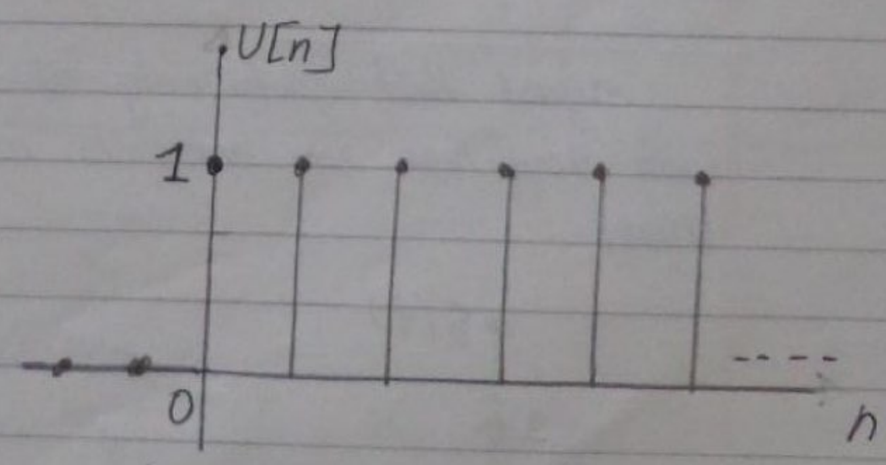
$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

Discrete time unit step signal is represented by,

$$u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n > 0 \end{cases}$$



Continuous time unit step signal



Discrete type unit step signal

Using half – maximum convention,

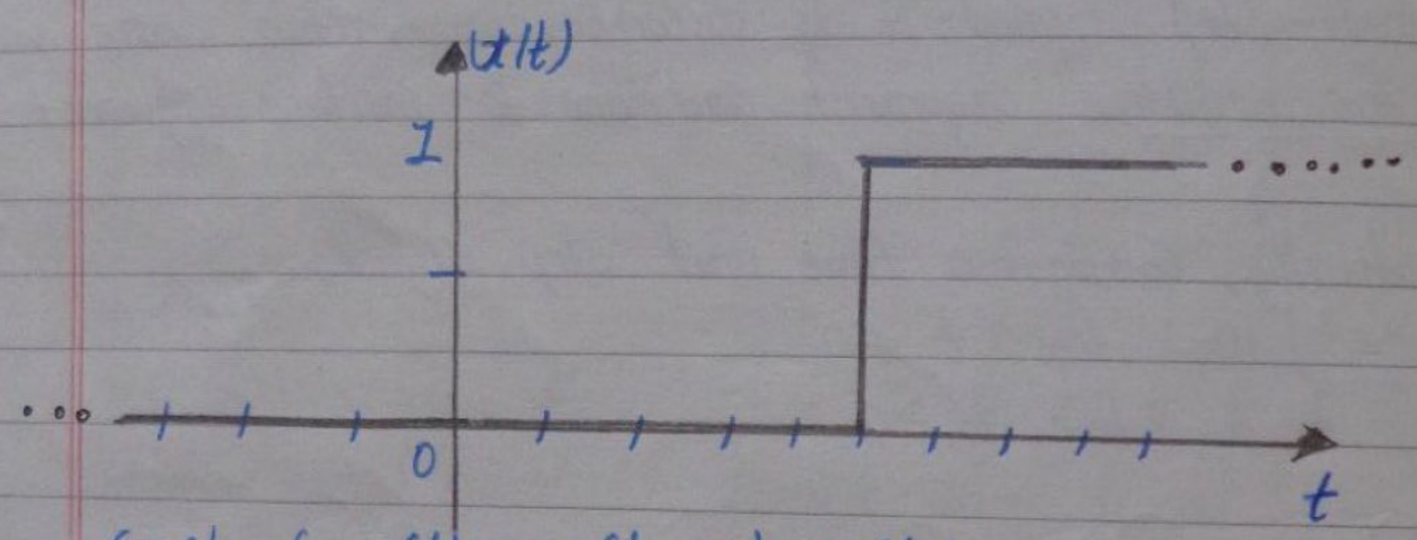
Scanned with CamScanner

$$u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1/2 & \text{for } n = 0 \\ 1 & \text{for } n > 0 \end{cases}$$

where n is an integer.

The distributional derivative of the unit step function is the dirac delta function.

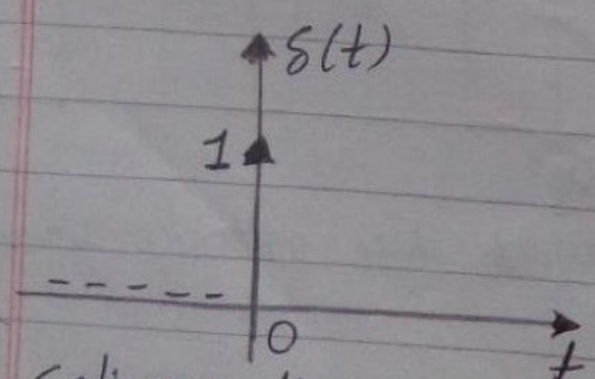
$$\frac{du(t)}{dt} = \delta(t)$$



Graph of $u(t) = u(t-5)$ shifted unit step signal.

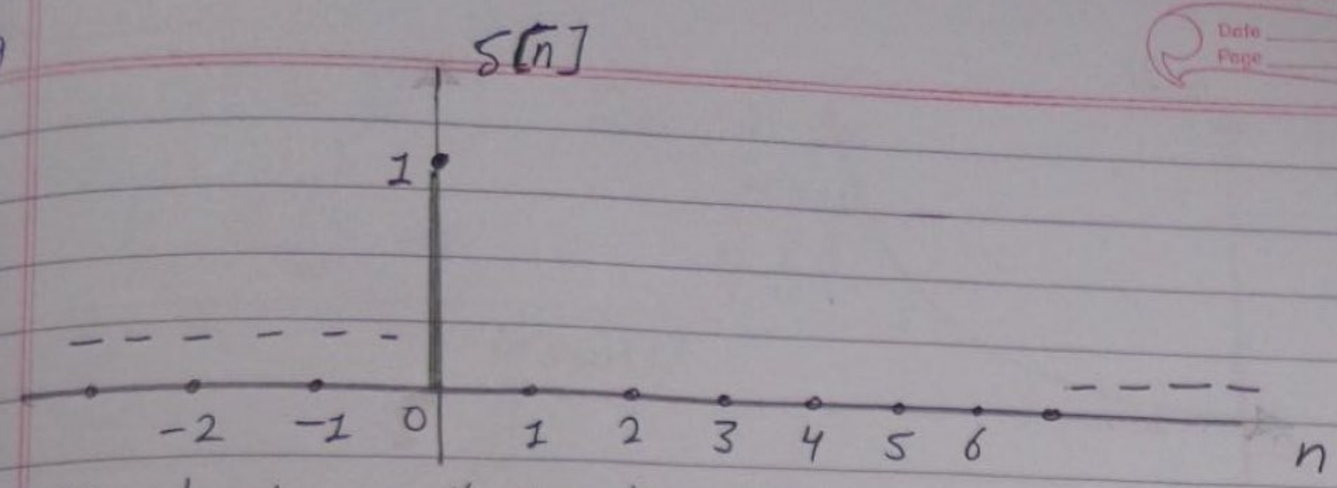
III Unit - impulse signal : Delta/dirac delta function: 2017/5

signal that exist only at the time instant zero with amplitude of unity is called unit impulse signal.



Continuous time

Scanned with CamScanner



Discrete time unit impulse signal

It is denoted by, $\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$

Also, $\int_{-\infty}^{\infty} \delta(t)\, d(t) = u(t)$

$\therefore \quad \delta(t) = \dfrac{du(t)}{dt}$

For discrete time,

$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0, & \text{otherwise} \end{cases}$

Unit impulse is also known as unit sample.

IV. Parabolic signals :

parabolic signal can be defined as,

$x(t) = \begin{cases} t^2/2 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$

In the time domain, it is represented by $\dfrac{t^2}{2}$.

Scanned with CamScanner

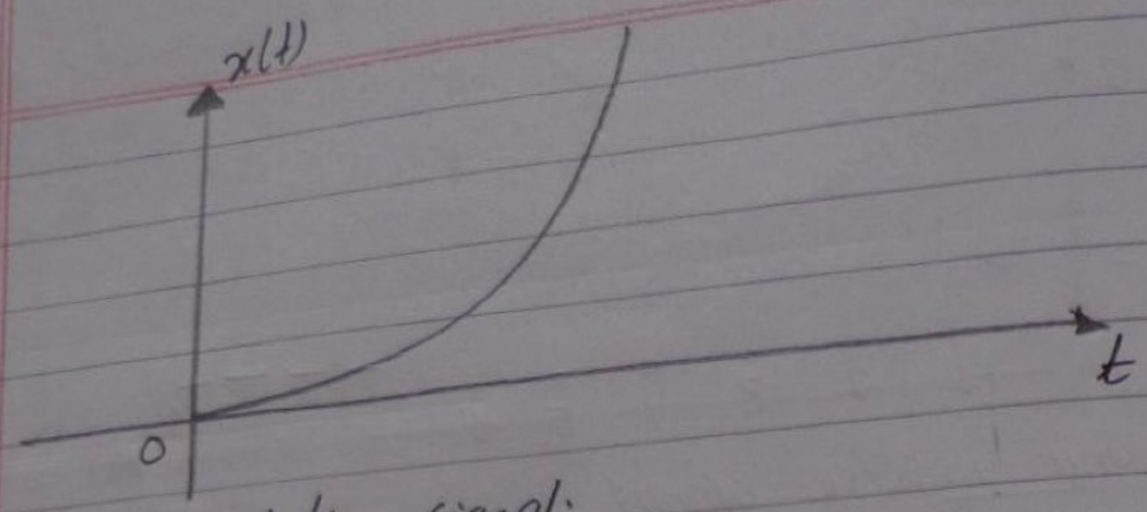


Fig: parabolic signal.

$$\iint v(t)\,dt = \int r(t)\,dt$$

$$= \int t\,dt$$

$$= \frac{t^2}{2} = \text{parabolic signal}$$

and, $v(t) = \dfrac{d^2(x(t))}{dt^2}$

$$\Rightarrow \quad r(t) = \frac{dx(t)}{dt}$$

V. Signum Function: 2016/F

signum function can be represented by,

$$sgn(t) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

or $sgn(t) = \lim_{a \to 0} \{ e^{-at} \cdot u(t) - e^{at} U(-t) \}$

and in discrete time as,

Scanned with CamScanner

11

$$sgn[n] = \begin{cases} 1 & \text{for } n > 0 \\ 0 & n = 0 \\ -1 & n < 0 \end{cases}$$

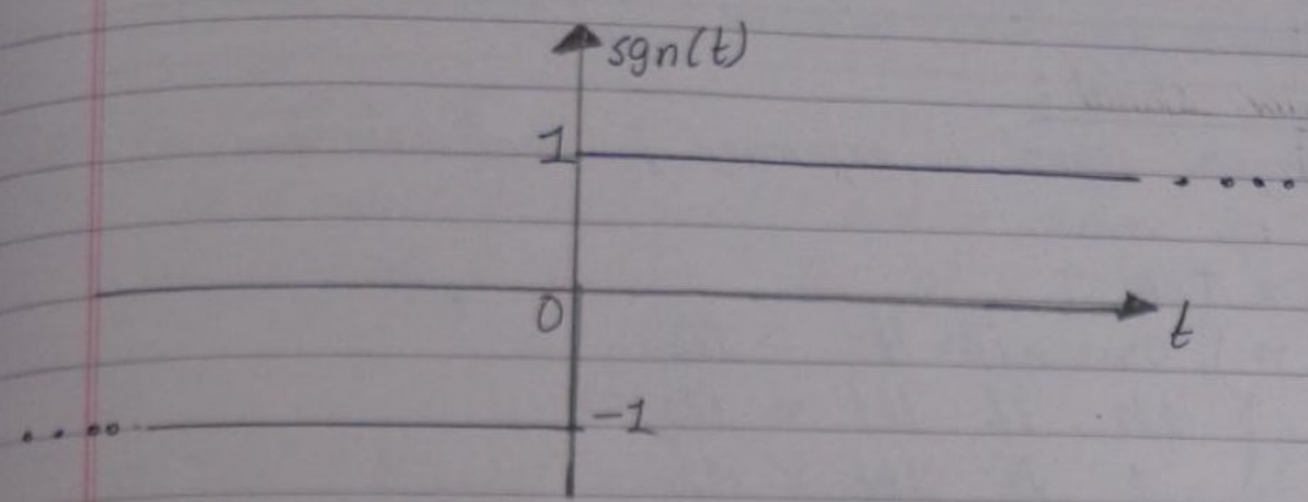


Fig: Continuous time signum pulse.

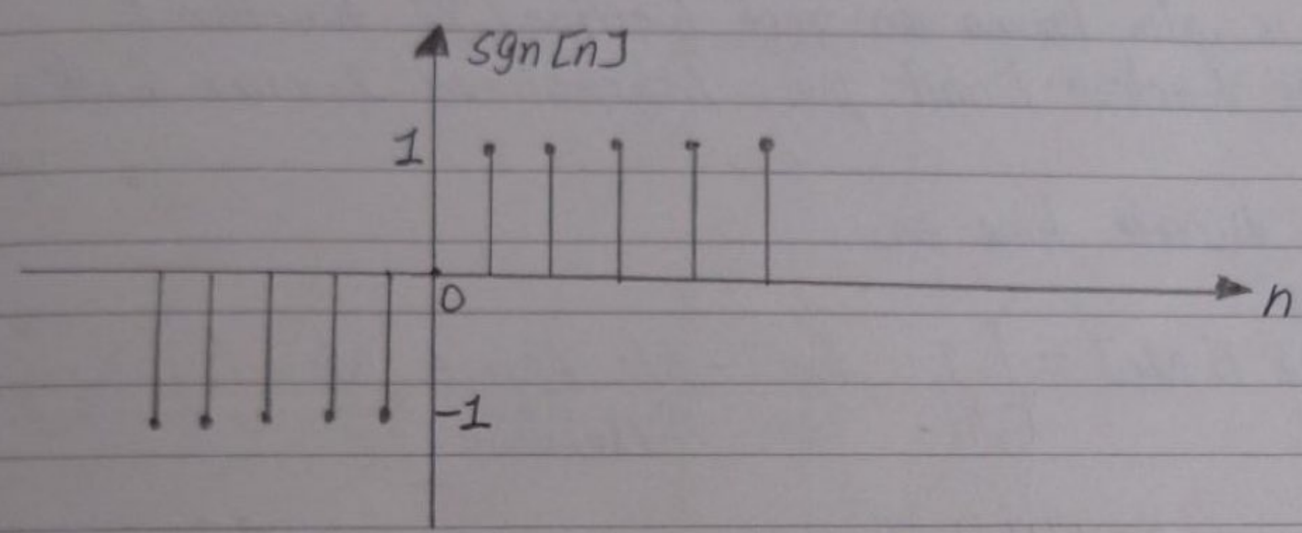


Fig: discrete time signum signal.

Any real number can be expressed as the product of its absolute value and its sign function.

$t = sgn(t) \cdot |t|$.

It follows that whenever t is not equal to 0,

$$sgn(t) = \frac{t}{|t|} = \frac{|t|}{t}$$

Scanned with CamScanner

The signum function is the derivative of absolute value function.

$$\frac{d|t|}{dt} = sgn(t) \quad \text{for } t \neq 0.$$

VI. Rectangular signal:

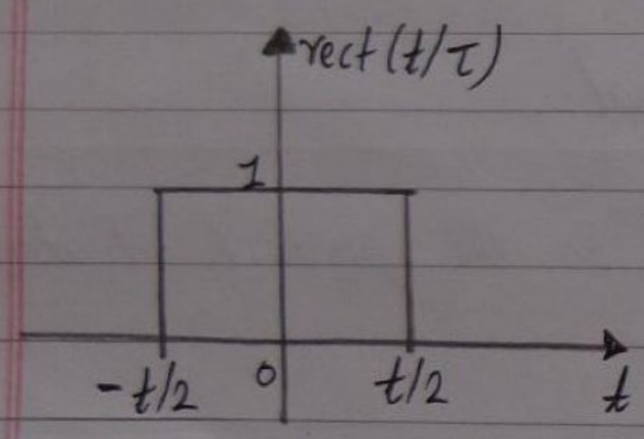
Rectangular signal is expressed in continuous time as,

$$rect(t) = \Pi(t) = \begin{cases} 0 & \text{if } |t| > 1/2 \\ 1/2 & \text{if } |t| = 1/2 \\ 1 & \text{if } |t| < 1/2 \end{cases}$$

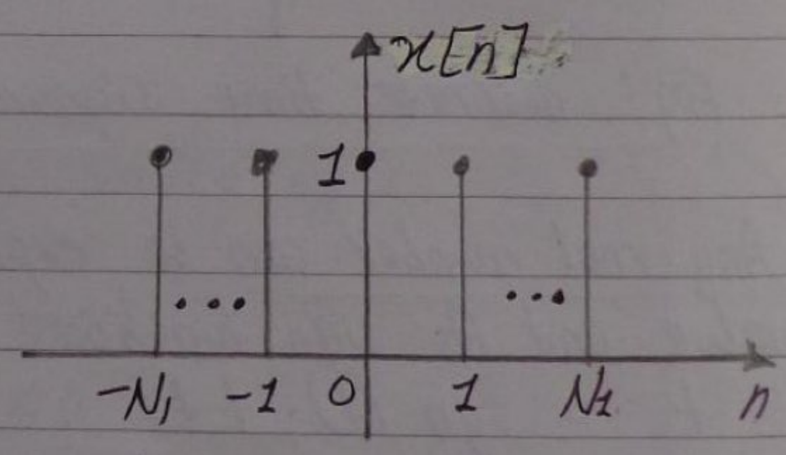
It is also known as rect function / Pi function / gate function / unit pulse / normalized Boxcar function.

In discrete time as,

$$rect[N_1] = \begin{cases} 1 & \text{for } -N_1 \leq n \leq N_1 \\ 0 & \text{otherwise.} \end{cases}$$



Continuous time rectangular



discrete time rect.

$$\Pi(x) = H\left(x + \frac{1}{2}\right) - H\left(x - \frac{1}{2}\right)$$

Scanned with CamScanner

13

$$\pi(x) = \frac{1}{2}\left[\operatorname{sgn}\left(x+\frac{1}{2}\right) - \operatorname{sgn}\left(x-\frac{1}{2}\right)\right]$$

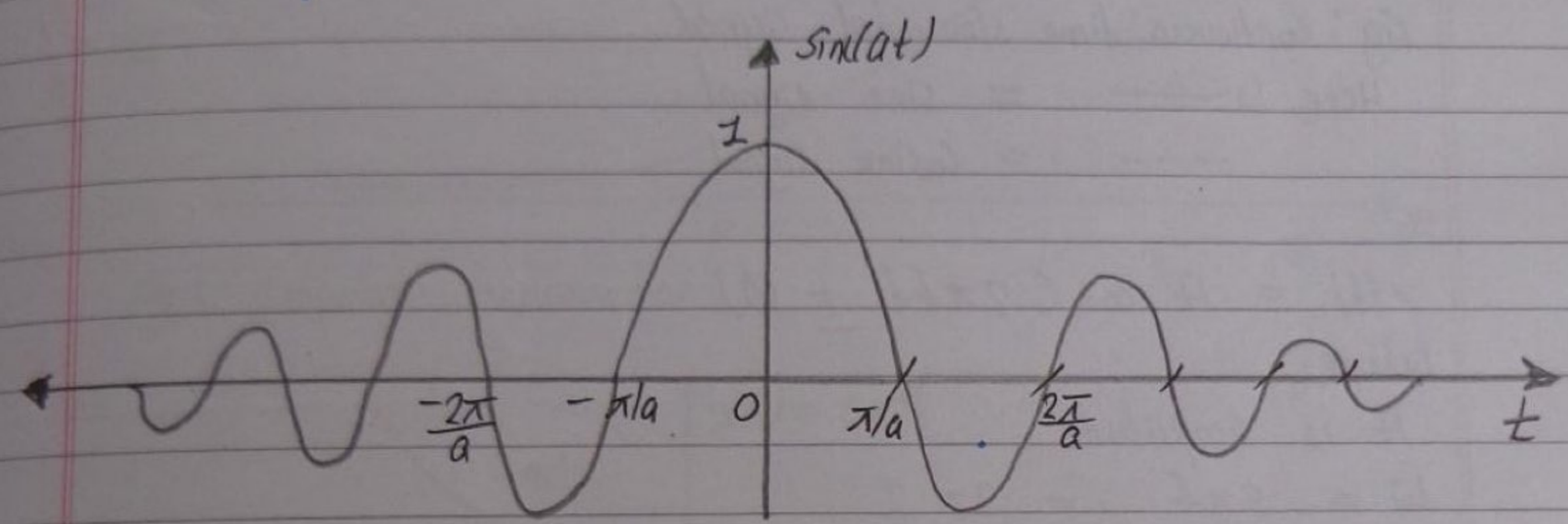
where,
H(x) is heaviside step function /unit step function.

VII. Sinc Function : 2016/F

It is denoted by sinc(t) and is defined as sinc.

$$\operatorname{sinc}(at) = \begin{cases} 0 & \text{for } at = \pm n\pi \\ \dfrac{\sin(at)}{at} & \text{elsewhere} \end{cases}$$

We use L'Hospital rule to compute the value of sinc(at) at t=0 which is equal to 1.



$$\int \operatorname{sinc}(t)\,dt = Si(t)$$

Where, Si(t) is the sine integral.

VIII. Sinusoidal signals :

The signals varying with sine function or cosine function of time are called as sinusoidal signals.

Scanned with CamScanner

Sinusoidal signal is in the form of,

$$x(t) = A \cos(\omega_0 \pm \phi)$$
$$= A \sin(\omega_0 \pm \phi)$$

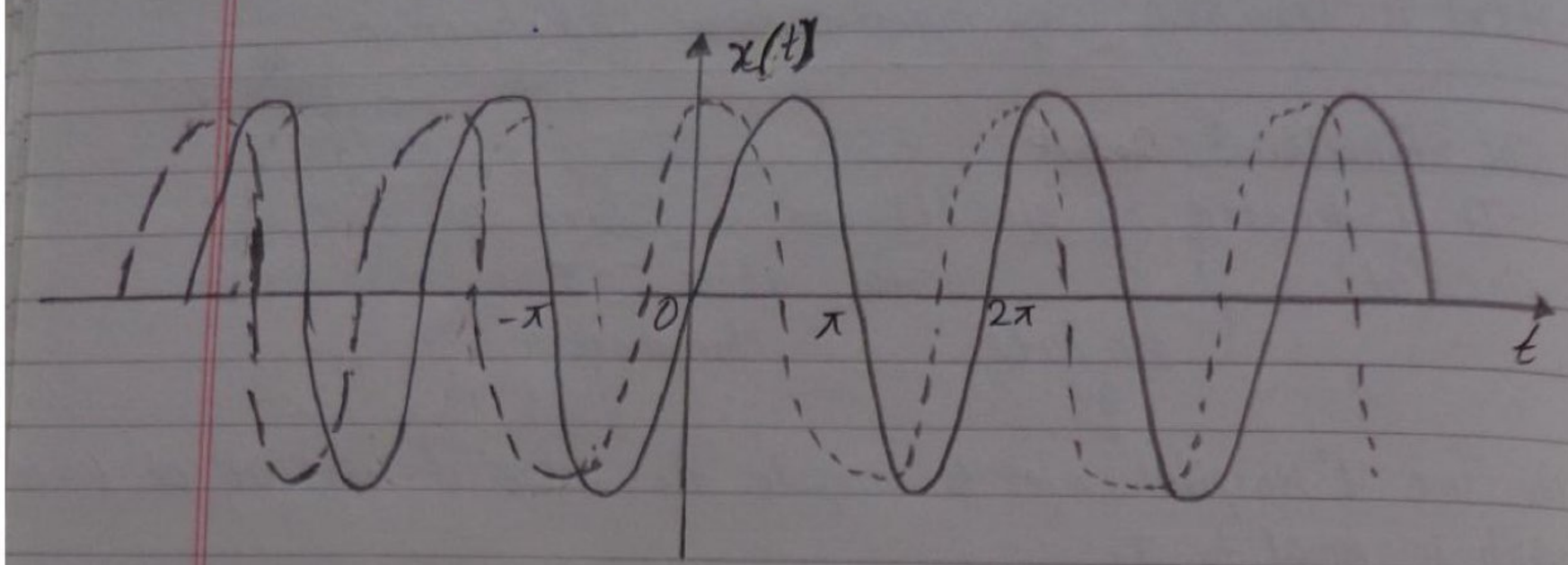


Fig: Continuous time sinusoidal signal

Here, ——— = sine signal.

--- = Cosine signal

$$x(t) = A \cos(2\pi f t_0 \pm \phi)$$

where,

A is amplitude

$\omega = 2\pi f = 2\pi \frac{1}{T}$

ϕ = initial phase

$T = 1/f$ is the time period.

IX. Exponential signal:

In continuous time signal, a real exponential signal may be expressed as,

$$x(t) = C \cdot e^{at}$$

Scanned with CamScanner

where both c and a are real numbers. There are two types of exponential signals :

a) Exponentially growing signal $(a>0)$
b) " decaying " $(a<0)$

Those signals which grow exponentially with the time are called as exponentially growing signals. and those which decay exponentially with time is called as exponentially decaying signal.

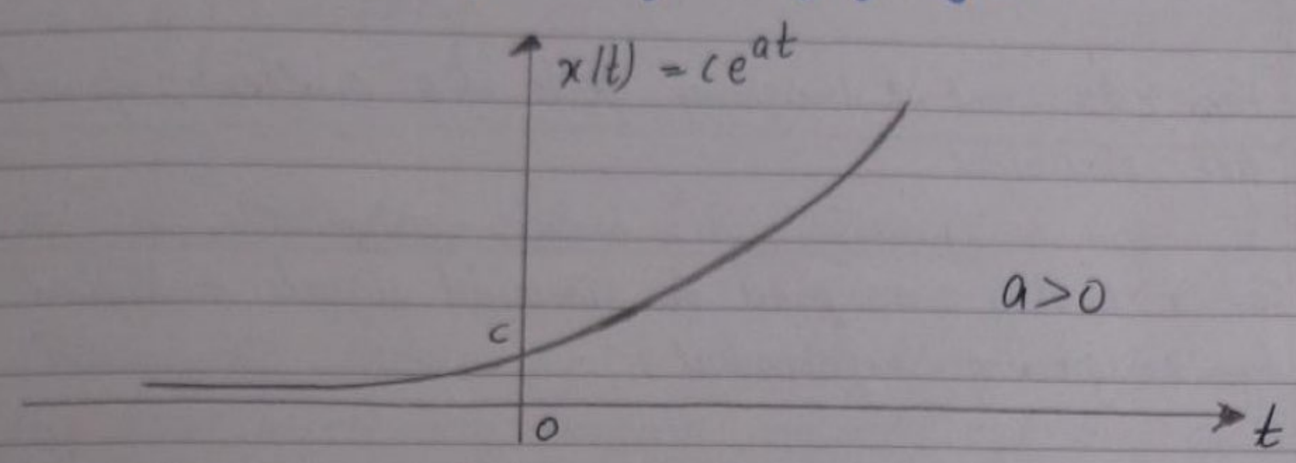


Fig: growing exponential.

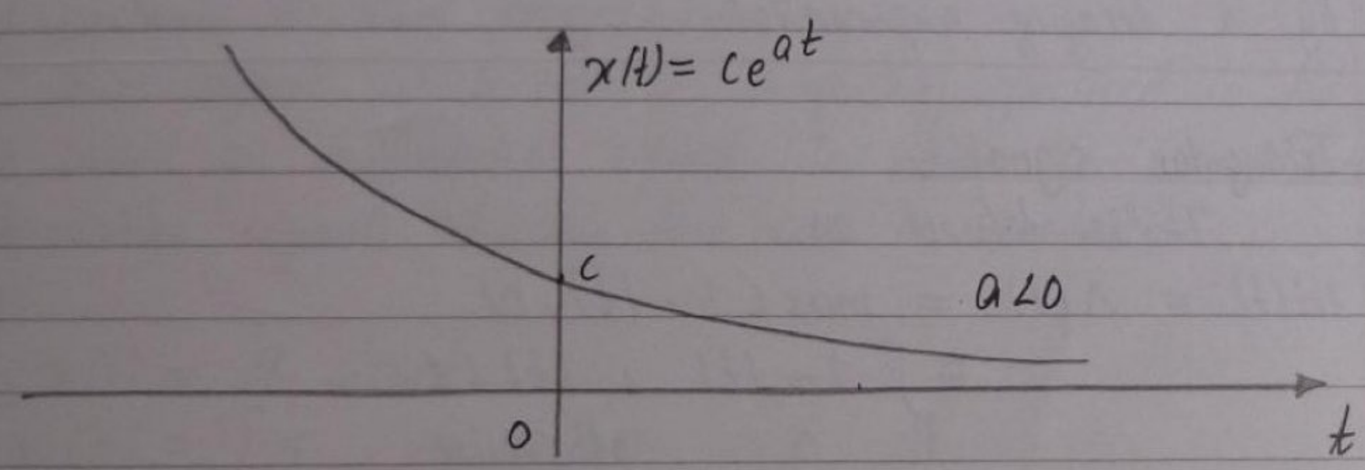


Fig: Decaying exponential.

General Complex exponential signals:

Consider a complex exponential signal ce^{at} where c is expressed in polar form and 'a' in rectangular form as,

Scanned with CamScanner

$c = |c| e^{j\theta}$

$a = r + j\omega_0$

Then,

$$ce^{at} = |c| e^{j\theta} \cdot e^{(r+j\omega_0)t}$$

$$\therefore ce^{at} = |c| e^{rt} \cdot e^{j(\omega_0 t + \theta)}$$

using Eulers relation,

$$\therefore ce^{at} = |c| e^{rt} \cos(\omega_0 t + \theta) + j|c| e^{rt} \sin(\omega_0 t + \theta)$$

For $r=0$, real & imaginary parts of a complex exponential are sinusoidal.

For $r>0$, they correspond to sinusoidal signals multiplied by a growing exponential.

For $r<0$, they correspond to sinusoidal signals multiplied by a decaying exponential.

X. <u>Triangular signal :</u>

It is defined as,

$$tri(t) = \Lambda(t) = \max(1-|t|, 0)$$

$$= \begin{cases} 1-|t| & , |t|<1 \\ 0 & \text{otherwise} \end{cases}$$

Also,

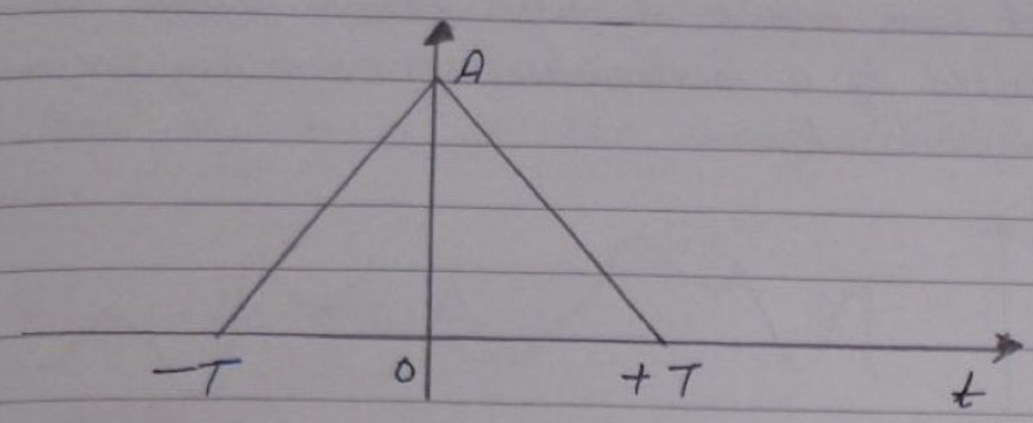
$$tri(t) = rect(t/2) * (1-|t|)$$

It has application in pulse code modulation as a pulse shape for transmitting digital signals. and as a

Scanned with CamScanner

17

matched filter for receiving the signals.



Also,

$$x(t) = A\left[1 - \frac{|t|}{T}\right]$$

XI. Deterministic and Non-deterministic signals:

Deterministic signals are a special class of stationary signals, and they have a relatively constant frequency and level content over a long time period. It is generated by rotating machines, musical devices, and electronic function generator.

Those signals which can be completely specified in the time are called as deterministic signals. The nature and amplitude of deterministic signals at any time can be predicted.

Eg:

i) $x(t) = A \cos \omega t$

ii) $x[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$

Non-deterministic signal is one which is of random nature. The pattern of such a signal is quite irregular. A typical example of this signals is the flickering of video signals that we see in

Scanned with CamScanner

TV sometimes. There is uncertainity with respect to its value at some instant of time. Random signals cannot be described by a mathematical equation. They are modelled in probabilistic term.

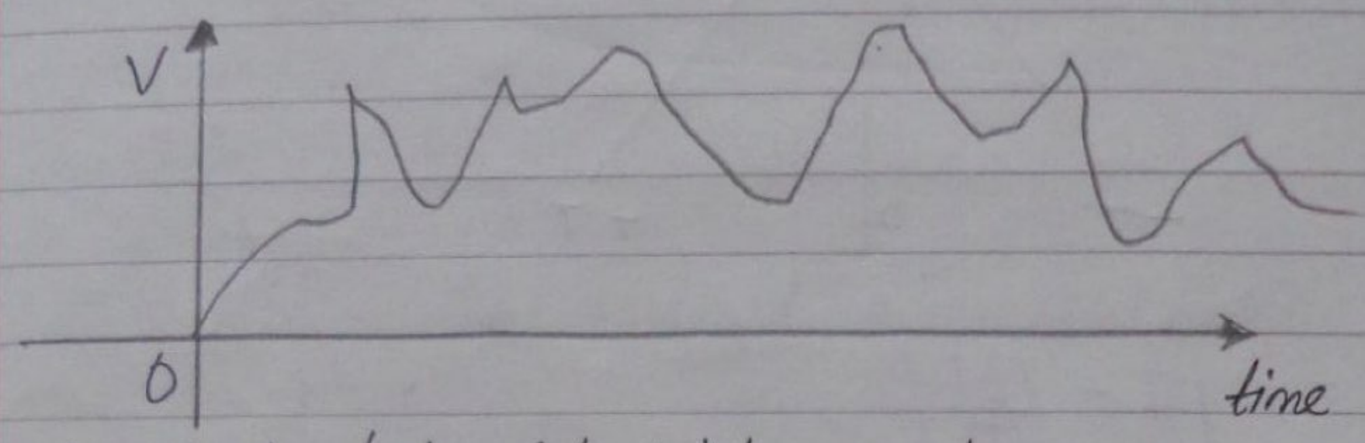


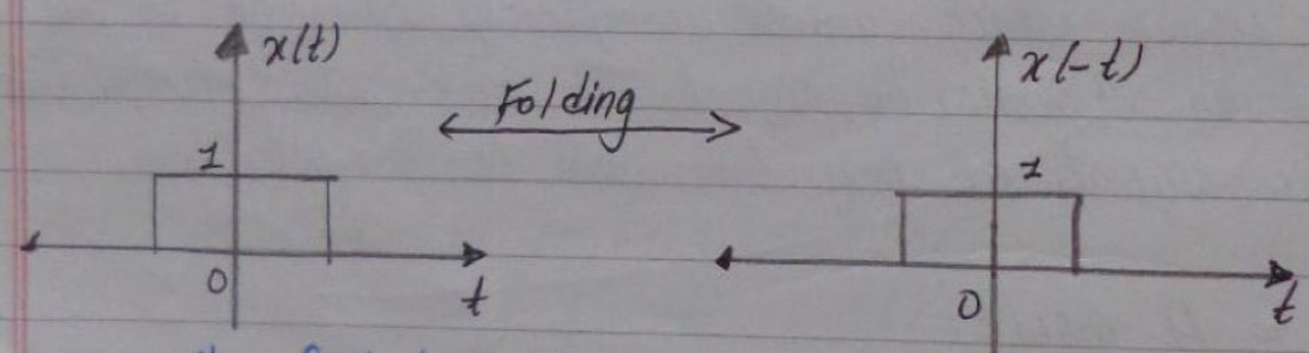
Fig: Random / Non-deterministic signal.

XII. Even and ODD signals: 2016/F, 2015/F, 2017/S

A signal is referred to as an even if it is identical to its time reversed counterpart.

ie, For continuous time signal, $x(t) = x(-t)$

" discrete " " , $x[n] = x[-n]$



Since the folded version of the signal $x(-t)$ and original signal $x(t)$ are identical, the signal is called even signal.

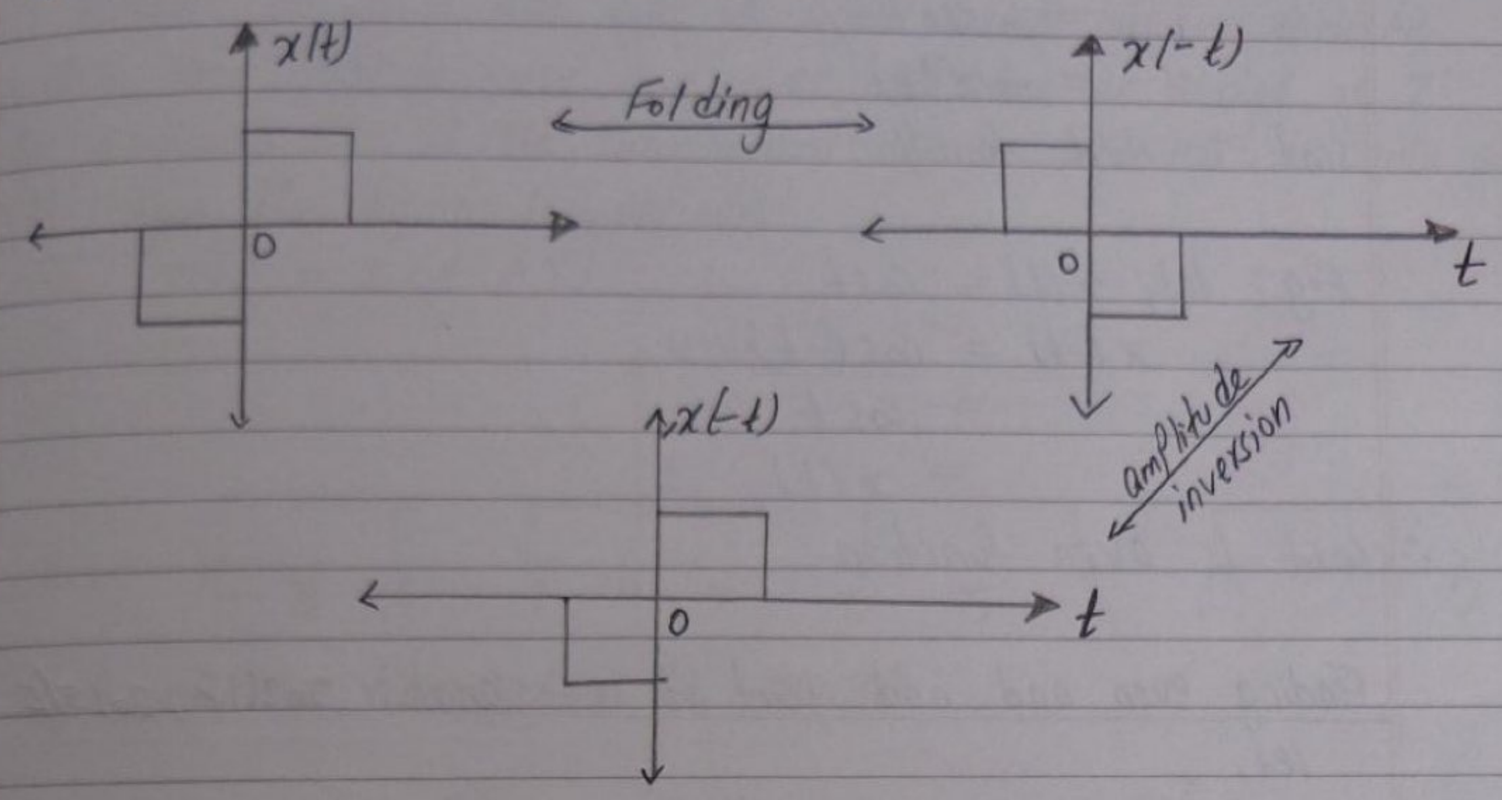
Eg: $\cos t$, t^2, $t \cdot \sin t$

Scanned with CamScanner

19

odd signals are not identical about origin. It satisfies the following condition
for continuous time signal, $x(t) = -x(-t)$
" discrete " " , $x[n] = x[-n]$
An odd signal must be 0 at $t=0$, the signal (odd) must pass through the origin.

Eg: t^3, $\sin t$, $t \cos t$



Geometrically speaking, the graph face of an even function is symmetric wrto y-axis, means that its graph remains unchanged after reflection about y-axis.

Some properties:

- The derivative of an even function is odd.
- " " " " odd " is even
- The maclaurin series of an even function include only even powers.
- " " " " " odd " " " odd powers.

Scanned with CamScanner

- The fourier series of a periodic even function includes only cosine terms.
- The fourier series of a periodic odd function includes only sine terms.

Eg: let, $x(t) = \sin t$

$$x(-t) = \sin(-t)$$
$$= -\sin t$$
$$= -x(t)$$

$\therefore$ sint is odd function

Eg: let, $x(t) = \cos t$

$$x(-t) = \cos(-t)$$
$$= \cos t$$
$$= x(t)$$

$\therefore$ cost is even function.

Finding even and odd part of a signal: 2016/F, 2015/F

let,

$x_e(t)$ = even part

$x_o(t)$ = odd part

$\therefore$ $x(t) = x_e(t) + x_o(t)$ — (I)

put $t = -t$,

or $x(-t) = x_e(-t) + x_o(-t)$

or $x(-t) = x_e(-t) - x_o(t)$ — (II)

adding (I) and (II),

$\therefore$ $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

Scanned with CamScanner

21 subtracting (i) from (ii),

$$\therefore x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

XIII. Periodic and Aperiodic signals :

Periodic signal is that type of signal which has a definite pattern and repeats over and over with time or period of T. In continuous time, if the transformed signal $x(t+T)$, is same as $x(t)$, then the signal is, periodic.

i.e, $x(t) = x(t+T)$

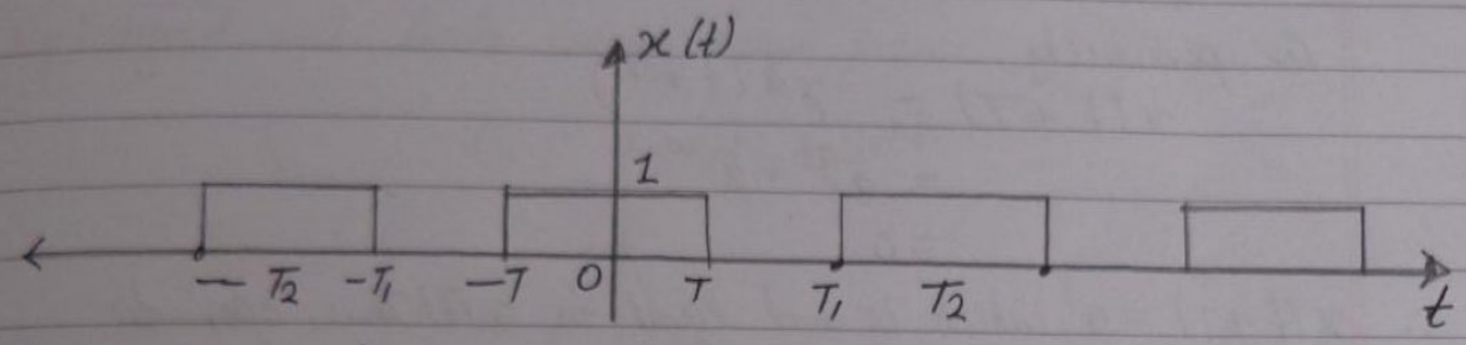


Fig: Continuous time periodic signal.

In discrete time, if the signal $x[n]$ is periodic,

$x[n] = x[n+N]$ for all integer n and m.

Here, N is called the sampling period of discrete time signal.

A signal is said to be aperiodic if it doesnot repeat. Sometime aperiodic signals are said to have a period equal to infinity. They cannot be represented by any mathematical equation. They are random signals.

E.g: sound signals from radio, all types of noise signals.

Scanned with CamScanner

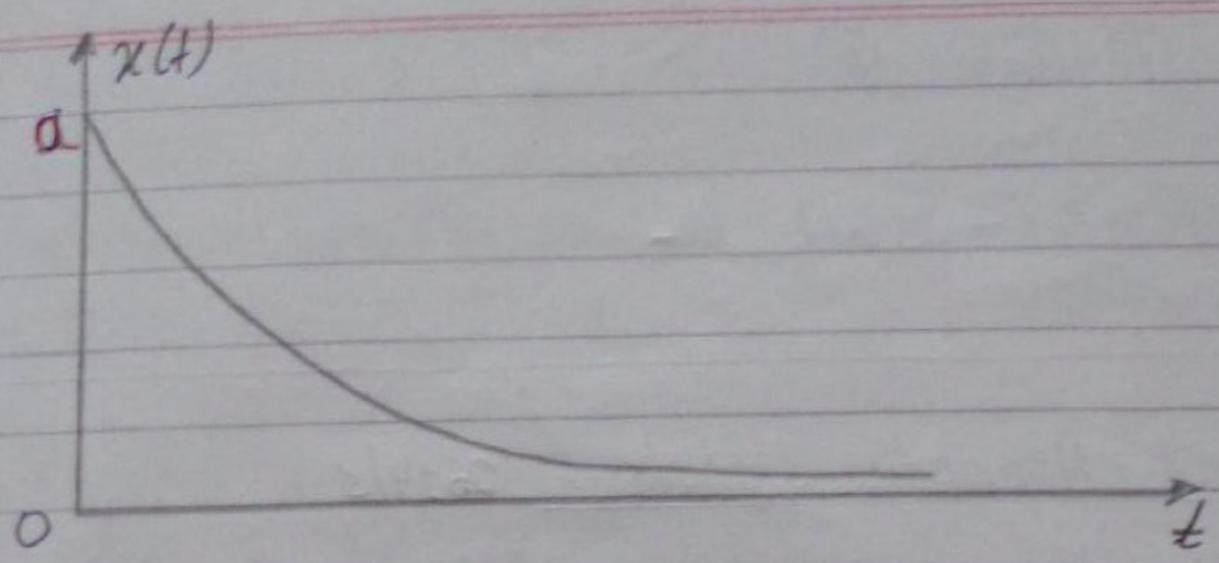


Fig: decaying exponential signal.

It is an aperiodic signal. This is a decaying exponential signal given by eqⁿ,

$$x(t) = e^{-at} \text{ for } a > 0$$

For periodicity,

$$x(t+T) = e^{-a(t+T)}$$
$$= e^{-at} \cdot e^{-\infty}$$
$$= 0$$

$\therefore$ $x(t+T) = 0$ which is not equal to $x(t)$. Thus, aperiodic signal.

XIV. Energy and power signal : 2015/F, 2017/S

The energy signal is one which has finite energy and zero average power. Hence, $x(t)$ is an energy signal if its energy E is finite i.e, $0 < E < \infty$ and average power P_{av} is zero.

$$\text{Energy, } E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

For a complex-valued signal, energy is,

Scanned with CamScanner

23

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

Similarly, energy in discrete-time signal $x[n]$ is,

$$E = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

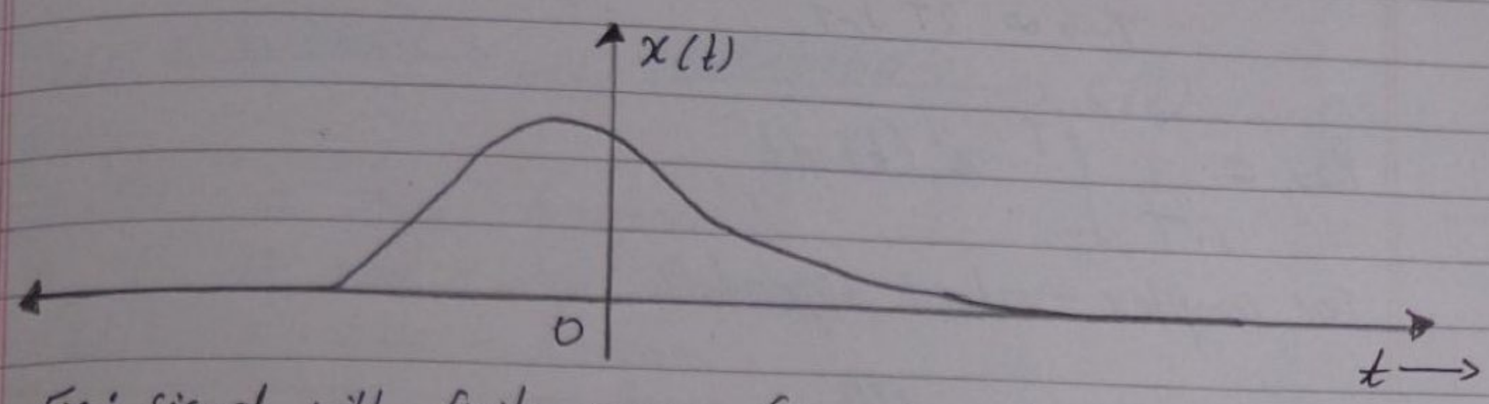


Fig: signal with finite energy (zero power).

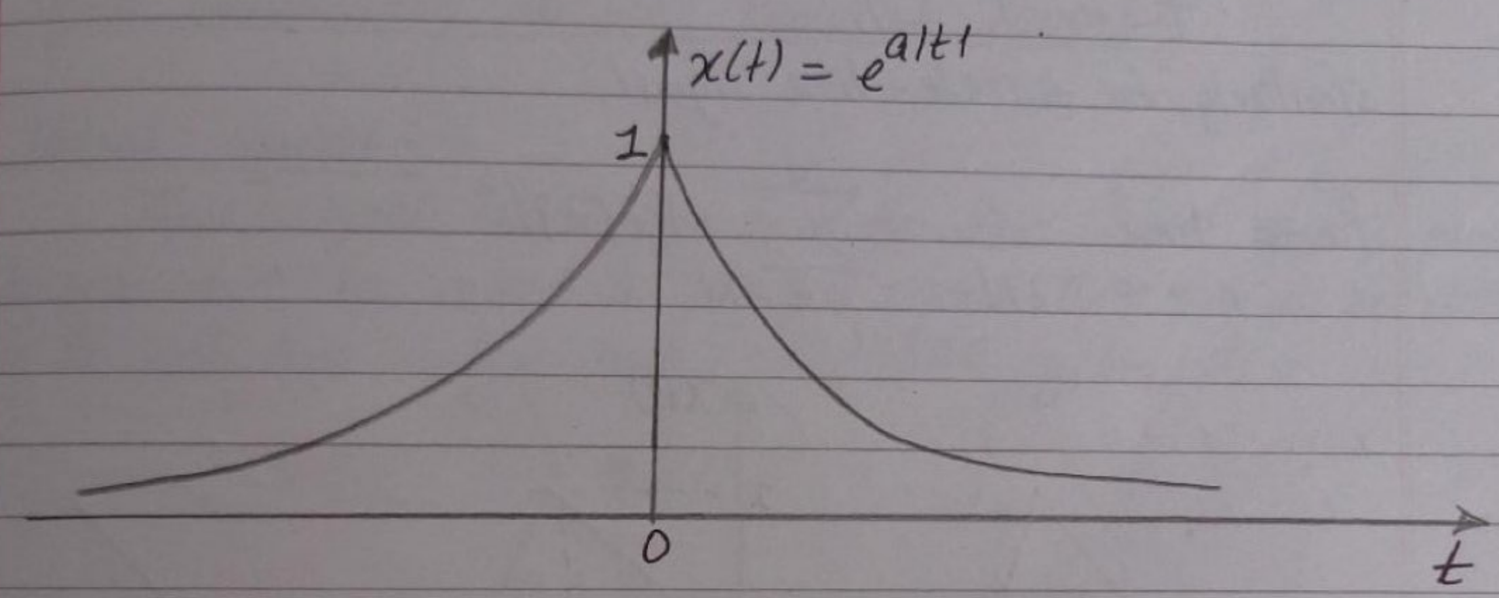


Fig: signal with finite energy

Here the signal amplitude becomes zero as $|t| \to \infty$, so this is an energy signal.

The power signal is one which has finite average power and infinite energy. Hence, $x(t)$ is an power signal if signal power P_{av}

Scanned with CamScanner

is finite i.e, $0 < P_{av} < \infty$ and energy of signal E is zero. i.e $E=0$. periodic signals are power signals. However if the signal doesnot satisfies any of the above two conditions, then it is neither an energy nor a power signal.

$$\text{Power, } P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\, dt$$

$$P_{av} = \frac{1}{T} \int_{0}^{T} x^2(t)\, dt$$

For complex-valued signal,

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

Similarly, for discrete-time signal,

$$P_{av} = \lim_{n \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

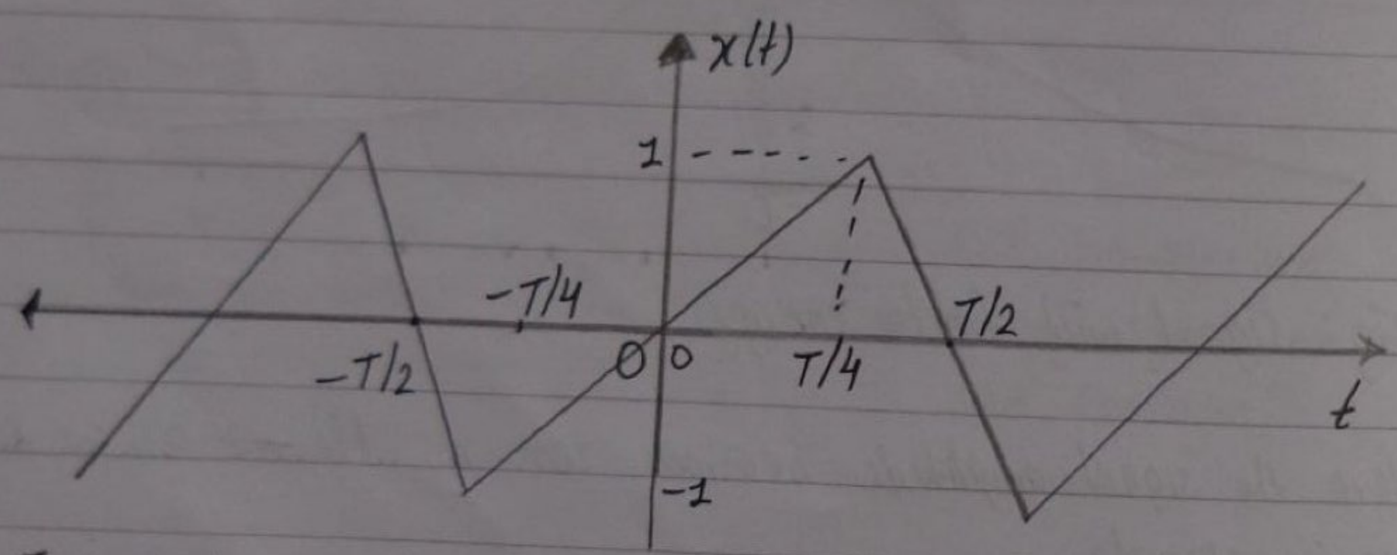


Fig: Continuous time power signal

A signal cannot be both energy and power simultaneously.

Scanned with CamScanner

27

Also, a signal may be neither energy nor power signal.

Finally,

i) $\delta(t) \xrightarrow{\text{integrate}} u(t) \xrightarrow{\text{integrate}} r(t)$

ii) $r(t) \xrightarrow{\text{differentiate}} u(t) \xrightarrow{\text{differentiate}} \delta(t)$

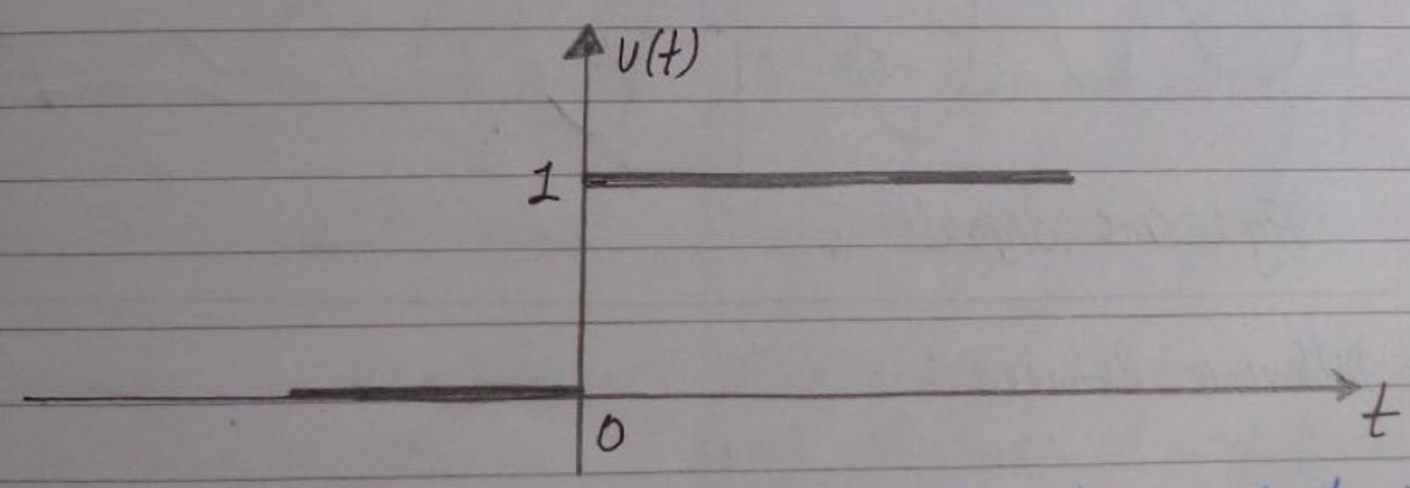
where,

$r(t)$ = ramp function.
$u(t)$ = unit - step function
$\delta(t)$ = delta - function

2.A. Signal classification on the Basis of causality:

i Causal signal:

Those signal which begin from the time instant zero & move towards $+\infty$ are called as causal signal. These signals are used in the real time systems such as RADAR system, ECG etc.

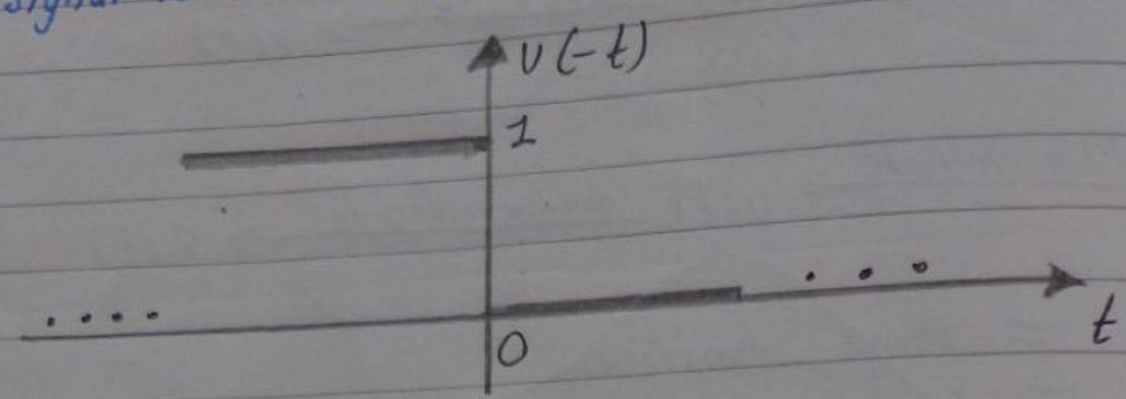


Signals which appear only in the positive time instants is called as causal signal.

Scanned with CamScanner

II. Anti- causal signal:

Those signals which occur only on negative time instants are called as anti-causal signal. Folded unit step signal is one example of anti-causal signal.



III. Non- causal signal:

Those signals which occur only on both positive and negative time instants are called as non-causal signal.

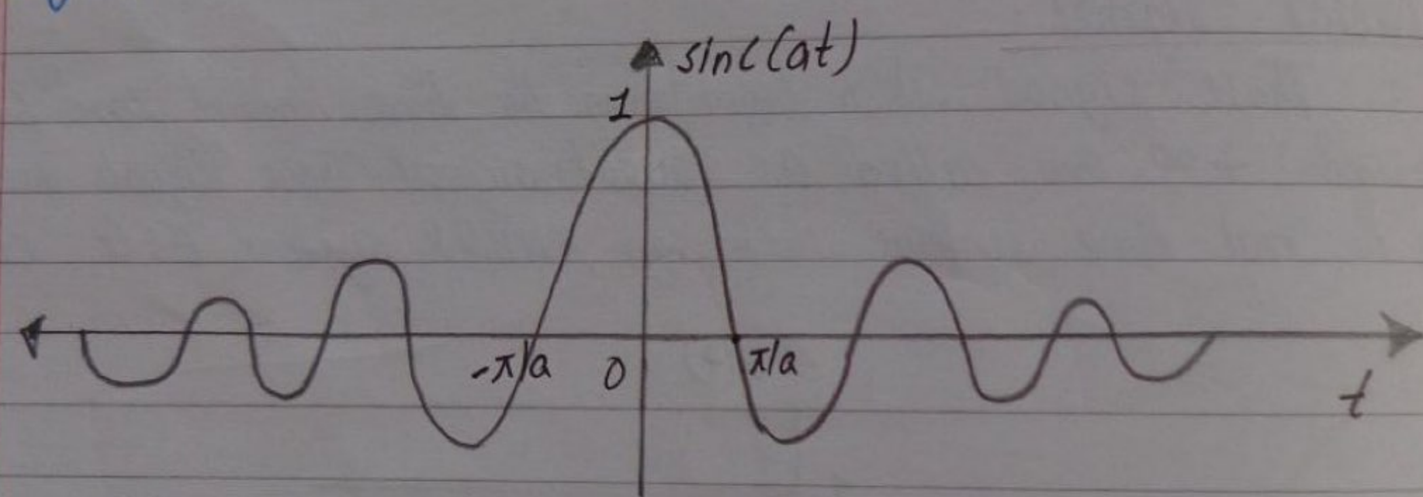


Fig: sinc signal.

3.A. Difference Between:

a)	Energy signal	Power signal
i)	Total normalized energy is	Normalized average power

Scanned with CamScanner

29

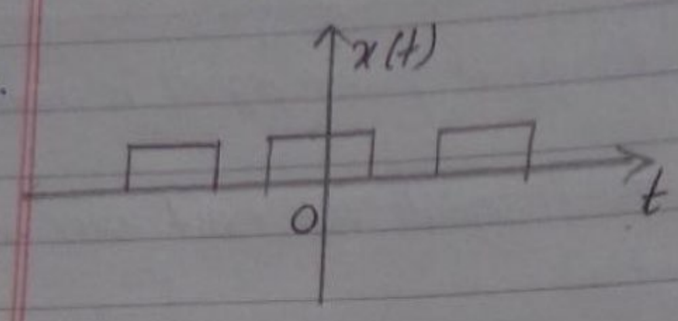
finite and non-zero.	is finite and non-zero.
ii. Power of energy signal is 0.	Energy of power signal is infinite.
III. These signals are time limited.	These signals can exist over infinite time.
IV. Non-periodic signals are generally energy signals.	Practical periodic signals are power signals.
V. $E = \int_{-\infty}^{\infty} x^2(t)\,dt$	$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,dt$

6. Periodic signal	Aperiodic signal
i. A signal which repeat itself after a specific interval of time is called periodic signal.	A signal which doesnot repeat itself after a specific interval of time is called aperiodic signal.
II. Eg: sine, cosine, square signals.	Eg: all types of noise signal.
III. They are deterministic signals.	They are random signals.
IV. Their value can be determined at any point of time.	Their value cannot be determined with certainity at any given point of time.
V. can be represented by mathe-	cannot be represented by mathematical

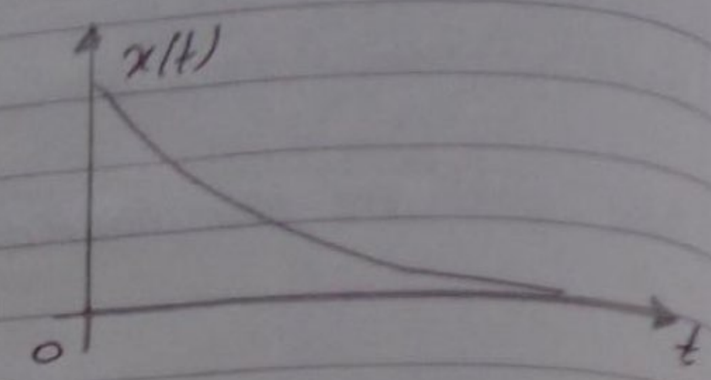
Scanned with CamScanner

matical equation.

vi.



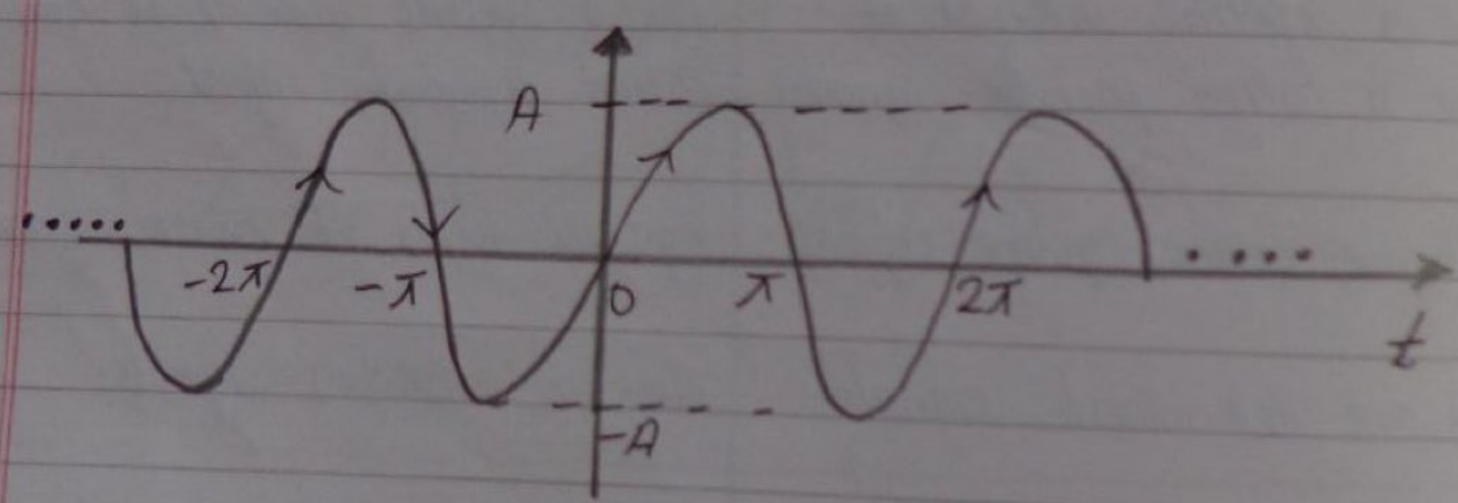
equation.



Numericals :

1.A sketch the following signal $x(t) = A \cdot \sin(t)$ for $-\infty < t < \infty$
Also check whether the $x(t)$ is a power signal or an energy signal or neither.

$\Rightarrow$ Sol^n,
Given signal, $x(t) = A \sin t$ for $-\infty < t < \infty$.



Since the signal is periodic, let us proceed with average power
The average power of the signal is,

$$P_{av} = \frac{1}{T}\int_0^T x^2(t)\, dt$$

Scanned with CamScanner

31

$$= \frac{1}{2\pi}\int_0^{2\pi} A^2 \cdot \sin^2 t \, dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} A^2 \left(\frac{1-\cos 2t}{2}\right) dt$$

$$= \frac{A^2}{4\pi}\int_0^{2\pi} 1 \cdot dt - \frac{A^2}{4\pi}\int_0^{2\pi} \cos 2t \, dt$$

$$= \frac{A^2}{4\pi} \cdot 2\pi - \frac{A^2}{4\pi}\left[\frac{\sin(2t)}{2}\right]_0^{2\pi}$$

$$= \frac{A^2}{2} - \frac{A^2}{8\pi}(\sin 4\pi - \sin 0)$$

$$\therefore P_{av} = \frac{A^2}{2}$$

Total energy is, $E = \int_{-\infty}^{\infty} x^2(t)\, dt$

$$= \int_{-\infty}^{\infty} A^2 \sin^2 t \, dt.$$

$$= \int_{-\infty}^{\infty} A^2 \left(\frac{1-\cos 2t}{2}\right) dt$$

$$= \frac{A^2}{2}\int_{-\infty}^{\infty} dt - \frac{A^2}{2}\int_{-\infty}^{\infty} \cos 2t \, dt$$

$$\therefore E = \infty$$

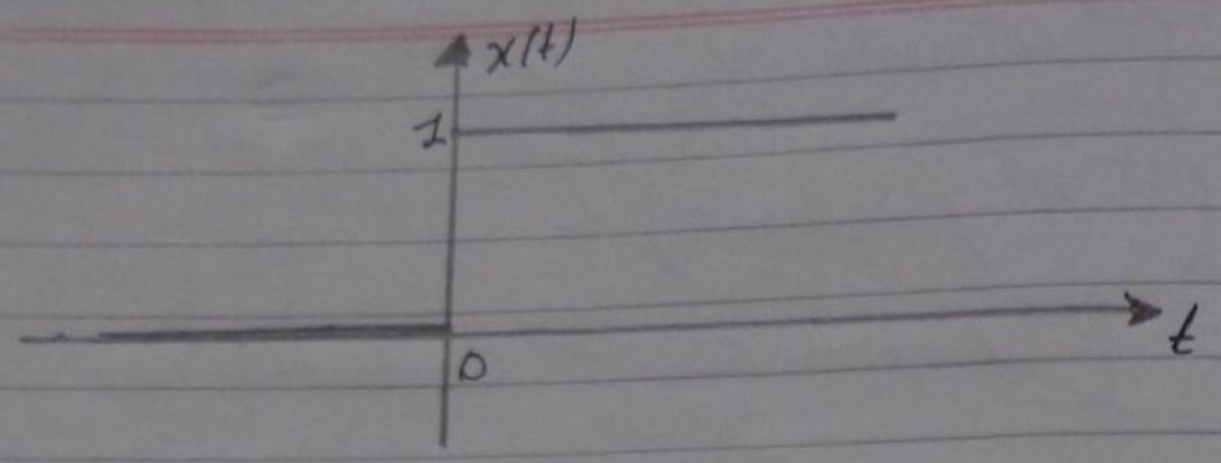
This shows finite power and infinite energy, hence this is a power signal.

---

8. Determine the total energy and total average power of continuous time unit step signal and decide whether it is energy signal or power or neither.

⇒ Sol^n,

Scanned with CamScanner



Given,

Continuous time unit step signal is,

$$u(t) = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t>0 \end{cases}$$

So, total energy of unit step signal is,

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\, dt$$

$$= \lim_{T\to\infty} \int_{0}^{T/2} (1)^2\, dt$$

$$= \lim_{T\to\infty} \frac{T}{2}$$

$$E = \infty$$

Total average power of unit step signal is,

$$P_{av} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} (1)^2\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \cdot \frac{T}{2}$$

Scanned with CamScanner

33

$\therefore P_{av} = \frac{1}{2}$

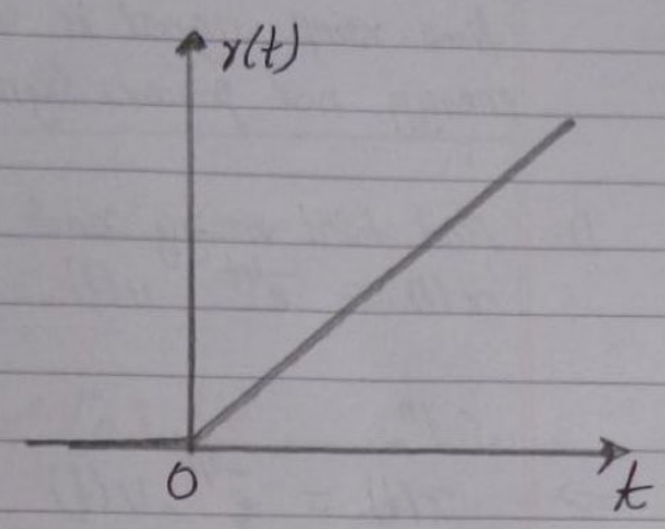
Thus, total energy of unit step signal is infinite and total average power is finite. Hence, it is power signal.

6. Determine the total energy and total average power of continuous time ramp signal and decide whether it is energy or power signal or neither.

⇒ soln,

Continuous time ramp signal is,

$$r(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$



So, total energy of ramp signal is,

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} x^2(t)\, dt$$

$$= \lim_{T \to \infty} \int_{0}^{T/2} (t^2)\, dt$$

$$= \lim_{T \to \infty} \left[\frac{t^3}{3} \right]_0^{T/2}$$

$$= \lim_{T \to \infty} \frac{T^3}{24}$$

$$= \infty$$

Total average power of ramp signal is,

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$$

Scanned with CamScanner

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^{T/2} (t^2)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \cdot \times \frac{T^3}{24}$$

$$= \lim_{T\to\infty} \frac{T^2}{24}$$

$\therefore P_{av} = \infty$

Since, both total energy and total average power of continuous time ramp signal is ∞, hence ramp signal is neither energy nor power signal.

D. Find total energy and average power of,
$x(t) = e^{-4t} \cdot u(t)$

Sol^n,

$\Rightarrow$ $x(t) = e^{-4t} \cdot u(t)$

Energy, $E = \int_{-\infty}^{\infty} x^2(t)\,dt$

$$= \int_0^{\infty} e^{-4t}\,dt$$

$$= \left[\frac{e^{-4t}}{-4}\right]_0^{\infty}$$

$$E = \frac{1}{4}$$

power of the signal, $P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$

$$= \lim_{T\to\infty} \frac{E}{T} = 0$$

Scanned with CamScanner

3

since the energy of the signal is finite and power is infinite so given signal is energy signal.

E. Given an aperiodic signal $x[n] = \{6, 5, 4, 3, 2, 1, 2, 3, 4, 5, 6\}$, find out the energy and average power.

Sol^n,

$\Rightarrow$ since, it is told that the given signal is aperiodic, we proceed by calculating the energy,

$$E = \sum_{n=0}^{10} x^2[n]$$

$$= 36 + 25 + 16 + 9 + 4 + 1 + 4 + 9 + 16 + 25 + 36$$

$$= 181.$$

Average power, $P_{av} = \frac{1}{N} \sum_{n=0}^{10} x^2[n]$

$$= \frac{181}{10}$$

$$= 18.1$$

Since the energy of the signal is finite and average power of the signal is also finite. The given signal is neither an energy nor power signal.

F. Find the total energy and average power of $x[n] = \{3, \underset{\uparrow}{1}, 0, 2+2j, 7\}$.

Sol^n,

$\Rightarrow$ $x[n] = \{3, \underset{\uparrow}{1}, 0, 2+2j, 7\}$

we know,

$$E = \sum_{n=-1}^{3} x^2[n]$$

Scanned with CamScanner

$= 9 + 1 + 0 + (2^2 + 2^2) + 7^2$

$= 67$

Average power, $P_{av} = \frac{1}{N} \sum_{n=-1}^{3} x^2[n]$

$= \frac{67}{5}$

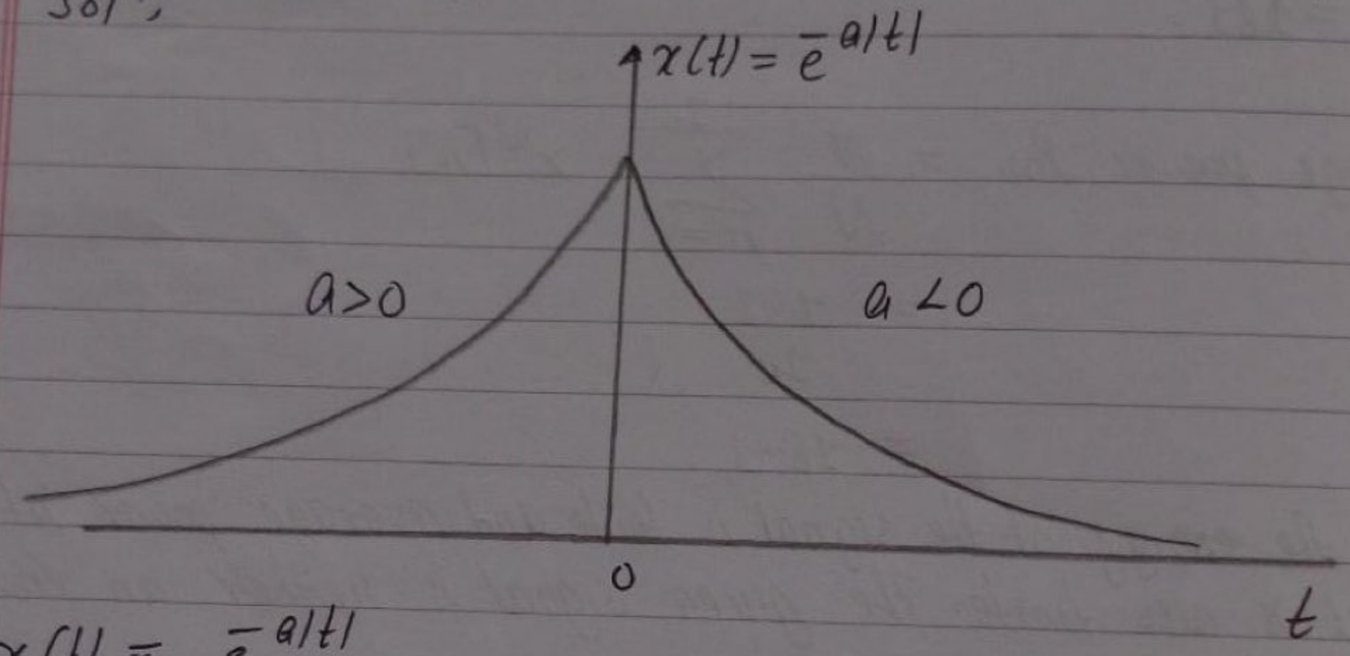
$= 13.4$

Since the energy of the signal is finite and average power of signal is also finite, it is neither energy nor power signal.

Eg Find the energy and power of following signal, $x(t) = e^{-a|t|}$ for $a>0$ and state whether it is energy or power signal.

Sol^n,

$\Rightarrow$



$x(t) = e^{-a|t|}$

$= \begin{cases} e^{-at} & \text{if } t>0 \\ e^{at} & \text{if } t<0 \end{cases}$

Since, given signal is aperiodic and amplitude of the signal becomes zero in future.

$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

Scanned with CamScanner

37

$$= \int_{-\infty}^{0} e^{2at}\,dt + \int_{0}^{\infty} e^{-2at}\,dt$$

$$= \left[\frac{e^{2at}}{2a}\right]_{-\infty}^{0} + \left[\frac{e^{-2at}}{-2a}\right]_{0}^{\infty}$$

$$= \frac{1}{2a} + \frac{1}{2a}$$

$$= \frac{1}{a}$$

Average power of the signal is,

$$P_{av} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\,dt$$

$$= \lim_{T\to\infty} \frac{E}{T}$$

$$= 0$$

Since, energy is finite and power of signal is zero, hence given signal is energy signal.

G. BASIC ELEMENTARY SIGNALS:

I Unit - Step Function
II Unit - impulse Function.
III Unit - ramp function.
IV exponential function.
V sinusoidal Function.

H. What are elementary signals? Explain periodicity properties of discrete time complex exponential signals and continuous time complex exponential signals.

Scanned with CamScanner

Elementary signals are the basic building blocks for constructing more complex signals. It includes sinusoidal, exponential signals, step unit function, impulse function and ramp function.

Periodic complex exponential signals :

2015/5

Consider,

$$x(t) = e^{j\omega_0 t}$$

It will be periodic with period T if,

$$e^{j\omega_0 t} = e^{j\omega_0 (t+T)}$$

or, $$e^{j\omega_0 (t+T)} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$$

For periodicity, we must have,

$$e^{j\omega_0 T} = 1$$

If $\omega_0 = 0$, then $x(t) = 1$ which is periodic for any value of T.
If $\omega_0 \neq 0$, then the fundamental period of T_0 of $x(t)$ - i.e, smallest positive value of T for which,

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have same fundamental period.

By Euler's relation,

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

Thus, Both the real and imaginary parts of the complex exponential signal are sinusoidal signals.

Scanned with CamScanner

A discrete time complex exponential signal is expressed by,

$$x[n] = e^{j\omega_0 n}$$

and,

Euler's relation gives, $e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$

In order for the signal $e^{j\omega_0 n}$ to be periodic with period $N>0$, we must have,

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n}$$

or equivalently, $e^{j\omega_0 N} = 1$

To hold $\omega_0 N$ must be a multiple of 2π. i.e, there must be an integer m such that,

$$\omega_0 N = 2\pi m$$

i.e, $\frac{\omega_0}{2\pi} = \frac{m}{N}$

Hence, the signal $e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number and is non-periodic otherwise.

	$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
i)	periodic for any choice of ω_0	periodic only if $\omega_0 = 2\pi m/N$ for some integer $N>0$ and m
ii	Fundamental frequency ω_0.	Fundamental frequency ω_0/m
iii	Fundamental period, $\omega_0 = 0$; undefined, $\omega_0 \neq 0$; $\frac{2\pi}{\omega_0}$	Fundamental period $\omega_0 = 0$: undefined, $\omega_0 \neq 0$; $m\left(\frac{2\pi}{\omega_0}\right)$
iv	Distinct signal for distinct values of ω_0.	Identical signals for values of ω_0 separated by multiples of 2π

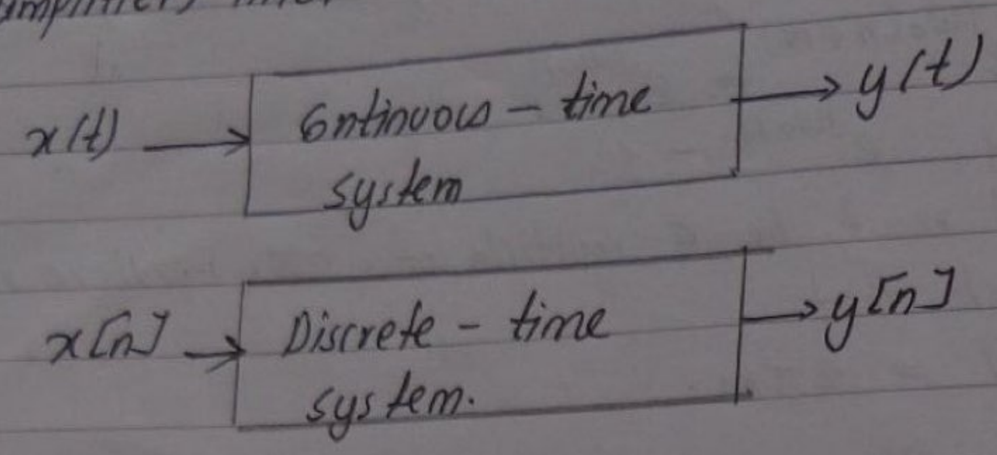
Scanned with CamScanner

2012/F, 2015/S, 2016/F

I. Define systems. Explain all the properties of systems.

⇒. A system can be defined as an interconnection of components devices or subsystems that processes particular input signals to produce the desired outputs.

Eg: amplifier, filter.



Properties of systems:

i) memory:

A system is said to be memoryless if its output signal depends only on the present value of the input signal. A resistor is memory less system.

A system is said to be with memory if its output signal depends on the past or future values of the input signal. A capacitor is an example of a continuous time system with memory. Since the voltage across it depends on the past values of the current.

i.e, $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(t)\,dt$.

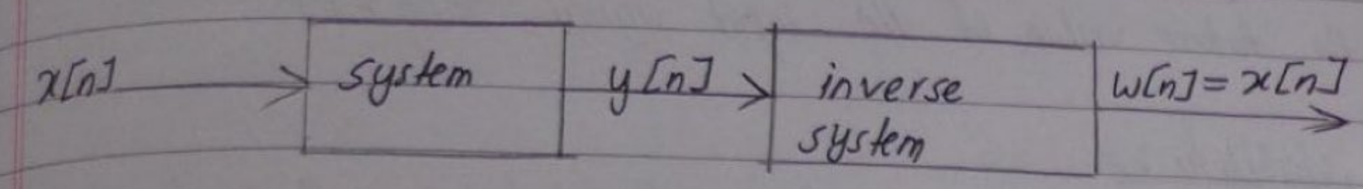
and

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Scanned with CamScanner

II. Invertibility :

A system is said to be invertible if the input of the system can be recovered from the output.



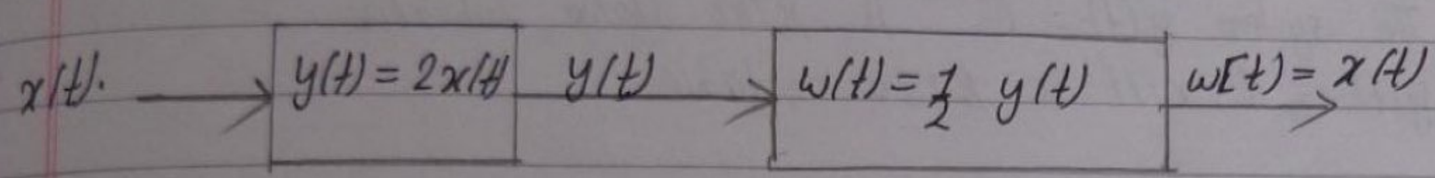
Example of invertible system are, $y[n] = 0$, $y(t) = x^2(t)$.

An example of an invertible continuous time system is,

$$y(t) = 2x(t)$$

for which the inverse system is,

$$w(t) = \frac{1}{2} y(t).$$



III. Causality :

A system is said to be causal if the present value of the output signal depends only on the present and or past values of the input signal. If the present value of the output signal depends also on the future value of the input system, then the system is referred to as the non-causal system.

All the real-time systems and memory less systems are causal. Eg: resistor, inductor, capacitor, accumulator etc.

The moving average system described by,

Scanned with CamScanner

$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$

is non-causal, since the output signal $y[n]$ depends also on the future value of the input signal, $x[n+1]$.

IV. Stability:

A system is said to be Bounded input - Bounded output (BIBO) stable if the output is bounded for bounded input.

i.e,

$|y(t)| \leq M_y < \infty$ for all t.

for input signal,

$|x(t)| \leq M_x < \infty$ for all t.

where, M_x and M_y are finite positive numbers.

The system $y(t) = e^{x(t)}$ is BIBO stable whereas $y(t) = t \cdot x(t)$ is BIBO unstable.

V. Time invariance:

A system is said to be time invariant if its behaviour doesnot change over time.

ie, if $x(t) \longrightarrow y(t)$, then,

$x(t-t_0) \longrightarrow y(t-t_0)$

VI. Linearity:

A system is said to be linear if it satisfies the superposition theorem. That means if an input consists of the weighted sum of several signals then the output is the weighted sum of the responses of the system to each of those signals.

Scanned with CamScanner

Let, $x_1(t)$ and $x_2(t)$ be input signals and $y_1(t)$ and $y_2(t)$ be the corresponding output signals,

$x_1(t) \longrightarrow y_1(t)$

$x_2(t) \longrightarrow y_2(t)$.

then system is linear if,

$x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$ (additive property).

and,

$ax_1(t) \longrightarrow ay_1(t)$ [homogeneity property].

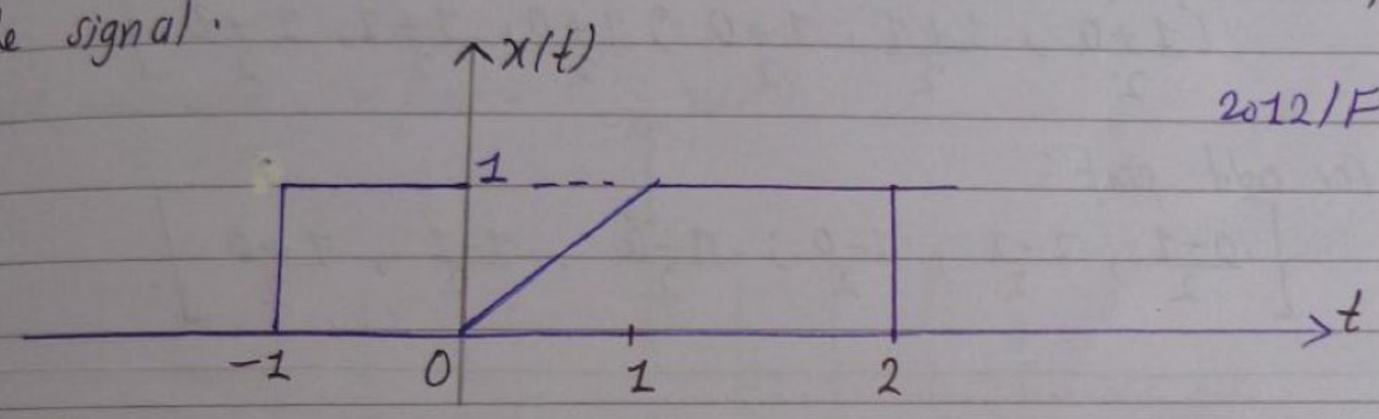
Combining two properties,

$ax_1(t) + bx_2(t) \longrightarrow ay_1(t) + by_2(t)$

where a and b are constant.

J. From the given signal $x(t)$, find and sketch the even & odd pairs of the signal.

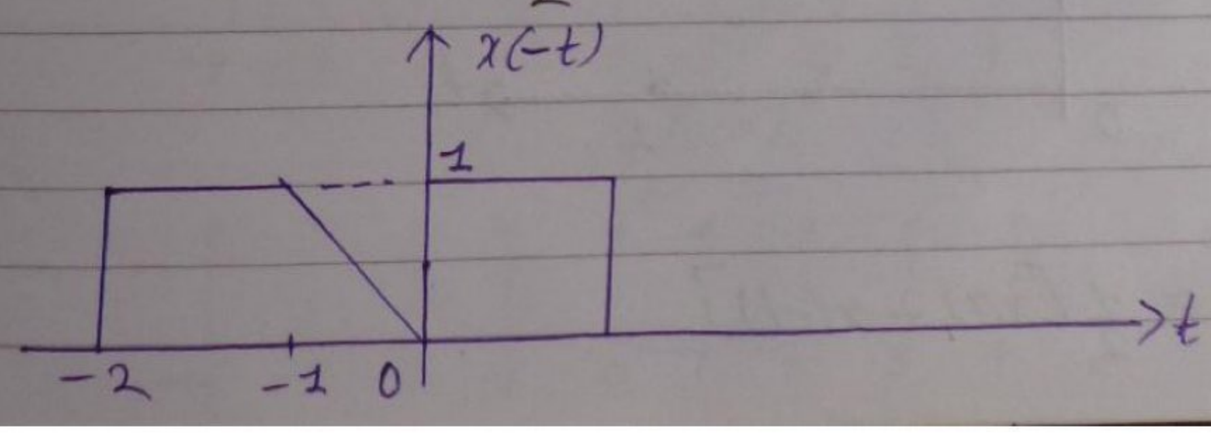
2012/F



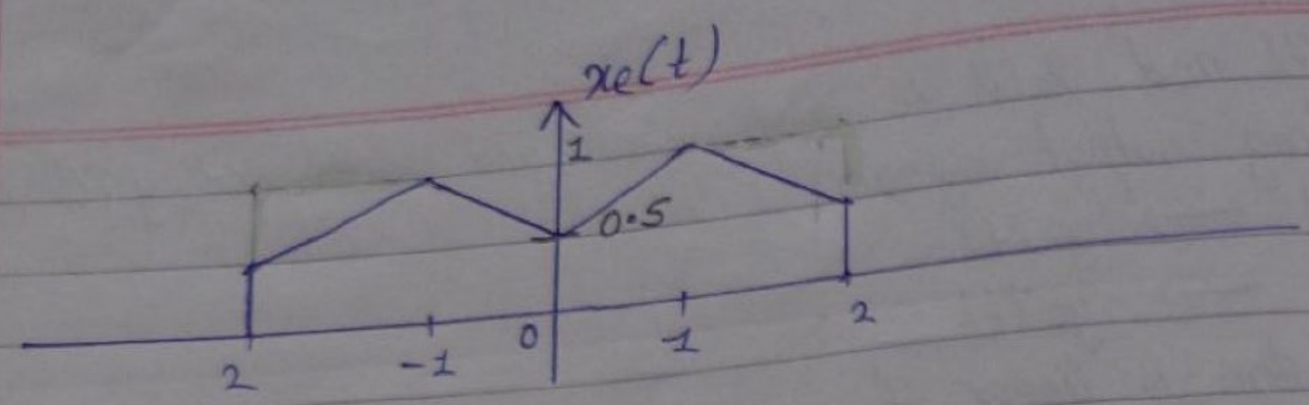
Sol^n,

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$



Scanned with CamScanner



even part:

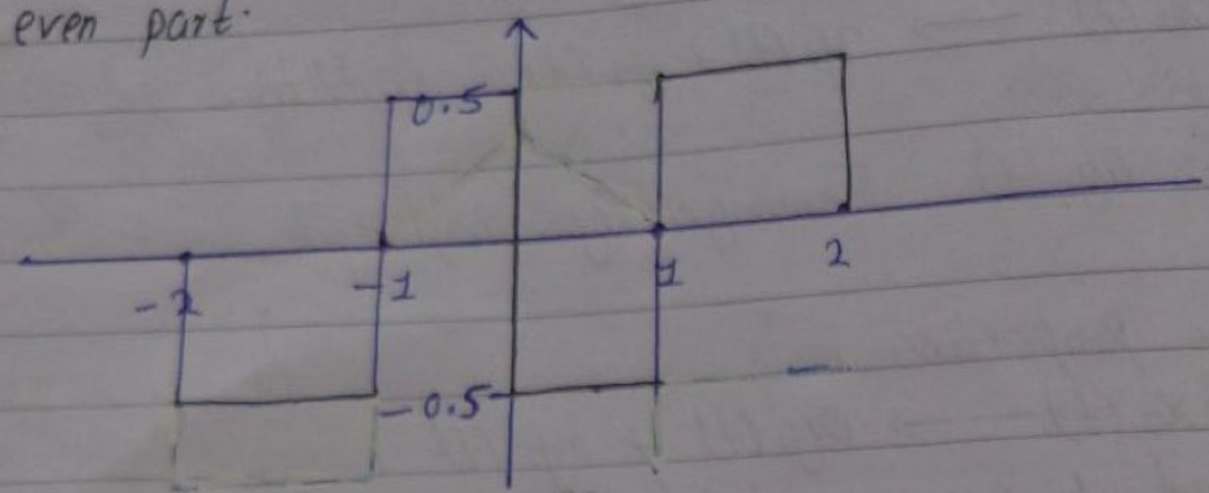


Fig: odd part of $x(t)$.

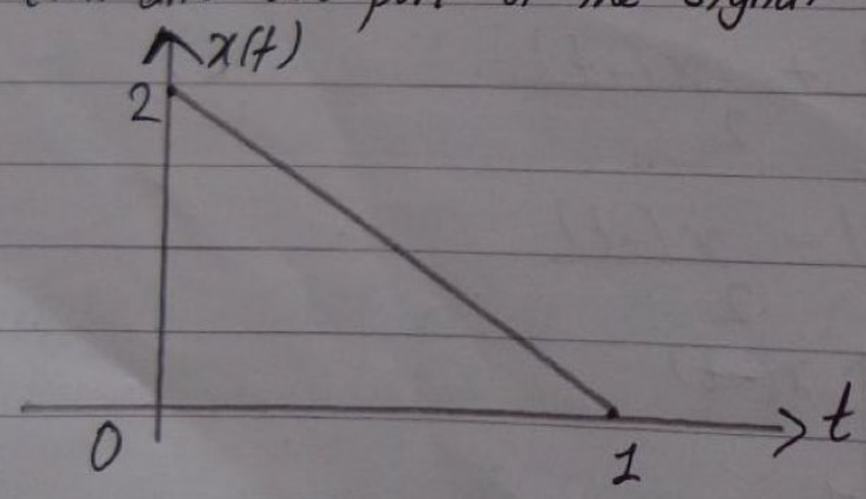
Trick: For even part: [amplitude]

$$\left[\frac{1+0}{2}, \frac{1+1}{2}, \frac{1+0}{2}; \frac{1+0}{2}, \frac{1+1}{2}, \frac{1+0}{2}\right]$$

For odd part:

$$\left[\frac{0-1}{2}, \frac{1-1}{2}, \frac{1-0}{2}; \frac{0-1}{2}, \frac{1-1}{2}, \frac{1+0}{2}\right]$$

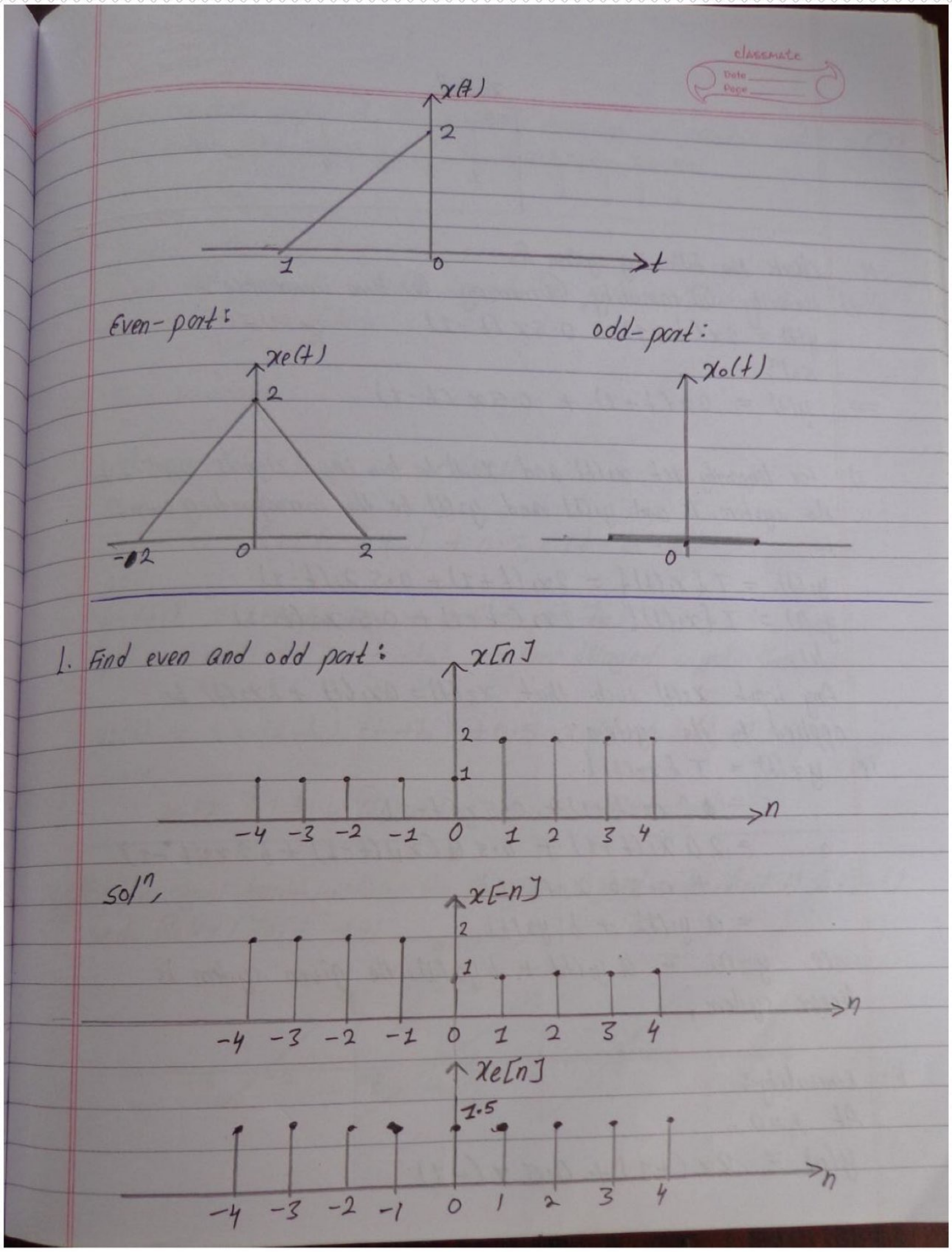
K. Find the even and odd part of the signal



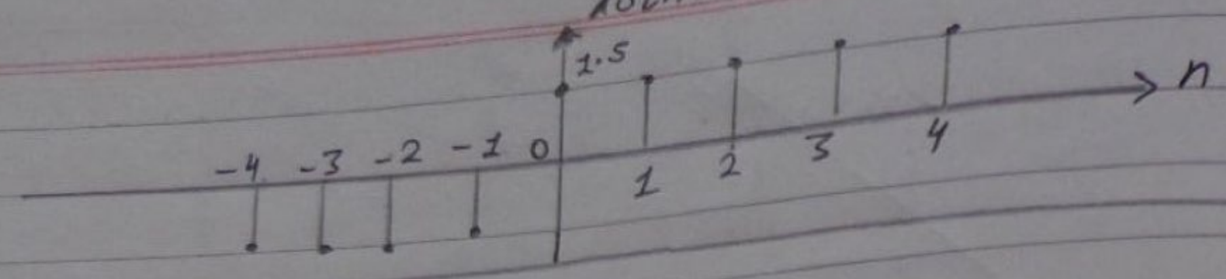
Sol^n,

$$\Rightarrow \quad x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

Scanned with CamScanner



Scanned with CamScanner



M. Check the following system for:
i) Linearity (ii) causality (iii) memory (iv) time invariance.

2014/F

$y(t) = 2x(t+1) + 0.5x(t-1)$

Sol^n,

$\Rightarrow \quad y(t) = 2x(t+1) + 0.5x(t-1)$

i) For linearity, let $x_1(t)$ and $x_2(t)$ be two input signals applied to the system T and $y_1(t)$ and $y_2(t)$ be the corresponding output.

i.e,

$y_1(t) = T\{x_1(t)\} = 2x_1(t+1) + 0.5\,x_1(t-1)$

$y_2(t) = T\{x_2(t)\} = 2x_2(t+1) + 0.5\,x_2(t-1)$

let,

Any input $x_3(t)$ such that $x_3(t) = ax_1(t) + bx_2(t)$ be applied to the system;

ie, $y_3(t) = T\{x_3(t)\}$

$= 2x_3(t+1) + 0.5x_3(t-1)$

$= 2a\,x_1(t+1) + 0.5\,a(x_1)(t-1) + b\,2x_2(t+1) + 0.5b\,x_2(t-1)$

$= a\,y_1(t) + b\,y_2(t)$

since, $y_3(t) = a\,y_1(t) + b\,y_2(t)$, the given system is linear system.

ii. causality:

At $t=0$,

$y(0) = 2x(+1) + 0.5\,x(-1)$

Scanned with CamScanner

since, present output of the system is depending on past value $x(-1)$ and future input value $(+1)$. So, it is non causal.

iii. memory:

$y(t) = 2x(+1) + 0.5x(-1)$

since, for present output depends on past and future input, the system isn't memoryless.

iv. Time invariance:

$y(t) = 2x(t+1) + 0.5x(t-1)$

Now,

$y_1(t-t_0) = 2x(t_0-t_0+1) + 0.5x(t-t_0-1)$

and,

$y_2(t_1) = 2x_2(t+1) + 0.5x_2(t-1)$

put, $x_2(t) = x_1(t-t_0)$ → time delayed input, then,

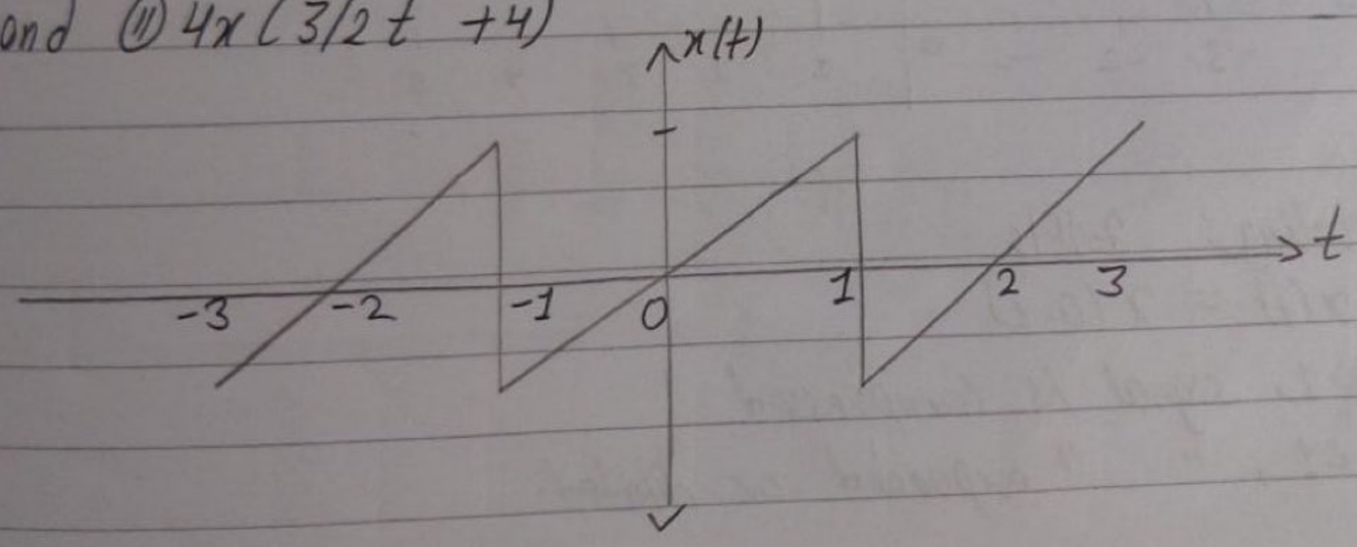
$y_2(t) = 2x_1(t-t_0)(t+1) + 0.5x_1(t-t_0)(t-1)$

since,

$y_1(t-t_0) \neq y_2(t)$, it is time variant system.

N. Explain signal transformations. For the given signal $x(t)$, find i) $x(3-t)$ and ii) $4x(3/2\,t+4)$

2008/5



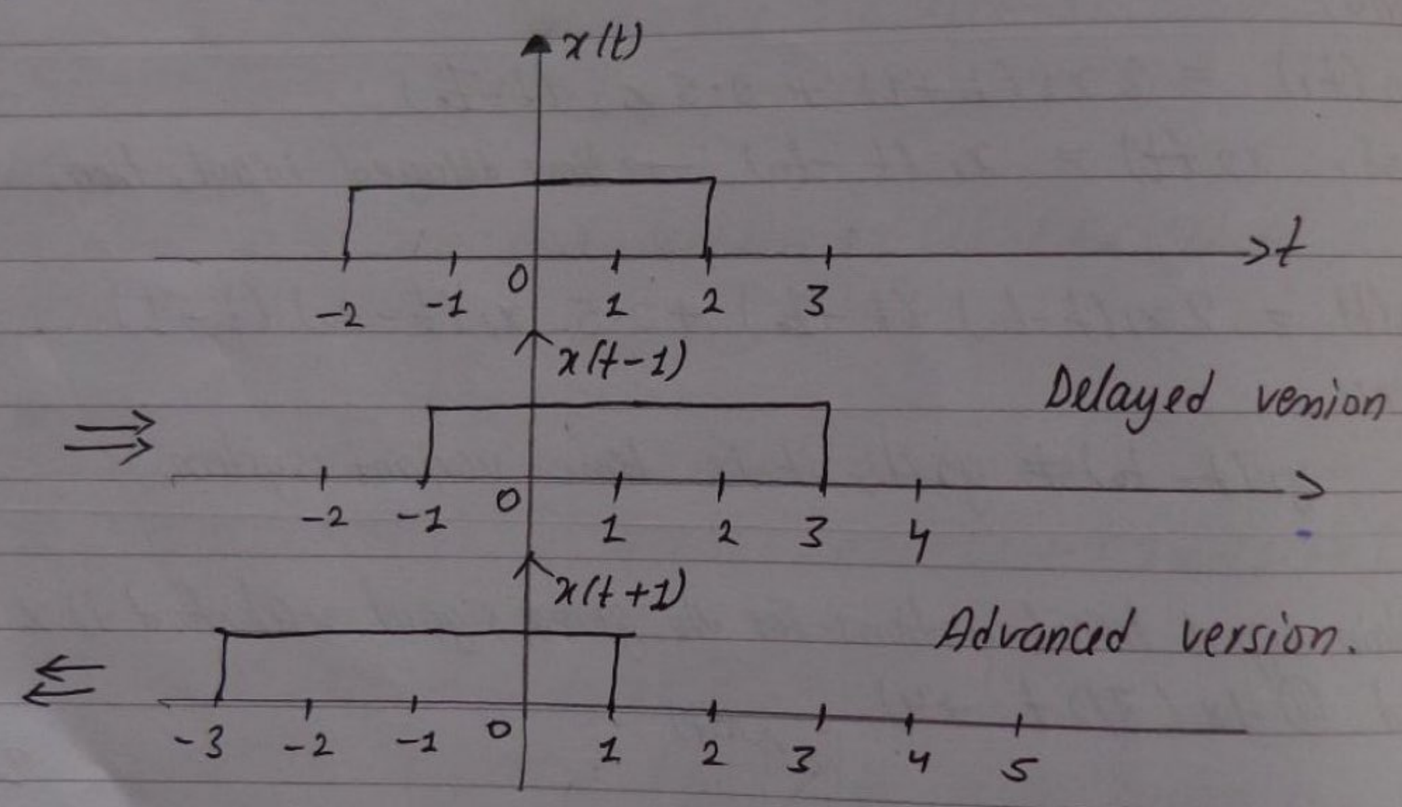
Scanned with CamScanner

When we process a signal, this signal may undergo several manipulations involving the independent variable. Some transformations of independent variable are;

i) Time - shifting
ii) Time - scaling
iii) Time - inversion.

i) Time-shifting:

It involves advancement and Delay. If the transformed signal begins earlier than the original signal then it is called as advancement of the signal. and if the transformed signals begin later than the original signal, then it is called as delay of the signal and signal is called delayed signal.



ii) Time scaling: 2016/s

let, $x(t) = x(a \cdot t)$

if $a > 1$, signal is compressed

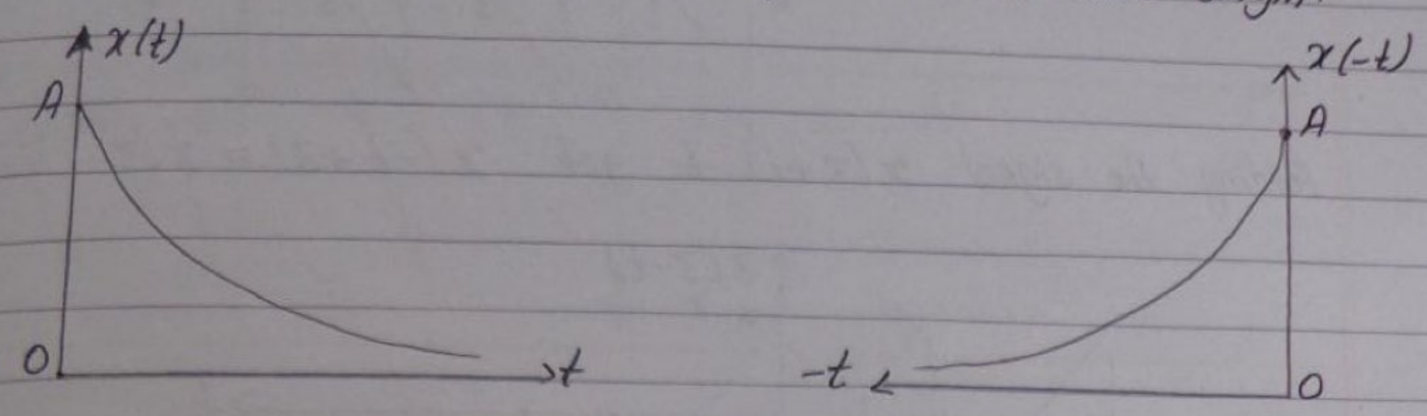
if $a < 1$, " " expanded or dilated.

Scanned with CamScanner

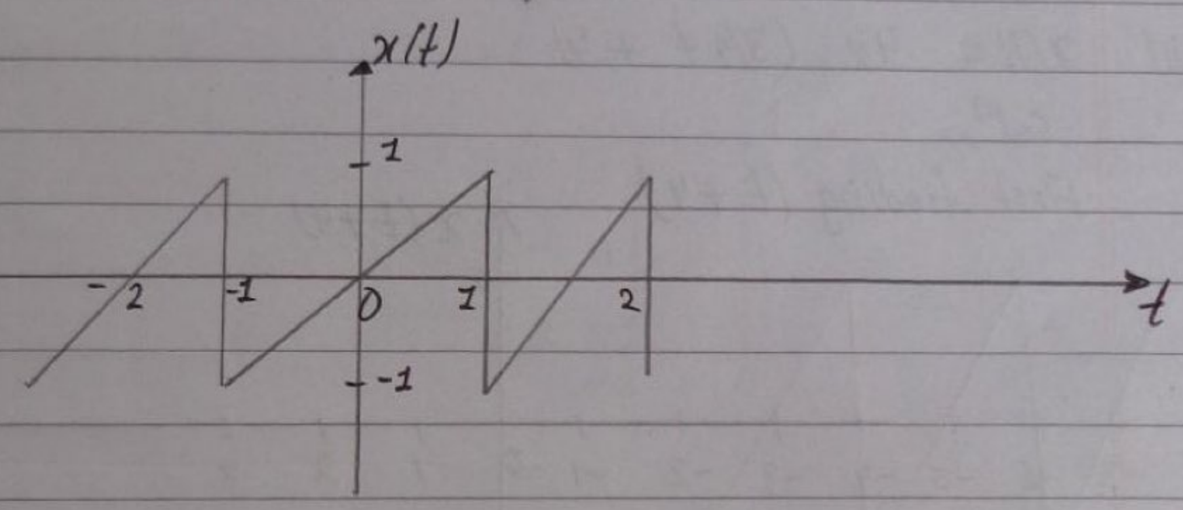
Assuming that 'a' Constant is never a complex number.

III. Time inversion: or Folding:

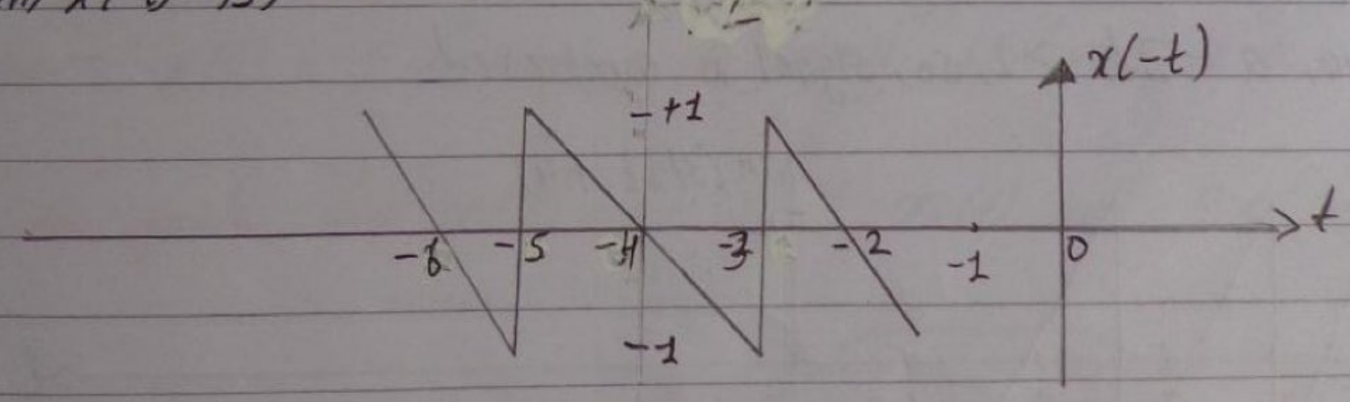
Time inversion is achieved by replacing the independent variable t by $-t$. It is also known as reflection of the signal about the time origin.



Sol^n,
Given,
$x(t)$:



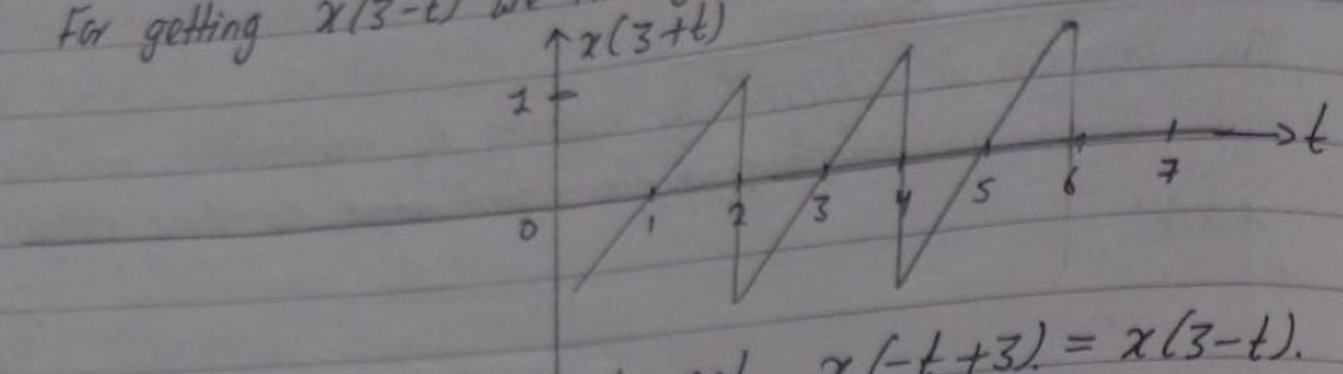
Then, $x(-t)$ is,



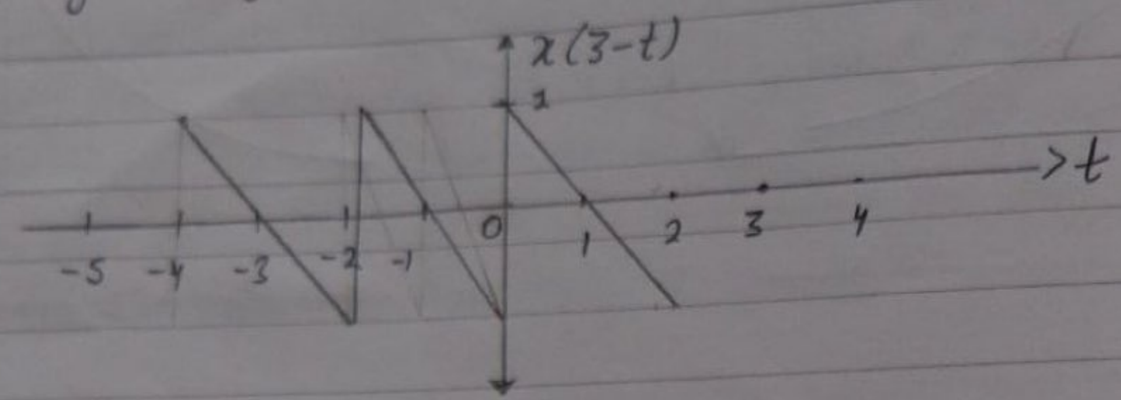
Scanned with CamScanner

i) For $x(3-t)$:

For getting $x(3-t)$ we first get $x(3+t)$ by advancing signal.



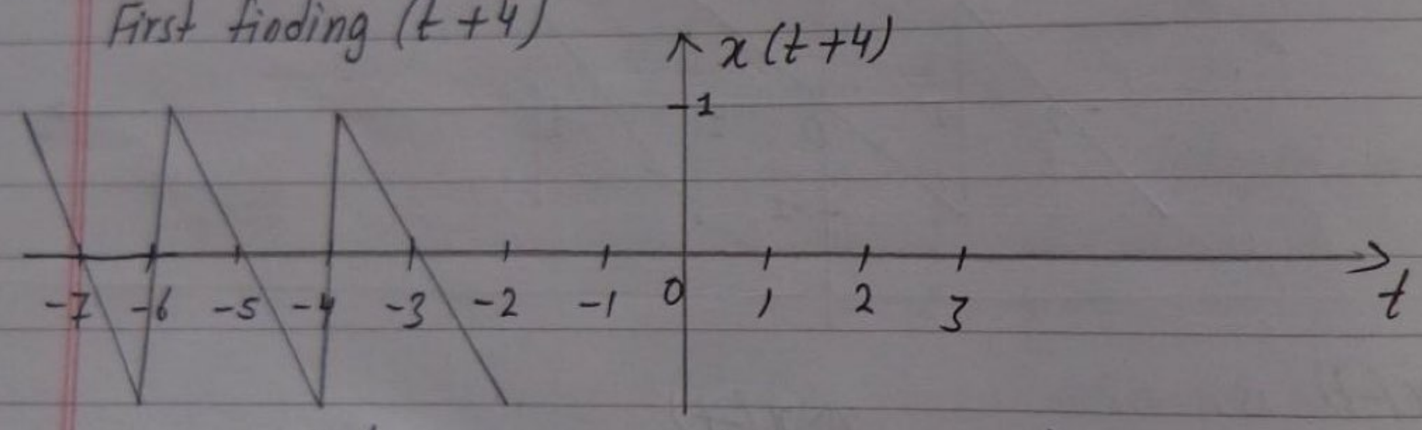
Folding the signal $x(3+t)$ to get $x(-t+3) = x(3-t)$.



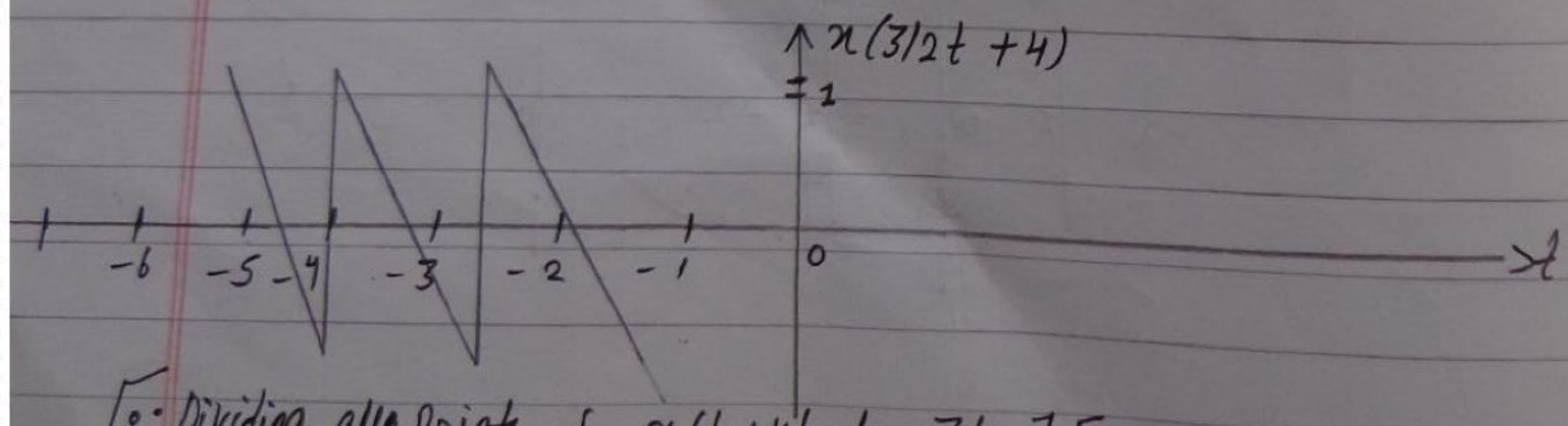
ii) $x(t) = 4x\ (3/2\, t + 4)$

Sol^n,

First finding $(t+4)$



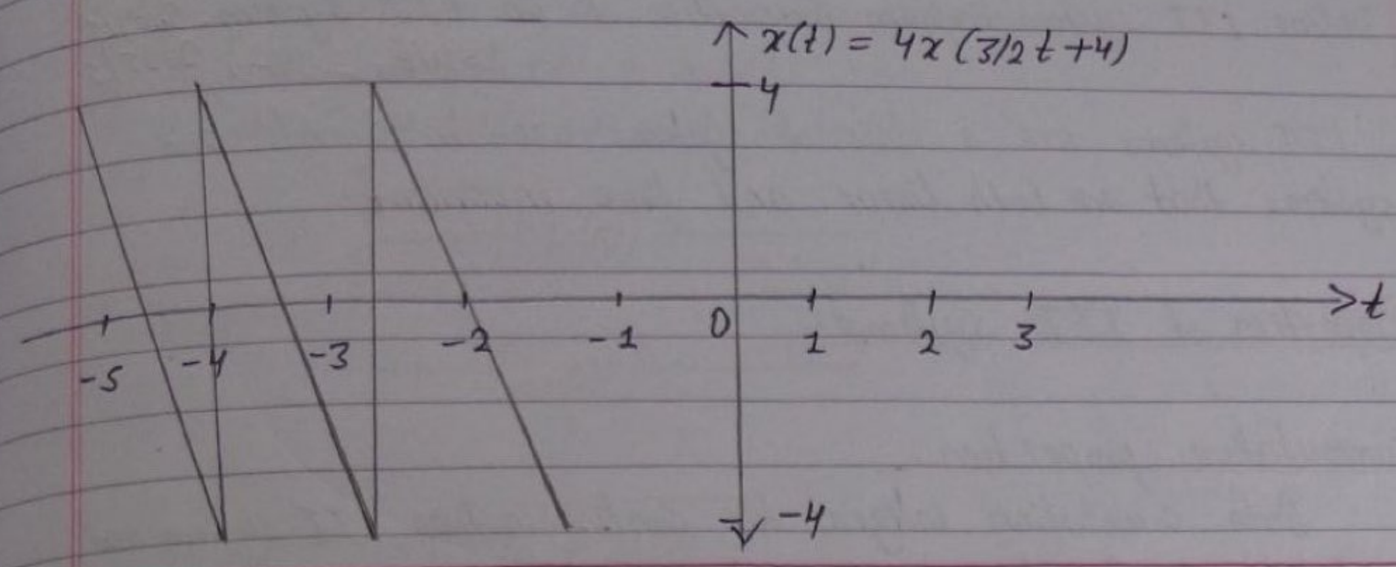
Since, $a = 3/2 > 1$, so, signal is compressed.



[$\because$ Dividing all points of $x(t+4)$ by $3/2$] [$\because$ Compression]

Scanned with CamScanner

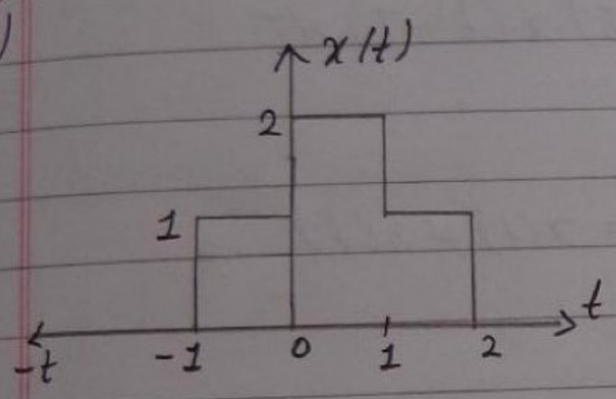
Now, amplitude scaling by 4 for $x(t) = 4x(3/2t + 4)$.



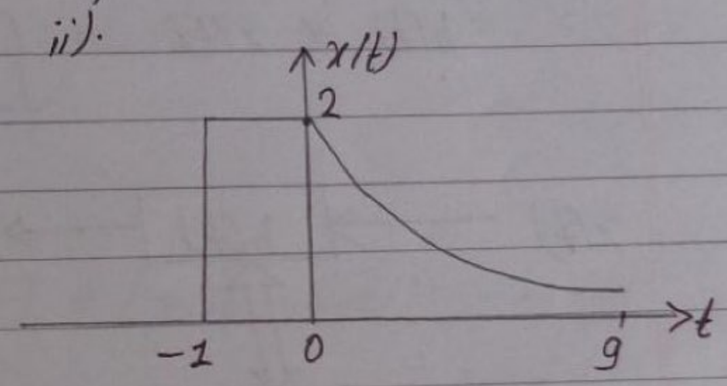
0. Practice Questions:

a) Find and sketch the even and odd parts of:

i)



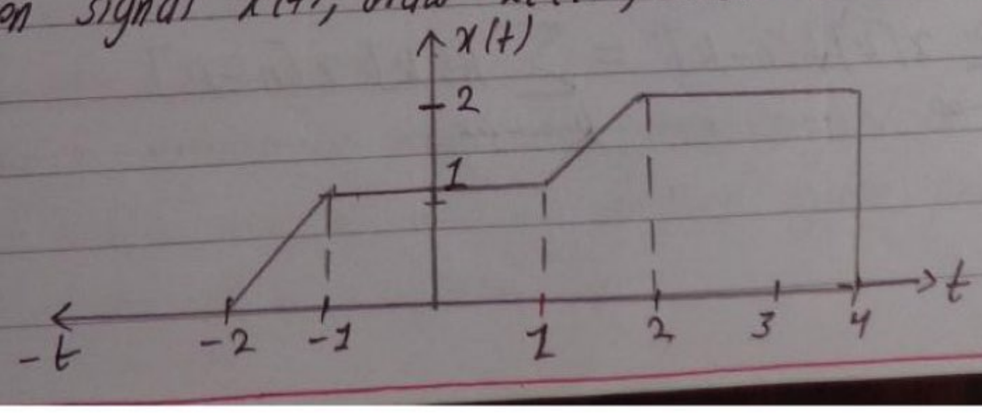
ii).



b. Determine whether the given signal is energy or power signal or neither; $x(t) = 5\cos(\pi t) + \sin(5\pi t)$, $-\infty < t < \infty$ 2013/F

c. For the given signal $x(t)$, draw $x_e(t)$, $x_o(t)$ and $2x(0.6t - 3)$.

2010/F



Scanned with CamScanner

Linear - time invariant systems.

chapter = 2

1. A. Define LTI system. Explain properties of an LTI system. 2009/S 2016/F, 2014/F, 2015/S

=> LTI systems are a class of systems used in signals and systems that are both linear and time invariant. 2017/S

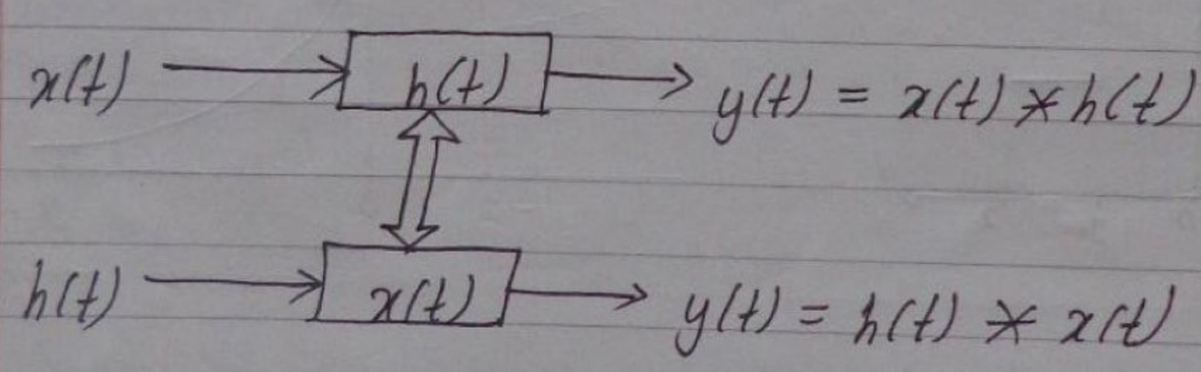
Properties of LTI systems:

i) Cummutative properties:

Both convolution integral for continuous time LTI system and convolution sum for discrete time LTI systems are cummutative. For a continuous LTI system, according to this property,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

$$= h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$$



For discrete time LTI system,

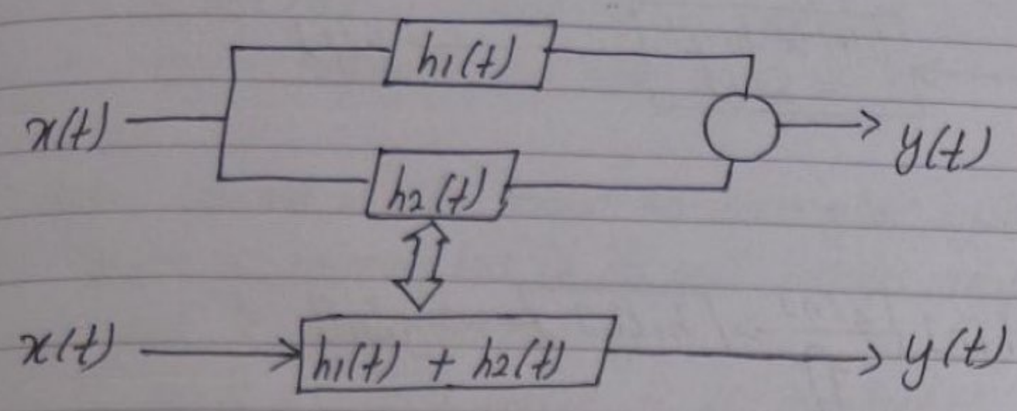
$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

Scanned with CamScanner

For continuous time, the distributive property is,

$$y(t) = x(t) * [h_1(t) + h_2(t)]$$
$$= x(t) * h_1(t) + x(t) * h_2(t)$$

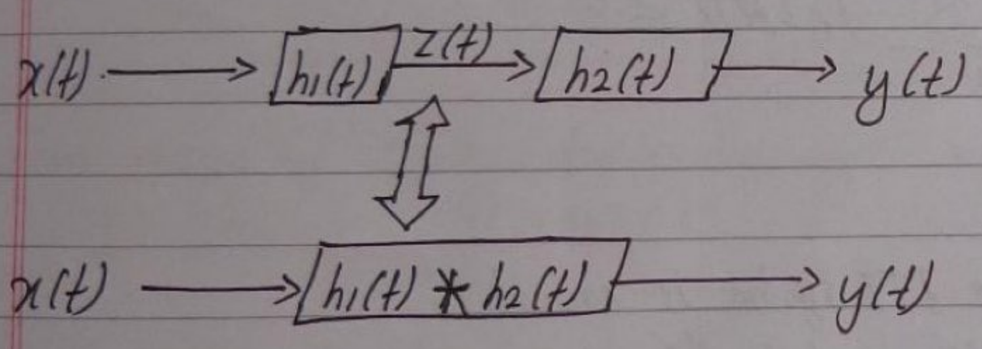


For discrete system,

$$y[n] = x[n] * [h_1[n] + h_2[n]]$$
$$= x[n] * h_1[n] + x[n] * h_2[n]$$

III. Associative property:

$$y(t) = x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

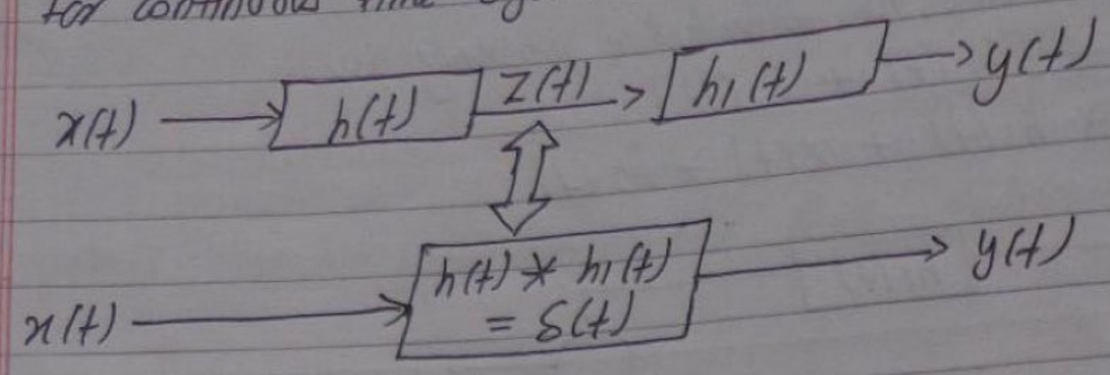


IV. Invertibility of LTI system:

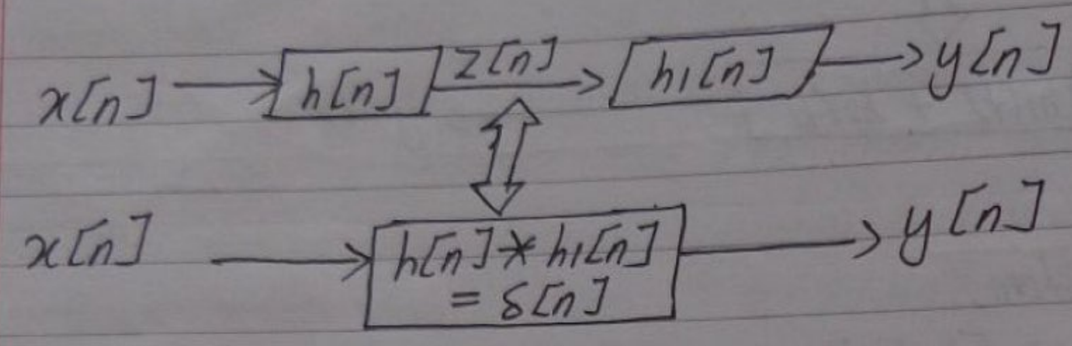
A system is known as invertible system only if an inverse system exists which when cascaded with the original system, produces an output equal to the input at first system.

Scanned with CamScanner

For continuous time system;



For discrete,



V. stability :

A necessary and sufficient condition of stability of continuous time LTI system is $X = \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$

and for discrete time is,

$$y[n] = \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Vi causality :

A continuous time LTI system is causal if $h(t) = 0$ for $t < 0$ and in this case,

$$y(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) d\tau \neq \int_{0}^{\infty} h(\tau) x(t-\tau) d\tau$$

For discrete time, $h[n] = 0$ for $n < 0$.

Scanned with CamScanner

vii. LTI system with and without memory:

A discrete time LTI system is memoryless if its unit impulse response $h[n]$ is zero for $n \neq 0$ and is characterized by;
$y[n] = k\,x[n]$ where k is constant.
and its impulse response is $h[n] = k\,\delta[n]$.

A static systems is also known as memoryless system. A continuous time LTI system is memory less if its unit impulse response $h(t)$ is zero for $t \neq 0$ and is characterized by,
$y(t) = k\,x(t)$, where k is constant.
and its impulse response is $h(t) = k\,\delta(t)$.

viii. unit step-response of LTI system:

It is the output of LTI system for input equal to unit step function or sequence. It is defined for continuous time system as
$s(t) = u(t) * h(t)$.

Similarly, for discrete time LTI system,
$s[n] = u[n] * h[n]$

1.B). For an LTI system, derive the condition to be: 2012/F / 2015/S, 2010/F, 2011/F
i) BIBO stable (ii) causal:

i) causality of LTI systems:

A time causal system is one which only depends on the present & past inputs but doesnot depends on future inputs for the output.
The system is causal if its impulse response is causal i.e $h[n] = 0$

Scanned with CamScanner

for $n < 0$.

proof :

output of discrete time LTI system is,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

output at any instant n_0 is,

$$y[n]|_{n=n_0} = \sum_{k=-\infty}^{\infty} h[k]x[n_0-k]$$

$$= \sum_{k=-\infty}^{-1} h[k]x[n_0-k] + \sum_{k=0}^{\infty} h[k]x[n_0-k]$$

$$= \ldots\ldots + h[-2]x[n_0+2] + h[-1]x[n_0+1] + h_0[0]x[n_0] + h[1]x[n_0-1] + h[2]x[n_0-2] \ldots\ldots$$

Here, the above expression consists of future input values and present and past values of inputs also. To make the system causal, future values of input should be cancelled out.

For that, $h[n] < 0$

so, limit of integration changes for causal system as;

$$\therefore y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

similarly,

a continuous time LTI system is causal if, $h(t)=0$ for $t<0$.

so,

$$y(t) = \int_0^{\infty} h(\tau)x(t-\tau)d\tau$$

Scanned with CamScanner

ii. BIBO stable :

If input signal $x(t)$ be bounded in magnitude then $|x(t)| < M$ for all t.
Now, from convolution integral, output is

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

So, magnitude of output $y(t)$ is given by,

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(t)\, x(t-\tau)\, d\tau \right|$$

$$= \int_{-\infty}^{\infty} |h(\tau)|\, |x(t-\tau)|\, d\tau$$

Now substituting value of $|x(t-\tau)| < M$ for all values of t and τ,

$$|y(t)| = M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau \quad \text{for all values of } t.$$

Hence, if the impulse response $h(t)$ is absolutely integrable then the output of a continuous time LTI system is bounded in magnitude, & hence the system is BIBO stable.
A necessary and sufficient condition for stability of continuous time LTI system is given by,

$$\therefore \quad X = \int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty.$$

Similarly for discrete time, to be BIBO stable,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

c. Define linearity and time invariance for a LTI system. Also, check the

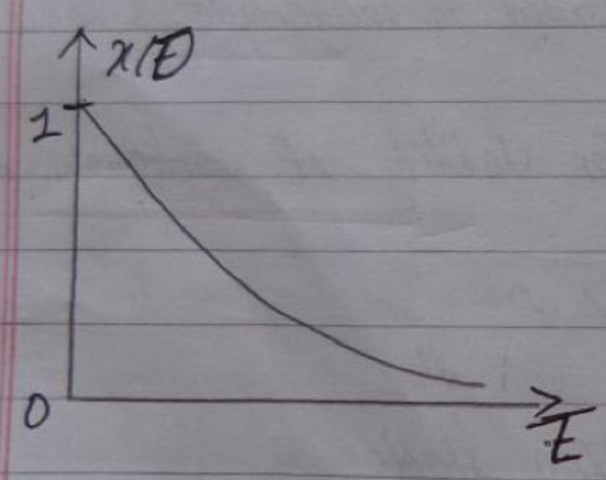
Scanned with CamScanner

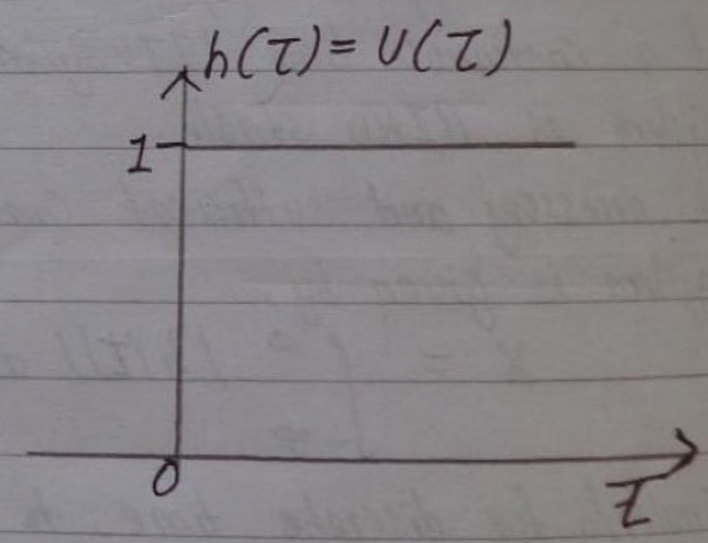
system $y(t) = c\,x(t^2-1)$ for (i) memory (ii) stability.
c is arbitary finite constant. 2011/F.

$\Rightarrow$ $y(t) = c\,x(t^2-1)$
At $t=0$
$y(0) = c\,x(-1)$
since, output is dependent upon past input, the system has memory.

For checking stability, A constant input $x(t^2-1) = 1$ yields $y(t) = c$ which is bounded. Hence, system is stable.

D. Find the convolution of the two continuous time signals, $x(t) = e^{-at}u(t)$, $a>0$ and $h(t) = u(t)$.
sol^n,
$\Rightarrow$ $x(t) = e^{-at}\cdot u(t)$
and, $h(t) = u(t)$

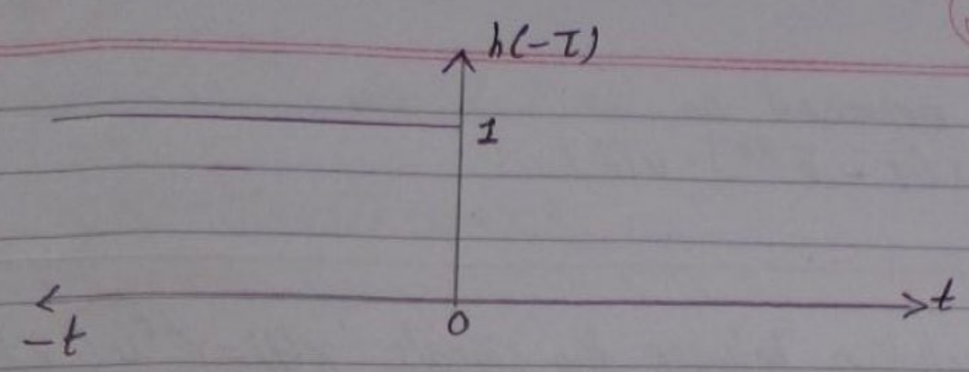


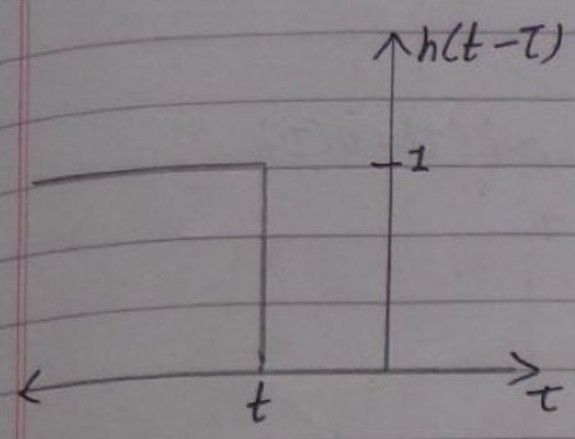


We know that the output $y(t)$ of a continuous time LTI system is given by the convolution of the applied input & unit impulse response as;

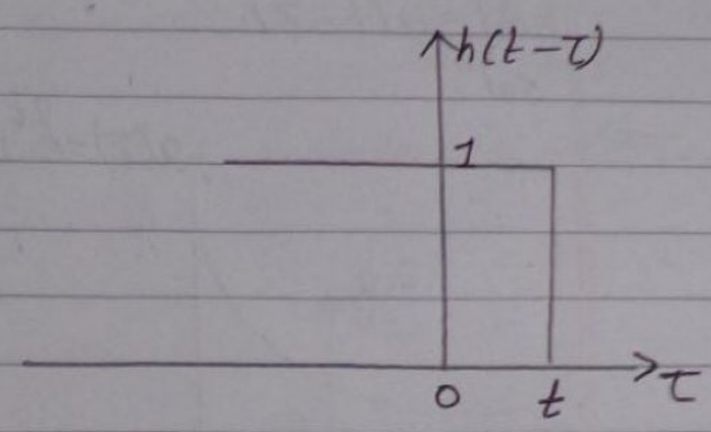
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\,d\tau$$

Scanned with CamScanner





For $h(t-\tau)$ for $t<0$



$h(t-\tau)$ for $t>0$

For $t<0$, we see from above figure that the product of $x(\tau)$ & $h(t-\tau)$ is zero and consequently $y(t)$ is zero.

For $t>0$, $x(\tau)\,h(t-\tau) = \begin{cases} e^{-a\tau} & 0<\tau<t \\ 0 & \text{otherwise.} \end{cases}$

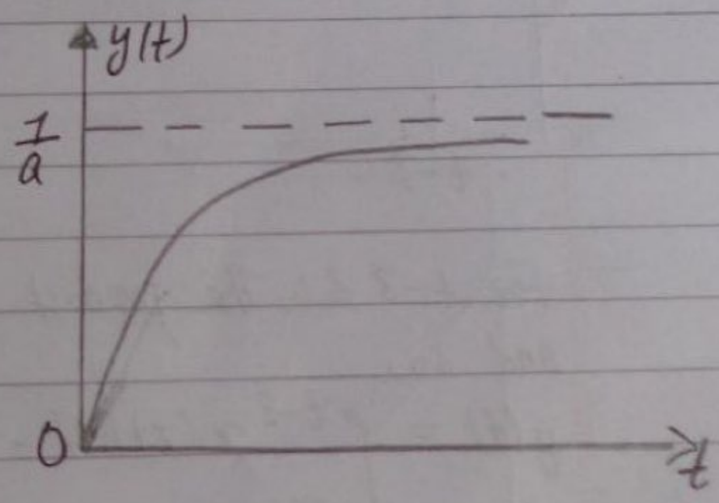
So, $y(t)$ for $t>0$ is,

$$y(t) = \int_0^t e^{-a\tau}\cdot 1\, d\tau$$

$$= \int_0^t e^{-a\tau}\cdot d\tau$$

$$= \left[\frac{e^{-a\tau}}{-a}\right]_0^t$$

$$\therefore y(t) = \frac{1}{a}\left(1-e^{-at}\right)$$



So,

$$\therefore y(t) = \begin{cases} 1/a\,(1-e^{-at}) & \text{for } t>0 \\ 0 & \text{for } t<0 \end{cases}$$

Scanned with CamScanner

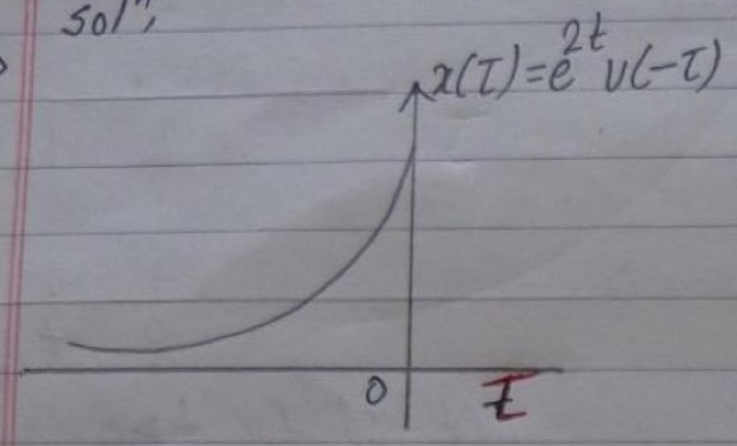
This can be expressed as,

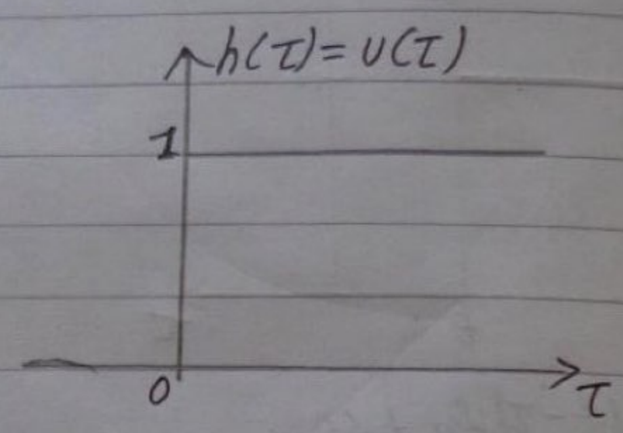
$\therefore \; y(t) = \frac{1}{a}\left(1 - e^{-at}\right) \cdot u(t)$

E. Find the convolution between two signals $x(t) = e^{2t} u(-t)$ and $h(t) = u(t-3)$.

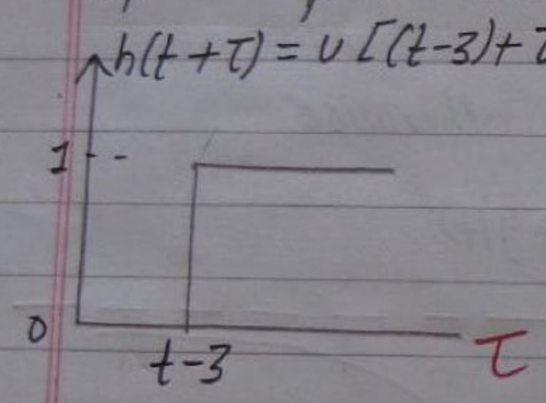
Soln,

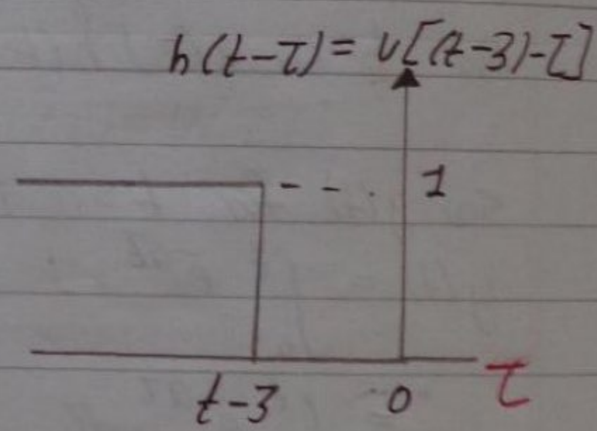
$\Rightarrow$





We know, that the output $y(t)$ of a continuous time LTI system is given by the convolution of the applied input and unit impulse response as;





For $t-3 \le 0$, the product $x(\tau)h(t-\tau)$ is non zero for $-\infty < \tau < t-3$, and so,

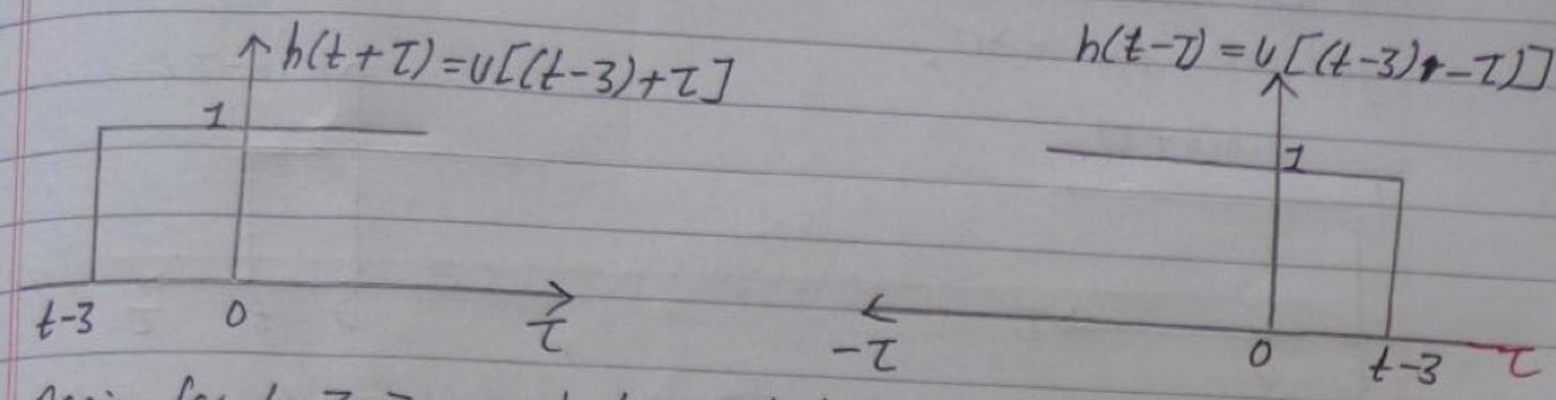
$$y(t) = \int_{-\infty}^{t-3} x(\tau)h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{t-3} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{t-3} e^{2\tau} \cdot 1 \, d\tau$$

$$= \left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{3}$$

Scanned with CamScanner

$\therefore y(t) = \frac{1}{2} e^{2(t-3)}$ for $t-3 \leq 0$



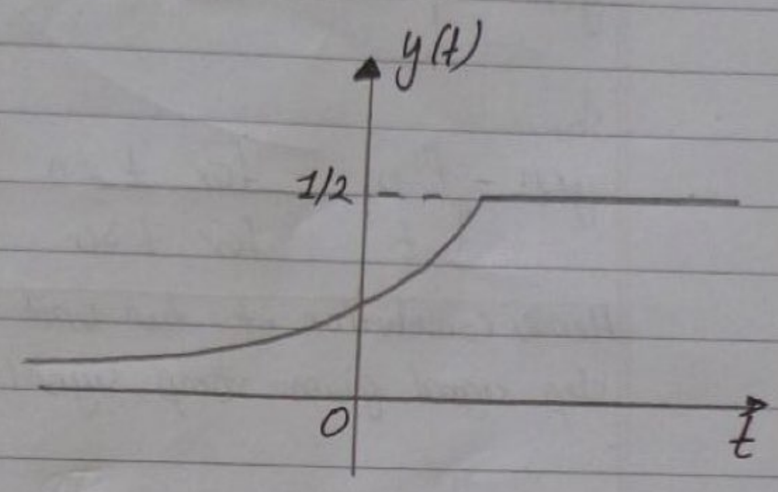
Again for $t-3 \geq 0$, product $x(\tau) \cdot h(t-\tau)$ is non-zero for $-\infty < \tau < 0$, and so,

$$y(t) = \int_{-\infty}^{0} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau}\, d\tau$$

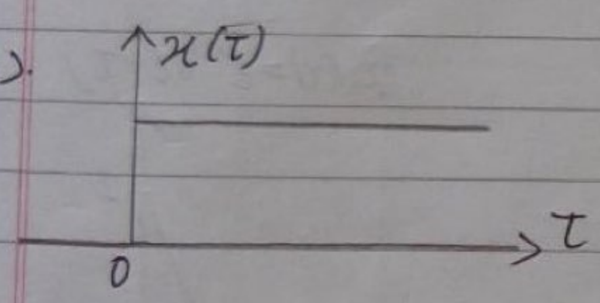
$$= \left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{0}$$

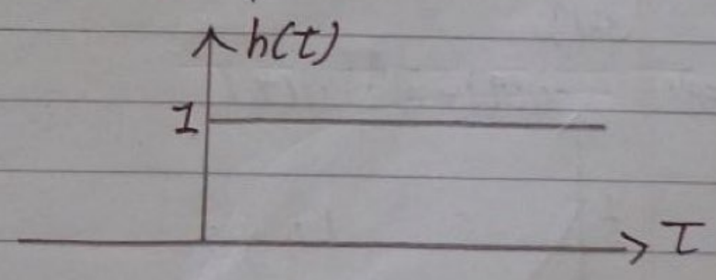
$\therefore y(t) = \frac{1}{2}$ for $t-3 \geq 0$



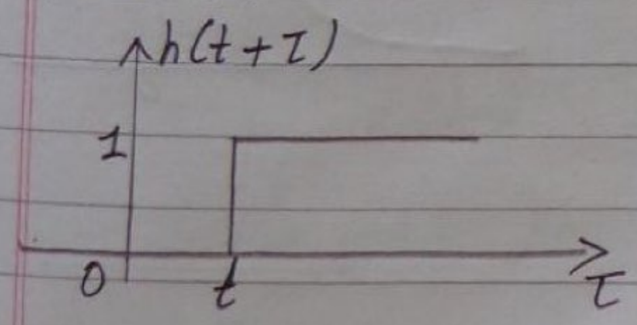
F Find the convolution of two unit step signals.

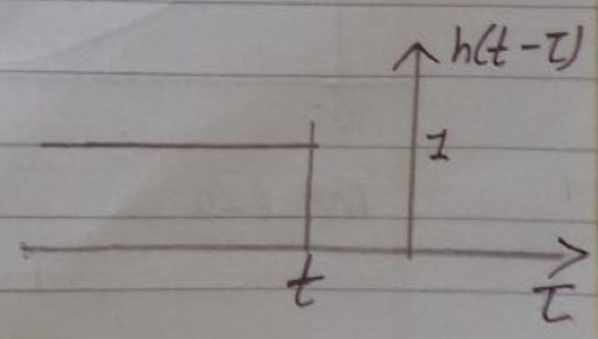
=>





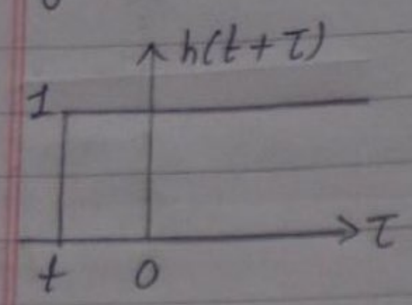
For $t < 0$,

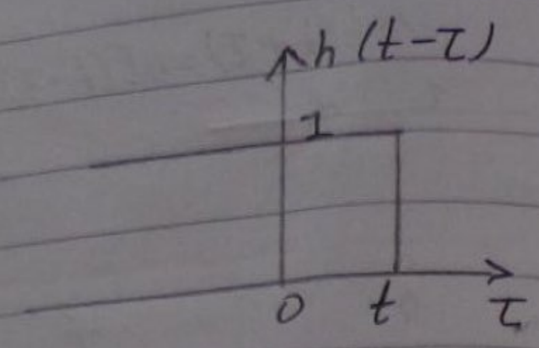




Scanned with CamScanner

In this case, there is no overlapping between $x(\tau)$ & $h(t-\tau)$, so
$y(t) = 0$, for $t > 0$,





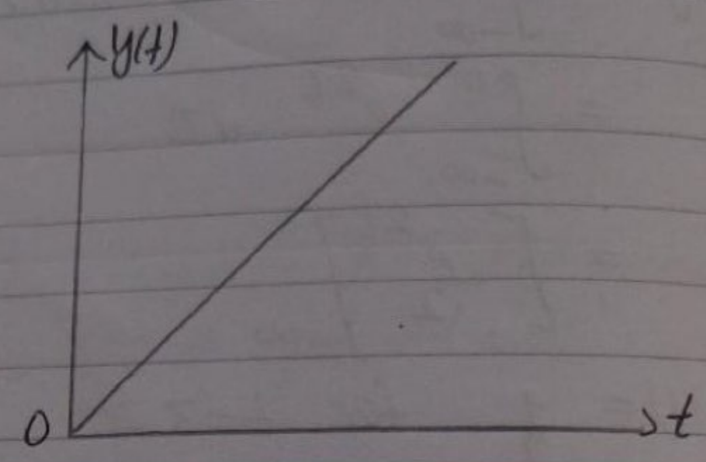
There is overlapping between $x(\tau)$ and $h(t-\tau)$, from 0 to t,

$$y(t) = \int_0^t 1 \cdot 1 \, d\tau = t$$

So,

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ t & \text{for } t > 0 \end{cases}$$

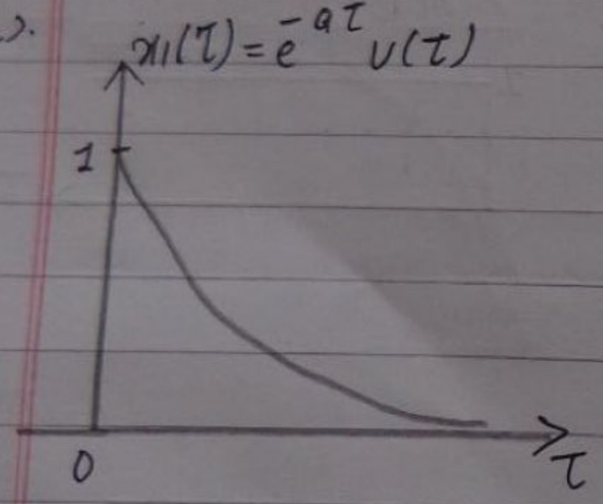
Hence, Convolution of two unit step signal gives ramp signal.

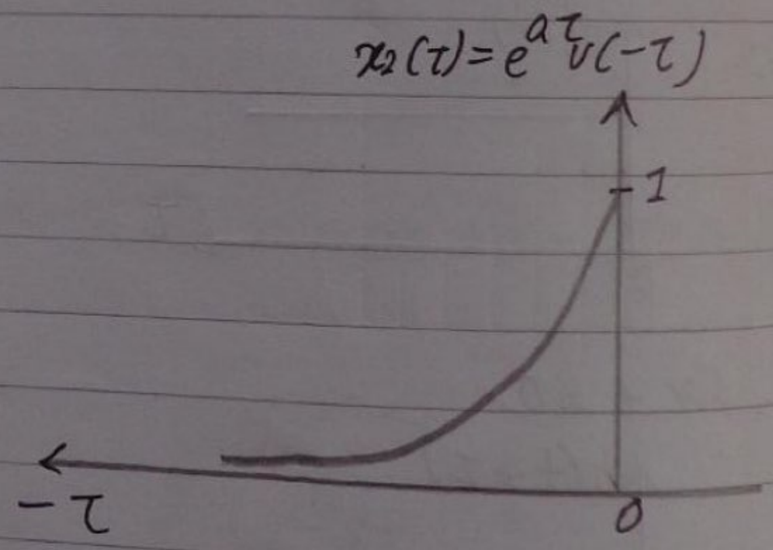


G. Find convolution between two signals $x_1(t) = e^{-at} u(t)$, $a > 0$ & $x_2(t) = e^{at} u(-t)$, $a > 0$.

Sol^n,

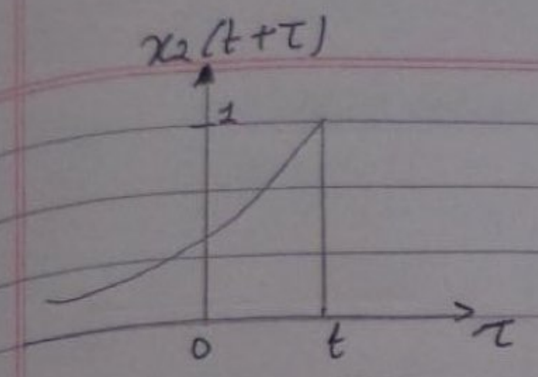
$\Rightarrow$

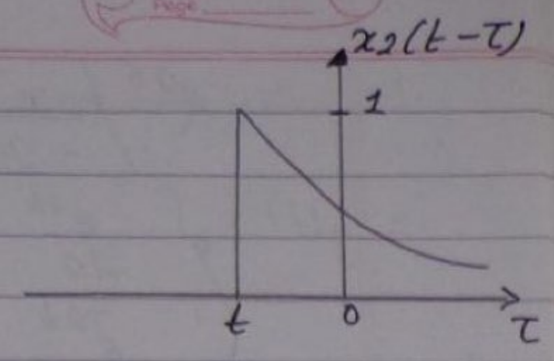




For $t < 0$

Scanned with CamScanner





So, $y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$

$= \int_{0}^{\infty} e^{-a\tau}\, e^{a(t-\tau)}\, d\tau$

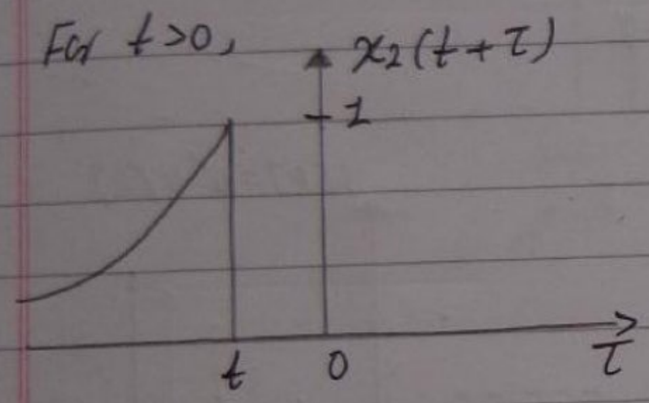
$= \int_{0}^{\infty} e^{-a\tau} \cdot e^{at} \cdot e^{-a\tau}\, d\tau$

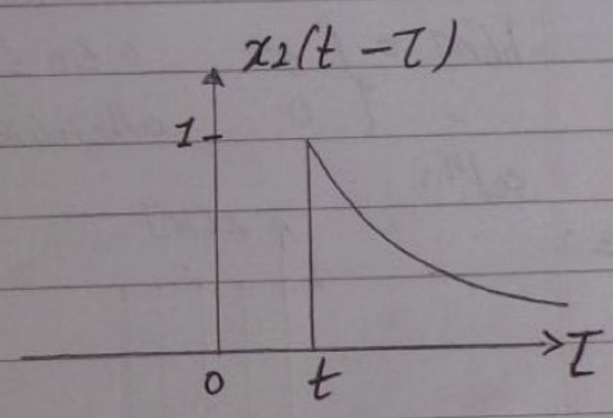
$= e^{at} \int_{0}^{\infty} e^{-2a\tau}\, d\tau$

$= e^{at} \left[\frac{e^{-2a\tau}}{-2a} \right]_{0}^{\infty}$

$\therefore\ y(t) = \frac{e^{at}}{2a}$

For $t>0$,





Here,

$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$

$= \int_{t}^{\infty} e^{-a\tau}\, e^{a(t-\tau)}\, d\tau$

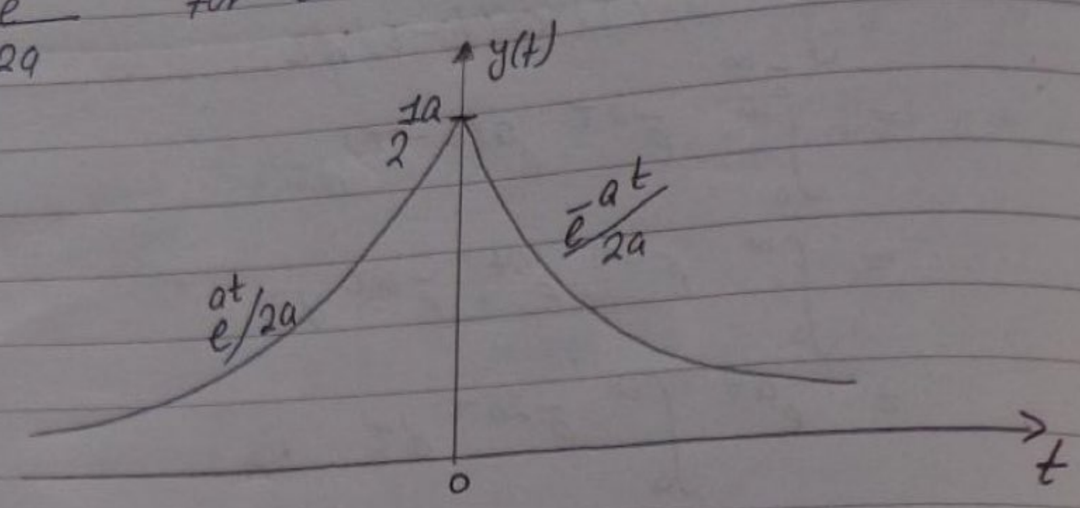
$= \int_{0}^{\infty} e^{-a\tau} \cdot e^{at} \cdot e^{-a\tau}\, d\tau$

Scanned with CamScanner

$$= e^{at}\left[\frac{e^{-2a\tau}}{-2a}\right]_0^{\infty} = \frac{e^{-at}}{2a}$$

$$\therefore y(t) = \begin{cases} \frac{e^{at}}{2a} & \text{for } t < 0 \\ \frac{e^{-at}}{2a} & \text{for } t > 0. \end{cases}$$

output :



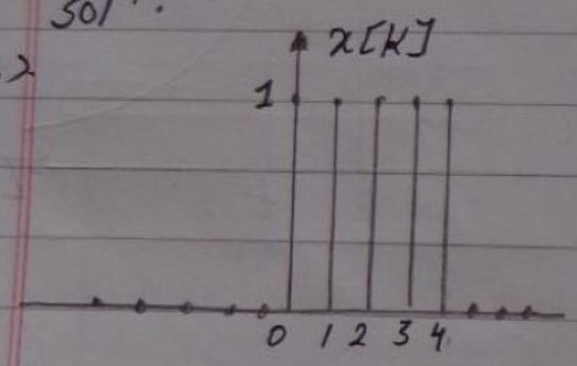
4. Find convolution of two signals $x[n]$ and $h[n]$ where,

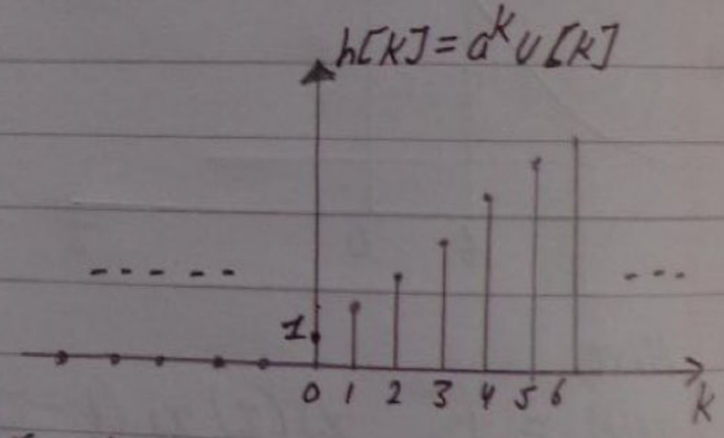
$$x[n] = \begin{cases} 1 & 0 \le n \le 4 \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$$h[n] = \begin{cases} a^n & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases} \quad a > 1.$$

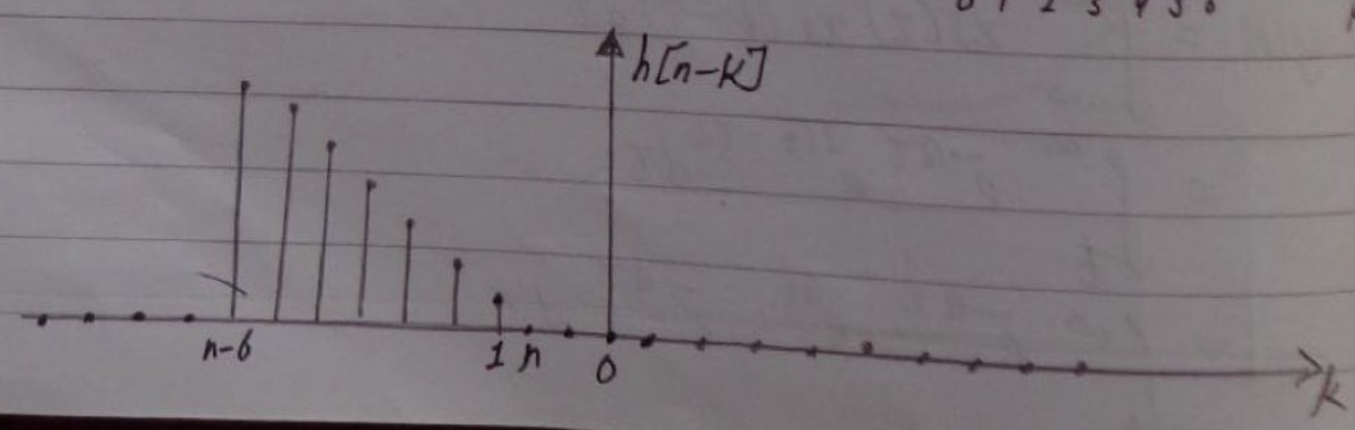
sol^n:

$\Rightarrow$





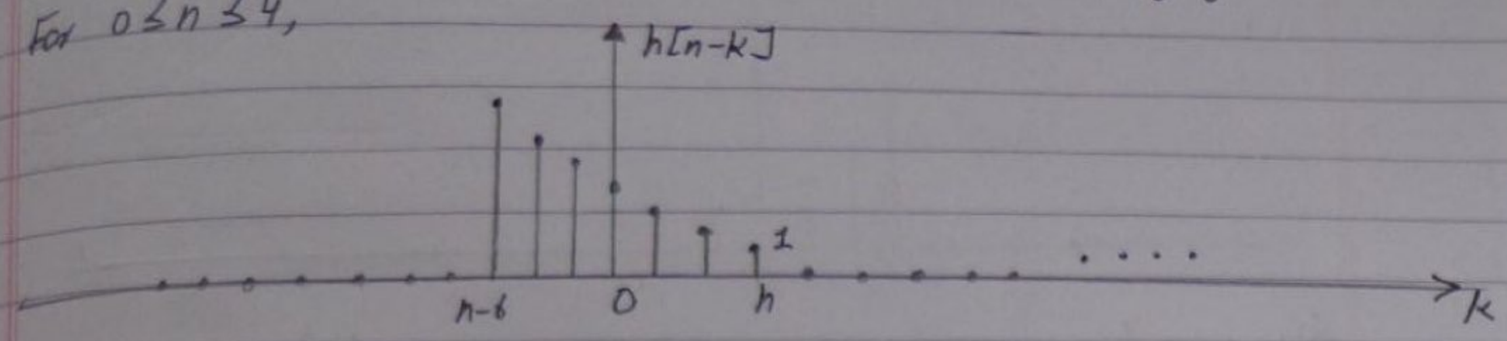
For $n < 0$



Scanned with CamScanner

There is no overlap between $x[k]$ and $h[n-k]$ and consequently $y[n]=0$.

For $0 \le n \le 4$,



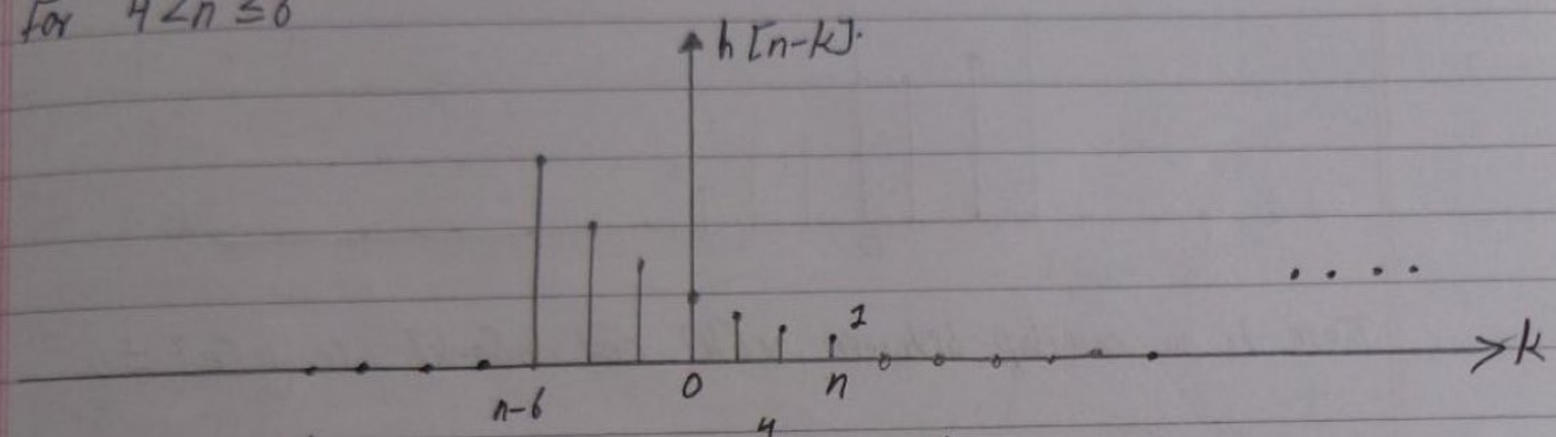
So,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{n} 1 \cdot a^{n-k} = a^n \sum_{k=0}^{n} a^{-k}$$

Replacing $-k$ by l, then

$$y[n] = a^n \sum_{l=-k}^{0} a^{l} = a^n \frac{a^{-n}-a}{1-a} = a^n \sum_{k=0}^{n} a^{-k}$$

For $4 < n \le 6$



So, $$y[n] = \sum_{k=0}^{4} 1 \cdot a^{n-k} = \sum_{k=0}^{4} a^{n-k}$$

$$= a^n \sum_{k=0}^{4} a^{-k} = a^n \sum_{l=-4}^{0} a^{l}$$

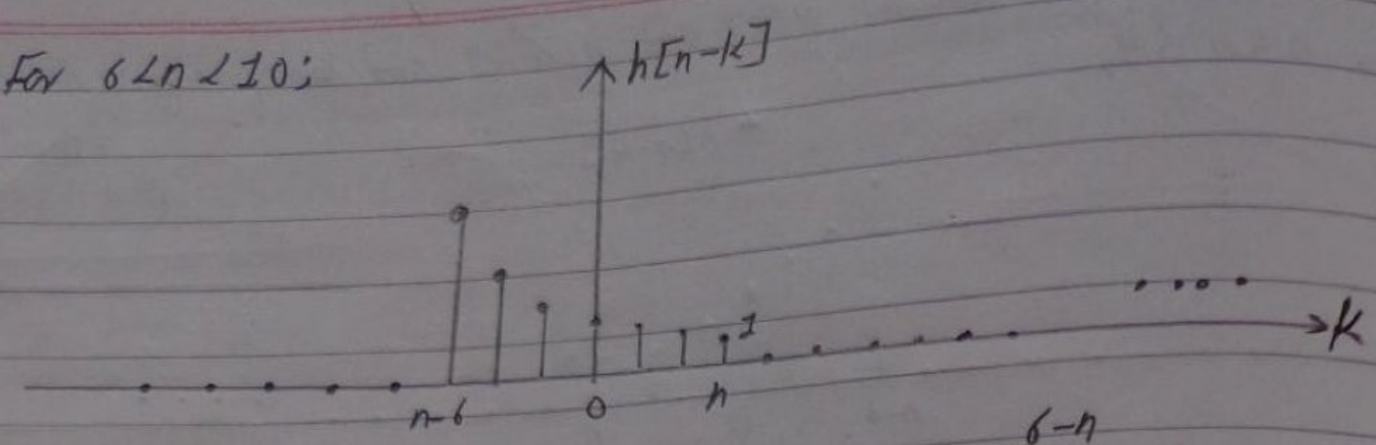
$$= a^n \cdot \frac{a^{-4}-a}{1-a}$$

$$= \frac{a^{n-4} - a^{n+1}}{1-a}$$

Now,

Scanned with CamScanner

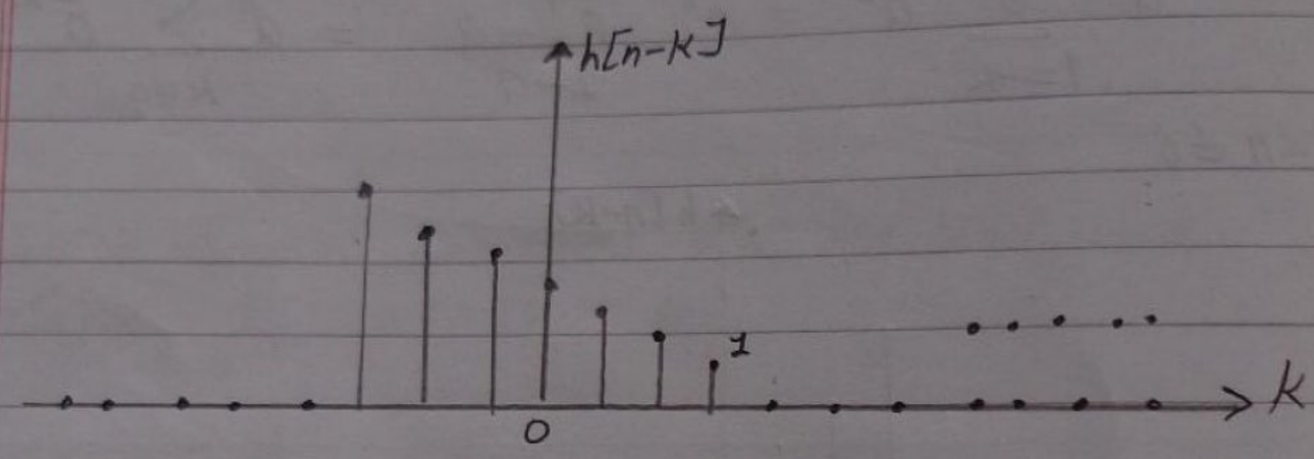
For $6 < n < 10$;



So,

$$\therefore \; y[n] = \sum_{k=n-6}^{4} 1 \cdot a^{n-k} = a^n \sum_{k=n-6}^{4} a^{-k} = a^n \sum_{l=-4}^{6-n} a^l$$

$$= a^n \cdot \frac{a^{-4} - a^{6-n+1}}{1-a} = \frac{a^{n-4} - a^l}{1-a}.$$



There is no overlap between $x[k]$ and $h[n-k]$, so $y[n] = 0$

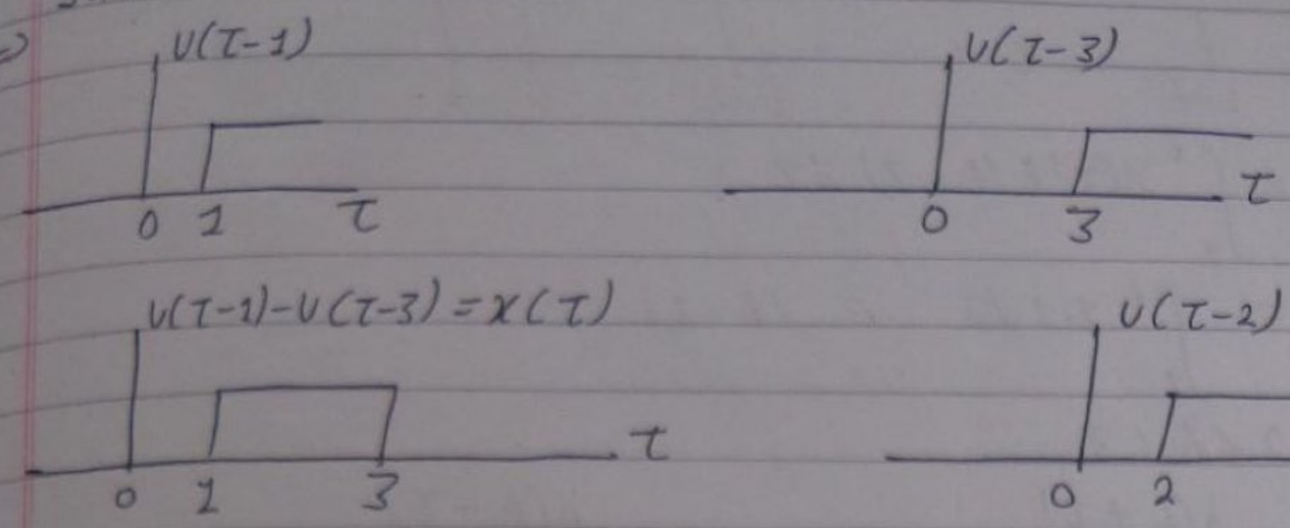
$$\therefore \; y[n] = \begin{cases} 0 & \text{for } n < 0 \\ \dfrac{1-a^{n+1}}{1-a} & \text{for } 0 \le n \le 4 \\ \dfrac{a^{n-4} - a^{n+1}}{1-a} & \text{for } 4 < n \le 6 \\ \dfrac{a^{n-4} - a^{7}}{1-a} & \text{for } 6 < n \le 10 \\ 0 & \text{for } n > 10. \end{cases}$$

Scanned with CamScanner

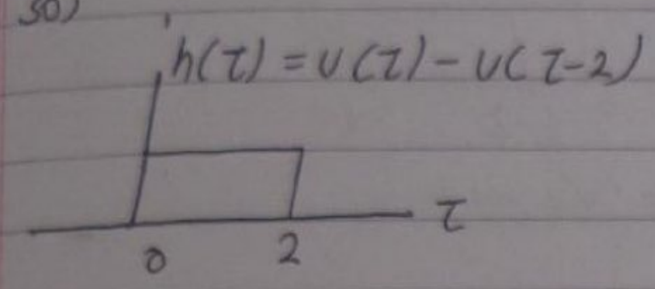
I. Find convolution between two signals $x(t) = u(t-1) - u(t-3)$ and $h(t) = u(t) - u(t-2)$ 2015/5

Sol^n

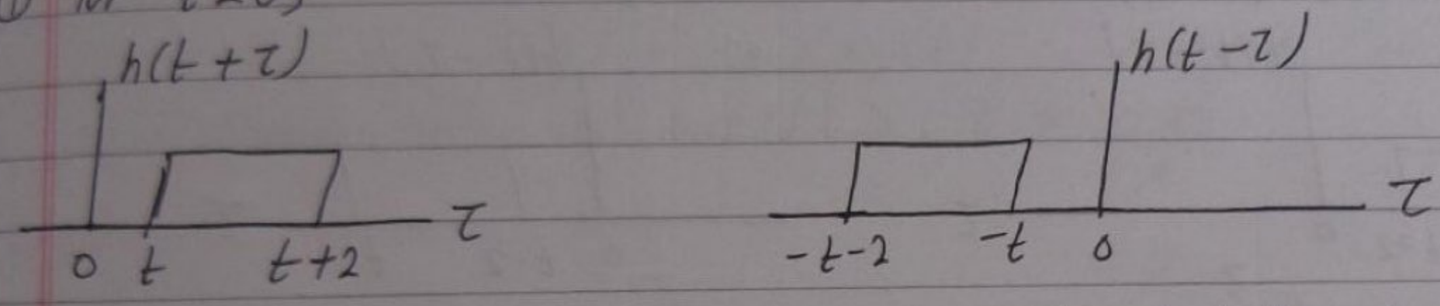
$\Rightarrow$



So,

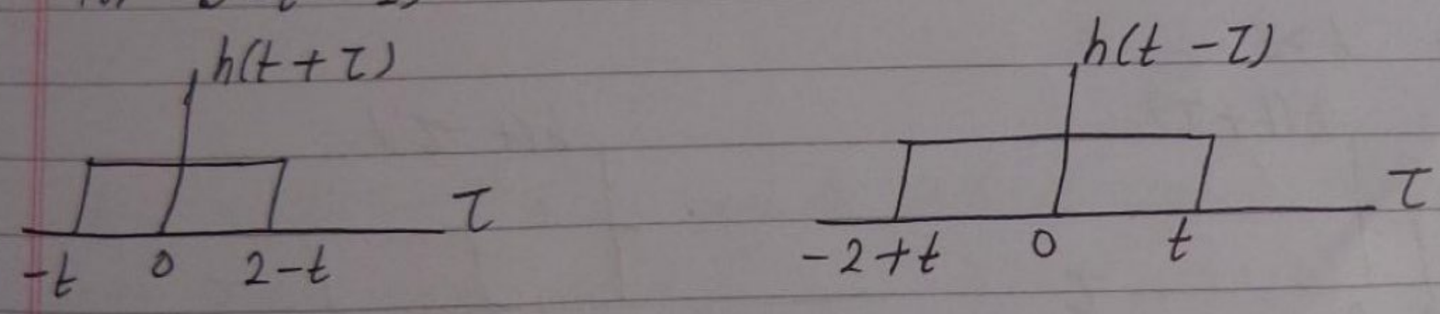


① for $t < 0$,



So, $y(t) = 0$, since no overlapping between $x(\tau)$ and $h(t-\tau)$.

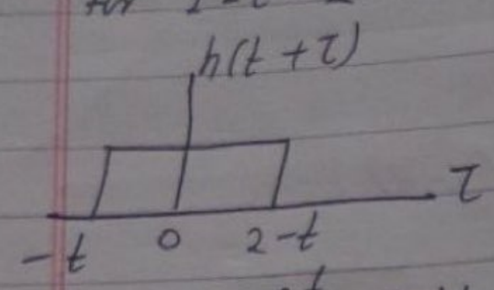
For $0 < t \leq 1$,

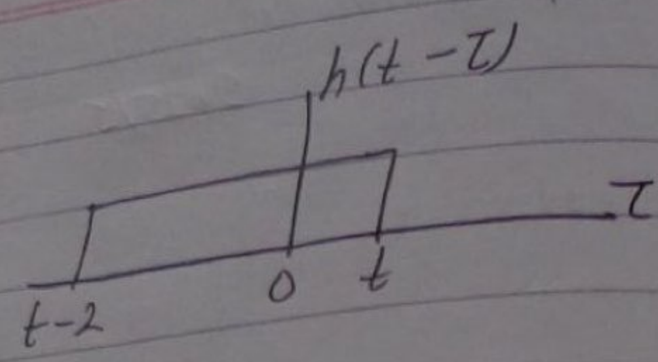


$\therefore y(t) = 0$, no overlapping

Scanned with CamScanner

For $1 < t \le 2$

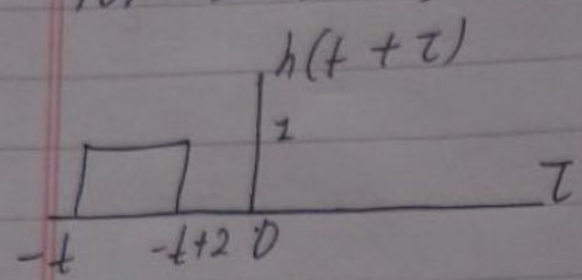


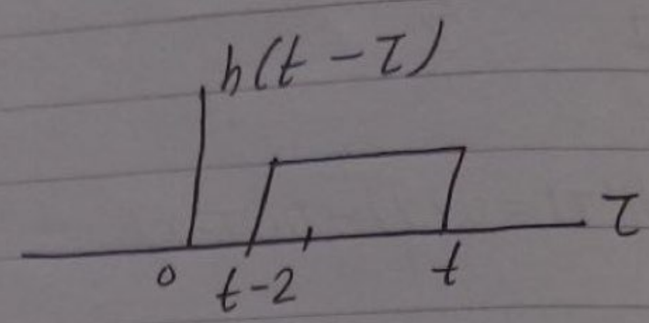


$$y(t) = \int_1^t x(\tau)h(t-\tau)\,d\tau$$

$$= \int_1^t 1 \cdot 1\,d\tau = (t-1)$$

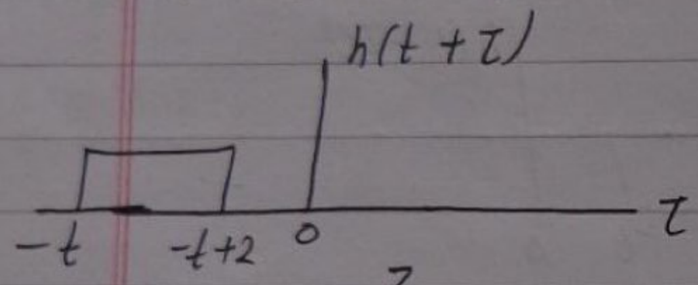
For $2 < t \le 3$,

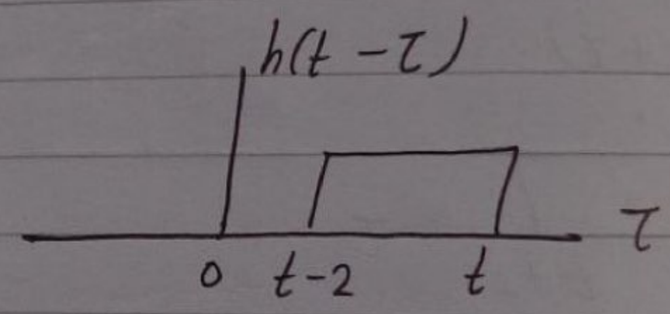




$$y(t) = \int_1^t x(\tau)h(t-\tau)\,d\tau = (t-1)$$

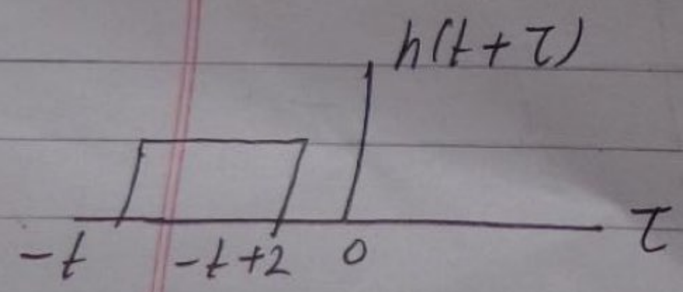
For $3 < t \le 5$,

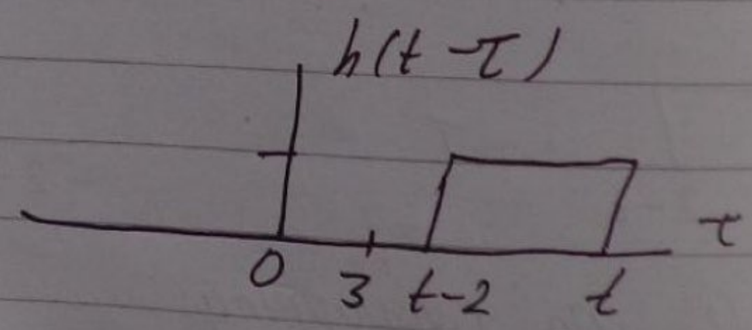




$$y(t) = \int_{t-2}^{3} x(\tau)\,h(t-\tau)\,d\tau = 3-(t-2) = 5-t$$

For $t > 5$,



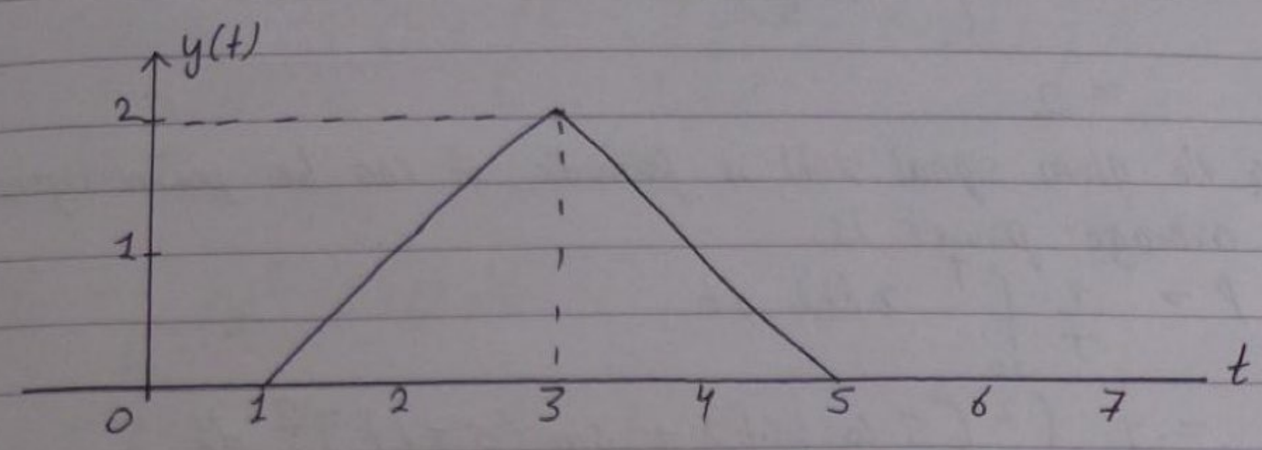


$y(t) = 0$, no overlapping.

Scanned with CamScanner

Hence,

$$y(t) = \begin{cases} 0 & \text{for } t \le 1 \\ t-1 & 1 < t \le 2 \\ t-1 & 2 < t \le 3 \\ 5-t & 3 < t \le 5 \\ 0 & t > 5 \end{cases}$$



J. Determine whether given signal is energy signal or power signal or neither.
2013/F

⇒ sol^n,

$x(t) = 5\cos(\pi t) + \sin(5\pi t), \quad -\infty < t < \infty$

Fundamental period of $\cos(\pi t)$ is, $T_1 = \frac{2\pi}{\pi}$

$= 2$

Fundamental period of $\sin(5\pi t)$ is $T_2 = \frac{2\pi}{5\pi}$

$= \frac{2}{5}$

For a signal $x(t)$ to be periodic, both the signals $\cos(\pi t)$ & $\sin(5\pi t)$ must have common period T,

i.e, $T = m \cdot T_1 = n T_2$

where, m and n are integers.

Scanned with CamScanner

So, $\frac{T_1}{T_2} = \frac{n}{m} = \frac{2}{2/5} = 5$

Since the ratio, T_1/T_2 is the rational number, the given signal is the periodic signal with fundamental period,

$$T = \text{lcm}(T_1, T_2)$$
$$= \text{lcm}\left(2, \frac{2}{5}\right)$$
$$= 2$$

Since the given signal $x(t)$ is periodic, it can be power signal.

The average power is,

$$P = \frac{1}{T}\int_0^T x(t)\,dt$$
$$= \frac{1}{2}\int_0^2 [5\cos(\pi t) + \sin(5\pi t)]^2\,dt$$
$$= \frac{1}{2}\int_0^2 [25\cos^2(\pi t) + 10\cos(\pi t)\cdot\sin(5\pi t) + \sin^2(5\pi t)\,dt$$
$$= \frac{25}{2}\int_0^2 \cos^2\pi t\,dt + 5\int_0^2 \cos\pi t\cdot\sin(5\pi t)\,dt + \frac{1}{2}\int_0^2 \sin^2(5\pi t)\,dt$$

let,

$$I_1 = \int_0^2 \cos^2(\pi t)\,dt$$
$$I_2 = \int_0^2 \cos(\pi t)\sin(5\pi t)\,dt$$
$$I_3 = \int_0^2 \sin^2(5\pi t)\,dt$$

Now,

$$I_1 = \int_0^2 \cos^2(\pi t)\,dt = \int_0^2 \left[\frac{1+\cos(2\pi t)}{2}\right]dt$$
$$= \left[\frac{t}{2} + \frac{1}{4\pi}\sin(2\pi t)\right]_0^2$$

Scanned with CamScanner

$= 1$

$$I_2 = \int_0^2 \cos(\pi t)\sin(5\pi t)\,dt = \frac{1}{2}\int_0^2 \sin(6\pi t) + \sin(4\pi t)\,dt$$

$$= \frac{1}{2}\int_0^2 \sin(6\pi t)\,dt + \frac{1}{2}\int_0^2 \sin(4\pi t)\,dt$$

$$= \left[\frac{-1}{12\pi}\cos 6\pi t\right]_0^2 - \left[\frac{1}{8\pi}\cos(4\pi t)\right]_0^2$$

$$= 0$$

$$I_3 = \int_0^2 \sin^2(5\pi t)\,dt$$

$$= \frac{1}{2}\int_0^2 (1-\cos(10\pi t))\,dt$$

$$= \left[\frac{t}{2} - \frac{1}{20\pi}\sin(10\pi t)\right]_0^2$$

$$= 1$$

Hence,
total average power is,

$$P = 1 + 0 + \frac{25}{2}\times 1$$

$$= 1\times I_3 + I_2\times 0 + I_1\times\frac{25}{2}$$

$$\therefore\ P = \frac{27}{2}$$

and Energy:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt.$$

$$= \infty$$

Scanned with CamScanner

Hence, given signal is Power signal.

K. What is the output of the LTI system whose impulse response is $h(t) = u(t) - u(t-5)$ and input is defined by,

$$x(t) = \begin{cases} e^{t} & , 0 \le t \le 3 \\ 0 & \text{otherwise} \end{cases}$$

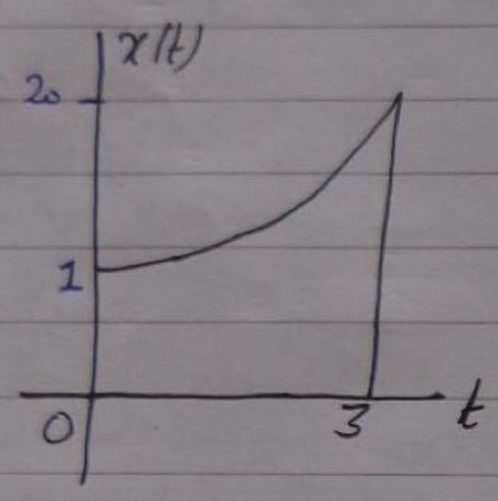
2014/F

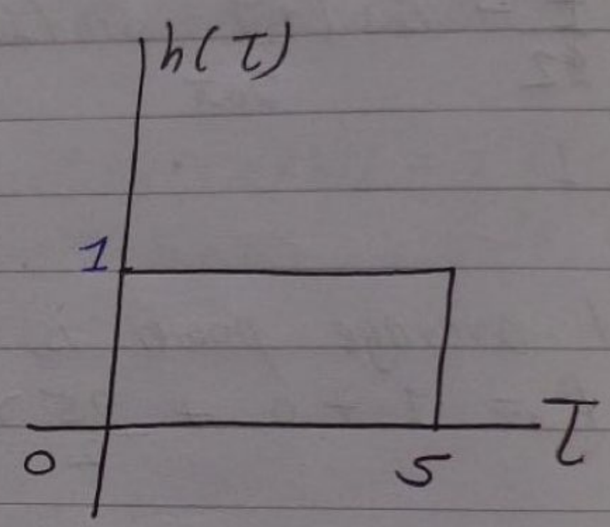
Sol^n,

$\Rightarrow$ We know that the output of an LTI system can be computed by the convolution integral as given by,

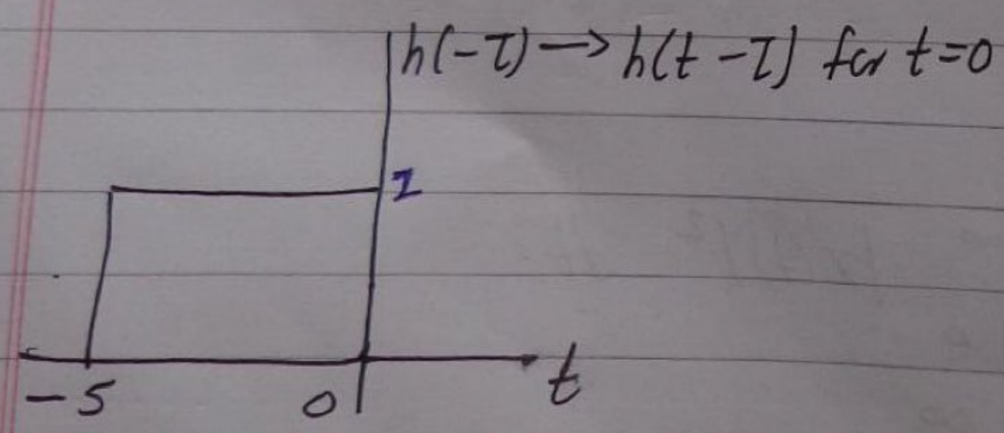
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

plotting $x(t)$ & $h(t)$ as function of τ, we get,





Flipping $h(\tau)$ over time,

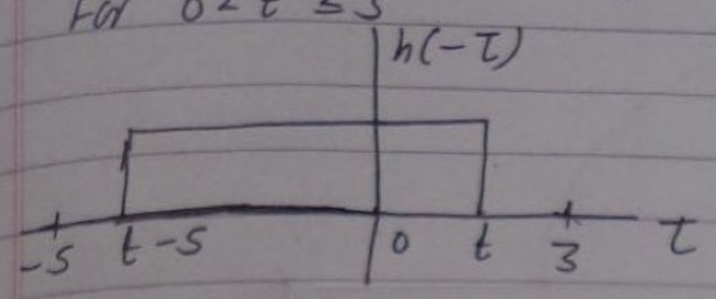


Scanned with CamScanner

For $t \le 0$,

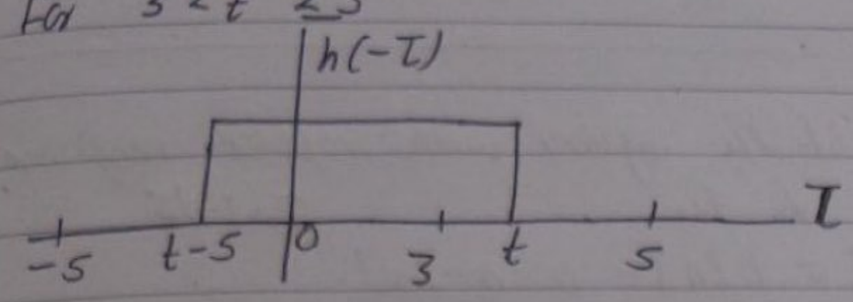
$$y(t) = 0$$

For $0 < t \le 3$



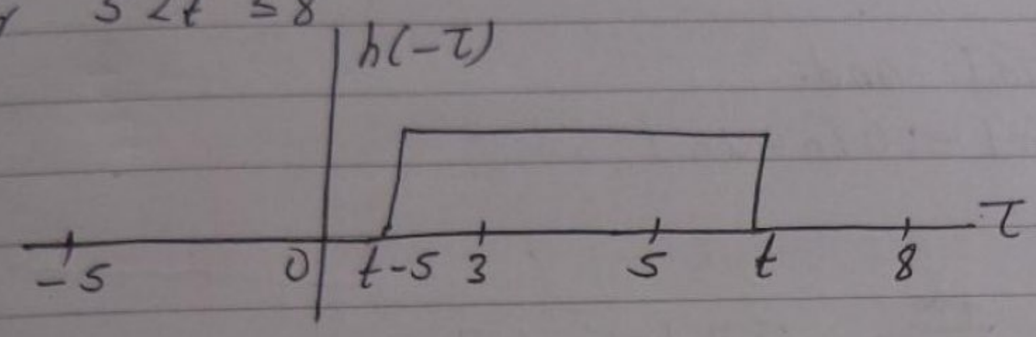
$$y(t) = \int_0^t e^{\tau} d\tau = e^{t} - 1$$

For $3 < t \le 5$



$$y(t) = \int_0^3 e^{\tau} d\tau$$
$$= e^{3} - 1$$

For $5 < t \le 8$



$$y(t) = \int_{t-5}^{3} e^{\tau} d\tau = e^{3} - e^{t-5}$$

For $t > 8$,

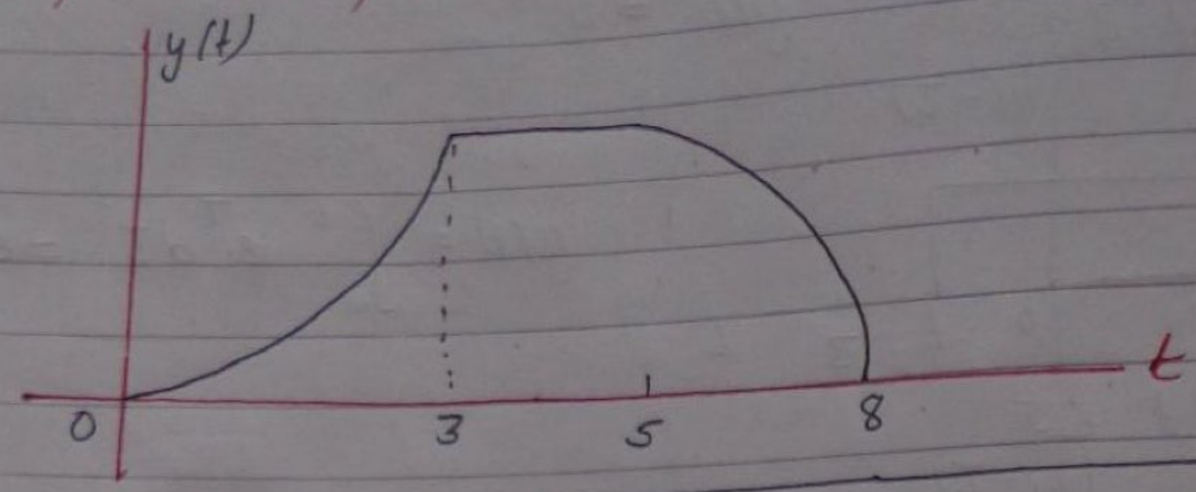
$$y(t) = 0$$

Hence,

$$y(t) = \begin{cases} 0 & , t \le 0 \\ e^{t} & , 0 < t \le 3 \\ e^{3} - 1 & , 3 < t \le 5 \\ e^{3} - e^{t-5} & , 5 < t \le 8 \\ 0 & , t > 0 \end{cases}$$

Scanned with CamScanner

output can be plotted as,



1. Determine the response of the system with impulse response. $h[n] = a^n u[n]$ to the input signal, $x[n] = u[n] - u[n-10]$. 2014/F

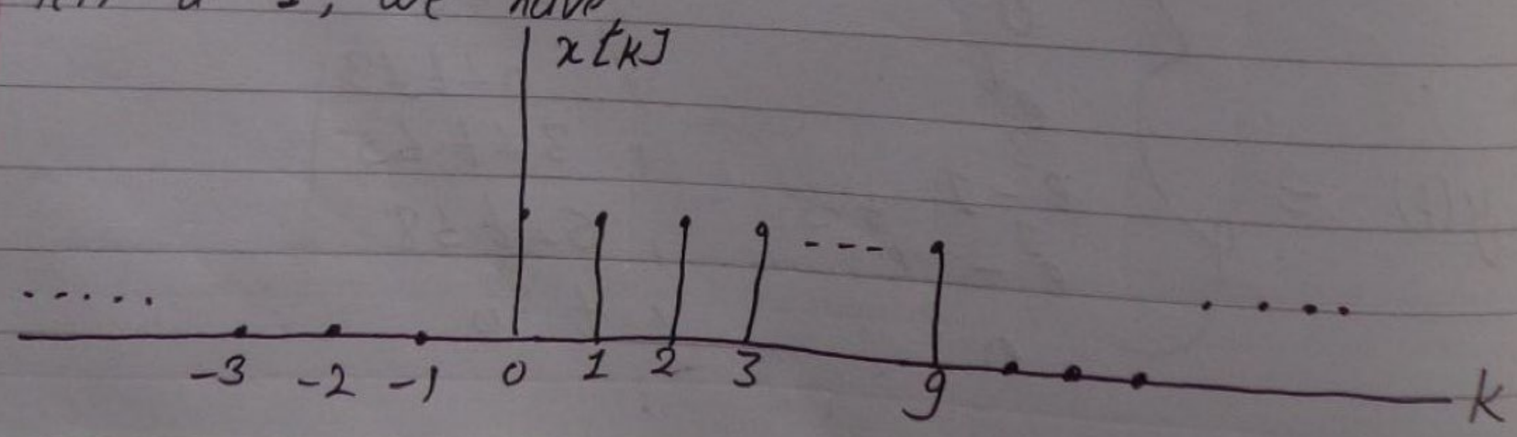
sol^n,

$\Rightarrow$ $h[n] = a^n u[n]$ and
$x[n] = u[n] - u[n-10]$,
$y[n] = ?$
we have,

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

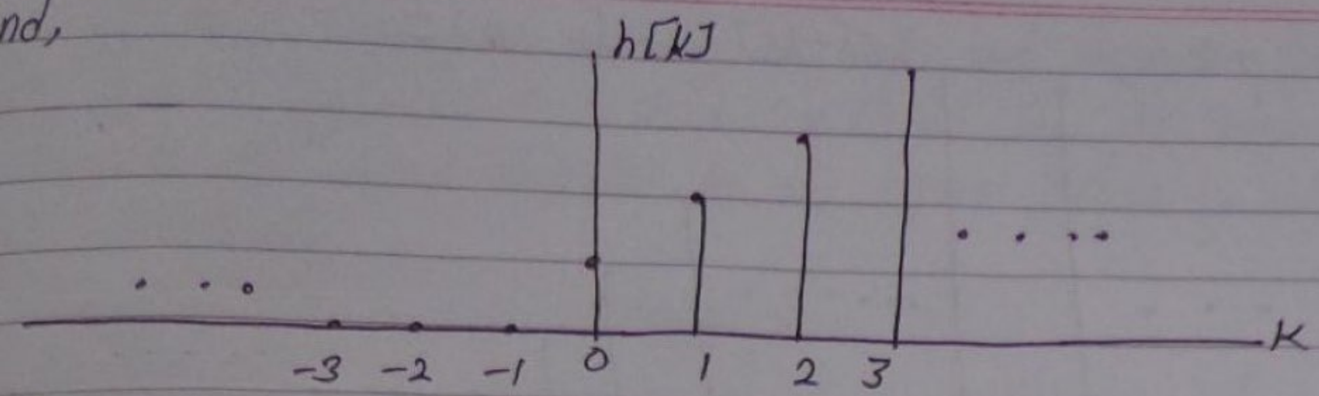
let's represent impulse response & input signal as function of k.

i.e, $h[k] = a^k u[k]$ and
$x[k] = u[k] - u[k-10]$
let, $a > 1$, we have

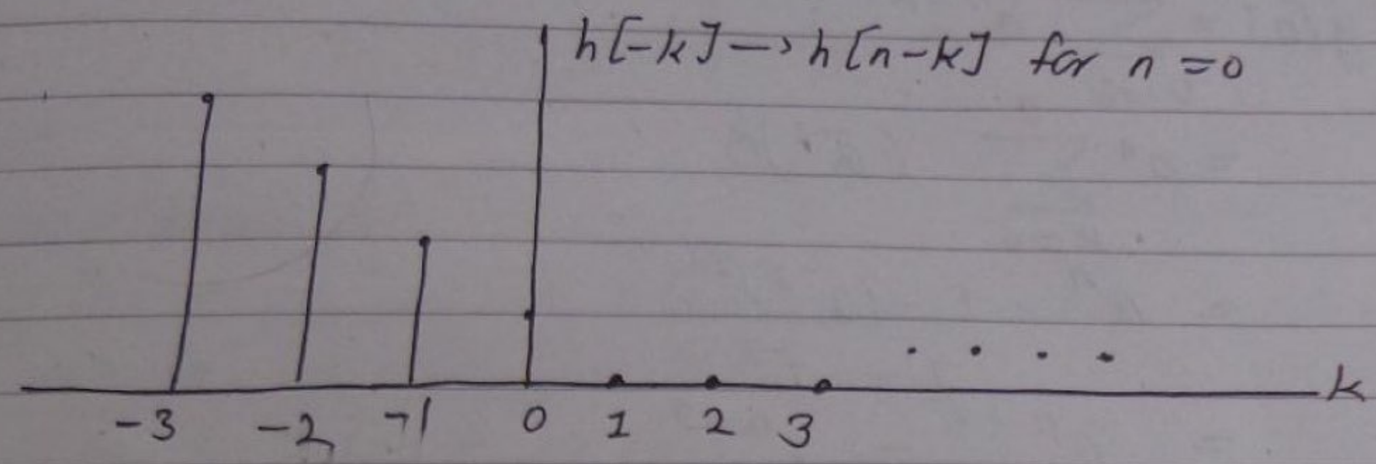


Scanned with CamScanner

and,



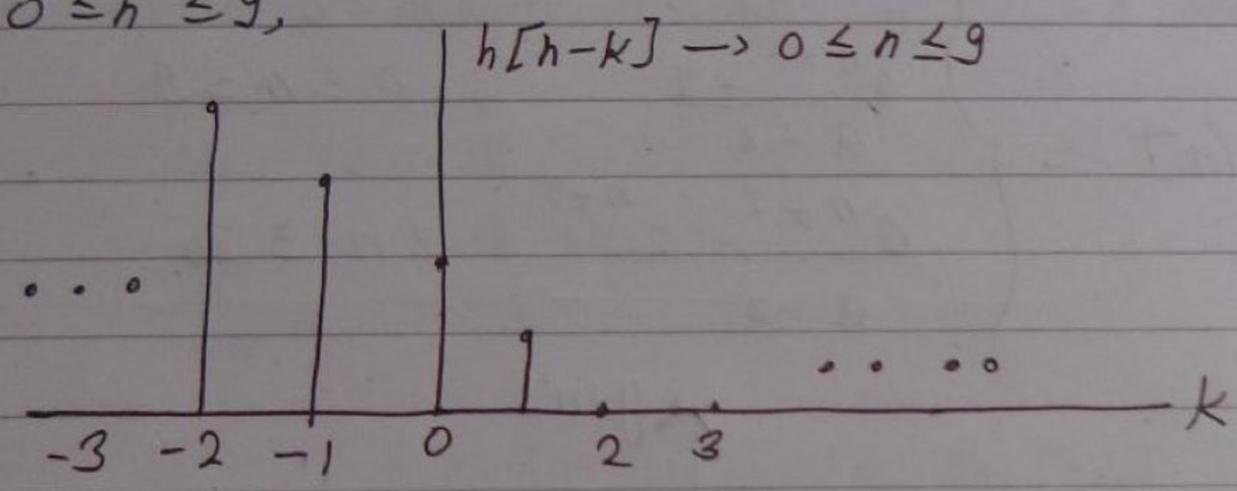
Folding $h[k]$ over time axis, we get.



For $n < 0$,

$$y[n] = 0$$

For $0 \le n \le 9$,



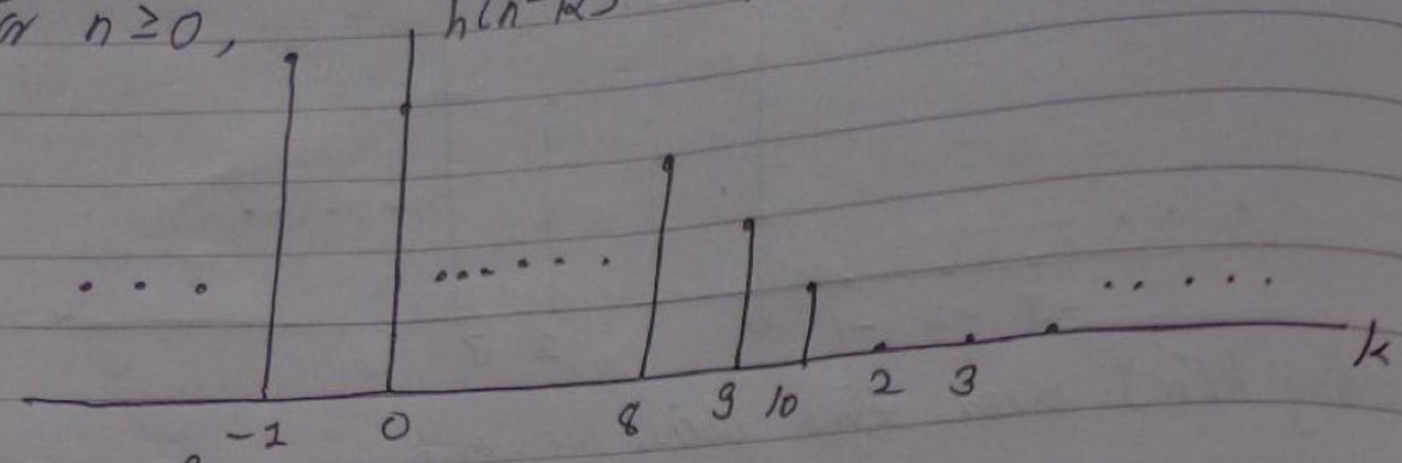
$$y[n] = \sum_{k=0}^{n} a^{n-k} = a^{n} \sum_{k=0}^{n} (a^{-1})^{k}$$

Using geometric series, formula, we get.

$$y[n] = a^{n} \frac{(1-a^{-1})^{n+1}}{1-a^{-1}} = \frac{a^{n+1}-1}{a-1}$$

Scanned with CamScanner

For $n \geq 0$, $h[n-k] \rightarrow$ for $n \geq 10$



$$y[n] = \sum_{k=0}^{9} a^{n-k}$$

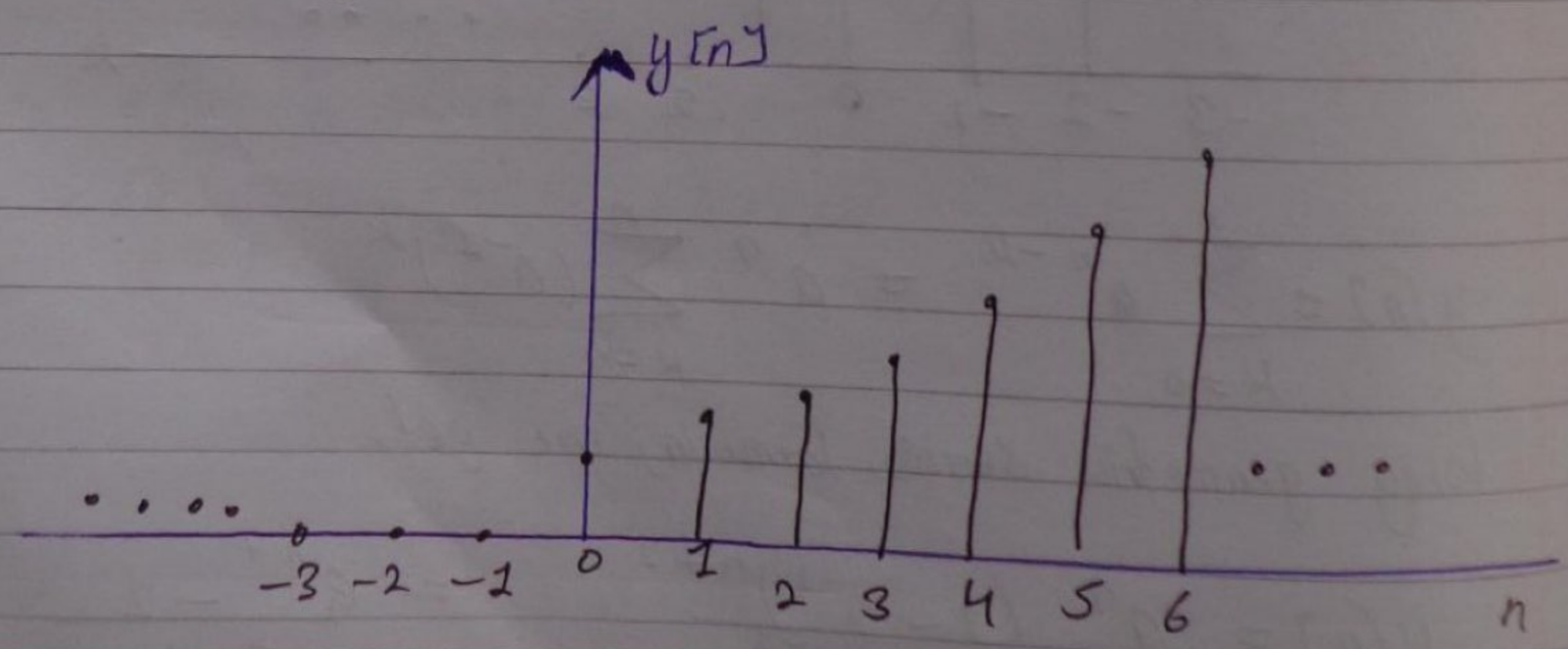
$$= a^n \sum_{k=0}^{9} (a^{-1})^k$$

$$= a^n \frac{1-(a^{-1})^{10}}{1-a^{-1}}$$

$$= \frac{a^{n+1} - a^{n-9}}{a-1}$$

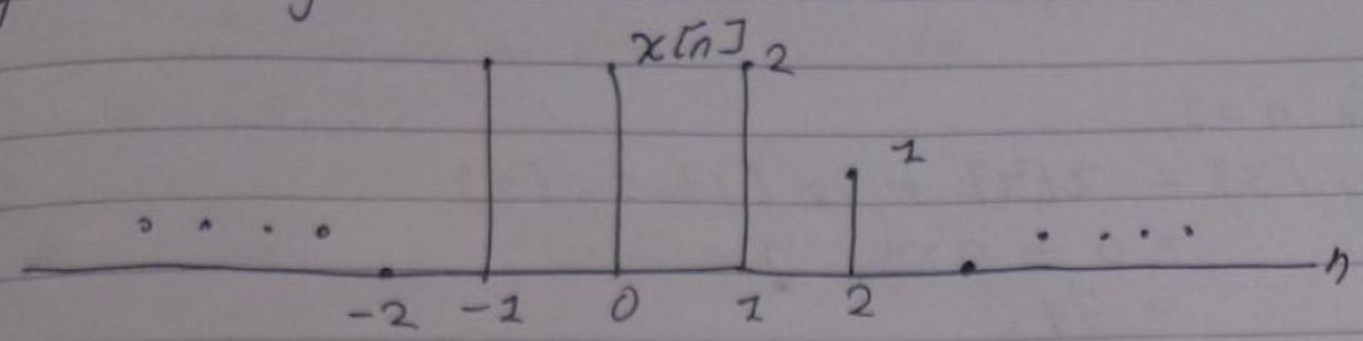
Hence, output is,

$$y[n] = \begin{cases} 0, & , n < 0 \\ \dfrac{a^{n+1}-1}{a-1} & , 0 \leq n \leq 9 \\ \dfrac{a^{n+1} - a^{n-9}}{a-1} & , n \geq 10 \end{cases}$$



Scanned with CamScanner

M. Consider the LTI system initially at rest and described by difference eq^n $y[n] + 2y[n-1] = x[n] + 2x[n-2]$. find the respome of the system to the input depicted in the figure below by solving difference eq^n recursively.



sol^n,

$\Rightarrow$ Given, difference eq^n,

$$y[n] + 2y[n-1] = x[n] + 2x[n-2]$$

The eq^n in recursive form is,

$$y[n] = x[n] + 2x[n-2] - 2y[n-1]$$

given input is,

since the system is initially at rest, $y[n] = 0$ for $n \leq -2$.

For $n = -1$, $y[-1] = x[-1] + 2x[-3] - 2y[-2]$
$= 2 + 0 + 0$
$= 2$

For $n = 0$, $y[0] = x[0] + 2x[-2] - 2y[-1]$
$= 2 + 0 - 2(2)$
$= -2$

For $n = 1$, $y[1] = x[1] + 2x[-1] - 2y[0]$
$= 2 + 2 \times 2 - 2 \times (-2)$
$= 10$

Scanned with CamScanner

For $n=2$,

$y[2] = x[2] + 2x[0] - 2y[1]$

$= 1 + 2(2) - 2(10)$

$= -15$

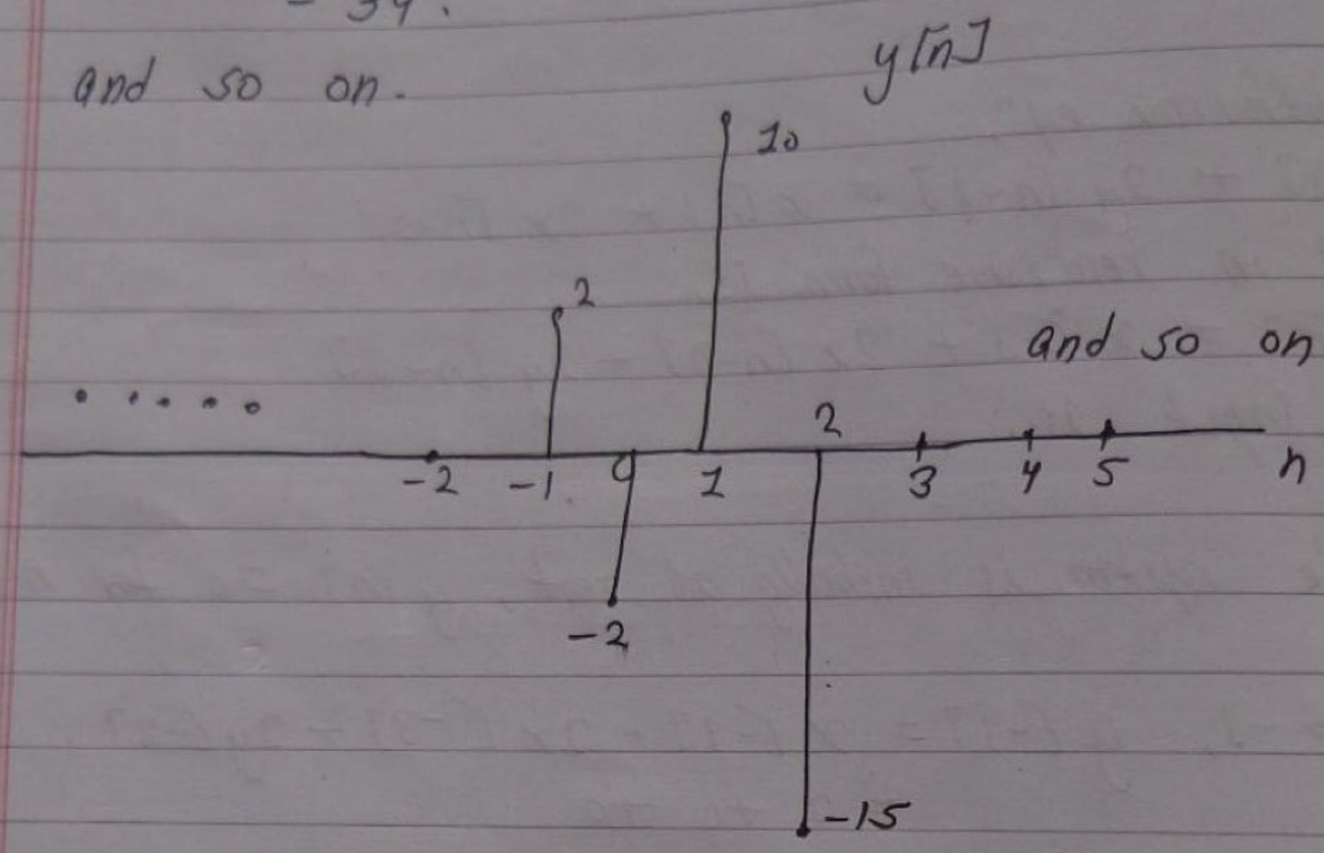
For $n=3$,

$y[3] = x[3] + 2x[1] - 2y[2]$

$= 0 + 2\times2 - 2\times(-15)$

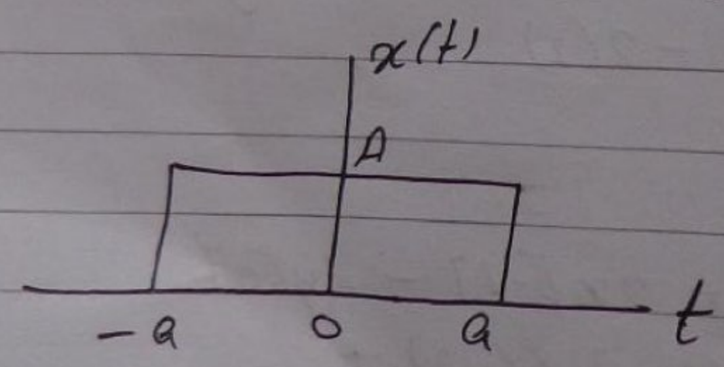
$= 34.$

and so on.



#i Check whether given signal is energy or power or neither signal.

$x(t) = A\,[u(t+a) - u(t-a)]$



Sol^n,

Scanned with CamScanner

since, this signal is of finite duration, this can be energy signal,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{-a} 0 \, dt + \int_{-a}^{a} A^2 \, dt + \int_{a}^{\infty} 0 \, dt$$

$$\therefore E = 2a A^2$$

&

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)^2| \, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \times E$$

$$\therefore P = 0$$

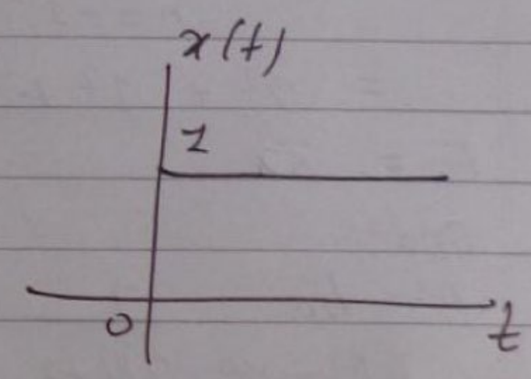
since, total energy is finite & average power is 0, the given signal is energy signal.

ii $x(t) = u(t)$.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{0}^{\infty} dt$$

$$= \infty$$



$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \times T$$

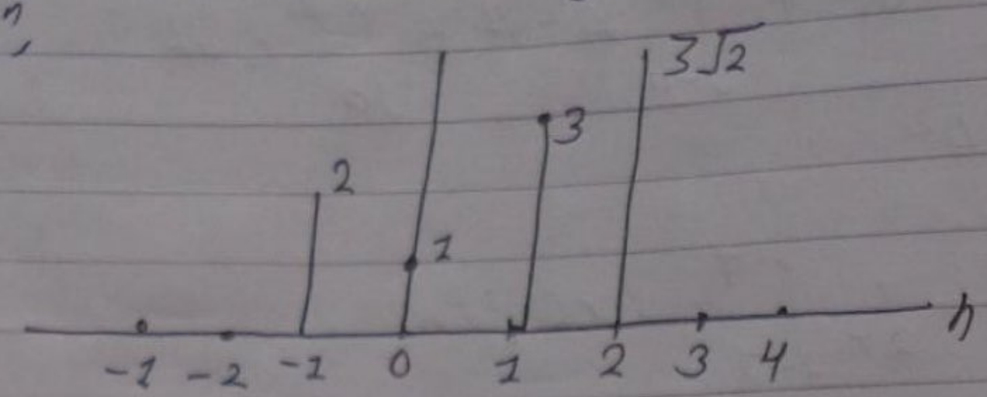
$$= \frac{1}{2}$$

Scanned with CamScanner

Hence, given signal is power signal.

iii $x[n] = \{2, 1, 3, 3+3j\}$

Sol^n,



Since, this is the sequence of finite duration, this can be energy signal,

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= 0 + \sum_{n=-1}^{2} |x[n]|^2 + 0$$

$$= 2^2 + 1^2 + 3^2 + (3\sqrt{2})^2 \qquad [\because \sqrt{3^2+3^2} = 3\sqrt{2}]$$

$\therefore E = 32.$

and

$$P = \lim_{N\to\infty} \frac{1}{2N+2} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+2} \sum_{n=-1}^{2} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \cdot E$$

$$= 0$$

Hence, given signal is eneg signal.

Scanned with CamScanner

iv. check for the system : linear

a) $y(t) = t \cdot x(t)$

let,

$x_1(t) \longrightarrow y_1(t) = t \cdot x_1(t)$

$x_2(t) \longrightarrow y_2(t) = t x_2(t)$

let,

$x_3(t) = a x_1(t) + b x_2(t)$, then

$y_3(t) = T\{x_3(t)\}$

$= T\{a x_1(t) + b x_2(t)\}$

$= a t x_1(t) + b t x_2(t)$

$= a y_1(t) + b y_2(t)$

Hence, this system is linear as $y_3(t) = x_3(t)$

b. $y[n] = (x[n])^2$

sol^n,

$\Rightarrow$ $x_1[n] \longrightarrow y_1[n] = [x_1[n]]^2$

$x_2[n] \longrightarrow y_2[n] = [x_2[n]]^2$

let,

$x_3[n] = a x_1[n] + b x_2[n]$, then,

$y_3[n] = [a x_1[n] + b x_2[n]]^2$, then,

$= a^2 [x_1[n]]^2 + 2ab x_1[n] x_2[n] + b^2 [x[n]]^2$

$\neq a y_1[n] + b y_2[n]$

Hence, system is non-linear.

v. check for time variant or not

a) $y(t) = t^2 x(t-1)$

Sol^n,

$x_1(t) \longrightarrow y_1(t) = t^2 x(t-1)$ — ①

$x_2(t) \longrightarrow x_1(t-t_0) \longrightarrow y_2(t) = t^2 x_1(t-1-t_0)$

Scanned with CamScanner

$= t^2 x_1(t-t_0-1)$ — ①

put $t = t-t_0$ in eq^n ①,

$y_1(t-t_0) = (t-t_0)^2 x_1(t-t_0-1)$ — ③

since, $y_1(t-t_0) \neq y_2(t)$, given system is time variant.

6. $y[n] = n[x[n]]^2$

sol^n,

$\Rightarrow$ $x_1[n] \longrightarrow y_1[n] = n[x_1[n]]^2$ — ①

$x_2[n] = x_1[n-n_0] \longrightarrow y_2[n] = n[x_2[n]]^2$

$= n[x_1[n-n_0]]^2$ — ②

put $n = n-n_0$ in eq^n ①.

$y_1[n-n_0] = [n-n_0][x_1[n-n_0]]^2$ — ③

since, $y_1[n-n_0] \neq y_2[n]$, given system is time variant.

1) check if stable or unstable.

i) $y(t) = t \cdot x(t)$.

sol^n,

$\rightarrow$ given system is unstable, since a constant (bounded) input $x(t) = 1$ yields $y(t) = t$, which is unbounded.

ii $y(t) = x(t)(\cos \omega(t))$

sol^n,

$\rightarrow$ $|y(t)| = |x(t) \cos(\omega t)| \leq |x(t)|$

$|\cos \omega t| \leq 1$,

Hence, system is stable as Bounded input gives bounded output.

iii $y[n] = n\, x[n]$

sol^n,

Scanned with CamScanner

let, $x[n]$ be bounded and has a constant value of 1
ie, $x[n] = 1$, then,

$y[n] = n$

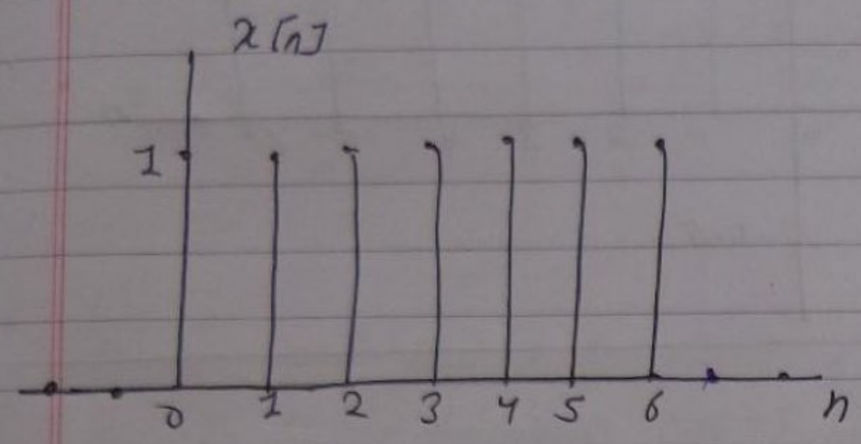
The output $y[n]$ is not bounded since it grows as time increases.
Hence system is unstable.

2. Decompose signal $x[n] = u[n] - u[n-7]$ into even & odd parts.

Sol^n,

=> Given signal,

$x[n] = u[n] - u[n-7]$

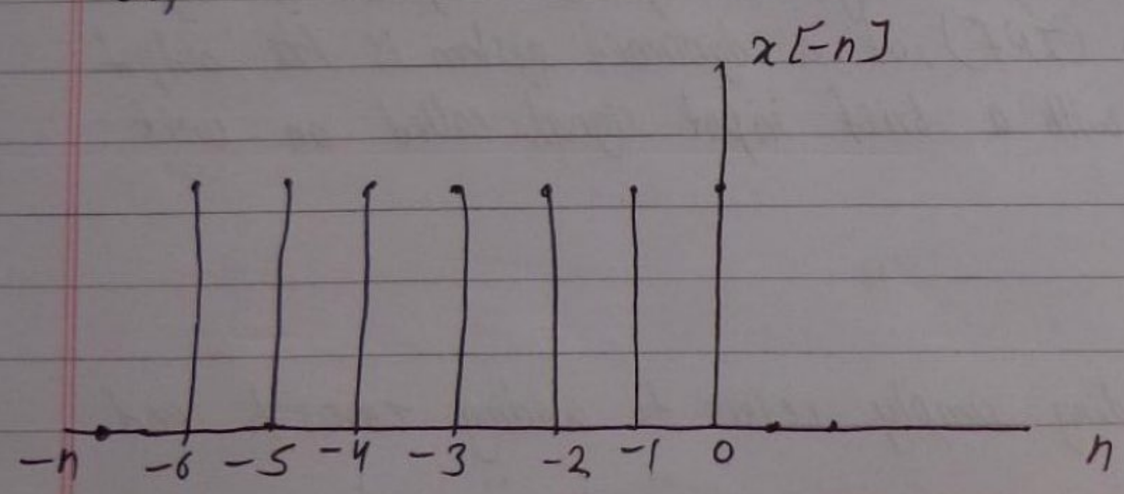


Any signal can be decomposed into even & odd parts,

even part : $x_e[n] = \frac{1}{2}\left[x[n] + x[-n]\right]$

odd part : $x_o[n] = \frac{1}{2}\left[x[n] - x[-n]\right]$

So,



Scanned with CamScanner

Hence, odd and even parts of the given signal is,

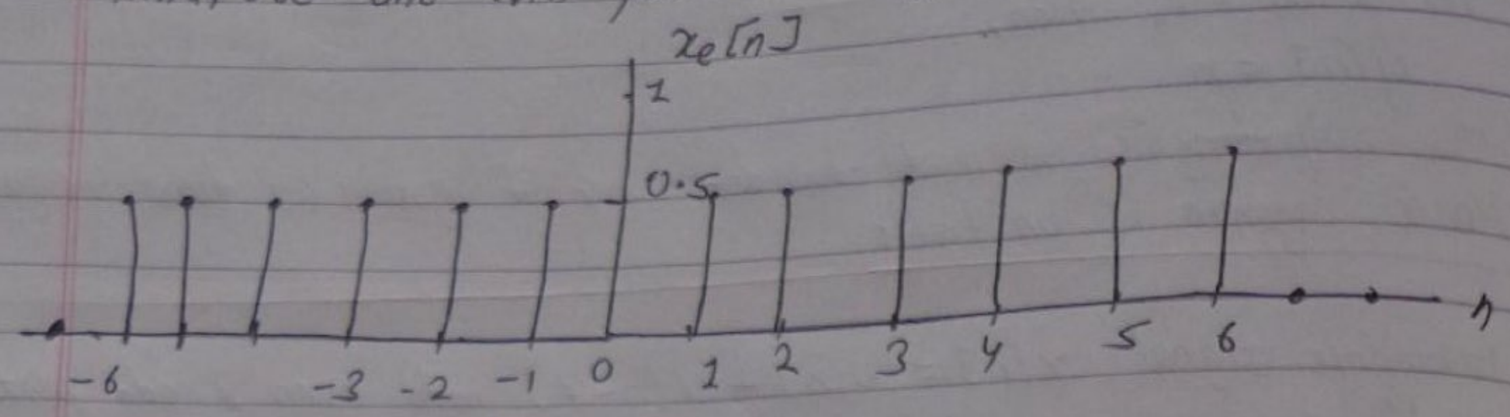
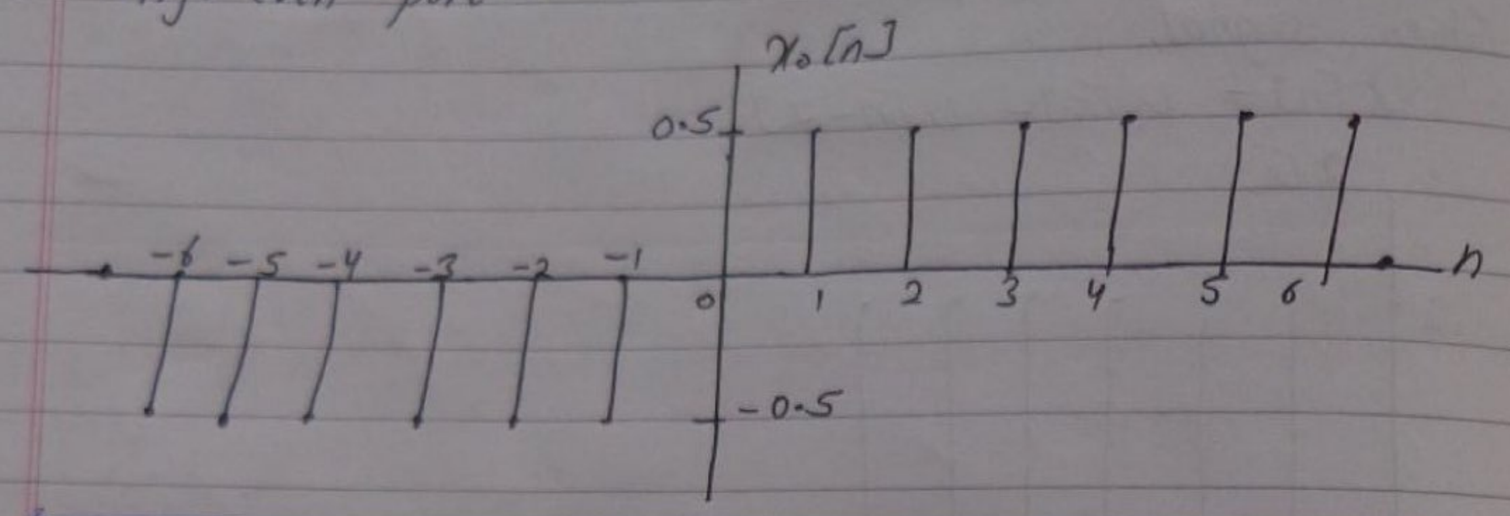


Fig: even part.



a. Unit step Response :

The response of a system (with all condition (initial) equal to zero at $t = 0^-$ i.e, a zero state response) to the unit step input is called the unit step response.

b. Unit impulse Response :

In signal processing, the impulse response or impulse response function (IRF), of a dynamic system is its output when presented with a brief input signal, called an unit impulse.

c. Zero padding :

Zero padding simply refers to adding zeros to end

Scanned with CamScanner

of time domain signal to increase its length. zero padding in the time domain gives band limited interpolation in the frequency domain. Similarly, zero padding in the frequency domain gives bandlimited interpolation in the time domain.

d. DFT:

The discrete fourier transform transforms a sequence of N complex numbers $x_0, x_1, x_2, \ldots, x_{N-1}$ into another sequence of complex numbers; $X_0, X_1, X_2, \ldots, X_{N-1}$ which is defined by,

$$X_K = \sum_{n=0}^{N-1} x_n \cdot e^{-j 2\pi \frac{Kn}{N}}$$

e. FFT:

A Fast fourier transform (FFT) algorithm computes the discrete fourier tramform (DFT) of a sequence or its inverse. If N samples are present, DFT takes N^2 operations while FFT takes only $N \times \log(N)$ operation. Hence, FFT is much faster than DFT.

FFT is a simpler and fastest method of implementing DFT. This is very useful when the value of N is large.

Scanned with CamScanner

Fourier series & Transform

Chapter = 3

1. A) $x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\left(\frac{2\pi}{T}\right)t}$ — (i)

where, $a_k = \frac{1}{T}\int_0^T x(t) \cdot e^{-jk\omega_0 t}\, dt$ — (ii)

eqn (i) is referred to as the synthesis equation and eqn (ii) as the analysis eqn. The set of coefficients a_k are often called the Fourier series coefficients or the spectral coefficients of $x(t)$. The coefficients a_0 is the DC or constant component of $x(t)$ and is given by,

$a_0 = \frac{1}{T}\int_0^T x(t)\, dt$.

B. For the signal $x(t) = \sin\omega_0 t$, whose fundamental frequency is the ω_0, find the fourier series coefficients.

Soln,

$\Rightarrow \sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$

Comparing the right hand side with

$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$

$\therefore a_1 = \frac{1}{2j}$, $a_{-1} = \frac{-1}{2j}$

C. What are the conditions that are required for the existence of the fourier series? Derive the alternative forms of the fourier series.

1 2012/F, 2013/F

$\Rightarrow$ There is Dirichlet conditions that are very essential for the convergence

Scanned with CamScanner

of Fourier series.

Condition ①:

- over any period, $x(t)$ must be absolutely integrable, i.e, $\int_0^T |x(t)|dt < \infty$

Condition ②:

In any finite interval of time, $x(t)$ is of bounded variation, ie, there should be a finite number of maxima and minima during any single period of the signal.

An example which meets condition ① but not 2 is, $x(t) = \sin(2\pi/t)$ of boundary $0 \le t \le 1$.

For this function, which is periodic with $T=1$, $\int_0^1 |x(t)|dt < 1$

The function has infinite number of maxima and minima in the interval.

Condition ③:

In any finite interval of time, there are only a finite number of discontinuities, furthermore each of these discontinuities is finite.

Alternative forms of the Fourier series:

① Compact Trigonometric Fourier series:

For periodic signal with period T with Fourier series representation given by,

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_0 t + b_n \sin w_0 t)$$

let, $C_0 = a_0$, $C_n = \sqrt{a_n^2 + b_n^2}$, $\theta = \tan^{-1}\left(\frac{b_n}{a_n}\right)$

Then,

$$x(t) = C_0 + \sum_{n=1}^{\infty} [(C_n \cos\theta) \cos n w_0 t + (C_n \sin\theta) \sin n w_0 t]$$

$$\therefore x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n w_0 t - \theta)$$

Scanned with CamScanner

This is compact trigonometric Fourier series representation. The compact Fourier series coeff can be obtained by, $a_0 = C_0$, $a_n = C_n \cos\theta$ and $b_n = C_n \sin\theta$.

(ii) Complex exponential Fourier series: 2017/s

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_0 t + b_n \sin w_0 t)$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \left(\frac{e^{jnw_0 t} + e^{-jnw_0 t}}{2} \right) + b_n \left(\frac{e^{jnw_0 t} - e^{-jnw_0 t}}{2j} \right) \right)$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\frac{(a_n - jb_n) e^{jnw_0 t}}{2} + \frac{(a_n + jb_n) e^{-jnw_0 t}}{2} \right]$$

Let, $C_n = \frac{a_n - jb_n}{2}$

$C_{-n} = \frac{a_n + jb_n}{2}$

$C_0 = a_0$

Then,

$$x(t) = C_0 + \sum_{n=1}^{\infty} (C_n e^{jnw_0 t} + C_{-n} e^{-jnw_0 t})$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{jnw_0 t} + \sum_{n=1}^{\infty} C_{-n} e^{-jnw_0 t}$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{jnw_0 t} + \sum_{n=-\infty}^{-1} C_{-n} e^{-jnw_0 t}$$

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} C_n \cdot e^{jnw_0 t}$$

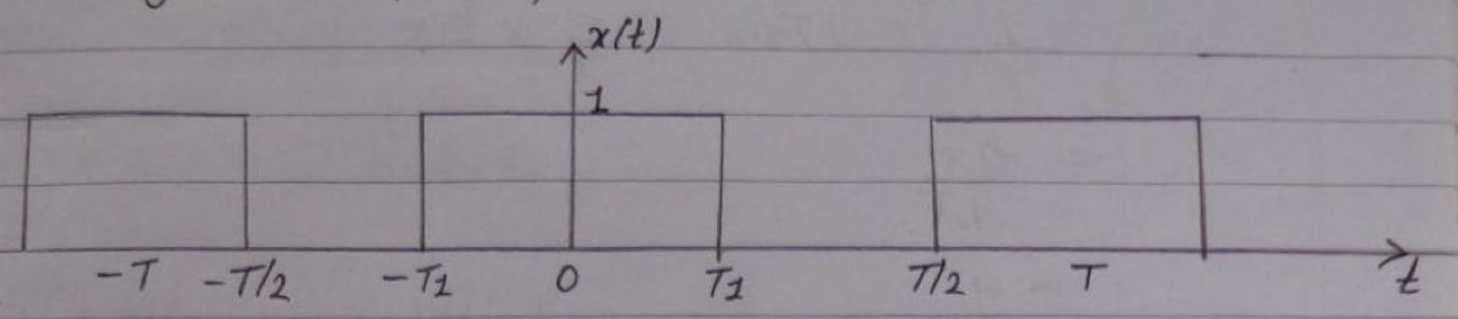
These are two alternative forms of fourier series.

Scanned with CamScanner

D. The periodic square wave, sketched as below and defined over one period as

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < T/2 \end{cases}$$

Plot the magnitude and phase spectra for $T = 4T_1$.



Sol^n,

$\Rightarrow$ we have,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$$

where, $$a_k = \frac{1}{T}\int_{-T_1}^{T_1} 1 \cdot e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1}$$

$$= \frac{1}{T}\left[\frac{e^{+jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{+jk\omega_0}\right]$$

$$= \frac{1}{T_0 k\omega_0} \times 2\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T_0}$$

$$\therefore a_k = \frac{\sin(k\omega_0 T_1)}{k\pi} \qquad \left[\because T_0 = \frac{2\pi}{\omega_0}\right]$$

For $T = 4T_1$, $x(t)$ is square wave that is unity for half period and zero for half period. In this case, $\omega_0 T_1 = \frac{\pi}{2}$ and so on,

Scanned with CamScanner

$\therefore a_k = \frac{\sin(k\pi/2)}{k\pi} \quad k \neq 0$ — ①

and, $a_0 = \frac{1}{T}\int_{-T_1}^{T_1} 1 \cdot dt$

$= \frac{1}{T} \times 2T_1 = \frac{1}{T} \times 2\pi$

$= \frac{2\pi}{4\pi}$

$= \frac{1}{2}$

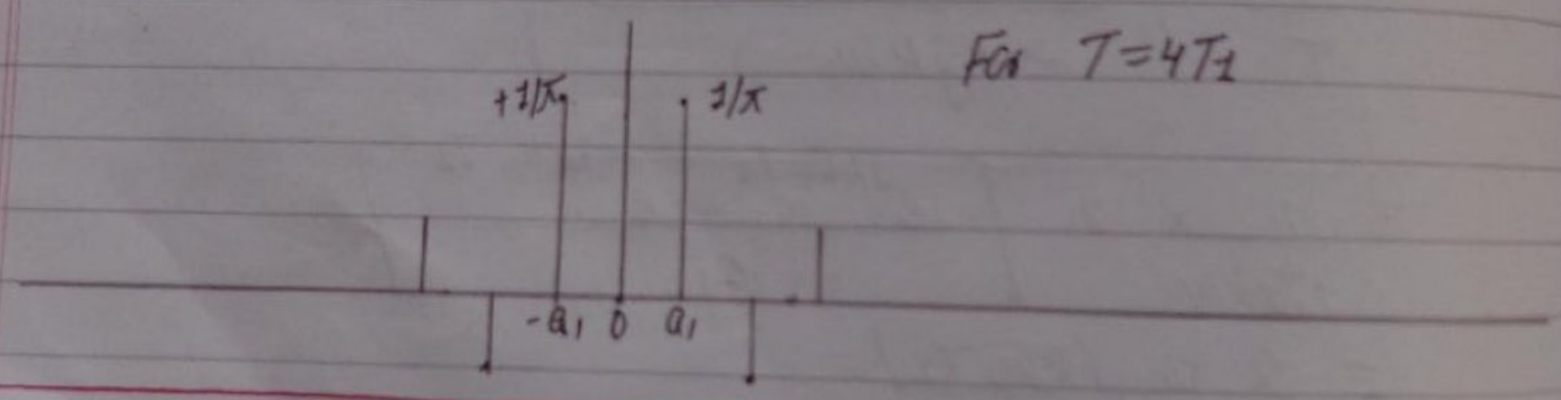
From eqn ①, $a_k = 0$ for k even and non-zero. Also, $\sin(k\pi/2)$ alternates between ± 1 for successive odd values of k. Hence,

$a_1 = a_{-1} = \frac{1}{\pi}$

$a_3 = a_{-3} = \frac{-1}{3\pi}$

$a_5 = a_5 = \frac{1}{5\pi}$ and so on.

For $T = 4T_1$



E. State and prove time scaling property of CTFT. 2078/F

⇒ Fourier transform of $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

and inverse fourier transform of $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$.

Scanned with CamScanner

Time scaling property:

If

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

then,

$$y(t) = x(at) \xleftrightarrow{F.T} Y(j\omega) = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

where a is a real constant.

proof:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

putting $at = \tau$, then,

$$Y(j\omega) = \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\frac{\omega}{a}\tau}\, d\tau, & a>0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\cdot e^{j\frac{\omega}{a}\tau}\, d\tau, & a<0 \end{cases}$$

$$\therefore Y(j\omega) = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

proved.

F. Time shifting and Frequency shifting of Fourier Transform: 2014/F, 2017/F, 2017/S

Time shifting:

If $x(t) \xleftrightarrow{F.T} X(j\omega)$

then,

$$y(t) = x(t-t_0) \xleftrightarrow{F.T} Y(j\omega) = e^{-j\omega t_0} X(j\omega)$$

proof:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt.$$

Scanned with CamScanner

$$= \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

put, $t - t_0 = \tau$, then,

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau + t_0)}\, d\tau$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega \tau}\, d\tau$$

$$\therefore Y(j\omega) = e^{-j\omega t_0}\, X(j\omega)$$

Similarly,

if $x[n] \xleftrightarrow{F.T} X(e^{j\omega})$

then,

$$x[n-n_0] \xleftrightarrow{F.T} e^{-j\omega n_0}\, X(e^{j\omega})$$

Frequency shifting:

If

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

then,

$$y(t) = e^{j\omega_0 t}\, x(t) \xleftrightarrow{F.T} Y(j\omega) = X(j(\omega - \omega_0))$$

proof:

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t}\, x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega - \omega_0)t}\, dt$$

$$\therefore Y(j\omega) = X(j(\omega - \omega_0))$$

Similarly,

If $x[n] \xleftrightarrow{F.T} X(e^{j\omega})$

Scanned with CamScanner

then, $e^{jw_0 n} x[n] \xleftrightarrow{F.T} X(e^{j(w-w_0)})$

G. Derive the harmonically related complex exponentials for continuous time signals.
2011/S, 2012/F

⇒ A continuous time periodic complex exponential signal with fundamental period T and fundamental frequency w_0 is defined by,

$$x(t) = e^{jw_0 t} \quad \text{— (1)}$$

where, $w_0 = \frac{2\pi}{T}$

A set of harmonically related complex exponential can be expressed as,

$$\phi_k(t) = e^{jkw_0 t}, \quad k = 0, \pm 1, \pm 2, \text{-----}$$

Each of these signals has a fundamental frequency that is an integer multiple of w_0 and each is periodic with common period T.

i.e,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\left(\frac{2\pi}{T}\right)t}$$

In general, the terms for $k = -N$ and $k = +N$ are collectively known as N^{th} harmonic components.

For finding a_k, multiplying both sides of eqn (1) by $e^{-jnw_0 t}$ & integrating from 0 to T,

$$\int_0^T x(t) e^{-jnw_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k \cdot e^{jkw_0 t} \cdot e^{-jnw_0 t} dt$$

interchanging the order of integration, & summation,

$$\int_0^T x(t) e^{-jnw_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \cdot \left[\int_0^T e^{j(k-n)w_0 t} dt\right] \quad \text{— (3)}$$

Scanned with CamScanner

using Euler's identity,

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \int_0^T \cos((k-n)\omega_0 t) + j\int_0^T \sin((k-n)\omega_0 t)\,dt \quad \text{— (11)}$$

For $k \neq n$, $(\cos(k-n)\omega_0 t)$ and $(\sin(k-n)\omega_0 t)$ are periodic signals with fundamental period $\frac{T}{|k-n|}$. Since, T is an integer multiple of $\frac{T}{|k-n|}$, then, for $k \neq n$, both of the integrals on the RHS of eqⁿ (11) are 0. For $k = n$, the integral on LHS of eqⁿ (11) results T,

Thus,

$$\int_0^T e^{j(k-n)\omega_0 t}\,dt = \begin{cases} T & , k = n \\ 0 & , k \neq n \end{cases}$$

And, RHS of eqⁿ (3) reduces to $T a_n$,

i.e, $\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt = T a_n$.

or, $$a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt.$$

or $$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\,dt$$

or, $$x(t) = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\left(\frac{2\pi}{T}\right)t} \quad \text{— (4)}$$

is known as synthesis eqⁿ.

where,

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\,dt$$

For $k = 0$, the coefficient,

$\therefore$ $a_0 = \frac{1}{T}\int_T x(t)\,dt$. is DC or constant component of $x(t)$.

Scanned with CamScanner

H. with an example show that how can you obtain the Fourier transform of discrete-time periodic signals. 2014/F

$\Rightarrow$ we have, DTFT, $X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$ which is known as analysis eqn.

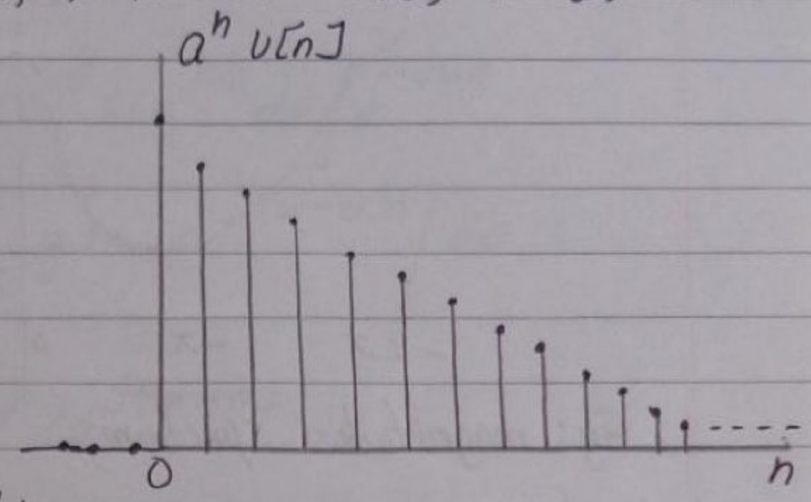
For example: consider a signal, $x[n] = a^n u[n]$, $|a| < 1$,

Now,

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$$= \sum_{n=0}^{\infty} a^n e^{-jwn}$$

$$\therefore \quad = \sum_{n=0}^{\infty} (ae^{-jw})^n$$



For magnitude & phase spectra,

$$X(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

$$= \frac{1}{1 - a\cos(w) + ja\sin(w)}$$

$$= \frac{1}{[1 - a\cos(w) + ja\sin(w)]} \times \left[\frac{1 - a\cos(w) - ja\sin(w)}{1 - a\cos(w) - ja\sin(w)}\right]$$

$$= \frac{1 - a\cos(w) - ja\sin(w)}{(1 - a\cos(w))^2 + (a\sin(w))^2}$$

$$= \frac{1 - a\cos(w)}{(1 - a\cos(w))^2 + (a\sin(w))^2} - j\frac{a\sin(w)}{(1 - a\cos(w))^2 + (a\sin(w))^2}$$

Therefore,

$$|X(e^{jw})| = \frac{1}{\sqrt{(1 - a\cos(w))^2 + (a\sin(w))^2}}$$

Scanned with CamScanner

$$= \frac{1}{\sqrt{1-2a\cos(w)+a^2\cos^2(w)+a^2\sin^2(w)}}$$

$$= \frac{1}{\sqrt{1-2a\cos(w)+a^2}}$$

and,

$$\angle X(e^{jw}) = -\tan^{-1}\left(\frac{a\sin(w)}{1-a\cos w}\right)$$

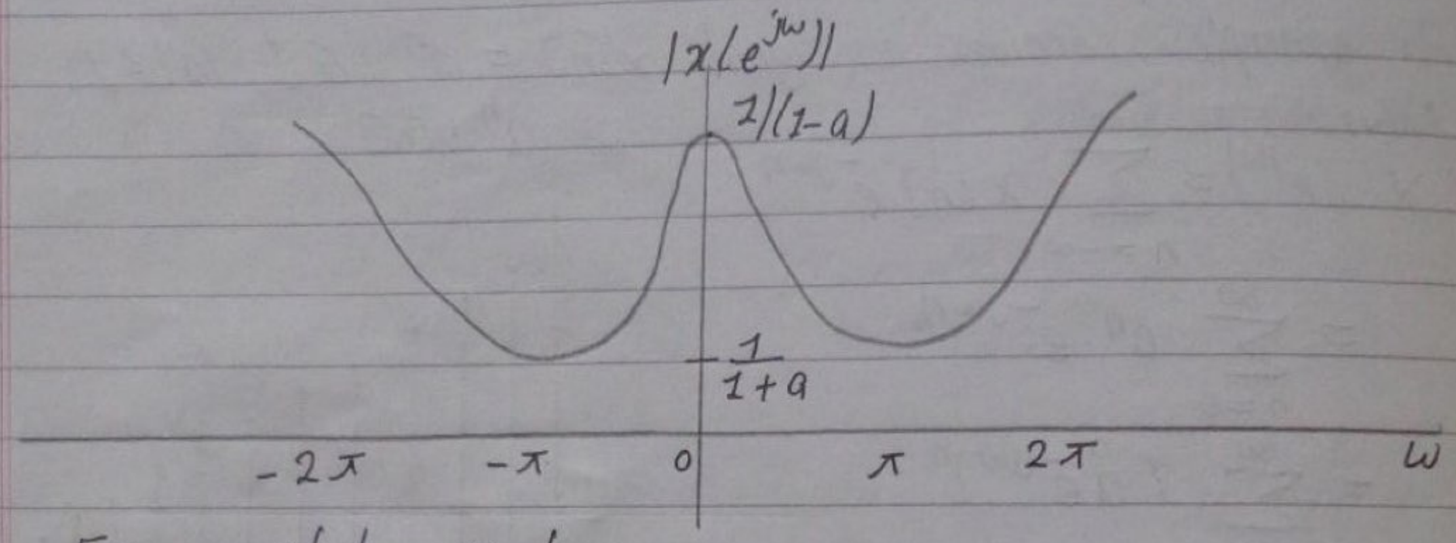


Fig: magnitude spectrum.

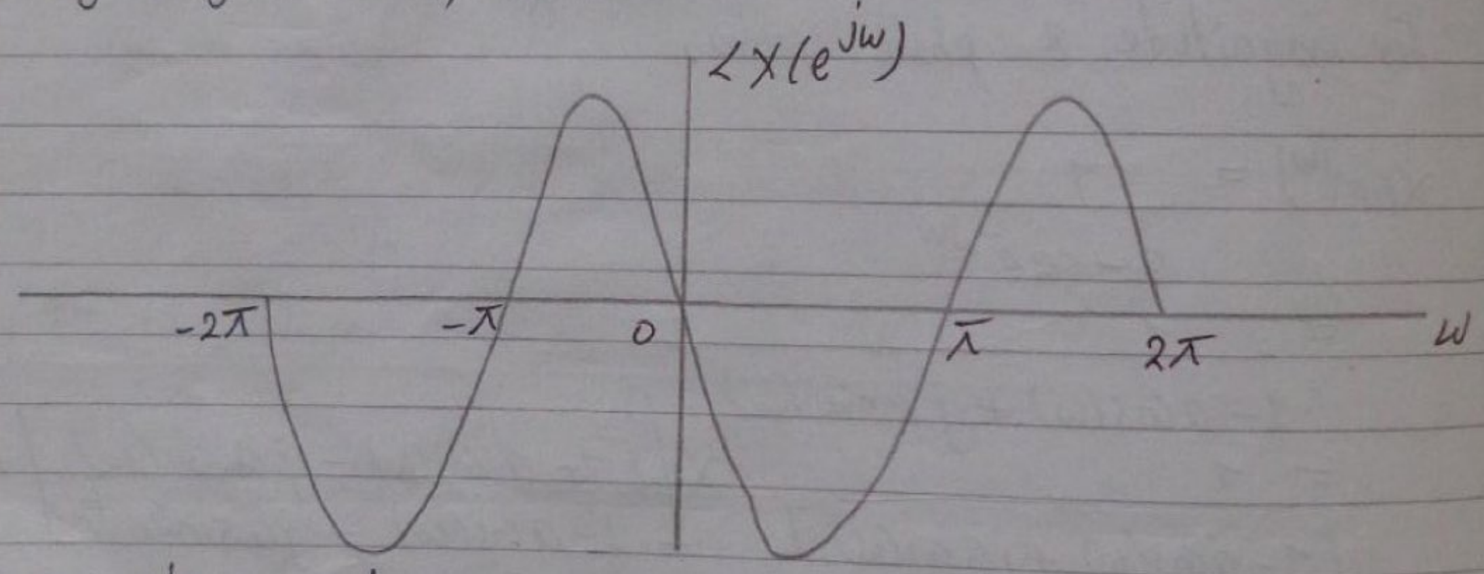


Fig: phase spectrum.

I. State and prove the multiplication property of (DTFT). 2015/5
modulation property, 2017/5

$\Rightarrow$ If $x[n] \xleftrightarrow{F.T} X(e^{jw})$

and,

Scanned with CamScanner

$y[n] \xleftrightarrow{F.T} Y(e^{jw})$

then,

$$z[n] = x[n]y[n] \xleftrightarrow{F.T} Z(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(w-\theta)})d\theta$$

Proof:

$$Z(e^{jw}) = \sum_{n=-\infty}^{\infty} z[n]e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} y[n]\left[\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})(e^{j\theta n}d\theta\right]e^{-jwn}$$

$$= \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})\left[\sum_{n=-\infty}^{\infty} y[n]e^{-j(w-\theta)n}\right]d\theta$$

$$\therefore Z(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(w-\theta)})d\theta$$

proved.

Parseval's theorem for:

① Continuous – Time periodic signals: 2010/F

If

$x(t) \xleftrightarrow{F.S} a_k$

Then, parseval's relation for continuous time periodic signal is defined by,

$$P = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Proof:

Average power of periodic signal is, $P = \frac{1}{T}\int_T |x(t)|^2 dt$

Scanned with CamScanner

$$P_{av} = \sum_{k=\langle N \rangle} |a_k|^2$$

(iii) Continuous time – aperiodic signals: 2075/F

If

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

then,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

proof:

The total energy of a continuous time aperiodic signal is,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t}\, d\omega\right] dt$$

interchanging order of integration,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega) \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\, X(j\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Hence,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

Here, $(|X(j\omega)|^2)$ is often refered as energy density spectrum.

Scanned with CamScanner

$$= \frac{1}{T}\int_T x(t)\cdot x^*(t)\,dt$$

$$= \frac{1}{T}\int_T x(t)\left[\sum_{k=-\infty}^{\infty} a_k^*\cdot e^{-jk\omega_0 t}\right]dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^*\left[\frac{1}{T}\int_T x(t)\,e^{-jk\omega_0 t}\right]dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^*\cdot a_k$$

$$P_{av} = \sum_{k=-\infty}^{\infty} |a_k|^2$$

This relation states that the total average power in a periodic signal equals the sum of average power in all of its harmonic components.

(ii) For Discrete time periodic signals: 2015/S, 2016/F

If

$$x[n] \xleftrightarrow{F.S} a_k$$

then,

$$P = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

proof:

Average power, $P = \frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2$

$$= \frac{1}{N}\sum_{n=\langle N\rangle} x[n] \times x^*[n]$$

$$= \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\cdot \sum_{k=\langle N\rangle} a_k^*\cdot e^{-jk\frac{2\pi}{N}\cdot n}$$

$$= \sum_{k=\langle N\rangle} a_k^*\left[\frac{1}{N}\sum_{n=\langle N\rangle} x[n]\cdot e^{-jk\frac{2\pi}{N}\cdot n}\right]$$

$$= \sum_{k=\langle N\rangle} a_k^*\cdot a_k$$

Scanned with CamScanner

(iv) For Discrete-time Aperiodic signals:

If

$$x[n] \xleftrightarrow{F.T} X(e^{j\omega})$$

then,

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Proof:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot x^*[n]$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left[\frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega}) \cdot e^{-j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega})\left[\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\right] d\omega$$

$$= \frac{1}{2\pi}\int_{2\pi} X^*(e^{j\omega}) X(e^{j\omega}) d\omega$$

$$\therefore E_t = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

According to parseval's relation, the total energy of a DTA periodic signal $x[n]$ can also be found by integrating the energy per unit frequency $|X(e^{j\omega})|^2/2\pi$, over 2π interval.

K. Prove that convolution of two aperiodic signals in time domain is equivalent to multiplication of their spectra in frequency domain.

2014/F.

=> The multiplication property tell us that multiplication in time domain corresponds to convolution in the frequency domain. This is

Scanned with CamScanner

also known as modulation property.

Multiplication Property:

If $x(t) \xleftrightarrow{F.T} X(j\omega)$

and,

$y(t) \xleftrightarrow{F.T} Y(j\omega)$

then,

$$z(t) = x(t) \cdot y(t) \xleftrightarrow{F.T} Z(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * Y(j\omega)\right]$$

Proof:

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega')\, e^{j\omega' t}\, d\omega'\right] e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} Y(j\omega')\, e^{-j(\omega-\omega')t}\, d\omega'\right] dt$$

put, $\sigma = \omega - \omega'$, then,

$$Z(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} Y(j(\omega-\sigma))\, e^{j\sigma t}\, d\sigma\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j(\omega-\sigma)) \left[\int_{-\infty}^{\infty} x(t)\, e^{j\sigma t}\, dt\right] d\sigma$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j(\omega-\sigma))\, X(j\sigma)\, d\sigma$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\sigma)\, Y(j(\omega-\sigma))\, d\sigma$$

$$\therefore Z(\omega) = \frac{1}{2\pi}\left[X(j\omega) * Y(j\omega)\right]$$

Scanned with CamScanner

and Convolution property is,

if

$$x(t) \xleftrightarrow{F.T} X(j\omega)$$

and $h(t) \xleftrightarrow{F.T} H(j\omega)$

then,

$$y(t) = x(t) * h(t) \xleftrightarrow{F.T} Y(j\omega) = X(j\omega) \cdot H(j\omega).$$

proof:

we have,

$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

2075/F Taking Fourier transform on both sides,

$$Y(j\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

interchanging order of integration,

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

By using time-shifting property,

$$\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt = e^{-j\omega \tau} H(j\omega)$$

Hence,

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega \tau} H(j\omega)\, d\tau$$
$$= H(j\omega) \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega \tau}\, d\tau$$

$$\therefore Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

Hence, when two fourier transform are multiplied in frequency

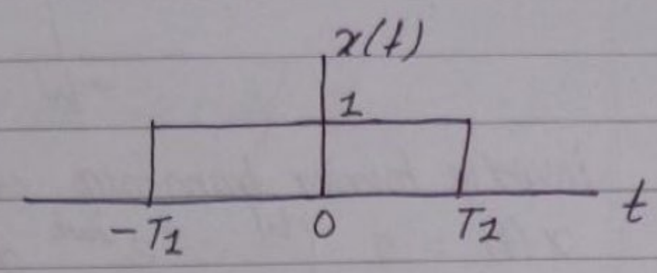
Scanned with CamScanner

domain, than they are transformed into the convolution of their individual signal in discrete- time domain.

1. Prove that Fourier transform of a rectangular pulse is the sinc function. 2015/5.

⇒ we have, Rectangular pulse as,

$$x(t) = \begin{cases} 1 & , |t| < T_1 \\ 0 & , |t| \geq T_1 \end{cases}$$



Fourier transform of $x(t)$ can be found as,

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\, dt$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-T_1}^{T_1}$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right]$$

$$= \frac{2}{\omega}\left[\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2j}\right]$$

$$\therefore X(j\omega) = 2T_1 \cdot \frac{\sin(\omega T_1)}{\omega T_1}$$

The spectrum $X(j\omega)$ is,

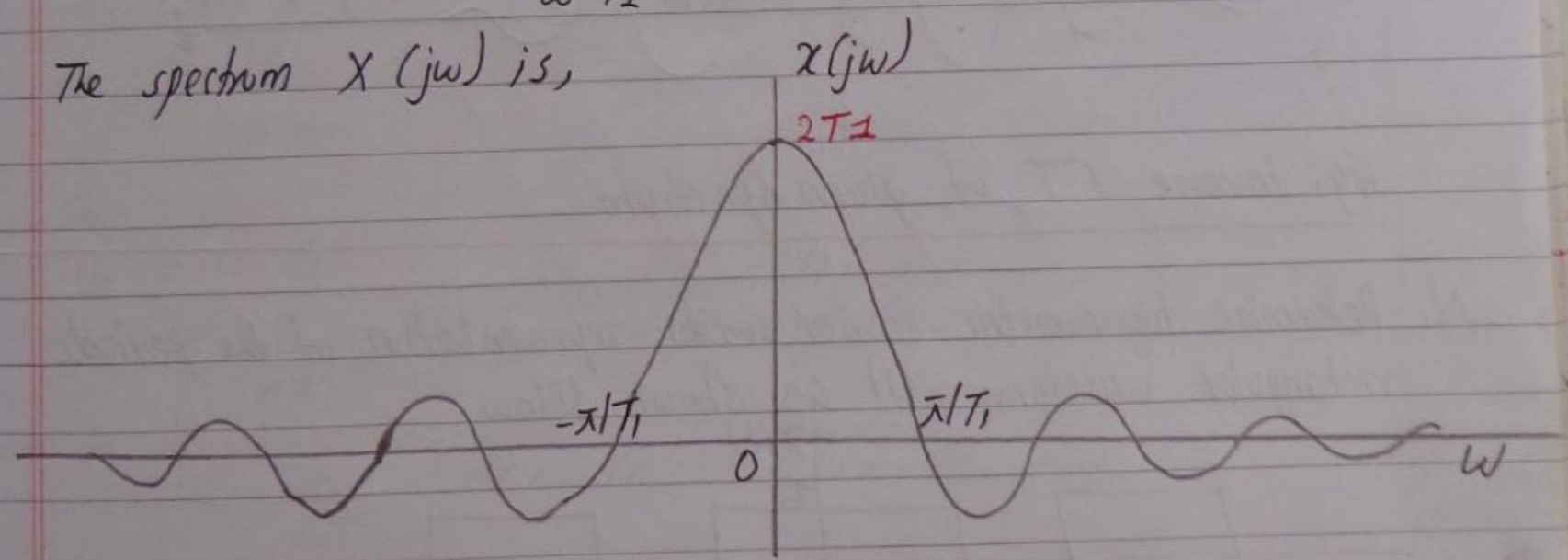


Fig: F.T of rectangular pulse ⟹ sinc function.

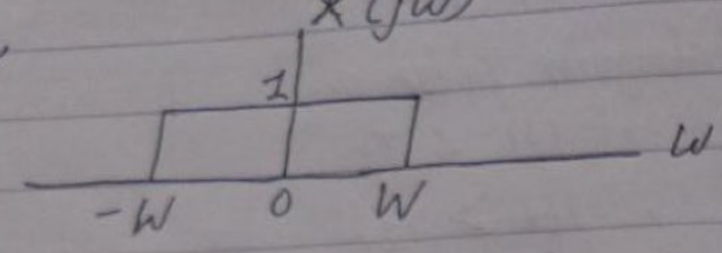
Scanned with CamScanner

M. Determine the inverse Fourier transform of:

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

Sol^n,

$\Rightarrow$ Given spectrum,



inverse fourier transform of given spectrum is,

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\,\frac{1}{jt}\left[e^{j\omega t}\right]_{-W}^{W}$$

$$= \frac{1}{\pi t}\left[\frac{e^{jWt} - e^{-jWt}}{2j}\right]$$

$$x(t) = \frac{W}{\pi}\,\frac{\sin(Wt)}{Wt}$$

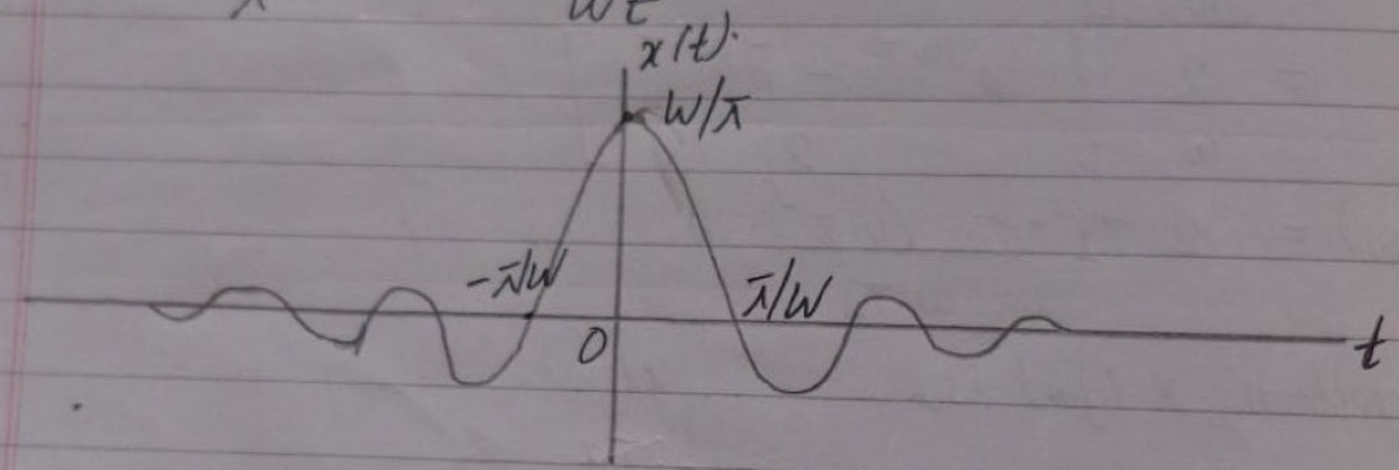
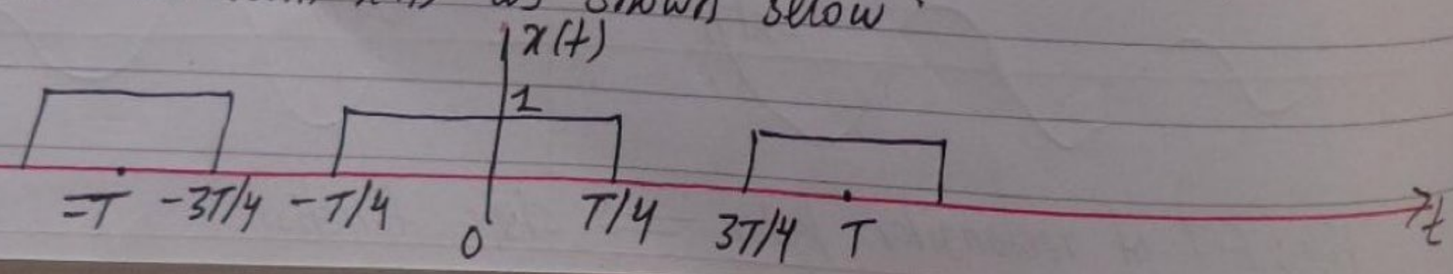


Fig: inverse F.T of given spectrum.

N. Determine trigonometric fourier series representation of the periodic rectangular waveform x(t) as shown below.



Scanned with CamScanner

Sol^n,

Here, $x(t) = \begin{cases} 0 & \text{for } -T/2 < t < -T/4 \\ 1 & -T/4 < t < T/4 \\ 0 & T/4 < t < T/2 \end{cases}$

Average value of signal $x(t)$ is,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} (0)\, dt + \int_{-T/4}^{T/4} (1)\, dt + 0\, dt\right]$$

$$= \frac{1}{T}\left[t\right]_{-T/4}^{T/4}$$

$$= \frac{T}{2}$$

Now, $a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t) \cos n\omega t\, dt$

$$= \frac{2}{T}\left[\int_{-T/2}^{-T/4} x(t)\cos n\omega t\, dt + \int_{-T/4}^{T/4} x(t)\cos n\omega t\, dt + \int_{T/4}^{T/2} x(t)\cos n\omega t\, dt\right]$$

$$= \frac{2}{T}\left[\frac{\sin \omega t}{n\omega}\right]_{-T/4}^{T/4}$$

$$= \frac{2}{n\pi T}\left[\sin\frac{n\omega T}{4} - \sin\frac{(-n\omega T)}{4}\right]$$

$$= \frac{4}{n\pi T}\sin\frac{n\omega T}{4}$$

So,

$a_n = \begin{cases} 0 & \text{for } n = 2, 4, 6 \ldots \\ \frac{2}{n\pi} & \text{for } n = 1, 5, 9 \ldots \\ \frac{-2}{n\pi} & \text{for, } n = 3, 7, 11, \ldots \end{cases}$

Scanned with CamScanner

Since, given signal is even signal, $b_n = 0$,

Hence,

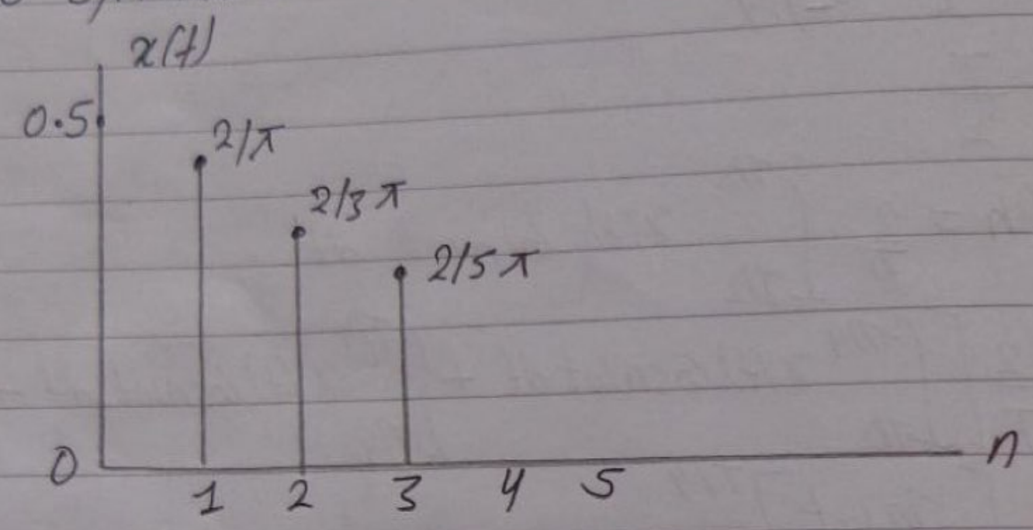
$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

$$= \frac{1}{2} + a_1 \cos\omega t + a_2 \cos\omega 2t + a_3 \cos\omega 3t + \text{-----}$$

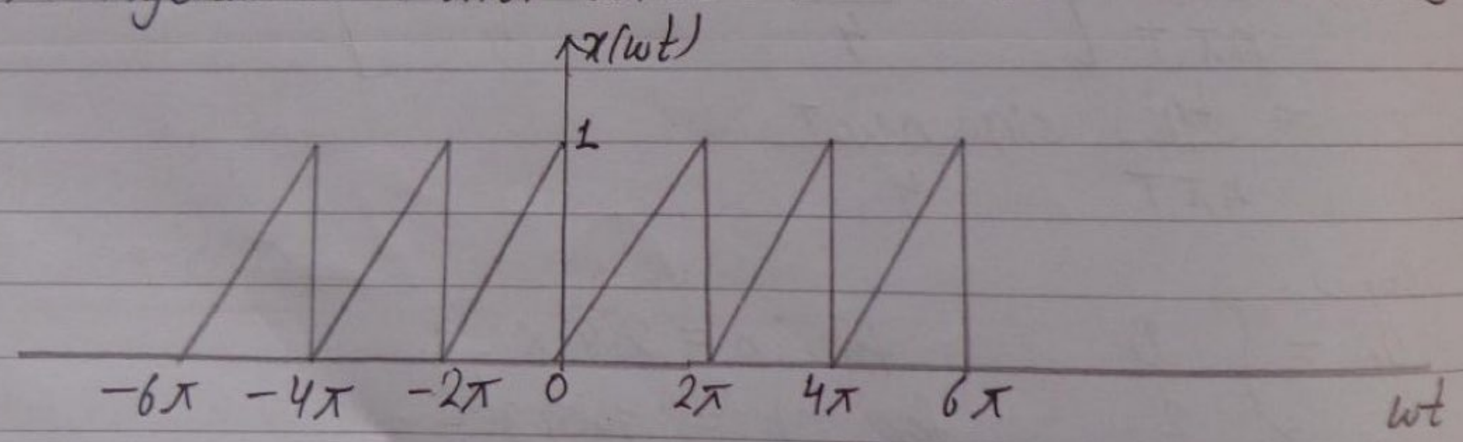
Hence,

$$x(t) = 0.5 + \frac{2}{\pi}\left[\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos\omega t + \text{----}\right]$$

Amplitude spectrum:



Q Find trigonometric Fourier series for the saw tooth forms as



sol^n,

By using two point formula between 0 & 2π,

$$x(\omega t) = \frac{\omega t}{2\pi} \quad \text{for } 0 < t < 2\pi$$

Scanned with CamScanner

The trigonometric Fourier series representation of signal $x(t)$ is,

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \quad \text{——} \quad (1)$$

The average value of signal $x(t)$ is,

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} x(\omega t)\, d(\omega t)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\omega t}{2\pi}\right) d(\omega t)$$

$$= \frac{1}{2\pi} \left[\frac{(\omega t)^2}{2}\right]_{\omega t=0}^{2\pi}$$

$$= \frac{1}{2\pi} \left[\frac{(2\pi)^2}{2}\right]$$

$$= \frac{1}{2}$$

$$a_n = \frac{2}{2\pi} \int_0^{2\pi} x(\omega t) \cos n\omega t\, d(\omega t) = \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\omega t}{2\pi}\right) \cos n\omega t\, d(\omega t)$$

$$= \frac{1}{2\pi^2} \left[\frac{\omega t}{n} \sin \omega t + \frac{\cos n\omega t}{n^2}\right]_0^{2\pi} = 0$$

and,

$$b_n = \frac{2}{2\pi} \left[\int_0^{2\pi} x(\omega t) \sin n\omega t\, d(\omega t)\right]$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\omega t}{2\pi}\right) \sin n\omega t\, d(\omega t)$$

$$= \frac{1}{2\pi^2} \left[\frac{-\omega t}{n} \cos n\omega t + \frac{\sin n\omega t}{n^2}\right]_0^{2\pi}$$

$$= \frac{-1}{n\pi}$$

Scanned with CamScanner

Hence, Replacing value of a_0, a_n, b_n,

$$\therefore x(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin n\omega t}{n\pi}$$

This is required Trigonometric Fourier series representation of a saw tooth wave.

For amplitude spectrum,

$$C_0 = a_0 = \frac{1}{2}$$

$$C_n = \sqrt{(a_n)^2 + (b_n)^2}$$
$$= \sqrt{0^2 + \left(-\frac{1}{n\pi}\right)^2} = \frac{1}{n\pi}$$

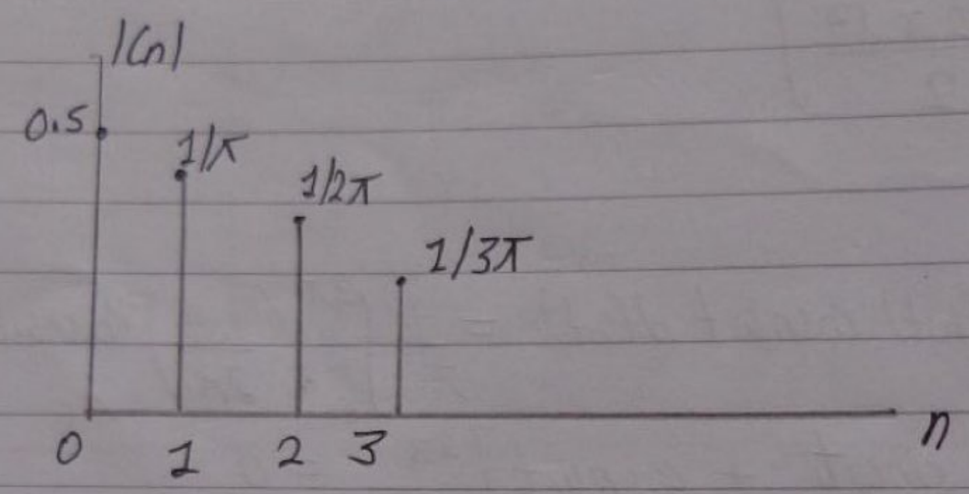
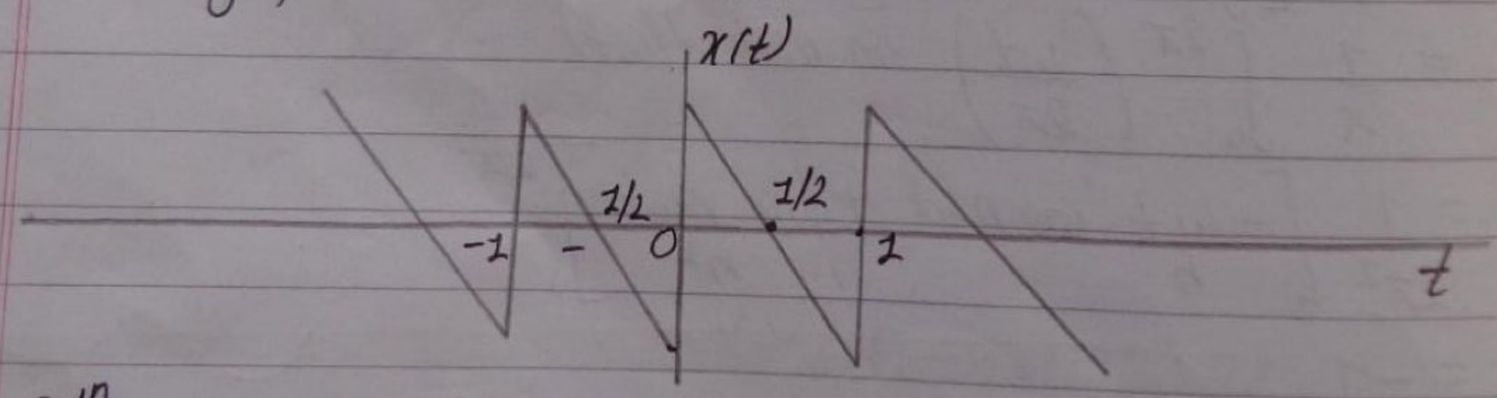


Fig: amplitude spectrum of saw-tooth wave.

P. Determine the trigonometric Fourier series representation of the following periodic sawtooth wave.



Sol^n,

Scanned with CamScanner

we have,

$$x(t) = b_0 + \sum_{k=1}^{\infty} [b_k \cos(k\omega_0 t) + c_k \sin(k\omega_0 t)]$$

Fundamental period $= T = 1$ sec

" frequency, $\omega_0 = 2\pi$ rad/sec

since, given signal $x(t)$ is odd; Hence, $b_0 = 0$, $b_k = 0$, and,

$$c_k = \frac{4}{T}\int_0^{1/2} x(t) \sin(k\omega_0 t)\, dt$$

$$= 4\int_0^{1/2} x(t) \sin(k\omega_0 t)\, dt.$$

For $0 < t < 1/2$, using two point formula,

$$x(t) - 1 = \frac{0-1}{1/2 - 0}(t-0)$$

$$\therefore x(t) = 1 - 2t$$

Now,

$$c_k = 4\int_0^{1/2} (1-2t) \sin(k\omega_0 t)\, dt$$

$$= 4\int_0^{1/2} \sin(k\omega_0 t)\, dt - 8\int_0^{1/2} t \sin(k\omega_0 t)\, dt$$

$$= 4\int_0^{1/2} \sin(k2\pi t)\, dt - 8\int_0^{1/2} t \sin(k2\pi t)\, dt$$

$$= \frac{-4}{k2\pi}\cos(2\pi t)\Big|_0^{1/2} - 8\left(t\int \sin(k2\pi t)\, dt - \int\left(\frac{dt}{dt}\int \sin(k2\pi t)\, dt\, dt\right)\right)$$

$$= \frac{-4}{k2\pi}(\cos(k\pi) - 1) - 8\left(\frac{-t}{k2\pi}\cos(k2\pi t)\Big|_0^{1/2} + \frac{1}{k2\pi}\int_0^{1/2} \cos(k2\pi t)\, dt\right)$$

$$= \frac{-4}{k2\pi}((-1)^k - 1) - 8\left(\frac{-1}{k4\pi}\cos(4\pi) + \frac{1}{(k2\pi)^2}\sin(k2\pi)\Big|_0^{1/2}\right)$$

$$= \frac{-2}{k\pi}(-1)^k + \frac{2}{k\pi} + \frac{2}{k\pi}(-1)^k$$

Scanned with CamScanner

$\therefore C_k = \frac{2}{k\pi}$

Hence,

$$\therefore \quad x(t) = 0 + \sum_{k=1}^{\infty} \left(\frac{2}{k\pi}\right) \sin(k\omega_0 t)$$

Q. Determine the Fourier transform of the gaussian pulse defined as,

$$x(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$ 2013/F

Sol^n,

$\Rightarrow$ we have, Fourier transform of $x(t)$ is,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \cdot e^{j\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(t^2/2 + j\omega t)}\, dt$$

let,

$$\frac{t^2}{2} + j\omega t = \left(\frac{t}{\sqrt{2}}\right)^2 + 2 \cdot \frac{t \cdot j\omega}{\sqrt{2} \cdot \sqrt{2}} - \left(\frac{j\omega}{\sqrt{2}}\right)^2$$

$$= \left(\frac{t}{\sqrt{2}} + \frac{j\omega}{\sqrt{2}}\right)^2 + \frac{\omega^2}{2}$$

Now,

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\left(\frac{t}{\sqrt{2}} + \frac{j\omega}{\sqrt{2}}\right) + \frac{\omega^2}{2}\right)}\, dt$$

Scanned with CamScanner

$$= \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2} \int_{-\infty}^{\infty} e^{-\left(\frac{t}{\sqrt{2}} + \frac{j\omega}{\sqrt{2}}\right)^2} dt$$

let,

$$\frac{t}{\sqrt{2}} + \frac{j\omega}{\sqrt{2}} = u$$

then, $dt = \sqrt{2}\, du$

So,

$$X(j\omega) = \frac{1}{\sqrt{\pi}} e^{-\omega^2/2} \int_{-\infty}^{\infty} e^{-u^2} du$$

Since, e^{-u^2} is even function,

$$X(j\omega) = \frac{2}{\sqrt{\pi}} e^{-\omega^2/2} \int_{0}^{\infty} e^{-u^2} du$$

we know,

$$\int_{+0}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

Hence,

$$X(j\omega) = \frac{2}{\sqrt{\pi}} e^{-\frac{\omega^2}{2}} \cdot \frac{\sqrt{\pi}}{2} = e^{-\omega^2/2}$$

This is required Fourier transform of given gaussian pulse.

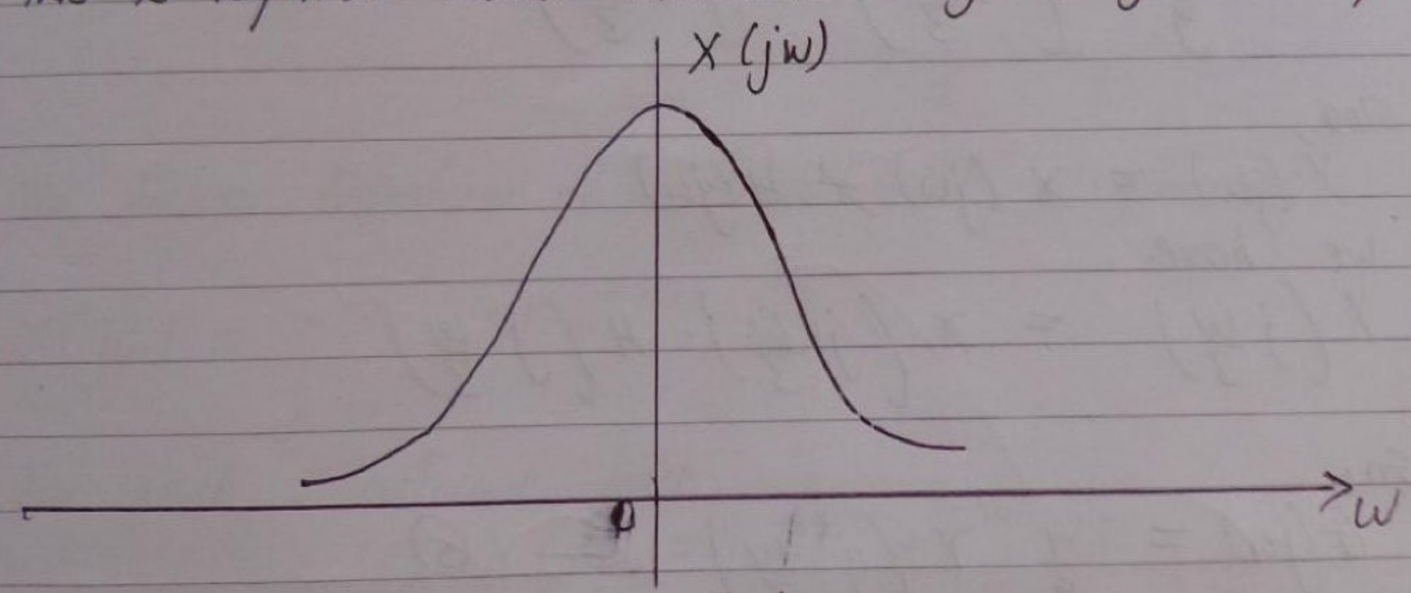


Fig: Gaussian pulse plot.

Scanned with CamScanner

R. Given the relationship $y(t) = x(t) * h(t)$ and $g(t) = x(3t) * h(3t)$ and given that $x(t)$ has fourier transform of $X(j\omega)$ and $h(t)$ has fourier transform $H(j\omega)$. use fourier transform properties to show that $g(t)$ has form $g(t) = A\,y\,B(t)$. Determine values of A and B.

2013/F

Sol^n,

$\Rightarrow$ Given,

$$g(t) = x(3t) * h(3t) \quad \text{— (i)}$$

using time-scaling property of Fourier transform,

$$x(3t) \xleftrightarrow{F.T} \frac{1}{3} X\left(j\frac{\omega}{3}\right) \quad \text{— (ii)}$$

and

$$h(3t) \xleftrightarrow{F.T} \frac{1}{3} H\left(j\frac{\omega}{3}\right) \quad \text{— (iii)}$$

using convolution property of fourier transform for eq^n (i),

$$G(j\omega) = \frac{1}{3} X\left(j\frac{\omega}{3}\right) * \frac{1}{3} H\left(j\frac{\omega}{3}\right)$$

$$= \frac{1}{9} X\left(j\frac{\omega}{3}\right) \cdot H\left(j\frac{\omega}{3}\right) \quad \text{— (4)}$$

since,

$$Y(j\omega) = X(j\omega) * H(j\omega)$$

we have.

$$Y\left(j\frac{\omega}{3}\right) = X\left(j\frac{\omega}{3}\right) \cdot H\left(j\frac{\omega}{3}\right)$$

So,

$$G(j\omega) = \frac{1}{9} Y\left(j\frac{\omega}{3}\right) \quad \text{— (5)}$$

Scanned with CamScanner

given that,

$$g(t) = A\,y(Bt) \quad — ⑥$$

Taking fourier transform of eqn ⑥ on both sides,

$$G(j\omega) = \frac{A}{B}\, Y\left(\frac{j\omega}{B}\right) \quad — ⑦$$

Comparing eqn ⑤ & ⑦,
we get,

$$B = 3$$

and,

$$\frac{A}{B} = \frac{1}{9}$$

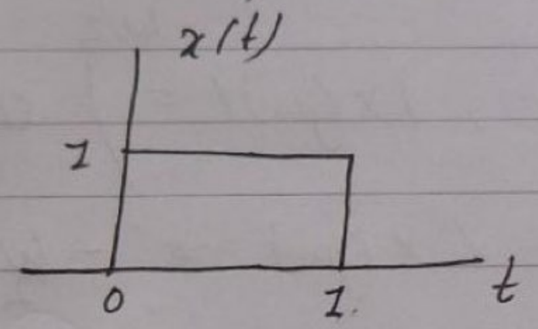
$$\therefore \quad A = \frac{1}{3}$$

$$\therefore \quad B = 3.$$

5. Find the Fourier transform of the spectrum of $x(t) = u(t) - u(t-1)$

Sol^n,

⇒ Given signal is,



The fourier transform of a signal $x(t)$ is given as,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

For given signal, we have,

$$X(j\omega) = \int_{0}^{1} e^{-j\omega t}\, dt$$

Scanned with CamScanner

$$= \left| \frac{1}{-jw} e^{-jwt} \right|_0^2$$

$$= \frac{1}{-jw} (e^{-w} - 1)$$

multiplying numerator & denominator by $e^{jw/2}$ we get,

$$X(jw) = \frac{1}{-jw} \frac{1}{e^{jw/2}} \cdot (e^{-jw/2} - e^{-jw/2})$$

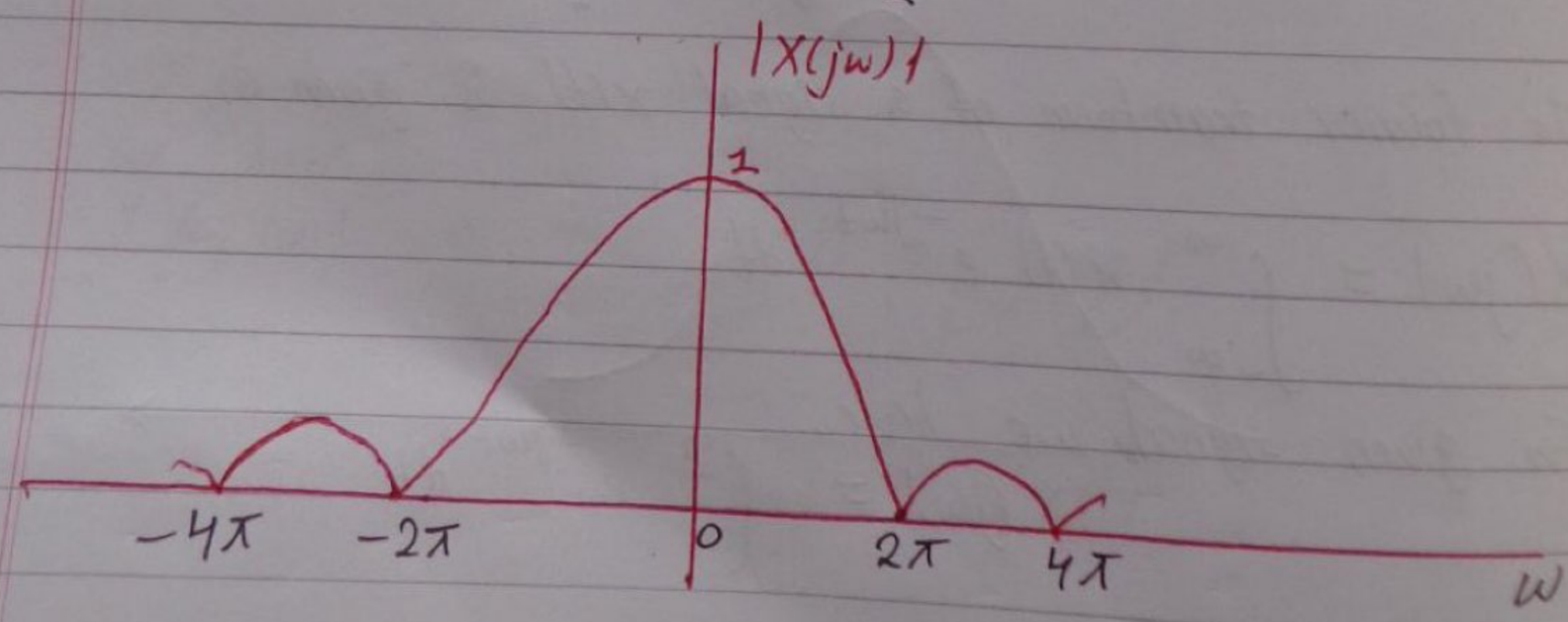
$$= \frac{2e^{-jw/2}}{w} \cdot \frac{(e^{jw/2} - e^{-jw/2})}{2j}$$

$$= e^{-jw/2} \cdot \frac{\sin(w/2)}{w/2}$$

$$\therefore X(jw) = e^{-jw/2} \cdot \frac{\sin(w/2)}{w/2}$$

magnitude, $|X(jw)| = \left| \frac{\sin(w/2)}{w/2} \right|$

phase : $\angle X(jw) = -\frac{w}{2}$



Scanned with CamScanner

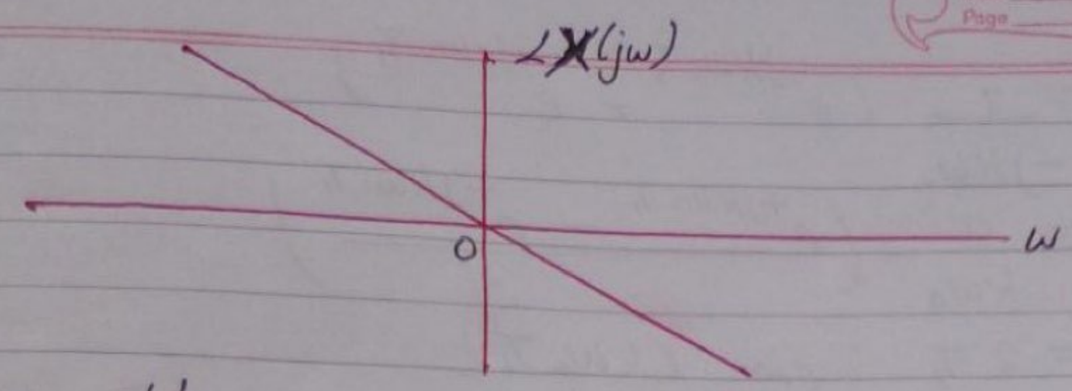
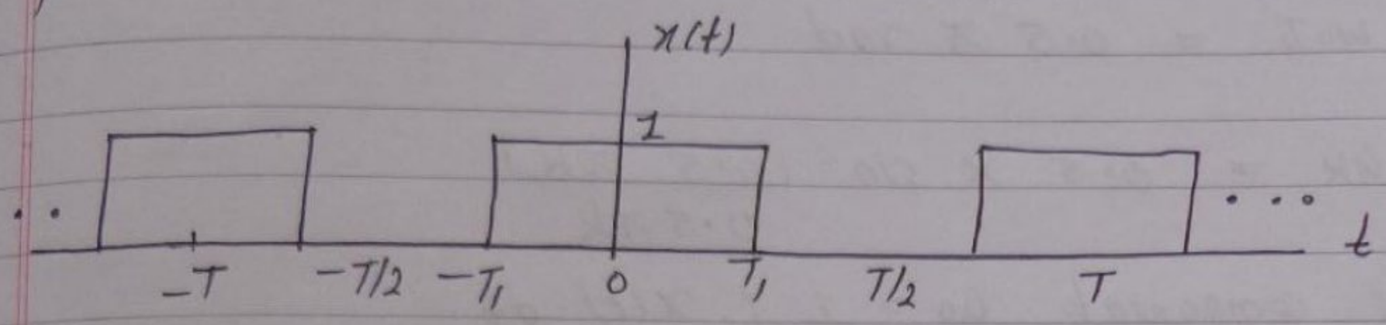


Fig : phase plot.

t Determine the Fourier series coeff of the following periodic rectangular pulse for $T = 4T_1$ where $T_1 = 0.25$ sec.



Sol^n,

$\Rightarrow$ Fundamental period, $T = 4T_1$

$= 4 \times 0.25$

$= 1$ sec

Fundamental frequency, $\omega_0 = \frac{2\pi}{T}$

$= \frac{2\pi}{1} = 2\pi$ rad/sec

We know,

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \int_{-T_1}^{T_1} e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{-jk\omega_0} e^{-jk\omega_0 t} \Big|_{-T_1}^{T_1}$$

Scanned with CamScanner

$$= \frac{1}{-jk\omega_0}\left(e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}\right)$$

$$= \frac{2}{k\omega_0}\left(\frac{e^{+jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right)$$

$$= 2T_1 \sin\frac{(k\omega_0 T_1)}{k\omega_0 T_1}$$

since, $\omega_0 = 2\pi$ rad/sec

$T_1 = 0.25$ sec

then,

$\omega_0 T_1 = 0.5\pi$ rad

$$\therefore a_k = 0.5 \times \sin\frac{(0.5\pi k)}{0.5\pi k}$$

DC component, $a_0 = \frac{1}{T}\int_T x(t)\,dt$

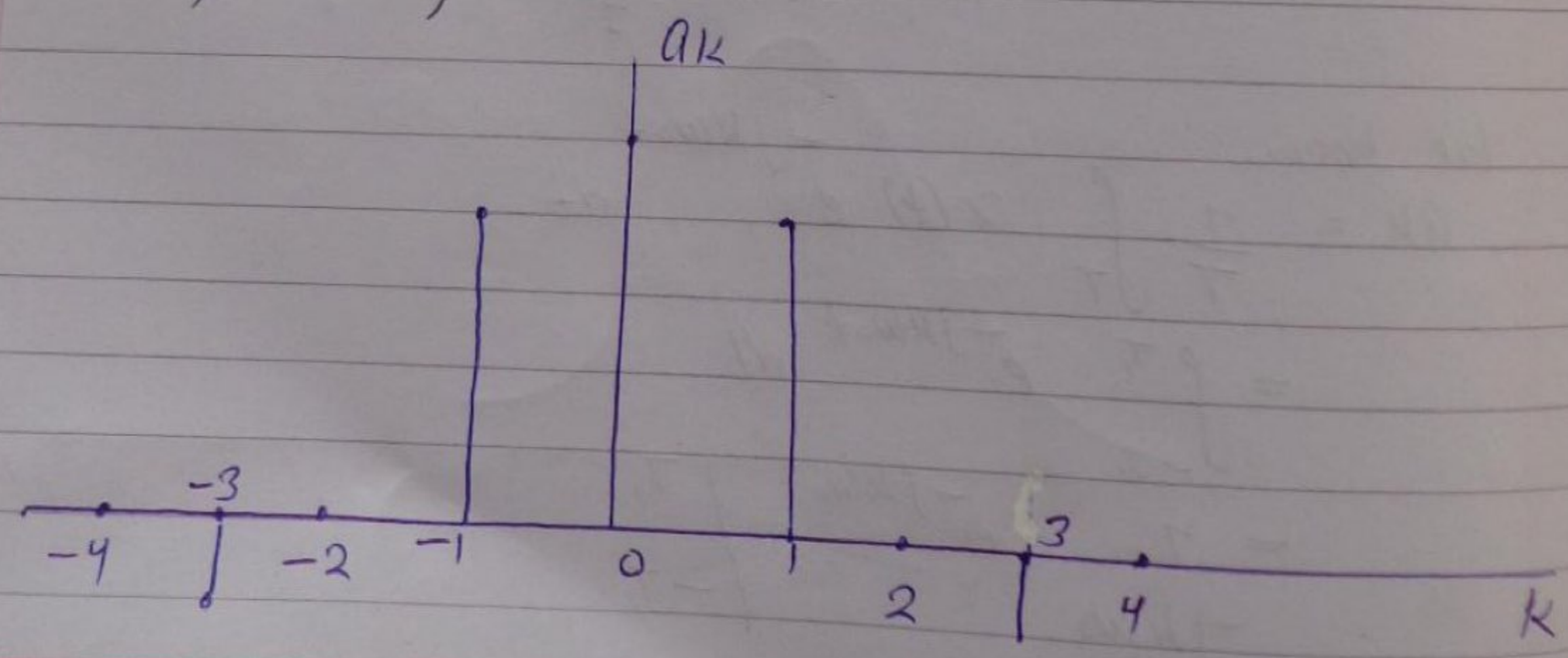
$$= \int_{-T_1}^{T_1} 1.dt$$

$$= 2T_1$$

$$= 2 \times 0.25$$

$$\therefore a_0 = 0.5$$

The spectrum plot is



Scanned with CamScanner

Energy and Power

chapter = 5

I.A. Define autocorrelation. state and prove properties of auto correlation function for power signal. 2015/S, 2015/F, 2012/F, 2017/S

⇒ Auto-correlation is defined as the measure of similarity between a signal $x(t)$ and its delayed version $x(t-\tau)$.

Properties of Autocorrelation Function for power signals:

① Auto correlation function exhibits conjugate symmetry.

i.e, $R(\tau) = R^*(-\tau)$

proof:

The auto correlation function for power signal $x(t)$ is,

$$R(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot x^*(t-\tau)\, dt$$

Taking Complex Conjugate,

$$R^*(\tau) = \frac{1}{T}\int_{-\infty}^{\infty} x^*(t)\, x(t-\tau)\, dt$$

So,

$$R^*(-\tau) = \frac{1}{T}\int_{-\infty}^{\infty} x^*(t) \cdot x(t+\tau)\, dt = R(\tau)$$

Hence, $R(\tau) = R^*(-\tau)$

② The value of auto correlation function at $\tau = 0$ i.e, at origin, is equal to average power of signal for power signal.

i.e, $R(0) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t)\, dt = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt = P$

proof:

$$R(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt$$

So, $R(0) = \frac{1}{T}\int_{-\infty}^{\infty} x(t)\, x^*(t-0)\, dt$

Scanned with CamScanner

$$= \frac{1}{T}\int_{-\infty}^{\infty} |x(t)|^2 \, dt = p.$$

(iii) Autocorrelation function has maximum value at origin
ie, $R(\tau) \leq R(0)$ for all τ.

(iv) For power signal, the autocorrelation function and power spectrum density forms a fourier transform pair.
ie, $R(\tau) \longleftrightarrow S(\omega)$
This is known as Wiener-Khintchine theorem.
Proof:
The power spectral density of an energy signal $g(t)$ is,

$$S(f) = \lim_{T\to\infty} \frac{|G(f)|^2}{T} = \lim_{T\to\infty} \frac{G(f)\cdot G^*(f)}{T}$$

where, $G(f)$ is the fourier transform of signal $g(t)$
For a signal $g(t)$, we have,

$$g(t) \otimes g(-t) \longleftrightarrow G(f)\cdot G^*(f)$$

Now, multiplying both sides by $\frac{1}{T}$ and taking limit as $T\to\infty$,

$$\lim_{T\to\infty}\left[\frac{1}{T} g(t) \otimes g(-t)\right] \xleftrightarrow{F.T} \lim_{T\to\infty} \frac{1}{T} G(f)\cdot G^*(f)$$

Using convolution integral,

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} g(t)\, g(t-\tau)\, d\tau \xleftrightarrow{F.T} \lim_{T\to\infty} \frac{1}{T} G(f)\cdot G^*(f)$$

$$\therefore R(t) \xleftrightarrow{F.T} S(f).$$

2009F, [redacted]/F

B. State and prove parseval's theorem for finite energy signal. 2014/F, 2016/S

⇒ Parseval's Theorem for energy signal states that the energy of continuous

Scanned with CamScanner

time signal $x(t)$ can be obtained with the help of its Fourier transform as, $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$

proof:

let us consider a function $x(t)$, then $|x(t)|^2 = x(t) \cdot x^*(t)$ where, $x^*(t)$ is a complex conjugate of $x(t)$, Now energy of a signal $x(t)$ is,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot x^*(t) \, dt$$

we know, $x(t)$ is given by inverse Fourier transform of $X(j\omega)$ as:

$$x(t) = F^{-1}[X(j\omega)]$$

$$\therefore x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t} \, d\omega$$

So,

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\left[\int_{-\infty}^{\infty} x^*(t) \cdot e^{j\omega t} \, dt\right] d\omega$$

since, $X(j\omega) = \int_{-\infty}^{\infty} x(t) \, e^{-j\omega t} \, dt$, then,

$$X(-j\omega) = \int_{-\infty}^{\infty} x^*(t) \, e^{j\omega t} \, dt$$

Hence, $E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \cdot X(-j\omega) \, d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \cdot X^*(j\omega) \, d\omega$$

$$\therefore E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

where, $|X(j\omega)|^2$ represents distribution of energy & is known as

Scanned with CamScanner

Energy density spectrum.

c. Define cross correlation. state and prove the properties of auto-correlation function for energy signals. 2011/F, 2013/F,

=> cross correlation is defined as the measure of similarity between a signal and the delayed version of another signal.

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t) \cdot x_2^*(t-\tau)\, dt$$

and cross-correlation between two power signals $x_1(t)$ & $x_2(t)$ is,

$$R_{12} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x_1(t) \cdot x_2^*(t-\tau)\, dt.$$

where x_1 and $x_2(t)$ are two different signals.

Properties of ACF for energy signals:

① Auto correlation function exhibits conjugate symmetry, that is,

$$R(\tau) = R^*(-\tau)$$

proof:

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t-\tau)\, dt$$

Taking complex conjugate on both sides,

$$R^*(\tau) = \int_{-\infty}^{\infty} x^*(t) \cdot x(t-\tau)\, dt$$

let, $\tau = -\tau$, then,

$$R^*(-\tau) = \int_{-\infty}^{\infty} x^*(t) \cdot x(t+\tau)\, dt$$

$$= \int_{-\infty}^{\infty} x(t+\tau) \cdot x^*(t)\, dt$$

$$\therefore R^*(-\tau) = R(\tau)$$

Scanned with CamScanner

(ii) value of ACF for $\tau = 0$ (at origin) is equal to the total energy of the signal. i.e,

$$R(0) = \int_{-\infty}^{\infty} |x(t)|^2 dt = E$$

Proof:

For $\tau = 0$,

$$R(0) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= E$$

(iii) The AC function has maximum value at origin, i.e,

$R(\tau) \leq R(0)$ for all τ

(iv) Autocorrelation function and Energy spectrum density form a fourier transform pair. i.e,

$$R(\tau) \xleftrightarrow{F.T} \psi(f)$$

proof:

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t-\tau)\, dt$$

we have, $x^*(t-\tau) = \int_{-\infty}^{\infty} X^*(f) \cdot e^{-j2\pi f(t-\tau)} df$

So,

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} X^*(f) \cdot e^{-j2\pi f(t-\tau)} df \right] dt$$

changing order of integration,

$$R(\tau) = \int_{-\infty}^{\infty} X^*(f)\, e^{2\pi j f \tau} \left[\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} dt \right] df$$

$$= \int_{-\infty}^{\infty} X^*(f) \cdot X(f)\, e^{2j\pi f \tau} df$$

Scanned with CamScanner

$$= \int_{-\infty}^{\infty} |X(f)|^2 e^{2\pi j f \tau} df$$

$$\therefore R(\tau) = \int_{-\infty}^{\infty} \psi(f) e^{j2\pi f \tau} df$$

Hence,

$$R(\tau) \xleftrightarrow{F.T} \psi(f).$$

D. Compare Energy and Power Density spectrum.

Energy density spectrum	Power density spectrum
(i) It gives the distribution of energy of a signal in frequency domain.	It gives the distribution of power of a signal in frequency domain.
(ii) It is denoted by $\psi(\omega)$ & given by, $\psi(\omega) = \lvert x(j\omega)\rvert^2$	It is denoted by $s(\omega)$ & given by, $s(\omega) = \lim_{\tau \to \infty} \frac{\lvert X_\tau(j\omega)\rvert^2}{\tau}$
(iii) Autocorrelation function for energy signal and energy density spectrum form a fourier transform pair.	ACF for power signal & power density spectrum form a fourier transform pair.
(iv) Total energy of a signal is given by the area under curve $\psi(\omega)$ & is $E = \int_{-\infty}^{\infty} (\omega)\, df$.	Total power of signal is given by area under the curve $s(\omega)$ and is, $P = \int_{-\infty}^{\infty} s(f)\, df$.
v) ESD of a energy signal $x(t)$ is a non-negative real-valued function of frequency. i.e, $\psi(f) \geq 0$.	PSD of a power signal $x(t)$ is a non-negative real valued function of frequency. i.e, $s(f) \geq 0$

Scanned with CamScanner

vi ESD of a real valued signal $x(t)$ is an even function of frequency. i.e, $\psi(f) = \psi(-f)$.	PSD of a real-valued power signal $x(t)$ is an even function of frequency. $S(f) = S(-f)$
vii. For a energy signal, $x(t)$, the auto correlation function and ESD function form a fourier transform pair. $R(\tau) \xleftrightarrow{F.T} \psi(f)$	The ACF and PSD function form a fourier transform pair. i.e, $R(\tau) \xleftrightarrow{F.T} S(f)$
VIII. $P = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{\infty} \lvert X_T(f)\rvert^2 df$ $= \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{\lvert X_T(f)\rvert^2}{T} df$ $= \int_{-\infty}^{\infty} S(f)\, df$ where, $S(f) = \lim_{T\to\infty} \frac{\lvert X_T(f)\rvert}{T}$ is called power spectral density (PSD) or power density spectrum of the power signal $x(t)$.	$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 d\omega$ $= \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$ $= \int_{-\infty}^{\infty} \psi(f)\, df$ where, $\psi(f) = \lvert X(f)\rvert^2$ is called energy density (ESD) of $x(t)$.

Scanned with CamScanner

Transmission of signals.

chapter = 6

1.A. Distortionless Transmission: 2015/F, 2013/F, 2014/F, 2016/F, 2016/S

Any undesired change in the waveshape of an electric signal passing through a physical system is known as distortion. The transmission of a signal through an LTI system is said to be distortionless if the output signal is an exact replica of the input signal. However a constant change in magnitude and a constant change in time delay in the output replica is not treated as distortion.

So the signal $x(t)$ is said to be transmitted without distortion if the output signal becomes,

$$y(t) = Kx(t - t_d) \quad \text{— (1)}$$

Taking fourier transform on both sides,

$$Y(j\omega) = K\,X(j\omega)\,e^{-j\omega t_d}$$

we know,

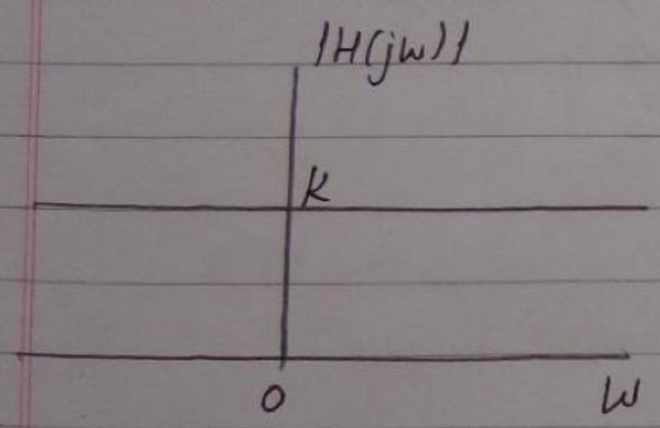
$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = Ke^{-j\omega t_d}$$

magnitude response is $|H(j\omega)| = K$

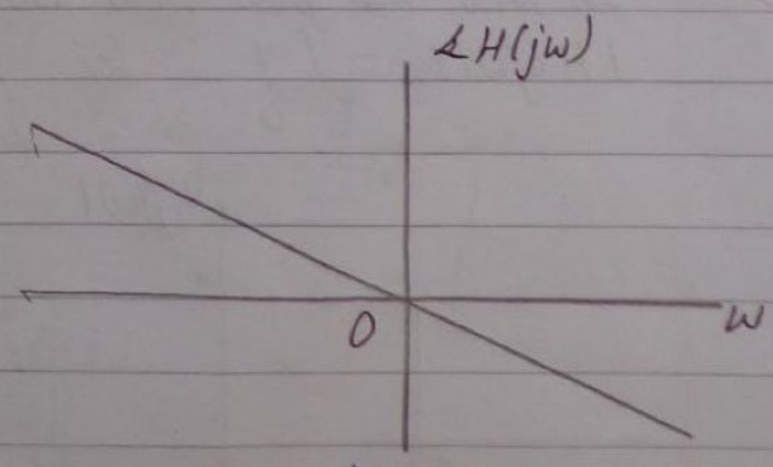
phase " is $\angle H(j\omega) = -\omega t_d$

Hence, for Distortionless transmission, Conditions needed are:

(1) magnitude response $|H(j\omega)|$ must be constant. i.e, all the frequency components must be equally attenuated or amplified and the system bandwidth must be infinite.



magnitude Plot



Phase response

Scanned with CamScanner

(ii) Phase response $\angle H(j\omega)$ must be directly proportional to frequency. Thus for distortionless transmission, the system Bandwidth must be infinite. But no practical system has infinite Bandwidth & therefore distortionless transmission cannot be achieved practically.

Following distortions occur during transmission of signals:

(i) phase distortion
(ii) amplitude "

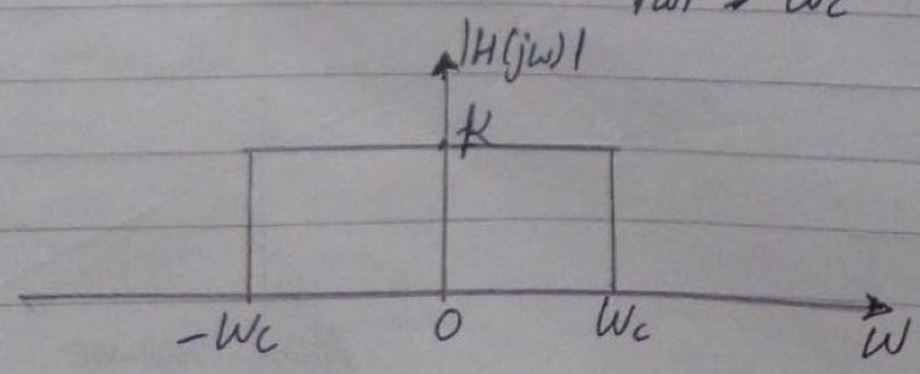
Amplitude distortion occurs when the magnitude response is not constant within the frequency band of interest and frequency components present in the input signal are transmitted with different gain or attenuation.

phase distortion occurs when the phase response is not linearly changing with frequency and the different frequency components in the input signal are subjected to different time delays during transmission.

B.) Define ideal low pass filter. prove that impulse response of an ideal lowpass filter is not causal. 2015/S, 2015/F, 2013/F, 2012/F, 2011/F 2010/F, 2016/F, 2014/S, 2017/S

⇒ An ideal low pass filter (LPF) is specified by,

$$|H(j\omega)| = \begin{cases} K & \text{for } |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$



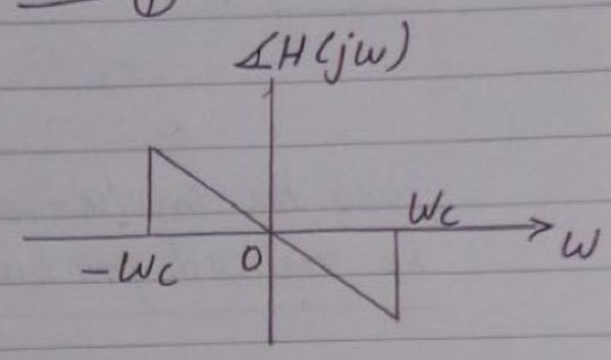
Scanned with CamScanner

Hence, an ideal LPF passes the frequency band from 0 to w_c without any kind of distortion and completely attenuates the frequencies above w_c, where the frequency w_c is called cutoff frequency.

Impulse response to Ideal LPF :

An ideal LPF is one that has transfer function as

$$H(jw) = \begin{cases} k e^{-jwt_d} & , |w| < w_c \\ 0 & , |w| > w_c \end{cases} \quad \text{— (1)}$$



Here, magnitude response is,

$$|H(jw)| = \begin{cases} k & \text{if } |w| < w_c \\ 0 & |w| > w_c \end{cases}$$

and phase response is,

$$\angle H(jw) = \begin{cases} -wt_d & , |w| < w_c \\ 0 & |w| > w_c \end{cases}$$

We know that the inverse fourier transform of frequency response is the impulse response of the LTI system.

Taking inverse fourier transform, we get,

$$h(t) = \frac{1}{2\pi} \int_{-w_c}^{w_c} k \cdot e^{-jwt_d} \cdot e^{jwt} \, dw$$

$$= \frac{k}{2\pi} \int_{-w_c}^{w_c} e^{jw(t-t_d)} \, dw$$

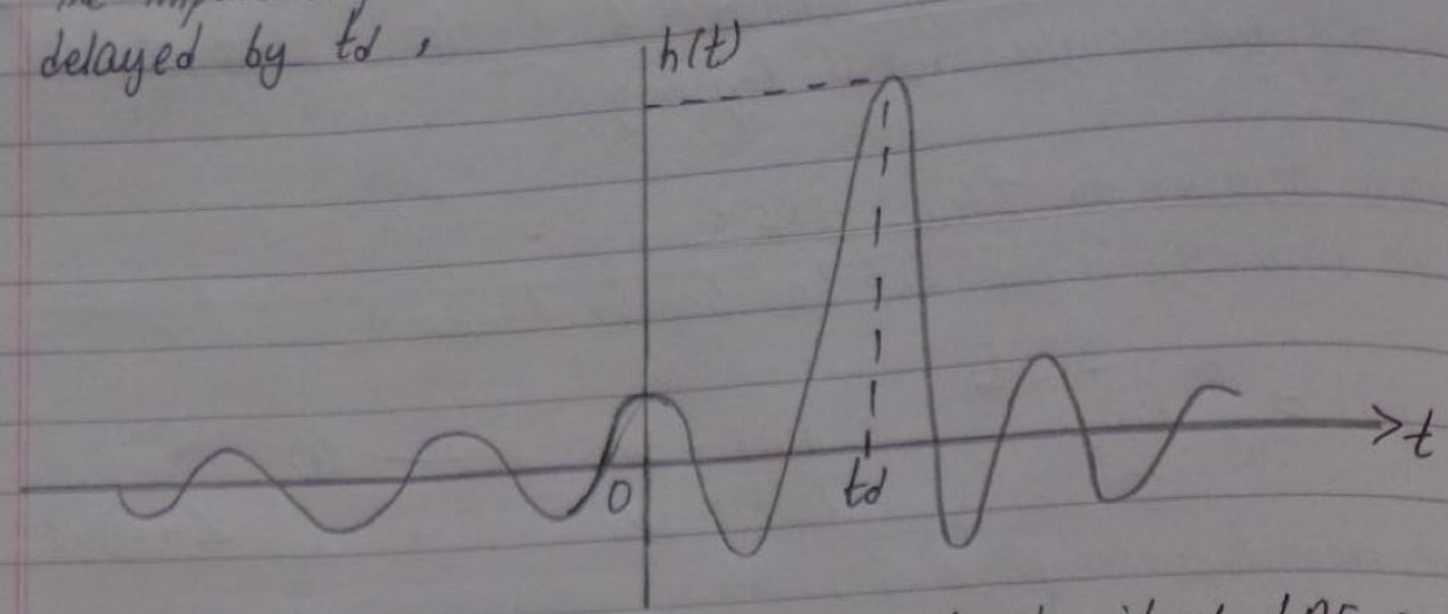
$$= \frac{k}{2\pi j (t-t_d)} \left[e^{jw(t-t_d)} \right]_{-w_c}^{w_c}$$

$$= \frac{k}{\pi (t-t_d)} \left[\frac{e^{jw_c(t-t_d)} - e^{-jw_c(t-t_d)}}{2j} \right]$$

$$\therefore h(t) = \frac{k w_c}{\pi} \cdot \frac{\sin(w_c(t-t_d))}{w_c(t-t_d)}$$

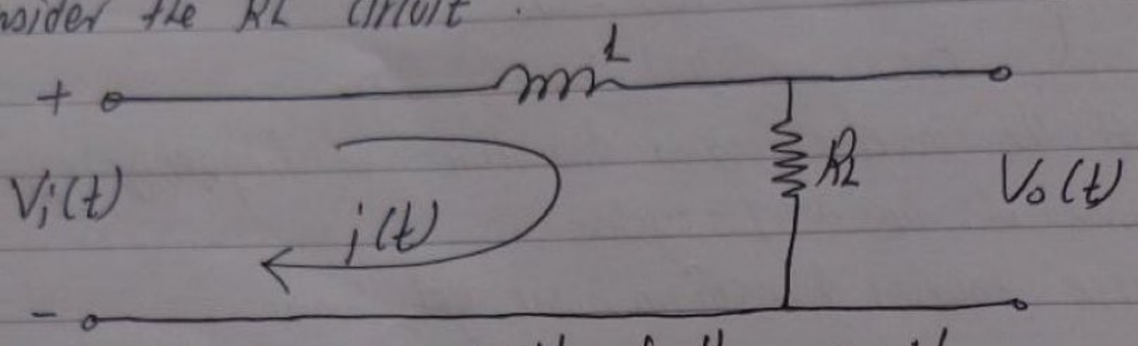
Scanned with CamScanner

The impulse response of an ideal LPF is the sinc function delayed by t_d,



Since the impulse response is not causal, the ideal LPF cannot be realized practically. ie, it is anti-causal.

C. Consider the RL circuit. 2012/F



(i) Find impulse response h(t) of the circuit
(ii) Find differential eqn relating Vo and Vi(t)

Soln:

⟹ Applying KVL,

$$V_i(t) = L\frac{di(t)}{dt} + V_o(t)$$

or $$V_i(t) = L\frac{d(V_o(t)/R)}{dt} + V_o(t)$$

or $$V_{in}(t) = \frac{L}{R}\frac{dV_o(t)}{dt} + V_o(t)$$

Scanned with CamScanner

or $V_o(t) + \frac{L}{R}\frac{dV_o(t)}{dt} = V_i(t)$ — (ii)

Taking Fourier transform on both sides of eqn (ii),

or $V_o(j\omega) + \frac{L}{R} j\omega V_o(j\omega) = V_{in}(j\omega)$

or $V_o(j\omega)\left(1 + j\omega\frac{L}{R}\right) = V_i(j\omega)$

Hence, frequency response of given RL ckt is,

$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1+j\omega\frac{L}{R}} = \frac{R/L}{j\omega + R/L} \quad \text{— (iii)}$$

since,

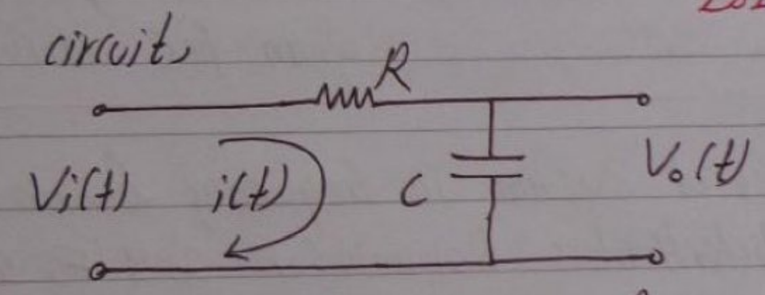
$$e^{-at}u(t) \xleftrightarrow{F.T} \frac{1}{j\omega + a}$$

& inverse fourier transform of eqn (iii) is,

$$\therefore \; h(t) = \frac{R}{L} e^{-\frac{R}{L}t} u(t)$$

2014/F

D. Consider an RC circuit,



Frequency response and impulse response Response of RC circuit

Sol^n,

$\Rightarrow$ Applying KVL,

$V_i(t) = i(t)R + V_o(t)$

or $V_i(t) = RC\frac{dV_o(t)}{dt} + V_o(t)$

Scanned with CamScanner

or $v_o(t) + RC \frac{d v_o(t)}{dt} = V_i(t)$ — (1)

Taking Fourier Transform on both sides,

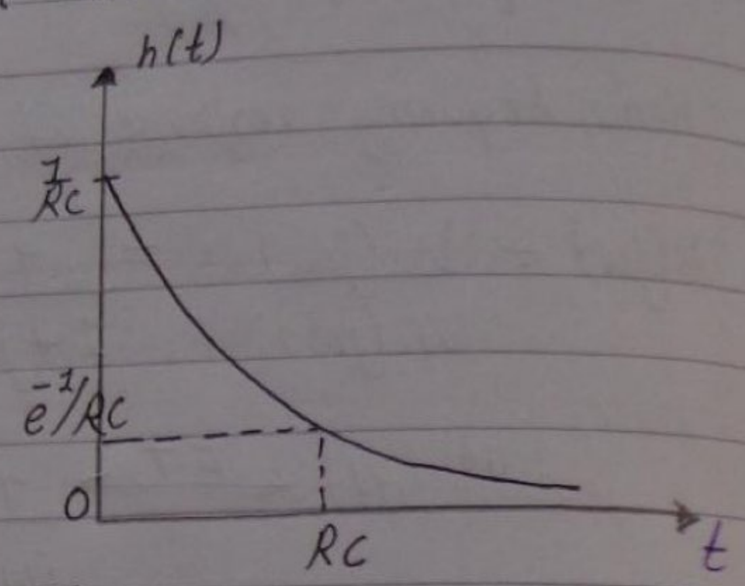
$V_o(j\omega) + RC\, j\omega\, V_o(j\omega) = V_i(j\omega)$

or $V_o(j\omega)(1 + j\omega RC) = V_i(j\omega)$

Hence,

Frequency response of given RC circuit is,

$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1 + j\omega RC} = \frac{1/RC}{j\omega + \frac{1}{RC}}$$



we have,

$$e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{a + j\omega}$$

Therefore, inverse fourier transform of $H(j\omega)$ is,

$\therefore h(t) = \frac{1}{RC} e^{-t/RC} u(t)$ which is impulse response of the given RC circuit.

E Explain response of system in frequency domain & Derive the condition for distortionless transmission. 2011/F, 2013/F, 2016/S

=> Let us consider an LTI system with impulse response $h(t)$. For any arbitrary input signal $x(t)$, the output signal $y(t)$ can be determined by performing convolution between input signal & impulse response as,

$$y(t) = x(t) * h(t) \quad \text{— (1)}$$

Scanned with CamScanner

let, $x(t) \xleftrightarrow{F.T} X(j\omega)$

$h(t) \longleftrightarrow H(j\omega)$

and,

$y(t) \xleftrightarrow{F.T} Y(j\omega)$

using convolution property of Fourier transform of eqn ①,

$$Y(j\omega) = X(j\omega) \cdot H(j\omega)$$

Now,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

is the transfer function or frequency response of the system. So, frequency response of a system is defined as the ratio of fourier transform of output signal to the Fourier transform of input signal,

let,

$$Y(j\omega) = |Y(j\omega)| e^{j\angle Y(j\omega)}$$

$$X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)}$$

and,

$$H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$$

then,

$$|H(j\omega)| e^{j\angle H(j\omega)} = \frac{|Y(j\omega)| e^{j\angle Y(j\omega)}}{|X(j\omega)| e^{j\angle X(j\omega)}} = \frac{|Y(j\omega)|}{|X(j\omega)|} e^{j(\angle Y(j\omega) - \angle X(j\omega))}$$

i.e,

$\therefore$ magnitude response of the system is given by,

$$|H(j\omega)| = \frac{|Y(j\omega)|}{|X(j\omega)|}$$

$\therefore$ Phase response is given by,

$$\angle H(j\omega) = \angle Y(j\omega) - \angle X(j\omega)$$

& For derivation for distortionless transmission :→ copy Q.N. 7.A (partially)

Scanned with CamScanner

Transmission of signals in Discrete-time systems:

chapter = 7

1. A) Transfer function or Frequency response of Discrete time system:

For a discrete time LTI system, with impulse response $h[n]$, output $y[n]$ for any arbitary input $x[n]$ can be determined by performing the convolution sum between $x[n]$ & $h[n]$ as

$$y[n] = x[n] * h[n] \quad \text{--- (1)}$$

let, $x[n] \xleftrightarrow{F.T} X(e^{j\omega})$

$h[n] \xleftrightarrow{F.T} H(e^{j\omega})$

and,

$y(n) \xleftrightarrow{F.T} Y(e^{j\omega})$

then applying convolution property of fourier transform,

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

Therefore,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

is the frequency response of a discrete time LTI system.

Also, applying Z transform on eqn (1),

$$Y(z) = X(z) \cdot H(z)$$

So,

$$H(z) = \frac{Y(z)}{X(z)}$$

is the frequency response of discrete time LTI system in term of Z-transform.

B) Types of Discrete time system or Digital Filters: 2011/F, 2013/F

(1) Finite-Duration Impulse Response (FIR) systems:

Scanned with CamScanner

If the impulse response is of finite duration, then the system is referred to as FIR system. For a discrete time LTI system, we have,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

For FIR system, the impulse response must be of finite length. Let M be the length of the impulse response. Then, causal FIR system is defined by,

$$y[n] = \sum_{k=0}^{M-1} h[k]\, x[n-k] \quad \text{--- (1)}$$

or $y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + \dots + h[M-1]x[n-M+1]$

This difference equation tells us that the output of a causal FIR system at any instant depends on the present and past inputs. Thus, a causal FIR system has a finite memory of length $M-1$, where M is length of impulse response.

Frequency response of FIR system:

From (1), Taking z transform on Both sides,

$$Y(z) = \sum_{k=0}^{M-1} h[k]\, X(z)\, z^{-k}$$

Hence,

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M-1} h[k]\, z^{-k}$$

is frequency response of the causal FIR system.

(ii) Infinite Duration impulse Response (IIR systems):

If the impulse response is of infinite duration, then the system is known as the IIR system. A causal IIR system can be expressed as,

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

Scanned with CamScanner

$= h[0]x[n] + h[1]x[n-1] + \cdots + h[\infty]x[n-\infty]$

Hence, the impulse response is of infinite duration & the IIR system requires infinite memory length.

Frequency response of IIR system:

$$H(z) = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}}$$

2013/F, 2010/S, 2010/F, 2016/F

B. Compare and contrast IIR & FIR. 2012/F, 2015/S, 2015/F, 2014/F,

	FIR system	IIR system.
(i)	Finite duration impulse response	Infinite duration impulse response.
(ii)	Has finite memory length.	Has infinite memory length.
(iii)	It can be realized with finite number of addition, multiplication & memory location.	It can't be realized with convolution sum because it requires infinite number of addition, multiplication & memory length. It is conveniently described by difference equation.
iv)	moving average system: $y[n] = \sum_{k=0}^{m} x[n-k]$ is an eg of FIR system	cumulative average given by, $y[n] = \frac{1}{n+1} \sum_{k=0}^{n} x[k]$ is an eg of IIR system.
v)	$h[n] = 0$ for ($n<0$ & $n \geq M-2$) then convolution output becomes, $y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$	$h[n]=0$ for $n<0$ so that convolution output becomes, $y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$.

Scanned with CamScanner

vi	It is non-recursive	It is recursive
vii	Filter produces zeros.	Filter produces poles
viii	FIR is always stable.	IIR can be unstable.
ix	FIR is dependent upon I/P only.	It is dependent on both I/P & O/P.
x	used for higher order tapping.	for lower-order tapping.
xi	FIR system are dependent upon linear phase characteristics.	IIR system are used for applications which are not linear.
xii	Requires more memory.	Requires less memory.
xiii	No limit cycles	Have limit cycles

C. Advantages, Disadvantages, uses of FIR system : 2012/F

- FIR system has amazing property called linear phase. Linear phase property implies that the phase is a linear function of frequency. It ensures that signals of all frequencies are delayed by the same amount of time, & thereby eliminating the possibility of phase distortion.

- FIR system are always stable. ie for finite input, output is always finite.

Demerits:

(i) A FIR system may have large number of coefficients.
(ii) FIRs can be limited in resolution at low frequencies, usually more complicated and time consuming than IIR system.

Uses:

(i) It is used in Digital communication in the IF stages of the receiver.
(ii)

Scanned with CamScanner

Implementation of IIR system :

(i) Direct Form - I structure

(ii) Direct Form - II "

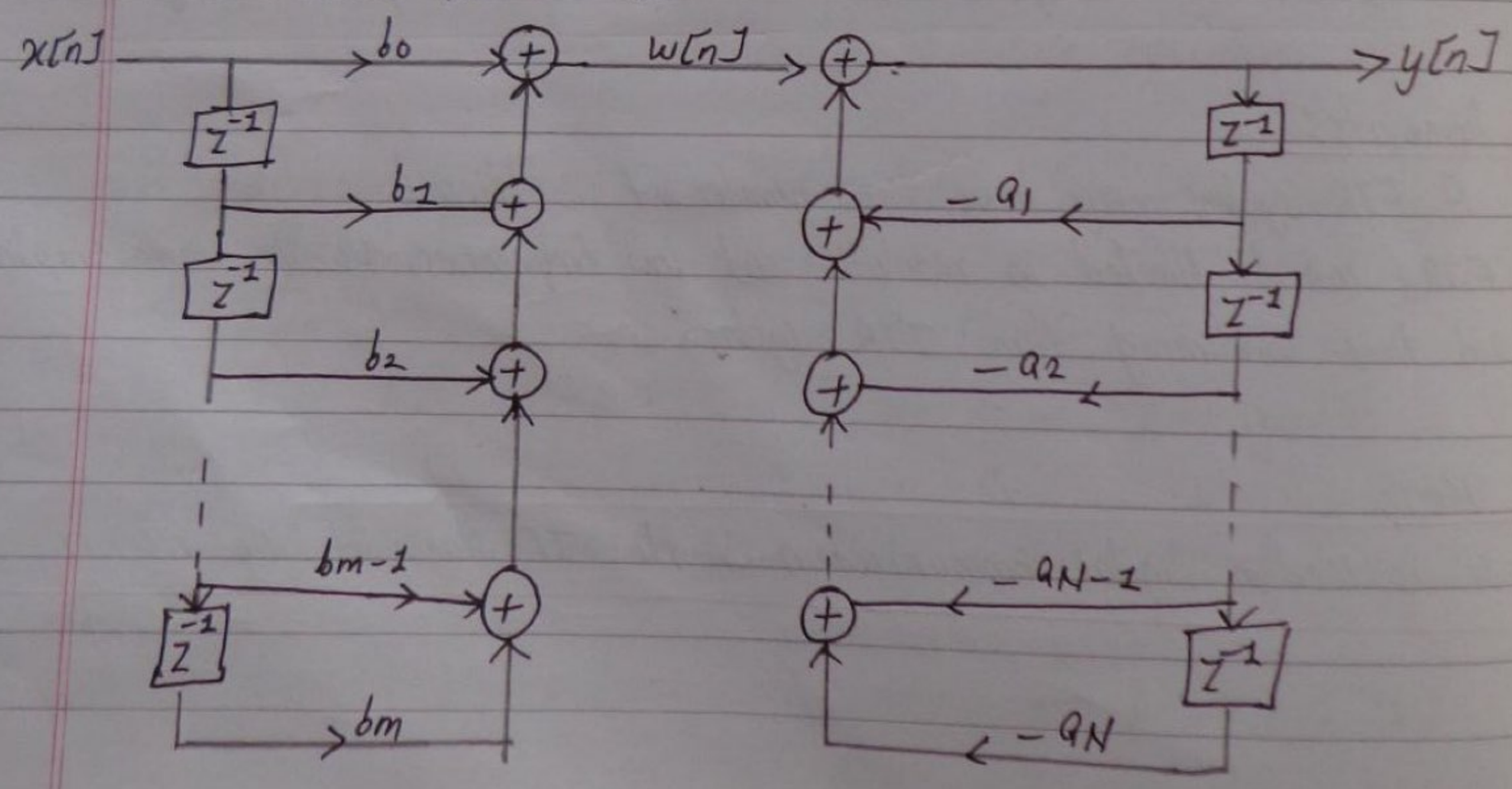
We have, frequency response as,

$$H(z) = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}} \quad \text{—— (ii)}$$

and, difference eqn as.

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k] - \sum_{k=1}^{N} a_k \cdot y[n-k] \quad \text{—— (i)}$$

The system described by direct form-I structure can be considered as a cascade of two LTI system one with input $x[n]$ and output $y[n]$ and the other with input $w[n]$ and output $y[n]$. We can interchange the order of these systems without changing the input-output behaviour of the cascade. for $M = N$.



Scanned with CamScanner

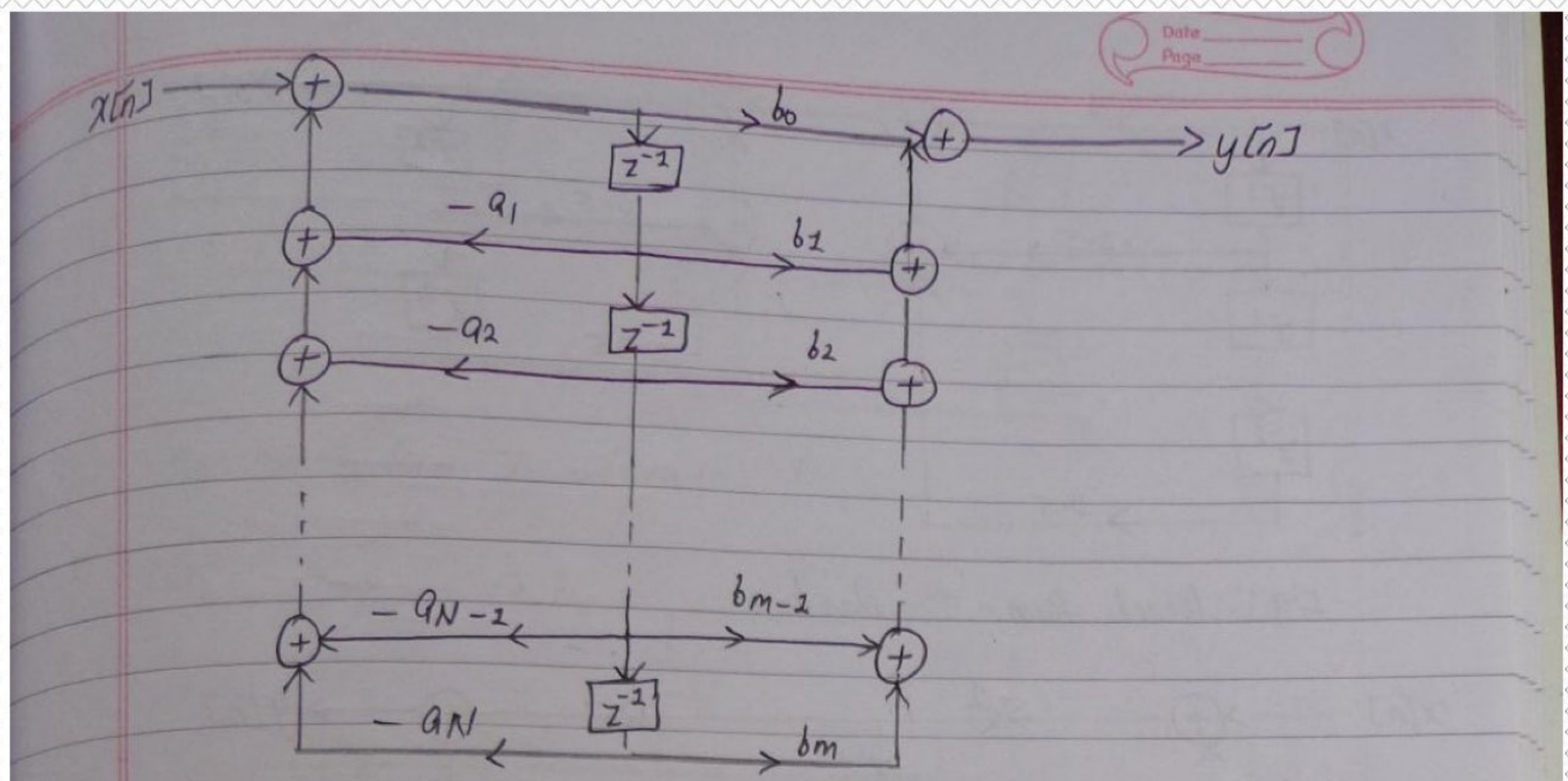


Fig: Direct form II structure.

① Realize the difference eqⁿ in direct I & direct form II structures
$2y[n] - y[n-1] + 0.5y[n-2] = 2x[n] + 3x[n-1] + x[n-3]$.
Solⁿ,

⇒ Rearranging the given difference eqⁿ in recursive form,
$y[n] = x[n] + 1.5x[n-1] + 0.5x[n-3] + 0.5y[n-1] - 0.25y[n-2]$ — ①
Taking z-transform on both sides,

or $Y(z) = X(z) + 1.5z^{-1}X(z) + 0.5z^{-3}X(z) + 0.5z^{-1}Y(z) - 0.25z^{-2}Y(z)$
or $Y(z)(1 - 0.5z^{-1} + 0.25z^{-2}) = X(z)(1 + 1.5z^{-1} + 0.5z^{-3})$
or $H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 1.5z^{-1} + 0.5z^{-3}}{1 - 0.5z^{-1} + 0.25z^{-2}}$ — (ii)

implementing eqⁿ ① or (ii) in direct form I & direct form II structures as shown in figure below,

Scanned with CamScanner

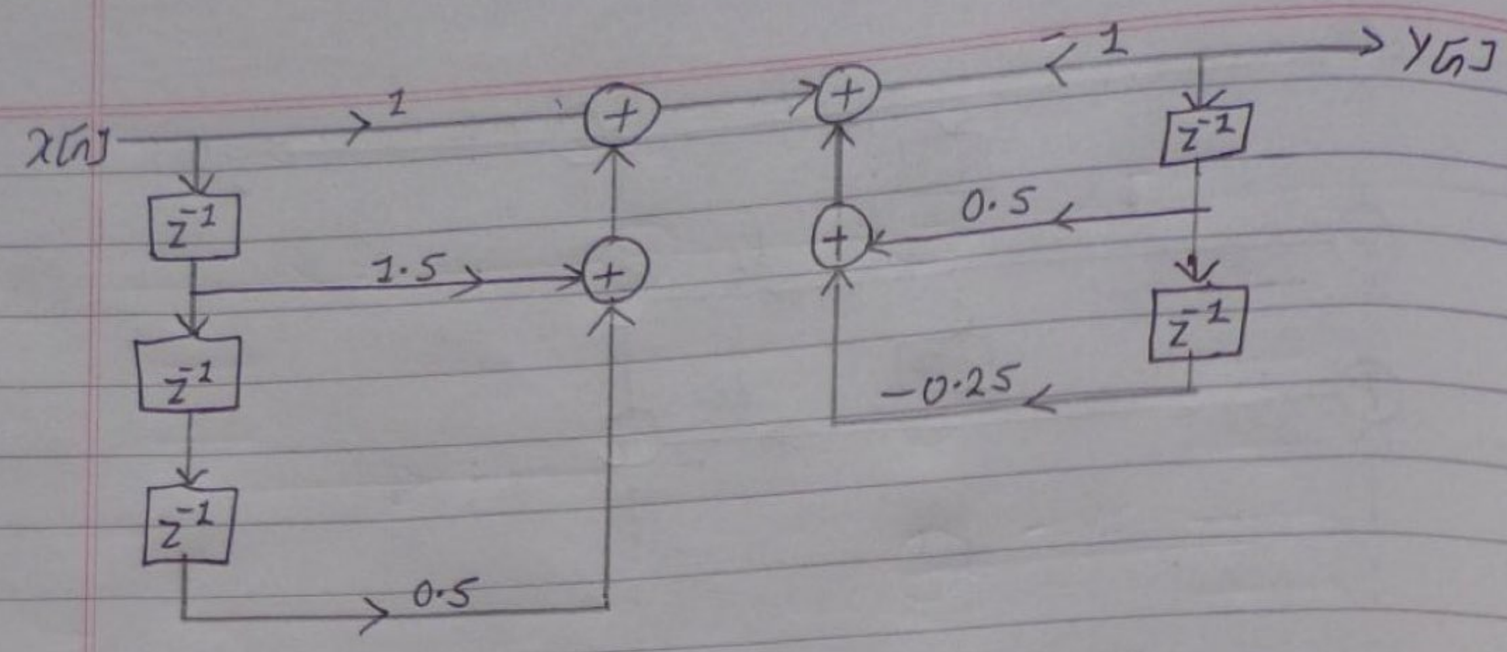


Fig: Direct form - I structure.

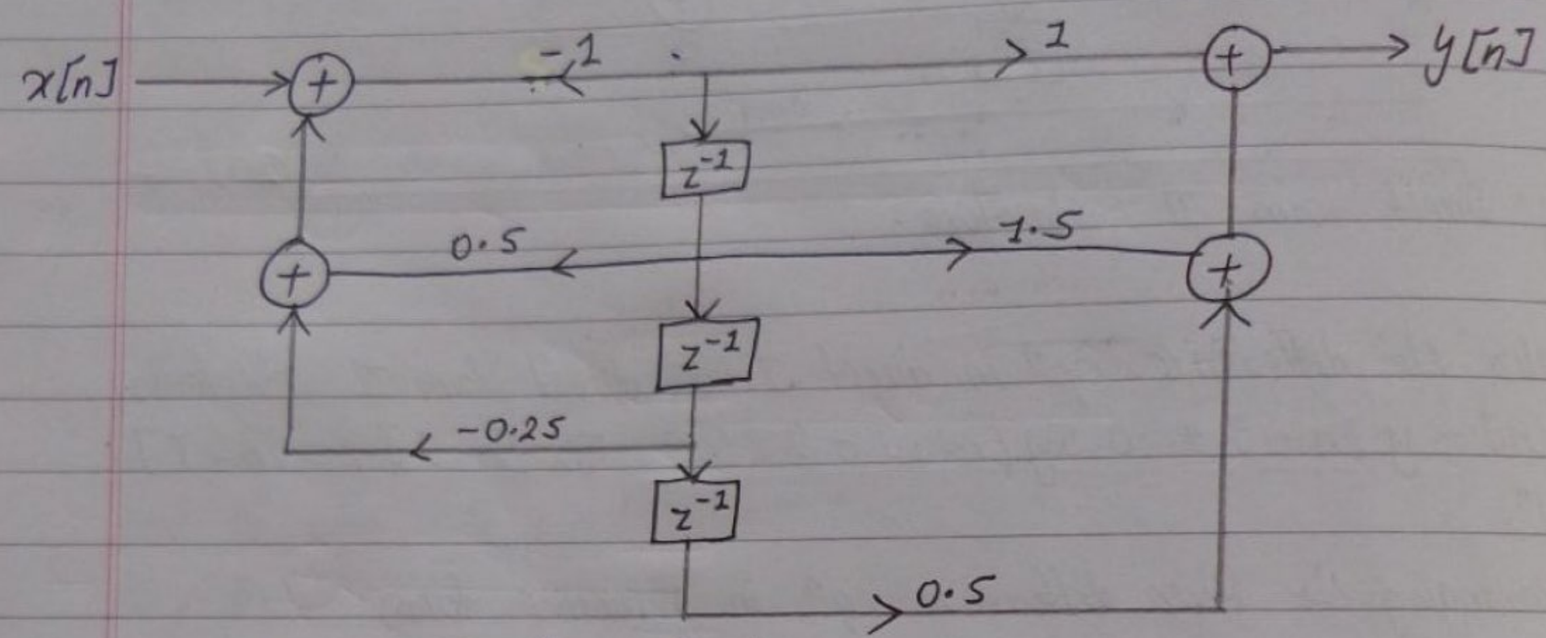


Fig: Direct II form structure.

(11) $y[n] = 0.5\,y[n-1] - 0.25\,y[n-2] + x[n] + 0.4x[n-1]$

Sol^n,

Given, diff eq^n is,

Taking Z-transform on both sides,

$Y(z) = 0.5z^{-1}Y(z) - 0.25z^{-2}Y(z) + X(z) + 0.4z^{-1}X(z)$

or $Y(z)(1 - 0.5z^{-1} + 0.25z^{2}) = X(z)(1 + 0.4z^{-1})$

or $H(z) = \dfrac{Y(z)}{X(z)}$

$\therefore H(z) = \dfrac{1 + 0.4z^{-1}}{1 - 0.5z^{-1} + 0.25z^{-2}}$

Scanned with CamScanner

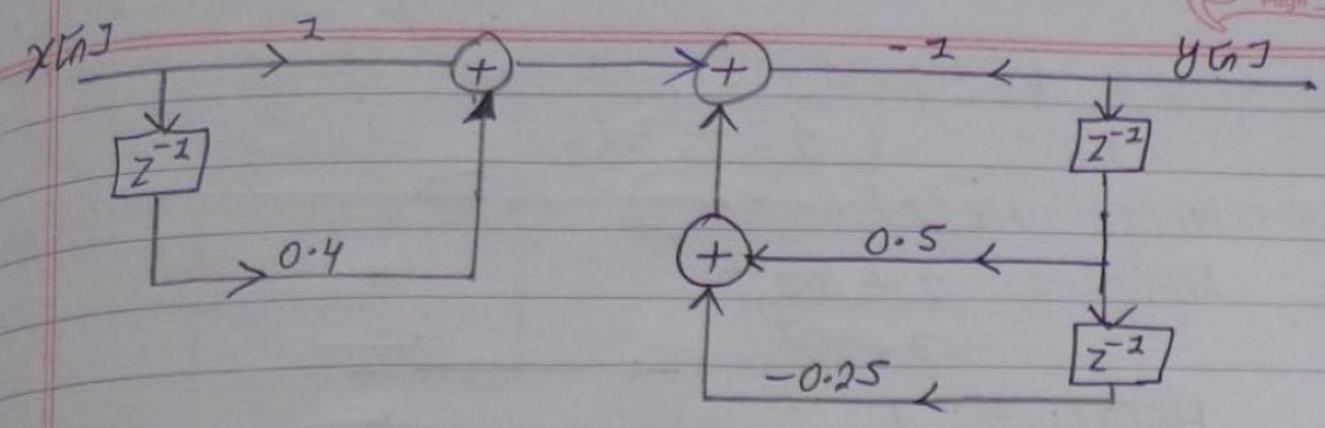


Fig: Direct form I structure.

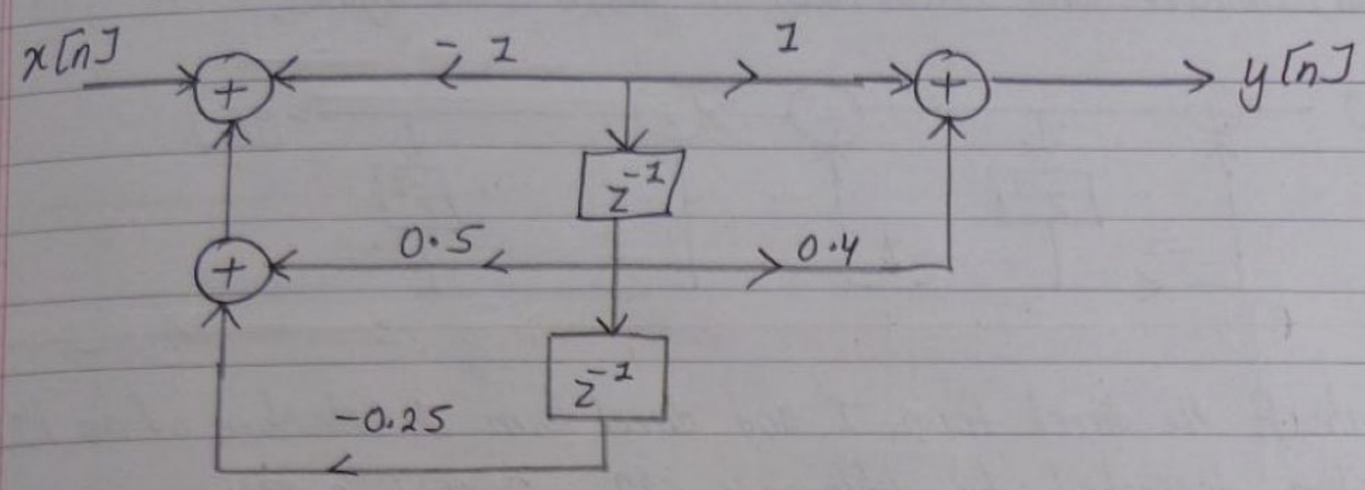


Fig: Direct form II structure.

(iii) Realize the following IIR system in cascade form structure.

$y[n] - 2y[n-1] - 3y[n-2] = x[n] + 4x[n-1]$

sol^n,

$\Rightarrow$ Taking Z-transform on both sides,

$Y(z) - 2z^{-1}Y(z) - 3z^{-2}Y(z) = X(z) + 4z^{-1}X(z)$

or $Y(z)(1 - 2z^{-1} - 3z^{-2}) = X(z)(1 + 4z^{-1})$

or $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 + 4z^{-1}}{1 - 2z^{-1} - 3z^{-2}}$

$= \dfrac{1 + 4z^{-1}}{1 + 3z^{-1} + z^{-1} - 3z^{-2}}$

$= \dfrac{1 + 4z^{-1}}{(1 + z^{-1})(1 + 3z^{-1})}$

Scanned with CamScanner

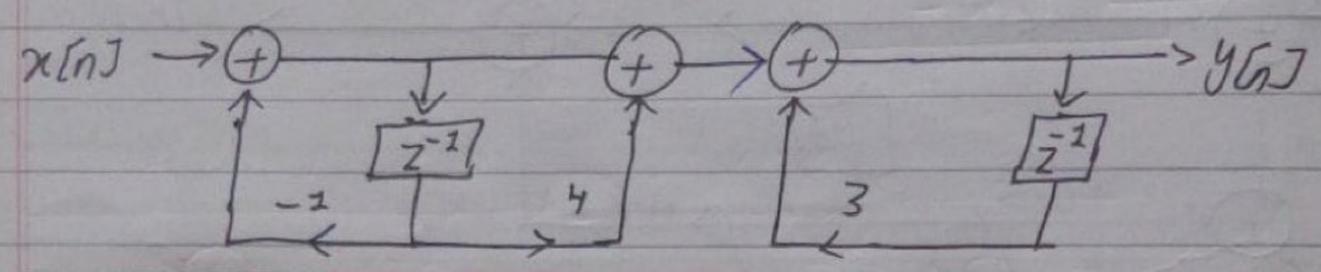
$$= \left(\frac{1+4z^{-1}}{1+z^{-1}}\right)\left(\frac{1}{1-3z^{-1}}\right)$$

$$= H_1(z) \cdot H_2(z)$$

where, $H_1(z) = \dfrac{1+4z^{-1}}{1+z^{-1}}$

$H_2(z) = \dfrac{1}{1-3z^{-1}}$

Now, cascade form realization of given IIR system,



iv Construct the direct form I and direct form II structure of an LTI system described by difference eqn, 2011/F, 2013/F, 2017/S

$2y[n] - 0.5y[n-1] + 0.25y[n-3] = 2x[n] + 3x[n-2]$ — (I)

Soln,

=> Rearranging given eqn is recursive form,

$2y[n] = 2x[n] + 3x[n-2] + 0.5y[n-1] - 0.25y[n-3]$

$y[n] = x[n] + 1.5x[n-2] + 0.25y[n-1] - 0.125y[n-3]$

Taking Z-transform,

$Y(z) = X(z) + 1.5z^{-2}X(z) + 0.25z^{-1}Y(z) - 0.125z^{-3}Y(z)$

or $Y(z)(1 + 0.125z^{-3} - 0.25z^{-1}) = X(z)(1 + 1.5z^{-2})$

or $\dfrac{Y(z)}{X(z)} = \dfrac{1 + 1.5z^{-2}}{1 + 0.125z^{-3} - 0.25z^{-1}}$ — (II)

$\therefore H(z) = \dfrac{Y(z)}{X(z)}$

Scanned with CamScanner

Implementing in Direct form I & direct form II structure,

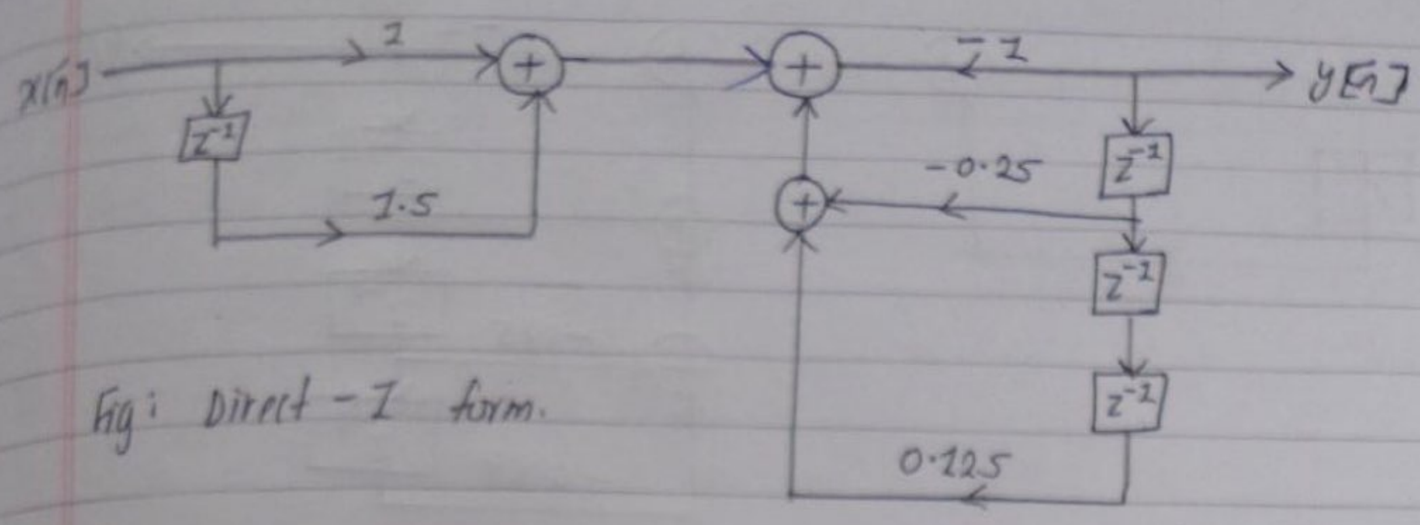


Fig: Direct - I form.

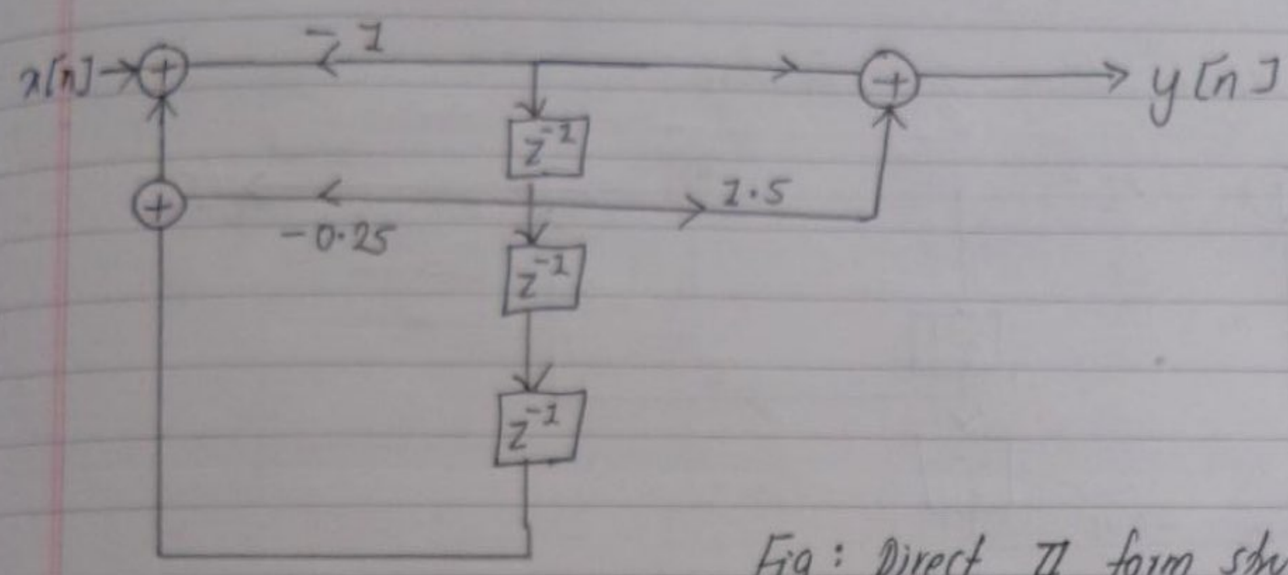


Fig: Direct II form structure.

V: Construct the Direct form - I and form - II Canonical structure of an LTI system described by difference eqn given below, 2015/5

$y[n] = -0.15\, y[n-2] - 0.84\, y[n-3] + x[n] + 2x[n-1]$

Which structure of the two is advantageous & why?

⇒ Soln,

Taking z-transform on both side,

$Y(z) = -0.15\, z^{-2} Y(z) - 0.84\, z^{-3} Y(z) + z^{-0} X(z) + 2 z^{-1} X(z)$

or $(1 + 0.15\, z^{-2} + 0.84\, z^{-3}) Y(z) = X(z)(1 + 2 z^{-1})$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 + 0.15 z^{-2} + 0.84 z^{-3}}$$

Scanned with CamScanner

Implementing in direct form-I & II structures :

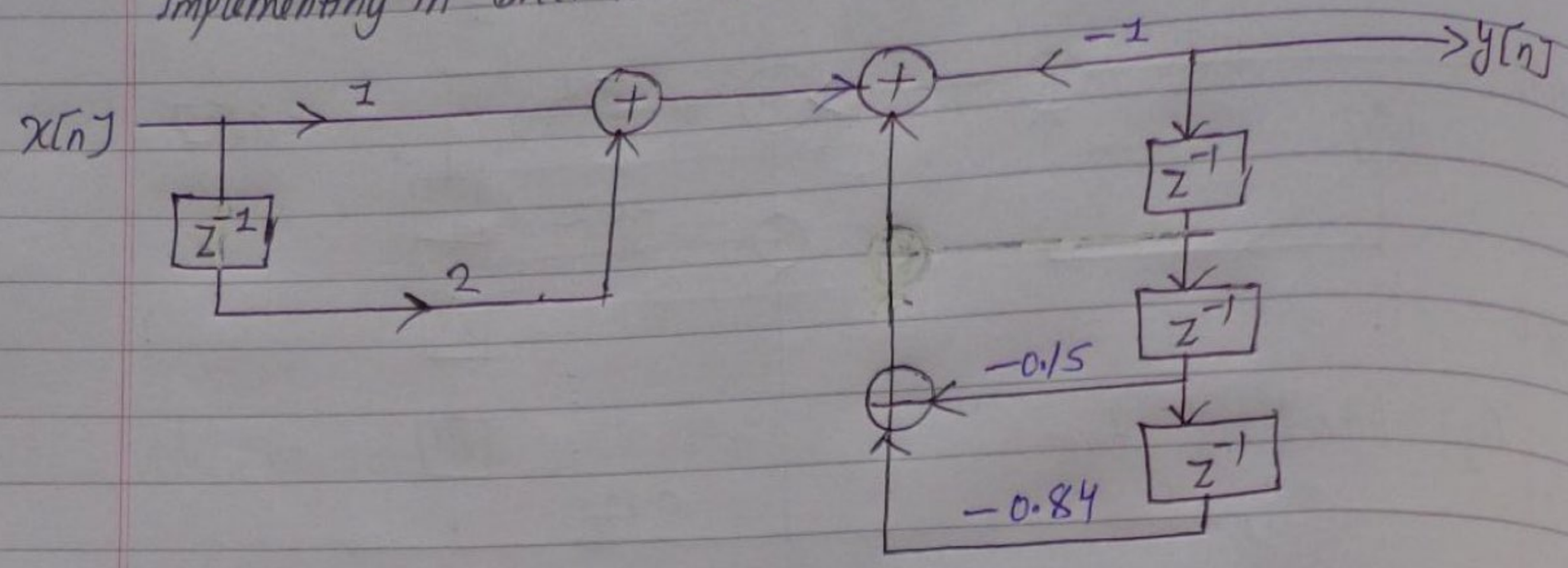


Fig: Direct Form I structure.

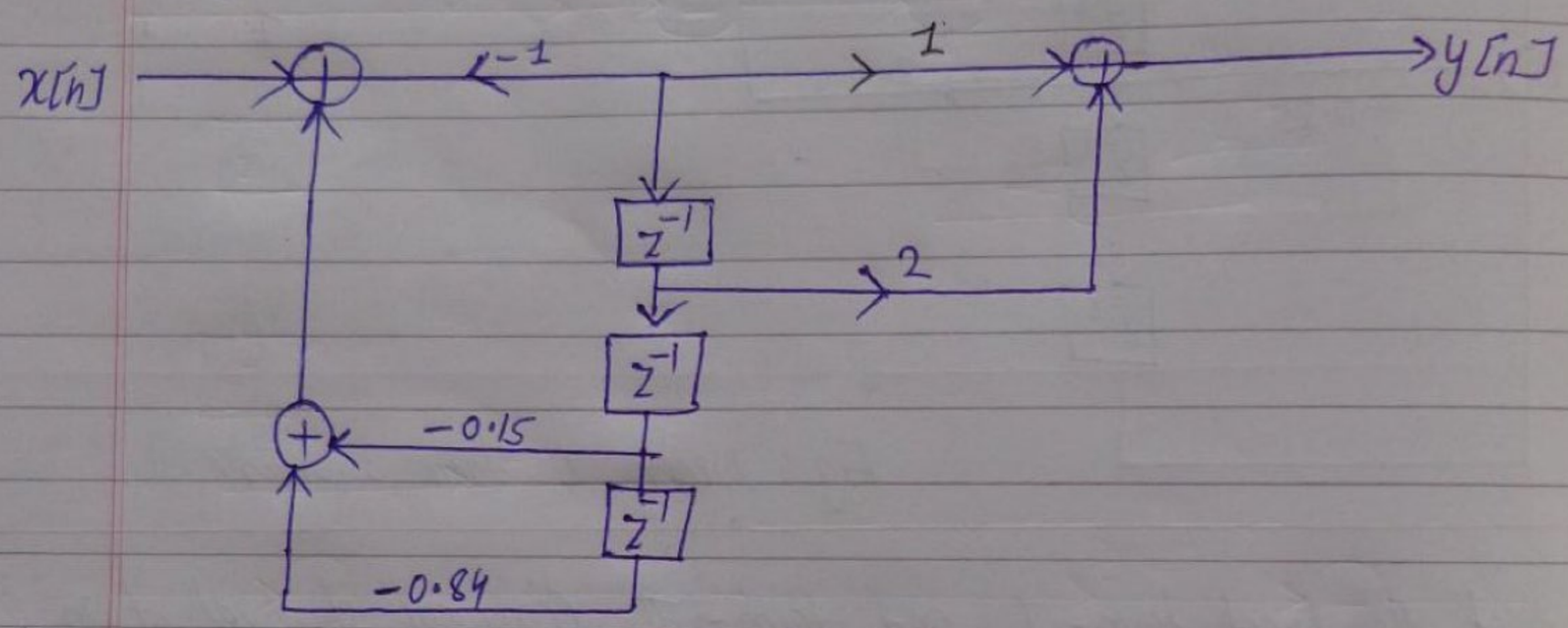


Fig: Direct form II structure.

In very general terms, Direct form I has advantages for fixed point implementations and direct form II has advantages for floating point implementations. Both have stability problems for higher order filters. Both are equivalent for non recursive filters.

Direct form I uses twice as many delay elements - it takes more memory to represent the state of the filter.

Scanned with CamScanner

Direct form II uses less delay elements, but may be more like to cause arithmetic overflow in a fixed point implementation.

Hence, Direct form II is more advantageous than direct form I.

vi) Draw direct form II structure of $H(z) = \dfrac{z^{-1} - 3z^{-3}}{10 + 5z^{-1} + 2z^{-3} + 5z^{-2}}$

2012/F.

Sol^n,

$\Rightarrow H(z) = \dfrac{z^{-1} + (-3z^{-3})}{10 + 5z^{-1} + 5z^{-2} + 2z^{-3}}$

Divide num & denominator by 10 to make it in standard form.

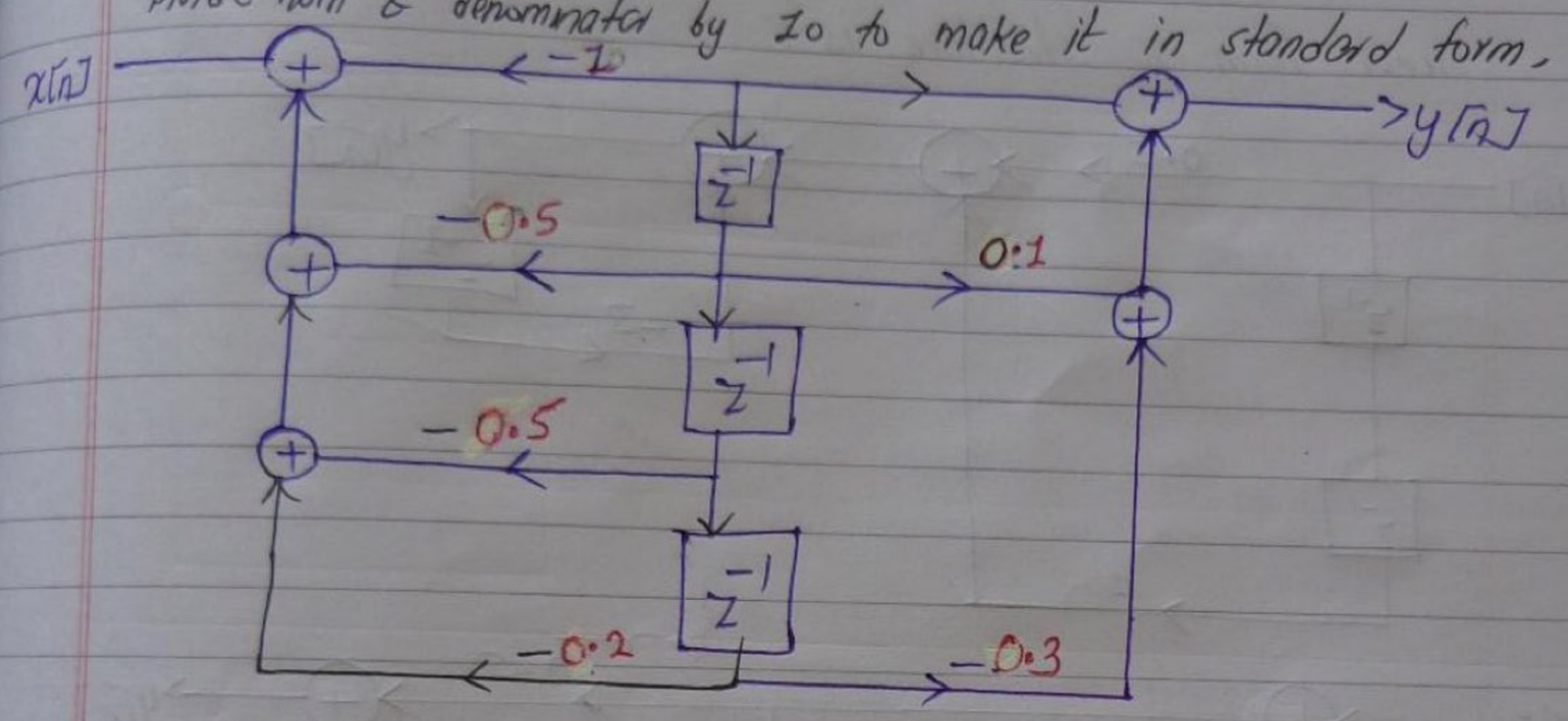


Fig: direct form II structure of H(z).

vii) Realize the LTI system described by the following difference eqn in direct form I and direct form II structure.

Sol^n,

$\Rightarrow y[n] + 3y[n-1] = 0.5\,x[n] + x[n-2]$

Rearranging given difference eqn,

$y[n] = 0.5\,x[n] + x[n-2] - 3y[n-1]$

Scanned with CamScanner

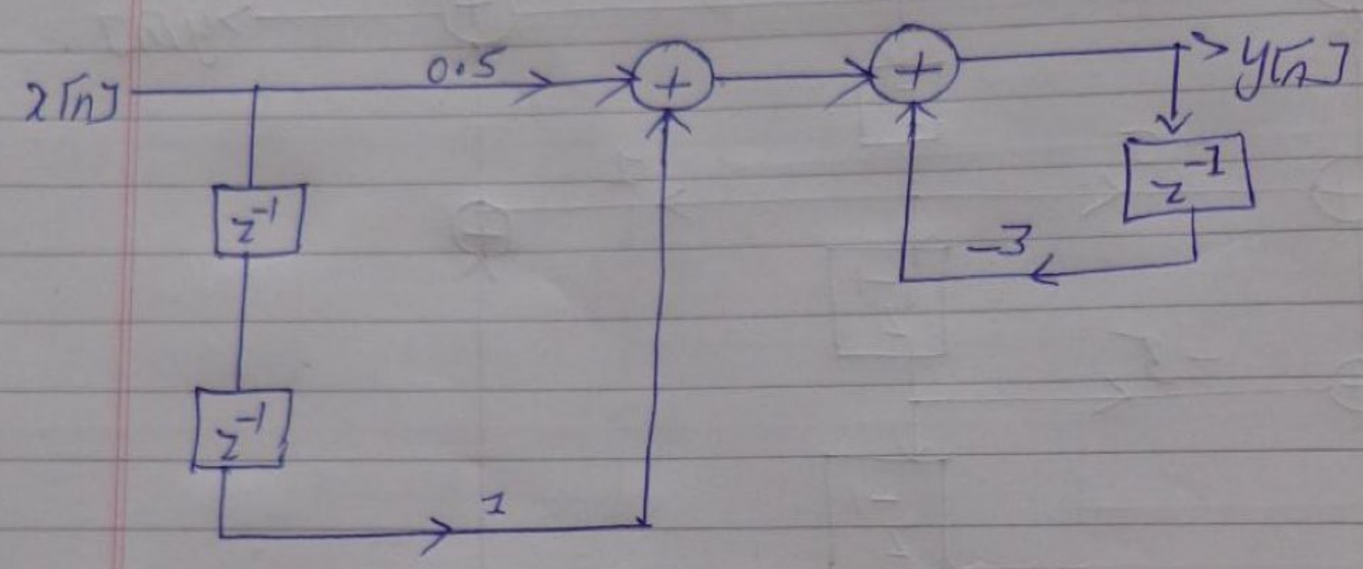
Taking z^{-1} on both sides,

$$Y(z) = 0.5\,X(z) + z^{-2}X(z) - 3\,z^{-1}Y(z)$$

or $$Y(z)\,(1 + 3z^{-2}) = X(z)\,[0.5 + z^{-2}]$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{0.5 + z^{-2}}{1 + 3z^{-2}}.$$

Direct- Form I structure is :



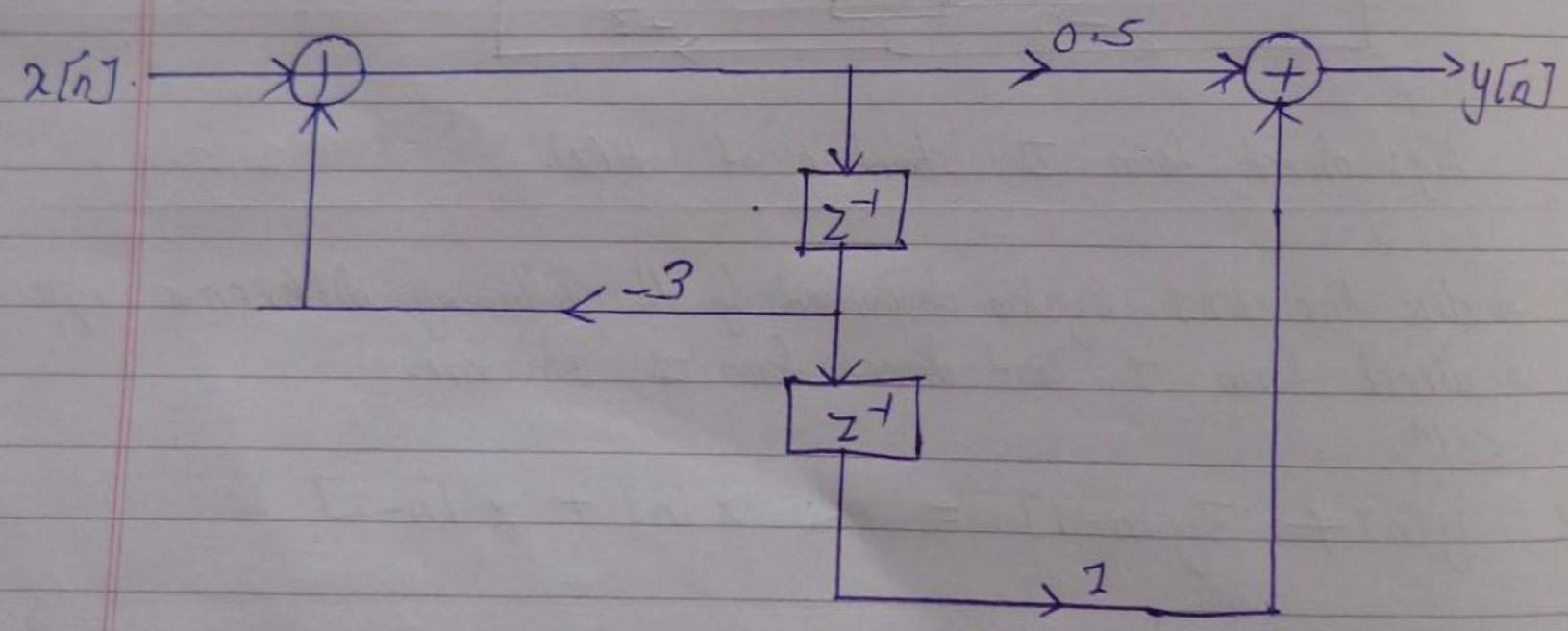


Fig: Direct form II structure.

Scanned with CamScanner

Discrete Fourier Transform (DFT):

Chapter 4:

1.A) Why do you need DFT although you have DTFT? Define DFT. 2017/5

⇒ The DFT is a powerful computational tool to perform the frequency domain analysis of discrete time signal. Frequency domain analysis of discrete time signals is usually performed on a digital signal processor. For performing the frequency domain analysis, the signal in time domain is converted into frequency domain. The frequency domain representation is given by the Fourier transform of the signal. Since, fourier transform is a continuous function of frequency, it cannot be computed in digital processor. To overcome this problem, we represent the signal by samples of it's spectrum. This kind of frequency domain representation leads to the Discrete fourier transform.

The DFT is the transform of a limited number of samples of a periodic signals. The DTFT is a transform of the entire sampled signal from $-\infty$ to $+\infty$ and input isnot necessarily periodic. It's output is continuous in frequency and period.

DTFT = discrete, Aperiodic

DFT = discrete, periodic.

B. DFT : $X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}\cdot n}$, $k = 0, 1, 2, \ldots N-1$

IDFT: $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{jk\frac{2\pi}{N}\cdot n}$, $n = 0, 1, 2, \ldots N-1$

C. DFT as a linear Transformation:

let us define an N-point vector x_N of signal $x[n]$, an N-point vector, X_N of frequency samples $X[k]$ and an $N \times N$ matrix, W_N

Scanned with CamScanner

as,

$$x_N = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}, \quad X_N = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{1} & W_N^{2} & \cdots & W_N^{N-1} \\ 1 & W_N^{2} & W_N^{4} & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Now, the N-point DFT may be expressed in matrix form as,

$$X_N = W_N \cdot x_N$$

where, W_N is the matrix of linear transformation. And IDFT may be matrix form as,

$$x_N = \frac{1}{N} \cdot W_N^{*} \cdot X_N.$$

where the matrix W_N^{*} is the complex conjugate of the matrix W_N.

---

eg ①: Compute the 4-point DFT of the sequence $x[n] = \{0, 1, 2, 3\}$ with and without linear transformation method.

Sol^n,

$\Rightarrow$ without linear transformation,

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}, \quad k = 0, 1, 2, \cdots N-1$$

$$= \sum_{n=0}^{3} x[n]\, e^{-jk\frac{2\pi}{N}\cdot n}$$

Scanned with CamScanner

$W_N = e^{-j\frac{2\pi}{N}}$ --- twiddle factor

For $k=0$: $X[0] = x[0] + x[1] + x[2] + x[3]$
$= 0 + 1 + 2 + 3$
$= 6$

For $k=1$: $X[1] : x[0] + x[1]e^{-j1\frac{2\pi}{4}\cdot 1} + x[2]e^{-j\cdot 1\frac{2\pi}{4}\cdot 2} + e^{-j1\cdot\frac{2\pi}{4}\cdot 3} \cdot x[3]$
$= 0 + (1)(-j) + 2(-1) + 3(j)$
$= -2 + 2j$

For $k=2$: $X[2] = 0 + (1)(-1) + 2 \times (1) + 3 \times (-1)$
$= -2$

For $k=3$; $X[3] = x[0] + x[1] * e^{-j3\cdot\frac{2\pi}{4}\cdot 1} + x[2]\cdot e^{-j3\frac{2\pi}{4}\cdot 2} +$
$x[3]e^{-j3\cdot\frac{2\pi}{4}\cdot 3}$
$= 0 + (1)(j) + (2)(-1) + 3 \times (-j)$
$= -2 - 2j$

Hence, $X[k] = \{6, -2+2j, -2, -2-2j\}$.

Using linear transformation,
we have, $X_N = W_N \times x_N$.

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

where,
$W_4 = e^{-j\frac{2\pi}{N}\cdot 1}$, $W_4^3 = j$, $W_4^9 = -j$
$W_4^1 = -j$ $W_4^4 = 1$
$W_4^2 = -1$ $W_4^6 = -1$

Scanned with CamScanner

Now,

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times0 + 1\times1 + 1\times2 + 1\times3 \\ 1\times0 - j\times1 - 1\times2 + j\times3 \\ 1\times0 - 1\times1 + 1\times2 - 1\times3 \\ 1\times0 + j\times1 - 1\times2 - j\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -2+j2 \\ -2 \\ -2-j2 \end{bmatrix}$$

Hence,

$X[k] = \{6, -2+2j, -2, (-2-2j)\}$.

D. Circular Convolution:

let, $x_1[n]$ and $x_2[n]$ be the two sequences and if,

$x_1[n] \xleftrightarrow[N]{D.F.T} X_1[k]$

and,

$x_2[n] \xleftrightarrow[N]{DFT} X_2[k]$

then,

$x_3[n] = x_1[n] \circledN x_2[n] \xleftrightarrow[N]{DFT} X_3[k] = X_1[k]\, X_2[k]$

Here, $x_3[m]$ is called the circular convolution between $x_1[n]$ & $x_2[n]$ and is expressed as,

$$x_3[m] = \sum_{n=0}^{N-1} x_1[n]\, x_2[(m-n)]_N, \quad m = 0, 1, 2, \ldots, N-1$$

Scanned with CamScanner

This property tell us that circular convolution of two sequence in time domain results multiplication of their corresponding DFT in frequency domain.

Eg: 1 Perform the circular convolution between $x_1[n] = \{1, 2, 3, 4\}$ and $x_2[n] = \{2, 3, 4, 5\}$.

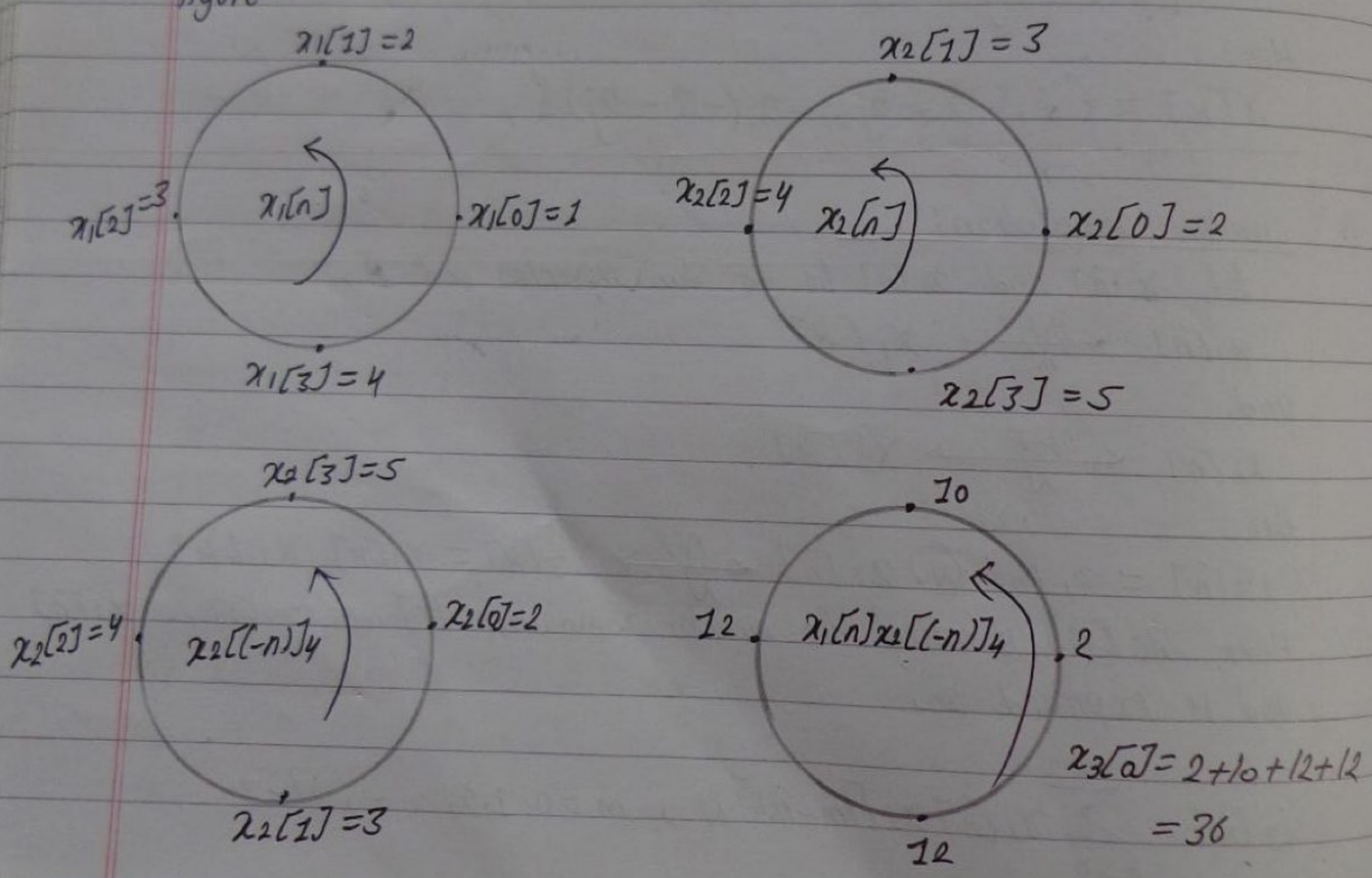
Sol^n,

$\Rightarrow$ The circular convolution between $x_1[n]$ and $x_2[n]$ can be computed by using the expression,

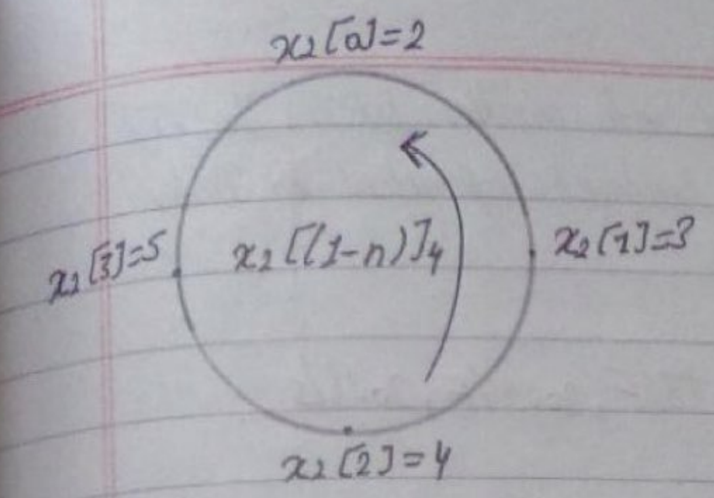
$$x_3[m] = \sum_{n=0}^{N-1} x_1[n] \cdot x_2[(m-n)]_N, \quad m = 0, 1, 2, \ldots N-1.$$

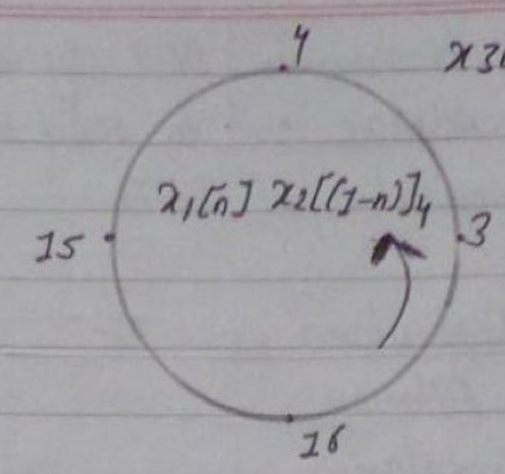
$$= \sum_{n=0}^{3} x_1[n] \cdot x_2[(m-n)]_4, \quad m = 0, 1, 2, \ldots N-1.$$

This expressed can be implemented graphically as shown in figure:

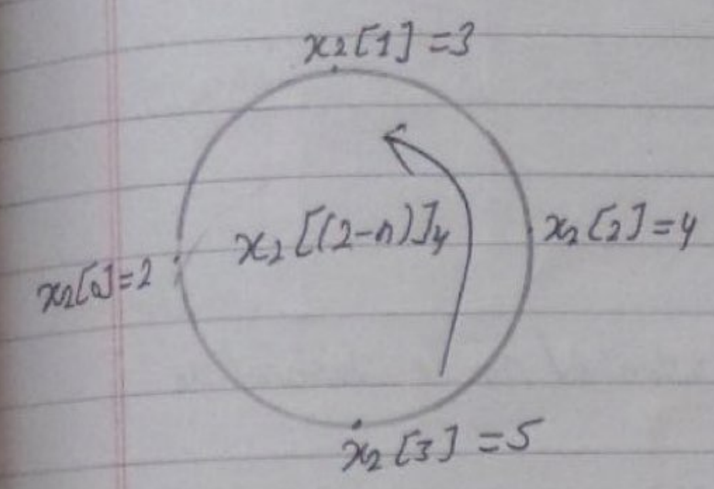


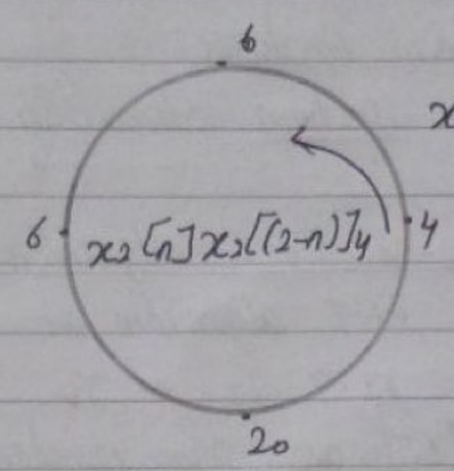
Scanned with CamScanner



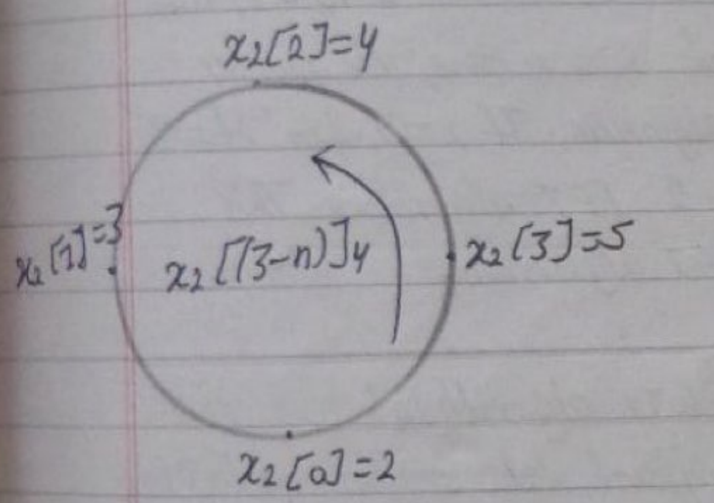


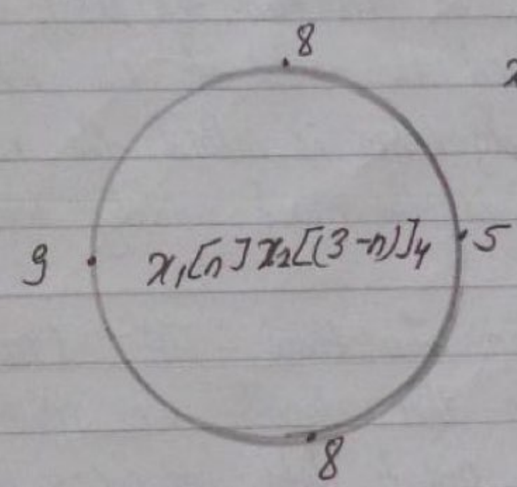
$x_3[1] = 3+4+15+16 = 38$





$x_3[2] = 4+6+6+20 = 36$





$x_3[3] = 5+8+9+8 = 30$

Hence the result is,

$$x_3[m] = \{36, 38, 36, 30\}.$$

2.A) Fast Fourier Transform: (FFT): 2017/s

FFT is an algorithm to compute the DFT faster. FFT algorithm reduces the number of arithmetic operations such as multiplication & additions so as to make the computation of DFT faster. The Basic idea of FFT algorithm

Scanned with CamScanner

is to compute DFT by decomposing it into smaller DFTs, The FFT algorithms are used to reduce the number of arithmetic operations for faster computation of DFT.

Applications of FFT: 2013/F, 2009/F, 2016/S

(i) Filtering
(ii) correlation
(iii) spectrum Analysis
(iv) power spectrum estimation.

* Radix-2 FFT algorithm:

FFT algorithm are applicable when N can be factored as

$N = r_1 \cdot r_2 \cdot r_3 \cdots r_v$

where $\{r_i\}$ are prime numbers. If $r_1 = r_2 = r_3 = \cdots = r_v = r$, then $N = r^v$, in this case, the DFTs are of size r. Here, the number r is called the radix of the FFT algorithm. If $r=2$, then, $N = 2^v$, and the algorithm is called the radix-2 FFT algorithm. This algorithm is the most widely used FFT algorithm.

Decimation in time - FFT (DIT-FFT) algorithm:

In this algorithms, we split N-point data sequence $x[n]$ into two N/2 point data sequences corresponding to the even numbered and odd numbered samples of $x[n]$. since the two N/2 points sequences are obtained by decimating $x[n]$, the algorithm is referred to as the decimation in time (DIT-FFT) algorithm.

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad k = 0, 1, 2, \ldots, N-1$$

$$= \sum_{n \text{ even}} x[n] W_N^{kn} + \sum_{n, \text{odd}} x[n] W_N^{kn}$$

Scanned with CamScanner

$$= \sum_{m=0}^{N/2-1} x[2m] W_N^{2km} + \sum_{m=0}^{N/2-1} x[2m+1] W_N^{k(2m+1)}$$

$$= \sum_{m=0}^{N/2-1} x[2m] W_N^{2km} + W_N^{k} \sum_{m=0}^{N/2-1} x[2m+1] W_N^{2km}$$

We know,

$$W_N^2 = (+e^{-j\frac{2\pi}{N}})^2 = e^{-j\frac{2\pi}{N/2}} = W_{N/2}, \text{ then,}$$

$$X[k] = \sum_{m=0}^{N/2-1} x[2m] W_{N/2}^{km} + W_N^{k} \sum_{m=0}^{N/2-1} x[2m+1] W_{N/2}^{km}$$

$$= G[k] + W_N^k \cdot H[k], \quad k = 0, 1, 2, \ldots N-1.$$

where,

$$G[k] = \sum_{m=0}^{N/2-1} x[2m] W_{N/2}^{km}$$

is the $N/2$ point DFT of even numbered samples &

$$H[k] = \sum_{m=0}^{N/2-1} x[2m+1] W_{N/2}^{km}$$

is the $N/2$ point DFT of odd numbered samples of $x[n]$.

Since, $X[k]$ is periodic with period N, $G[k]$ and $H[k]$ will also be periodic with period $N/2$, Also, $W_N^{k+N/2} = -W_N^k$, then,

$$X[k] = G[k] + W_N^k \cdot H[k], \quad k = 0, 1, 2, \ldots N/2 - 1.$$

$$X\left[k + \frac{N}{2}\right] = G[k] - W_N^k H[k], \quad k = 0, 1, 2, \ldots N/2 - 1.$$

In the radix-2 DIT-FFT algorithm, the total number of complex multiplication is reduced to $(N/2)\log_2 N$ & total number of complex addition is reduced to $N \log_2 N$.

Scanned with CamScanner

Fig: Block diagram of radix-2 DIT-FFT algorithm.

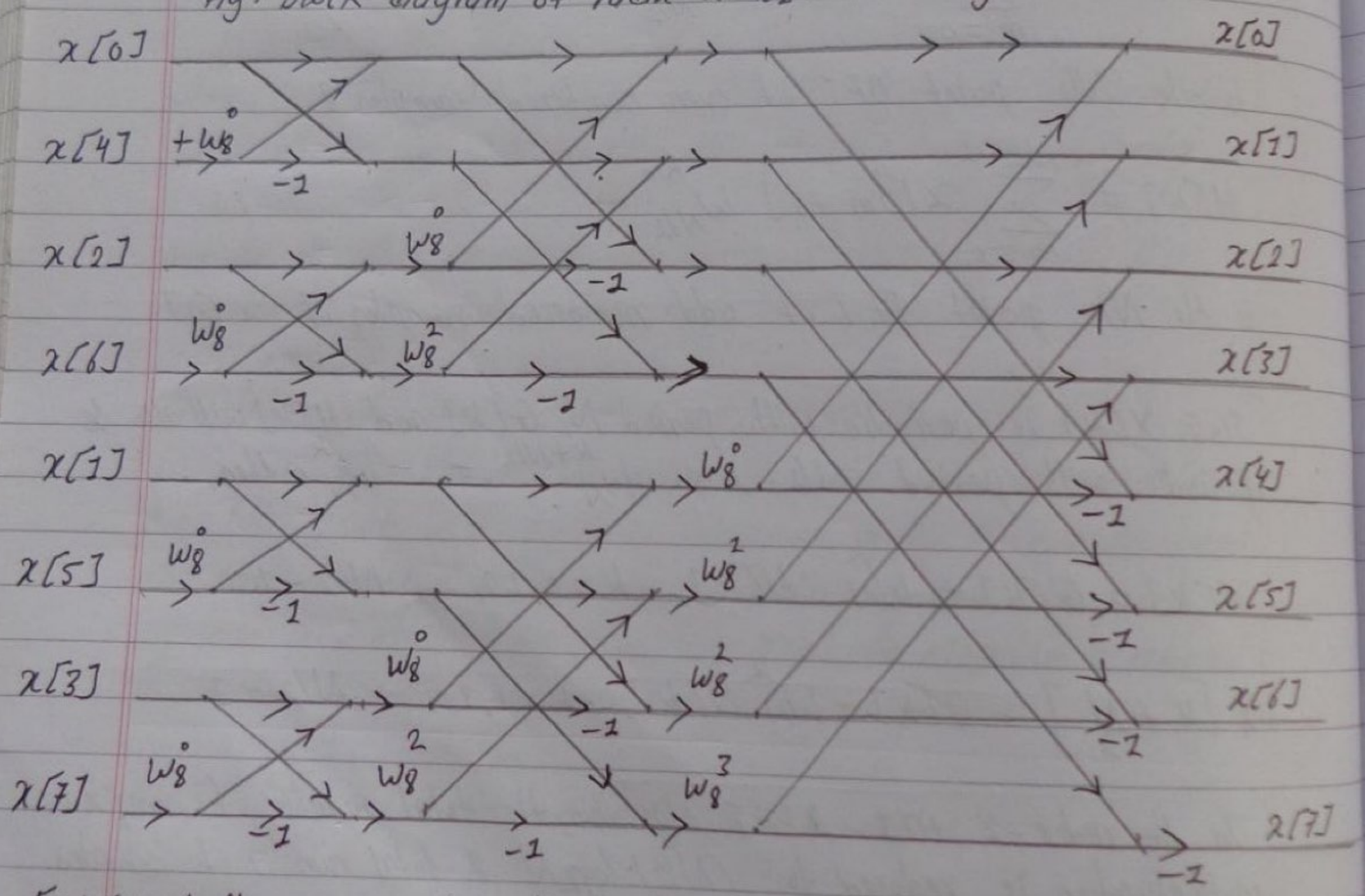


Fig: Signal flow graph of radix-2 DIT-FFT algorithm.

Scanned with CamScanner

(8) Bit-Reversal technique:

The shuffling of data can be done by bit reversal technique.

n	$n_2 n_1 n_0$	$n_0 n_1 n_2$	n
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

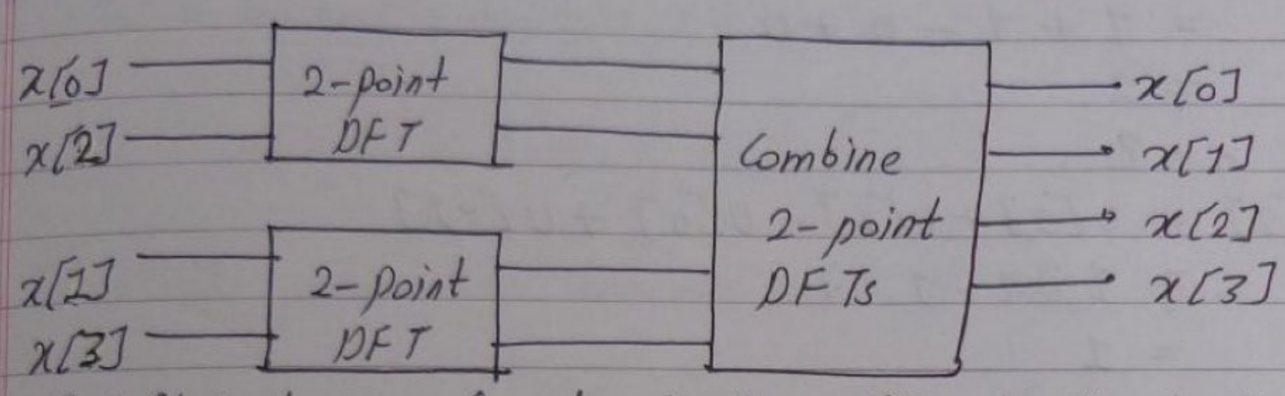


Fig: Block diagram of radix-2 DIT-FFT algorithm for 4-point DFT.

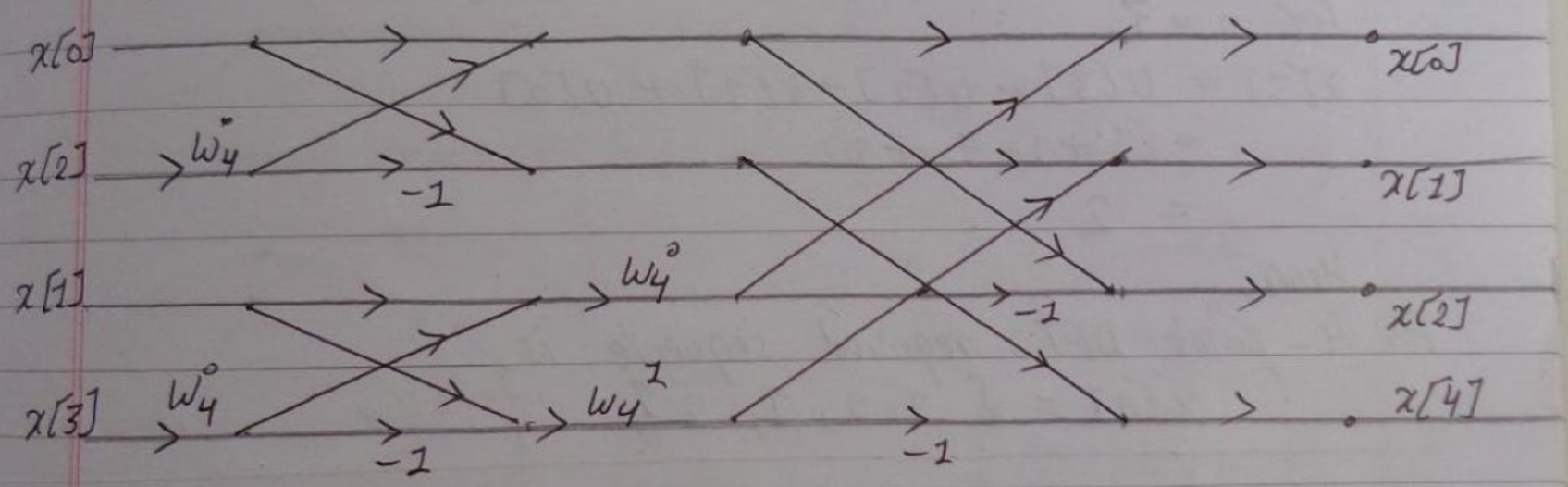


Fig: signal flow graph of radix-2 DIT-FFT algorithm for computing 4-point DFT.

Scanned with CamScanner

a. Compute 4-point DFT of the following sequence.
Sol^n, 2014/5, 2013/4, 2017/5

$\Rightarrow x[n] = u[n] + u[n-1] + u[n-3] - u[n-2]$

From the given sequence,
For $n = 0$,
$x[0] = u[0] + u[-1] - u[-2] + u[-3]$
$= 1 + 0 + 0 + 0$
$= 1$
For $n = 1$,
$x[1] = u[1] + u[0] + u[-2] - u[-1]$
$= 1 + 1 - 0 + 0$
$= 2$
For $n = 2$,
$x[2] = u[2] + u[1] - u[0] + u[-1]$
$= 1 + 1 - 1 + 0$
$= 1$

For $n = 3$,
$x[3] = u[3] + u[2] - u[1] + u[0]$
$= 1 + 1 - 1 + 1$
$= 2$
Hence,
For 4-point DFT, required sequence is,
$x[n] = \{1, 2, 1, 2\}$.

b. Find the linear convolution using convolution of the following sequences.
$x[n] = \{1, \underset{\uparrow}{2}, 2, 1\}$ and $h[n] = \{1, 2, 3, \underset{\uparrow}{2}, 1\}$

Scanned with CamScanner

Sol^n:

$\Rightarrow$ Here the length of $x[n]$ is 4 and length of $h[n]$ is 5. Hence, the length of linear convolution is $4+5-1 = 8$. padding zero to $x[n]$ & $h[n]$ to make the length 8, so,

$x[n] = \{1, \underset{\uparrow}{2}, 2, 1, 0, 0, 0, 0\}$

$h[n] = \{1, 2, 3, \underset{\uparrow}{2}, 1, 0, 0, 0\}$

we know that for circular convolution, both $x[n]$ and $h[n]$ are periodic with period 8 such that we can rearrange the sequence beginning from $n=0$ to $n=7$ as,

$x[n] = \{2, 2, 1, 0, 0, 0, 0, 1\}$.

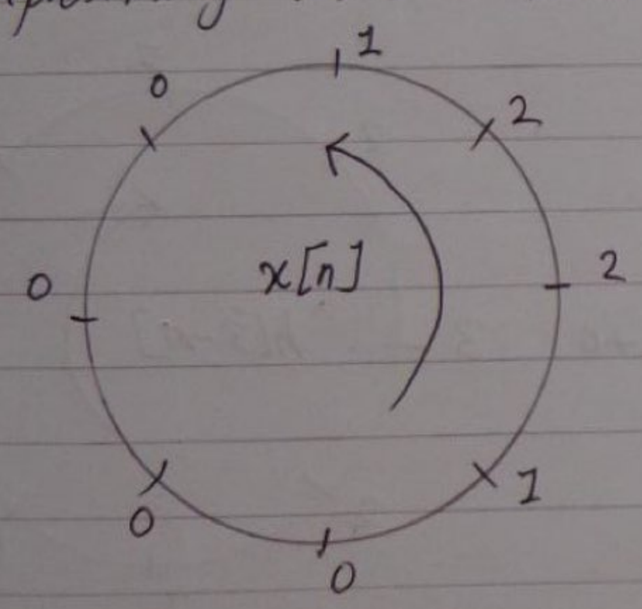
$h[n] = \{2, 1, 0, 0, 0, 1, 2, 3\}$.

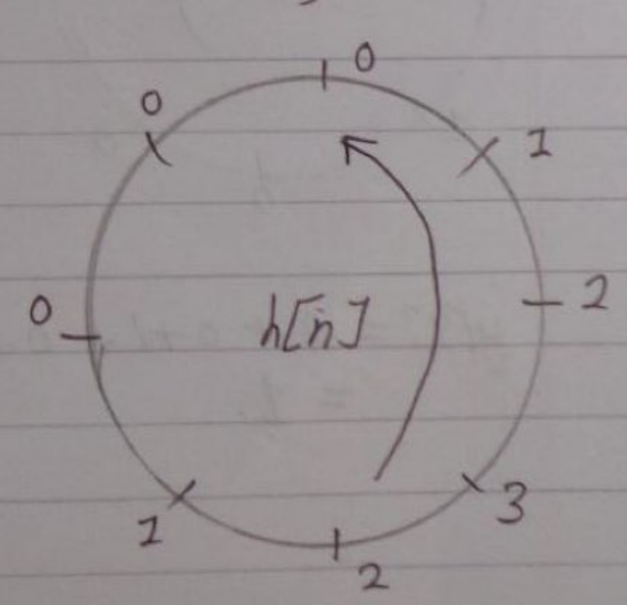
The circular convolution between $x[n]$ & $h[n]$ can be determined as,

$$y[m] = \sum_{n=0}^{7} x[n]\, h[(m-n)]_7$$

where, $m = 0, 1, 2, ---7$

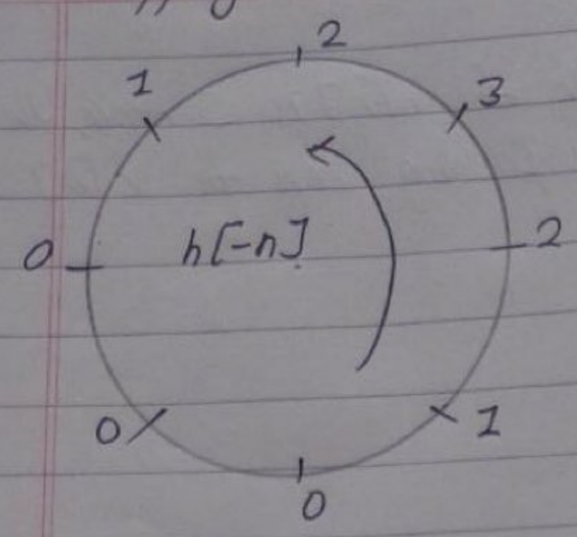
Representing $x[n]$ & $h[n]$ in circular fashions,





Scanned with CamScanner

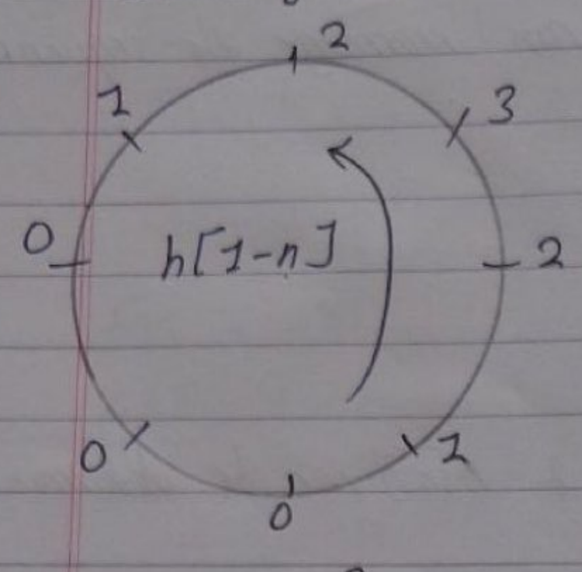
Flipping $h[n]$ over time, we get,



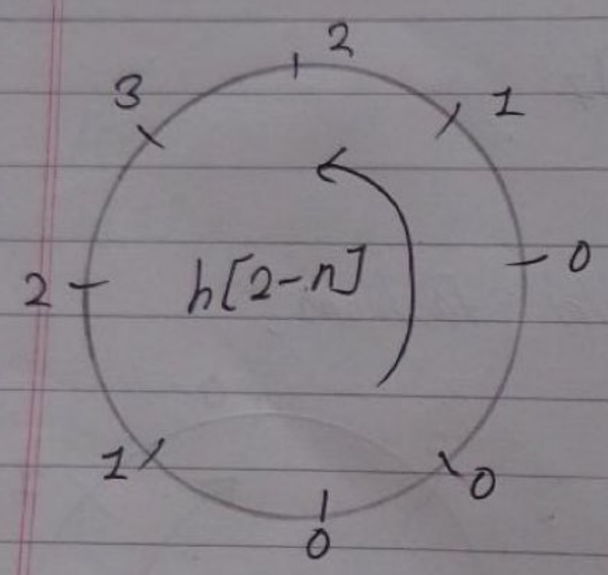
$\Rightarrow h[-n]\,x[-n]$

$y[0] = 4+6+2+0+0+0+1$
$= 13$

Similarly for $n = 1, 2, 3, 4, 5, 6, 7$,

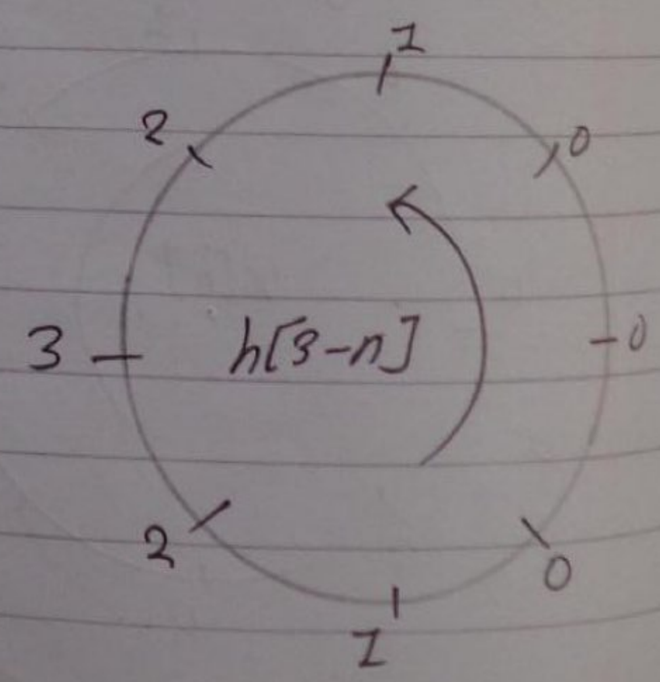


$y[1] = 2+4+6+3+0+0+0+0+0$
$= 9$

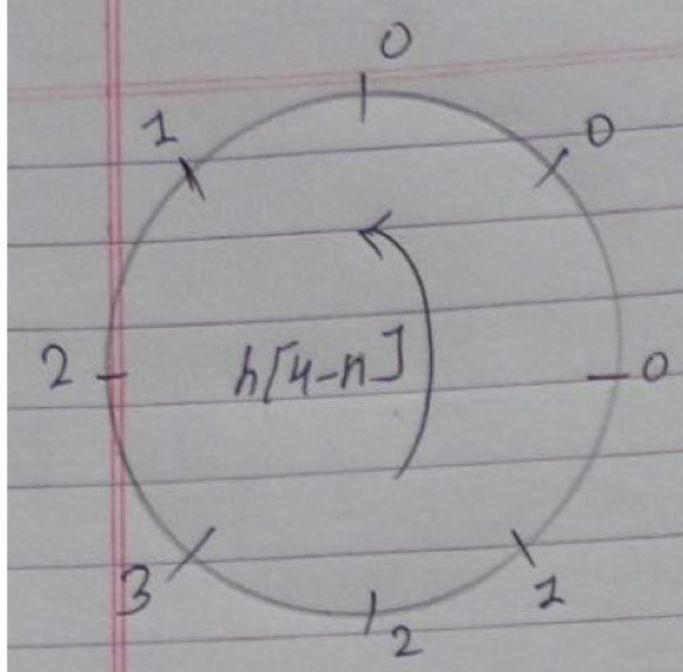


$y[2] = 0+2+2+0+0+0+0+0$
$= 4$

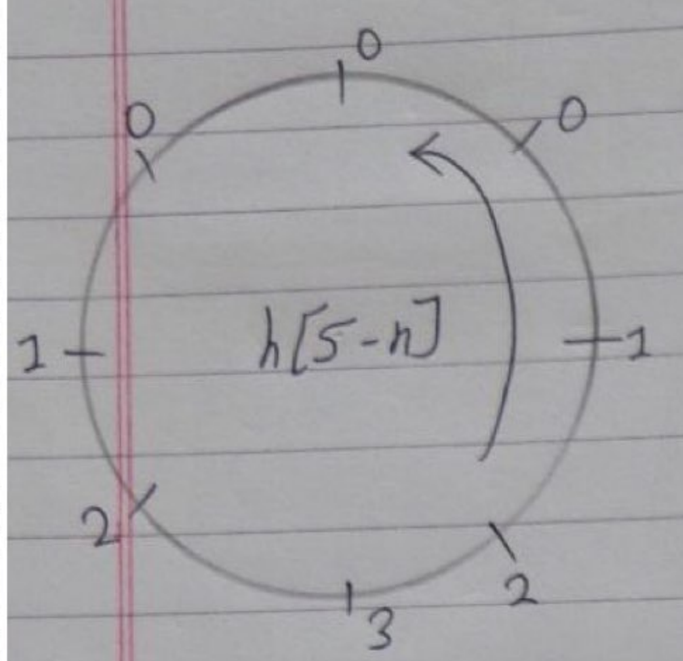
$y[3] = 0+0+1+0+0+0+0+0$
$= 1.$



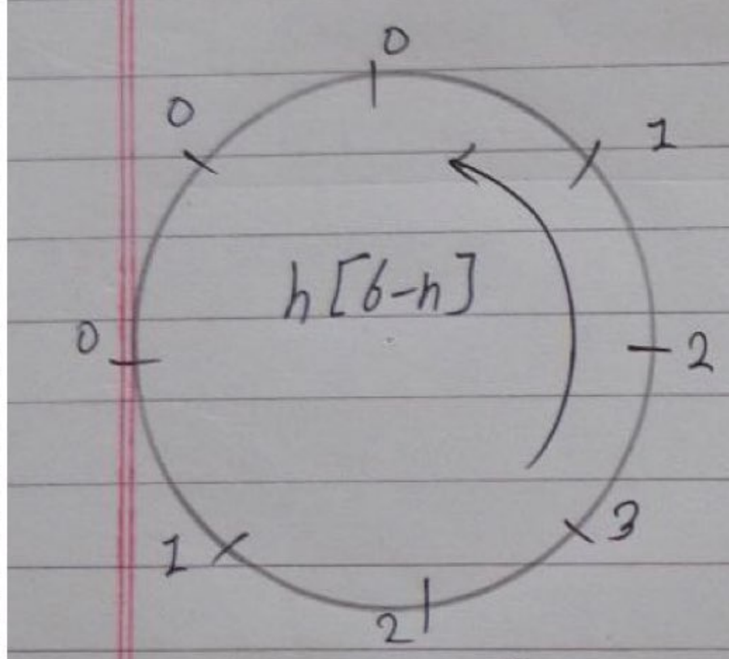
Scanned with CamScanner



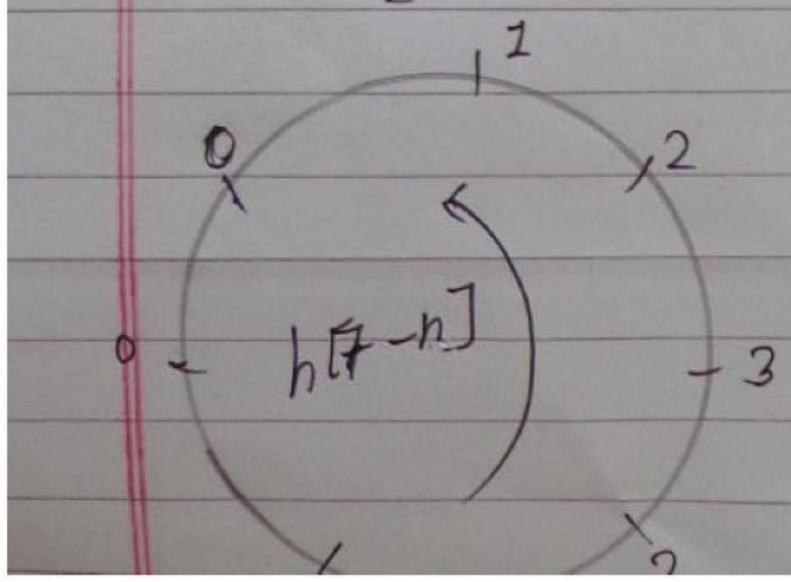
$y[4] = 0+0+0+0+0+0+0+1$
$= 7$



$y[5] = 2+0+0+0+0+0+0+2$
$= 4$



$y[6] = 4+2+0+0+0+0+0+3$
$= 9$



$y[7] = 6+4+1+0+0+0+0+2$
$= 13$

Scanned with CamScanner

Hence, result of circular convolution is,

$y[n] = \{13, 9, 4, 1, 1, 4, 9, 13\}$.

Since the linear convolution in this problem starts from -4 upto 3, we can shift $y[n]$ circularly by 4 to obtain the result of linear convolution as,

$y[n] = \{1, 4, 9, 13, 13, 9, 4, 1\}$.

c) Calculate DFT of the sequence $x[n] = \{5, 3, 2, 2\}$.

Sol^n,

$\Rightarrow$ Here, $N = 4$,

By using linear transformation,

$$X_N = W_N \cdot x_N$$

i.e

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

We know,

$$W_4 = e^{-j\frac{2\pi}{4}} = -j$$

Then,

$W_4^1 = -j$ $\qquad$ $W_4^3 = j$

$W_4^2 = -1$ $\qquad$ $W_4^4 = 1$

$W_4^6 = -1$ $\qquad$ $W_4^9 = -j$

Hence,

Scanned with CamScanner

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5+3+2+2 \\ 5+j3-2+2j \\ 5-3+2-2 \\ 5+3j-2-2j \end{bmatrix}$$

Hence,

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 12 \\ 3-j \\ 2 \\ 3+j \end{bmatrix}$$

The required N- point DFT is,

$x[k] = \{12, 3-j, 2, 3+j\}$.

d. Determine the output of an RC low pass filter for a unity magnitude rectangular pulse extending from +0 to 2.

Sol^n, $\Rightarrow$

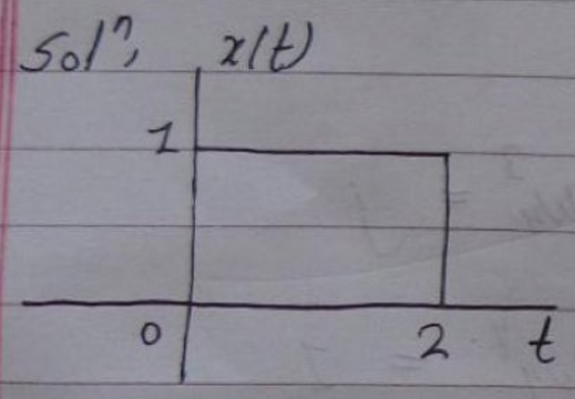


Fig: Rectangular pulse

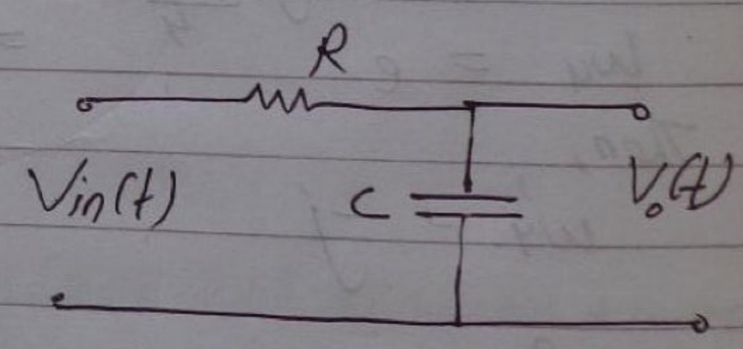


Fig: RC filter.

Impulse response of an RC filter is given by,

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t) \quad \text{--- (i)}$$

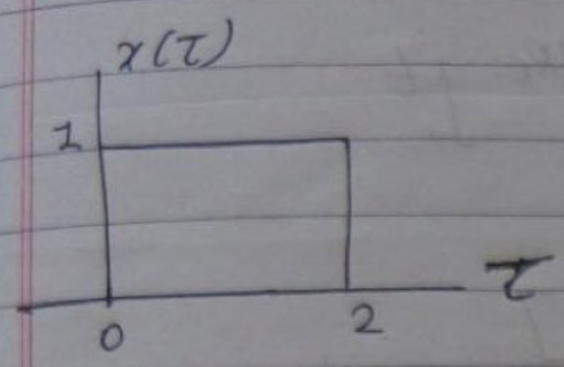
Scanned with CamScanner

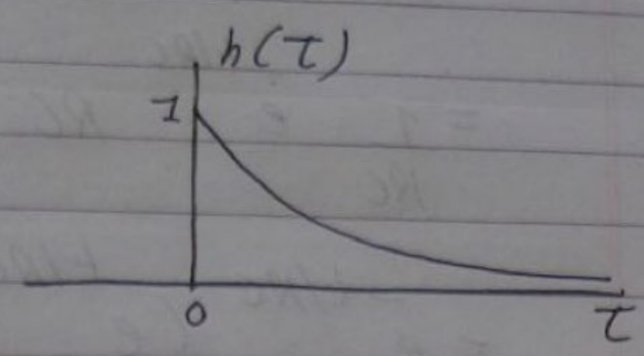
We have,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

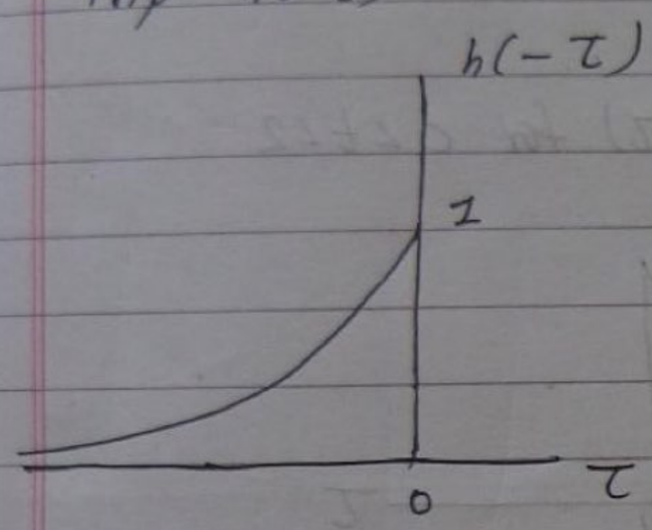
From ①, as a function of τ,

$$h(\tau) = \frac{1}{RC} e^{-\tau/RC} u(\tau)$$

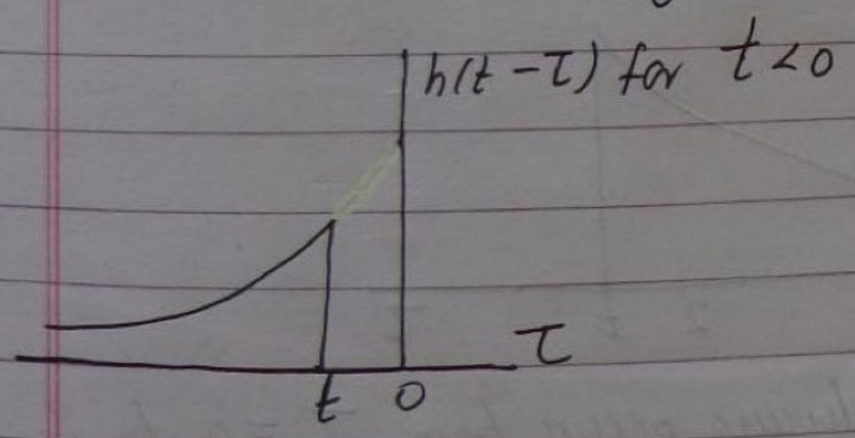




Flip $h(\tau)$



For $t<0$, $h(t-\tau)$ is the time shifted version of $h(-\tau)$ to the left by t time unit.



For $t<0$, $x(\tau) \times h(t-\tau) = 0$

Hence, $y(t) = 0$.

Scanned with CamScanner

For $0 < t < 2$,

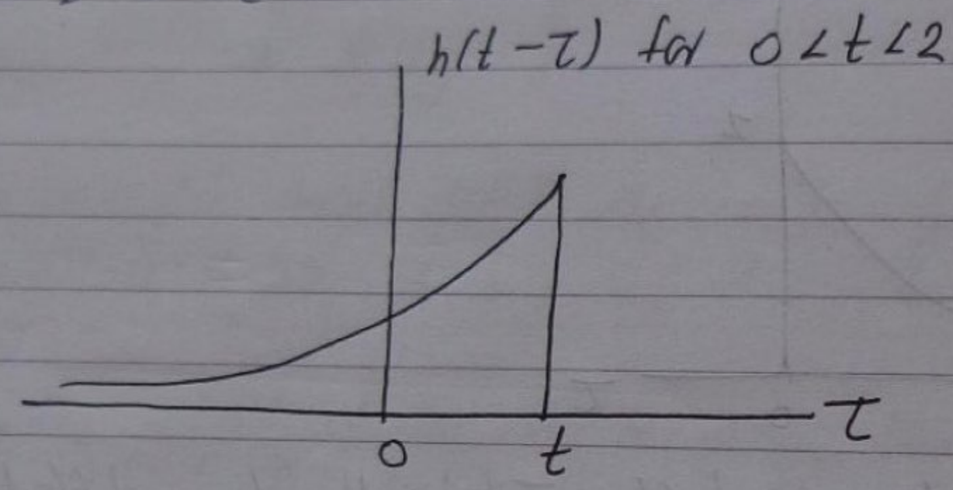
$$y(t) = \int_0^t \frac{1}{RC} e^{-(t-\tau)/RC} d\tau$$

$$= \frac{1}{RC} e^{-t/RC} \int_0^t e^{+\tau/RC} d\tau$$

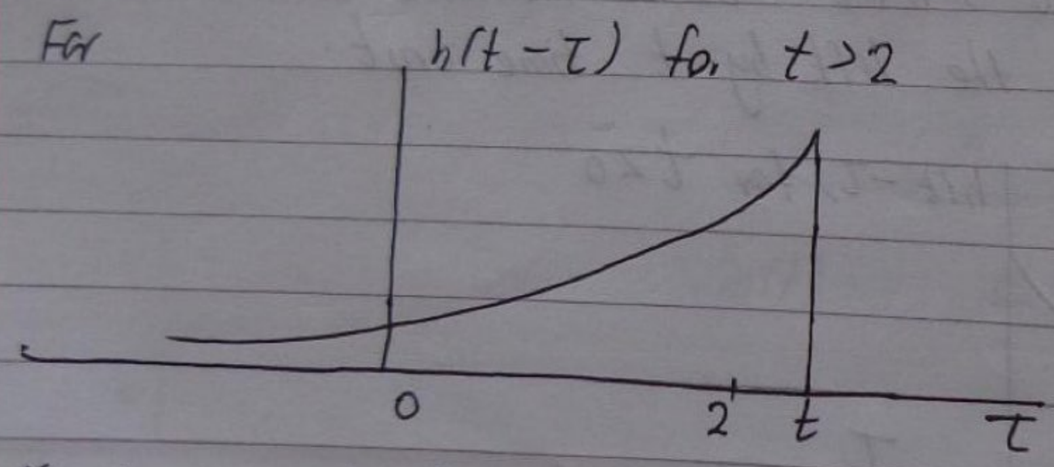
$$= \frac{1}{RC} e^{-t/RC} RC \left[e^{\tau/RC} \right]_0^t$$

$$= e^{-t/RC} \left[e^{t/RC} - e^0 \right]$$

$$\therefore \; y(t) = 1 - e^{-t/RC}$$



For



For $t > 2$, overlapping always occurs from $\tau = 0$ to 2.
Hence,

$$y(t) = \int_0^2 \frac{1}{RC} e^{-(t-\tau)/RC} d\tau$$

Scanned with CamScanner

$$= \frac{1}{RC} e^{-t/RC} \int_0^t e^{\tau/RC} d\tau$$

$$= \frac{1}{RC} e^{-t/RC} \cdot RC \left[e^{\tau/RC} \right]_0^2$$

$$\therefore y(t) = e^{-t/RC} \cdot \left(e^{2RC} - 1 \right)$$

Hence,

$$\text{output, } y(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-t/RC}, & 0 < t < 2 \\ e^{-t/RC} \left(e^{2/RC} - 1 \right), & t > 2 \end{cases}$$

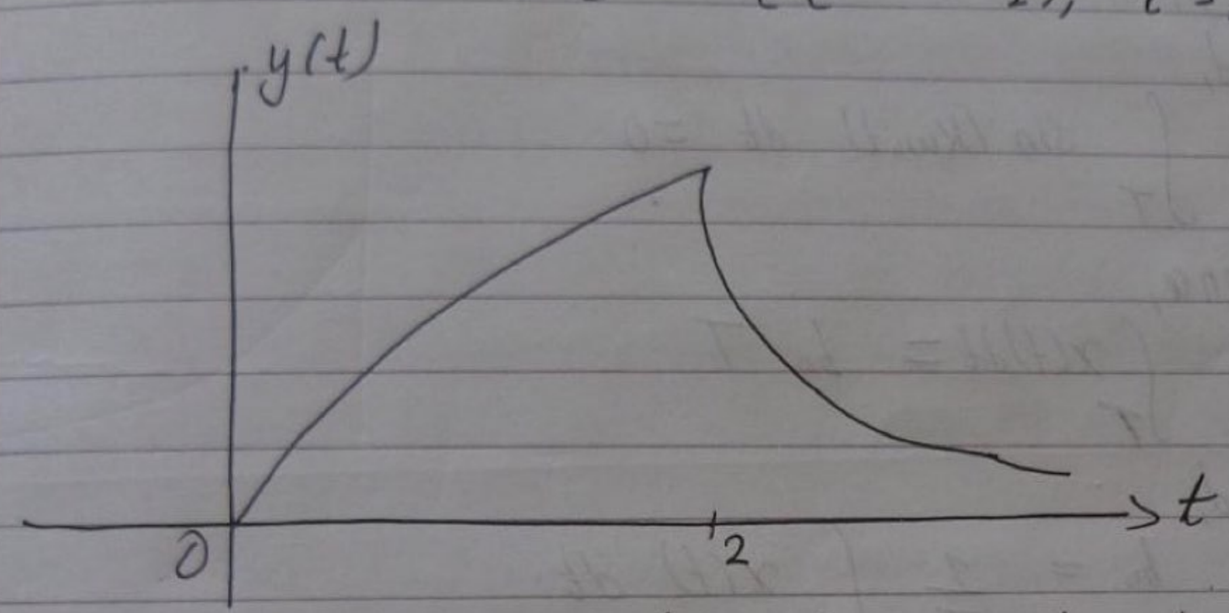


Fig: output of an RC filter for a rectangular pulse.

e: Trigonometric Fourier series:

A periodic signal $x(t)$ can be expressed by trigonometric Fourier series as,

$$x(t) = b_0 + \sum_{k=1}^{\infty} \left[b_k \cos(k\omega_0 t) + c_k \sin(k\omega_0 t) \right] \quad \text{—①}$$

Here fundamental period, $T = \frac{2\pi}{\omega_0}$ and b_k, c_k are

Scanned with CamScanner

Fourier series coefficients.

Evaluation of b_0 :
integrating both sides of eq^n ① over one period,

$$\int_T x(t)\,dt = \int_T b_0\,dt + \int \sum_{k=1}^{\infty} b_k \cos(k\omega_0 t) + c_k \sin(k\omega_0 t)]\,dt$$

$$= b_0 T + \sum_{k=1}^{\infty} b_k \int_T \cos(k\omega_0 t)\,dt + \sum_{k=1}^{\infty} c_k \int_T \sin(k\omega_0 t)\,dt$$

since,

$$\int_T \cos(k\omega_0 t)\,dt = 0$$

and,

$$\int_T \sin(k\omega_0 t)\,dt = 0$$

Hence,

$$\int_T x(t)\,dt = b_0 T$$

and,

$$b_0 = \frac{1}{T}\int_T x(t)\,dt.$$

This is dc component of the signal.
similarly,

$$b_k = \frac{2}{T}\int_T x(t)\cos(k\omega_0 t)\,dt$$

$$c_k = \frac{2}{T}\int_T x(t)\sin(k\omega_0 t)\,dt$$

If $x(t)$ is even, then,

Scanned with CamScanner

$$b_0 = \frac{2}{T}\int_0^{T/2} x(t)\, dt$$

$$b_k = \frac{4}{T}\int_0^{T/2} x(t)\cos(k\omega_0 t)\, dt$$

$$C_k = 0$$

If $x(t)$ is odd, then,

$$b_0 = 0$$
$$b_k = 0$$

$$\therefore C_k = \frac{4}{T}\int_0^{T/2} x(t)\sin(k\omega_0 t)\, dt.$$

E. Obtain the expression for convolution integral. 2016/s 2017/s

=> A continuous time LTI system is completely characterized by its impulse response, the output can be determined for any arbitary continuous time input signal from the knowledge of its impulse response.

In continuous time, an input signal $x(t)$ can be represented as a weighted integral of time shifted impulses as,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

Let, $h(t)$ be defined as the impulse response of the system for an impulse input $\delta(t)$. i.e,

$$\delta(t) \longrightarrow h(t)$$

Then using time invariance property, we have,

$$\delta(t-\tau) \longrightarrow h(t-\tau)$$

Scanned with CamScanner

Also, using linear property, the output of an LTI system is given as linear combination of time shifted impulse response given by,

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)\, x(\tau)\, d\tau \quad \text{— (ii)}$$

This eqⁿ is termed as convolution integral.

symbollically, we can write,

$$\therefore \quad y(t) = x(t) * h(t)$$

For discrete time,
convolution integral is given by,

$$y[n] = x[n] * h[n]$$

$$\therefore \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

F. For the lower order difference eqⁿ, $y[n] = \frac{1}{2} y[n] + 2x[n]$ characteristics LTI system. Find the system function and impulse response.

⇒ solⁿ,

$$y[n] = \frac{1}{2} y[n] + 2x[n]$$

we have,

$$y[n] \xleftrightarrow{F.T} x[n] * h[n]$$

let,

Scanned with CamScanner

$x[n] \xleftrightarrow{F.T} X(e^{j\omega})$

$h[n] \xleftrightarrow{F.T} H(e^{j\omega})$

and,

$y[n] \xleftrightarrow{F.T} Y(e^{j\omega})$

Applying convolution property of Fourier transform,

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

Hence,

system function, $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$

So,

$$y[n] - \frac{1}{2} y[n] = 2x[n] \quad \text{--- (1)}$$

or, $y[n]\left[1 - \frac{1}{2}\right] = 2\,x[n]$

or, $\dfrac{y[n]}{x[n]} = \dfrac{2 \times 2}{}$

$\therefore \dfrac{y[n]}{x[n]} = 4$

Taking Z-transform,

$$\frac{H(z)}{} = \frac{Y(z)}{X(z)}$$

$$\therefore H(z) = 4$$

From (1),

$$Y(e^{j\omega}) - 0.5\,e^{-}\,Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$\therefore H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$= \frac{2}{0.5} = 4$$

Scanned with CamScanner

G Find the fourier series coefficients and plot the magnitude and phase of the signal. 2014/F

Sol^n:

$\Rightarrow$ Given,

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

It has fundamental frequency ω_0.

Expanding $x(t)$ directly in terms of complex exponentials,

$$\text{or } x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\pi\omega_0 t + \pi/4)}\right]$$

Collecting terms, we get,

$$\text{or } x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}$$

Thus,

Fourier series coefficients are,

$a_0 = 1$

$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j$

$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1+j)$

$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j$

$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1-j)$

$a_k = 0, \ |k| > 2$

Scanned with CamScanner

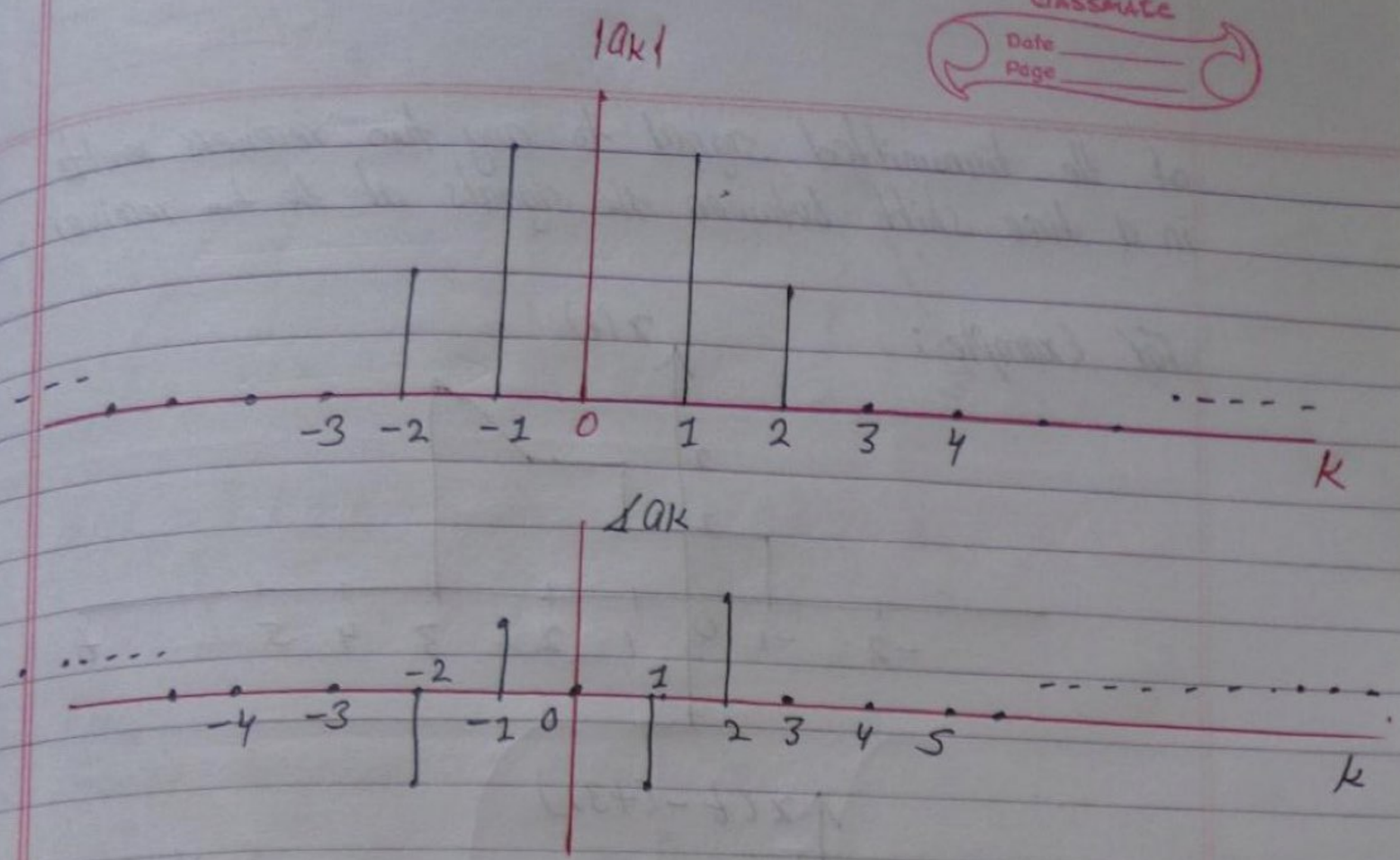


Fig: magnitude & phase plot of $x(t)$.

11. Effect of delay operation on signal: 2016 ls

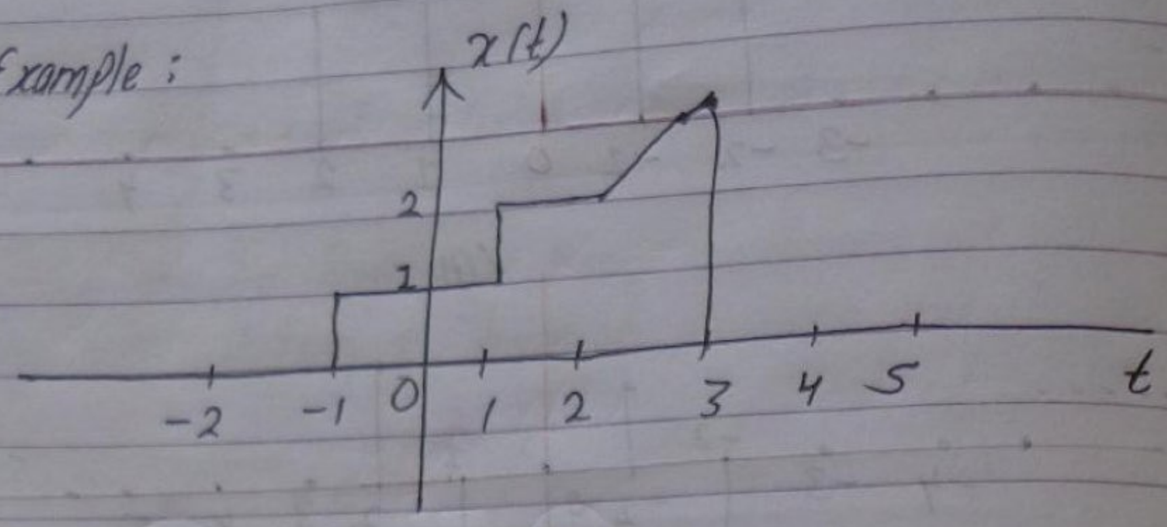
Time shifting of signals is probably the most important one and most widely used among all basic signal operation. It is generally used to fast-forward or delay a signal, as is necessary in most practical circumstances. Time shifting is mathematically expressed as, $y(t) = x(t - t_0)$, where $x(t)$ is the original signal and t_0 represents the shift in time. For a signal, $x(t)$, if the position shift $t_0 > 0$, then the signal is said to be right shifted or delayed. similarly, if $t_0 < 0$, it implies that the signal is left shifted or delayed.

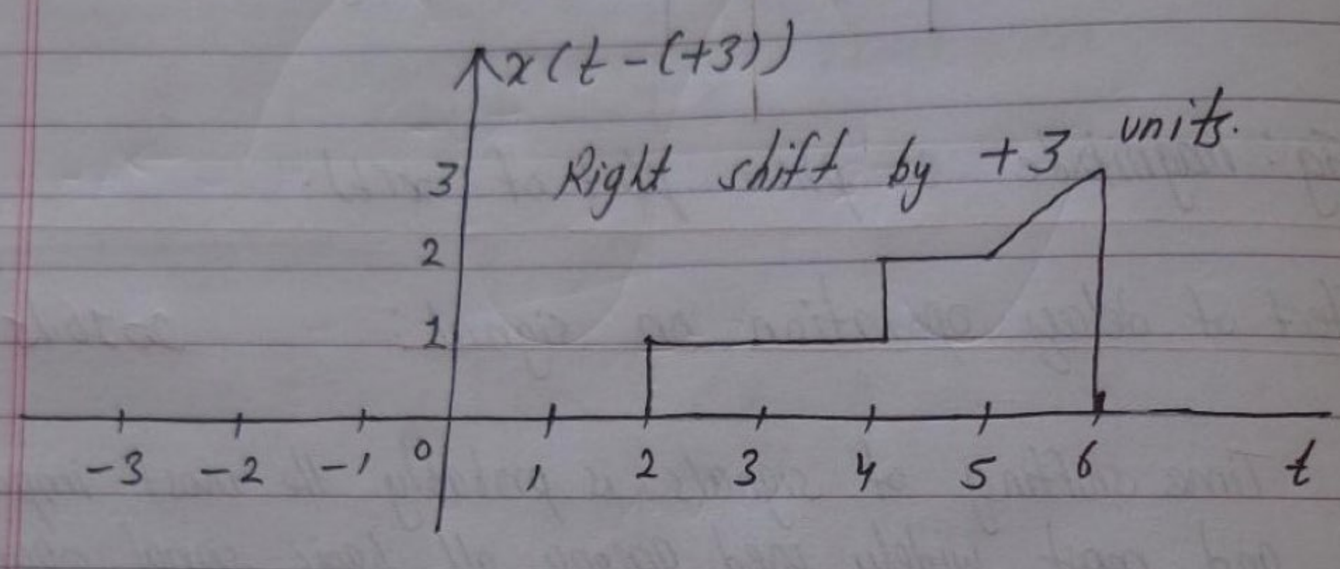
signals that are related in this fashion arise in applications such as radar, sonar, and seismic signal processing, in which several receivers at different locations observe a signal being transmitted through a medium (water, rock, air etc). In this case, the difference in propagation time from the point of origin

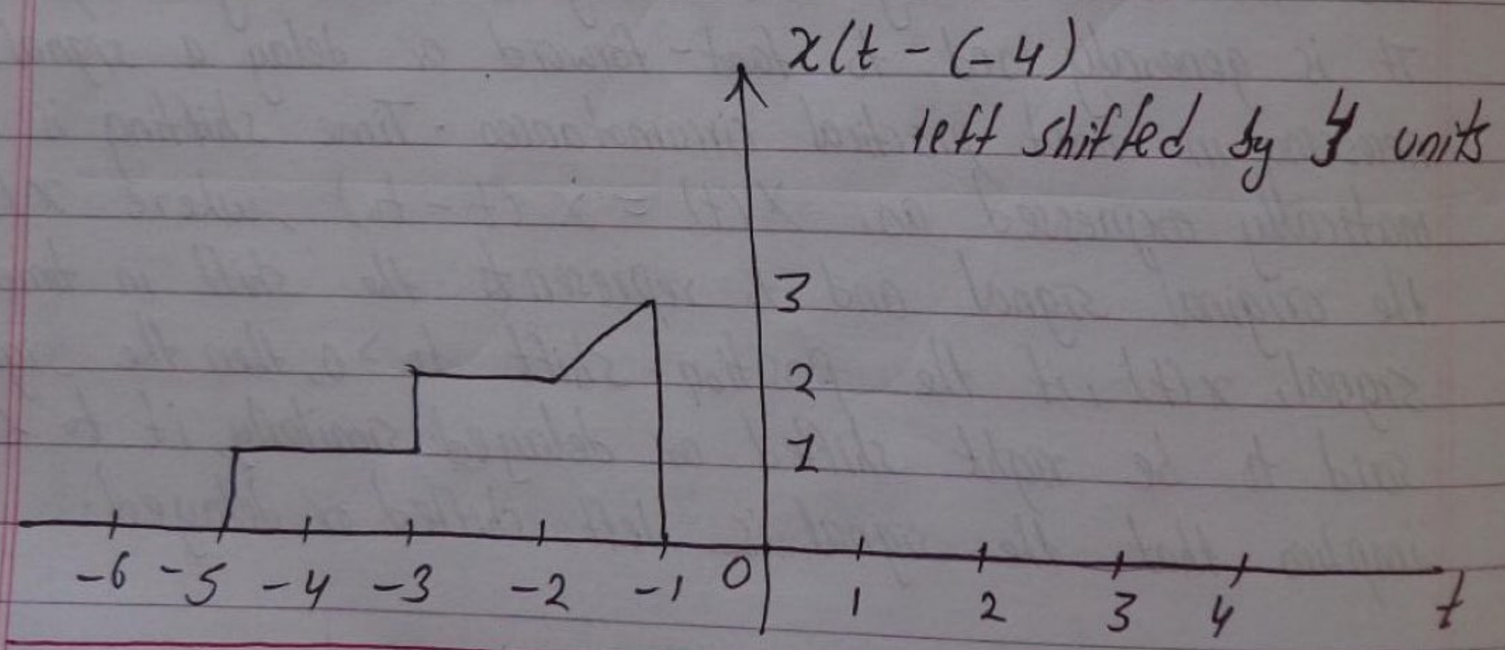
Scanned with CamScanner

of the transmitted signal to any two receivers results in a time shift between the signals at the two receiver.

For Example :



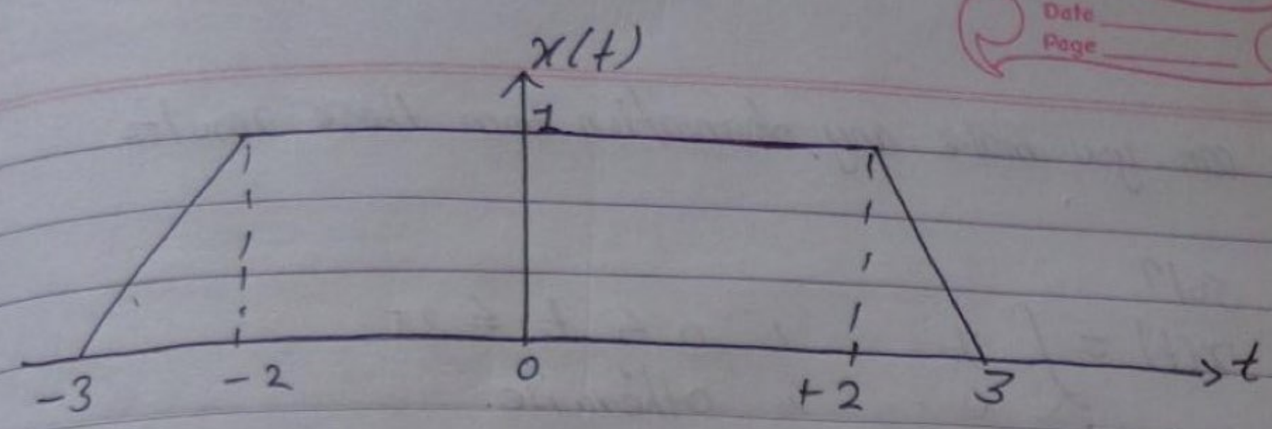




I. Find the fourier transform of the following signal x(t). 2011/F

=> sol^n,

Scanned with CamScanner



$\Rightarrow$

$$x(t) = \begin{cases} t+3 & \text{for } -3 < t < -2 \\ 1 & \text{for } -2 < t < 2 \\ 3-t & \text{for } 2 < t < 3 \end{cases}$$

Now,
Fourier transform of $x(t)$ is,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-3}^{-2} (t+3)\, e^{-j\omega t}\, dt + \int_{-2}^{2} 1 \cdot e^{-j\omega t}\, dt + \int_{2}^{3} e^{-j\omega t}(3-t)\, dt$$

$$= \left[\frac{t^2}{2} \frac{e^{-j\omega t}}{-j\omega}\right]_{-3}^{-2} + 3\frac{e^{-j\omega t}}{-j\omega}\Big|_{-3}^{-2} + \frac{e^{-j\omega t}}{-j\omega}\Big|_{-2}^{2} + \left[3\frac{e^{-j\omega t}}{-j\omega} - \frac{t^2}{2} \cdot e^{-j\omega t}\right]_{2}^{3}$$

= solve yourself :

J. Given a pair of signal $x(t)$ & $y(t)$ such that,

$$x(t) = \begin{cases} \sin t & , 0 \le t \le 2\pi \\ 0 & , \text{otherwise} \end{cases}$$

2014/5

$$y(t) = \begin{cases} 1 & , 0 \le t \le 2\pi \\ 0 & , \text{otherwise} \end{cases}$$

(i) sketch the signals $x(t)$ and $y(t)$.
(ii) Find the energies of the signals $x(t) + y(t)$ & $x(t) - y(t)$

Scanned with CamScanner

can you make any observation from these results.

=> soln

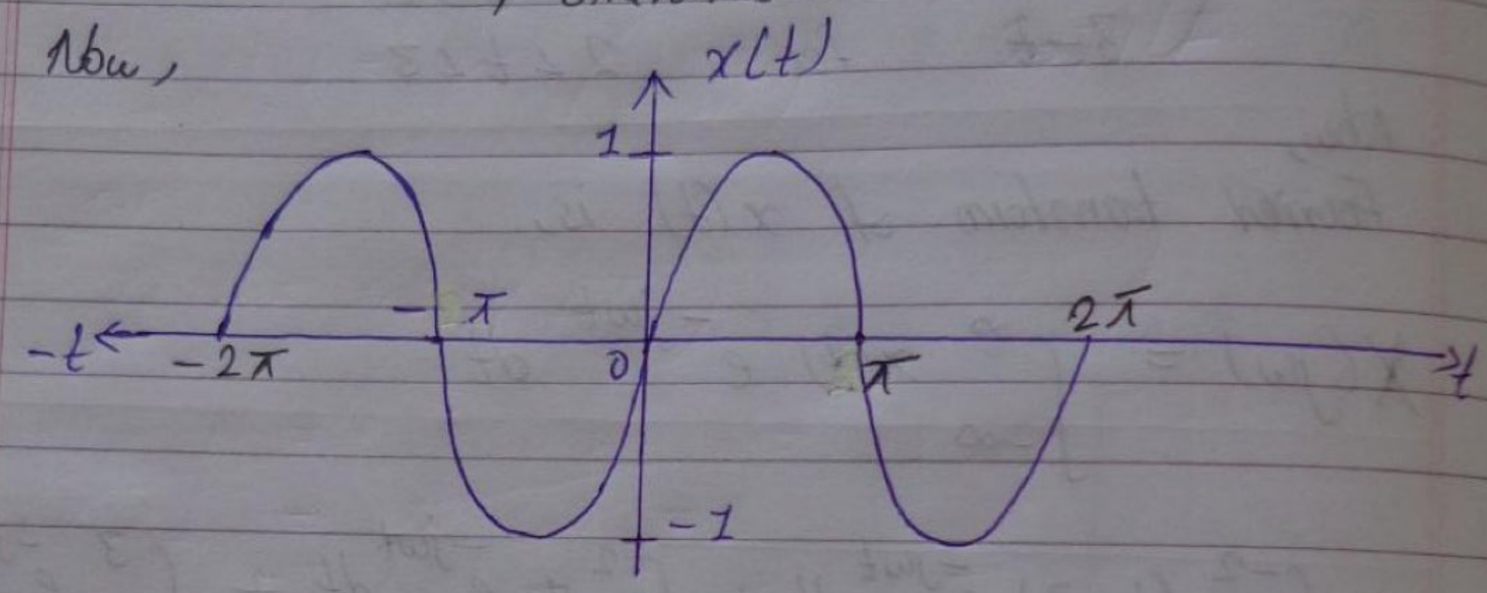
$$x(t) = \begin{cases} 1 & , 0 \le t \le 2\pi \\ 0 & , \text{otherwise} \end{cases}$$

and

$$y(t) = \begin{cases} 1 & , 0 \le t \le 2\pi \\ 0 & , \text{otherwise} \end{cases}$$

Now,

(i)



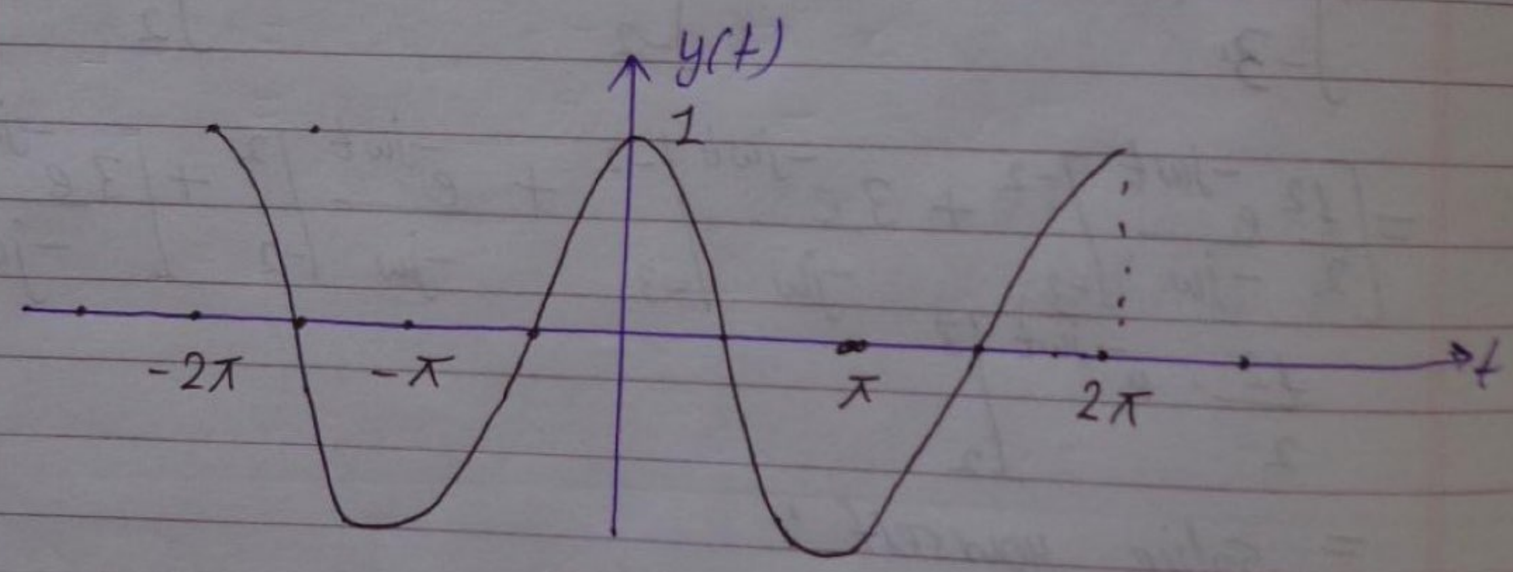


Fig: plot of $x(t)$ and $y(t)$

(ii) Energies of signals :

$x(t) + y(t) = ?$

we have,

Energy of signal $[x(t)] = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$

Scanned with CamScanner

$$= \int_{0}^{2\pi} \sin t \, dt$$

$$= 0$$

and,

$$\text{Energy}, [y(t)] = \int_{-\infty}^{\infty} |y(t)|^2 \, dt$$

$$= \int_{0}^{2\pi} 1 \cdot dt$$

$$= 2\pi$$

So,

$\therefore$ Energy of $[x(t) + y(t)] = 2\pi + 0 = 2\pi$

and,

$\therefore$ Energy of $[x(t) - y(t)] = 2\pi - 0 = 2\pi$

K. Determine whether the given signal is periodic or not. 2014/5 2017/5 (1.a eg):

$$x(t) = \begin{cases} \cos t & , t < 0 \\ \sin t & , t \geq 0 \end{cases}$$

Sol^n.

$$\Rightarrow E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{0} \cos t \, dt + \int_{0}^{\infty} \sin t \, dt$$

$$= \Big[+\sin t \Big]_{-\infty}^{0} + \Big[-\cos t \Big]_{0}^{\infty}$$

$\therefore E_{av} = \infty$

and

average power of the signal, is,

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

$$= \lim_{T \to \infty} \frac{E}{\infty}$$

Scanned with CamScanner

$\therefore P_{av} = \infty$

Since, both total energy and total average power of given $x(t)$ is ∞, hence given signal is neither energy nor power signal.

2. Find the 4-point DFT of the following signal using DIT - FFT,

$$x[n] = \cos\left(\frac{n\pi}{4}\right)$$ 2014/F

Sol^n,

$\Rightarrow$ $x[n] = \cos\left(\frac{n\pi}{4}\right)$

Here, $N = 4$, and $n = 0, 1, 2, 3$

$x[0] = \cos(0) = 1$

$x[1] = \cos(\pi/4) = 1/\sqrt{2}$

$x[2] = \cos(2\pi/4) = 0$

$x[3] = \cos(3\pi/4) = -1/\sqrt{2}$

$\therefore x[n] = \left\{1, \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right\}$.

Using linear transformation method,

$$X_N = W_N \cdot x_N.$$

or,
$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

we have,

$$W_4 = W_4^1 = e^{-j\frac{2\pi}{N}} = -j$$

Scanned with CamScanner

$W_4^2 = -1$ $W_4^3 = j$ $W_4^4 = 1$ $W_4^6 = -1$

$W_4^9 = -j$

So,

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 1/\sqrt{2} + 0 - 1/\sqrt{2} \\ 1 - j\,1/\sqrt{2} + 0 - j/\sqrt{2} \\ 1 - 1/\sqrt{2} + 0 + 1/\sqrt{2} \\ 1 + j\,1/\sqrt{2} + 0 + j\,1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 - j\sqrt{2} \\ 1 \\ 1 + j\sqrt{2} \end{bmatrix}$$

Hence,

$X(k) = \{1, 1 - j\sqrt{2}, 1, 1 + j\sqrt{2}\}$.

M. State and prove convolution property of discrete time LTI transform. 2014/5.

=> If $x[n] \xleftrightarrow{F.T} X(e^{j\omega})$

and

$h[n] \xleftrightarrow{F.T} H(e^{j\omega})$

then,

$y[n] = x[n] * h[n] \xleftrightarrow{F.T} Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$

Proof:

Scanned with CamScanner

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \right] e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \left[\sum_{n=-\infty}^{\infty} h[n-k]\, e^{-j\omega n} \right]$$

Using time-shifting property,

$$\sum_{n=-\infty}^{\infty} h[n-k]\, e^{-j\omega n} = e^{-j\omega k}\, H(e^{j\omega})$$

So,

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}\, H(e^{j\omega})$$

$$= H(e^{j\omega}) \sum_{k=-\infty}^{\infty} x[k]\, e^{-j\omega k}$$

$$\therefore \quad = H(e^{j\omega}) \cdot H(e^{j\omega})$$

N. Determine the Fourier transform of the Gaussian pulse defined as :

$$x(t) = e^{-a^2 t^2}$$

2014/5

Sol^n;

$\Rightarrow$ Given signal is,

$$x(t) = e^{-a^2 t^2} \quad \text{—— (1)}$$

we have Fourier transform of $x(t)$ is,

Scanned with CamScanner

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-a^2t^2}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-(a^2t^2 + j\omega t)}\, dt$$

let,

$$a^2t^2 + j\omega t \Rightarrow (at)^2 + \frac{2atj\omega}{2a} + \left(\frac{j\omega}{2a}\right)^2 - \left(\frac{j\omega}{2a}\right)^2$$

$$= \left(at + \frac{j\omega}{2a}\right)^2 - \left(\frac{j\omega}{2a}\right)^2$$

$$= \left(at + \frac{j\omega}{2a}\right)^2 + \frac{\omega^2}{4a}$$

Now,

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-\left[\left(at + \frac{j\omega}{2a}\right)^2 + \frac{\omega^2}{4a}\right]}\, dt$$

$$= \int_{-\infty}^{\infty} e^{\frac{-\omega^2}{4a}} \cdot e^{-\left(at + \frac{j\omega}{2a}\right)^2}\, dt.$$

let,

$$at + \frac{j\omega}{2a} = u$$

then, $dt = \frac{1}{a} \cdot du$

So,

$$X(j\omega) = e^{-\omega^2/4a} \cdot \frac{1}{a} \int_{-\infty}^{\infty} e^{-u^2}\, du$$

Since, e^{-u^2} is even function,

$$X(j\omega) = \frac{1}{a} \cdot e^{-\omega^2/4a} \int_{0}^{\infty} e^{-u^2}\, du$$

Scanned with CamScanner

We know, $\int_0^\infty e^{-v^2} dv = \frac{\sqrt{\pi}}{2}$

Hence,
$$X(j\omega) = \frac{1}{a} \cdot e^{-\omega^2/4a} \cdot \frac{\sqrt{\pi}}{2}$$

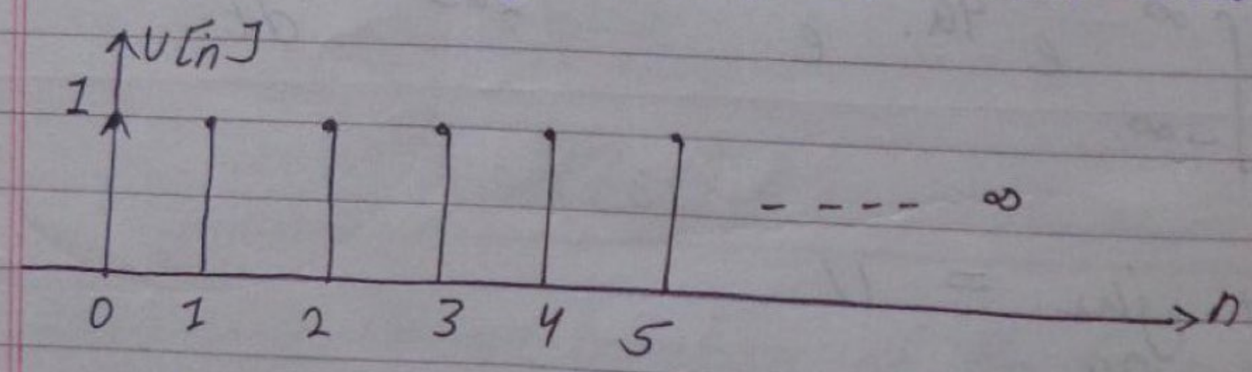
$$\therefore \quad X(j\omega) = \frac{1}{a} \cdot \frac{\sqrt{\pi}}{2} \cdot e^{-\omega^2/4}$$

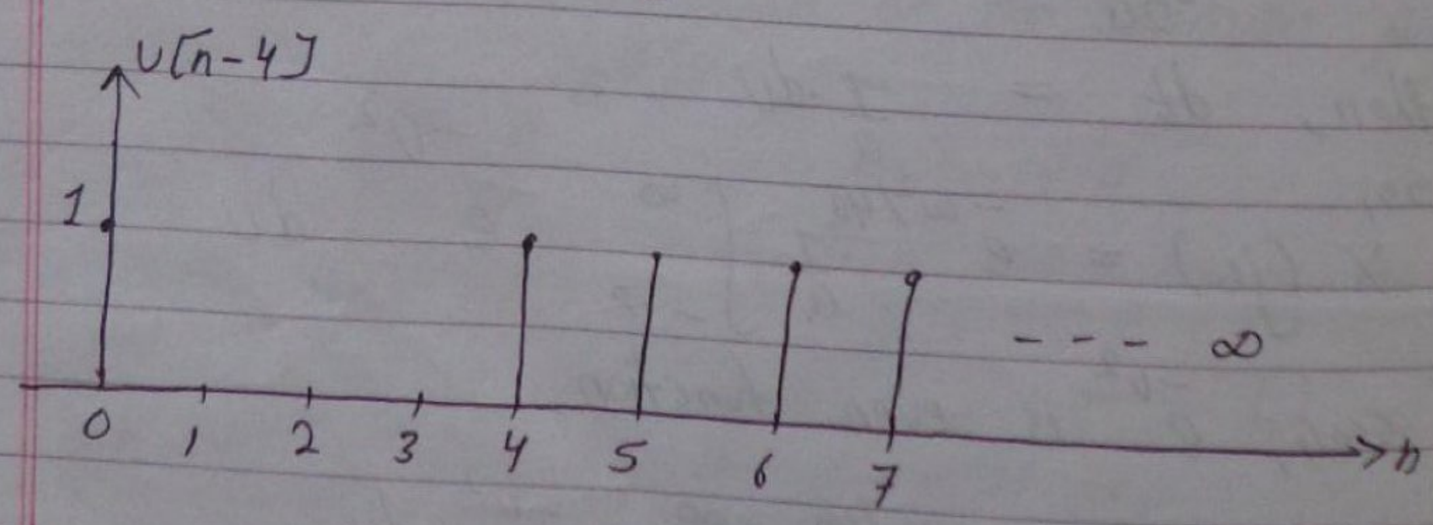
This is required Fourier transform of given gaussian pulse.

Q. Find circular convolution between $x_1[n] = \{1, 2\}$ and $x_2[n] = u[n] - u[n-4]$ 2015/5, 2017/5

Sol^n,

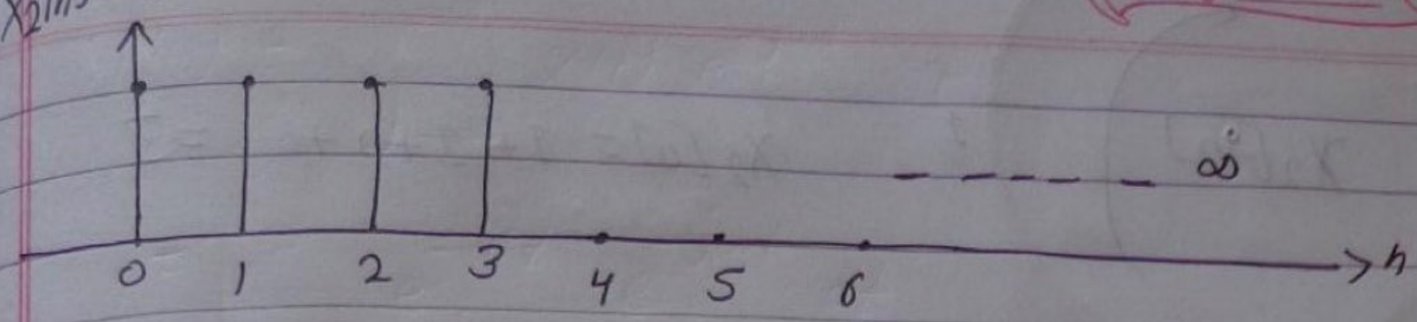
$\Rightarrow$ To find out circular convolution between $x_1[n]$ & $x_2[n]$ first of all we have to find out the elements $x_2[n]$. The steps of finding $x_2[n]$ are shown below:





Scanned with CamScanner

$x_2[n] = u[n] - u[n-4]$



So,

$x_2[n] = \{1, 1, 1, 1\}$.

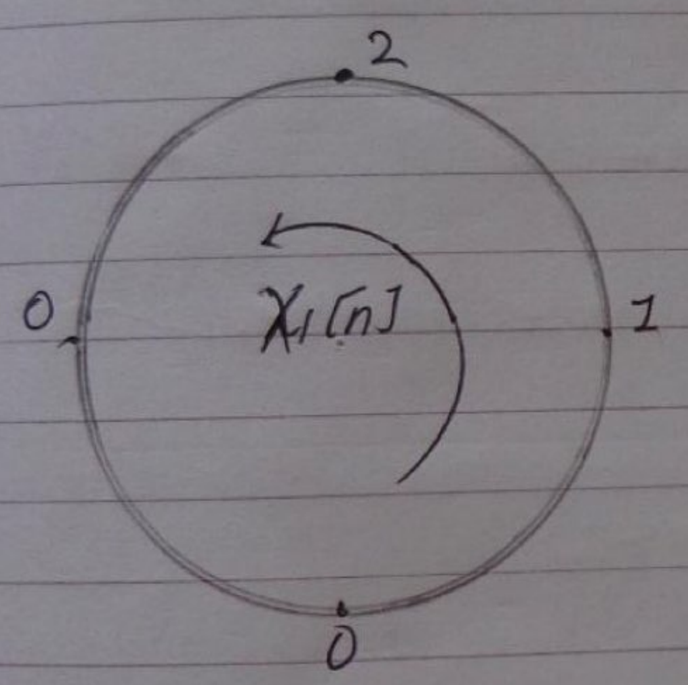
Since length of $x_1[n]$ & $x_2[n]$ are not equal, we need to pad zeros to the sequence $x_1[n]$.

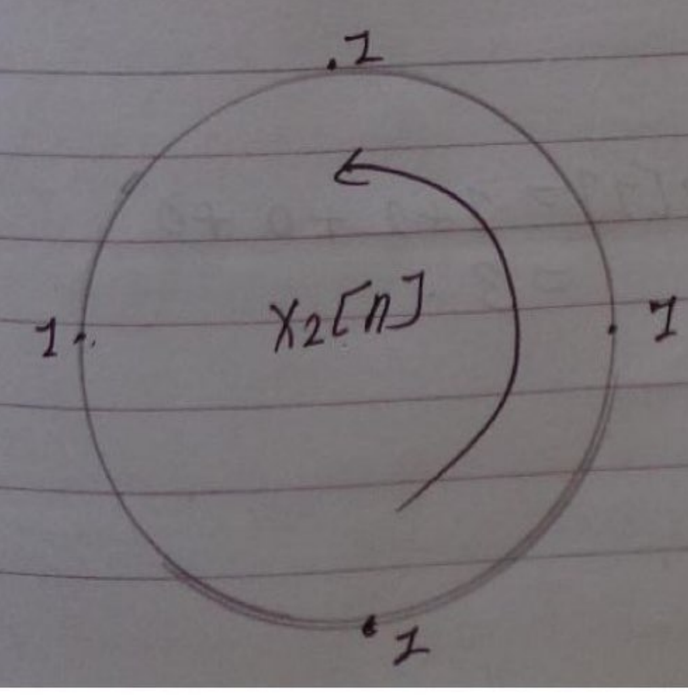
So, $x_1[n] = \{1, 2, 0, 0\}$.

let the circular convolution of $x_1[n]$ and $x_2[n]$ is $x_3[m]$

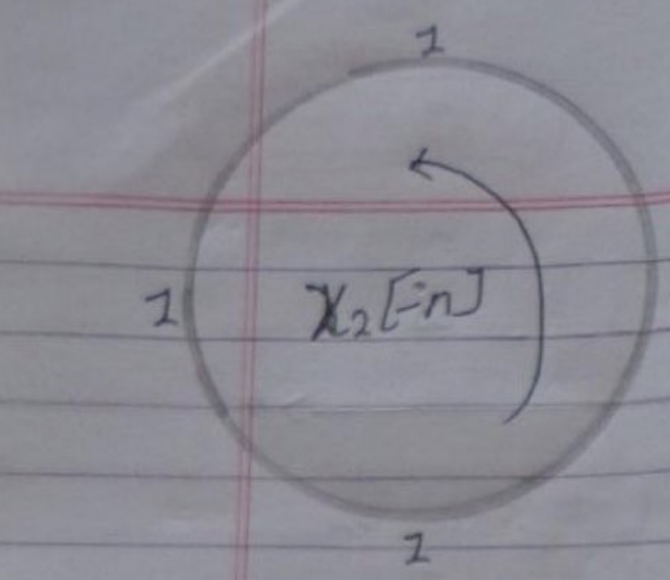
Thus,

$$x_3[m] = \sum_{n=0}^{N-1} x_1[n]\, x_2[(m-n)]_4 \quad , \quad m = 0, 1, 2, 3, \ldots N-1$$

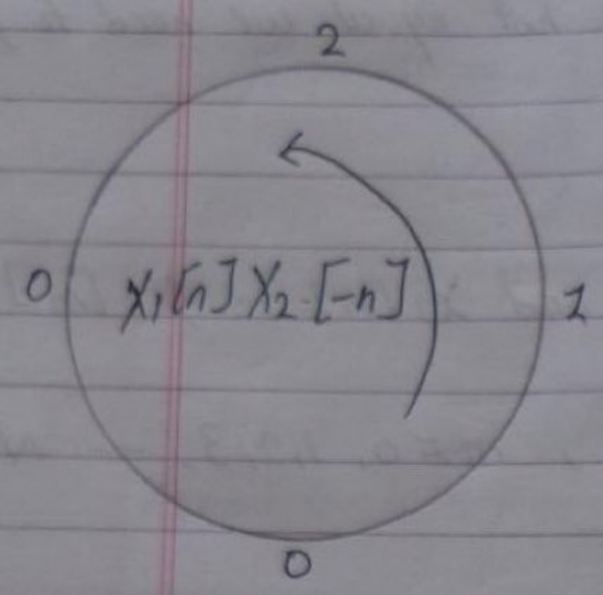




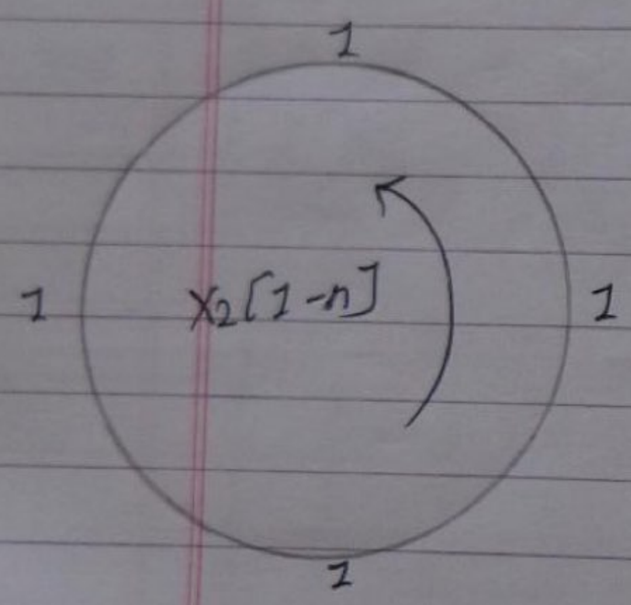
Scanned with CamScanner

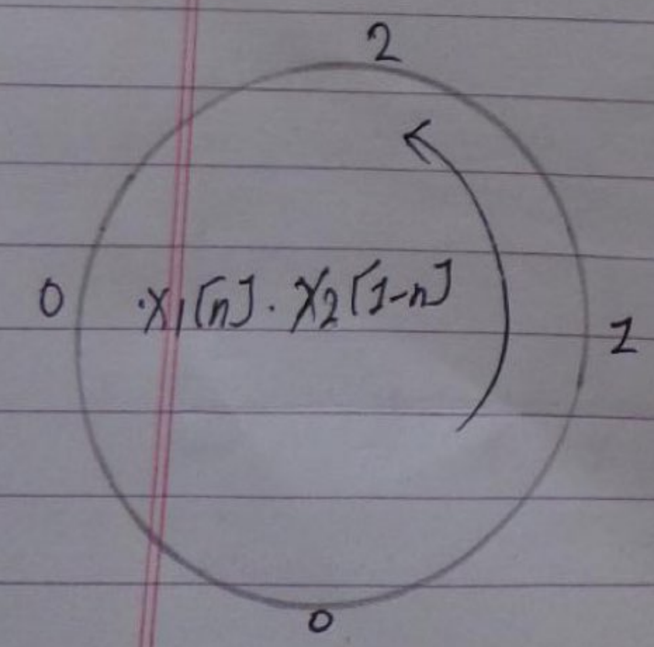


$x_2[0] = 1+2+0+0 = 3$



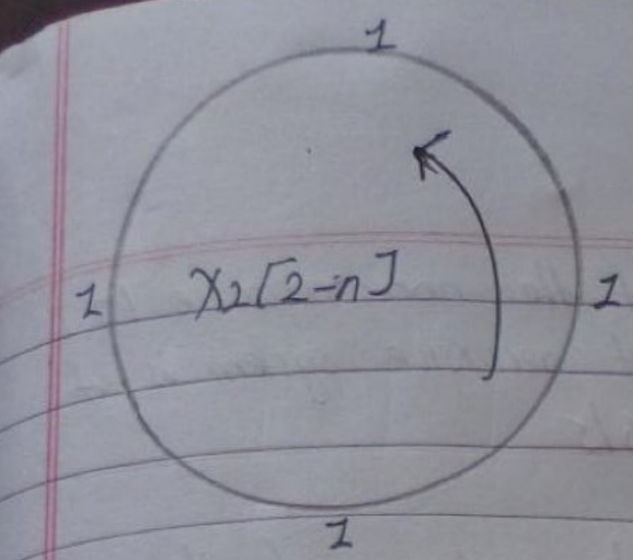
$x_3[1] = 1+2+0+0$
$= 3.$



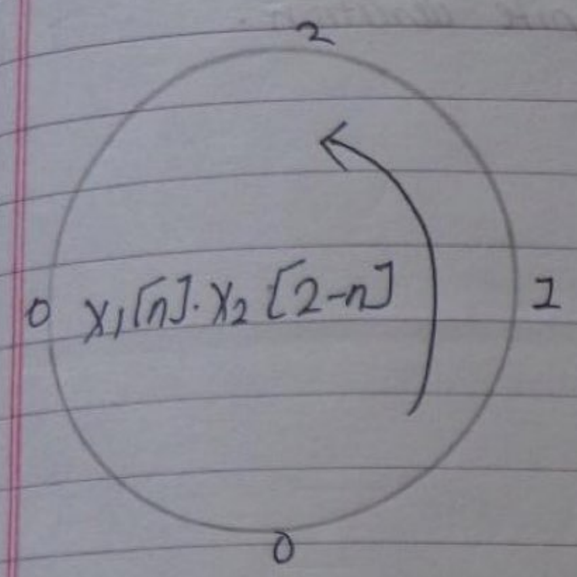


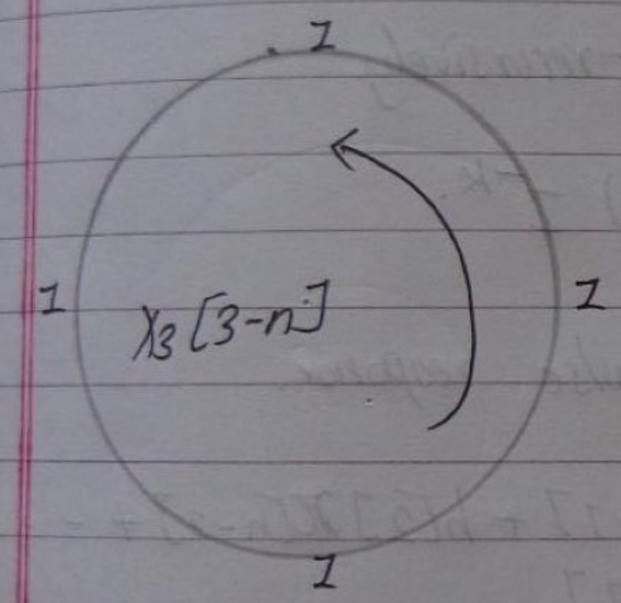
$\therefore x_3[1] = 1+2+0+0$
$= 3.$

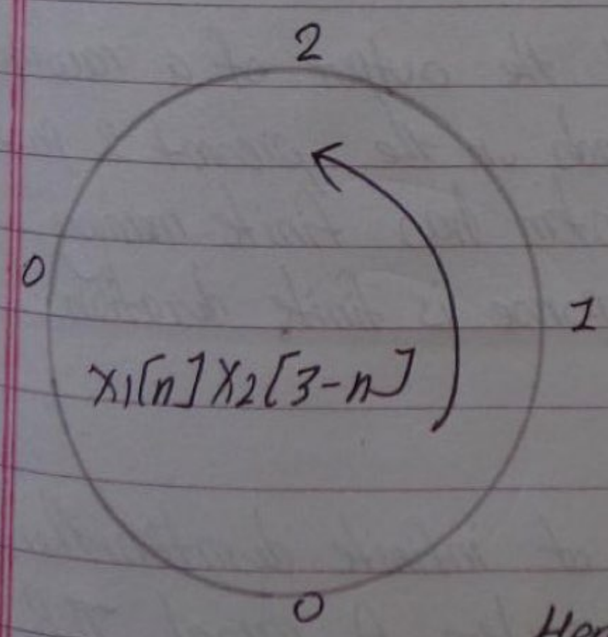
Scanned with CamScanner



$x_3[2] = 1+2+0+0$
$= 3$







$x_3[3] = 1+2+0+0$
$= 3.$

Hence, $x_3[m] = \{3, 3, 3, 3\}$.

Scanned with CamScanner

P. Prove that the impulse response of the non-recursive LTI system is of finite duration but recursive system is of infinite duration. 2016/5

=> We have to prove that impulse response of the non-recursive LTI system is of finite duration.

Finite impulse response:

$$H[z] = \sum_{k=0}^{N-1} h[k]\, z^{-k}$$

And

$$y[n] = \sum_{k=0}^{N-1} h[k]\, x[n-k]$$

$h[n]$: finite duration, non-recursively.

And,

$$Y(z) = \sum_{k=0}^{N-1} h[k]\, X(z)\, z^{-k}.$$

where, N is the length of impulse response.

$$y[n] = h[0]\,x[n] + h[1]\,x[n-1] + h[2]\,x[n-2] + \dots + h[N-1]\,x[n-m+1]$$

This difference eq^n tell us that the output of a causal FIR system at any instant depends on the present & past values. Thus, a causal FIR system has finite memory length N-1. Hence, impulse response is finite duration.

Infinite impulse Response:

If the impulse response is of infinite duration, then the system is known as IIR system. A causal IIR system can be expressed as,

Scanned with CamScanner

$$y[n] = \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

$$= h[0]\,x[n] + h[1]\,x[n-1] + h[2]\,x[n-2] + \cdots + h[\infty]\,x[n-\infty]$$

Here, the impulse response is of infinite duration and IIR system requires infinite memory length.

$$H(z) = \frac{\sum_{k=0}^{M} b_i z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}}$$

and,

$$y[n] = \sum_{i=0}^{M} b_i\, x(n-i) - \sum_{i=1}^{N} a_i\, y(n-i)$$

$$= \sum_{k=0}^{\infty} h[k]\, x[n-k]$$

Here,

$h[n]$ = infinite length, Recursively.

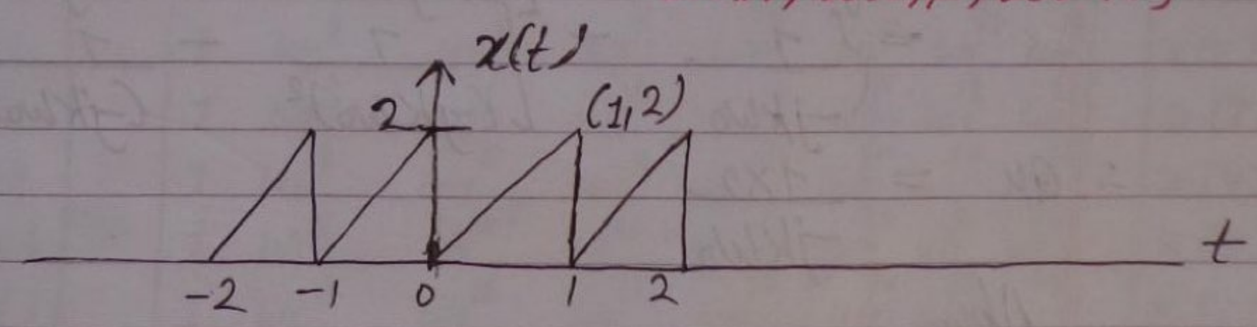
---

① Determine the exponential Fourier series of the following wave form. plot the magnitude & phase spectrum.

2014/F, 2013/F, 2015/F,

Sol^n

⇒



⇒ Here, $T = 1$ sec

$$a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

Here,

Scanned with CamScanner

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or $$y - 0 = \frac{2-0}{1-0}(t-0)$$

$$\therefore y = x(t) = 2t$$

Thus,

$$a_k = \frac{1}{T}\int_0^1 +2t\, e^{-jk\omega_0 t}\, dt$$

or $$a_k = \int_0^1 2t\, e^{-jk\omega_0 t}\, dt$$

$$= \left[\frac{t e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^1 - \int \frac{dt}{dt}\left(\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right) dt \Big] \times 2$$

$$= \left[\frac{t e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^1 - \left[\frac{e^{-jk\omega_0 t}}{(-jk\omega_0)(-jk\omega_0)}\right]_0^1 \times 2$$

$$= \left\{\frac{e^{-jk\omega_0}}{-jk\omega_0} - 0 - \left[\frac{e^{-jk\omega_0}}{(-jk\omega_0)^2} - \frac{1}{(-jk\omega_0)^2}\right]\right\} \times 2$$

$$= \left\{\frac{e^{-jk2\pi}}{-jk\omega_0} - \left[\frac{e^{-jk2\pi}}{(-jk\omega_0)^2} - \frac{1}{(-jk\omega_0)^2}\right]\right\} \times 2$$

$$= \left\{\frac{1}{-jk\omega_0} - \left[\frac{1}{(-jk\omega_0)^2} - \frac{1}{(-jk\omega_0)^2}\right]\right\} \times 2$$

$$\therefore a_k = \frac{1 \times 2}{-jk\omega_0}$$

Now,

DC component $a_0 = \int_T x(t)\, dt$

$$= \int_0^1 2t \cdot dt$$

Scanned with CamScanner

$$= \left[\frac{t^2 \times 2}{2}\right]_0^2$$

$$= 1 - 0$$

$$\therefore \quad l_{eo} = 1$$

R. Determine the 8-point DFT of the following sequences.

$x[n] = 1$ for $-3 \le n \le 3$

$= 0$ for otherwise 2016/5

$\Rightarrow$ Sol^n,

$$x[n] = \begin{cases} 1 & \text{for } -3 \le n \le 3 \\ 0 & \text{for else where,} \end{cases}$$

$x[n] = \{-3, -2, -1, 0, 1, 2, 3\}$

Zero-padding to $x[n]$ to make its length 8

$\therefore x[n] = \{-3, -2, -1, 0, 1, 2, 3, 0\}$.

Now,

using linear transformation,

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ 1 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ 1 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ 1 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ 1 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ 1 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \times \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

Now, $W_8^1 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} j$

Scanned with CamScanner

$W_8^2 = -j$

$W_8^3 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$

$W_8^4 = -1$

$W_8^5 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$

$W_8^6 = 1j$

$W^7 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$

$W^8 = 1$

$W^{10} = -j$

$W^9 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$

$W^{12} = -1$

$W^{14} = j$

$W^{15} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$

$W^{16} = 1$

$W^{18} = -j$

$W^{21} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j$

$W^{20} = -1$

$W^{24} = 1$

$W^{25} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$

$W^{28} = -1$

$W^{30} = j$

$W^{35} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$

$W^{36} = -1$

$W^{42} = -j$

$W^{49} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j$

Scanned with CamScanner

Thus,

$$\dot{X}(0) = 1\times(-3) - (2\times 1) + 1\times(-1) + 1\times 0 + 1\times 1 + 1\times 2 + 1\times 3 + 1\times 0$$
$$= -3 - 2 - 1 + 0 + 1 + 2 + 3 + 0$$
$$= 0$$

$$x[1] = 1\times(-3) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right)\times(-2) + (-1)\times(-j) + \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right)\times 0$$
$$+ 1\times(-1) + 2\times j + 3\times\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)$$

= Continue similarly like 4-point DFT.

5. For the lower order difference eq^n, $y[n] = \frac{1}{2}y[n-1] + 2x[n]$. characteristics LTI system. Find the system function and impulse response.

2016/5

Sol^n,

=> Given,

$$y[n] = \frac{1}{2}y[n-1] + 2x[n].$$

Taking Z-transform,

or, $Y(z) = \frac{1}{2}z^{-1}X(z) + 2X(z)$

or $Y(z)\left[1 - \frac{1}{2}z^{-1}\right] = 2X(z)$

or $\frac{Y(z)}{X(z)} = \frac{2}{1-0.5z^{-1}}$

i.e,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-0.5z^{-1}}$$

is the transfer function or system function.

Scanned with CamScanner

For impulse response,

$$H(z) = 2 \times \frac{1}{1-0.5z^{-1}}$$

We have,

$$\frac{1}{1-\alpha \cdot z^{-1}} \xrightarrow{\text{transform}} \alpha^n \cdot u[n] \text{ for } |z| > |\alpha|$$

and,

$$\frac{1}{1-\alpha \cdot z^{-1}} \xrightarrow{\text{transform}} -\alpha^n u[-n-1] \text{ for } |z| < |\alpha|$$

Thus,

$$\therefore h[n] = 2\left(+\frac{1}{2}\right)^n u[n] \quad \text{for } |z| > |\alpha|$$

and

$$\therefore h[n] = -2\left(+\frac{1}{2}\right)^n u[-n-1] \text{ for } |z| < |\alpha|$$

T. obtain the direct form I and Direct form II realization of the IIR system with transfer function,

$$H(z) = \frac{3(0.33z^2 + z + 1)}{(2z+1)(z+2)}$$

2016/5

Given,

$$\Rightarrow H(z) = \frac{0.99z^2 + 3z + 3}{2z^2 + 5z + 2}$$

$$H(z) = \frac{Y(z)}{X(z)}$$

Thus,

$$H(z) = \frac{z^{+2}[0.99 + 3z^{-1} + 3z^{-2}]}{z^{+2}[2 + 5z^{-1} + 2z^{-2}]}$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{3(0.33 + z^{-1} + z^{-2})}{2 + 5z^{-1} + 2z^{-2}}$$

Scanned with CamScanner

Expressing in general form, dividing numerator & denominator by 2,

$$\text{or } \frac{Y(z)}{X(z)} = \frac{1.5(0.33 + z^{-1} + z^{-2})}{1 + 2.5z^{-1} + z^{-2}} \quad \text{— (1)}$$

$$\text{or, } Y(z) + 2.5z^{-2}Y(z) + Y(z)z^{-2} = 0.495X(z) + 1.5z^{-1}X(z) + z^{-2}\cdot 1.5X(z)$$

Taking inverse of Z

$$\text{or } y[n] + 2.5y[n-1] + y[n-2] = 0.495x[n] + 1.5x[n-1] + 1.5x[n-2] \quad \text{— (2)}$$

eqn (2) is difference eqn.

From (1), for direct form I & II structure,

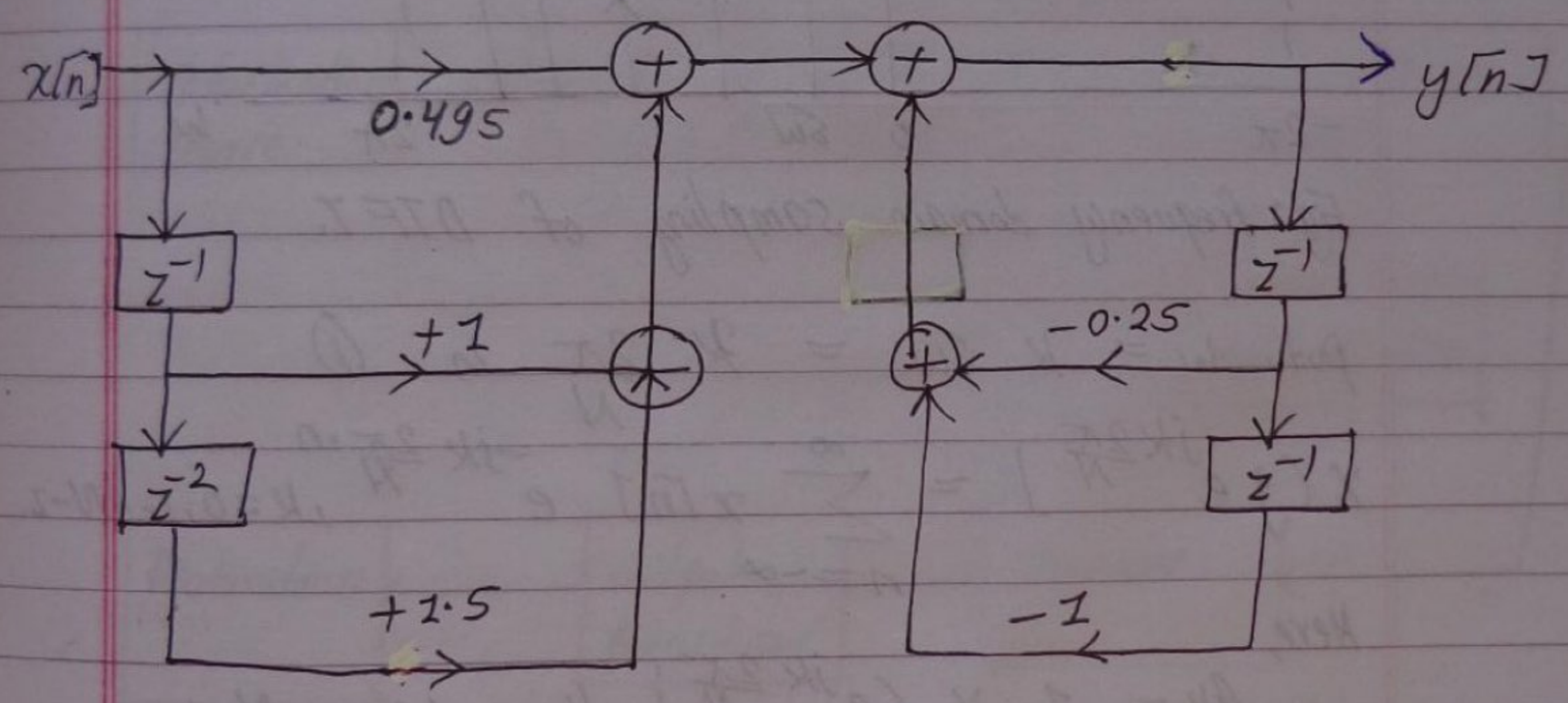


Fig: Direct form I structure.

V. Frequency domain sampling and Reconstruction: 2017/F

Let us consider a discrete time aperiodic signal x[n]

Scanned with CamScanner

with fourier transform,

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn} \quad \text{— (1)}$$

let us take sample $X(e^{jw})$ in frequency at a spacing of δw between successive samples. since $X(e^{jw})$ is periodic with period 2π, only samples in the interval of 2π are sufficient.

let us take N equidistant samples in the interval $0 \le w < 2\pi$ with spacing $\delta w = \frac{2\pi}{N}$.

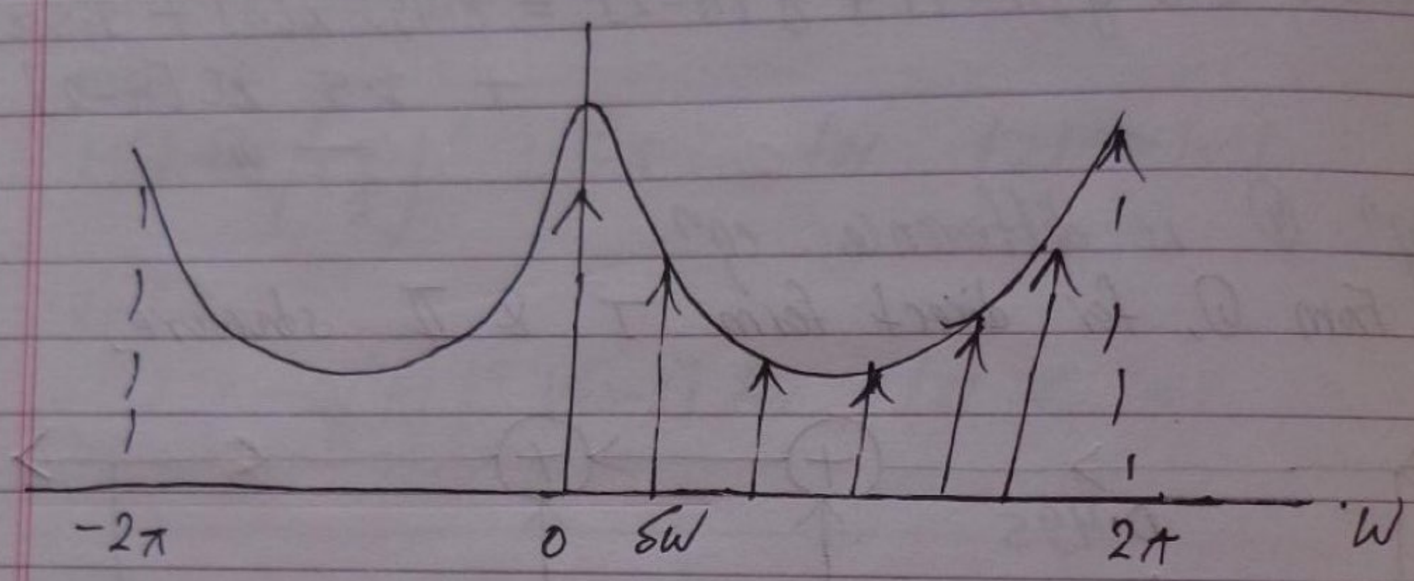


Fig: frequency domain sampling of DTFT.

put, $w = k\,\delta w = k\,\frac{2\pi}{N}$ in (1)

$$X\left(e^{jk\frac{2\pi}{N}}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\frac{2\pi}{N}\cdot n}, \quad k = 0, 1 \ldots N-1.$$

Here,

$$a_k = \frac{1}{N} X\left(e^{jk\frac{2\pi}{N}}\right), \quad k = 0, 1, 2, \ldots N-1$$

and,

$$x_p[n] = \sum_{k=0}^{N-1} a_k \cdot e^{jk\frac{2\pi}{N}\cdot n}, \quad n = 0, 1, 2, \ldots N-1,$$

thus,

Scanned with CamScanner

$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{jk\frac{2\pi}{N}}\right) e^{jk\frac{2\pi}{N}\cdot n}, \quad n = 0,1,2 \ldots N-1.$$

This eqn reconstructs the periodic signal $x_p[n]$ from the sample of $X\left(e^{jk\frac{2\pi}{N}}\right)$ of the spectrum $X(e^{j\omega n})$. However, $x[n]$ can be recovered from $x_p[n]$ only if there is no aliasing in time domain.

V. Describe with neat diagram the process of transmission in communication system. 2017/F

=> Communication is the process of conveying an information-bearing signal from one point (source) to another point (destination). For this different electronic devices such as transducer, amplifier, modulator, demodulator, filter, antenna etc are used to form communication system.

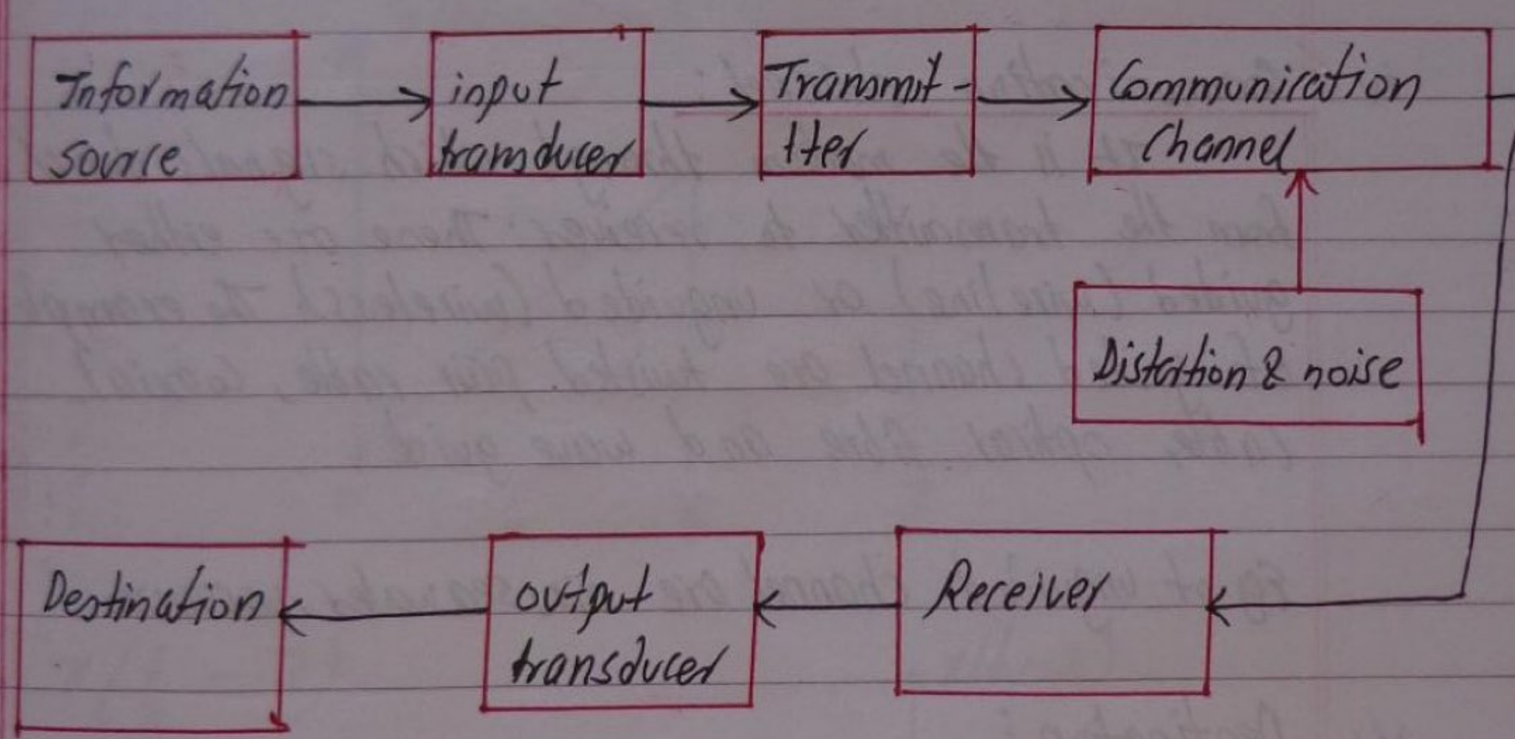


Fig: Block diagram of communication system.

i) Information source:

It originates a message such as human voice, television picture, or data. Eg: microphone.

Scanned with CamScanner

ii Input transducer:

The function of transducer is to change one form of energy into another form. The message produced by the information source may or may not be in electrical form. An input transducer is used to convert message into an electrical waveform.

iii Transmitter:

It converts the message signal into a suitable form for reliable communication over a channel. Different processes are performed in the transmitter depending upon the type of communication system. Modulation is the most important process performed in transmitter. Amplification and filtration may be required before transmission.

iv Communication channel:

It is the medium through which signals travel from the transmitter to receiver. These are either guided (wireline) or unguided (wireless). The example of guided channel are twisted pair cable, coaxial cable, optical fibre and wave guide.

Eg of unguided channel are air, seawater, vacuum.

v. Destination:

It is the point where message is finally delivered.

vi Receiver:

It receives signals from the channel and recovers

Scanned with CamScanner

the original message signal in electrical form by the process known as demodulation. other operation performed are filtering and amplifications.

Q) Sketch the signal $x(t) = \begin{cases} t & \text{for } 0 < t < 2 \\ 1 & \text{for } 2 < t < 3 \\ 0 & \text{elsewhere} \end{cases}$

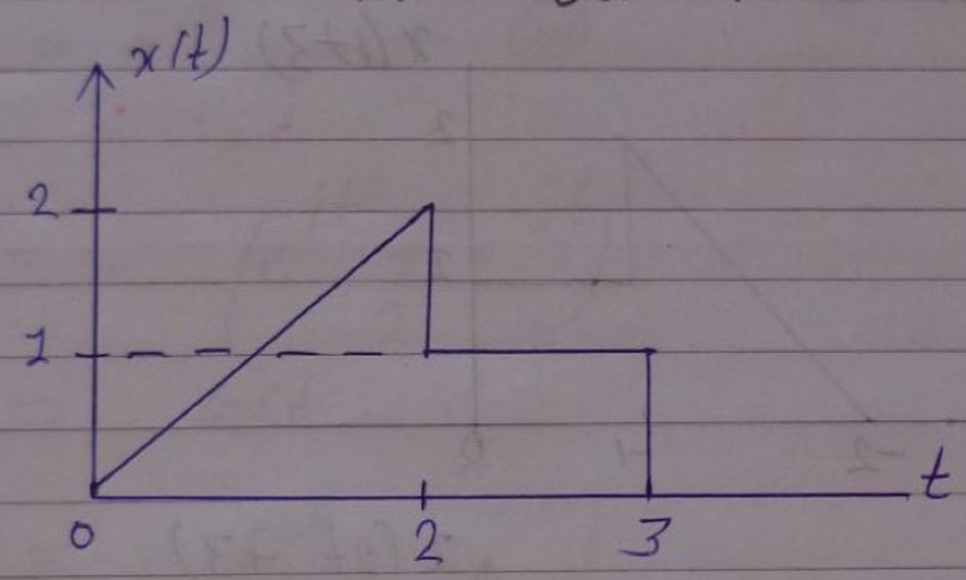
& transform signal into.

i) $x\left(\frac{t}{2} - 2\right)$

ii) $x(2t + 3)$

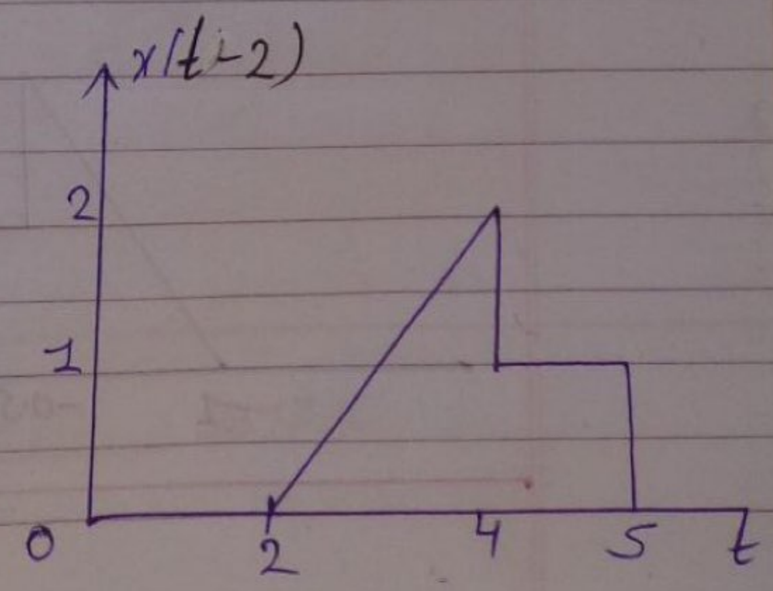
Soln,

$\Rightarrow x(t) = \begin{cases} t & \text{for } 0 < t < 2 \\ 1 & \text{for } 2 < t < 3 \\ 0 & \text{for elsewhere.} \end{cases}$



① $x\left(\frac{t}{2} - 2\right)$

$\Rightarrow$ shifting $x(t)$ to $x(t-2)$



Scanned with CamScanner

For time scaling, $a = \frac{1}{2} = 0.5 < 1$,

So, expansion occurs by factor 0.5,

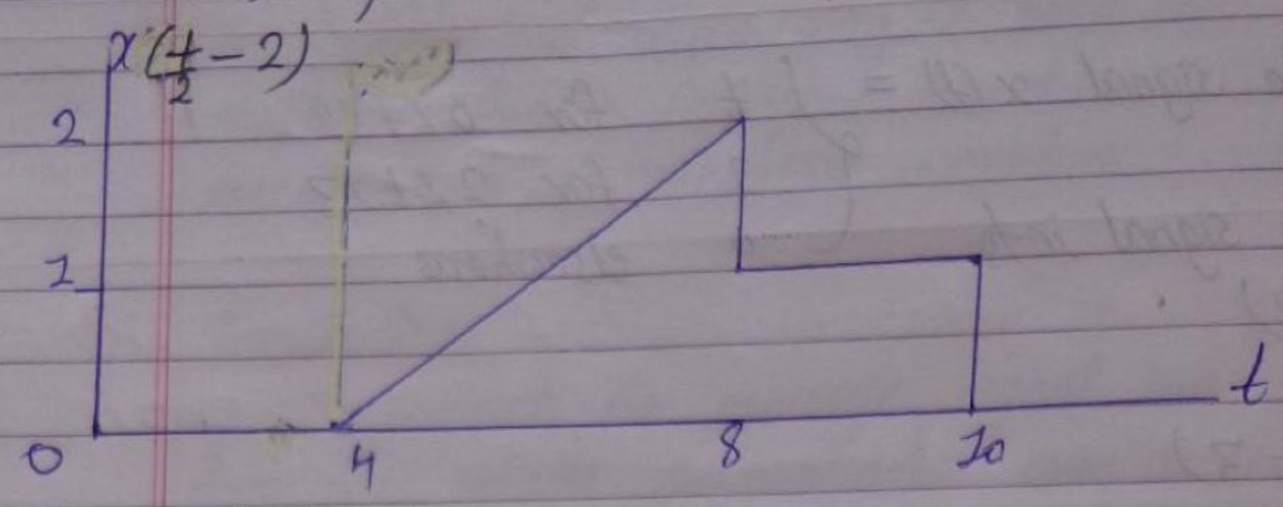
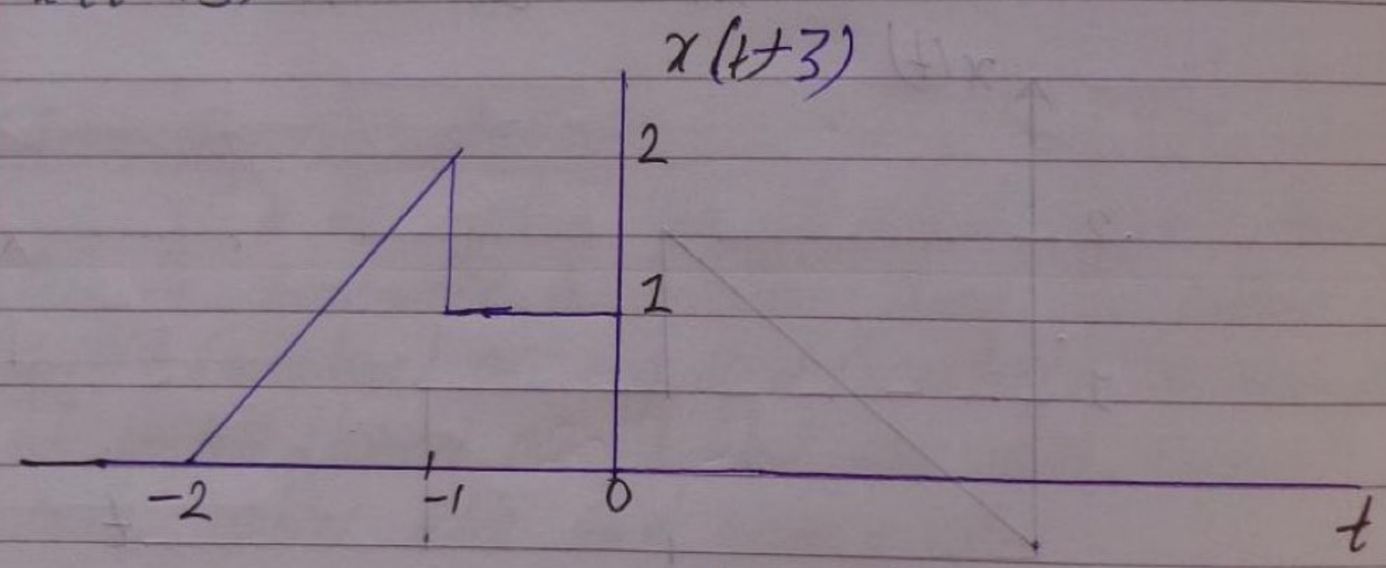
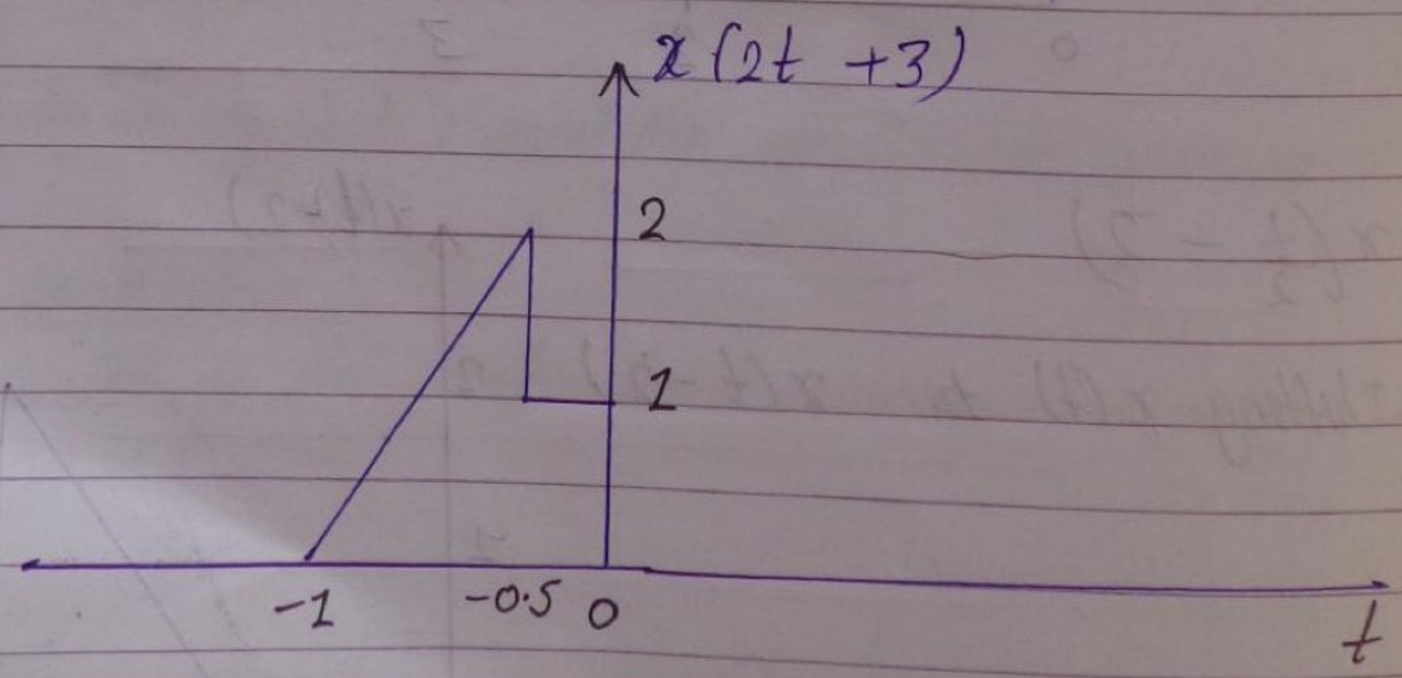


Fig: $x\left(\frac{t}{2} - 2\right)$ plot.

(ii) $x(t+3)$:





Scanned with CamScanner

X. Find even & odd parts of signal $x(t) = e^{jt}$
Sol^n, 2017/F

$\Rightarrow$ $x(t) = x_e(t) + x_o(t)$ — (I)
$x_e(t)$ = even parts
$x_o(t)$ = odd parts

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

put $t = -t$ in eq^n (II),

$$x(t) = e^{jt} \quad \text{— (II)}$$

$$x(-t) = e^{-jt} \quad \text{— (III)}$$

Thus,

$$\therefore x_o(t) = \left(\frac{e^{jt} + e^{-jt}}{2}\right)$$
$$= \cos t$$

$$\therefore x_e(t) = \left(\frac{e^{jt} - e^{-jt}}{2}\right)$$
$$= j\left(\frac{e^{jt} - e^{-jt}}{2j}\right)$$

$$\therefore x_e(t) = j \sin t$$

Y. Multiplication property of DFT: 2017/F

Scanned with CamScanner

$$DFT[x_1[n] \cdot x_2[n]] = \frac{1}{N} X_1[k] \circledN X_2[k]$$

If $X[k] = DFT\{x[n]\}$

$Y[k] = DFT\{y[n]\}$.

then,

$$\frac{1}{N} X[k] \circledast Y[k] = DFT[x[n] \cdot y[n]]$$

Also, if.

$x[n] = DFT^{-1}[X[k]]$

and

$y[n] = DFT^{-1}[Y[k]]$

then,

$$x[n]\,y[n] = DFT^{-1}\left[\frac{1}{N} X[k] \circledast Y[k]\right]$$

---

2) Significance of time and frequency domain representation of signal: 2017/5

⇒ In the linear case, the frequency domain analysis provides easy stability tests (Bode plot, Nyquist criterion, root locus) and relative stability tests. Frequency domain is a method used to analyze data. This refers to analyzing a mathematical function or a signal with respect to the frequency. Frequency domain analysis is widely used in fields of control system Engineering, electronics and statistics. It is mostly used to signal or function that are periodic over time. This doesnot mean that frequency

Scanned with CamScanner

domain analysis cannot be used in signals that are not periodic. Transformation is used to convert a time domain function to a frequency domain and vice-versa.

Electrical signals have both time and frequency domain representations. In the time domain, voltage or current is expressed as a function of time. signals can also be represented by a magnitude & phase as a function of frequency. signals that repeat periodically in time are represented by power spectrum. Signals that are time limited are represented by an energy spectrum. An operation that is hard to perform in time domain may be very simple in frequency domain.

Actually time domain gives some issue when we design systems, solving mathematics problems like convolution filtering, reconstruct signals and reduce noise. However every, domain has advantage & disadvantage. As example, In control theory, we can't know if system is stable or not if we use time domain, so we convert the function of system to frequency domain or s- domain to know that.

Time domain signals just provide the information regarding the value of function / signal at any given instance. They do not convey the information as to the rate at which the signal is varying.

Scanned with CamScanner

Important Question solution

1).A Determine the fundamental period of the following signals:

(i) $x(t) = e^{j3\pi t/5}$

Sol^n,

$\Rightarrow$ If $x(t)$ is a periodic signal, then there exists $T > 0$, such that $x(t) = x(t+T)$, then,

or $x(t) = x(t+T)$

or, $e^{j3\pi t/5} = e^{j3\pi(t+T)/5}$

or $1 = e^{j3\pi T/5}$

or $e^{j2k\pi} = e^{j3\pi T/5}$

or $T = \frac{10}{3}$ $(k=1)$

$\therefore T = \frac{10}{3}$

(ii) $x[n] = e^{j3\pi n/5}$

Sol^n,

$\Rightarrow$ Let, $x[n] = e^{j3\pi n/5}$, if $x[n]$ is a periodic signal, then there exists an integer $N>0$, such that $x[n] = x[n+N]$,

So, $x[n] = x[n+N]$

$\Rightarrow e^{j3\pi n/5} = e^{j3\pi(n+N)/5}$

$\Rightarrow 1 = e^{j3\pi N/5}$

$\Rightarrow e^{j2k\pi} = e^{j3\pi N/5}$

$\therefore T = 10$ $(k=3)$

Scanned with CamScanner

B. Determine given signal is power or energy signal.

Sol^n,

$\Rightarrow \quad x(n) = (-0.5)^n u[n]$

Now,

$$E = \sum_{n=-\infty}^{\infty} x^2[n]$$

$$= \sum_{n=-\infty}^{\infty} (0.25)^n = \frac{1}{1-0.25} = \frac{4}{3} < \infty$$

and,

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$$

$$= \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} (0.25)^n$$

$$= \frac{1}{2\infty+1} \sum_{n=0}^{N} (0.25)^n$$

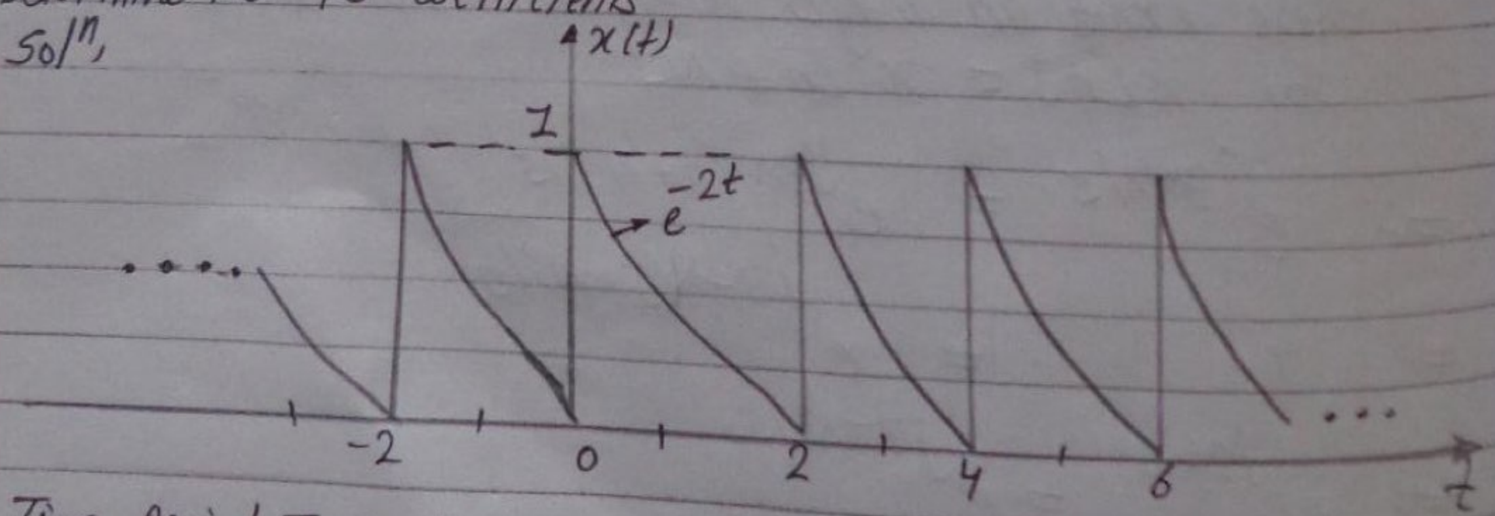
$$= 0$$

Hence, power is zero and energy is finite, Hence, it is an energy signal.

C. Determine the FS Coefficients.

Sol^n,

$\Rightarrow$



Time period, $T=2$, Hence,

Scanned with CamScanner

$$w_0 = \frac{2\pi}{2} = \frac{2\pi}{T} = \pi$$

on the interval $0 \le t \le 2$, one period of $x(t)$ is expressed as $x(t) = e^{-2t}$, so,

$$x[k] = \frac{1}{2}\int_0^2 e^{-2t} \cdot e^{-jk\pi t}\, dt$$

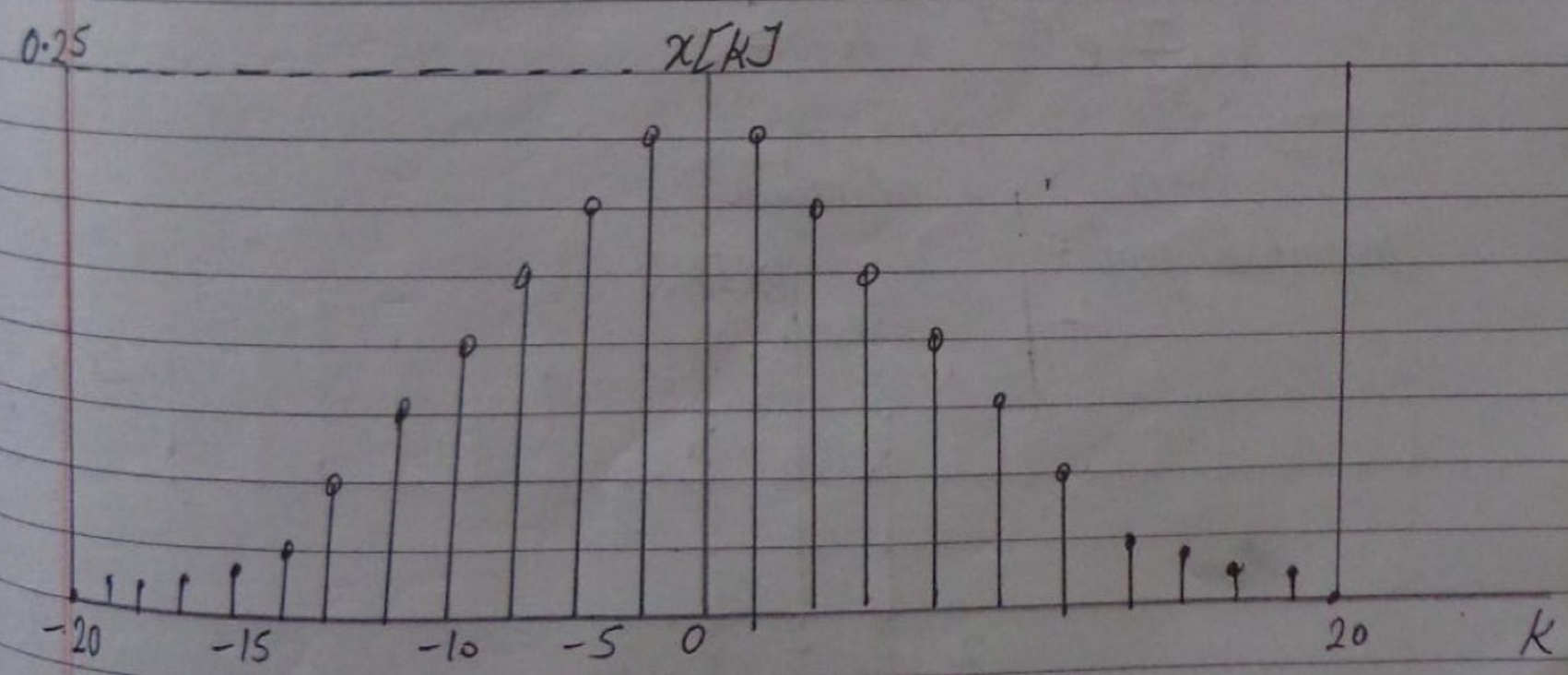
$$= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\, dt$$

evaluating the integral,

$$\text{or, } x[k] = \frac{-1}{2(2+jk\pi)} e^{-(2+jk\pi)t}\Big|_0^2$$

$$= \frac{1}{4+jk2\pi}\left(1 - e^{-4} \cdot e^{-jk2\pi}\right)$$

$$\therefore x[k] = \frac{1-e^{-4}}{4+jk2\pi}$$



Scanned with CamScanner

D. Determine the FS representation of the signal,

$x(t) = 3\cos(\pi t/2 + \pi/4)$

sol^n,

$\Rightarrow$ Time period, $T = 4$, so,

$$\omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk2\pi t/2}$$

using Euler's formula,

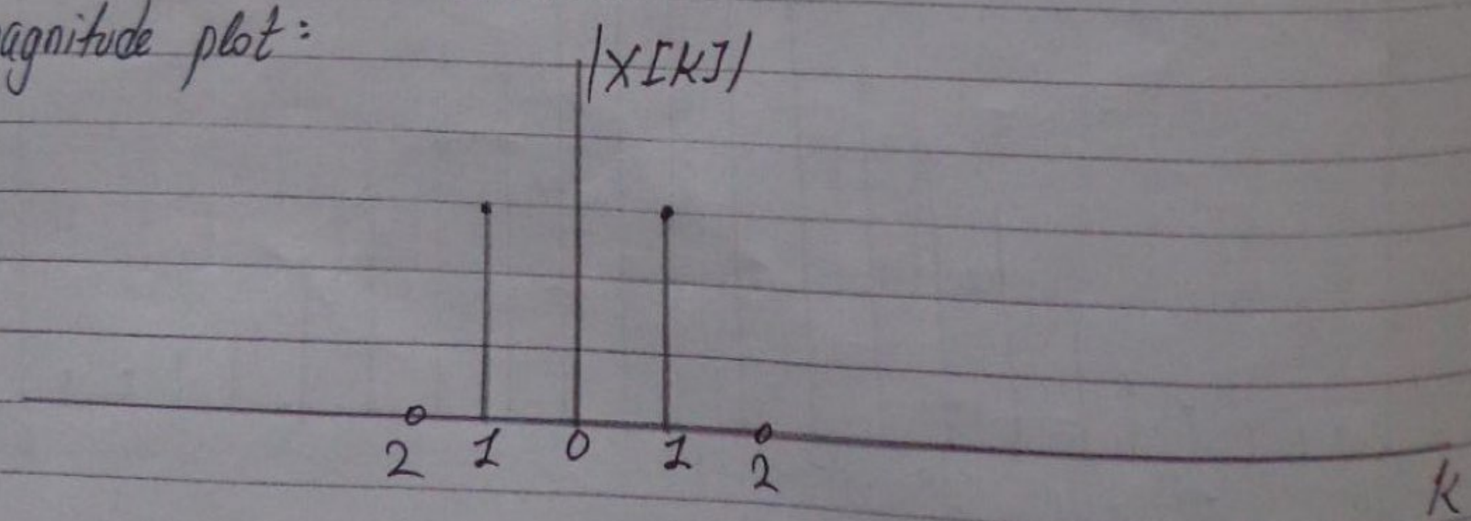
$$x(t) = 3 \cdot \frac{X\, e^{j(\pi t/2 + \pi/4)} + e^{-j(\pi t/2 + \pi/4)}}{2}$$

$$= \frac{3}{2} e^{j\pi/4} \cdot e^{j\pi/2} + \frac{3}{2} e^{-j\pi/4} \cdot e^{-j\pi/2}$$

equating each term in this expansion to the terms in eq^n of $x(t)$ gives the FS coefficient

$$\therefore x[k] = \begin{cases} \frac{3}{2} e^{-j\pi/4}, & k = -1 \\ \frac{3}{2} e^{j\pi/4}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

magnitude plot:



Scanned with CamScanner

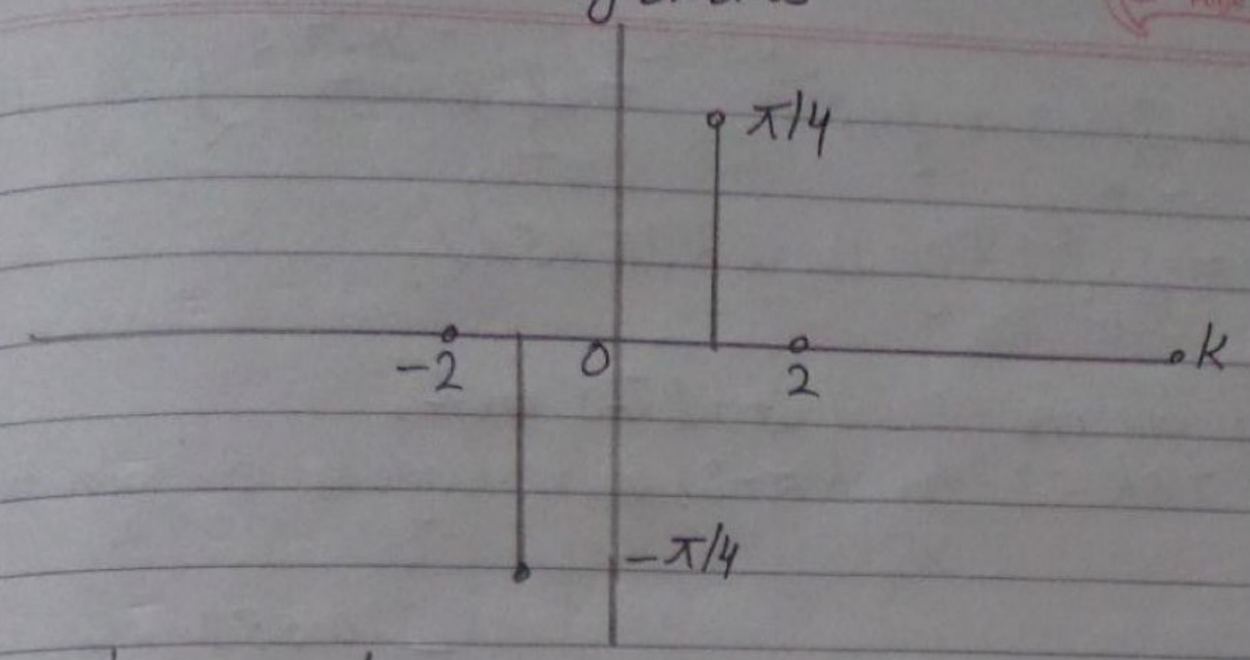
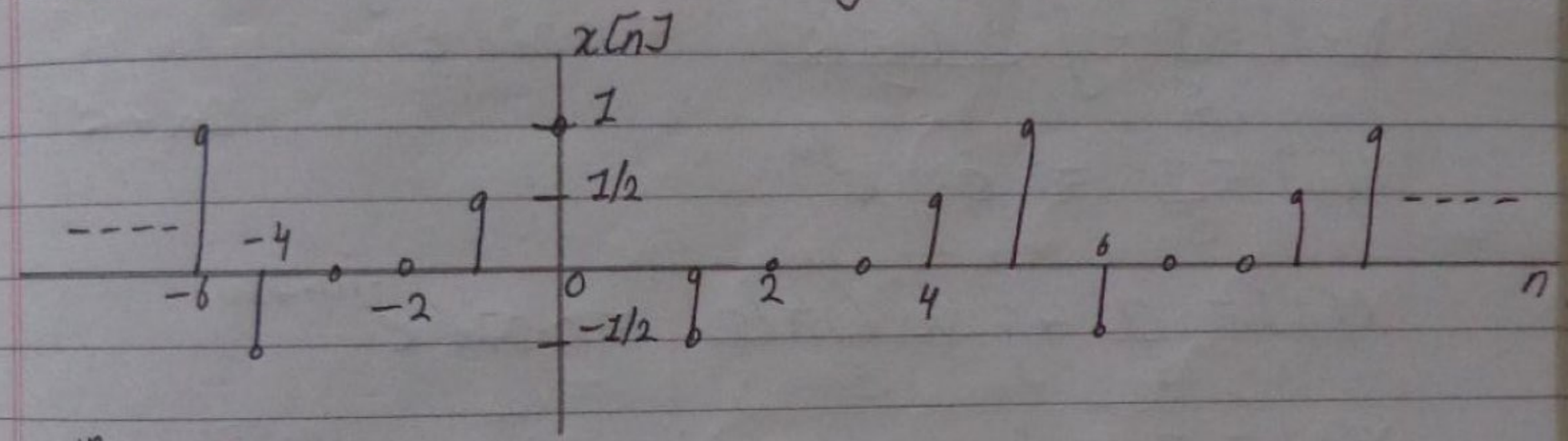


Fig: Phase spectrum.

E. Determine the DTFS coefficient for signal $x[n]$ shown,



Sol^n,

$\Rightarrow$ The signal has period, $N = 5$, so, $\omega_0 = \frac{2\pi}{5}$, As the signal also has odd symmetry,
it can be sum over $n = -2$ to $n = 2$ in the eq^n,

$$X[k] = \frac{1}{5}\sum_{n=-2}^{2} x[n]\, e^{-jk2\pi n/5}$$

$$= \frac{1}{5}\left[x[-2]e^{jk4\pi/5} + x[-1]e^{jk2\pi/5} + x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5}\right]$$

Using the values of $x[n]$, we get,

Scanned with CamScanner

or $X[k] = \frac{1}{5}\left\{1 + \frac{1}{2} e^{jk2\pi/5} - \frac{1}{2} e^{-jk2\pi/5}\right\}$

$= \frac{1}{5}\left\{1 + j \sin(k2\pi/5)\right\}$.

we identify one period of DTFS coeff, $X[k]$ i, $k=2$ to $k=-2$, in rectangular & polar coordinates as,

$X[-2] = \frac{1}{5} - j\frac{\sin(4\pi/5)}{5} = 0.232\, e^{-j0.531}$

$X[-1] = \frac{1}{5} - j\frac{\sin(2\pi/5)}{5} = 0.276\, e^{-j0.760}$

$X[0] = \frac{1}{5} = 0.2\, e^{j0}$

$X[1] = \frac{1}{5} + j\frac{\sin(2\pi/5)}{5} = 0.276\, e^{j0.760}$

$X[2] = \frac{1}{5} + j\frac{\sin(4\pi/5)}{5} = 0.232\, e^{j0.531}$

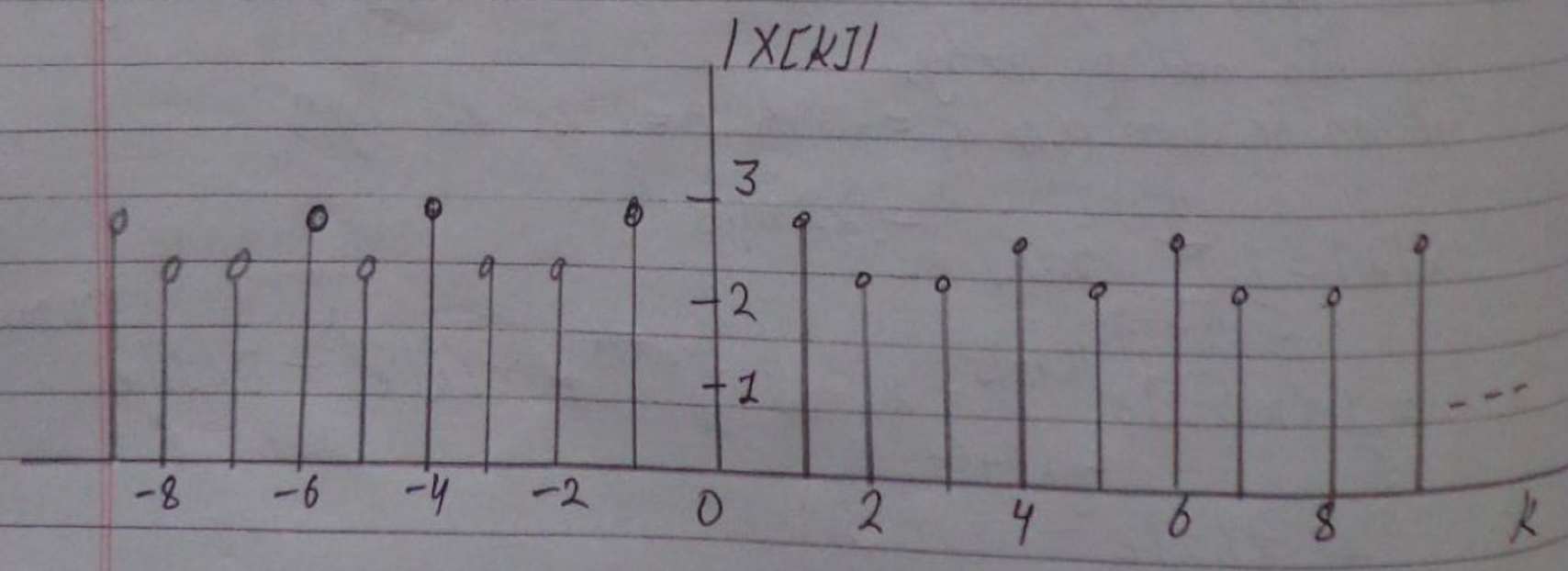
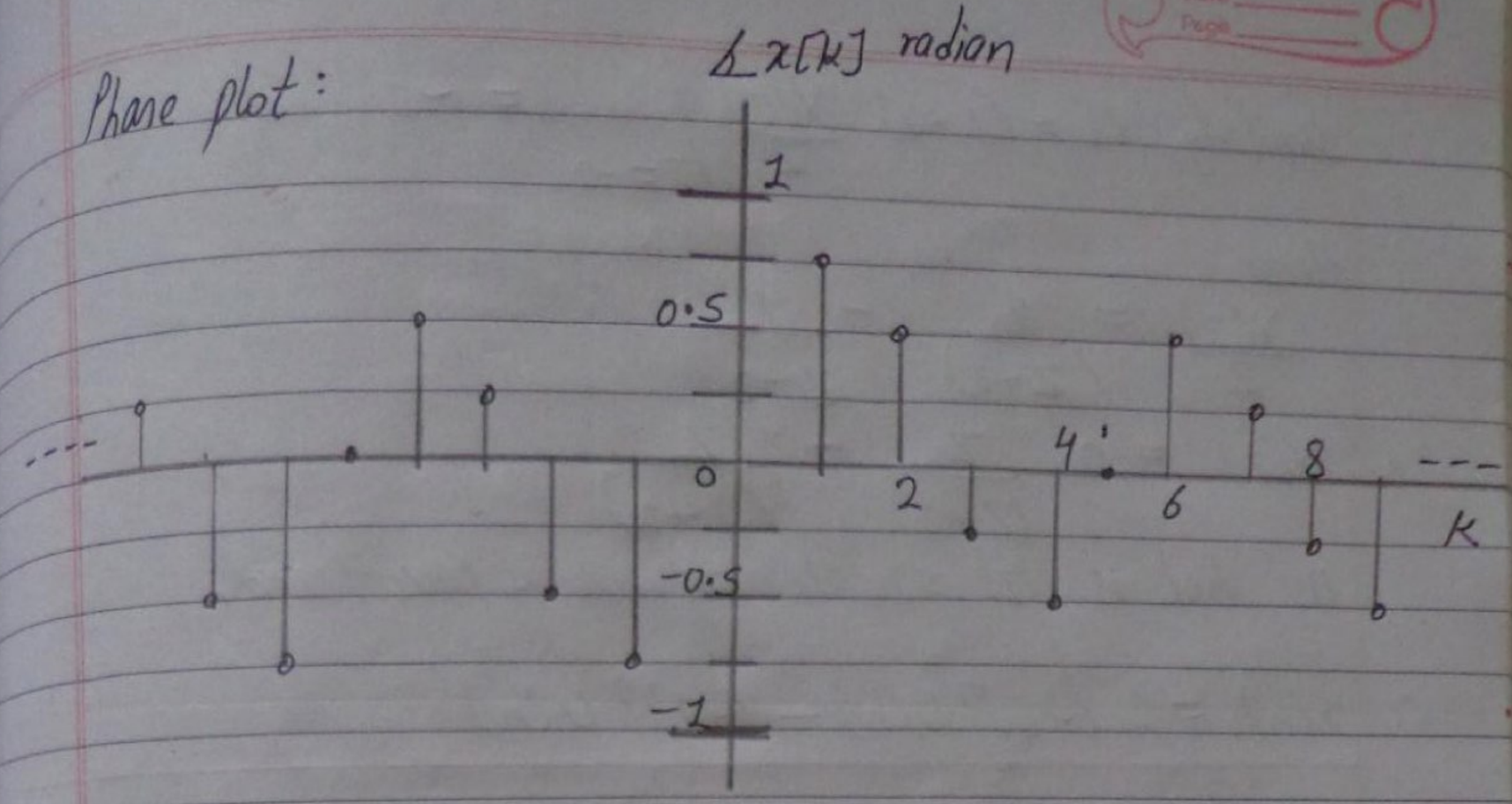


Fig: magnitude plot.

Scanned with CamScanner

Phase plot:



F. Find inverse Z transform of,

$$X(z) = \frac{1 + 3z^{-1}}{1 + 3z^{-1} + 2z^{-2}} \quad |z| > 2$$

Sol^n,

$\Rightarrow$ eliminate negative power by multiplying numerator and denominator by z^2,

$$X(z) = \frac{z(3+z)}{z^2 + 3z + 2}$$

Dividing $X(z)$ by z, we get,

$$\frac{X(z)}{z} = \frac{z+3}{(z+1)(z+2)}$$

writing in partial fraction form,

$$\text{or } \frac{X(z)}{z} = \frac{C_1}{z+1} + \frac{C_2}{z+2}$$

where, $C_1 = (z+1)\frac{X(z)}{z}\Big|_{z=-1} = 2$

Scanned with CamScanner

Similarly, $C_1 = (z+2)\frac{X(z)}{z}\Big|_{z=-2} = -1$

Hence,

$$\frac{X(z)}{z} = \frac{2}{(z+1)} - \frac{1}{z+2}$$

or $X(z) = 2\cdot\frac{z}{(z+1)} - \frac{z}{(z+2)}$

As ROC is $|z| > 2$, the sequence is causal and,

$\therefore x(n) = 2(-1)^n u(n) - (-2)^n u(n)$

6. Verify Parseval's theorem for the energy signal $x(t) = e^{-at}\cdot u(t)$ $(a>0)$ 2017/5

$\Rightarrow$ Sol^n, $x(t) = e^{-at} u(t)$, $a>0$

Therefore,

$$E_f = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2+a^2} = \frac{1}{2a}.$$

The F.T is,

$$F(\omega) = \frac{1}{j\omega + a}$$

This can actually be verified by integrating,

$$E_f = \frac{1}{\pi}\int_0^{\infty} \frac{1}{\omega^2+a^2}\, d\omega$$

$$= \frac{1}{\pi a}\tan^{-1}\frac{\omega}{a}\Big|_0^{\infty}$$

$$= \frac{1}{2a}$$

Scanned with CamScanner

and,

$$E_f = \int f^2(t)\, dt$$

$$= \int_0^\infty e^{-2at}\, dt$$

$$\therefore E_f = \frac{1}{2a}$$

Hence, parsevals theorem is verified.

In the given problem : Find bandwidth w such that 95% of the energy is contained in frequencies below w.

=> To find 95% containment bandwidth, we solve,

or $$\frac{0.95}{2a} = \frac{1}{\pi}\int_0^w \frac{dw}{w^2+a^2}$$

for w, then,

or $$\frac{0.95}{2a} = \frac{1}{\pi a} \tan^{-1} \frac{w}{a}\Big|_0^w$$

we get,

$$\frac{0.95}{2} = \tan^{-1}\left(\frac{w}{a}\right)$$

so,

$$\therefore w = 12.706 \text{ radian.}$$

H. Write the difference between fourier series and fourier transform in term of frequency domain representation. 2017/5

=>

① Fourier series	Fourier Transform.
$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

Scanned with CamScanner

Continuous and aperiodic in time	Continuous and aperiodic in time
$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\, dt$	$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$
discrete and aperiodic in frequency	continuous and aperiodic in frequency.
(II) $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n}\, d\omega$
discrete and periodic in time.	discrete and aperiodic in time
$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$
Continuous and aperiodic in frequency.	Continuous and periodic in frequency.

I. Why is zero padding necessary in signal processing? 2017/5

=> Zero padding is a simple concept, it simply refers to adding zeros to end of a time domain signal to increase its length. Zero padding consists of appending zeros to a signal. It maps a length N signal to a length $m > N$ signal but, m need not be an integer multiple of N.

$$\text{Zeropad.}_{N,m}(x) = \triangleq \begin{cases} x(m); & 0 \le m \le N-1 \\ 0, & N \le m \le m-1 \end{cases}$$

Using fourier theorem, we will able to show that zero padding in the time domain gives bandlimited interpolation in frequency domain and similarly, zero padding in the frequency domain

Scanned with CamScanner

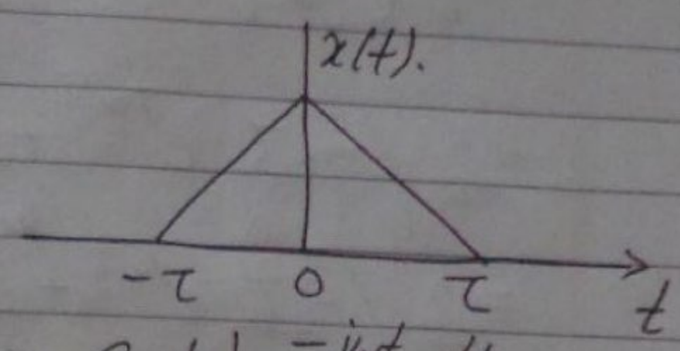
gives bandlimited interpolation in the time domain. This is how ideal sampling rate conversion is accomplished.

7. Find the fourier transforms of the triangular pulse $x(t)$

Sol^n,

$\Rightarrow$ we have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$



$$= \int_{-\tau}^{0} \left(2 + \frac{2}{\tau} t\right) e^{-j\omega t}\, dt + \int_{0}^{\tau} \left(2 - \frac{2}{\tau} t\right) e^{-j\omega t}\, dt.$$

$$= 2\int_{-\tau}^{0} e^{-j\omega t}\, dt + \frac{2}{\tau}\int_{-\tau}^{0} t e^{-j\omega t}\, dt + 2\int_{0}^{\tau} e^{-j\omega t} - \frac{2}{\tau}\int_{0}^{\tau} t e^{-j\omega t}\, dt$$

Here,

$$I_1 = 2\int_{-\tau}^{0} e^{-j\omega t}\, dt$$

$$= 2\, \frac{1}{-j\omega} \left[e^{-j\omega t}\right]_{-\tau}^{0}$$

$$= 2 \times \frac{1}{-j\omega} \left(e^{0} - e^{-j\omega\tau}\right)$$

$$= j\,\frac{2}{\omega}\left(1 - e^{j\omega\tau}\right)$$

$$= j\,\frac{2}{\omega} - j\,\frac{2}{\omega}\, e^{j\omega\tau}$$

Again, $I_2 = \frac{2}{\tau}\int_{-\tau}^{0} t\, e^{-j\omega t}\, dt$

$$= \frac{2}{\tau}\left(t\int e^{-j\omega t}\, dt - \int\left(\frac{dt}{dt}\int e^{-j\omega t}\, dt\right) dt\right)$$

Scanned with CamScanner

$$= \frac{2}{\tau}\left(\left[t - \frac{1}{-j\omega} \cdot e^{-j\omega t}\right]_{-\tau}^{0} - \frac{1}{-j\omega}\int e^{-j\omega t}\,dt\right)$$

$$= \frac{2}{\tau}\left(\frac{-1}{j\omega}\left(0 + \tau e^{+j\omega\tau}\right) - \frac{1}{(-j\omega)^2}\left[e^{-j\omega t}\right]_{-\tau}^{0}\right)$$

$$= \frac{2}{\tau}\left(-\frac{\tau}{j\omega} e^{j\omega\tau} + \frac{1}{\omega^2}\left(e^{0} - e^{j\omega\tau}\right)\right)$$

$$= \frac{2}{\tau}\left(\frac{-\tau}{j\omega} e^{j\omega\tau} + \frac{1}{\omega^2} - \frac{1}{\omega^2} e^{j\omega\tau}\right)$$

$$\therefore I_2 = j\frac{2}{\omega} e^{j\omega\tau} + \frac{2}{\tau\omega^2} - \frac{2}{\omega^2\tau} e^{j\omega\tau}$$

Again,

$$I_3 = 2\int_0^{t} e^{-j\omega t}\,dt$$

$$= 2 \cdot \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_0^{\tau}$$

$$= 2 \cdot \frac{1}{-j\omega}\left[e^{-j\omega\tau} - e^{0}\right]$$

$$= \frac{2}{j\omega}\left(1 - e^{-j\omega\tau}\right)$$

$$= \frac{-j \cdot 2}{\omega} + j\frac{2}{\omega} e^{-j\omega\tau}$$

Again,

$$I_4 = \frac{2}{\tau}\int_0^{\tau} t e^{-j\omega t}\,dt$$

$$= \frac{2}{\tau}\left(t\int e^{-j\omega t}\,dt - \int\left(\frac{dt}{dt}\int e^{-j\omega t}\,dt\right)dt\right)$$

Scanned with CamScanner

$$= \frac{2}{\tau}\left(\left[t \cdot \frac{1}{-j\omega} e^{-j\omega t}\right]_0^{\tau} - \frac{1}{-j\omega}\int e^{-j\omega t}\,dt\right)$$

$$= \frac{2}{\tau}\left(\frac{-1}{j\omega}\left(\tau e^{-j\omega\tau} - 0\right) - \frac{1}{(-j\omega)^2}\left[e^{-j\omega t}\right]_0^{\tau}\right)$$

$$= \frac{2}{\tau}\left(\frac{-1}{j\omega}\left[e^{-j\omega t}\right] + \frac{1}{\omega^2}\left(e^{-j\omega t} - e^{0}\right)\right)$$

$$= \frac{2}{\tau}\left(\frac{-1}{j\omega}\tau e^{-j\omega\tau} + \frac{1}{\omega^2}e^{-j\omega\tau} - \frac{1}{\omega^2}\right)$$

$$\therefore I_4 = j\frac{2}{\omega}e^{-j\omega\tau} + \frac{2}{\omega^2\tau}e^{-j\omega\tau} - \frac{2}{\omega^2\tau}$$

So,

$$X(j\omega) = I_1 + I_2 + I_3 - I_4$$

$$= \frac{4}{\omega^2\tau} - \frac{2}{\omega^2\tau}\left(e^{j\omega\tau} + e^{-j\omega\tau}\right)$$

$$= \frac{4}{\omega^2\tau} - \frac{4}{\omega^2\tau}\cos\omega t$$

$$= \frac{4}{\omega^2\tau}\left(1 - \cos\left(2 \cdot \omega\tau/2\right)\right)$$

$$= \frac{4}{\omega^2\tau} \cdot 2 \cdot \sin^2\left(\frac{\omega\tau}{2}\right)$$

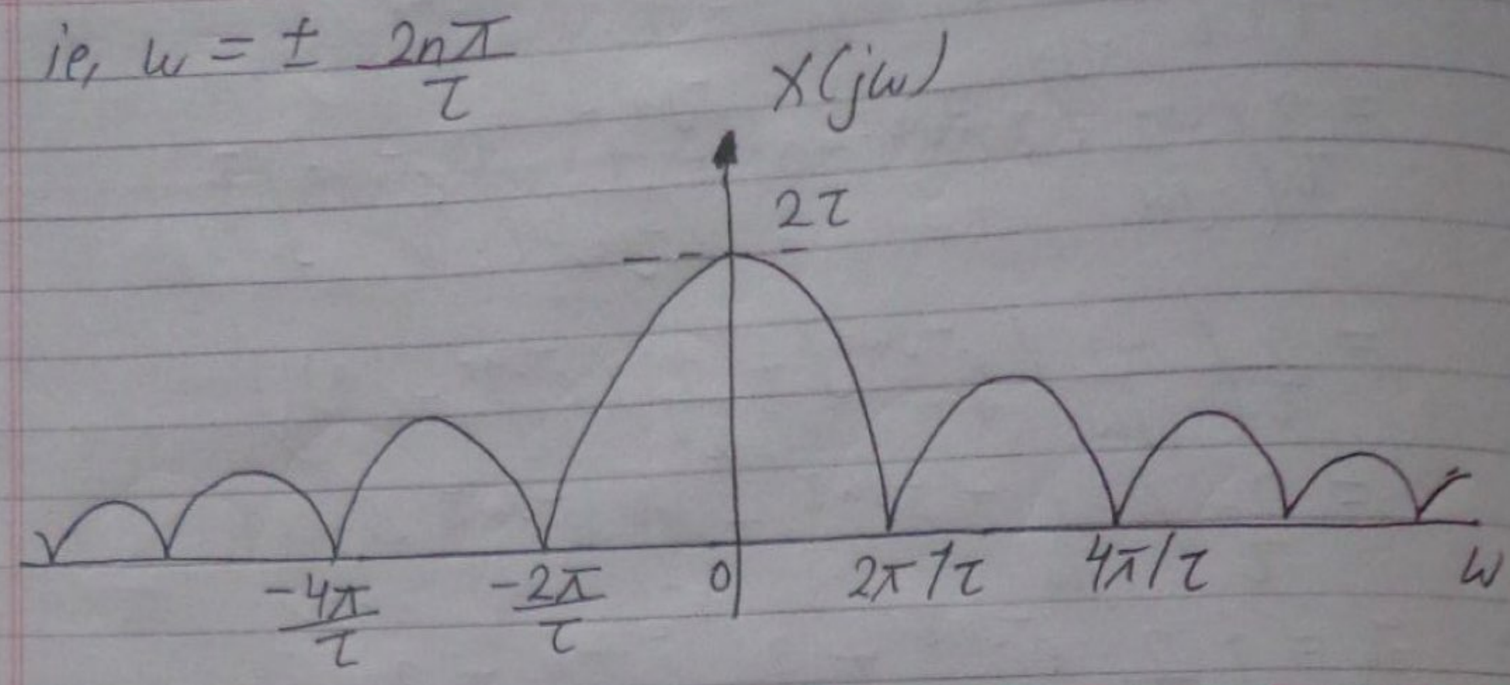
$$= 2\tau\,\frac{\sin^2(\omega\tau/2)}{\left(\frac{\omega\tau}{2}\right)^2}$$

$$\therefore X(j\omega) = 2\tau\,\mathrm{sinc}^2\left(\frac{\omega\tau}{2}\right)$$

$$X(j\omega) = 0 \quad \text{for} \quad \frac{\omega\tau}{2} = \pm n\pi$$

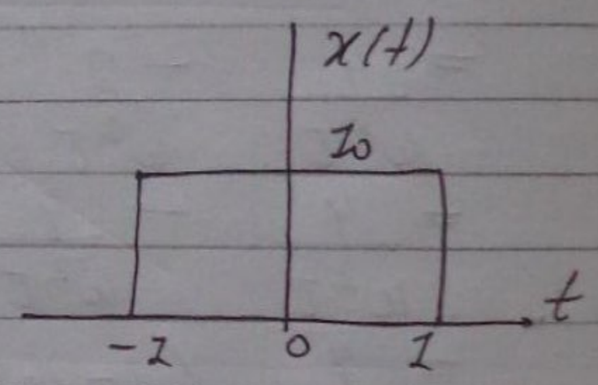
Scanned with CamScanner

ie, $\omega = \pm \frac{2n\pi}{\tau}$



K. Find the fourier transform of a rectangular pulse of duration 2 seconds and having magnitude of 10 v ① starting from -1 sec and ⑪ starting from origin.

Sol^n,



$\Rightarrow$

① We have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-1}^{1} 10\, e^{-j\omega t}\, dt$$

$$= 10 \cdot \frac{1}{-j\omega} \left[e^{-j\omega t} \right]_{-1}^{1}$$

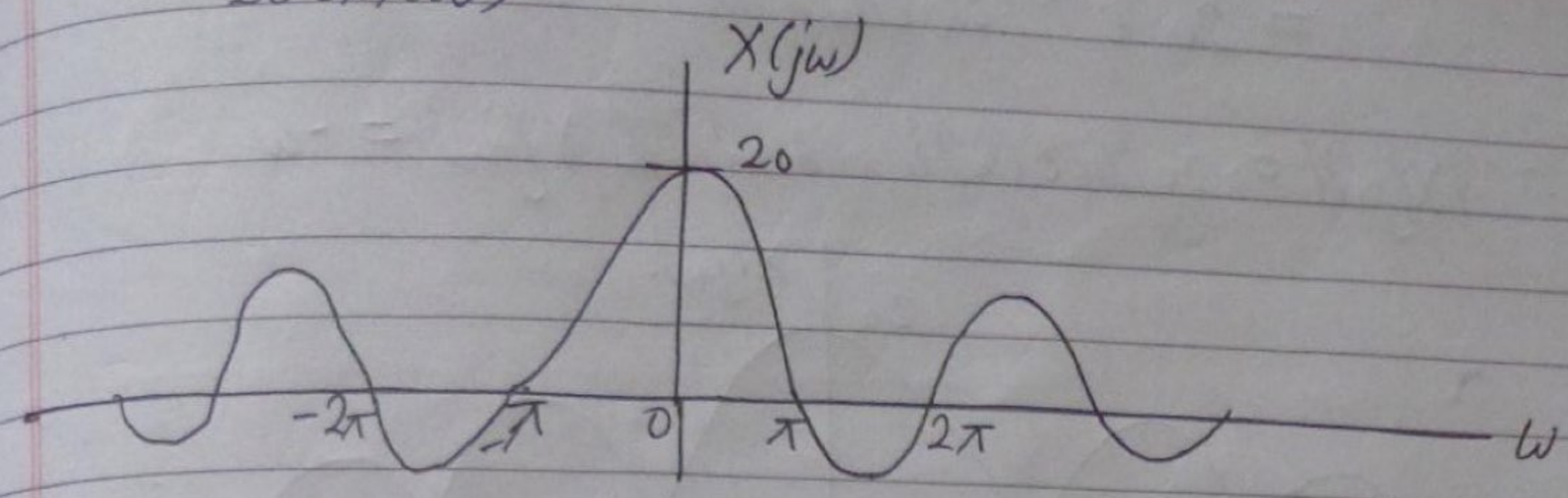
$$= \frac{10}{-j\omega} \left(e^{-j\omega} - e^{j\omega} \right)$$

$$= \frac{20}{\omega} \, \frac{e^{j\omega} - e^{-j\omega}}{2j}$$

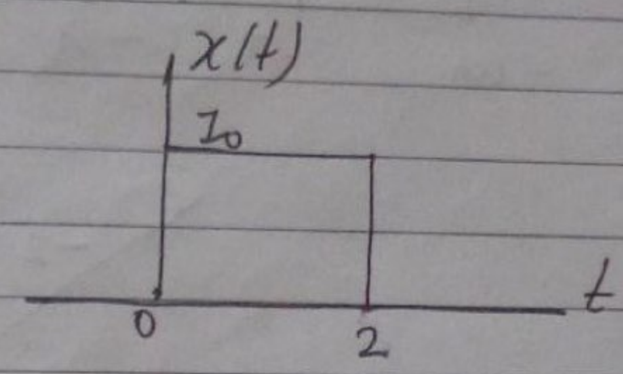
$$= \frac{20}{\omega} \sin(\omega)$$

Scanned with CamScanner

$= 20 \sin(w)$



① we have,



$$X(jw) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt$$

$$= \int_0^2 I_0\, e^{-jwt}\, dt$$

$$= I_0 \cdot \frac{1}{-jw} \left[e^{-jwt} \right]_0^2$$

$$= I_0 \cdot \frac{-1}{jw} \left(e^{-j2w} - e^{0} \right)$$

$$= I_0 \cdot \frac{1}{jw} \left(1 - e^{-j2w} \right)$$

$$= I_0 \cdot \frac{1}{jw} \frac{e^{jw}}{e^{jw}} \left(1 - e^{j2w} \right)$$

$$= I_0 \cdot \frac{1}{jw} \cdot \frac{-1}{e^{jw}} \left(e^{jw} - e^{-jw} \right)$$

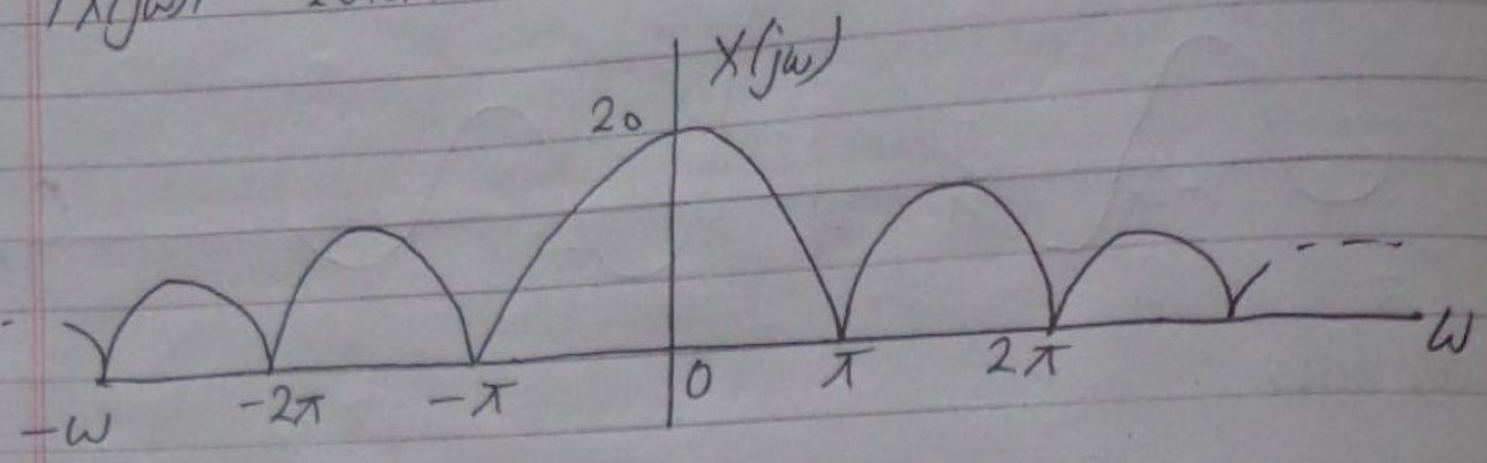
$$= 2_0 \frac{e^{-jw}}{w} \cdot \frac{e^{jw} - e^{-jw}}{2j}$$

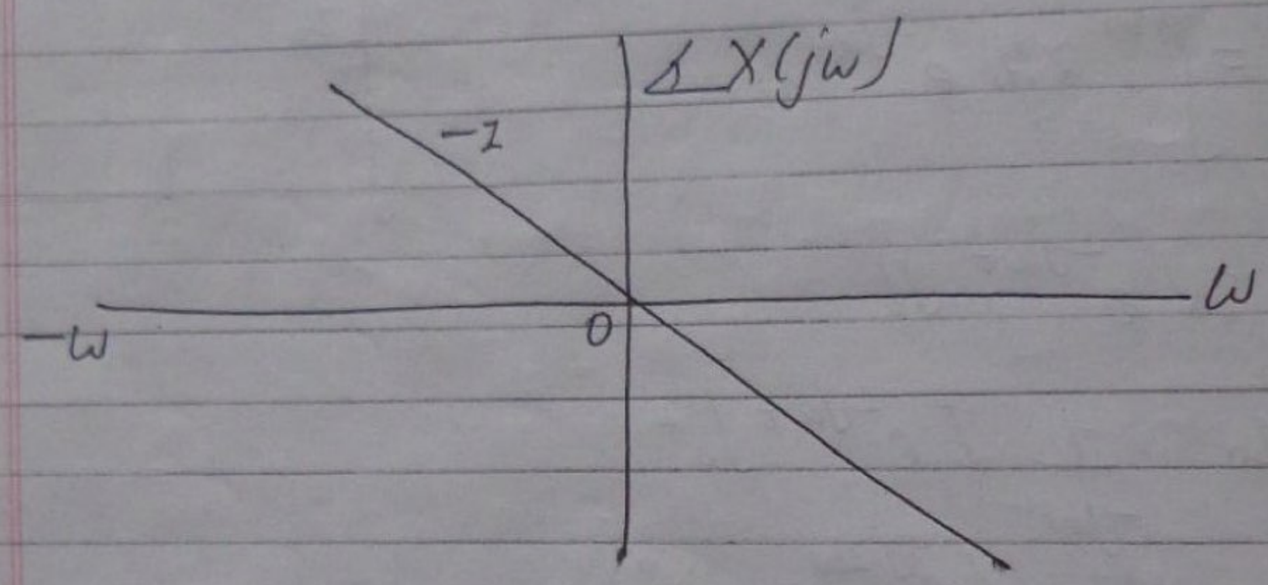
$$= 2_0 \frac{e^{-jw}}{w} \cdot \sin(w)$$

Scanned with CamScanner

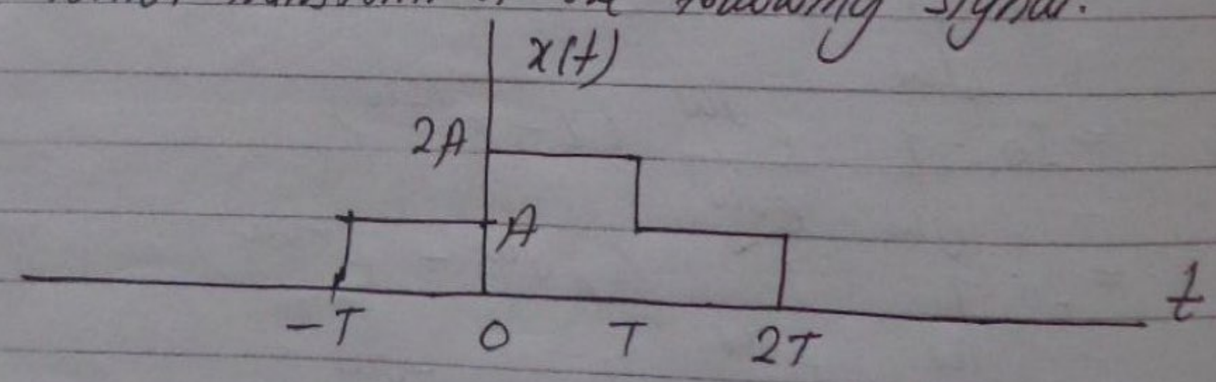
$$= 20\, e^{-j\omega} \sin c(\omega)$$

$\therefore\ |X(j\omega)| = 20\,|\sin c(\omega)|$ and $\angle X(j\omega) = -\omega$





1. Find the fourier transform of the following signal.



Sol^n,

$$\Rightarrow X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{0} A e^{-j\omega t}\, dt + \int_{0}^{T} 2A e^{-j\omega t}\, dt + \int_{T}^{2T} A e^{-j\omega t}\, dt$$

Scanned with CamScanner

$$= \frac{A}{-j\omega}\left[e^{-j\omega t}\right]_{-T}^{0} + \frac{2A}{-j\omega}\left[e^{-j\omega t}\right]_{0}^{T} + \frac{A}{-j\omega}\left[e^{-j\omega t}\right]_{T}^{2T}$$

$$= \frac{A}{-j\omega}\left(1-e^{-j\omega t}\right) + \frac{2A}{-j\omega}\left(e^{-j\omega T}-1\right) + \frac{A}{-j\omega}\left(e^{-j\omega 2T} - e^{-j\omega T}\right)$$

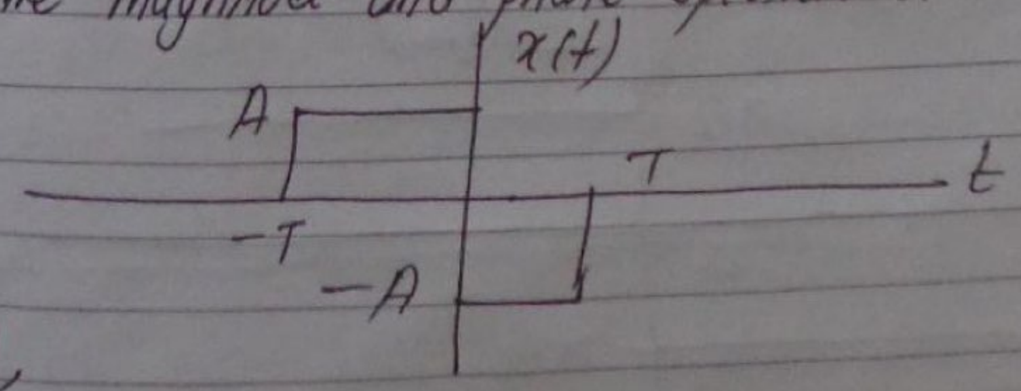
$$= \frac{A}{-j\omega}\cdot\frac{1}{e^{-j\omega T/2}}\left(e^{-j\omega T/2}-e^{-j\omega T/2}\right) + \frac{2A}{-j\omega}\cdot\frac{1}{e^{j\omega T/2}}\left(e^{-j\omega T/2} - e^{j\omega}\right) + \frac{A}{-j\omega}e^{-j\omega T}\left(e^{j\omega T}-1\right)$$

$$= \frac{A}{+j\omega}e^{j\omega T/2}\left(e^{j\omega T/2}-e^{-j\omega T/2}\right) + \frac{2A}{j\omega}e^{-j\omega T/2}\left(e^{j\omega T/2} - e^{-j}\right) + \frac{+A}{-j\omega}e^{-j\omega T}\cdot\frac{1}{j\omega T/2}\left(e^{-j\omega T/2}-e^{j\omega T/2}\right)$$

$$= \frac{2A}{\omega}e^{j\omega T/2}\sin(\omega T/2) + \frac{4A}{\omega}e^{-j\omega T/2}\sin(\omega T/2) + \frac{2A}{\omega}\cdot e^{-j\omega T\frac{3}{2}}\sin(\omega T/2)$$

$$\therefore X(j\omega) = TAe^{j\omega T/2}\sin(\omega T/2) + 2TAe^{-j\omega T/2}\sin(\omega T/2) + TAe^{-j3/2\omega T}\sin(\omega T/2)$$

11. Plot the magnitude and phase spectrum of the signal



Sol^n,

Scanned with CamScanner

Sol^n,

$\Rightarrow$ we have,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{0} A e^{-j\omega t}\, dt + \int_{0}^{T} (-A) e^{-j\omega t}\, dt.$$

$$= \frac{A}{-j\omega}\left[e^{-j\omega t}\right]_{-T}^{0} - \frac{A}{-j\omega}\left[e^{-j\omega t}\right]_{0}^{T}$$

$$= \frac{A}{-j\omega}\left(1 - e^{+j\omega T}\right) + \frac{A}{j\omega}\left(e^{-j\omega T} - 1\right)$$

$$= j\frac{A}{\omega} - j\frac{A}{\omega} e^{j\omega T} - j\frac{A}{\omega} e^{-j\omega T} + j\frac{A}{\omega}$$

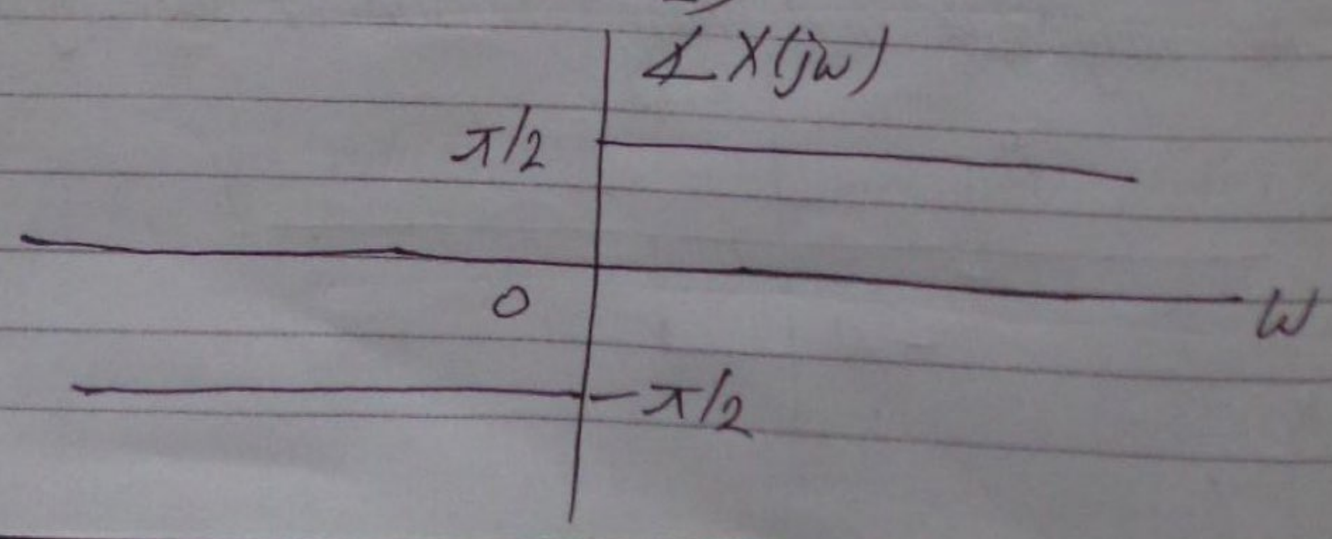
$$= j\frac{2A}{\omega} - j\frac{A}{\omega}\left(e^{j\omega T} + e^{-j\omega T}\right)$$

$$= j\frac{2A}{\omega} - j\frac{2A}{\omega}\cos(\omega T)$$

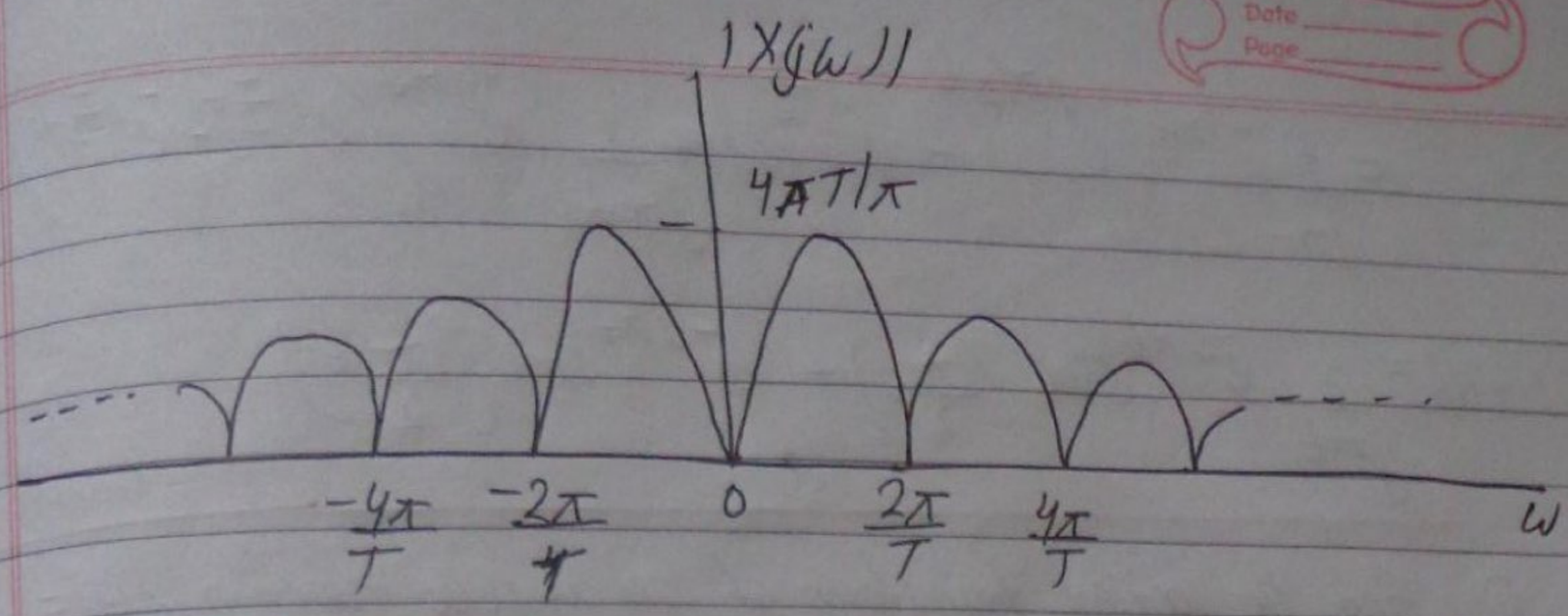
$$= j\left(\frac{2A}{\omega}\right) 2\sin^2\left(\frac{\omega T}{2}\right)$$

$$= j\, AT^2\omega\, \frac{\sin^2(\omega T/2)}{\left(\frac{\omega T}{2}\right)^2}$$

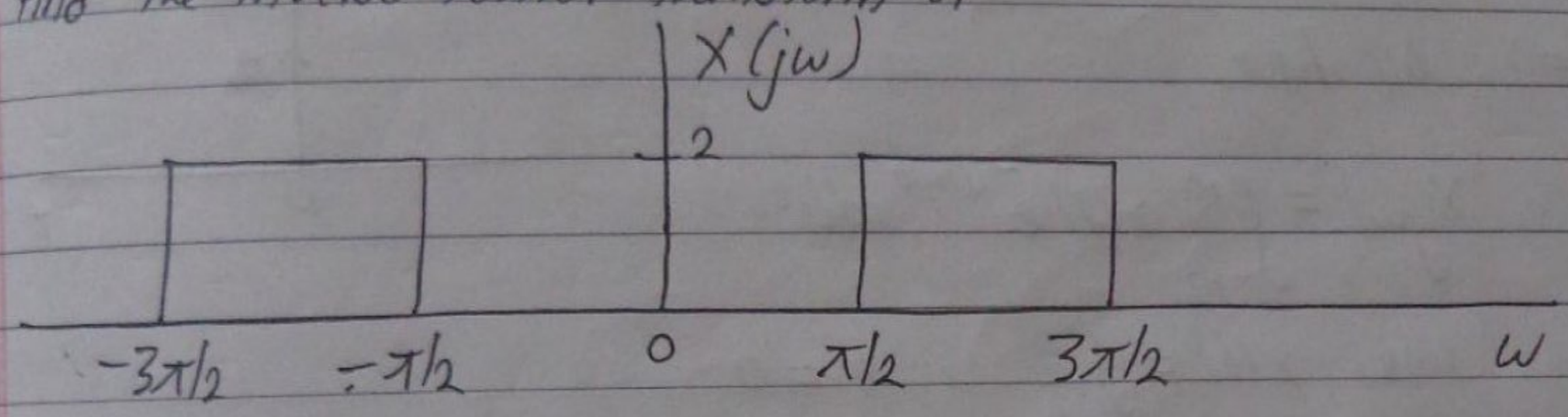
$$\therefore X(j\omega) = j\, AT^2\omega\, \mathrm{sinc}^2\left(\frac{\omega T}{2}\right)$$



Scanned with CamScanner



N. Find the inverse fourier transform of:



Sol^n,

$\Rightarrow$ we have,

inverse fourier transform of $X(j\omega)$ is,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-3\pi/2}^{-\pi/2} 2e^{j\omega t}\, d\omega + \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} 2e^{j\omega t}\, d\omega$$

$$= \frac{1}{\pi}\cdot\frac{1}{j}\left[e^{j\omega t}\right]_{-3\pi/2}^{-\pi/2} + \frac{1}{\pi}\cdot\frac{1}{j}\cdot\left[e^{j\omega t}\right]_{\pi/2}^{3\pi/2}$$

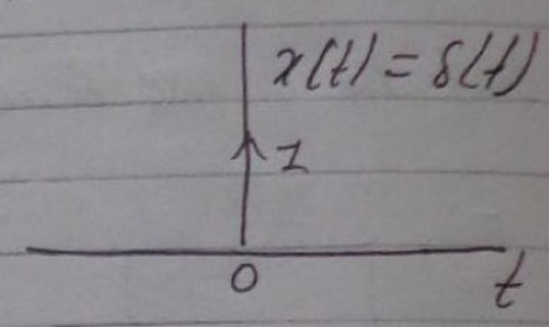
$$= \frac{1}{j\pi}\left(e^{-j\pi/2\, t} - e^{-j3\pi/2\, t}\right) + \frac{1}{j\pi}\left(e^{j3\pi/2\, t} - e^{j\frac{\pi}{2} t}\right)$$

Scanned with CamScanner

$$= \frac{1}{j\pi} e^{-j\pi t}\left(e^{j\pi/2\, t} - e^{-j\frac{\pi}{2}t}\right) + \frac{1}{j\pi t} e^{+j\pi t}\left(e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}\right)$$

$$\therefore x(t) = \frac{2}{j\pi t} e^{-j\pi t} \operatorname{sinc}\left(\frac{\pi}{2}t\right) + e^{-j\pi t} \operatorname{sinc}\left(\frac{\pi}{2}t\right)$$

Q. Find the fourier transform of an impulse function $x(t) = \delta(t)$. Also, plot the spectrum.

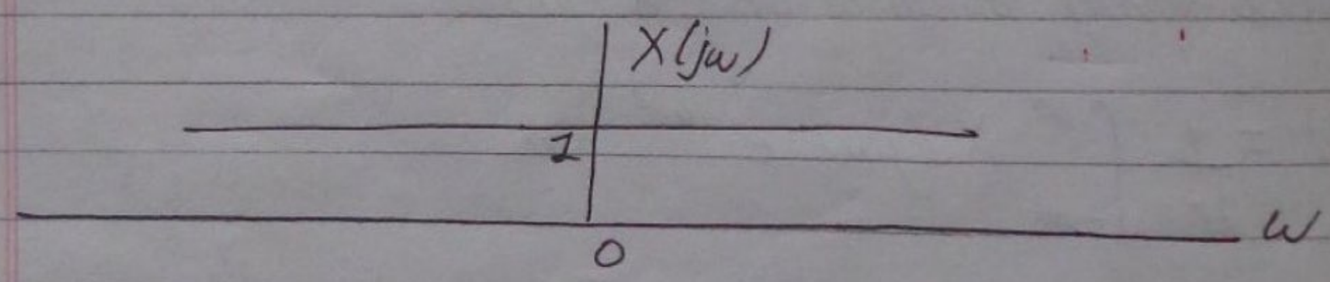


$\Rightarrow$ We have,

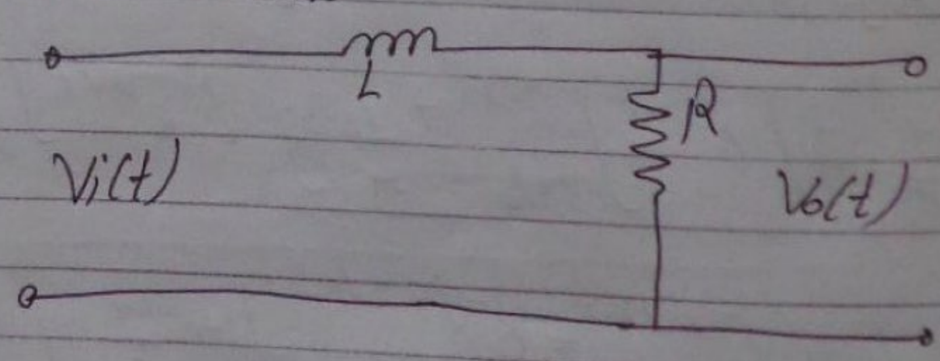
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt.$$

since, $x(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$

$$\therefore X(j\omega) = e^{-j\omega t}\Big|_{t=0} = 1$$



P. Consider the RL circuit shown,



Sol^n,

Scanned with CamScanner

using KVL,

$$V_i(t) = L\frac{di(t)}{dt} + V_o(t)$$

since, $i(t) = \frac{V_o(t)}{R}$

$$\therefore V_i(t) = \frac{L}{R}\cdot\frac{dV_o(t)}{dt} + V_o(t)$$

Applying laplace transform and zero initial conditions,

$$V_i(s) = \frac{L}{R}\cdot s\cdot V_o(s) + V_o(s)$$

or $H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{\frac{L}{R}\cdot s + 1}$

or $H(s) = \frac{R/L}{s + R/L}$

Applying inverse laplace transform, we get,

$$h(t) = \frac{R}{L}\cdot e^{-\frac{R}{L}\cdot t}\, u(t)$$

This is the required impulse response.

Q. Determine the zero state response of a discrete time systems described by the difference eqn,

$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n]$ where, $x[n] = 2^n$ for $n \geq 0$ — ①

Soln,

⇒ Assuming zero initial conditions,

$$y[n-1] = y[n-2] = 0$$

Scanned with CamScanner

Zero state response, $y_{zs}[n] = ?$

$y_h[n] - \frac{5}{6}[n-1] + \frac{1}{6}y_n[n-2] = 0$

Homogenous solutions is of the form, λ^n,

$\therefore \lambda^n - \frac{5}{6}\lambda^{n-1} + \frac{1}{6}\lambda^{n-2} = 0$

or $\lambda^{n-2}(\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6}) = 0$

or $\lambda^2 - \frac{5}{6}\lambda + \frac{1}{6} = 0$

or $6\lambda^2 - 3\lambda - 2\lambda + 1 = 0$

or $3\lambda(2\lambda - 1) - 1(2\lambda - 1) = 0$

or $(2\lambda - 1)(3\lambda - 1) = 0$

$\therefore \lambda = \frac{1}{2}$ or $\frac{1}{3}$

So,

$y_h[n] = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{3}\right)^n$ — ②

For particular solutions, $y_p[n]$,

$y_p[n] - \frac{5}{6}y_p[n-1] + \frac{1}{6}y_p[n-2] = x[n]$

let, $y_p[n] = k2^n$ - for $n \geq 0$

$\therefore k2^n - \frac{5}{6}k2^{n-1} + \frac{1}{6}k2^{n-2} = 2^n$

or $k2^n(1 - \frac{5}{6}\cdot 2^{-1} + \frac{1}{6}\cdot 2^{-2}) = 2^n$

Scanned with CamScanner

$\therefore K = \frac{8}{5}$

$\therefore y_p[n] = \frac{8}{5} \cdot 2^n$ for $n \geq 0$

Hence, $y_{zs}[n] = y_n[n] + y_p[n]$

$= \left[C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{3}\right)^n\right]u[n] + \frac{8}{5} \cdot 2^n u[n]$ — ③

For calculating value of C_1 and C_2,
For $n=0$, from eqn ①

or $y_{zs}[0] - \frac{5}{6} y_{zs}[-1] + \frac{1}{6} y[-2] = 2^0$

or $y[0] = 1$
For $n=1$,
$y_{zs}[1] - \frac{5}{6} y_{zs}[0] + \frac{1}{6} y_{zs}[-1] = 2^1$

or $y_{zs}[1] = 2 + \frac{5}{6} = \frac{17}{6}$

From ③ using $y_{zs}[0] = 1$ and $y_{zs}[-1] = \frac{17}{6}$

or $1 = C_1\left(\frac{1}{2}\right)^0 + C_2\left(\frac{1}{3}\right)^0 + \left(\frac{8}{5}\right)2^0$

or $1 = C_1 + C_2 + \frac{8}{5}$

or $C_1 + C_2 = -\frac{8}{5} + 1 = \frac{-3}{5}$ — ④

Now,

Scanned with CamScanner

or $\frac{17}{6} = C_1\left(\frac{1}{2}\right) + C_2\left(\frac{1}{3}\right) + \frac{8}{5}(2)$

or $3C_1 + 2C_2 = \frac{-11}{5}$ — (5)

solving eqn (4) and (5),

$C_1 = -1$

$C_2 = \frac{2}{5}$

Hence,

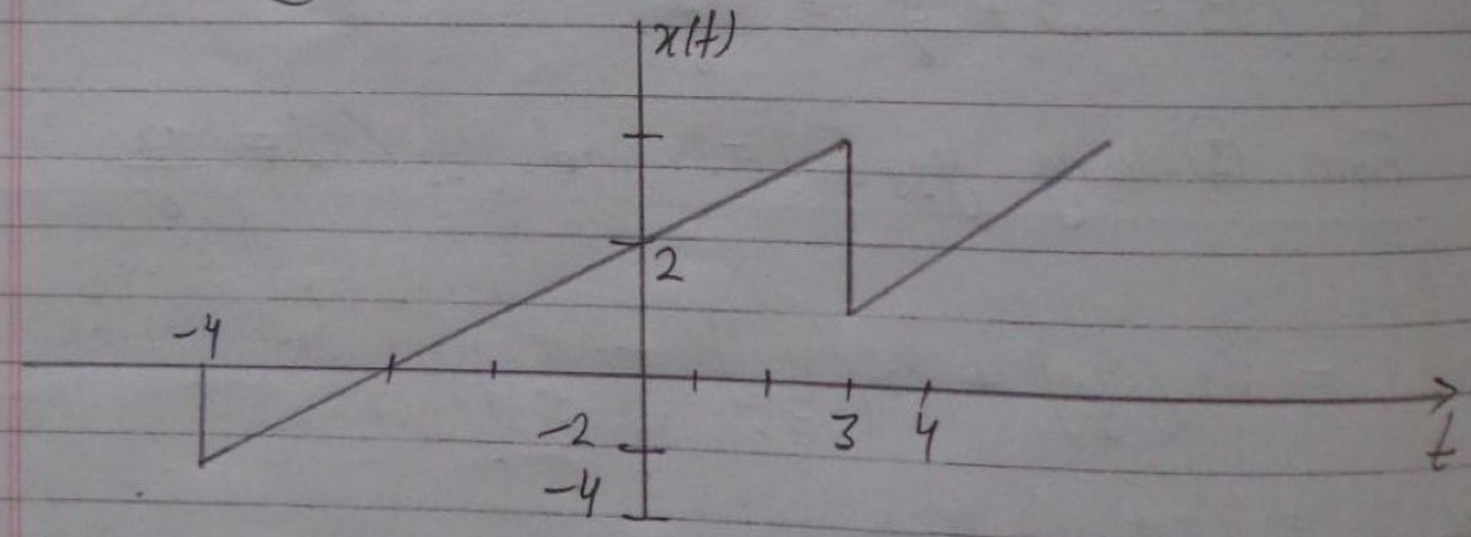
$\therefore y_{zs}[n] = \left(-\left(\frac{1}{2}\right)^n + \frac{2}{5}\left(\frac{1}{3}\right)^n + \left(\frac{8}{5}\right)2^n\right) u[n]$

R. Sketch the following signals.

i)

$$x(t) = \begin{cases} 0 & \text{if } t < -4 \\ t+2 & \text{if } -4 \le t < 3 \\ t-2 & \text{if } 3 \le t \end{cases}$$

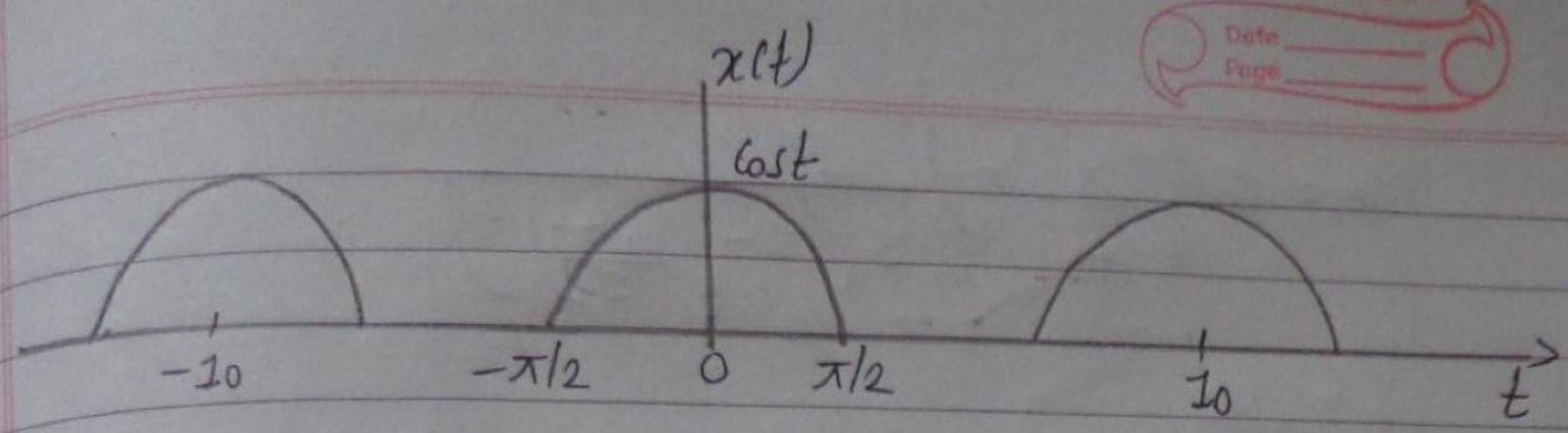
$\Rightarrow$



(11) Find the fourier series and plot the spectra versus frequency $\omega = n\omega_0$.

Sol^n,

Scanned with CamScanner



$\Rightarrow T = 10$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$x(t) = \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

$$C_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{10}\int_{-\pi/2}^{\pi/2} \cos t\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{20}\int_{-\pi/2}^{\pi/2} e^{jt(1-n\omega_0)} + e^{-jt(1+n\omega_0)}\, dt$$

$$= \frac{1}{20}\cdot\frac{1}{j(1-n\omega_0)}\left[e^{j\frac{\pi}{2}(1-n\omega_0)} - e^{-j\frac{\pi}{2}(1-n\omega_0)}\right] + \frac{-1}{20j(1+n\omega_0)}\left[e^{-\frac{\pi}{2}j(1+n\omega_0)} - e^{j\frac{\pi}{2}(1+n\omega_0)}\right]$$

$$= \frac{1}{10(1-n\omega_0)}\cos\left(\frac{\pi}{2}n\omega_0\right) + \frac{1}{10(1+n\omega_0)}\cos\left(\frac{\pi}{2}n\omega_0\right)$$

$$= \cos\left(\frac{\pi}{2}n\omega_0\right) \times \frac{1}{10}\left[\frac{1}{1-n\omega_0} + \frac{1}{1+n\omega_0}\right]$$

$$C_n = \cos\left(\frac{\pi}{2}n\omega_0\right)\frac{1}{5(1-(n\omega_0)^2)}$$

for $\omega_0 = \frac{\pi}{5}$,

Scanned with CamScanner

$$C_n = \cos\left(\frac{\pi^2}{T_0} n\right) \frac{1}{5\left(1-\left(\frac{n\pi}{5}\right)^2\right)}$$

$$C_0 = 1/5$$

$$C_1 = \cos\left(\frac{\pi^2}{T_0}\right) \frac{1}{5\left(j-\left(\frac{\pi}{5}\right)^2\right)} = 0.1822$$

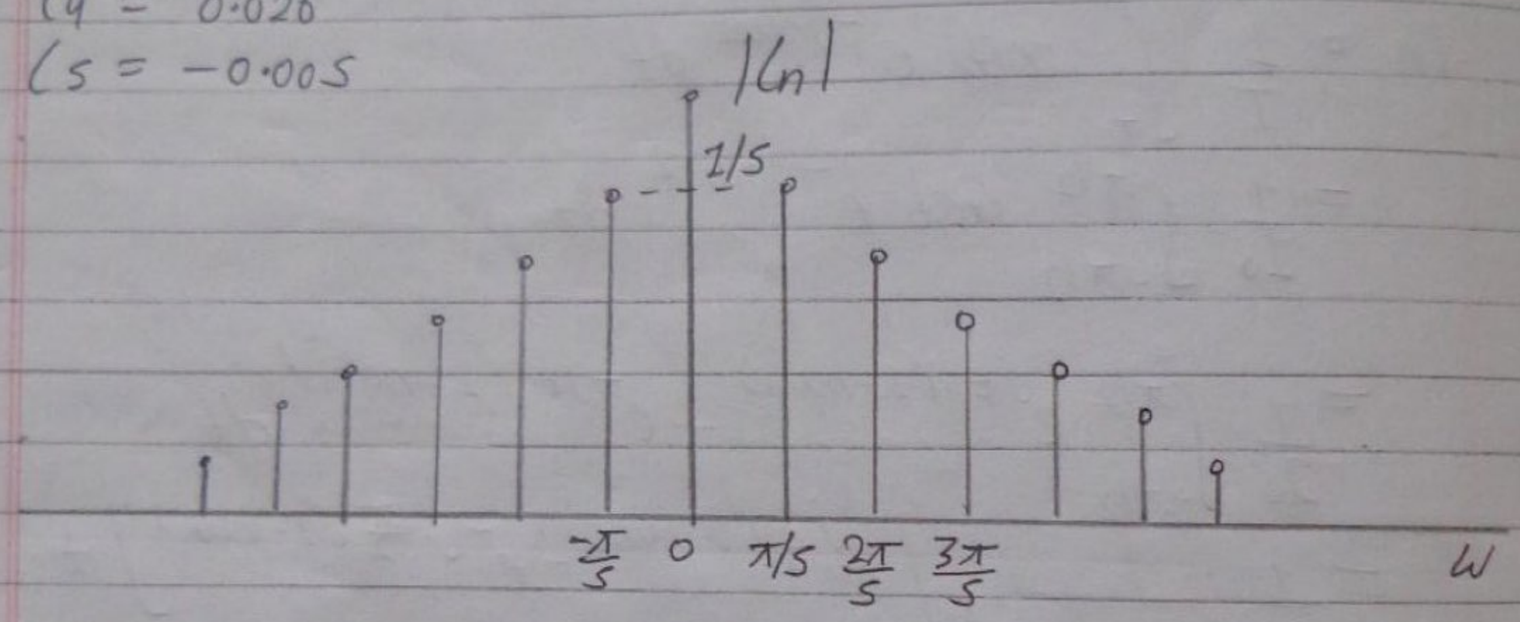
$C_2 = 0.1355$

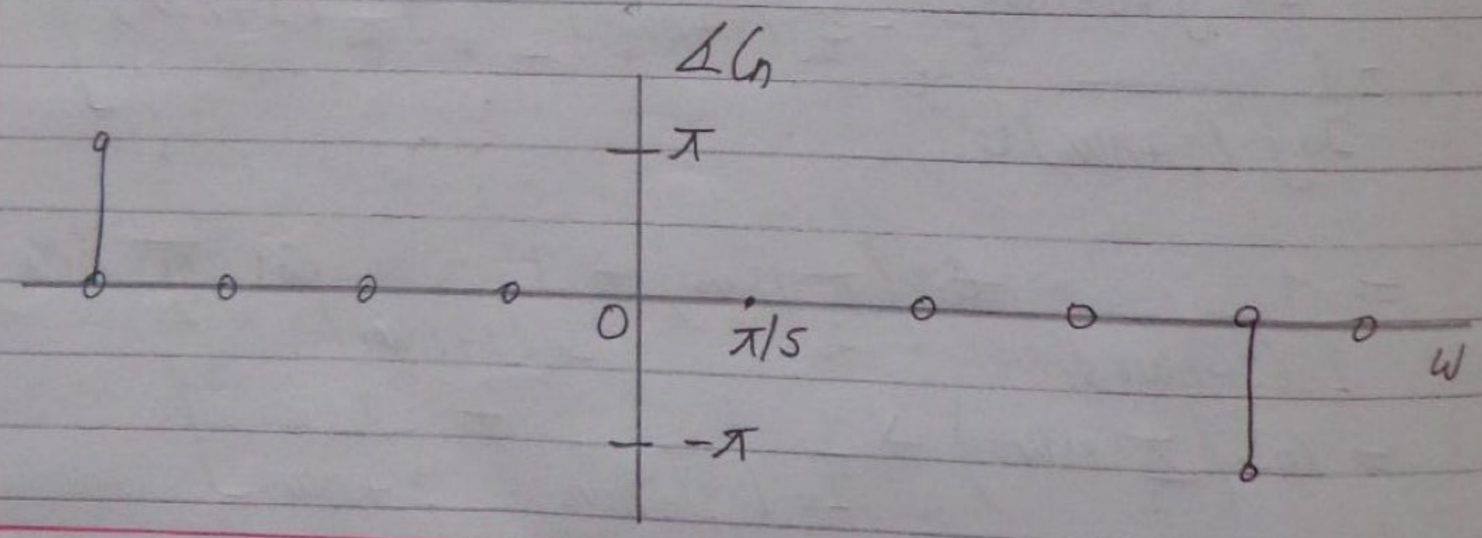
$C_3 = 0.0771$

$C_4 = 0.026$

$C_5 = -0.005$

and, $C_{-n} = \overline{C_n} = C_n$





(iii) simplify following expression. give your answers both in polar and in rectangular form.

Scanned with CamScanner

a) $c = 3e^{j\pi/4} + 4e^{-j\pi/2}$

Sol^n,

$\Rightarrow c = 3\cos\frac{\pi}{4} + 3j\sin\frac{\pi}{4} + 4\cos\left(-\frac{\pi}{2}\right) + 4j\sin\left(-\frac{\pi}{2}\right)$

$= \frac{3}{2}\cdot\sqrt{2} + j\frac{3\sqrt{2}}{2} - 4j$

$= 2.121 - 1.878j$

$= 2.833\, e^{-j0.725}$

b) $c = 2e^{j\pi/2} - 3e^{j\pi/3}$

Sol^n,

$\Rightarrow c = 2\cos\frac{\pi}{2} + 2j\sin\frac{\pi}{2} - 3\cos\frac{\pi}{3} - 3j\sin\frac{\pi}{3}$

$\Rightarrow c = 2j - \frac{3}{2} - 2.598j$

$= -1.5 - j0.598$

$= 1.615 \times e^{3.521j}$

iv) Find the Z-transform of the following signals:

a) $x[n] = u[n] - u[n-4]$

Sol^n,

$\Rightarrow X(z) = \sum_{n=0}^{3} z^{-n}$

$= 1 + z^{-1} + z^{-2} + z^{-3}$

b) $x[n] = 0.5^n\, u[n]$

$X(z) = \sum_{n=-\infty}^{\infty} 0.5^n\, u[n]\, z^{-n}$

Scanned with CamScanner

$$= \sum_{n=0}^{\infty} (0.5 Z^{-1})^n$$

using geometric series convergence,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{if } |a| < 1,$$

$$\therefore \quad X(z) = \frac{1}{1-0.5Z^{-1}}$$

c) $x[n] = [1 \quad 4 \quad 8 \quad 2]$

sol^n,

$$\Rightarrow \quad X(z) = 1 + 4Z^{-1} + 8Z^{-2} + 2Z^{-3}$$

d. $x[n] = 2(0.8)^n u[n]$

sol^n,

$$\Rightarrow \quad X(z) = \frac{2}{1-0.8Z^{-1}}$$

e) Find the inverse Z-transform of :

(1) $X(z) = \dfrac{(Z-1)(Z+0.8)}{(Z-0.5)(Z+0.2)}$

sol^n,

$$\Rightarrow \quad \frac{X(Z)}{Z} = \frac{C_1}{Z} + \frac{C_2}{Z-0.5} + \frac{C_3}{Z+0.2}$$

$$C_1 = \frac{X(Z)}{Z} \cdot Z \Big|_{Z=0} = 8$$

$$C_2 = \frac{X(Z)}{Z} (Z-0.5) \Big|_{Z=0.5} = \frac{(Z-1)(Z+0.8)}{Z \cdot (Z+0.2)} \Big|_{Z=0.5}$$

$$= -1.857$$

Scanned with CamScanner

$$C_3 = X(z)\frac{(z+0.2)}{z}\Big|_{z=-0.2} = -5.143$$

$$X(z) = 8 - \frac{1.857\,z}{z-0.5} - \frac{5.143\,z}{z+0.2}$$

$$\therefore\ x[n] = 8\,\delta[n] - 1.857\,(0.5)^n u[n] - 5.143\,(-0.2)^n\, u[n].$$

(ii) $$X(z) = \frac{z+0.8}{(z-0.5)(z+0.2)}$$

$$X(z) = \frac{1.857}{z-0.5} - \frac{0.8571}{z+0.2}$$

let, $$X_1(z) = \frac{1.857}{z-0.5}z - \frac{0.8571\,z}{z+0.2}$$

then,

$$x_1[n] = (1.857\,(0.5)^n - 0.8571\,(-0.2)^n)\,u[n]$$

Hence,

$$X(z) = z^{-1}X(z)$$

or,

$$\therefore\ x[n] = (1.857\,(0.5)^{n-1} - 0.8571\,(-0.2)^{n-1})\,u[n-1].$$

5. Find the necessary and sufficient condition on the impulse response $h[n]$ such that for any input, $x[n]$,

$$\max\{|x[n]|\} \geq \{|y[n]|\}.$$

where,

$$y[n] = x[n] * h[n].$$

Sol^n,

=> Given,

$$y[n] = x[n] * h[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k]$$

Scanned with CamScanner

or $|y[n]| = \left|\sum_{k=-\infty}^{\infty} x[n-k]\, h[k]\right|$

$$\max\{|y[n]|\} = \max\left\{\left|\sum_{k=-\infty}^{\infty} x[n-k]\, h[k]\right|\right\}$$

$$\leq \max \sum_{k=-\infty}^{\infty} |x[n-k]|\, |h[k]|$$

$$\leq \sum_{k=-\infty}^{\infty} \max\{|x[n-k]|\}\, |h[k]|$$

$$= \max\{|x[n]|\} \sum_{k=-\infty}^{\infty} |h[k]|$$

Hence,

$$\max\{|y[n]|\} \leq \max\{|x[n]|\} \sum_{k=-\infty}^{\infty} |h[k]|$$

that $\sum_{k=-\infty}^{\infty} |h[k]| \leq 1, \Rightarrow \max\{|y[n]|\} \leq \max\{|x[n]|\}$.

This means that $\sum_{k=-\infty}^{\infty} |h[k]| \leq 1$ is a sufficient condition. It is necessary because some $x[n]$ always exists that yields $y[n] = \sum_{k=-\infty}^{\infty} |h[k]| \cdot x[n]$ consists of a sequence of $+1$'s and -1's. Therefore, since, $\max\{x[n]\} = 1$, it is necessary that $\sum_{k=-\infty}^{\infty} |h[k]| \leq 1$ to ensure that $y[n] \leq \max\{|x[n]|\} = 1$.

Scanned with CamScanner

Click on wanted Notes/solutions to Download from website:

26)Engineering Chemistry
27)C programming 2015 Fall solution
28)OOPS
29)Electronic Device
30)Network theory
31)Electromagnetic Field and Theory
32)Logic circuit
33)Electronic circuits
34)Electrical Machine
35)Instrumentation
36)Advance Electrical Machine
37)Control system
38)Signals and systems
39)Probability and statistics
40)Transmission and distribution system
41)Microprocessor Board Questions solutions
42)Switch gear and protection system
43)Power electronics 6th semester Note
44)Research methodology
45)Power plant Equipment
46)Engineering Economics
47)Communication system Engineering
48)Power system analysis
49)C programming lecture Note
50)OOPS lecture Note

And other many Notes / solutions/Projects are available :)

Scanned with CamScanner

Signals & Systems

Solutions to Assessment 2014

1. **(a) Define the terms "signals" and "systems". Discuss briefly about unit impulse signal, unit step signal, exponential signal, sinusoidal signal, and sinc signal.**

Solution:

Signals:

A signal is defined as a function of one or more independent variables that carries the information about the nature of a physical phenomenon. In electrical engineering, time 't' is the independent variable. Examples of signals include speech, picture, text, etc.

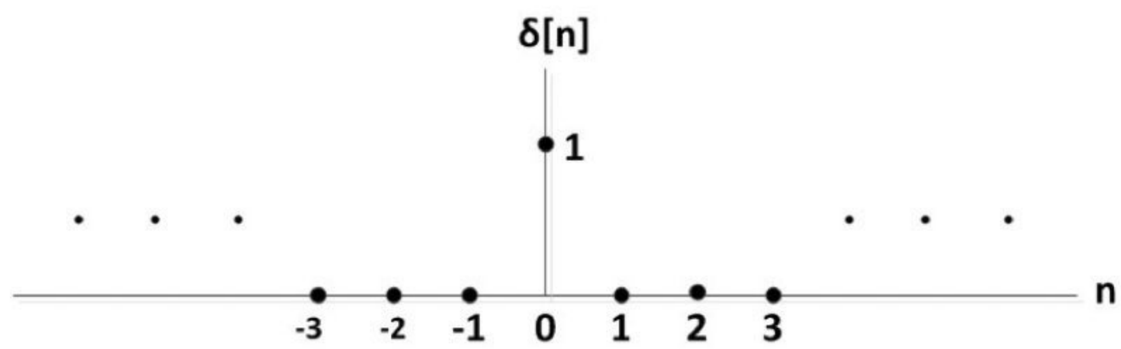
Systems:

A system is defined as an element or an interconnection of several elements or devices that performs a particular task to produce the desired output signal for the particular input. Amplifiers, filters, etc. are the examples of systems.

Unit Impulse Signal:

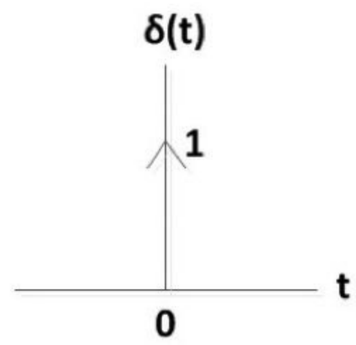
A discrete-time unit impulse signal is defined as,

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



A continuous-time unit impulse signal is defined as,

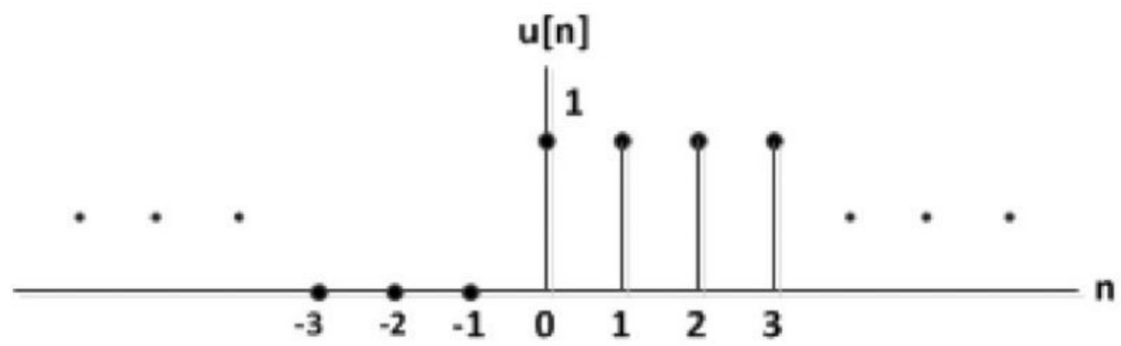
$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$



Unit Step Signal:

A discrete-time unit step signal is defined as,

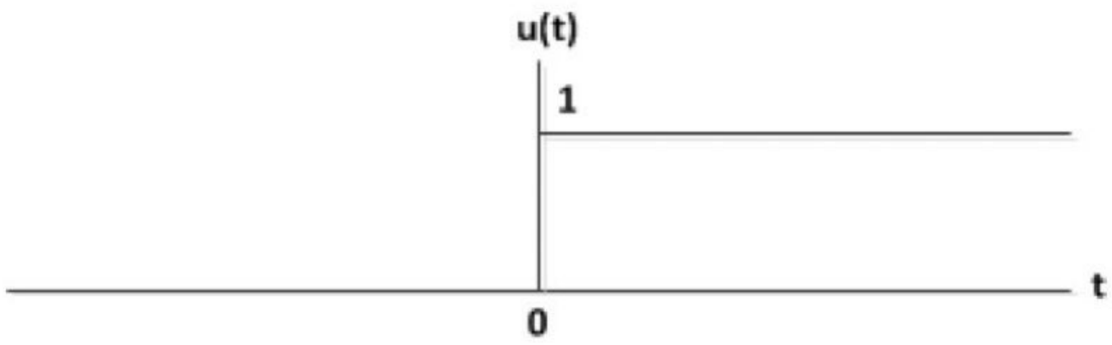
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



A continuous-time unit step signal is defined as,

Scanned with CamScanner

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

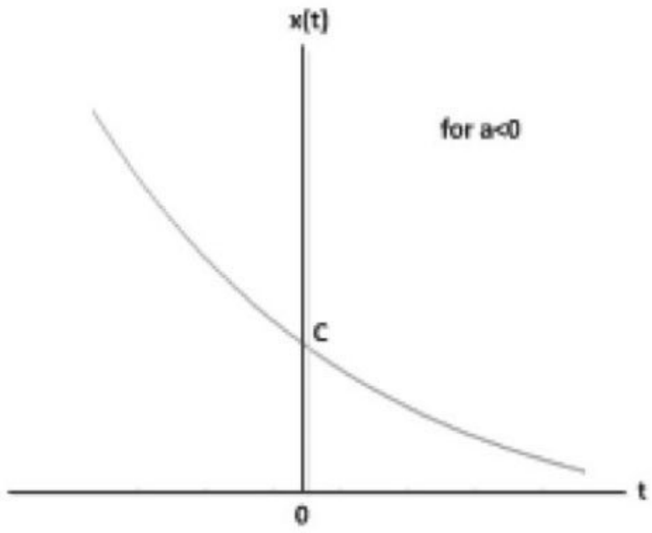


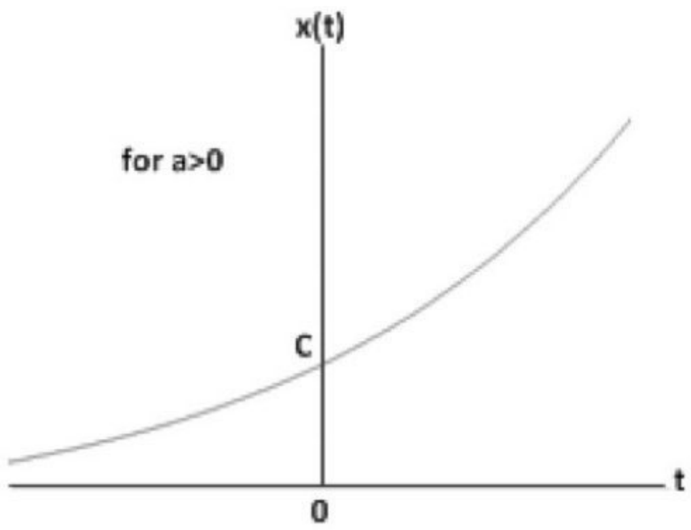
Exponential Signal:

A continuous-time real exponential signal is defined as,

$$x(t) = Ce^{at},$$

where C and a are real constants. Depending upon the sign of $'a'$, we have two forms: decaying exponential for $a < 0$ and growing exponential for $a > 0$.



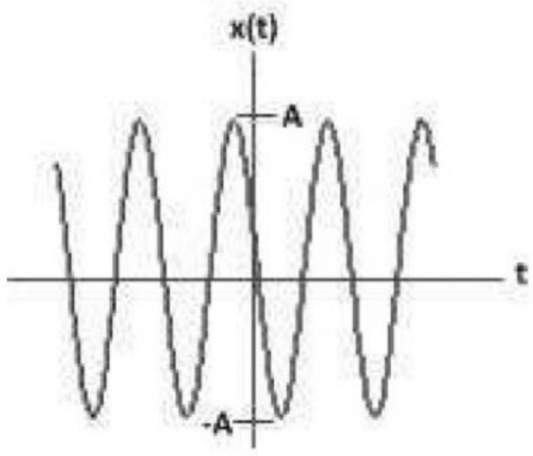


Sinusoidal Signal:

A continuous-time sinusoidal signal is defined as,

$$x(t) = A\cos(2\pi ft + \emptyset),$$

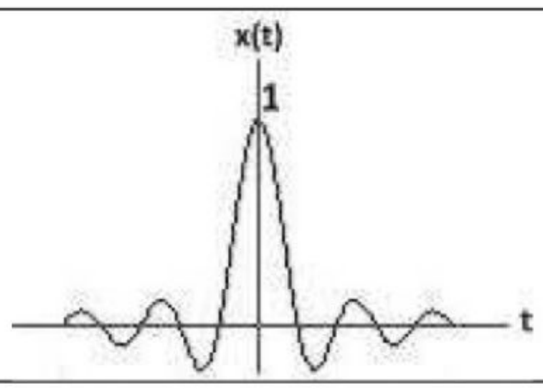
where A is the peak amplitude (V), f is the frequency (Hz), and $\emptyset$ is the relative phase angle (rad.).



Sinc Signal:

A continuous-time sinc signal is defined as,

$$\text{sinc}(t) = \begin{cases} 1, & t = 0 \\ 0, & t = \pm n \\ \dfrac{\sin(\pi t)}{\pi t}, & \text{otherwise} \end{cases}$$



(b) What do you mean by even and odd signals? Decompose the signal $x[n] = u[n] - u[n-7]$ into even and odd parts.

Solution:

Even Signals and Odd Signals:

A signal $x(t)$ is said to be an even signal if it satisfies the following condition:

Scanned with CamScanner

$$x(-t) = x(t), \text{ for all } t$$

A signal $x(t)$ is said to be an odd signal if it satisfies the following condition:

$$x(-t) = -x(t), \text{ for all } t$$

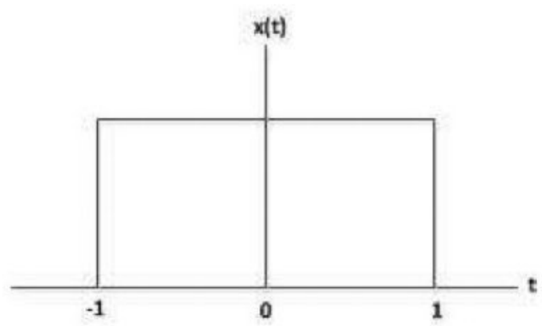


Figure Example of an even signal.

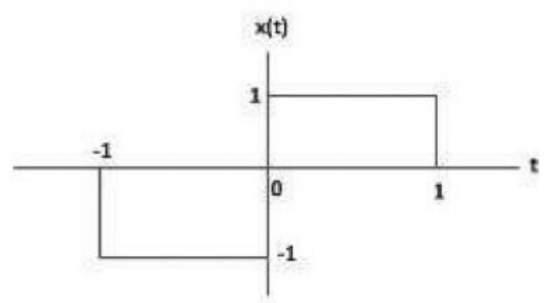
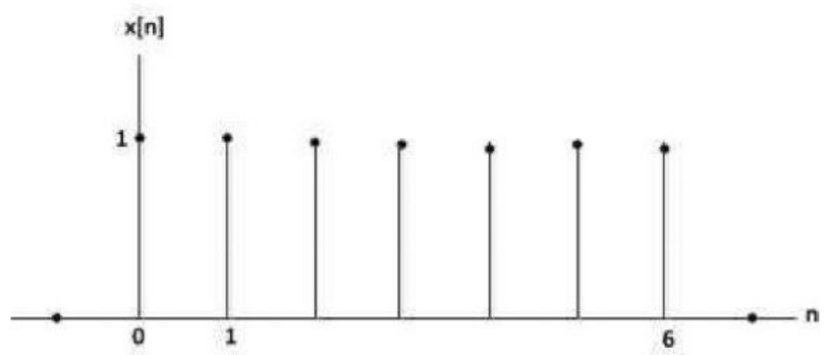


Figure Example of an odd signal.

Given signal is,

$x[n] = u[n] - u[n] - u[n-7]$ which is shown in the following figure.

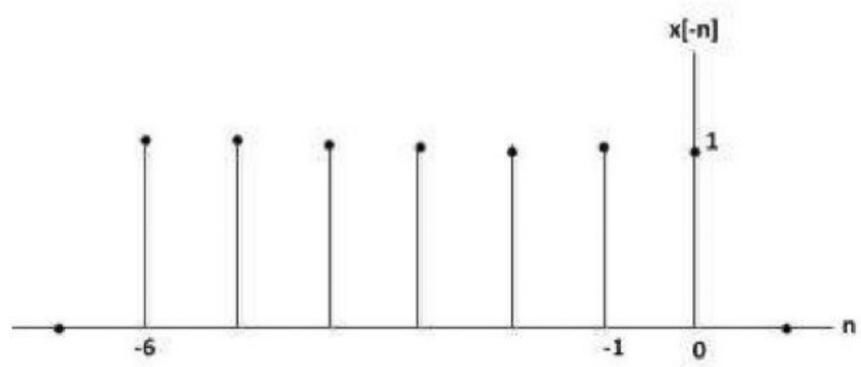


Any signal can be decomposed into even and odd parts as,

Even part: $x_e[n] = \frac{1}{2}[x[n] + x[-n]]$

And, odd part: $x_o[n] = \frac{1}{2}[x[n] - x[-n]]$

We have,



Therefore, the even and odd parts of the given signal will be

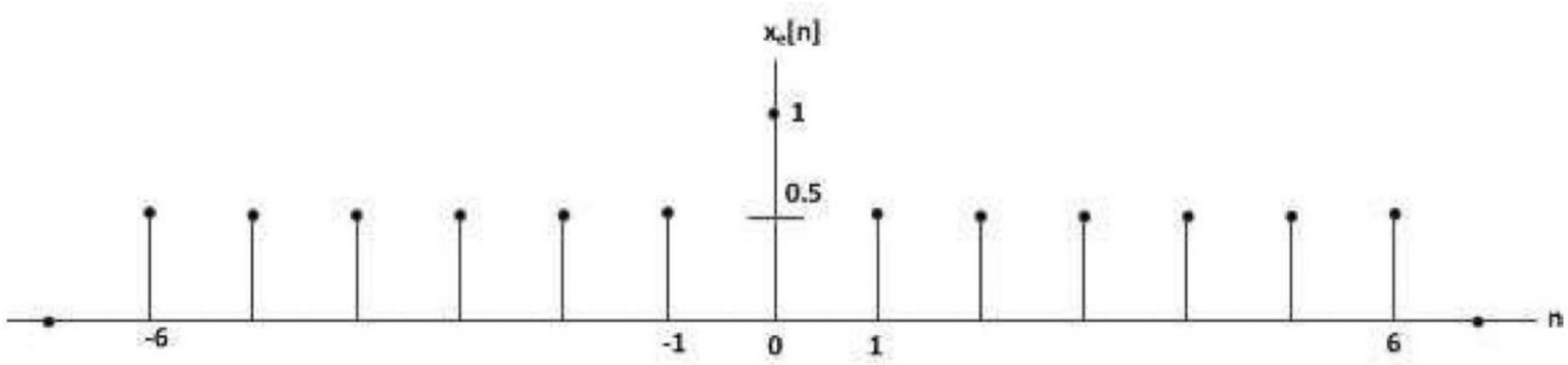


Figure Even part.

Scanned with CamScanner

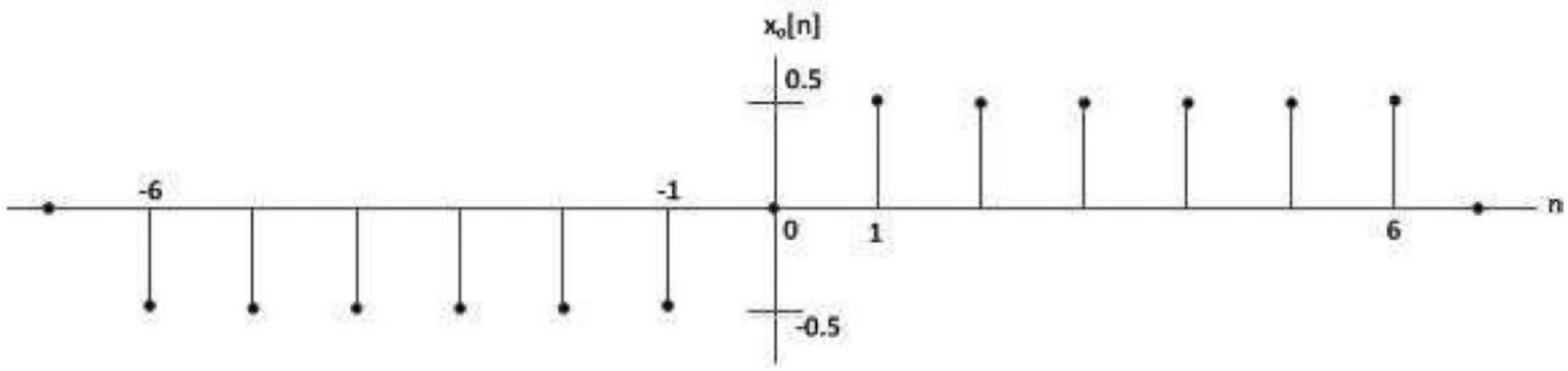
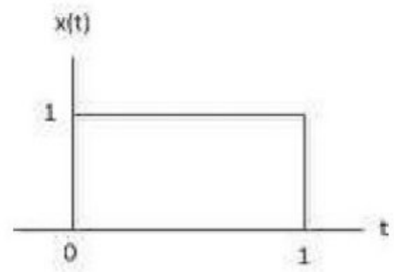


Figure Odd part.

2. **(a) Find the Fourier transform and plot the spectrum of $x(t) = u(t) - u(t-1)$.**

Solution:

The given signal is,



The Fourier transform of a signal $x(t)$ is given as,

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

For the given signal, we have

$$X(jw) = \int_0^1 e^{-jwt}dt$$

$$= \frac{1}{-jw} e^{-jwt}\Big|_0^1$$

$$= \frac{1}{-jw}\left(e^{-jw} - 1\right)$$

Multiplying numerator and denominator by $e^{jw/2}$, we get

$$X(jw) = \frac{1}{-jw}\frac{1}{e^{jw/2}}\left(e^{-jw/2} - e^{jw/2}\right)$$

$$= \frac{2e^{-jw/2}}{w}\frac{\left(e^{jw/2} - e^{-jw/2}\right)}{2j}$$

$$\therefore X(jw) = e^{-jw/2}\frac{\sin(w/2)}{w/2}$$

The magnitude is,

$$|X(jw)| = \left|\frac{\sin(w/2)}{w/2}\right|$$

And, the phase is,

$$\measuredangle X(jw) = -\frac{w}{2}$$

The magnitude plot and the phase plot are shown in the following figures.

Scanned with CamScanner

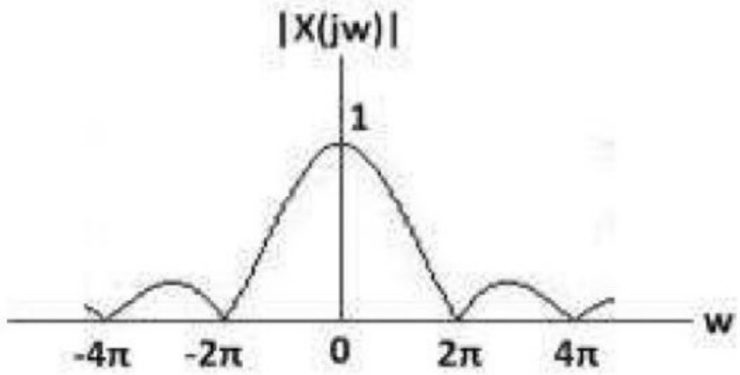


Figure Magnitude plot.

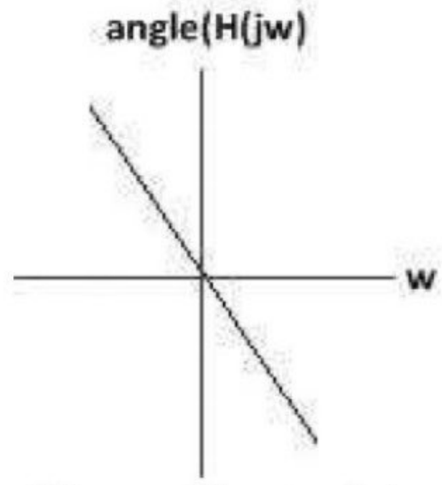
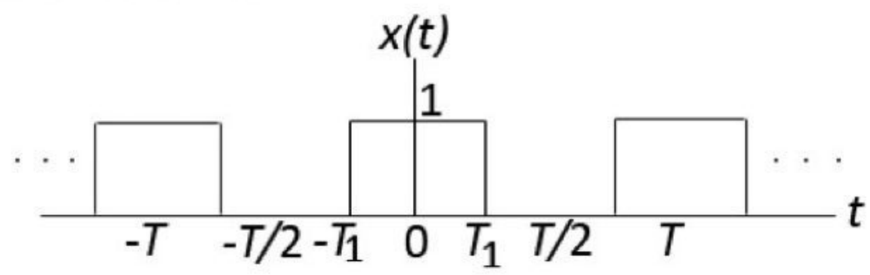


Figure Phase plot.

(b) Determine and plot the Fourier series coefficients of the following periodic rectangular pulse for $T = 4T_1$ where $T_1 = 0.25$ seconds.



Solution:

Here,

The fundamental period, $T = 4T_1 = 4 * 0.25 = 1$second

And, the fundamental frequency, $w_0 = {}^{2\pi}/_{T} = 2\pi$ rad./sec.

We know that the Fourier series coefficients of a periodic signal are given by,

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt$$

$$= \int_{-T_1}^{T_1} e^{-jkw_0t}dt$$

$$= \frac{1}{-jkw_0}e^{-jkw_0t}\Big|_{-T_1}^{T_1}$$

$$= \frac{1}{-jkw_0}\left(e^{-jkw_0T_1} - e^{jkw_0T_1}\right)$$

$$= \frac{2}{kw_0}\frac{e^{jkw_0T_1} - e^{-jkw_0T_1}}{2j}$$

$$= 2T_1\frac{\sin(kw_0T_1)}{kw_0T_1}$$

Since $w_0 = 2\pi$ rad./sec. and $T_1 = 0.25$sec., then $w_0T_1 = 0.5\pi$ rad.

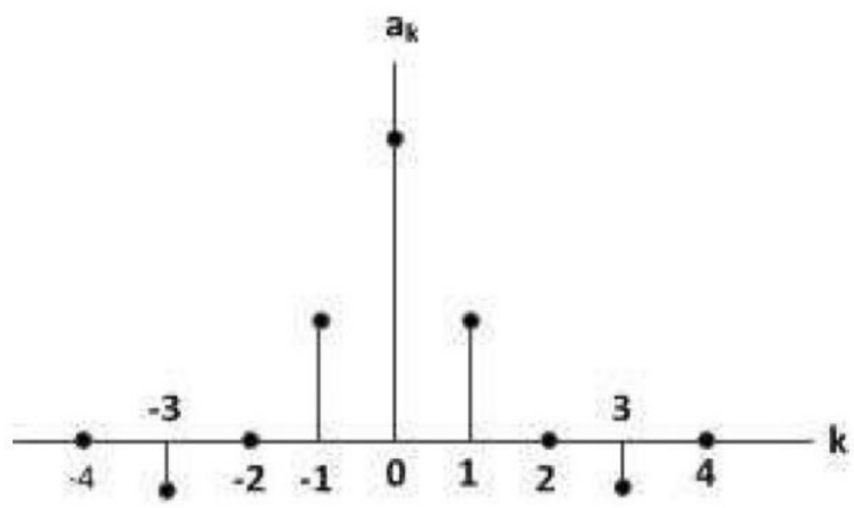
$$\therefore a_k = 0.5\frac{\sin(0.5\pi k)}{0.5\pi k}$$

And, the dc component is

$$a_0 = \frac{1}{T}\int_T x(t)dt = \int_{-T_1}^{T_1} dt = 2T_1 = 2 * 0.25 = 0.5$$

The spectrum plot is shown in the following figure.

Scanned with CamScanner



3. (a) Discuss the several properties of Fourier transform.

Solution:

There are several properties of Fourier transform. Let $x_1(t)$and $x_2(t)$ be the energy signals and $X_1(jw)$ and $X_2(jw)$ be their corresponding Fourier transforms. That means,

$$x_1(t) \overset{\text{FT}}{\leftrightarrow} X_1(jw)$$
$$x_2(t) \overset{\text{FT}}{\leftrightarrow} X_2(jw)$$

Then, we can discuss the following properties of Fourier transform.

(i) <u>Linearity:</u>

$$Ax_1(t) + Bx_2(t) \overset{\text{FT}}{\leftrightarrow} AX_1(jw) + BX_2(jw)$$

(ii) <u>Time Shifting and Frequency Shifting:</u>

$$x(t - t_0) \overset{\text{FT}}{\leftrightarrow} X(jw)e^{-jwt_0}$$
$$x(t)e^{jw_0t} \overset{\text{FT}}{\leftrightarrow} X(j(w - w_0))$$

(iii) <u>Conjugation:</u>

$$x^*(t) \overset{\text{FT}}{\leftrightarrow} X^*(-jw)$$

(iv) <u>Differentiation and Integration:</u>

$$\frac{dx(t)}{dt} \overset{\text{FT}}{\leftrightarrow} jwX(jw)$$
$$\int_{-\infty}^{t} x(\tau)d\tau \overset{\text{FT}}{\leftrightarrow} \frac{1}{jw}X(jw) + \pi X(0)\delta(w)$$

(v) <u>Time and Frequency Scaling:</u>

$$x(at) \overset{\text{FT}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{jw}{a}\right)$$

(vi) <u>Parseval's Relation:</u>

Parseval's relation states that the total energy of a signal can be computed also in terms of its Fourier transform. That is,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

(vii) <u>The Convolution Property:</u>

The convolution between two signals in time domain is equivalent to the multiplication of their Fourier transforms in frequency domain. That is,

Scanned with CamScanner

$$x_1(t) * x_2(t) \overset{\text{FT}}{\leftrightarrow} X_1(jw)X_2(jw)$$

(viii) <u>The Multiplication Property:</u>

The multiplication between two signals in time domain is equivalent to the convolution between their Fourier transforms in frequency domain. That is,

$$x_1(t)x_2(t) \overset{\text{FT}}{\leftrightarrow} \frac{1}{2\pi}[X_1(jw) * X_2(jw)]$$

(b) Calculate the DFT of the sequence $x[n] = \{5, 3, 2, 2\}$.

Solution:

Here, $N = 4$. By using linear transformation method,

$$X_N = W_N x_N$$

That is,

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w_4^1 & w_4^2 & w_4^3 \\ 1 & w_4^2 & w_4^4 & w_4^6 \\ 1 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

We know,

$$w_4 = e^{-j2\pi/4} = -j$$

Then $w_4^1 = -j, w_4^2 = -1, w_4^3 = j, w_4^4 = 1, w_4^6 = -1, w_4^9 = -j$

So,

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5+3+2+2 \\ 5-j3-2+j2 \\ 5-3+2-2 \\ 5+j3-2-j2 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 12 \\ 3-j \\ 2 \\ 3+j \end{bmatrix}$$

The required N-point DFT is therefore

$$X[k] = \{12, 3-j, 2, 3+j\}.$$

4. (a) Perform the circular convolution between $x_1[n] = \{2, 1, 2, 3\}$ and $x_2[n] = \{2, 3, 4, 5\}$.

Scanned with CamScanner

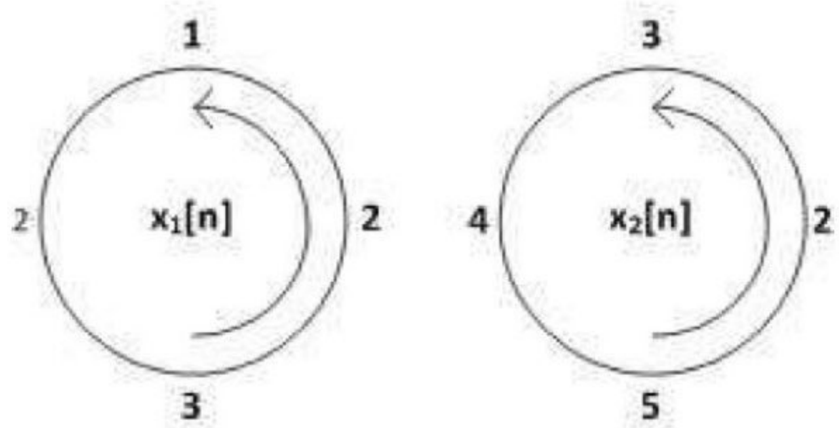
Solution:

The circular convolution between two finite-length sequences $x_1[n]$ and $x_2[n]$ each of length N can be computed by

$$x_3[m] = \sum_{n=0}^{N-1} x_1[n]\, x_2[(m-n)]_N$$

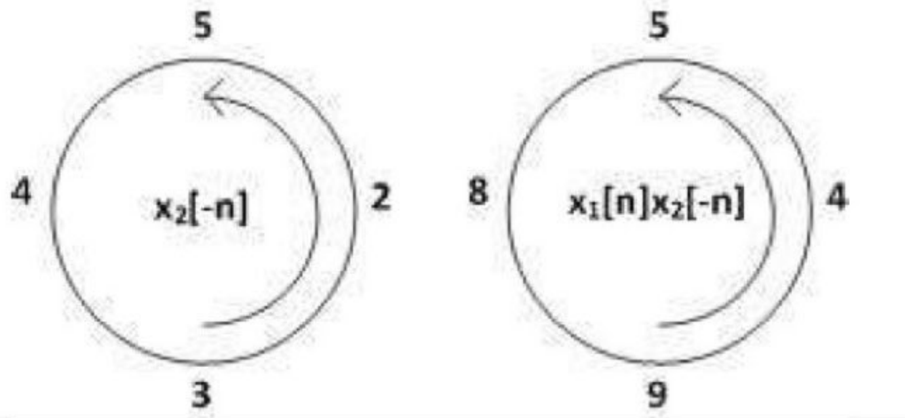
where $m = 0, 1, \ldots, N-1$. In our case, $N = 4$. Orienting $x_1[n]$ and $x_2[n]$ in circular fashion, we get

For $m = 0$,



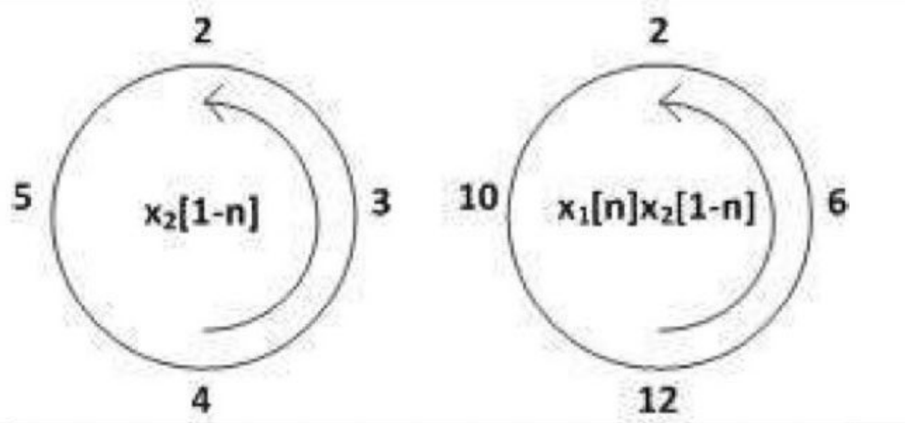
$$x_3[0] = \sum_{0}^{3} x_1[n]\, x_2[(-n)]_4$$

Reversing $x_2[n]$ in circular fashion and calculating $x_3[0]$ we get

	$x_3[0] = 4 + 5 + 8 + 9 = 26$

For $m = 1$,

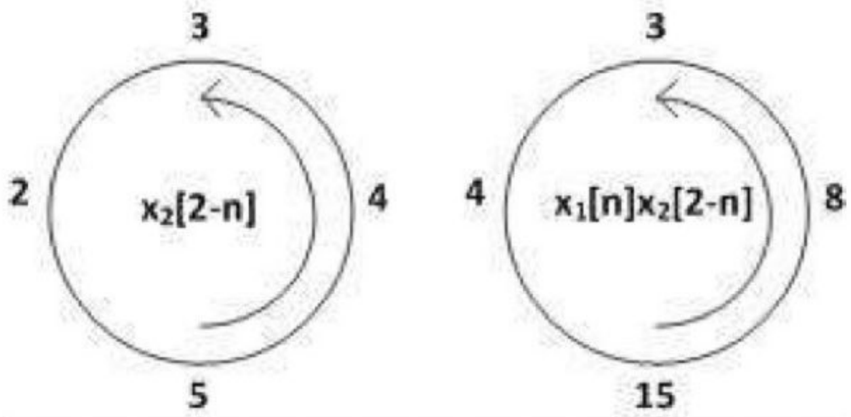
$$x_3[1] = \sum_{0}^{3} x_1[n]\, x_2[(1-n)]_4$$

	$x_3[1] = 6 + 2 + 10 + 12 = 30$

For $m = 2$,

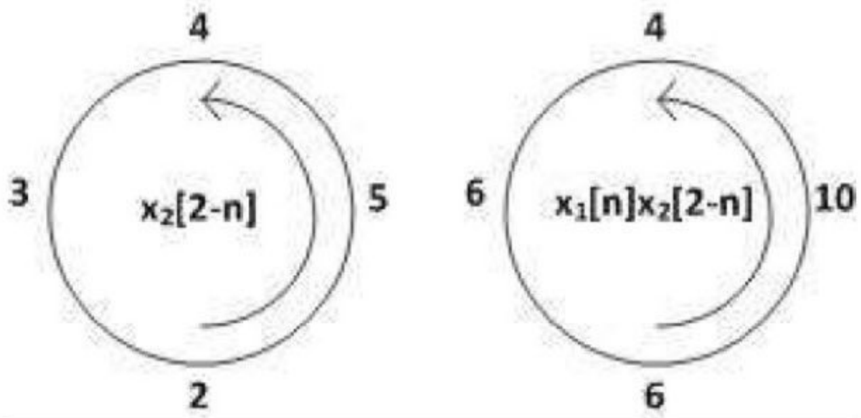
$$x_3[2] = \sum_{0}^{3} x_1[n]\, x_2[(2-n)]_4$$

Scanned with CamScanner

	$x_3[2] = 8 + 3 + 4 + 15 = 30$

For $m = 3$,

$$x_3[3] = \sum_{0}^{3} x_1[n]\, x_2[(3-n)]_4$$

	$x_3[3] = 10 + 4 + 6 + 6 = 26$

Hence, the desired result is

$$x_3[m] = \{26, 30, 30, 26\}$$

(b) What do you mean by auto-correlation function? Determine the value of auto-correlation function of the energy signal $x(t) = u(t+1) - u(t-1)$.

Solution:

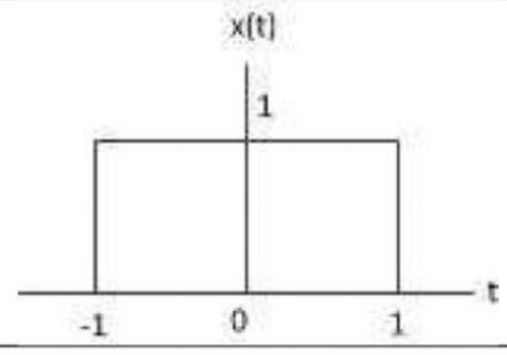
Auto-correlation function is defined as the function that measures the similarity between a signal and its delayed version. For a complex-valued energy signal $x(t)$, the auto-correlation is defined as,

$$R(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)dt$$

If $x(t)$ is real, then

$$R(\tau) = \int_{-\infty}^{+\infty} x(t)x(t-\tau)dt$$

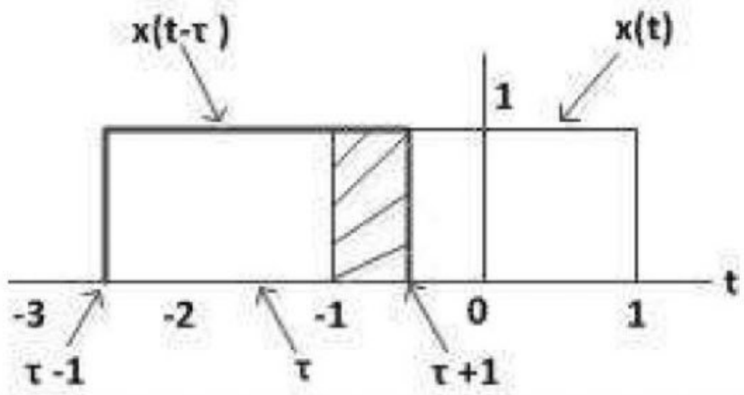
The given energy signal is

$x(t) = u(t+1) - u(t-1)$	

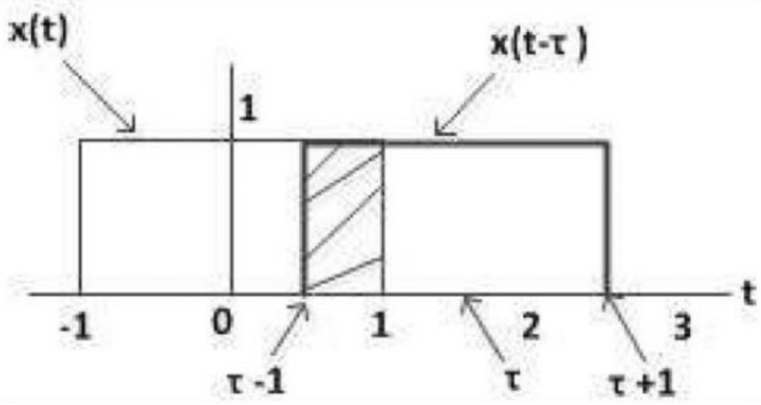
<u>For $\tau < -2$ and $\tau > 2$</u> $x(\tau)$ and $x(t-\tau)$ do not overlap. So, $R(\tau) = 0$ for this range of τ.

<u>For $-2 < \tau < 0$</u>,

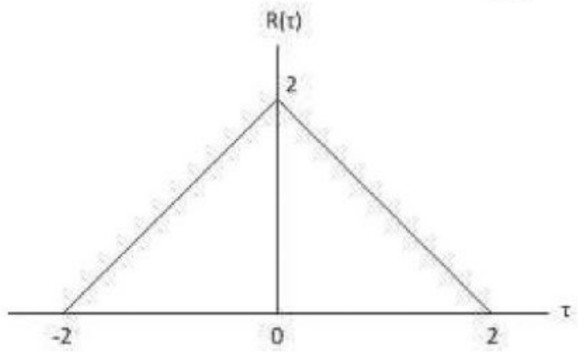
Scanned with CamScanner

	$R(\tau) = \int_{-1}^{\tau+1} dt = \tau + 2$

<u>For $0 < \tau < 2$,</u>

	$R(\tau) = \int_{\tau-1}^{1} dt = 2 - \tau$

Hence, the value of auto-correlation function of the given energy signal is,



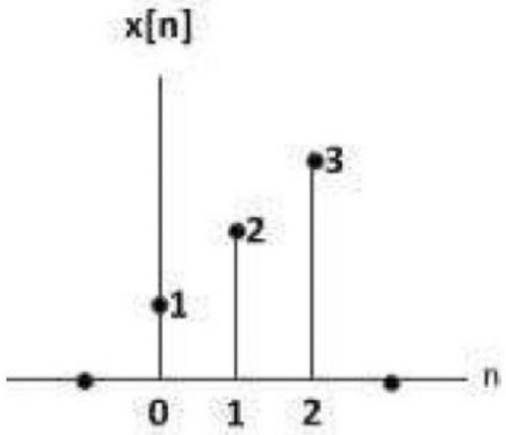
5. (a) If the impulse response of a discrete-time LTI system is $h[n] = \{1, 2, -1\}$. Then, determine the output of the system for the input $x[n] = \{1, 2, 3\}$.

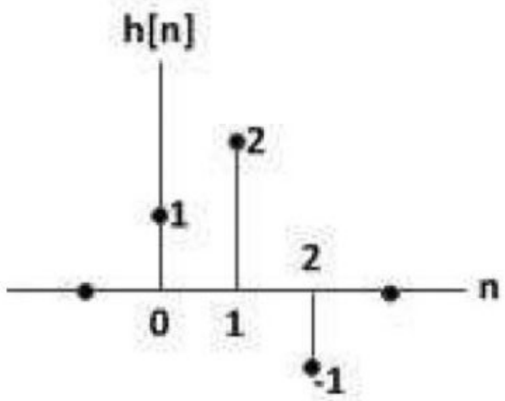
Solution:

The output of a discrete-time LTI system can be obtained by performing the convolution sum between the input signal $x[n]$and the impulse response $h[n]$ as given by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

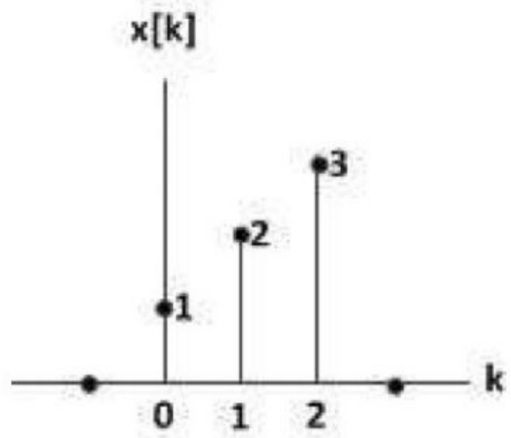
The given $x[n]$ and $h[n]$ are plotted as shown below.

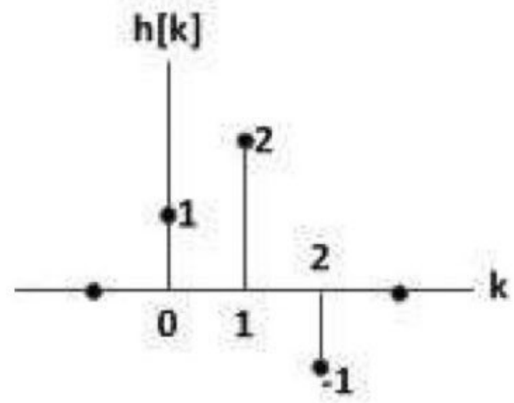




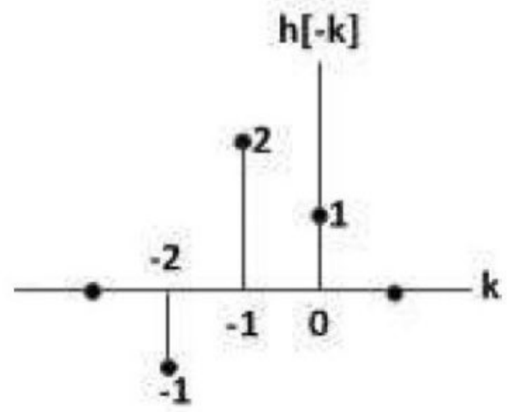
Step 1: Express input signal and impulse response as functions of k

Scanned with CamScanner





Step 2: Flip $h[k]$ over k

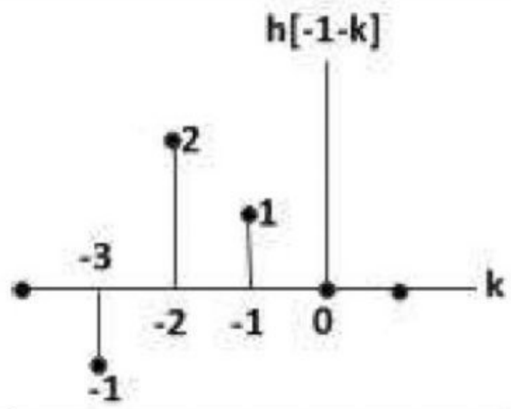


Step 3:

<u>For $n = 0$</u>

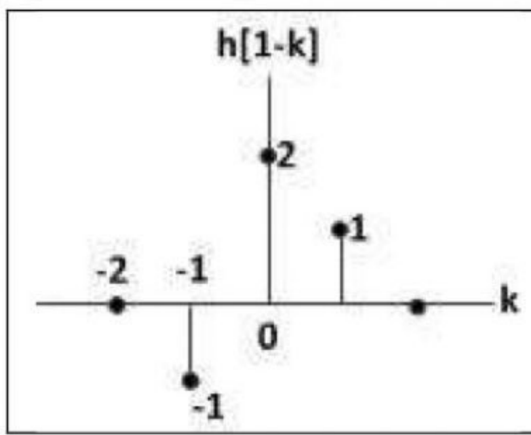
$$y[0] = \sum_{k=-\infty}^{+\infty} x[k]h[-k] = 0 + 1 * 1 + 0 = 1$$

<u>For $n = -1$</u>

	$y[-1] = \sum_{k=-\infty}^{+\infty} x[k]h[-1-k] = 0$

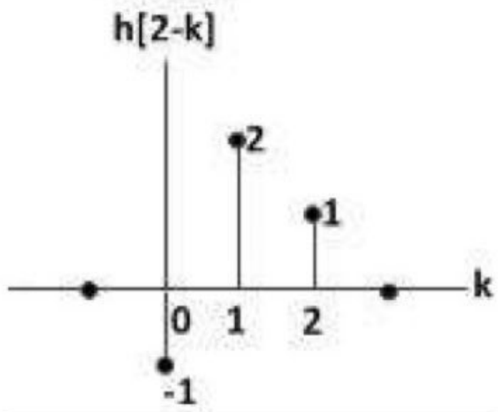
<u>Similarly, for $n < -1, y[n] = 0$</u>

<u>For $n = 1$</u>

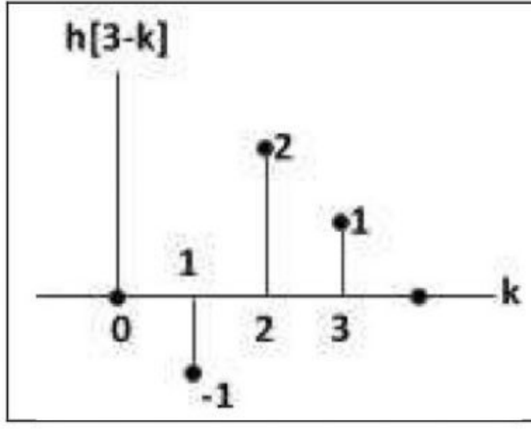
	$y[1] = \sum_{k=-\infty}^{+\infty} x[k]h[1-k]$ $= 0 + 1 * 2 + 2 * 1 + 0 = 4$

<u>For $n = 2$</u>

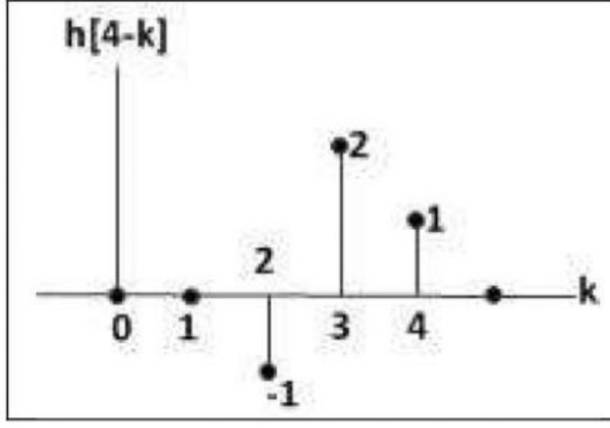
Scanned with CamScanner

	$y[2] = \sum_{k=-\infty}^{+\infty} x[k]h[2-k]$ $= 0 + (1)(-1) + 2 * 2 + 3 * 1 + 0 = 6$

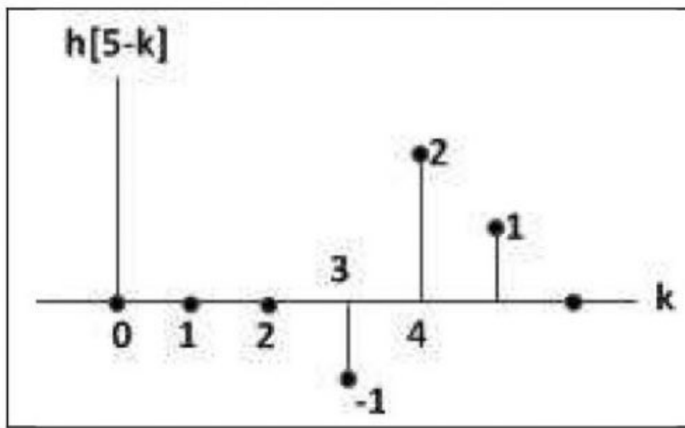
For $n = 3$

	$y[3] = \sum_{k=-\infty}^{+\infty} x[k]h[3-k]$ $= 0 + (2)(-1) + 3 * 2 + 0 = 4$

For $n = 4$

	$y[4] = \sum_{k=-\infty}^{+\infty} x[k]h[4-k]$ $= 0 + (3)(-1) + 0 = -3$

For $n = 5$

	$y[5] = \sum_{k=-\infty}^{+\infty} x[k]h[5-k] = 0$

Similarly, for $n > 5, y[n] = 0$

Hence, the desired output is

$$y[n] = \{\dots 0, \underset{\uparrow}{1}, 4, 6, 4, -3, 0, \dots\}$$

(b) Discuss briefly the properties of LTI systems.

Solution:

The properties of an LTI system are described in brief as below. For simplicity, we will consider discrete-time LTI systems.

(i) The Commutative Property:

It states that the output of an LTI system does not change even if the order of input signal and impulse response is changed. That is,

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

(ii) The Distributive Property:

Scanned with CamScanner

The distributive property can be written as

$$y[n] = x[n] * (h_1[n] + h_2[n]) = x[n] * (h_1[n] + h_2[n])$$

where $h_1[n]$ and $h_2[n]$ are the impulse responses of two LTI systems. This property states that a parallel combination of LTI systems can be replaced by a single LTI system whose impulse response is the sum of impulse responses of all the systems.

(iii) <u>The Associative Property:</u>

The associative property is defined by

$$y[n] = x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

This property states that the series combination of two LTI systems can be replaced by a single system whose impulse response is the convolution between their impulse responses.

(iv) <u>Memory:</u>

An LTI system is said to be memoryless if

$$h[n] = 0, \;\; n \neq 0.$$

Then, the output of memoryless LTI system is given as

$$y[n] = Kx[n]$$

where K is constant.

(v) <u>Causality:</u>

For an LTI system to be causal,

$$h[n] = 0, \;\; n < 0.$$

Then the output is,

$$y[n] = \sum_{k=0}^{+\infty} h[k]x[n-k].$$

(vi) <u>Invertibility:</u>

For an LTI system with impulse response $h[n]$ to be invertible, there must exist another LTI system with impulse response $h_1[n]$ such that their cascade becomes the identity system; that is,

$$h[n] * h_1[n] = \delta[n].$$

(vii) <u>Stability:</u>

An LTI system is said to be BIBO (Bounded-Input-Bounded-Output) stable if its impulse response is absolutely summable. That is,

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty.$$

6. (a) What is distortionless transmission? Derive the conditions for distortionless transmission.

Solution:

The transmission of a signal is said to be distortionless if the output signal is the exact replica of the input signal. But, signal may get amplified or attenuated by some amount and it may also suffer from some time delay which we do not treat as distortion. Therefore, for distortionless transmission, the output signal $y(t)$ can be expressed as

$$y(t) = Kx(t - t_d)$$

where $x(t)$ is the input signal, K is amplitude scaling constant, and t_d is the time delay constant.

Taking Fourier transform on both sides of the above equation, we get

$$Y(jw) = KX(jw)e^{-jwt_d}$$

So, the frequency response of the system is

Scanned with CamScanner

$$H(jw) = \frac{Y(jw)}{X(jw)} = Ke^{-jwt_d}$$

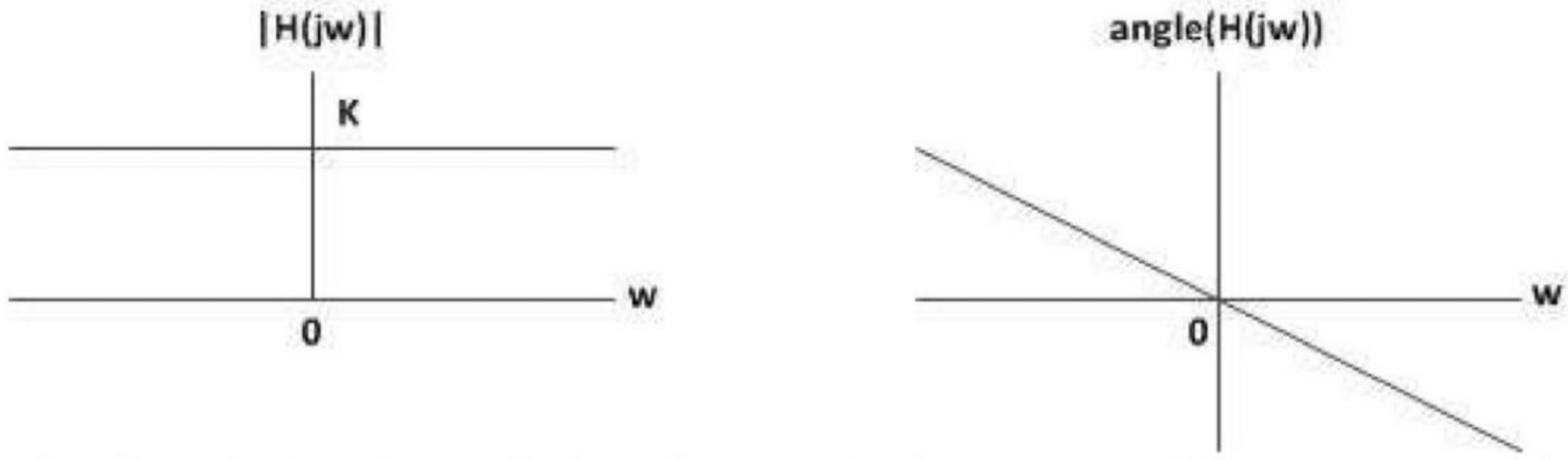
The magnitude response is

$$|H(jw| = K$$

And, the phase response is

$$\measuredangle H(jw) = -wt_d$$

The magnitude response and the phase response are plotted as below.



Therefore, for distortionless transmission, the magnitude response of the system must be constant and the phase response must be linearly proportional to the frequency.

(b) Realize the LTI system described by the following difference equation in Direct-Form I and Direct-Form II (canonical) structures.

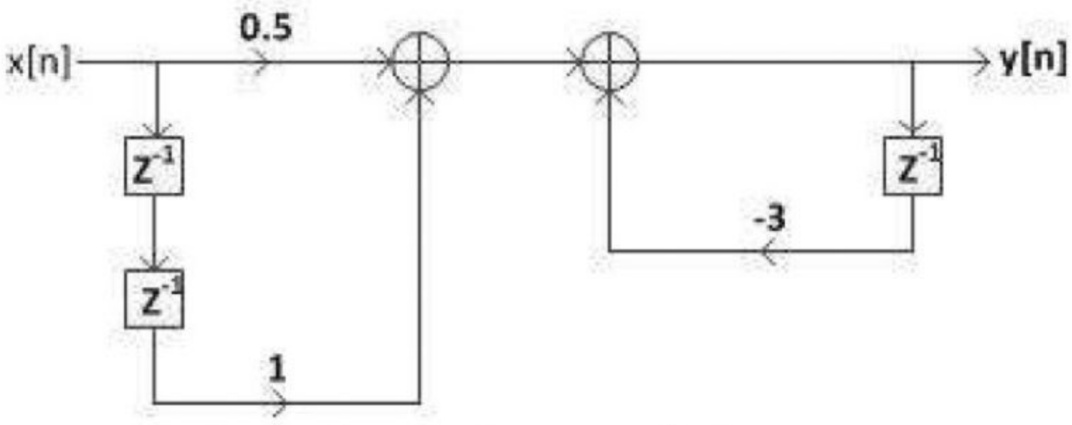
$$\boldsymbol{y[n] + 3y[n-1] = 0.5x[n] + x[n-2]}.$$

Solution:

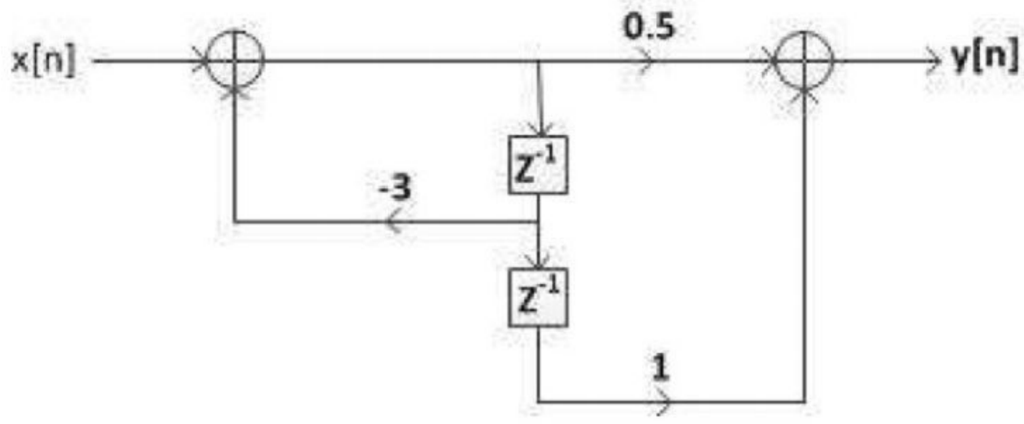
Rearranging the given difference equation, we get

$$y[n] = 0.5x[n] + x[n-2] - 3y[n-1]$$

We can make the Direct-Form I structure as shown below.



And, Direct-Form II or canonical structure is drawn as below.



7. Attempt any of two.

(a) Parseval's theorem for continuous-time energy signal

Parseval's theorem for continuous-time energy signal states that the total energy of a signal can be determined also from its Fourier transform as given below

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

Scanned with CamScanner

Proof:

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$$

We have inverse Fourier transform as

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

So,

$$E = \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)e^{-jwt}dw\right]dt$$

Interchanging the order of integration, we get

$$E = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)\left[\int_{-\infty}^{+\infty} x(t)e^{-jwt}dt\right]dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(jw)X(jw)dw$$

$$\therefore E = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(jw)|^2 dw$$

Hence, the Parseval's theorem is proved.

(b) Fast Fourier Transform (FFT)

Fast Fourier Transform (FFT) is an algorithm to compute the DFT efficiently. To calculate N-point DFT, we have

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-jk\frac{2\pi}{N}n}$$

where $k = 0, 1, \dots, N-1$.

To calculate N-point DFT by direct method, we need N^2 number of complex multiplications and $N(N-1)$ number of complex additions. The main aim of FFT algorithm is to reduce the number of multiplications and additions so that the DFT calculation is faster.

There are different FFT algorithms; radix-2 FFT algorithm is the most widely used one. In this algorithm, N-point DFT is broken into two N/2-point DFTs, each N/2-point DFT is again broken into two N/4-point DFTs and so on such that the smallest DFT will be 2-point DFT. With this algorithm, for N-point DFT calculation, we need N/2log2(N) number of complex multiplications and Nlog2(N) number of complex additions. For an example, if N = 8, then the direct computation of N-point DFT, $N^2 = 64$ number of multiplications and $N(N-1) = 8 * 7 = 56$ number of additions is required. But, the DFT calculation with radix-2 FFT algorithm will require $N/2\log_2 N = 4 * 3 = 12$ number of multiplications and $N\log_2 N = 8 * 3 = 24$ number of additions, thus saving huge computational time.

(c) FIR and IIR Systems

FIR Systems

FIR stands for finite-duration impulse response. If the impulse response of an LTI system is of finite length, then the system is referred to as the FIR system. If M be the length of the impulse response, then the output of a causal FIR system is expressed as

Scanned with CamScanner

$$y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]$$

This difference equation tells us that the output of a causal FIR system depends upon the present and past inputs and this system requires $M-1$number of memory locations. The moving average system is an example of FIR system which is given as

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

<u>IIR Systems</u>

IIR stands for infinite-duration impulse response. If the impulse response of an LTI system is of infinite length, then the system is referred to as the IIR system. For causal IIR system, the output may be expressed as

$$y[n] = \sum_{k=0}^{+\infty} h[k]x[n-k]$$

It seems that we need infinite number of memory locations and this is practically infeasible. However, by using recursive form of equation (that is, by providing feedback), we can reduce the number of memory locations and make the system design feasible. Normally, the output of IIR system can be expressed as

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

Thus the output of IIR system depends upon present input, past inputs, and past outputs unlike in FIR system. The cumulative average system is an example of IIR system which is expressed as

$$y[n] = \frac{1}{n+1}\sum_{k=0}^{n} x[k].$$

Scanned with CamScanner

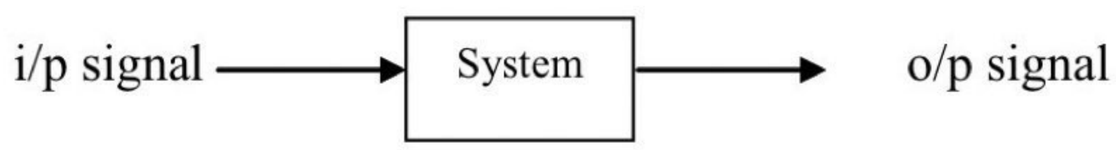
Introduction to Signals:

What is a Signal ?

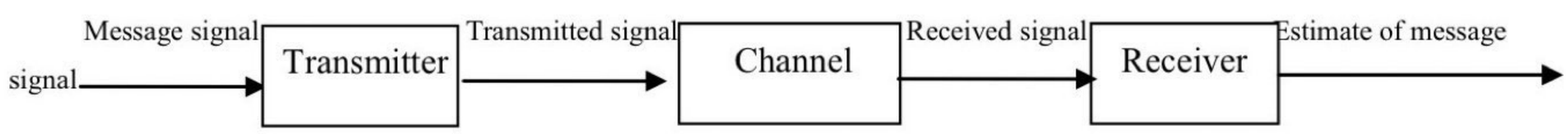
- A signal is formally defined as a function of one or more variables that conveys information on the nature of a physical phenomenon.
- When the function depends on a single variable, the signal is said to be one dimensional. E.g.; Speech signal (Amplitude varies with respect to time)
- When the function depends on two or more variables, the signal is said to be multidimensional. E.g.; Image – 2D (Horizontal & vertical coordinates of the images are two dimensional)

What is a System ?

- A system is formally defined as an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals.



e.g.; In a communication system the input signal could be a speech signal or computer data. The system itself is made up of the combination of a transmitter, channel and a receiver. The output signal is an estimate of the information contain in the original message.



The examples of other systems are control systems, biomedical signal processing system, audio system, remote sensing system, microelectro mechanical system etc.

General signal characteristics:

(a) *Multichannel & multidimensional signals:*

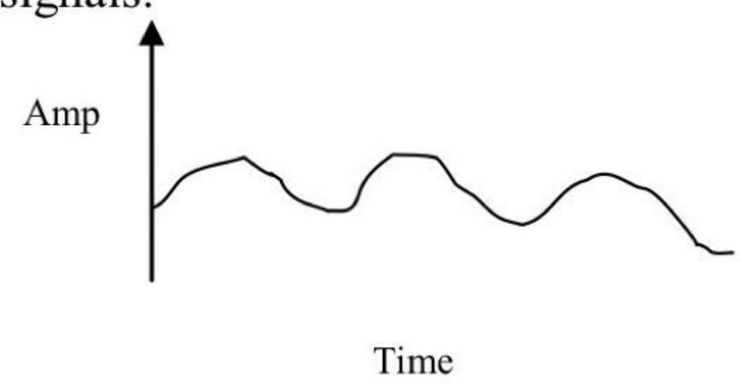
- A signal is described by a function of one or more independent variables.
- The value of the function (dependent variable) can be real valued scalar quantity, a complex valued quantity or perhaps a vector.

  Real valued signal $x_1(A) = A\ sin3\pi t$

  Complex valued signal $x_2(A) = Ae^{j3\pi t} = A\ cos3\pi t + jAsin3\pi t$
- In some applications, signals are generated by multiple sources or multiple sensors. Such signals can be represented in vector form and we refer such a vector of signal as a multichannel signal.

  E.g.; In electrocardiography, 3-lead & 12-lead electrocardiograms (ECG) are often used, which result in 3-channel & 12-channel signals.

One dimensional: If the signal is a function of a single independent variable, the signal is called 1-D signal.
e.g.; Speech signal



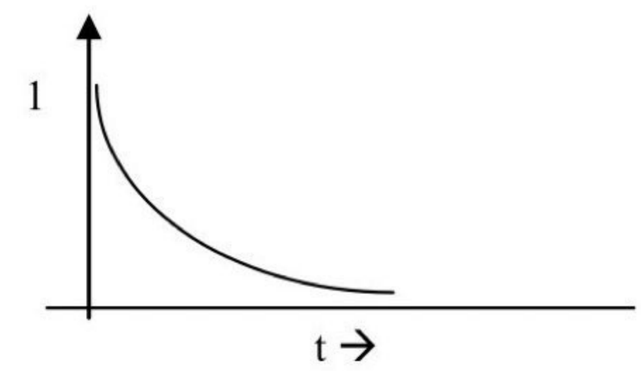
Scanned with CamScanner

Multidimensional signal: Signals can be functions of more than one variable, e.g., image signals (2D), Colour image (3D), etc.

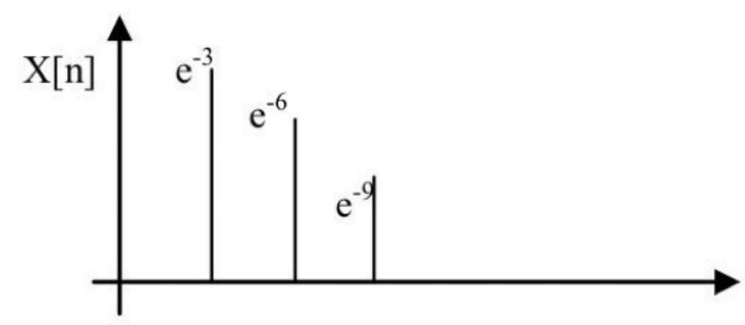
Classification of signals

Broadly we classify signals as:

1. ***Continuous-time signal***: A signal $x(t)$, is said to be continuous-time signal if it is defined for all time t, where t is a real-valued variable denoting time.
   Ex: $x(t) = e^{-3t}u(t)$



   Discrete-time signal: A signal $x(n)$, is said to be discrete-time signal; if it is defined only at discrete instant of time, where n is an integer-valued variable denoting the discrete samples of time. We use square brackets [·] to denote a discrete-time signal.
   Ex: $x[n] = e^{-3n}u[n]$



2. ***Even and odd signal***:

A continuous-time signal $x(t)$ is even, if $x(-t) = x(t)$
and it is odd if $x(-t) = -x(t)$.

A discrete-time signal $x[n]$ is even if $x[-n] = x[n]$
and is odd if $x[-n] = -x[n]$.

Example 1: $x(t) = t^2 - 40$ is even.
Example 2: $x(t) = 0.1t^3$ is odd.
Example 3: $x(t) = e^{0.4t}$ is neither even nor odd.

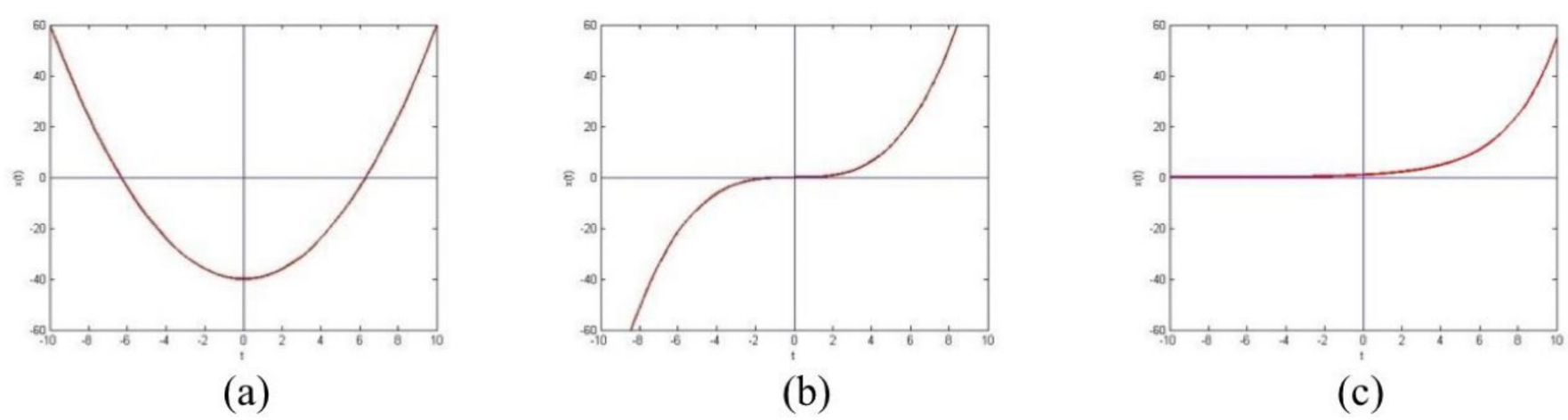


Figure: Illustrations of odd and even functions. (a) Even; (b) Odd; (c) Neither.

Scanned with CamScanner

Decomposition Theorem

Every continuous-time signal $x(t)$ can be expressed as:

$$x(t) = y(t) + z(t)$$

where $y(t)$ is even, and $z(t)$is odd.

$$y(t) = \frac{x(t) + x(-t)}{2}$$

and

$$z(t)\text{m} = \frac{x(t) - x(-t)}{2}$$

3. *Periodic & non-periodic signals*:

A continuous time signal $x(t)$ is periodic if there is a constant $T > 0$, such that

$x(t) = x(t+ T)$, for all t

A discrete time signal $x[n]$ is periodic if there is an integer constant $N > 0$, such that

$x[n] = x[n + N]$, for all n

Signals do not satisfy the periodicity conditions are called non-periodic signals.

Note: The smallest value of T (N) that satisfies the above equations is called fundamental period

Example: Determine the fundamental period of the following signals:

(a) $e^{j3\pi t/5}$

(b) $e^{j3\pi n/5}$

Solution:

(a) Let $x(t) = e^{j3\pi t/5}$. If $x(t)$ is a periodic signal, then there exists $T > 0$ such that $x(t) = x(t+ T)$. Therefore,

$$x(t) = \text{x}(t+ T)$$

$$\Rightarrow e^{j3\pi t/5} = e^{j3\pi(t+T)/5}$$

$$\Rightarrow 1 = e^{j3\pi T/5}$$

$$\Rightarrow e^{j2k\pi} = e^{j3\pi T/5}$$

$$\Rightarrow T = \frac{10}{3} \qquad (k = 1)$$

(b) Let $\text{x[n]} = \text{e}^{\text{j}3\pi\text{n}/5}$. If x[n] is a periodic signal, then there exists an integer $\text{N} > 0$ such that $x[n] = x[n + N]$. So,

$$x[n] = x[n + N]$$

$$\Rightarrow e^{j3\pi n/5} = e^{j3\pi(n+N)/5}$$

$$\Rightarrow 1 = e^{j3\pi N/5}$$

$$\Rightarrow e^{j2k\pi} = e^{j3\pi N/5}$$

$$\Rightarrow T = 10 \qquad (\text{k} = 3)$$

Scanned with CamScanner

4. Energy signals and power signals:

In electrical systems, a signal may represent a voltage or a current. Consider a voltage v(t)developed across a resistor R, producing a current i(t). The instantaneous power dissipated in this resistor is defined by

Define the *total energy* of the continuous-time signal *x(* t) as

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)dt$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} x^2(t)dt$$

and its *time-averaged,* or *average, power* as

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$$

From above equation, we readily see that the time-averaged power of a periodic signal *x(t)* of fundamental period T is given by

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$$

The square root of the average power *P* is called the *root mean-square* (rms) value of the periodic signal *x(* t).

In the case of a discrete-time signal *x[n],* the integrals in above equations are replaced by corresponding sums. Thus, the total energy of *x[n]* is defined by

$$E = \sum_{n=-\infty}^{\infty} x^2[n]$$

and its average power is defined by

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n]$$

A signal is referred to an energy signal if and only if the total energy is finite .i.e.,

$0 < E < \infty$

A signal is referred to an power signal if and only if the average power is finite .i.e.,

$0 < P < \infty$

Note: Energy signal has zero time average power and power signal has infinite energy.

Scanned with CamScanner

Example: $x(n) = (-0.5)^n u[n]$

Solution:

$$E = \sum_{n=-\infty}^{\infty} x^2[n] = \sum_{n=0}^{\infty} 0.25^n = \frac{1}{1-0.25} = \frac{4}{3} < \infty$$

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2[n] = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 0.25^n = \frac{1}{2\infty+1} \sum_{n=0}^{N} 0.25^n = 0$$

We got power zero and finite energy. Hence it is an energy signal.

5. *Deterministic signals and random signals:*

The deterministic signal is a signal about which there is no uncertainty with respect to its value at any time. The deterministic signals may be modeled as completely specified function of time.
Example: $x(t) = cos^2(2\pi t)$
A random signal is a signal about which there is uncertainty before it occurs. Example: The electrical noise generated in the amplifier of a radio or television receiver.

Basic Operations of Signals

Operation performed on independent variable:

Time Shift
For any t_0 and n_0, time shift is an operation defined as

$$x(t) \rightarrow x(t - t_0)$$
$$x[n] \rightarrow x[n - n_0].$$

If $t_0 > 0$, the time shift is known as "delay". If $t_0 < 0$, the time shift is known as "advance".

Example. In Fig. given below, the left image shows a continuous-time signal $x(t)$. A time- shifted version $x(t-2)$ is shown in the right image.

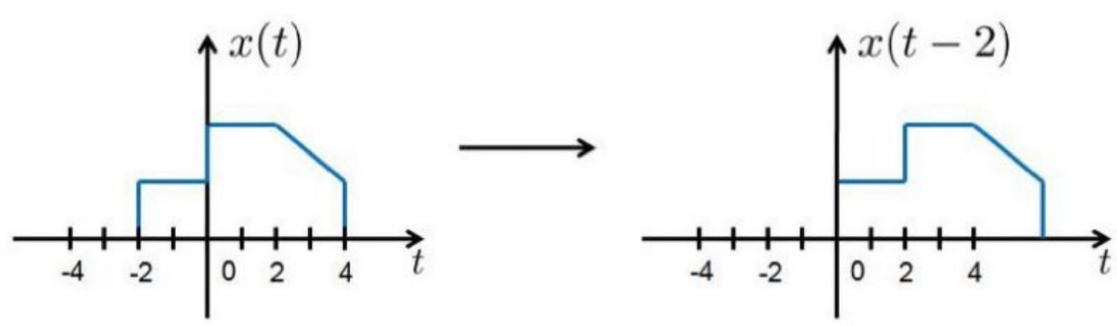


Figure: An example of time shift.

Time Reversal

Time reversal is defined as

$$x(t) \rightarrow x(-t)$$
$$x[n] \rightarrow x[-n],$$

which can be interpreted as the "flip over the y-axis".

Scanned with CamScanner

Example:

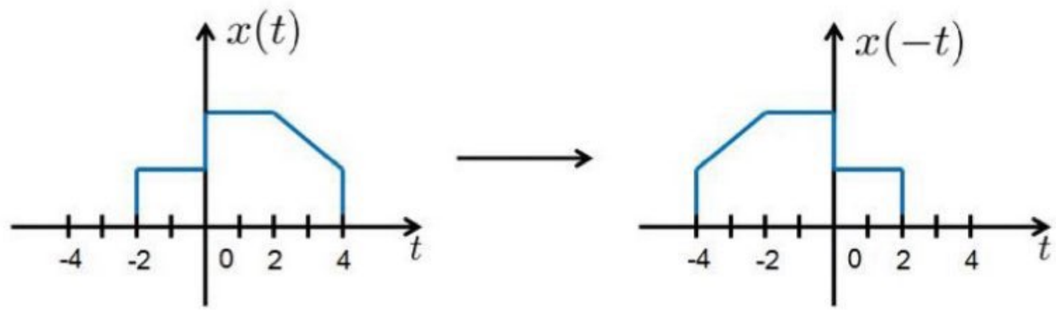


Figure: An example of time reversal.

Time Scaling

Time scaling is the operation where the time variable t is multiplied by a constant a:

$$x(t) \rightarrow x(at), \quad a > 0$$

If a > 1, the time scale of the resultant signal is "*decimated*" (speed up).

If 0 < a < 1,
the time scale of the resultant signal is "*expanded*" (slowed down).

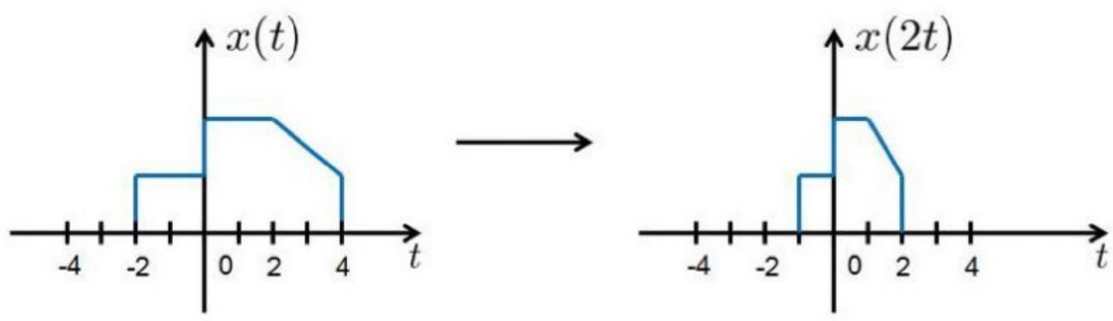


Figure : An example of time scaling.

Decimation and Expansion

Decimation and expansion are standard discrete-time signal processing operations.

Decimation is defined as

$$y_D[n] = x[Mn], \text{ for some integers M}$$

Where, M is the decimation factor.

Expansion is defined as

$$y_E[n] = \begin{cases} x[\frac{n}{L}], & n = \text{integer multiple of L} \\ 0, & \text{otherwise.} \end{cases}$$

Where, L is the expansion factor.

Scanned with CamScanner

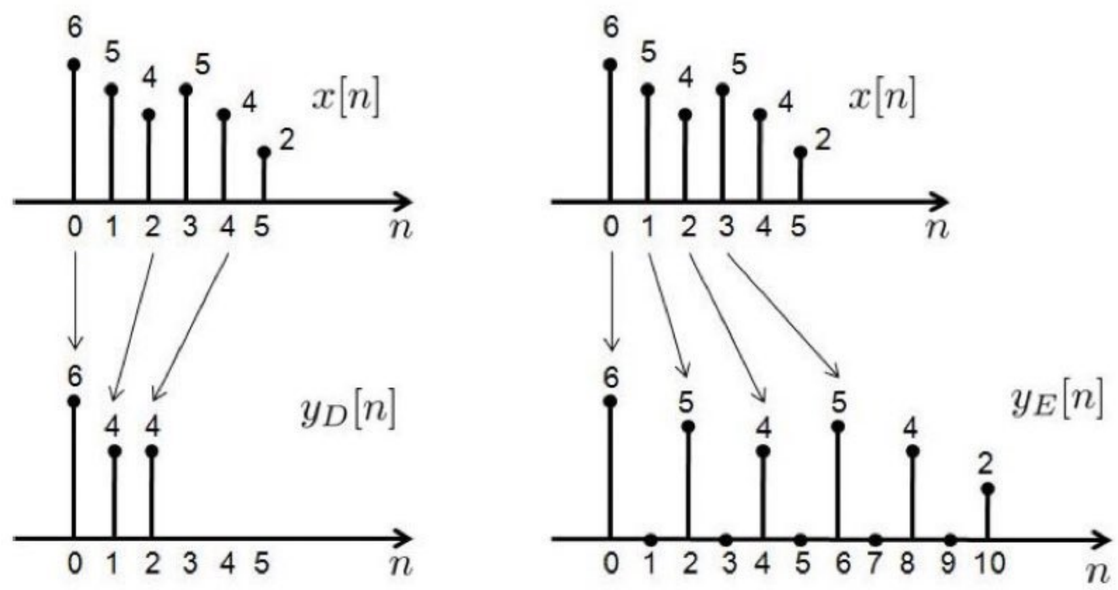


Figure 1.8: Examples of decimation and expansion for M = 2 and L = 2.

Combination of Operations

Generally, linear operation (in time) on a signal x(t) can be expressed as $y(t) = x(at-b)$. The recommended method is "Shift, then Scale".

Example: The signal x(t) shown in Figure of sketch $x(3t-5)$.

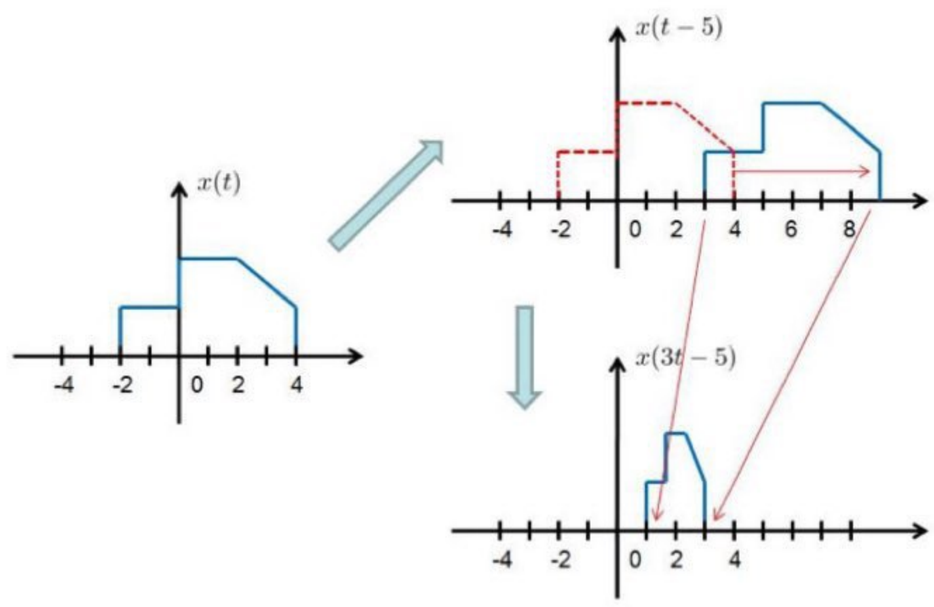


Figure: An example of Shift, then Scale

Operation performed on dependent variable:

Amplitude scaling:

Let x(t) denote a continuous time signal

By amplitude scaling, we get $y(t) = cx(t)$

Where, c is the scaling factor.

Example: An electronic amplifier, a device that performs amplitude scaling.

For discrete time signal $y[n] = cx[n]$

Amplitude addition:

Let $x_1(t)$ and $x_2(t)$ is a pair of continuous time signal

By adding these two signals, we get $y(t) = x_1(t) + x_2(t)$

Example: An audio mixture

For discrete time signal, $y[n] = x_1[n] + x_2[n]$

Scanned with CamScanner

Amplitude multiplication:

Let $x_1(t)$ and $x_2(t)$ is a pair of continuous time signal

By multiplying these two signals, we get $y(t) = x_1(t)\, x_2(t)$

Example: An AM radio signal, in which

$x_1(t)$ is an audio signal

$x_2(t)$ is an sinusoidal carrier wave

For discrete time signal, $y[n] = x_1[n]\, x_2[n]$

Differentiation:

$y(t) \frac{d}{dt}\, x(t)$

Example: Voltage across an inductor L → $v(t) \frac{d}{dt}\, i(t)$

Integration:

$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$

Example: Voltage across a capacitor C → $y(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau) d\tau$

Elementary Signals

Several elementary signals feature prominently in the study of signals and systems. These are exponential and sinusoidal signals, the step function, the impulse function, and the ramp function, all of which serve as building blocks for the construction of more complex signals

Exponential Signals

A real exponential signal, in its most general form, is written as

$x(t) = Be^{at}$,

where both B and a are real parameters. The parameter B is the amplitude of the exponential signal measured at time $t = 0$. Depending on whether the other parameter a is positive or negative, we may identify two special cases:

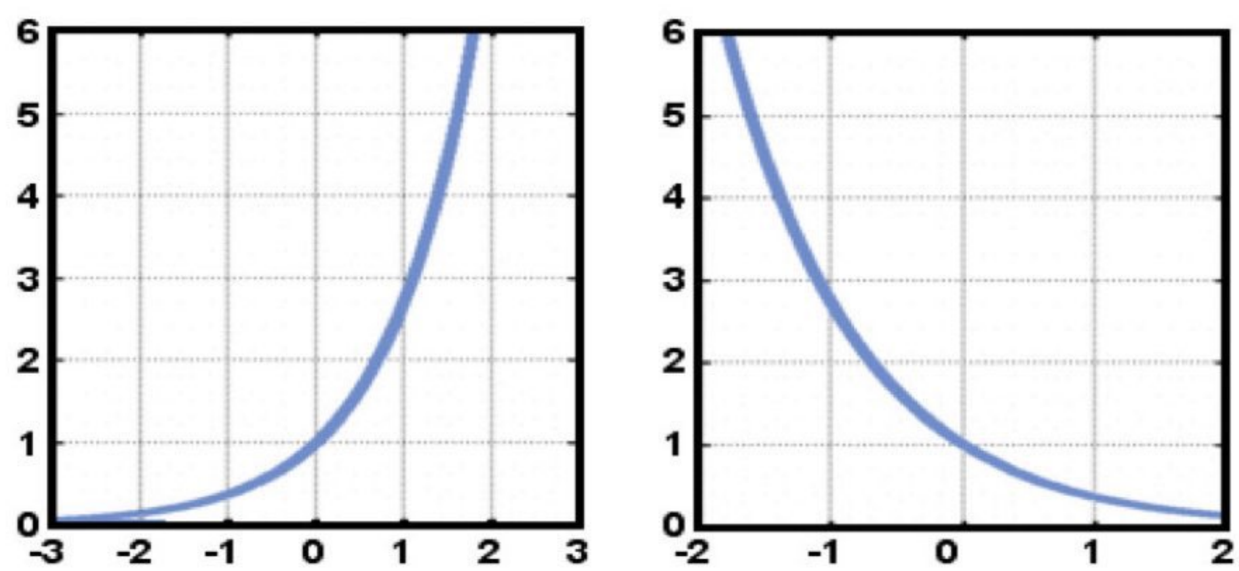
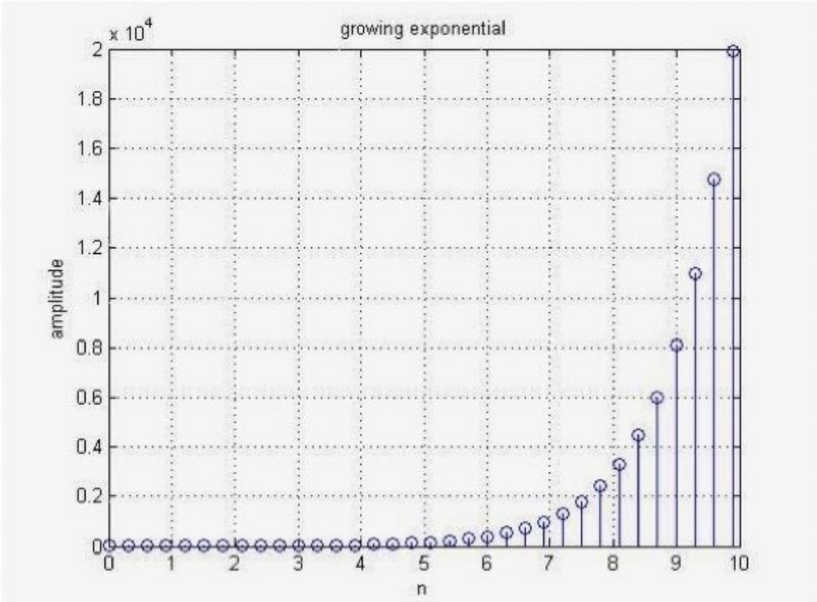


Fig: Growing exponential, for $a > 0$ *Decaying exponential, for* $a < 0$

In discrete time, it is common practice to write a real exponential signal as $x[n] = Br^n$

Scanned with CamScanner



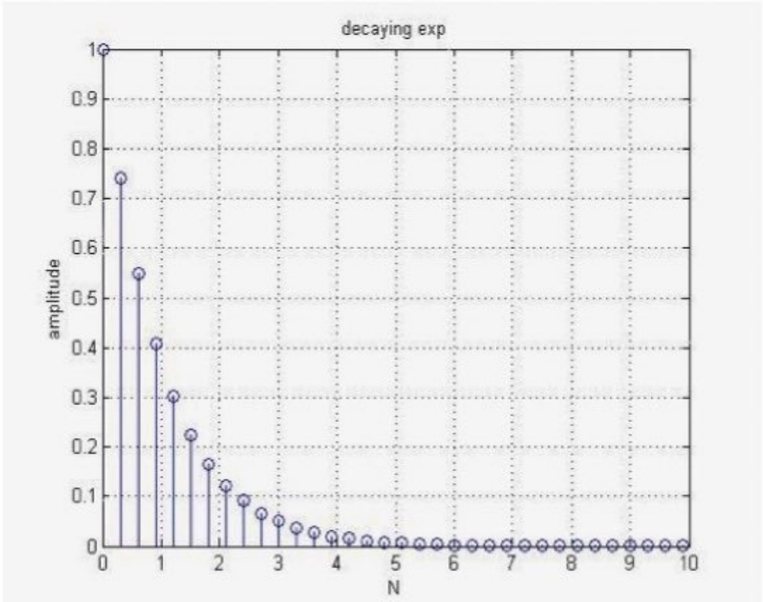


Fig: Growing exponential for r > 1 *Decaying exponential for 0 < r <1*

Impulse functions

The discrete-time version of the *unit impulse* is defined by

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$$

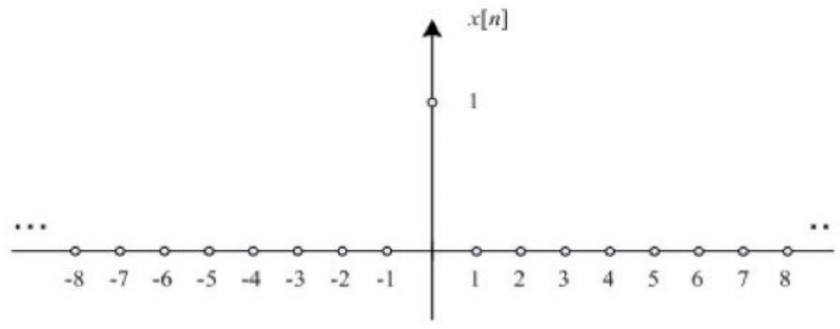


Fig: Discrete time form of unit impulse

The continuous-time version of the unit impulse is defined by the following pair of relations:

$$\delta(t) = 0 \text{ for } t \neq 0$$

and $\int_{-\infty}^{\infty} \delta(t)dt = 1$

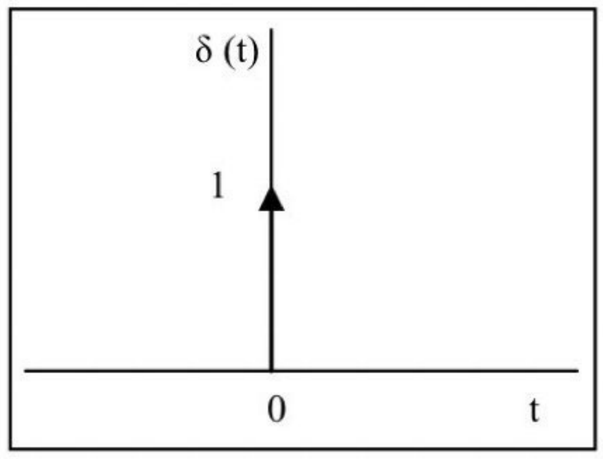


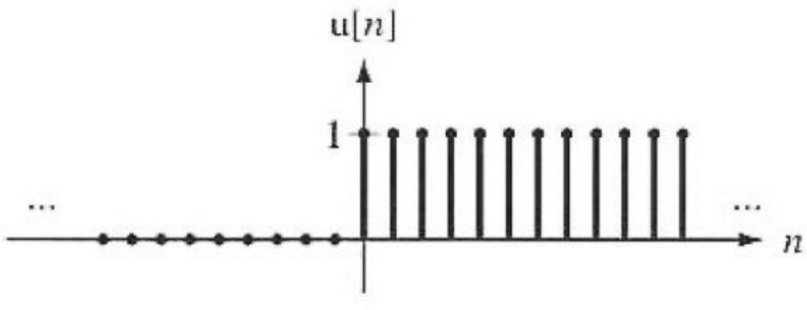
Fig: Continuous time form of unit impulse

Above equation says that the impulse δ *(t)* is zero everywhere except at the origin. Equation says that the total area under the unit impulse is unity. The impulse δ *(t)* is also referred to as the *Dirac delta function.*

Step function:

The discrete-time version of the unit- step function is defined by:

$$\boxed{u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}}$$



Scanned with CamScanner

The continuous-time version of the unit- step function is defined by: $u(t) = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$

Ramp function:

The integral of the step function u(t) is a ramp function of unit slope.

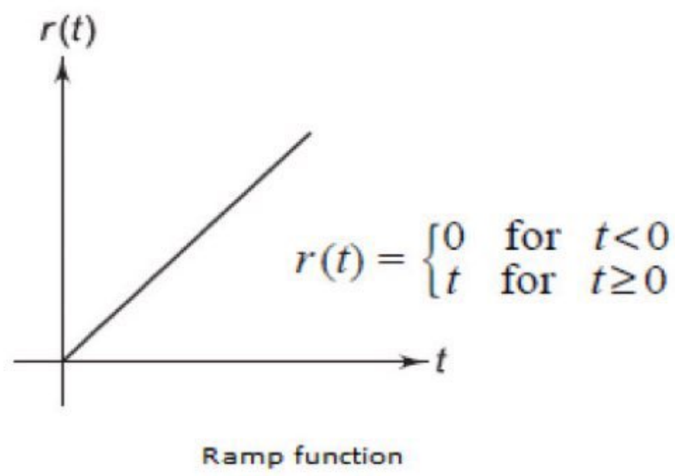
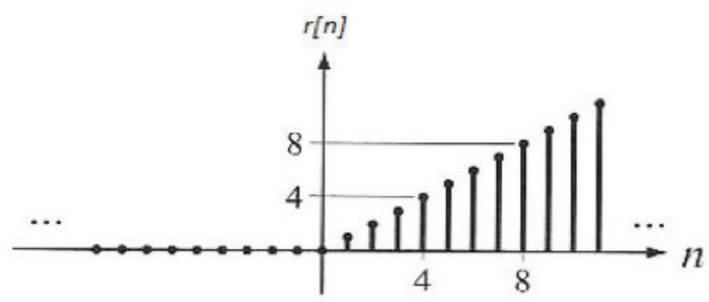


Fig: Ramp function of unit slope

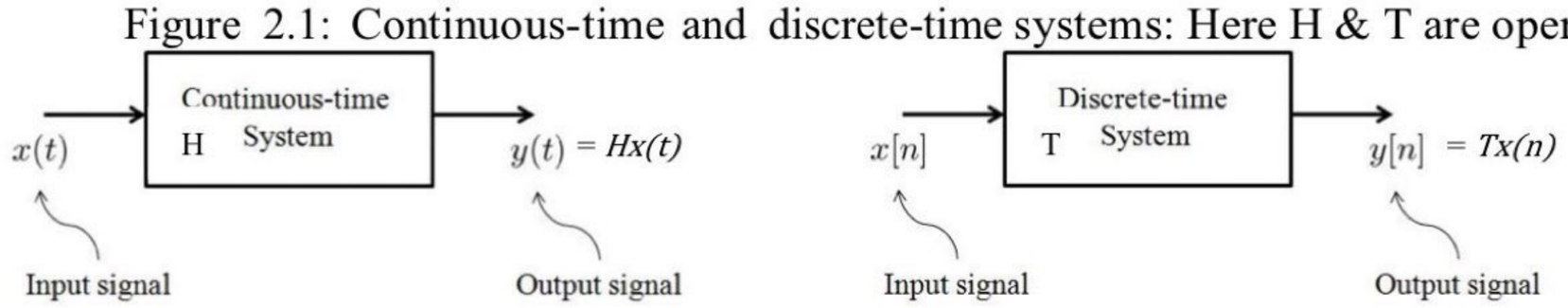
The discrete-time version of the unit- ramp function is defined by: $r[n] = \begin{cases} n, & n\geq 0 \\ 0, & n<0 \end{cases}$



Introduction to Systems

- Systems are used to process signals to allow modification or extraction of additional information from the signal.
- A system may consist of physical components (hardware realization) or an algorithm (operator) that computes the output signal from the input signal.
- A physical system consists of inter-connected components which are characterized by their input-output relationships.

Figure 2.1: Continuous-time and discrete-time systems: Here H & T are operators.



Scanned with CamScanner

Properties of systems: (Classification of systems):

1 Static (Memoryless) & Dynamic (with memory):

Static: A system is static if the output at time t (or n) depends only on the input at time t (or n).

Examples:

1. $y(t) = (2x(t) - x^2(t))^2$ is memoryless, because y(t) depends on x(t) only. There is no $x(t-1)$, or $x(t+1)$ terms, for example.
2. $y[n] = x^2[n]$ is memoryless. In fact, this system is passing the input to output directly, without any processing.
3. Current flowing through a resistor i.e., $i(t) = \frac{1}{R} v(t)$

Dynamic: A system is said to possess memory if its output signal depends on past or future values of input.

Example:

1. Inductor and capacitor, since the current flowing through the inductor at time 't' depends on the all past values of the voltage v(t) i.e., $i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$ and $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$
2. The moving average system given by $y(n) = \frac{1}{3}(x[n]+x[n-1]+x[n-2])$

2 Stable & unstable system:

- A system is said to be bounded-input, bounded-output (BIBO) stable if and only if every bounded input results in a bounded output, otherwise it is said to be unstable.
- If for $|x(t)| \le M_x < \infty$ for all t, output is $|y(t)| \le M_y < \infty$ for all t; where M_x & M_y are some finite positive number.

Example:
1. $y(t) = x(t-3)$ is a stable system.
2. $y(t) = t\,x(t)$ is an unstable system.
3. $y[n] = e^{x[n]}$ is a stable system.

Assume that $|x(n)| \le M_x < \infty$, for all 't'

$y[n] = e^{x[n]} = e^{Mx} =$ finite $\rightarrow$ Stable

4. $y[n] = r^n x[n]$, where $r > 1$

Assume that $|x(n)| \le M_x < \infty$, for all 't', then

$|y[n]| = |r^n x[n]| = |r^n|\,|x[n]|$

as 'n' $\rightarrow \infty$ $|r^n| \rightarrow \infty$

so $y[n] \rightarrow \infty$ hence unstable.

3 Causal and non-Causal system:

Causal: A system is said to be causal if the present value of output signal depends only on the present or past values of the input signal. A causal system is also known as physical or non-anticipative system.

Example:
1. The moving average system given by $y(n) = \frac{1}{3}(x[n]+x[n-1]+x[n-2])$
2. $y(t) = x(t)\cos 6t$

Scanned with CamScanner

Note: i) Any practical system that operates in real time must necessarily be causal.
ii) All static systems are causal.

Non-Causal: A system is said to be non-causal if the present value of output signal depends on one or more future values of the input signal.

Example:1. The moving average system given by $y(n)= \frac{1}{3}(x[n]+x[n-1]+x[n+2])$

4 Time invariant and time variant system:

Time invariant: A system is time-invariant if a time-shift of the input signal results in the same time-shift of the output signal.

That is, if

$$x(t) \rightarrow y(t),$$

then the system is time-invariant if

$$x(t-t_0) \rightarrow y(t-t_0), \text{ for any } t_0 .$$

.

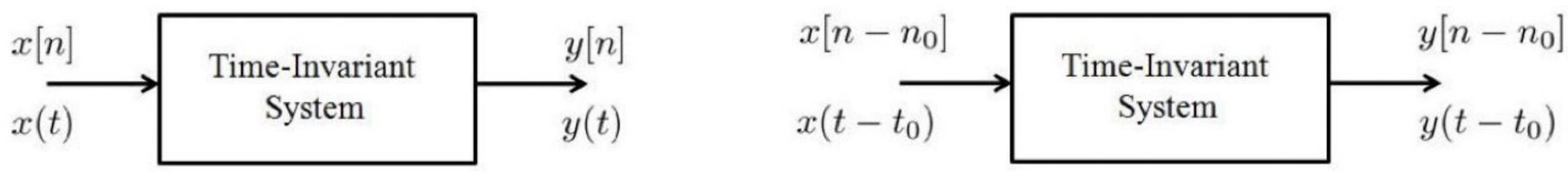


Figure 2.2: Illustration of a time-invariant system.

Example 1.
The system $y(t) = \sin[x(t)]$ is time-invariant

Proof. Let us consider a time-shifted signal $x_1(t) = x(t-t_0)$. Correspondingly, we let $y_1(t)$ be the output of $x_1(t)$. Therefore,

$$y_1(t) = \sin[x_1(t)] = \sin[x(t-t_0)].$$

Now, we have to check whether $y_1(t) = y(t-t_0)$. To show this, we note that

$$y(t-t_0) = \sin[x(t-t_0)],$$

which is the same as $y_1(t)$. Therefore, the system is time-invariant. □

Time variant: A system is time-variant if its input –output characteristic changes with time.

Example 2:
The system $y[n] = nx[n]$ is time-variant.

Proof: Output for a time shifted input is

$$y[n] \mid_{x(n-k)} = nx(n-k)$$

then the same time shifted output is

$$y(n-k) = (n-k)x(n-k)$$

the above two equations are not same. Hence it is time variant.

Scanned with CamScanner

4 *Linear and non-linear system:*

***Linear system**:* A system is said to be linear if it satisfies two properties i.e.; superposition & homogeneity.

Superposition: It states that the response of the system to a weighted sum of signals be equal to the corresponding weighted sum of responses (Outputs of the system to each of the individual input signal.

For an input $x(t) = x_1(t)$, the output $y(t) = y_1(t)$

and input $x(t) = x_2(t)$, the output $y(t) = y_2(t)$

then, the system is linear if & only if

$$\mathcal{T}[a_1x_1(t) + a_2x_2(t)] = a_1\mathcal{T}[x_1(t)] + a_2\mathcal{T}[x_2(t)]$$

Homogeneity: If the input x(t) is scaled by a constant factor 'a', then the output y(t) is also scaled by exactly the same constant factor 'a'.

For an input $x(t) \rightarrow$ output $y(t)$

and input $x_1(t) = ax(t) \rightarrow$ output $y_1(t) = ay(t)$

Example 1:

The system $y(t) = 2\pi x(t)$ is linear. To see this, let's consider a signal

$$x(t) = ax_1(t) + bx_2(t),$$

where $y_1(t) = 2\pi x_1(t)$ and $y_2(t) = 2\pi x_2(t)$. Then

$$\begin{aligned} ay_1(t) + by_2(t) &= a(2\pi x_1(t)) + b(2\pi x_2(t)) \\ &= 2\pi[ax_1(t) + bx_2(t)] = 2\pi x(t) = y(t). \end{aligned}$$

Example 2.

The system $y[n] = (x[2n])^2$ is not linear. To see this, let's consider the signal

$$x[n] = ax_1[n] + bx_2[n],$$

where $y_1[n] = (x_1[2n])^2$ and $y_2[n] = (x_2[2n])^2$. We want to see whether $y[n] = ay_1[n] + by_2[n]$. It holds that

$$ay_1[n] + by_2[n] = a(x_1[2n])^2 + b(x_2[2n])^2.$$

However,

$$y[n] = (x[2n])^2 = (ax_1[2n] + bx_2[2n])^2 = a^2(x_1[2n])^2 + b^2(x_2[2n])^2 + 2abx_1[n]x_2[n].$$

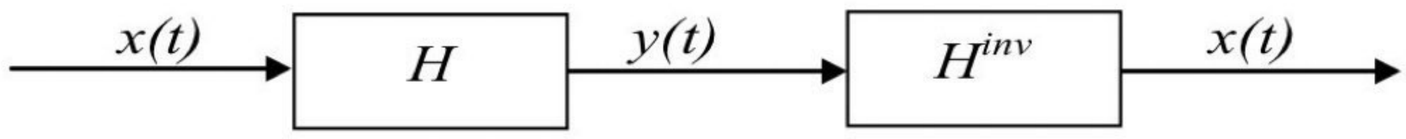
5 *Invertible and non-invertible system:*

A system is said to be invertible if the input of the system can be recovered from the output.

Let the set of operations needed to recover the input represents the second system which is connected in cascade with the given system such that the output signal of the second system is equal to the input applied to the given system.

Scanned with CamScanner

Let H →the continuous time system
$x(t)$ →input signal to the system
$y(t)$ →output signal of the system
H^{inv} →the second continuous time system



The output signal of the second system is given by

$$H^{inv}\{y(t)\} = H^{inv}\{Hx(t)\}$$
$$= H^{inv}H\{x(t)\}$$

For the output signal to equal to the original input, we require that

$$\boxed{H\,H^{inv} = I}$$

Where 'I' denotes the identity operator.

The system whose output is equal to the input is an identity system. The operator H^{inv} must satisfy the above condition for H to be an invertible system. Cascading a system, with its inverse system, result in an identity system.

Example:

An inductor is described by the relation

$$y(t) = \frac{1}{L}\int_{-\infty}^{t} x(\tau)d\tau \text{ is an invertible system}$$

because, by rearranging terms, we get

$$x(t) = L\frac{d}{dt}y(t),$$

which is the inversion formula.

Note:i) A system is not invertible unless distinct inputs applied to the system produce distinct outputs.

ii) There must be a one to one mapping between input and output signal for system to be invertible.

Non-invertible System: When several dibfferent inputs results in the same output, it is impossible to obtain the input from output. Such system is called a non-invertible system.

Example: A square-law system described by the input output relation

$$y(t) = x^2(t), \text{ is non-invertible,}$$

because distinct inputs x(t) & -x(t) produce the same output y(t) [not distinct output].

Linear –time convolution system (LTI)

Linear time invariant (LTI) systems are good models for many real-life systems, and they have properties that lead to a very powerful and effective theory for analyzing their behavior. The LTI systems can be studied through its characteristic function, called the impulse response. Further, any arbitrary input signal can be decomposed and represented as a weighted sum of unit sample sequences. As a consequence of the linearity and time invariance properties of the system, the response of the

Scanned with CamScanner

system to any arbitrary input signal can be expressed in terms of the unit sample response of the system. The general form of the expression that relates the unit sample response of the system and the arbitrary input signal to the output signal, called the convolution sum, is also derived.

Resolution of a Discrete-time signal into impulses:

Any arbitrary sequence x(n) can be represented in terms of delayed and scaled impulse sequence δ(n). Let x(n) is an infinite sequence as shown in figure below.

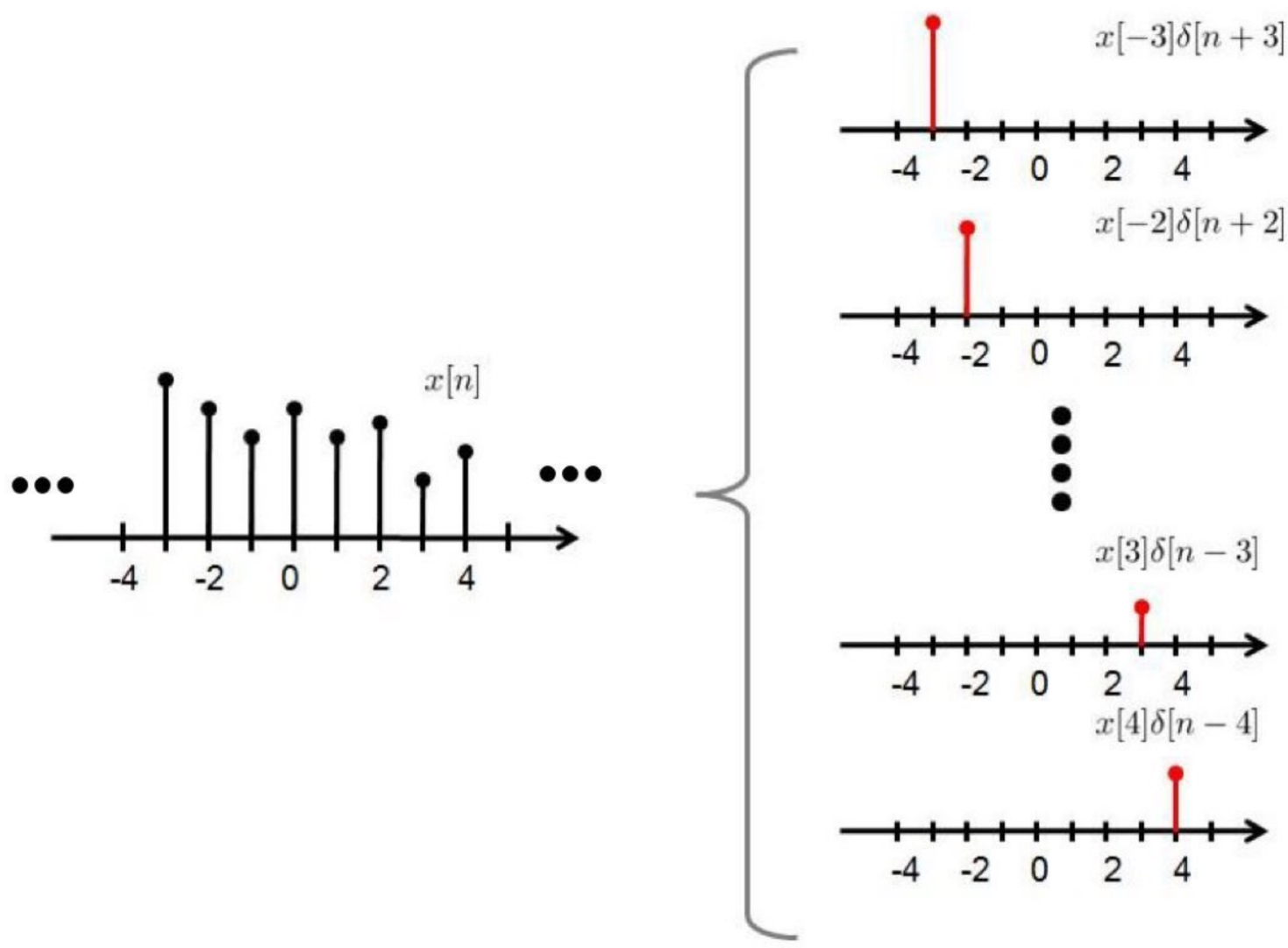


Figure 1.13: Representing of a signal *x[n]* using a train of impulses δ[n − k].

The sample x(0) can be obtained by multiplying x(0), the magnitude, with unit impulse δ(n)

$$\text{i.e., } x[n]\,\delta[n] = \begin{cases} x(0), & n=0 \\ 0, & n\neq 0 \end{cases}$$

Similarly, the sample x(-3) can be obtained as shown in the figure.

$$\text{i.e., } x[-3]\,\delta[n+3] = \begin{cases} x(-3), & n=-3 \\ 0, & n\neq -3 \end{cases}$$

In the same way we can get the sequence *x[n]* by summing all the shifted and scaled impulse function

$$\text{i.e., } x[n] = \ldots\, x[-3]\,\delta[n+3] + x[-2]\,\delta[n+2] + \ldots + x[0]\,\delta[n] + \ldots + x[4]\,\delta[n-4] \ldots$$

$$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Impulse response and convolution sum:

Impulse response: A discrete-time system performs an operation on an input signal based on predefined criteria to produce a modified output signal. The input signal x[n] is

Scanned with CamScanner

the system excitation, and y[n] is the system response. The transform operation is shown in the figure below.

If the input to the system is the unit impulse i.e., x[n] = δ[n], then the output of the system is known as impulse response represented by h[n] where

$$h[n] = T\,[\delta[n]]$$

Response of LTI system to arbitrary inputs: The convolution sum

From the above discussion, we get the response of an LTI system to an unit impulse as the impulse response h[n] i.e.,

$$\delta[n] \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow h[n-k], \text{ by time invariant property}$$

$$x(k)\delta[n-k] \longrightarrow x(k)h[n-k], \text{ by homogeneity principle}$$

$$\sum_{k=-\infty}^{\infty} x(k)\delta[n-k] \longrightarrow \sum_{k=-\infty}^{\infty} x(k)h[n-k], \text{ by super position}$$

As we know the arbitrary input signal is a weighted sum of impulse, the LHS = x[n] having a response in RHS = y[n] known as convolution summation.

i.e., x[n] $\longrightarrow$ y[n]

In other words, given a signal x[n] and the impulse response of an LTI system h[n], the convolution between x[n] and h[n] is defined as

$$y[n] = \sum_{k=-\infty}^{\infty} x(k)h[n-k]$$

We denote convolution as $y[n] = x[n] * h[n]$.

- Equivalent form: Letting $m = n - k$, we can show that

$$\sum_{k=-\infty}^{\infty} x(k)h[n-k] = \sum_{m=-\infty}^{\infty} x(n-m)h[m] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Properties of convolution:

The following "standard" properties can be proved easily:

1. Commutative: $x[n] * h[n] = h[n] * x[n]$
2. Associative: $x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$
3. Distributive: $x[n] * (h_1[n] + h_2[n]) = (x(t) * h_1[n]) + (x[n] * h_2[n])$

Scanned with CamScanner

How to Evaluate Convolution?

To evaluate convolution, there are four basic steps:

1. Fold
2. Shift
3. Multiply
4. Summation

Example1: Consider the signal x[n] and the impulse response h[n] shown below.

Let's compute the output y[n] one by one. First, consider y[0]:

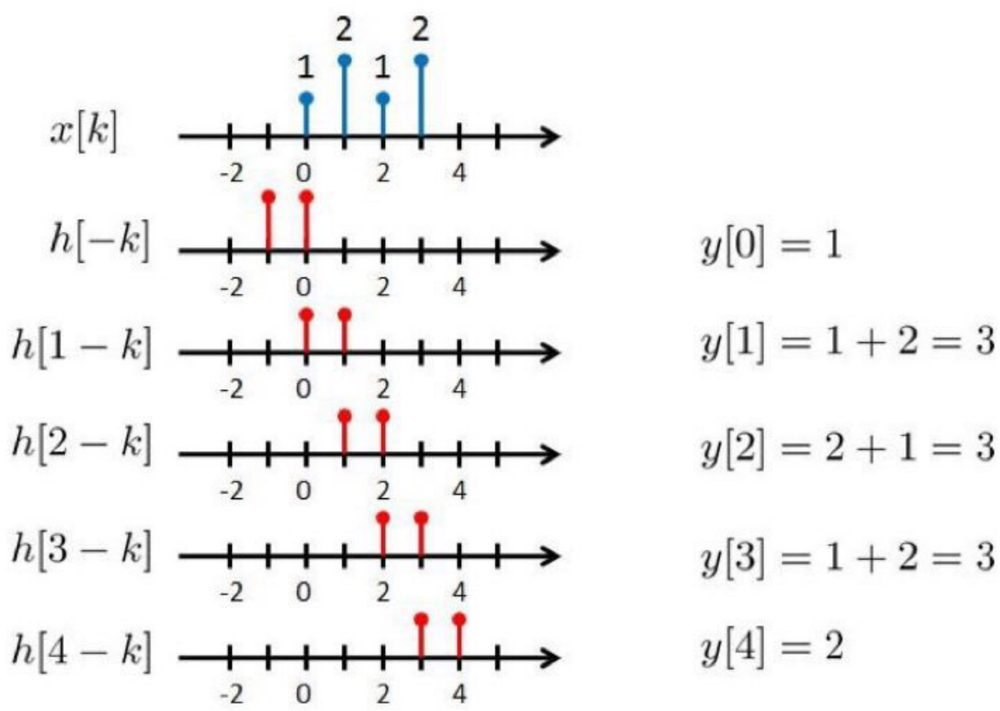
$$y[o] = \sum_{k=-\infty}^{\infty} x(k)h[0-k] = \sum_{k=-\infty}^{\infty} x(k)h[-k] = 1$$

Note that h[−k] is the flipped version of h[k], and $\sum_{k=-\infty}^{\infty} x(k)h[-k] = 1$ is the multiply-add between x[k] and h[−k].

To calculate y[l], we flip h[k] to get h[−k], shift h[−k] go get h[l−k], and multiply-add to get $\sum_{k=-\infty}^{\infty} x(k)h[1-k]$. Therefore

$$y[1] = \sum_{k=-\infty}^{\infty} x(k)h[1-k] = \sum_{k=-\infty}^{\infty} x(k)h[1-k] = 1 \times 1 + 2 \times 1 = 3$$

The calculation is shown in the figure below.



System Properties

With the notion of convolution, we can now proceed to discuss the system properties in terms of impulse responses.

Scanned with CamScanner

Memoryless

A system is memoryless if the output depends on the current input only. An equivalent statement using the impulse response h[n] is that:

An LTI system is memoryless if and only if

$$h[n] = a\delta[n], \text{ for some } a.$$

Invertible

An LTI system is invertible if and only if there exist g[n] such that

$$h[n] * g[n] = \delta[n].$$

Causal

An LTI system is causal if and only if

$$h[n] = 0, \quad \text{for all } n < 0.$$

Stable

An LTI system is stable if and only if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

Proof: Suppose that $\sum_{k=-\infty}^{\infty}|h[k]| < \infty$. For any bounded signal $|x[n]| \leq B$, the output is

$$|y[n]| \leq \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right|$$

$$= \sum_{k=-\infty}^{\infty} |x[k]|.|h[n-k]|$$

$$\leq B. \sum_{k=-\infty}^{\infty} |h[n-k]|$$

Therefore, $y[n]$ is bounded.

Scanned with CamScanner

Continuous-time Convolution

The continuous-time case is analogous to the discrete-time case. In continuous-time signals, the signal decomposition is

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)d\tau$$

and consequently, the continuous time convolution is defined as

$$(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)d\tau$$

Example: The continuous-time convolution also follows the three step rule: flip, shift, multiply- add. Let us consider the signal $x(t) = e^{-at}\,u(t)$ for $a > 0$, and impulse response $h(t) = u(t)$. The output $y(t)$ is

Case A: $t > 0$:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} e^{-ar}\,u(\tau)\,u(t-\tau)$$

$$= \int_{0}^{t} e^{-ar}\,d\tau$$

$$= \frac{1}{-a}\,[1\text{-}e^{-at}]$$

Case B: $t \leq 0$:

$$y(t) = 0.$$

Therefore,

$$y(t) = \frac{1}{a}\,[1 - e^{-at}]u(t)$$

Properties of CT Convolution

The following properties can be proved easily:

1. Commutative: $x(t) * h(t) = h(t) * x(t)$
2. Associative: $x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$
3. Distributive: $x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$

Continuous-time System Properties

The following results are analogous to the discrete-time case.

Memoryless.

An LTI system is memoryless if and only if

Scanned with CamScanner

$$h(t) = a\delta(t), \text{ for some } a$$

Invertible.

An LTI system is invertible if and only if there exist g(t) such that

$$h(t) * g(t) = \delta(t).$$

Causal.

A system is causal if and only if

$$h(t) = 0, \quad \text{for all } t < 0$$

Stable.

A system is stable if and only if

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

Interconnection of LTI systems:

1. **Parallel connection of LTI System:** Consider two LTI systems with impulse responses $h_1(t)$and $h_2(t)$ connected in parallel as shown in the figure below. The output of this connections of systems, $y(t)$, is the sum of the outputs of the two systems i.e.,

$$y(t) = y_1(t) + y_2(t) = x(t) * h_1(t) + x(t) * h_2(t)$$

$$= x(t) * [h_1(t) + h_2(t)]$$

Identical results hold for the discrete time case.

$$x(n) * h_1(n) + x(n) * h_2(n) = x(n) * [h_1(n) + h_2(n)]$$

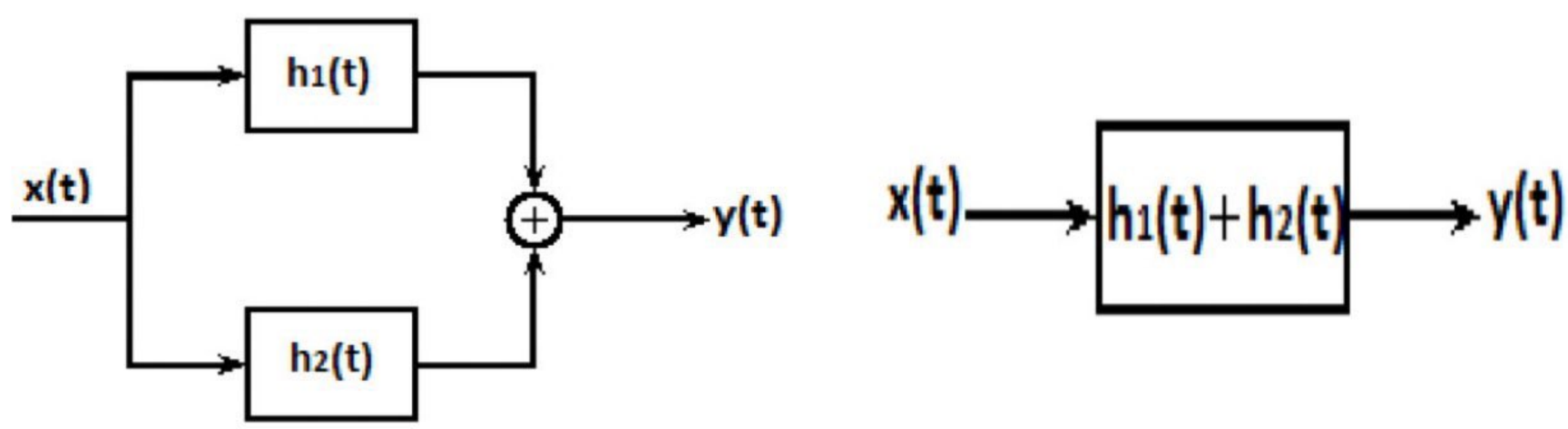


Fig: Parallel interconnection of two LTI system & its equivalent systems

2. **Cascade connection of LTI System:** Consider the cascade connection of two LTI systems as shown in the figure. The output of this connection of systems

$$y(t) = \{x(t) * h_1(t) * h_2(t)\}$$

Using associative property of convolution, we get

$$y(t) = x(t) * \{h_1(t) * h_2(t)\}$$

Scanned with CamScanner

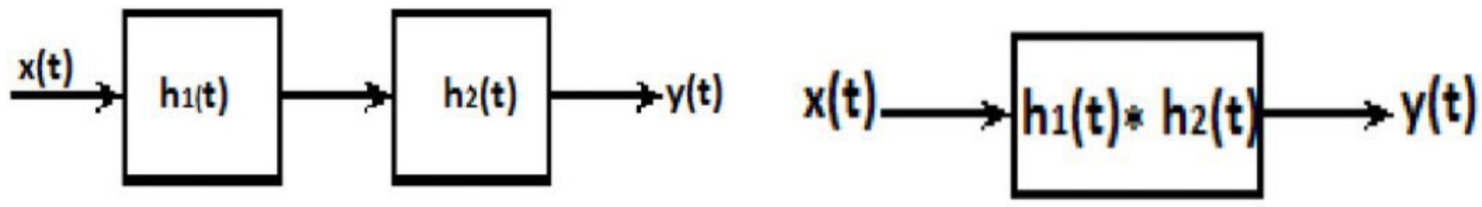


Fig: Cascade Interconnection of two LTI system & its equivalent systems

Step response: Step input response are often used to characterize the response of an LTI system to sudden changes in the input. It is defined as the output due to a unit step input signal.

Let $h[n]$ be the impulse response of a discrete-time LTI system and $s[n]$ be the step response.

Then, $$s[n] = h[n] * u[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]u[n-k]$$

Now, as u[n-k] = 0 for k > n and u[n-k] = 1 for k ≤ n,

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]$$

i.e., the step response is the running sum of the impulse response. Similarly, the step response s(t) of a continuous-time system is expressed as the running integral of the impulse response:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

Note: These relationships may be inverted to express the impulse response in terms of the step response as

$$h[n] = s[n] - s[n-1]$$

and, $$h(t) = \frac{d}{dt}\, s(t)$$

Scanned with CamScanner

Fourier Representations for Signals

In this chapter, the signal is represented as a weighted superposition of complex sinusoids. If such a signal is applied to an LTI system, then the system output is a weighted superposition of the system response to each complex sinusoid. Representing signal as superposition of complex sinusoids not only leads to a useful expression for the system output, but also provides an insightful characterization of the signals and systems. The study of signals and systems using sinusoidal representation is known as *Fourier analysis* named after Joseph Fourier.
Basing on the periodicity properties of the signal and whether the signal is discrete or continuous in time, there are four different types of Fourier representations, each applicable to a different class of signals.

Complex sinusoids and frequency response of LTI systems:

The response of an LTI system to a sinusoidal input leads to a characterization of system behavior termed as *frequency response* of the system. This characterization is obtained in terms of the impulse response by using convolution and a complex sinusoidal input signal.
Let us consider the output of a discrete-time LTI system with impulse response $h[n]$ and unit amplitude complex sinusoidal input $x[n] = e^{j\Omega n}$. This output is given by:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\Omega(n-k)}$$

We factor $e^{j\Omega n}$ from the sum to get

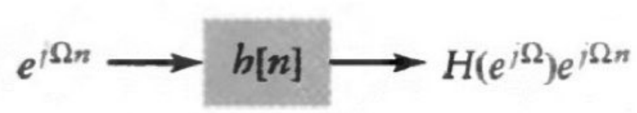
$$y[n] = e^{j\Omega n} \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\Omega k}$$

$$= \mathrm{H}(e^{j\Omega})e^{j\Omega n}$$

Where we have defined

$$H(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\Omega k}$$

Hence, the output of the system is a complex sinusoid of the same frequency as the input, multiplied by the complex number $H(e^{j\Omega})$. The relationship is shown in figure below:



The complex scaling factor $H(e^{j\Omega})$ is not a function of time n, but only is a function of frequency Ω and is termed the *frequency response* of the discrete-time system.
The results obtained for continuous-time LTI system is similar to the above.

Scanned with CamScanner

Let the impulse response of such a system be h(t) and the input be x(t) = $e^{j\omega t}$. Then the convolution integral gives the output as

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}\,d\tau$$
$$= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$$
$$= H(j\omega)e^{j\omega t},$$

Where we define,

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau.$$

The above equation is referred to as *frequency response* of the continuous time system. Writing the complex valued frequency response $H(j\omega)$ in polar form

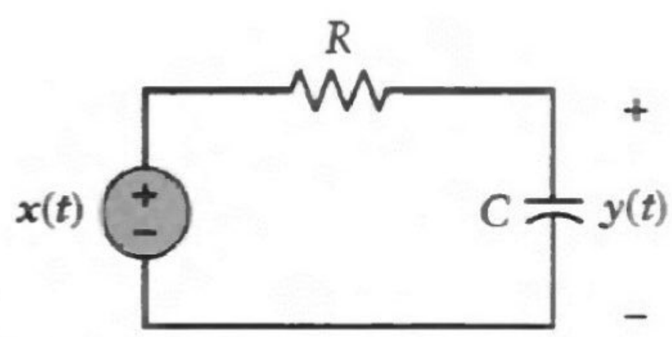
$$H(j\omega) = |H(j\omega)|e^{j\varphi}$$

Where,

$|H(j\omega)|$ => magnitude response
And, φ => phase response = arg$\{H(j\omega)\}$

Example: The impulse response of the system given the figure below is

$$h(t) = \frac{1}{RC}e^{-\frac{t}{RC}}u(t)$$



Find an expression for the frequency response and plot the magnitude and phase response.

Solution: Substituting h(t) in equation of $H(j\omega)$, we get

Scanned with CamScanner

$$
\begin{aligned}
H(j\omega) &= \frac{1}{RC}\int_{-\infty}^{\infty} e^{-\frac{\tau}{RC}}u(\tau)e^{-j\omega\tau}\,d\tau \\
&= \frac{1}{RC}\int_{0}^{\infty} e^{-\left(j\omega+\frac{1}{RC}\right)\tau}\,d\tau \\
&= \frac{1}{RC}\frac{-1}{\left(j\omega+\frac{1}{RC}\right)}e^{-\left(j\omega+\frac{1}{RC}\right)\tau}\Big|_{0}^{\infty} \\
&= \frac{1}{RC}\frac{-1}{\left(j\omega+\frac{1}{RC}\right)}(0-1) \\
&= \frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}.
\end{aligned}
$$

The magnitude response is:

$$|H(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2+\left(\frac{1}{RC}\right)^2}}$$

While the phase response is

$$\arg\{H(j\omega)\} = -\arctan(\omega RC)$$

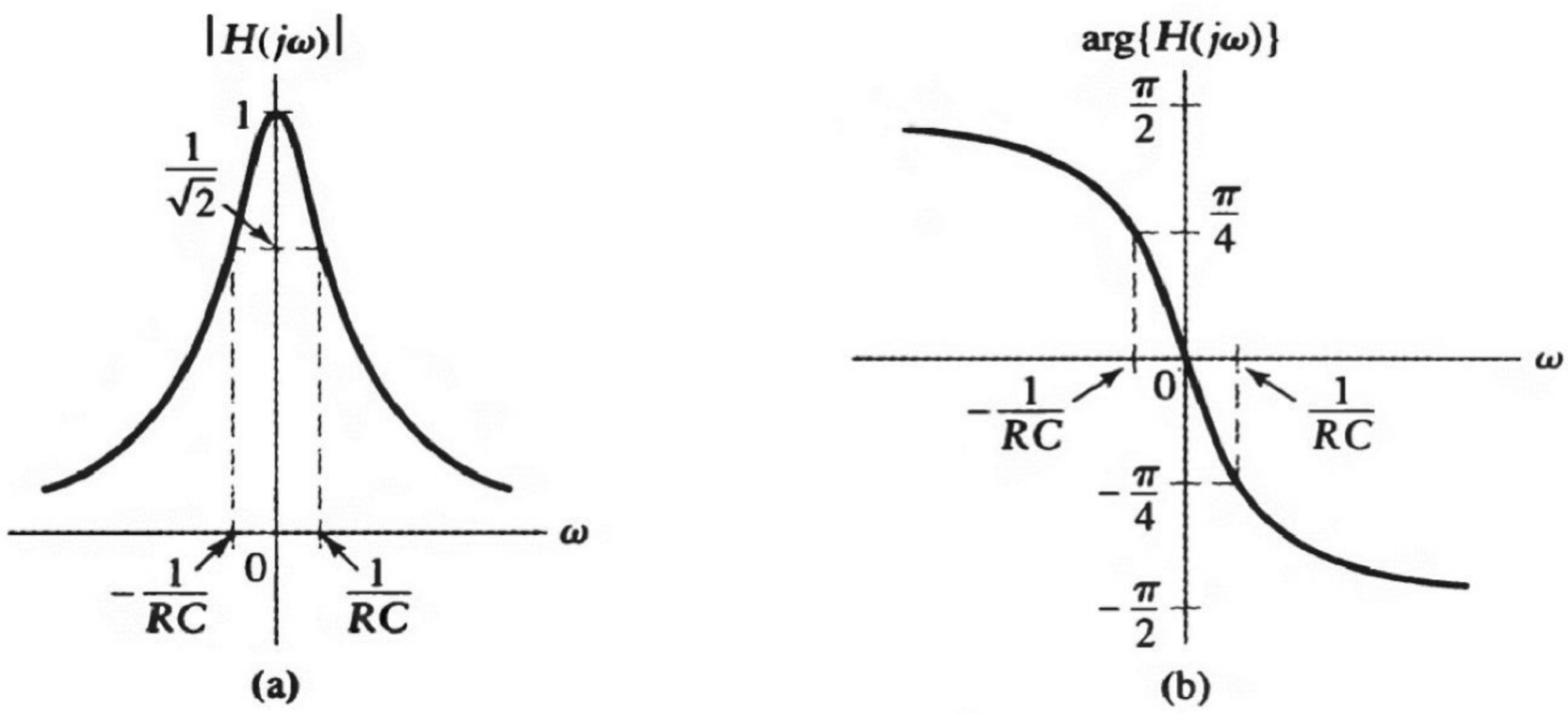


Fig: (a) Magnitude-response (b) Phase-response

Eigenvalue and Eigenfunctions of an LTI System

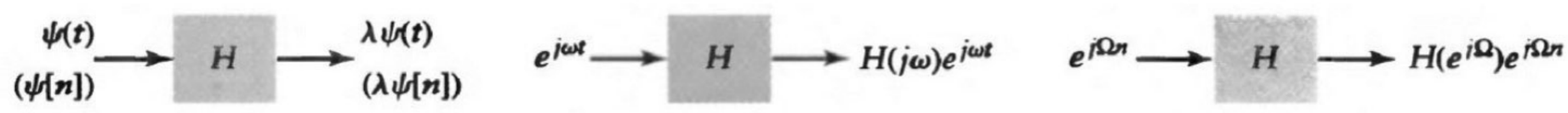
Definition: For an LTI system, if the output is a scaled version of its input, then the input function is called an *eigenfunction* of the system. The scaling factor is called the *eigenvalue* of the system.

We take the complex sinusoid $\psi(t) = e^{j\omega t}$ is an *eigenfunction* of the LTI system H associated with the *eigenvalue* $\lambda = H(j\omega)$, because ψ satisfies an eigenvalue problem described by

$$H\{\psi(t)\} = \lambda\psi(t)$$

Scanned with CamScanner

The effect of the system on an eigenfunction input signal is scalar multiplication. The output is given by the product of the input and a complex number. This eigen representation is shown in the figure below.



Fourier representation of four classes of signals:

- There are four distinct Fourier representations, each applicable to a different class of signals.
- The Fourier series (*FS*) applies to continuous time periodic signal, and the discrete - time Fourier series (*DTFS*) applies to discrete time periodic signal.
- The Fourier transform (FT) applies to a signal that is continuous in time and nonperiodic.
- The discrete-time Fourier transform (DTFT) applies to a signal that is discrete in time and nonperiodic.

Relationship between time properties of a signal and the appropriate Fourier representation is given below:

Time Property	*Periodic*	*Nonperiodic*
Continuous (t)	Fourier Series (FS)	Fourier Transform (FT)
Discrete $[n]$	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)

Continuous-time periodic signals: The Fourier series

Continuous-time periodic signals are represented by the Fourier series (FS). We may write the FS of a signal x(t) with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$, as

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t},$$

Where,

$$X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt$$

are the FS coefficient of the signal x(t). We say that $x(t)$ and X[k] are an FS pair and denote this relationship as

$$x(t) \xleftrightarrow{FS;\omega_o} X[k].$$

The Fourier series coefficients are known as the *frequency-domain representation of x(t)*,because each FS coefficient is associated with a complex sinusoid of different frequency.

In the representation of the periodic signal $x(t)$ by the Fourier series, the issue arises, is whether or not the series converges to $x(t)$ for each value of t, i.e., whether the signal $x(t)$ and its FS representation are equal at each value of t.

Scanned with CamScanner

The Dirichlet conditions guarantee that the FS will be equal to $x(t)$, except at the value of t for which $x(t)$ is discontinuous. At these values of t, FS converges to the mid-point of the discontinuity. The Dirichlet conditions are:

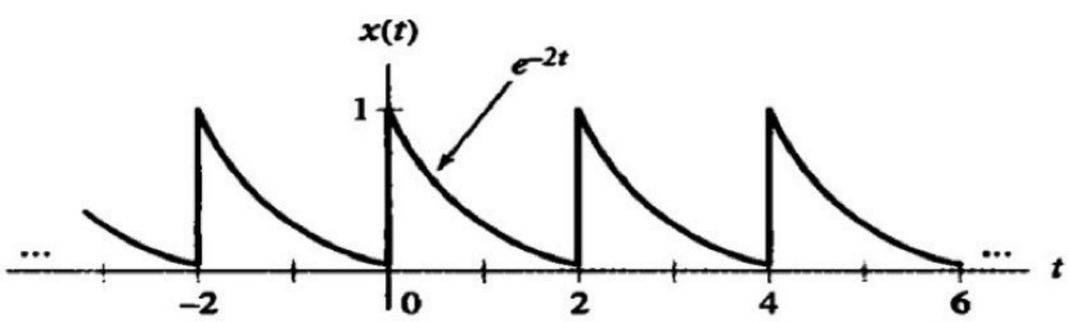
1. The signal $x(t)$ has a finite number of discontinuities in any period.
2. The signal contains a finite number of maxima and minima during any period.
3. The signal $x(t)$ is absolutely integrable (bounded) i.e.,

$$\int_T^{\infty} |x(t)|dt < \infty$$

If $x(t)$ is periodic and satisfies the Dirichlet condition, it can be represented in FS.

Direct calculation of FS coefficients:

Example: Determine the FS coefficient for signal $x(t)$



Solution: Time period T = 2, Hence, $\omega_0 = 2\pi/2 = \pi$. On the interval $0 \le t \le 2$, one period of $x(t)$ is expressed as $x(t) = e^{-2t}$. So,

$$\begin{aligned} X[k] &= \frac{1}{2}\int_0^2 e^{-2t}e^{-jk\pi t}\,dt \\ &= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\,dt. \end{aligned}$$

We evaluate the integral to get

$$\begin{aligned} X[k] &= \frac{-1}{2(2+jk\pi)}e^{-(2+jk\pi)t}\Big|_0^2 \\ &= \frac{1}{4+jk2\pi}(1-e^{-4}e^{-jk2\pi}) \\ &= \frac{1-e^{-4}}{4+jk2\pi}, \end{aligned}$$

Scanned with CamScanner

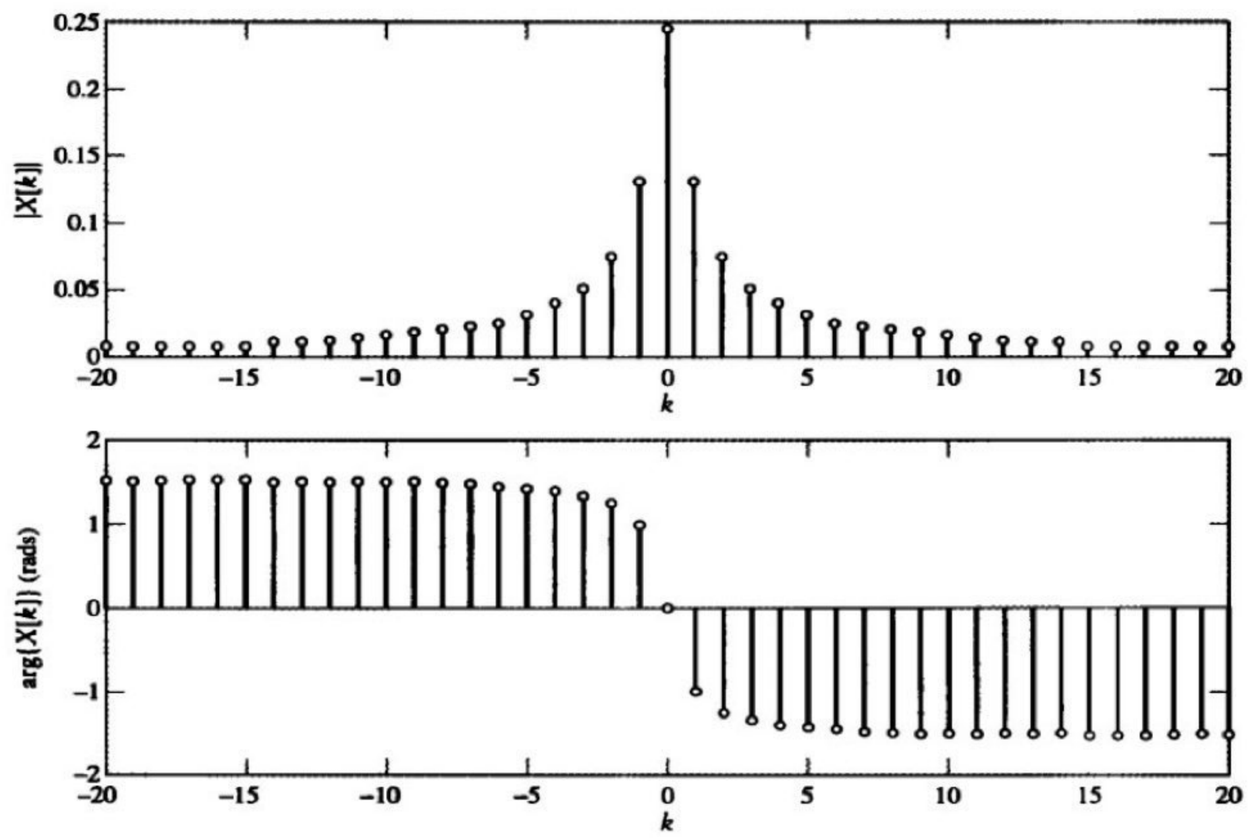


Fig: Magnitude and phase response of X[k]

Calculation of FS coefficients by inspection:

Example: Determine the FS representation of the signal

$$x(t) = 3\cos(\pi t/2 + \pi/4)$$

Solution: Time period $T = 4$, So, $\omega_0 = 2\pi/4 = \pi/2$. We may write FS of a signal x(t) is,

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\pi t/2}$$

Using Euler's formula to expand the cosine, gives

$$x(t) = 3\frac{e^{j(\pi t/2+\pi/4)} + e^{-j(\pi t/2+\pi/4)}}{2}$$
$$= \frac{3}{2}e^{j\pi/4}e^{j\pi t/2} + \frac{3}{2}e^{-j\pi/4}e^{-j\pi t/2}$$

Equating each term in this expansion to the terms in equation of x(t) gives the FS coefficient:

$$X[k] = \begin{cases} \frac{3}{2}e^{-j\pi/4}, & k = -1 \\ \frac{3}{2}e^{j\pi/4}, & k = 1 \\ 0, & \text{otherwise} \end{cases}$$

The magnitude and phase spectra are shown below.

Scanned with CamScanner

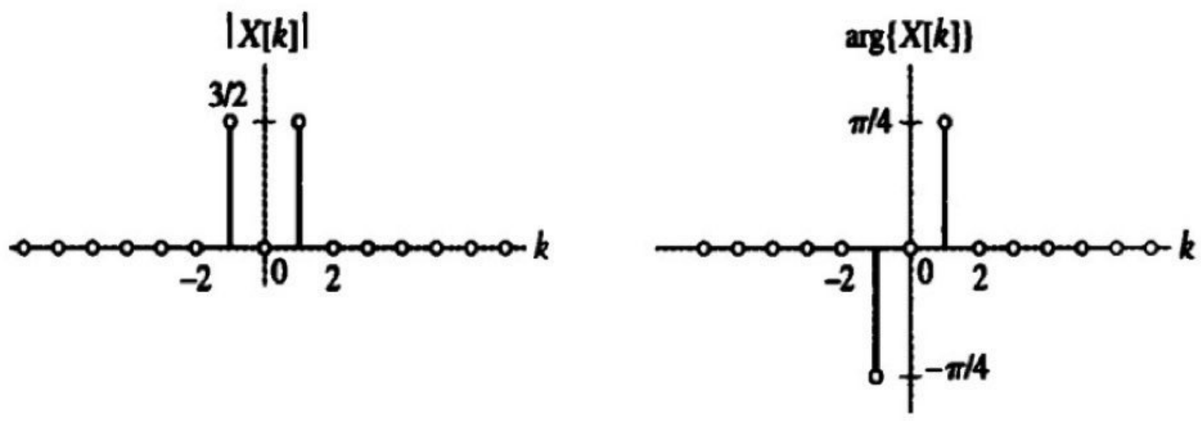


Figure: Magnitude and phase spectrum

Example: Find the time domain signal x(t) corresponding to the FS coefficient

$$X[k] = (1/2)^{|k|}e^{jk\pi/20}$$

Assume that fundamental period T = 2.

Solution: Substituting the values given for X[k] and $\omega_0 = 2\pi/2 = \pi$ into equation x(t) gives

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty}(1/2)^k e^{jk\pi/20}e^{jk\pi t} + \sum_{k=-1}^{-\infty}(1/2)^{-k}e^{jk\pi/20}e^{jk\pi t} \\ &= \sum_{k=0}^{\infty}(1/2)^k e^{jk\pi/20}e^{jk\pi t} + \sum_{l=1}^{\infty}(1/2)^l e^{-jl\pi/20}e^{-jl\pi t}. \end{aligned}$$

The second geometric series is evaluated by summing from $l = 0$ to $l = \infty$, and subtracting the $l = 0$ term. The result of summing both infinite geometric series is

$$x(t) = \frac{1}{1-(1/2)e^{j(\pi t+\pi/20)}} + \frac{1}{1-(1/2)e^{-j(\pi t+\pi/20)}} - 1$$

Putting the fractions over a common denominator results in

$$x(t) = \frac{3}{5-4\cos(\pi t+\pi/20)}$$

Discrete-time periodic signals: The discrete-time Fourier series

Discrete-time periodic signals are represented by the discrete-time Fourier series (DTFS). We may write the DTFS of a signal x[n] with fundamental period N and fundamental frequency $\Omega_0 = 2\pi/N$, as

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n}.$$

Where

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n}$$

Are the DTFS coefficient of the signal x[n].We say that x[n] and X[k] are a DTFS pair and denote this relationship as

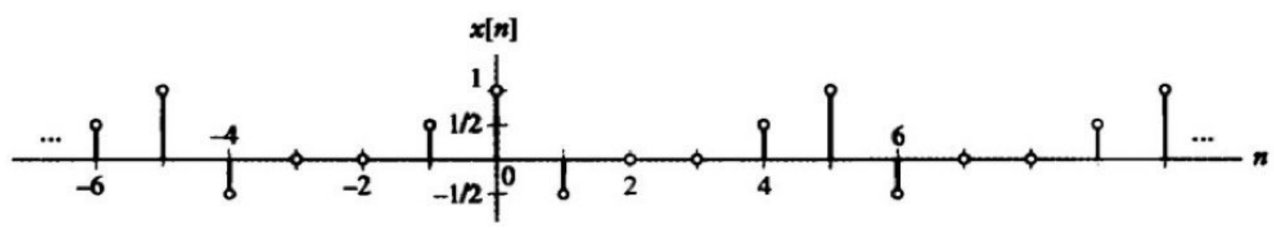
$$x[n] \xleftrightarrow{DTFS;\Omega_o} X[k].$$

Scanned with CamScanner

The DTFS coefficients are known as the *frequency-domain representation of* $x[n]$, because each DTFS coefficient is associated with a complex sinusoid of different frequency.

Direct calculation of DTFS coefficients:

Example: Determine the DTFS coefficient for signal x[n] shown



Solution: The signal has a period $N = 5$, so $\Omega 0 = 2\pi/5$. As the signal also has odd symmetry, it can be sum over n =-2 to n =2 in the equation and we get,

$$X[k] = \frac{1}{5}\sum_{n=-2}^{2} x[n]e^{-jk2\pi n/5}$$
$$= \frac{1}{5}\{x[-2]e^{jk4\pi/5} + x[-1]e^{jk2\pi/5} + x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5}\}$$

Using the values of $x[n]$, we get

$$X[k] = \frac{1}{5}\left\{1 + \frac{1}{2}e^{jk2\pi/5} - \frac{1}{2}e^{-jk2\pi/5}\right\}$$
$$= \frac{1}{5}\{1 + j\sin(k2\pi/5)\}.$$

From the above equation, we identify one period of DTFS coefficients $X[k]$, k=2 to k=-2, in rectangular and polar coordinates as

$$X[-2] = \frac{1}{5} - j\frac{\sin(4\pi/5)}{5} = 0.232e^{-j0.531}$$
$$X[-1] = \frac{1}{5} - j\frac{\sin(2\pi/5)}{5} = 0.276e^{-j0.760}$$
$$X[0] = \frac{1}{5} = 0.2e^{j0}$$
$$X[1] = \frac{1}{5} + j\frac{\sin(2\pi/5)}{5} = 0.276e^{j0.760}$$
$$X[2] = \frac{1}{5} + j\frac{\sin(4\pi/5)}{5} = 0.232e^{j0.531}.$$

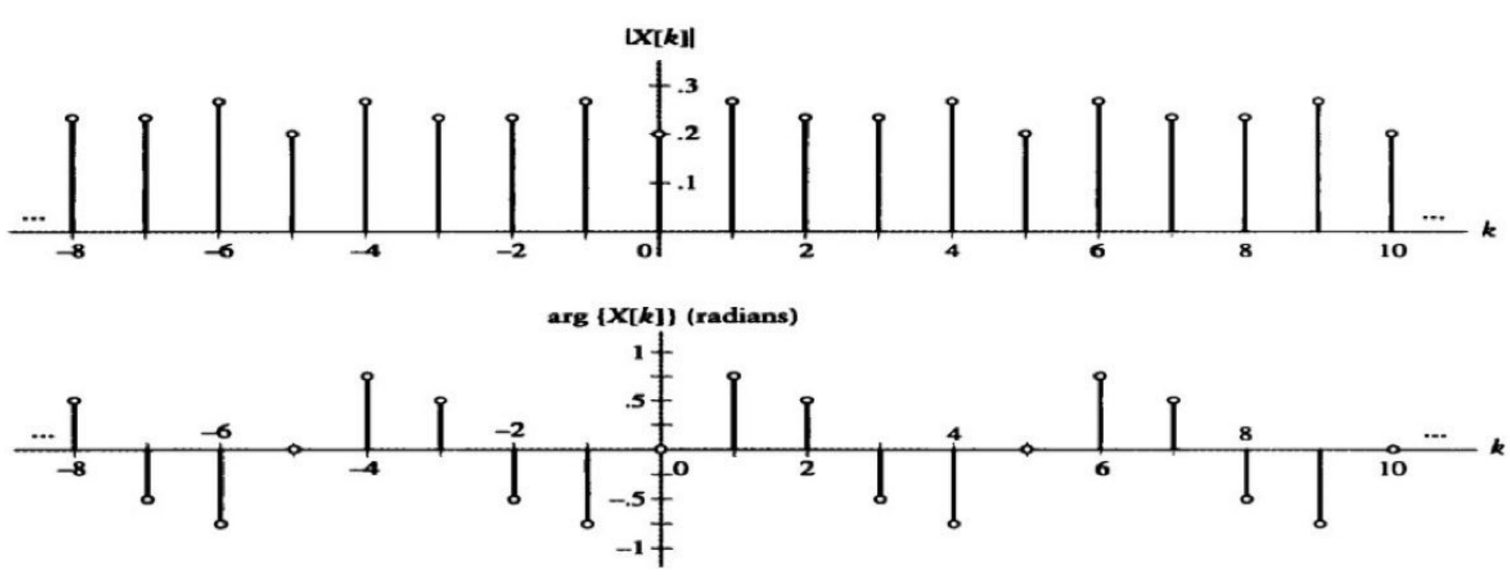


Fig: Magnitude and Phase Response of $X[k]$

Scanned with CamScanner

The above figure shows the magnitude and phase of $X[k]$ as functions of the frequency index k.

Now suppose we calculate $X[k]$ using $n = 0$ to $n = 4$ for the limits on the sum in eqn.of $X[k]$ to obtain

$$X[k] = \frac{1}{5}\{x[0]e^{j0} + x[1]e^{-j2\pi/5} + x[2]e^{-jk4\pi/5} + x[3]e^{-jk6\pi/5} + x[4]e^{-j8\pi/5}\}$$
$$= \frac{1}{5}\left\{1 - \frac{1}{2}e^{-jk2\pi/5} + \frac{1}{2}e^{-jk8\pi/5}\right\}.$$

Calculation of DTFS coefficients by inspection:

Example: Determine the DTFS coefficients of the signal

$$x[n] = \cos(\pi n/3 + \varphi)$$

Solution: Time period $N=6$.We expand the cosine by using Euler's formula as

$$x[n] = \frac{e^{j\left(\frac{\pi}{3}n+\phi\right)} + e^{-j\left(\frac{\pi}{3}n+\phi\right)}}{2}$$
$$= \frac{1}{2}e^{-j\phi}e^{-j\frac{\pi}{3}n} + \frac{1}{2}e^{j\phi}e^{j\frac{\pi}{3}n}.$$

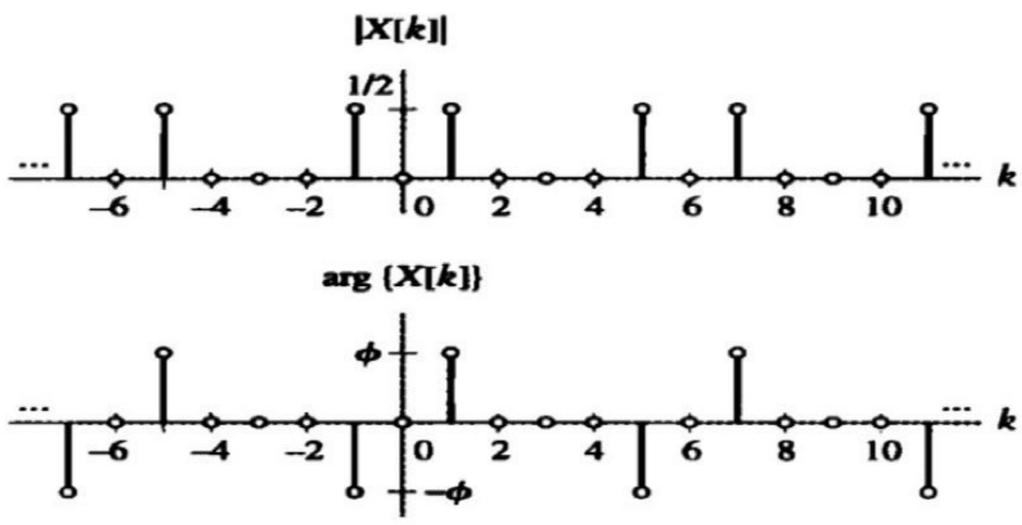
Now comparing with the DTFS equation with $\Omega_0 = 2\pi/6 = \pi/3$, written by summing from $k = -2$ to 3

$$x[n] = \sum_{k=-2}^{3} X[k]e^{jk\pi n/3}$$
$$= X[-2]e^{-j2\pi n/3} + X[-1]e^{-j\pi n/3} + X[0] + X[1]e^{j\pi n/3} + X[2]e^{j2\pi n/3} + X[3]e^{j\pi n}.$$

Equating the terms, we get

$$x[n] \xleftrightarrow{DTFS;\frac{\pi}{3}} X[k] = \begin{cases} e^{-j\phi}/2, & k = -1 \\ e^{j\phi}/2, & k = 1 \\ 0, & \text{otherwise on } -2 \le k \le 3 \end{cases}.$$

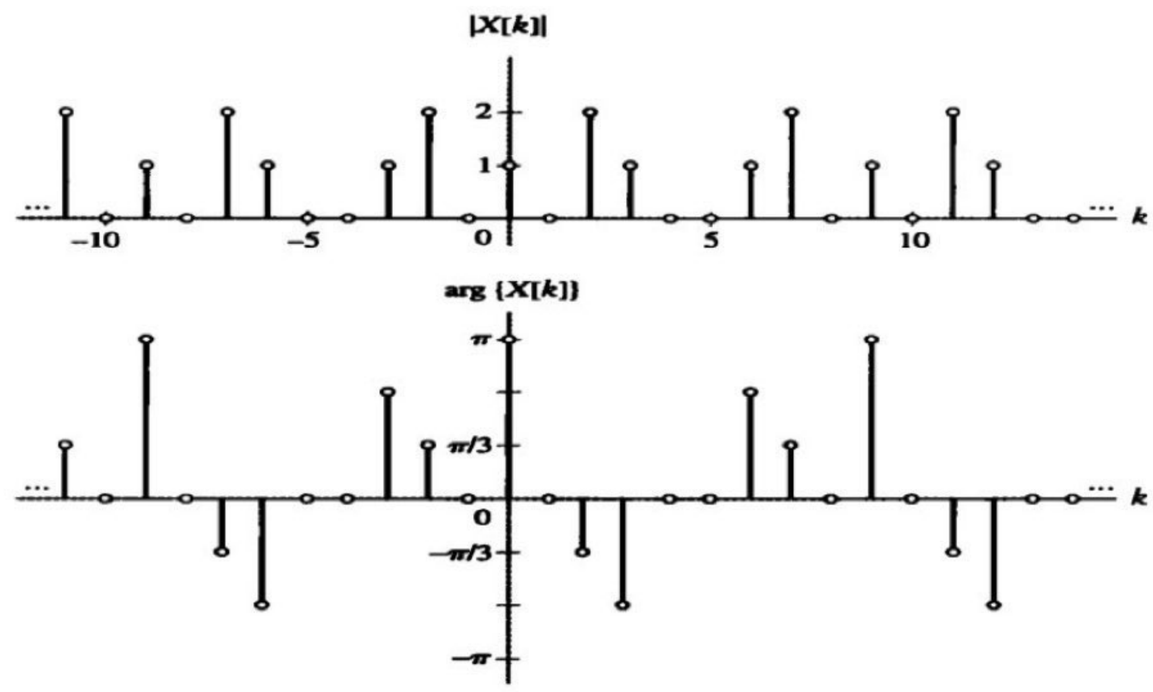
The magnitude and phase spectrum is given below



Scanned with CamScanner

The Inverse DTFS:

Example: Determine the time signal x[n] from the DTFS coefficients given in figure below



Solution: The DTFS coefficients have period = 9, hence $\Omega_0=2\pi/9$. It is convenient to evaluate *x[n]* over the interval k = -4 to k = 4 to obtain

$$\begin{aligned} x[n] &= \sum_{k=-4}^{4} X[k]e^{jk2\pi n/9} \\ &= e^{j2\pi/3}e^{-j6\pi n/9} + 2e^{j\pi/3}e^{-j4\pi n/9} - 1 + 2e^{-j\pi/3}e^{j4\pi n/9} + e^{-j2\pi/3}e^{j6\pi n/9} \\ &= 2\cos(6\pi n/9 - 2\pi/3) + 4\cos(4\pi n/9 - \pi/3) - 1. \end{aligned}$$

Continuous-time nonperiodic signals: The Fourier transform

The Fourier transform is used to represent a continuous-time nonperiodic signal as a superposition of complex sinusoids. We know that the continuous nonperiodic nature of a time signal implies that the superposition of complex sinusoids used in the Fourier representation of the signal involves a continuum of frequencies ranging from -∞ to ∞. So the FT representation of a continuous-time signal involves an intyegral over the entire frequency interval; i.e.,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

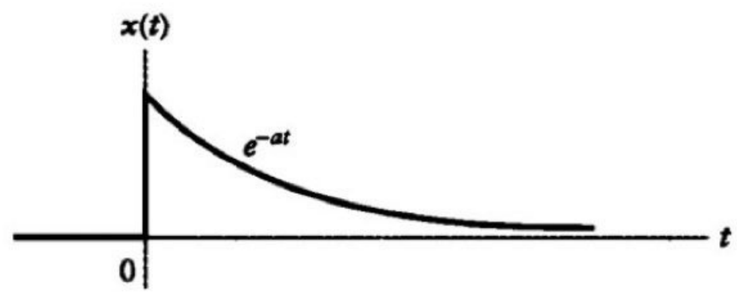
Where,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Example1. FT of a real decaying exponential:

Find the FT of $x(t) = e^{-at}u(t)$ shown in the figure below.

Scanned with CamScanner



Solution: The FT does not converge for a ≤ 0, since $x(t)$ is not absolutely integrable, i.e.;

$$\int_0^{\infty} e^{-at}\,dt = \infty \quad , \quad a \leq 0$$

For a > 0, we have

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt \\ &= \int_0^{\infty} e^{-(a+j\omega)t}\,dt \\ &= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} \\ &= \frac{1}{a+j\omega}. \end{aligned}$$

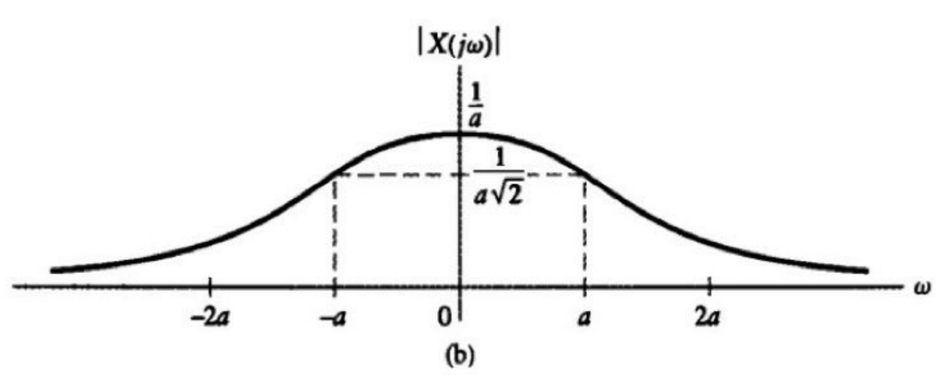
Converting to polar form, the magnitude and phase of $X(j\omega)$ are respectively given by

$$|X(j\omega)| = \frac{1}{(a^2+\omega^2)^{\frac{1}{2}}}$$

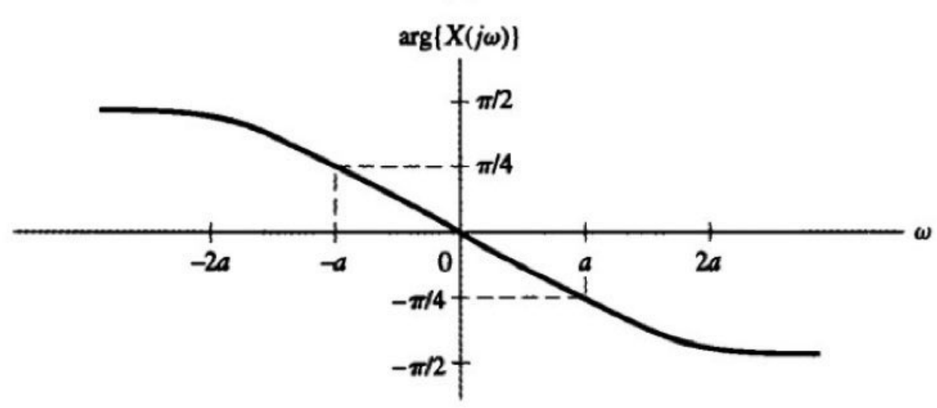
and

$$\arg\{X(j\omega)\} = -\arctan(\omega/a)$$

as shown in figure below.



(b)



(c)

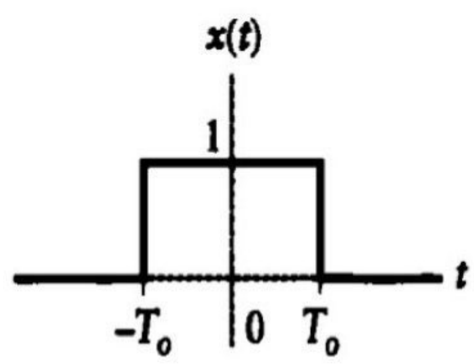
Scanned with CamScanner

The magnitude of $X(j\omega)$ plotted against ω is known as the magnitude spectrum of the signal $x(t)$, and the phase of $X(j\omega)$ plotted as a function of ω is known as the phase spectrum of $x(t)$.

***Example* 2: FT of a rectangular pulse**:

Consider the rectangular pulse shown figure below and defined as

$$x(t) = \begin{cases} 1, & -T_o < t < T_o \\ 0, & |t| > T_o \end{cases}.$$



Find the FT of $x(t)$.

Solution: The rectangular pulse $x(t)$ is absolutely integrable, provided that $T_0 < \infty$. So we have

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= \int_{-T_o}^{T_o} e^{-j\omega t}\,dt \\ &= -\frac{1}{j\omega}e^{-j\omega t}\Big|_{-T_o}^{T_o} \\ &= \frac{2}{\omega}\sin(\omega T_o), \qquad \omega \neq 0 \end{aligned}$$

For $\omega = 0$, the integral simplifies to $2T_0$. L'Hospital's rule straightforwardly shows that

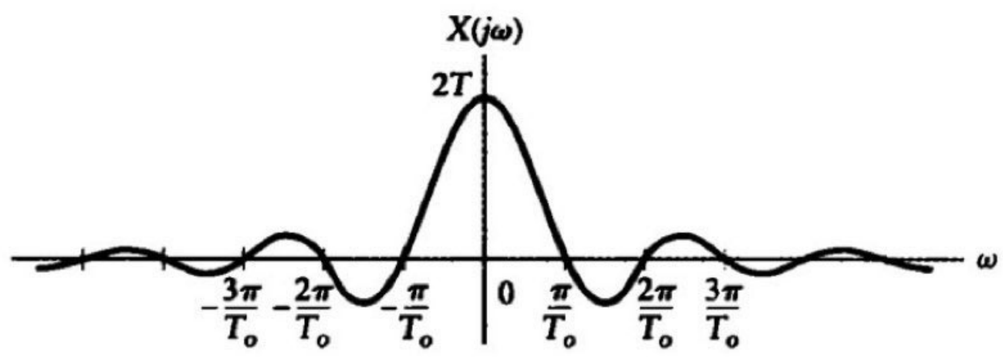
$$\lim_{\omega \to 0} \frac{2}{\omega}\sin(\omega T_0) = 2T_0$$

Thus, we usually write

$$X(j\omega) = \frac{2}{\omega}\sin(\omega\, T_0)$$

With the understanding that the value at ω = 0 is obtained by evaluating a limit. In this case $X(j\omega)$ is real and is shown in the figure below.

Scanned with CamScanner



The magnitude spectrum is

$$|X(j\omega)| = 2\left|\frac{\sin(\omega T_o)}{\omega}\right|$$

and the phase spectrum is

$$\arg\{X(j\omega)\} = \begin{cases} 0, & \sin(\omega T_o)/\omega > 0 \\ \pi, & \sin(\omega T_o)/\omega < 0 \end{cases}$$

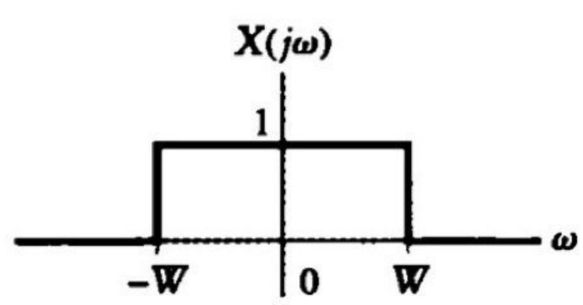
Using sinc function notation, we may write $X(j\omega)$ as

$$X(j\omega) = 2T_o \operatorname{sinc}(\omega T_o/\pi)$$

Inverse FT of a rectangular spectrum:

Example: Find the inverse FT of the rectangular spectrum (figure below) given by

$$X(j\omega) = \begin{cases} 1, & -W < \omega < W \\ 0, & |\omega| > W \end{cases}$$



Solution: Using equation for $x(t)$ for inverse FT yields

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega \\ &= \frac{1}{2j\pi t} e^{j\omega t}\Big|_{-W}^{W} \\ &= \frac{1}{\pi t}\sin(Wt), \qquad t \neq 0 \end{aligned}$$

When $t = 0$, the integral simplifies to W/π. As

$$\lim_{t\to 0}\frac{1}{\pi t}\sin(Wt) = W/\pi,$$

Scanned with CamScanner

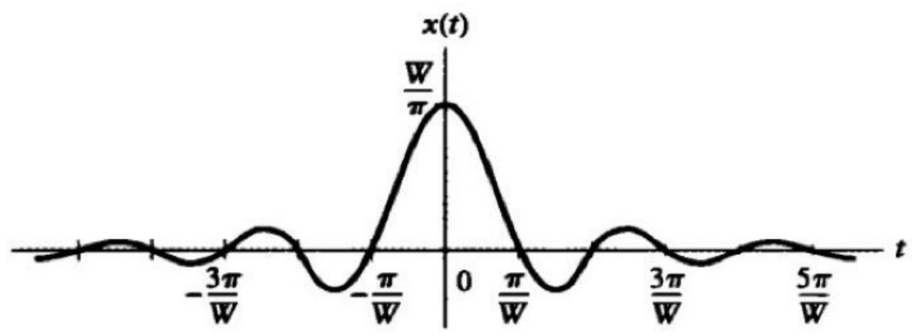
We usually write

$$x(t) = \frac{1}{\pi t}\sin(Wt)$$

or

$$x(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right)$$

The value at $t = 0$ is obtained as a limit. The $x(t)$ is shown in the following diagram.



Discrete-time nonperiodic signals: The Discrete-time Fourier transform

The DTFT is used to represent a discrete-time nonperiodic signal as a super position of complex sinusoids. As reasoned previously, the DTFT would involve a continuum of frequencies on the interval $-\pi < \Omega < \pi$, where Ω has unit of radians. Thus, the DTFT representation of a time-domain signal involves an integral over frequency, namely,

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$$

Where

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

is the DTFT of the signal $x[n]$. As $X(e^{j\Omega})$ and $x[n]$ are a DTFT pair, we can write

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}).$$

The transform $X(e^{j\Omega})$ describes the signal $x[n]$ as a function of a sinusoidal frequency Ω and is called the *frequency-domain* representation of $x[n]$. The equation for $x[n]$ is usually called the *inverse DTFT,* as it maps the frequency-domain representation back into the time-domain.

If

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

i.e.; if $x[n]$ is absolutely summable, then the sum in eqn. $X(e^{j\Omega})$ converges uniformly to a continuous function of ω.

Example: Find the DTFT of the sequence $x[n] = \alpha^n u[n]$.

Scanned with CamScanner

Solution: Using the above equation, we have

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \alpha^n u[n] e^{-j\Omega n}$$
$$= \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n}.$$

This sum diverges for $|\alpha| \geq 1$. For $|\alpha| \leq 1$, we have the convergent geometric series

$$X(e^{j\Omega}) = \sum_{n=0}^{\infty} (\alpha e^{-j\Omega})^n$$
$$= \frac{1}{1 - \alpha e^{-j\Omega}}, \qquad |\alpha| < 1$$

If α is real valued, the denominator of the above equation may be expanded. Using Euler's formula, we get

$$X(e^{j\Omega}) = \frac{1}{1 - \alpha \cos \Omega + j\alpha \sin \Omega}$$

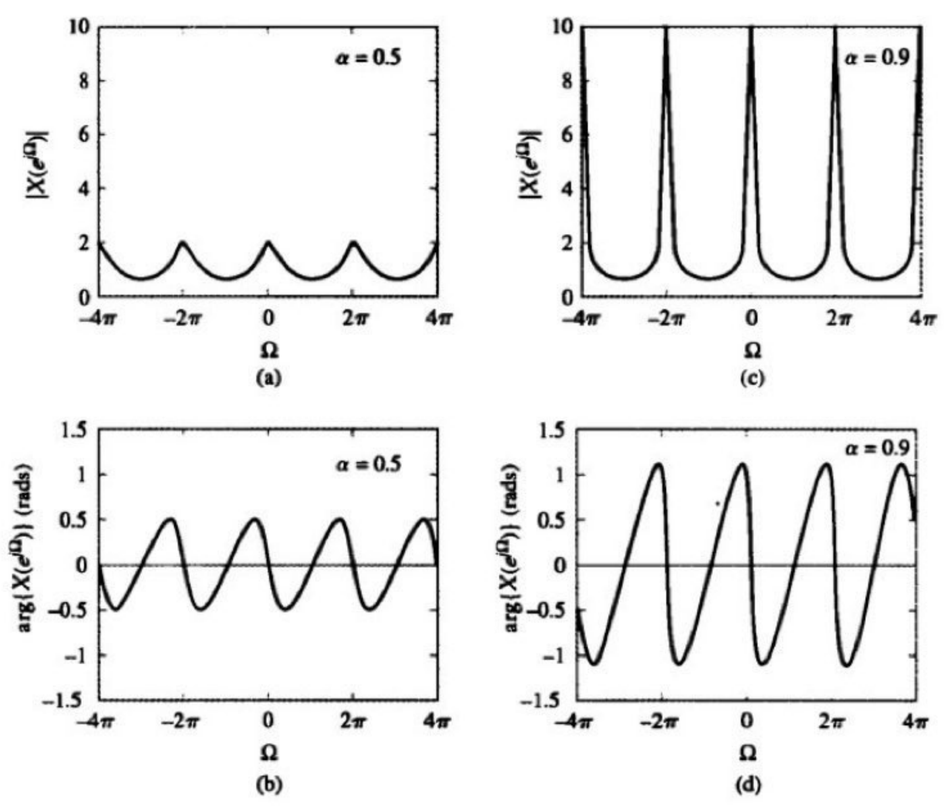
From this form, we see that the magnitude and phase spectra are given by

$$|X(e^{j\Omega})| = \frac{1}{((1 - \alpha \cos \Omega)^2 + \alpha^2 \sin^2 \Omega)^{1/2}}$$
$$= \frac{1}{(\alpha^2 + 1 - 2\alpha \cos \Omega)^{1/2}}$$

and

$$\arg\{X(e^{j\Omega})\} = -\arctan\left(\frac{\alpha \sin \Omega}{1 - \alpha \cos \Omega}\right),$$ respectively.

The magnitude and phase spectra for α = 0.5 and α = 0.9 are shown in the figure below. The magnitude is given and the phase is odd and both are 2π periodic.

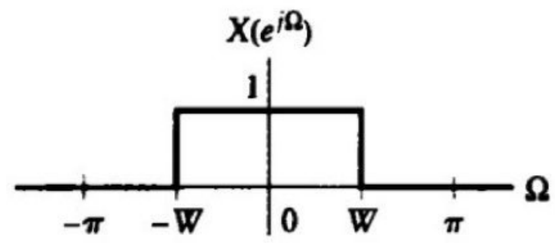


Inverse DTFT

Example: Find the Inverse DTFT of the following rectangular spectrum

$$X(e^{j\Omega}) = \begin{cases} 1, & |\Omega| < W \\ 0, & W < |\Omega| < \pi \end{cases}$$

Scanned with CamScanner



Solution: Substituting $X(e^{j\Omega})$ in DTFT representation, we get

$$x[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\Omega n}\, d\Omega$$
$$= \frac{1}{2\pi n j} e^{j\Omega n}\Big|_{-W}^{W}, \quad n \neq 0$$
$$= \frac{1}{\pi n}\sin(Wn), \quad n \neq 0.$$

For $n = 0$, the integrand is unity and we have $x[0]=W/\pi$. Using L'Hopital's rule, we can easily show that

$$\lim_{n\to 0}\frac{1}{\pi n}\sin(Wn) = \frac{W}{\pi}$$

And thus we usually write

$$x[n] = \frac{1}{\pi n}\sin(Wn)$$

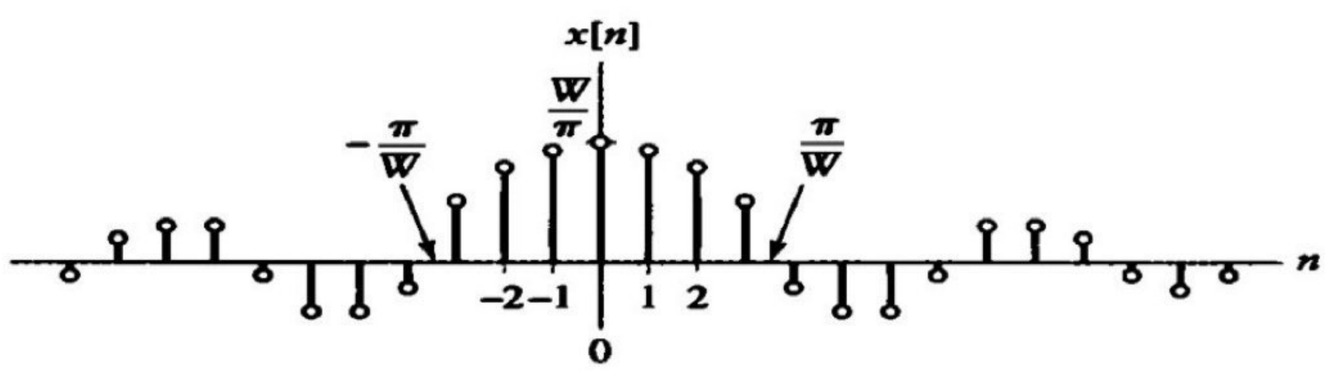


Fig: Inverse DTFT in Time domain

Properties of Fourier representation:

1. ***Linearity:***
   All four Fourier representations satisfy the linearity property.

$$z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(j\omega) = aX(j\omega) + bY(j\omega)$$
$$z(t) = ax(t) + by(t) \xleftrightarrow{FS;\omega_o} Z[k] = aX[k] + bY[k]$$
$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$$
$$z[n] = ax[n] + by[n] \xleftrightarrow{DTFS;\Omega_o} Z[k] = aX[k] + bY[k]$$

Scanned with CamScanner

In these relationships, we assume that the upper case symbol denotes the Fourier representation of the corresponding lower case symbol.

2. ***Symmetry:***

Representation	*Real-Valued Time Signals*	*Imaginary-Valued Time Signals*
FT	$X^*(j\omega) = X(-j\omega)$	$X^*(j\omega) = -X(-j\omega)$
FS	$X^*[k] = X[-k]$	$X^*[k] = -X[-k]$
DTFT	$X^*(e^{j\Omega}) = X(e^{-j\Omega})$	$X^*(e^{j\Omega}) = -X(e^{-j\Omega})$
DTFS	$X^*[k] = X[-k]$	$X^*[k] = -X[-k]$

3. ***Convolution:***

a) Convolution of nonperiodic signal: Convolution of two signals $h(t)$ & $x(t)$ in the time domain corresponds to multiplication of their FT, $H(j\omega)$ & $X(j\omega)$ in frequency domain.

$$y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)H(j\omega)$$

A similar property holds for convolution of discrete time non-periodic signal .If

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) \text{ and } h[n] \xleftrightarrow{DTFT} H(e^{j\Omega}), \text{then}$$

$$y[n] = x[n] * h[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega})\, H(e^{j\Omega})$$

b) Convolution of periodic signal: The periodic convolution of two periodic CT signals x(t) and z(t), each having period T , as

$$\begin{aligned} y(t) &= x(t) \circledast z(t) \\ &= \int_0^T x(\tau) z(t-\tau)\, d\tau \end{aligned}$$

Substituting the FS representation of z(t) into the convolution integral leads to the property

$$y(t) = x(t) \circledast z(t) \xleftrightarrow{FS;\ \frac{2\pi}{T}} Y[k] = TX[k]Z[k]$$

Similarly in DTFS

$$y[n] = x[n] \circledast z[n] \xleftrightarrow{DTFS;\ \frac{2\pi}{N}} Y[k] = NX[k]Z[k]$$

4) ***Differentiation and Integration:***

Scanned with CamScanner

a) *Differentiation in time:* Differentiating a signal in time domain corresponds to multiplying its FT by jω in the frequency domain i.e;

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

b) *Differentiation in frequency*: Differentiation of FT in frequency domain corresponds to multiplication of the signal by –jt in the time domain i.e;

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

Commonly used Differentiation and Integration properties:

$$\frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega)$$

$$\frac{d}{dt}x(t) \xleftrightarrow{FS;\omega_o} jk\omega_o X[k]$$

$$-jtx(t) \xleftrightarrow{FT} \frac{d}{d\omega}X(j\omega)$$

$$-jnx[n] \xleftrightarrow{DTFT} \frac{d}{d\Omega}X(e^{j\Omega})$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$$

5) Time Shift:

Let z(t) = x(t-t_0) be a time –shifted version of x(t). The goal is to relate the FT of *z*(t) to the FT of x(t)

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(t - t_o)e^{-j\omega t}\,dt.$$

Putting τ = t – t_0, we obtain

$$Z(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau + t_o)}\,d\tau$$

$$= e^{-j\omega t_o}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau$$

$$= e^{-j\omega t_o}X(j\omega).$$

Time-shift properties of Fourier representation:

$$x(t - t_o) \xleftrightarrow{FT} e^{-j\omega t_o}X(j\omega)$$

$$x(t - t_o) \xleftrightarrow{FS;\omega_o} e^{-jk\omega_o t_o}X[k]$$

$$x[n - n_o] \xleftrightarrow{DTFT} e^{-j\Omega n_o}X(e^{j\Omega})$$

$$x[n - n_o] \xleftrightarrow{DTFS;\Omega_o} e^{-jk\Omega_o n_o}X[k]$$

Scanned with CamScanner

Frequency-shift properties of Fourier representation:

$$e^{j\gamma t}x(t) \xleftrightarrow{FT} X(j(\omega - \gamma))$$

$$e^{jk_o\omega_o t}x(t) \xleftrightarrow{FS;\omega_o} X[k - k_o]$$

$$e^{j\Gamma n}x[n] \xleftrightarrow{DTFT} X(e^{j(\Omega-\Gamma)})$$

$$e^{jk_o\Omega_o n}x[n] \xleftrightarrow{DTFS;\Omega_o} X[k - k_o]$$

Multiplication property:

For nonperiodic signal:

$$y(t) = x(t)z(t) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{2\pi}X(j\omega) * Z(j\omega).$$

$$y[n] = x[n]z[n] \xleftrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{2\pi}X(e^{j\Omega}) \circledast Z(e^{j\Omega})$$

For periodic signal:

$$y(t) = x(t)z(t) \xleftrightarrow{FT;2\pi/T} Y[k] = X[k] * Z[k].$$

$$y[n] = x[n]z[n] \xleftrightarrow{DTFS;2\pi/N} Y[k] = X[k] \circledast Z[k].$$

Scaling properties:

Let us consider the effect of scaling the time variable on the frequency-domain representation of a signal. Beginning with the FT, let $z(t) = x(at)$, where a is a constant. By definition, we have

$$Z(j\omega) = \int_{-\infty}^{\infty} z(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(at)e^{-j\omega t}\,dt.$$

We effect the change of variable $\tau = at$ to get

$$Z(j\omega) = \begin{cases} (1/a)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}\,d\tau, & a > 0 \\ (1/a)\int_{\infty}^{-\infty} x(\tau)e^{-j(\omega/a)\tau}\,d\tau, & a < 0 \end{cases}.$$

These can be combined into a single integral

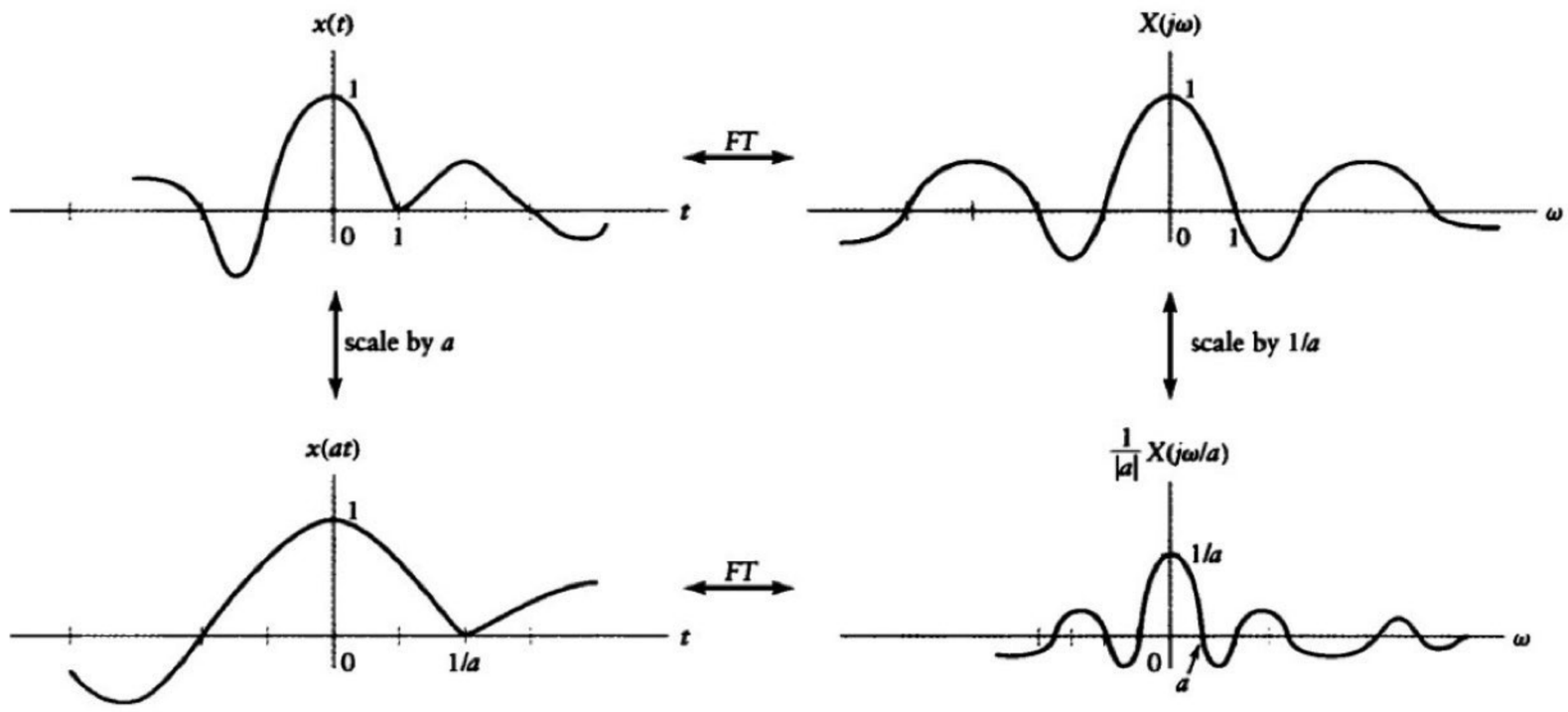
$$Z(j\omega) = (1/|a|)\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}\,d\tau.$$

We can conclude that

$$z(t) = x(at) \xleftrightarrow{FT} (1/|a|)X(j\omega/a)$$

Scaling the signal in time introduces the inverse scaling in the frequency-domain representation and an amplitude change, as shown in the given figure:

Scanned with CamScanner



Parseval Relationships:

It states that the energy or power in the time-domain representation of a signal is equal to the energy or power in the frequency-domain representation. So the energy and power are conserved in the Fourier representation.
The energy in a continuous-time non-periodic signal is

$$Wx = \int_{-\infty}^{\infty} |x\,(t)|^2 dt$$

Where it is assumed that $x(t)$ may be complex valued general. As $|x(t)|^2 = x(t)\ x^*(t)$, taking the conjugate of both sides of Eq, we may express $x^*(t)$ in terms of its FT X(jω) as

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}\,d\omega.$$

Substituting this formula into the expression for W_x, we obtain

$$W_x = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)e^{-j\omega t}\,d\omega\right]dt.$$

Now we interchange the order of integration:

$$W_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\left\{\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right\}d\omega.$$

Observing that the integral inside the braces is the FT of $x(t)$, we obtain

$$W_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)X(j\omega)\,d\omega$$

And so conclude that

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\,d\omega.$$

Hence, the energy in the time-domain representation of the signal is equal to the energy in the frequency-domain representation, normalized by 2π. The quantity $|X(j\omega)|^2$ plotted against ω is known as *energy spectrum* of the signal.
Analogous results hold for the periodic signal is known as *power density spectrum* of the signal.

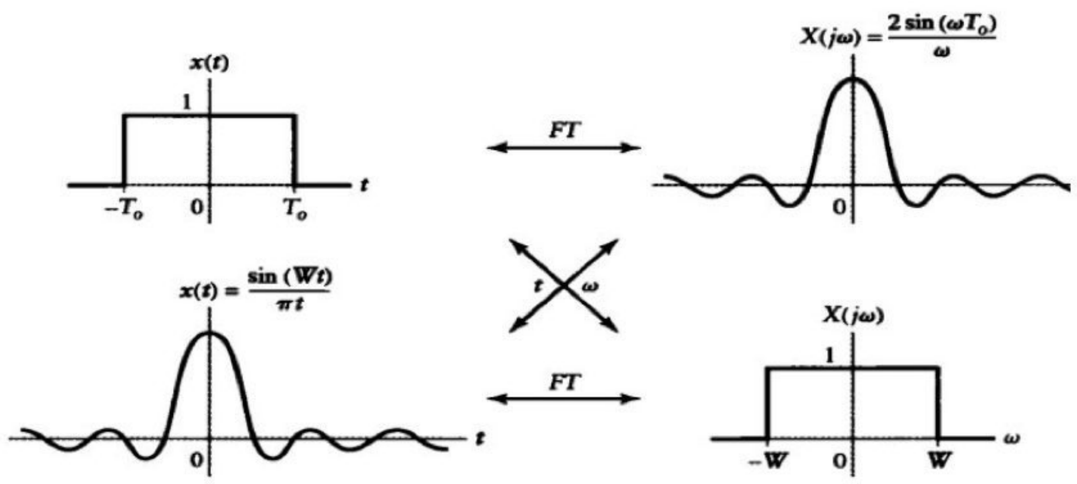
Scanned with CamScanner

The other three representations are summarized in the table below:

Representation	*Parseval Relation*
FT	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\, d\omega$
FS	$\frac{1}{T}\int_{0}^{T} \lvert x(t)\rvert^2\, dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$
DTFT	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\Omega})\rvert^2\, d\Omega$
DTFS	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert X[k]\rvert^2$

Duality:

In this chapter, we observed a consistent symmetry between the time and frequency domain representations of signals. For example, a rectangular pulse in either time or frequency corresponds to a sinc function in either frequency or time, as shown in the figure below.



We have also observed symmetries in Fourier representation properties i.e.; convolution in one domain corresponds to modulation in other domain; differentiation in one domain corresponds to multiplication by the independent variable in the other domain, and so on. So by the consequence of this symmetry, we may interchange time and frequency. This interchangeability property is termed *duality*.

Duality properties of Fourier representation is summarized in the table below:

FT	$f(t) \xleftrightarrow{FT} F(j\omega)$	$F(jt) \xleftrightarrow{FT} 2\pi f(-\omega)$
DTFS	$x[n] \xleftrightarrow{DTFS;\,2\pi/N} X[k]$	$X[n] \xleftrightarrow{DTFS;\,2\pi/N} (1/N)x[-k]$
FS–DTFT	$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$	$X(e^{jt}) \xleftrightarrow{FS;\,1} x[-k]$

Scanned with CamScanner

Applications of Fourier representations to mixed signal classes

- We now discuss the application of Fourier representations to mixed signals like
  - Periodic & nonperiodic signal
  - Continuous & discrete time signal
- Such mixing of signals occurs most commonly when one uses Fourier methods to
  - Analyze the interaction between signals & systems
  - Numerically evaluate properties of signal or the behavior of a system
- For example: If we apply a periodic signal to a stable LTI system, the convolution representation of the system output involves a mixing of nonperiodic (impulse response) and periodic (input) signals.
- Another example: A system that samples continuous time-signal involves both continuous & discrete-time signals.
- In order to use Fourier methods to analyze such interactions, we must build bridges between the Fourier representations of different classes of signals.
- DTFS is the only Fourier representation that can be evaluated numerically on a computer.

Fourier transform representation of periodic signals:

Neither FT nor DTFT converges for periodic signals. However, by incorporating impulses into the FT & DTFT in the appropriate manner, we may develop FT & DTFT representations of such signals.

Relating FT to FS:

The FS representation of a periodic signal x(t) is

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jkt\,\omega_0}$$

Where, ω_0 is the fundamental frequency of the signal.
As, $1 \xleftrightarrow{FT} 2\pi\delta(\omega)$, using frequency shift property, the inverse FT of a frequency shifted impulse $\delta\,(\omega\text{-}k\omega_0)$ is a complex sinusoid with frequency $k\omega_0$. i.e,;

$$e^{jk\omega_o t} \xleftrightarrow{FT} 2\pi\delta(\omega - k\omega_o)$$

If we find FT of $x(t)$, then by using linearity property of FT we obtain

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t} \xleftrightarrow{FT} X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\omega_o)$$

Thus the FT of a periodic signal is a series of impulses spaced by the fundamental frequency ω_0. The kth impulse has strength $2\pi X(k)$, where $X(k)$ is the kth FS coefficient.
The shape of $X(j\omega)$ is identical to that of $X(k)$. The FT obtained from the FS by placing impulses at integer multiples of ω_0 and weighing them by 2π times the corresponding FS coefficient.
Given an FT consisting of impulses that are uniformly spaced in ω, we obtain the corresponding FS coefficients by dividing the impulse strengths by 2π.

Scanned with CamScanner

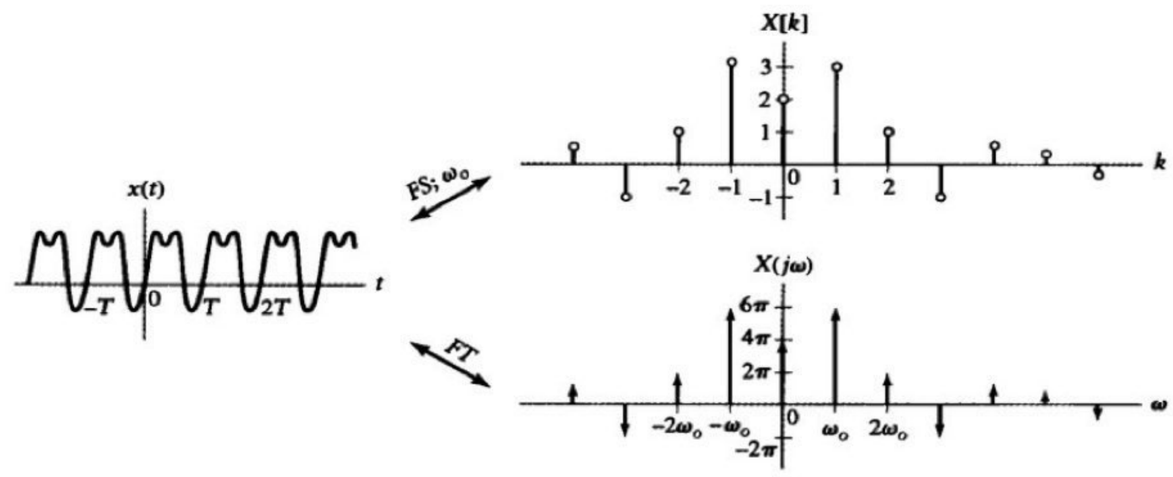


Fig: FS and FT representation of periodic continuous-time signals

Relating DTFT to DTFS:

The DTFS expression for a N periodic signal x[n] is

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n}.$$

The inverse DTFT of a frequency shifted impulse is a discrete-time complex sinusoid. The DTFT is a 2π periodic function of frequency. So we may express a frequency shifted impulse either by expressing one period such as

$$e^{jk\Omega_o n} \xleftrightarrow{DTFT} \delta(\Omega - k\Omega_o), \quad -\pi < \Omega \le \pi, \quad -\pi < k\Omega_o \le \pi.$$

Or, by using an infinite series of shifted impulses, separated by an interval of 2π, to obtain the 2π periodic function

$$e^{jk\Omega_o n} \xleftrightarrow{DTFT} \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_o - m2\pi)$$

The inverse DTFT equation is evaluated by means of the shifting property of the impulse function. We have

$$\frac{1}{2\pi} e^{jk\Omega_o n} \xleftrightarrow{DTFT} \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_o - m2\pi).$$

Hence we identify the complex sinusoid and the frequency shifted impulse as a DTFT pair using linearity property and substituting the above equation in equation of x[n], we get DTFT of periodic signal x[n]

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n} \xleftrightarrow{DTFT} X(e^{j\Omega}) = 2\pi \sum_{k=0}^{N-1} X[k] \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_o - m2\pi).$$

Since DTFT is 2π periodic, it follows that, DTFT of x[n] consists of a set of N impulses of strength $2\pi X(k), k=0,1,2\ldots\ldots,N-1$.

$$x[n] = \sum_{k=0}^{N-1} X[k]e^{jk\Omega_o n} \xleftrightarrow{DTFT} X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_o)$$

Given the DTFS coefficient and the fundamental frequency Ω_0, we obtain the DTFT representation by placing impulses at integer multiples of Ω_0 and weighting them by 2π times the corresponding DTFS coefficient.

Scanned with CamScanner

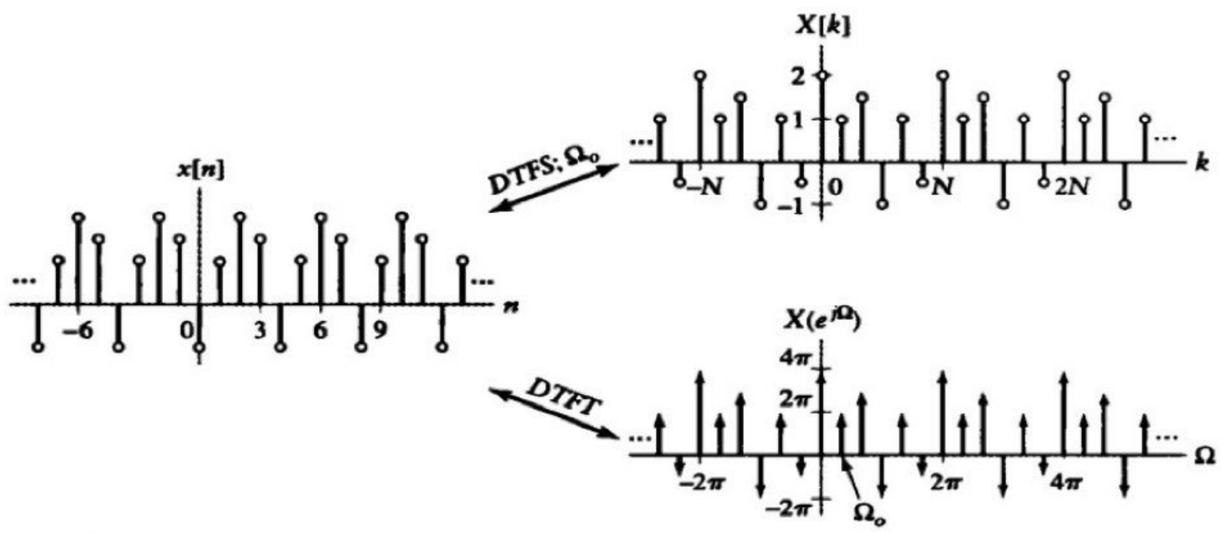


Fig: DTFS & DTFT representation of a periodic discrete-time signal

Fourier transform representation of Discrete-time signals:

In this section we derive, an FT representation of discrete-time signal by incorporating impulses into the description of the signal the appropriate manner, establishing a correspondence between the continuous frequency ω and the discrete-time frequency Ω.
Let us define the complex sinusoids

$$x(t) = e^{j\omega t} \text{ and } g[n] = e^{j\Omega n}$$

Suppose a force $g[n]$ equal to the samples of $x(t)$ taken at intervals of T_s i.e.;

$$g[n] = x(nT_s)$$

$$=> e^{j\Omega n} = e^{j\omega T_s n}$$

From which we conclude that $\Omega = \omega T_s$.

Relating FT to DTFT:

DTFT of an arbitrary discrete-time signal $x[n]$ is

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}.$$

We want to find out an FT pair $x_\delta(t) \xleftrightarrow{FT} X_\delta(j\omega)$ that corresponds to the DTFT pair $x[n] \xleftrightarrow{DTFT} X(e^{j\Omega})$. Substituting $\Omega = \omega T_s$, we obtain the following function of continuous time frequency ω.

$$X_\delta(j\omega) = X(e^{j\Omega})|_{\Omega=\omega T_s},$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n}.$$

Taking the inverse FT of $X_\delta(j\omega)$, using linearity and the FT pair

$$\delta(t - nT_s) \xleftrightarrow{FT} e^{-j\omega T_s n};$$

yields the continuous time signal description

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

Hence,

Scanned with CamScanner

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s) \xleftrightarrow{FT} X_\delta(j\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n}$$

The discrete-signal has values $x[n]$, while the corresponding continuous time signal consists of a series of impulses separated by T_s, with n^{th} impulse having strength $x[n]$. The DTFT $X(e^{j\Omega})$ is 2π periodic in Ω, while the FT $X_\delta(j\omega)$ is $2\pi / T_s$ periodic in ω.

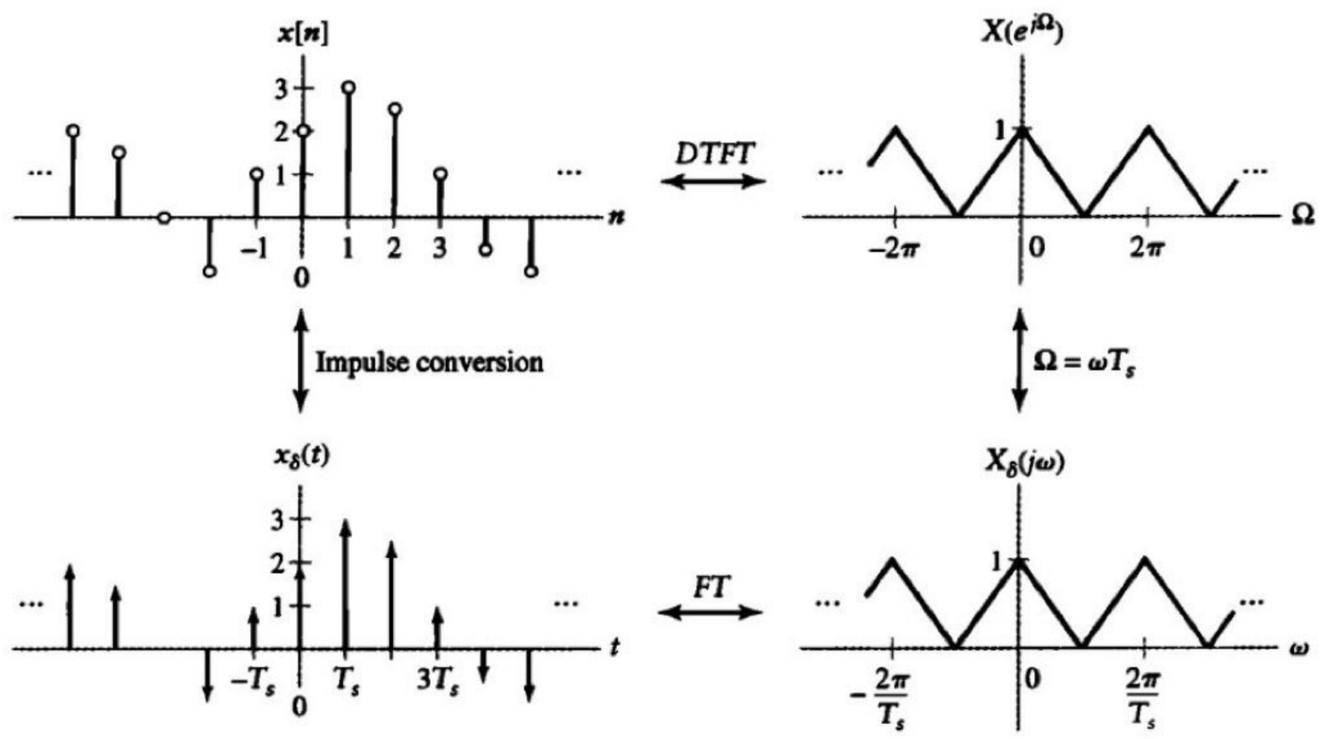


Fig: Relationship between FT and DTFT representation of a discrete-time signal

Relating FT to DTFS:

The DTFT representation of an N periodic signal $x[n]$ is given as

$$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\Omega - k\Omega_o)$$

Where $X[k]$ is DTFS coefficient.
Substituting $\Omega = \omega T_s$ into this eqn. yields the FT representation

$$\begin{aligned} X_\delta(j\omega) &= X(e^{j\omega T_s}) \\ &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(\omega T_s - k\Omega_o) \\ &= 2\pi \sum_{k=-\infty}^{\infty} X[k]\delta(T_s(\omega - k\Omega_o/T_s)) \end{aligned}$$

Using scaling property of impulse

$$\delta(a\nu) = (1/a)\delta(\nu)$$

We can write

$$X_\delta(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} X[k]\delta(\omega - k\Omega_o/T_s)$$

DTFS coefficients $X[k]$ are N-periodic function, which implies that $X_\delta(j\omega)$ is periodic with period $N\Omega_0 / T_s = 2\pi / T_s$.
Continuous time representation of discrete-time signal as derived in previous chapter

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

As $x[n]$ is N periodic, so $x_\delta(t)$ is also periodic with fundamental period NT_s

Scanned with CamScanner

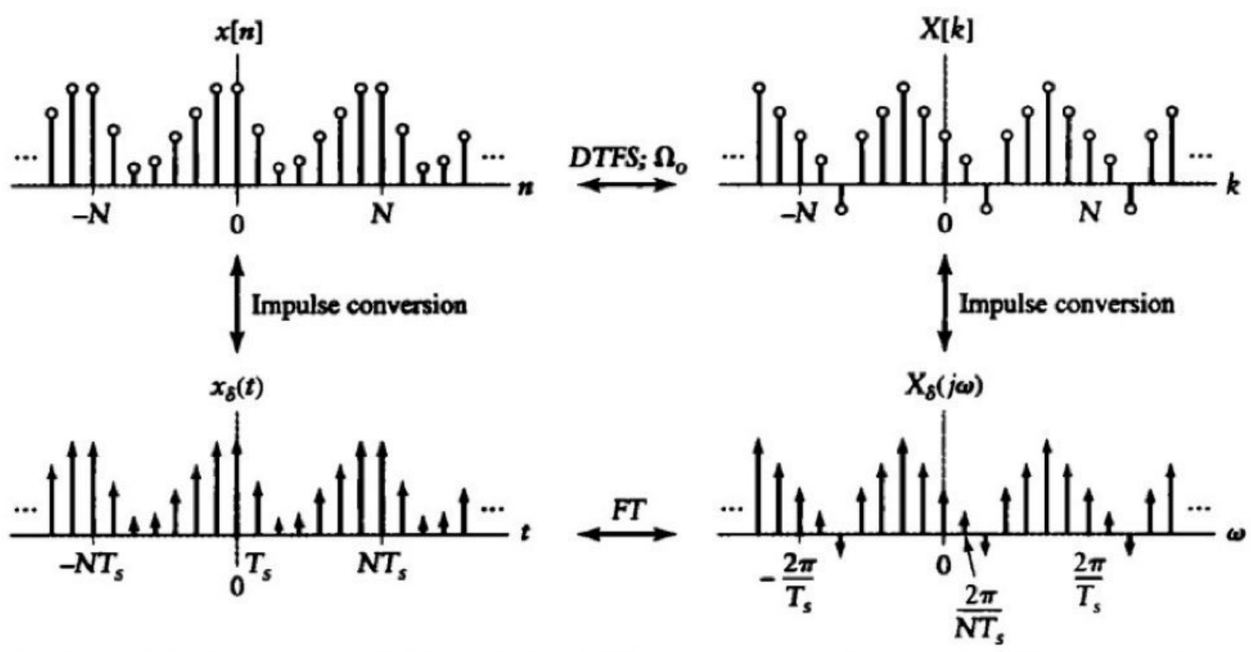


Fig: Relationship between FT and DTFS representation of a discrete-time signal

Sampling:

- The sampling operation generates a discrete-time signal from the continuous-time signal in order to manipulate the signal on a computer or microprocessor.
- Such manipulations are common in communication, control and signal processing systems.
- Sampling is also frequently performed on discrete-time signal to change the effective data rate, an operation termed *subsampling*.

Sampling continuous-time signals:

- Let $x(t)$ be a continuous-time signal. To define a discrete-time signal $x[n]$ which is equal to the *samples* of $x(t)$ at integer multiples of a sampling interval T_s, i.e.;

$$x[n] = x[nT_s]$$

- The impact of sampling is elevated by relating the DTFT of $x[n]$ to the FT of $x(t)$.
- The continuous-time representation of discrete-time signal $x[n]$ is given by

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

- Substituting $x[nT_s]$ for $x[n]$ in above eqn. we get

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Since

$$x(t)\delta(t - nT_s) = x(nT_s)\delta(t - nT_s)$$

So we may write

$$x_\delta(t) = x(t)p(t)$$

Where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- The above eqn. implies that we may mathematically represent the sample signal as the product of original continuous-time signal and impulse train.

Scanned with CamScanner

- This representation is commonly termed as *impulse sampling* and is a mathematical tool used only to analyze sampling.
- The effect of sampling is determined by relating FT of $x_\delta(t)$.to FT of $x(t)$.
- As multiplication in the time domain corresponds to the convolution in the frequency domain, so by multiplication property:

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

- As impulse train is continuous periodic function, so its FS is given as

$$P[k] = \frac{1}{T_s}\int_{-T_{s/2}}^{T_{s/2}} \delta(t) e^{-jk\,\omega_0 t}\, dt = \frac{1}{T_s}$$

By using FT representation of FS

$$P(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} P[k]\delta(\omega - k\omega_s)$$

We get FT of impulse train as

$$P(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

Where $\omega_s = 2\pi/T_s$ is the sampling frequency.

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

Now we convolve $X(j\omega)$ with each of the frequency shifted impulse to get

$$X_\delta(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

- The FT of the sampled signal is given by an infinite sum of shifted versions of the original signal's FT.
- The shifted versions are offset by integer multiples of ω_s.
- The shifted versions of $X(j\omega)$ may overlap with each other if ω_s is not large enough compared with the frequency extent or bandwidth of $X(j\omega)$.
- Let the frequency component of the signal x(t) is assumed to lie within the frequency band $-W < \omega < W$ for the purpose of illustration.

Scanned with CamScanner

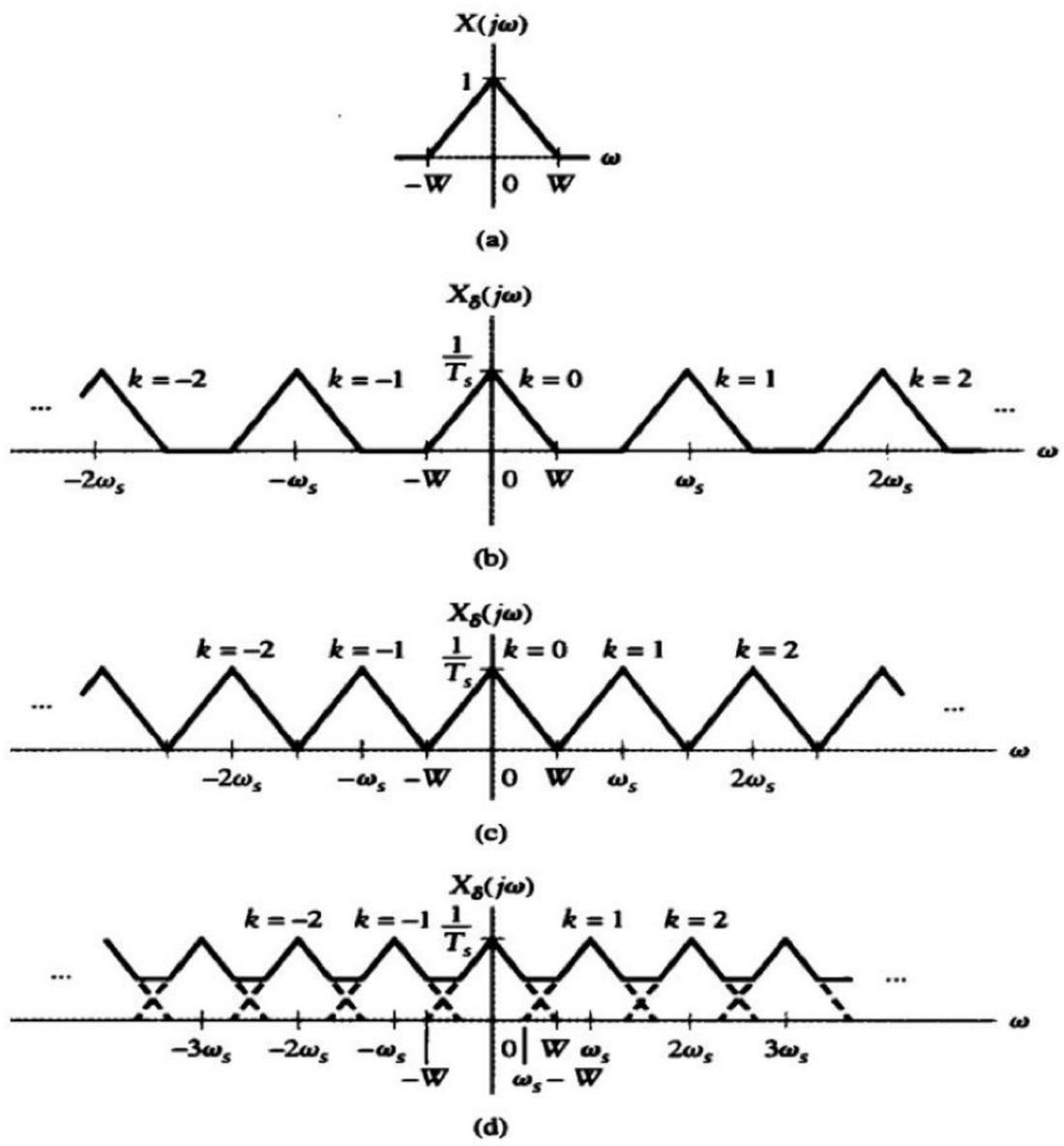


Fig: The FT of a sampled signal for different sampling frequencies (a) Spectrum of a continuous-time signal, (b) Spectrum of sampled signal when $\omega_s = 3W$, (c) Spectrum of sampled signal when $\omega_s = 2W$, (d) Spectrum of sampled signal when $\omega_s = 1.5W$

- Note that, as T_s increases and ω_s decreases, the shifted replicas of $X(j\omega)$ move close together, finally overlapping one another when $\omega_s < 2W$.
- Overlap in the shifted replicas of the original spectrum is termed *aliasing*.
- Aliasing distorts the spectrum of the sampled signal.
- The spectrum of the sampled signal no longer has a one to one correspondence with that of the original continuous-time signal.
- This means that we cannot use the spectrum of the sampled signal to analyze the continuous-time signal and we cannot uniquely reconstruct the original continuous-time signal from its samples.
- The DTFT of the sampled signal is obtained from $X_\delta(j\omega)$ by using the relationship $\Omega = \omega T_s$

$$x[n] \xleftrightarrow{DTFT} X(e^{j\Omega}) = X_\delta(j\omega)\big|_{\omega=\frac{\Omega}{T_s}}$$

- The scaling of the independent variable implies that $\omega = \omega_s$ corresponds to $\Omega = 2\pi$.
- The FTs have period ω_s, while DTFTs have period 2π.

Scanned with CamScanner

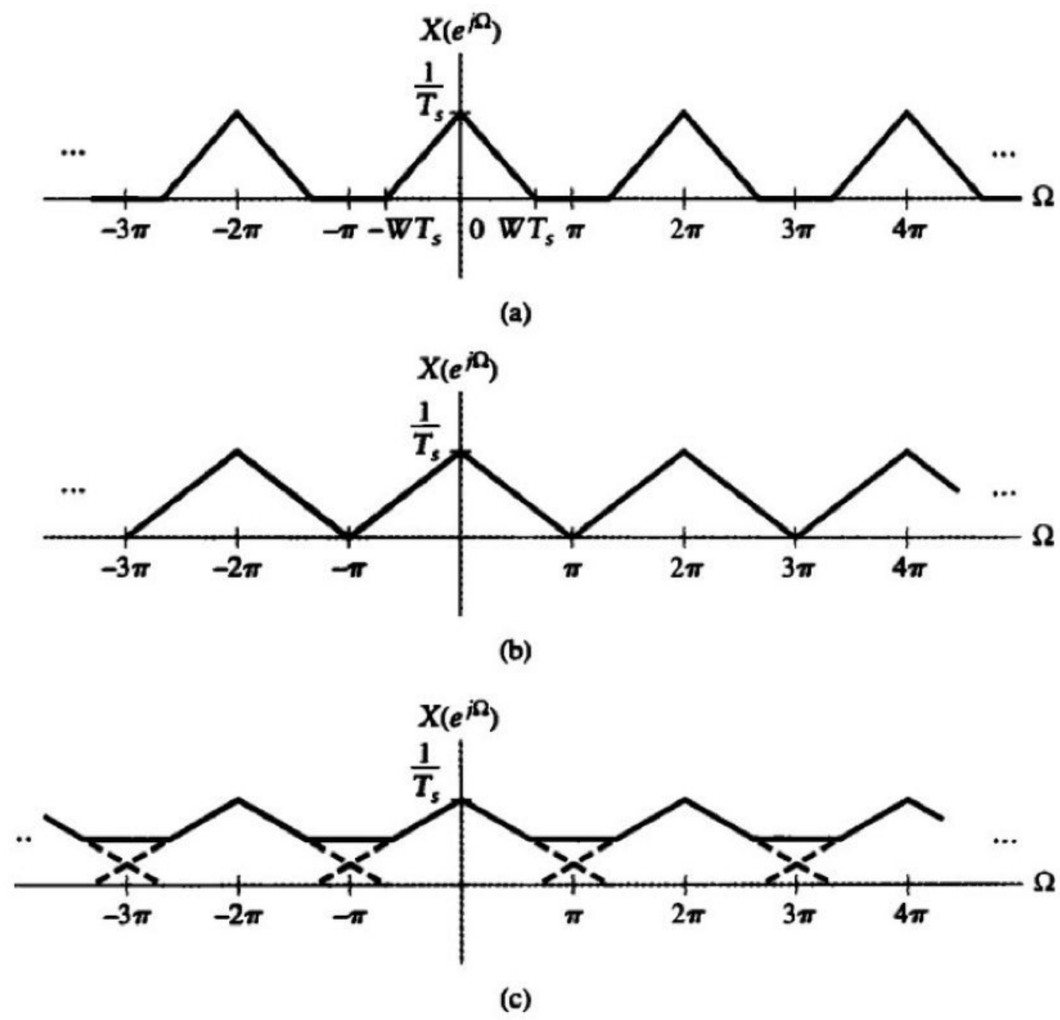
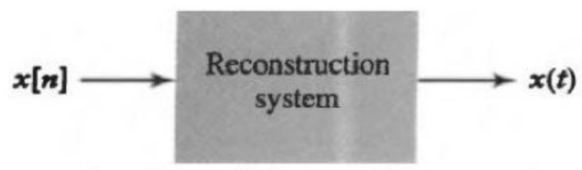


Fig: The DTFTs corresponding to the FTs depicted in Fig:1((b)-(d)). (a) $\omega_s = 3W$, (b) $\omega_s = 2W$, (c) $\omega_s = 1.5W$

Reconstruction of continuous-time signals from samples:

- The problem of reconstructing a continuous-time signal from samples involving a mixture of continuous & discrete-time signals
- A device that performs reconstruction has a discrete-time input signal and a continuous-time output signal.



- The FT is well suited for analyzing this problem, since it may be used to represent both continuous & discrete-time signals.
- We first consider the conditions that must be met in order to uniquely reconstruct a continuous-time signal from its samples.

<u>*Sampling Theorem:*</u>

- *The samples of a signal do not always uniquely determine the corresponding continuous-time signal.*
- For example, the figure below shows, two different continuous-time signals having the same set of samples $x[k] = x_1(nT_s) = x_2(nT_s)$

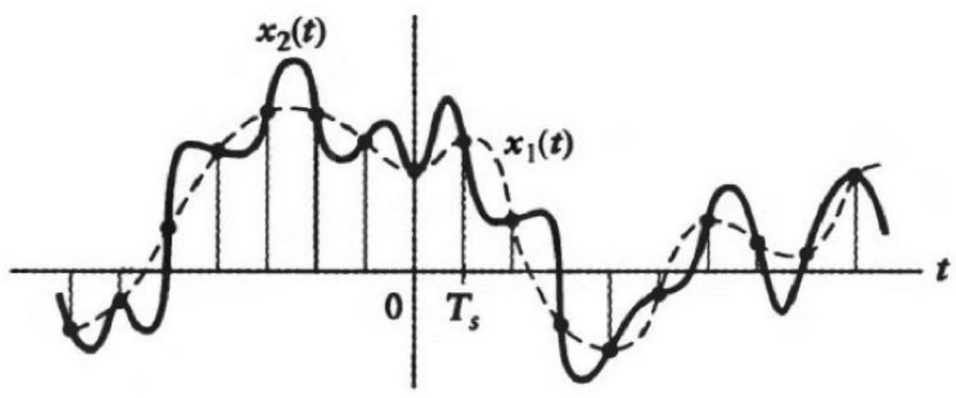


Fig: Two continuous-time signals $x_1(t)$ (dashed line) and $x_2(t)$ (solid line) that have the same set of samples

Scanned with CamScanner

- Note that the samples do not tell us anything about the behavior of the signal in between the times it is sampled.
- In order to determine how the signal behaves in between those times, we must specify additional constraints on the continuous-time signal.
- One such set of constraints, that is very useful in practice, involves requiring the signal to make smooth transitions from one sample to another.
- The smoothness, or rate at which the time domain signal changes, is directly related to the maximum frequency that is present in the signal.
- So, constraining smoothness in the time domain corresponds to limiting the bandwidth of the signal.
- Because there is one to one correspondence between the time domain and frequency domain representations of a signal, we may also consider the problem of reconstructing the continuous-time signal in the frequency domain.
- To reconstruct a continuous-time signal uniquely from its samples, there must be a unique correspondence between the FTs of the continuous-time signal and the sampled signal.
- The FTs are uniquely related if the sampling process does not introduce aliasing.
- Aliasing distorts the spectrum of the original signal and destroys one-to-one relationship between the FTs of the continuous-time signal and the sampled signal.
- Prevention of aliasing requires satisfying the sampling theorem.
- Let $x(t) \xleftrightarrow{FT} X(j\omega)$ represent a band-limited signal so that $X(j\omega) = 0$ for $|\omega| > \omega_m$

  If $\omega_s > 2\ \omega_m$, where $\omega_s = 2\pi\ /\ T_s$ is the sampling frequency, then x(t) is uniquely determined by its samples $x(nT_s)$, n=0,±1,±2,……
- The minimum sampling frequency, $2\ \omega_m$, is termed the *Nyquist sampling rate* or *Nyquist rate*. The actual sampling frequency, ω_s, is commonly referred to as the Nyquist frequency.
- If

$$f_m = \omega_m / 2\pi \text{ and } f_s > 2 f_m$$

$$\Rightarrow 1/T_s > 2 f_m$$

$$\Rightarrow T_s < 1/2 f_m$$

Scanned with CamScanner

Ideal Reconstruction:

- The sampling theorem indicates how fast we must sample a signal so that the samples uniquely represent the continuous-time signal.
- If $x(t) \xleftrightarrow{FT} X(j\omega)$, then the FT representation of the sampled signal is given by:

$$X_\delta(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s).$$

- The goal of reconstruction is to apply some operation to $X_\delta(j\omega)$ that converts it back to $X(j\omega)$.
- Any such operation must eliminate the replicas, images of $X(j\omega)$ that are centered at $k\omega_s$.
- This is accomplished by multiplying $X_\delta(j\omega)$ by

$$H_r(j\omega) = \begin{cases} T_s, & |\omega| \le \omega_s/2 \\ 0, & |\omega| > \omega_s/2 \end{cases}$$, as shown in the figure below

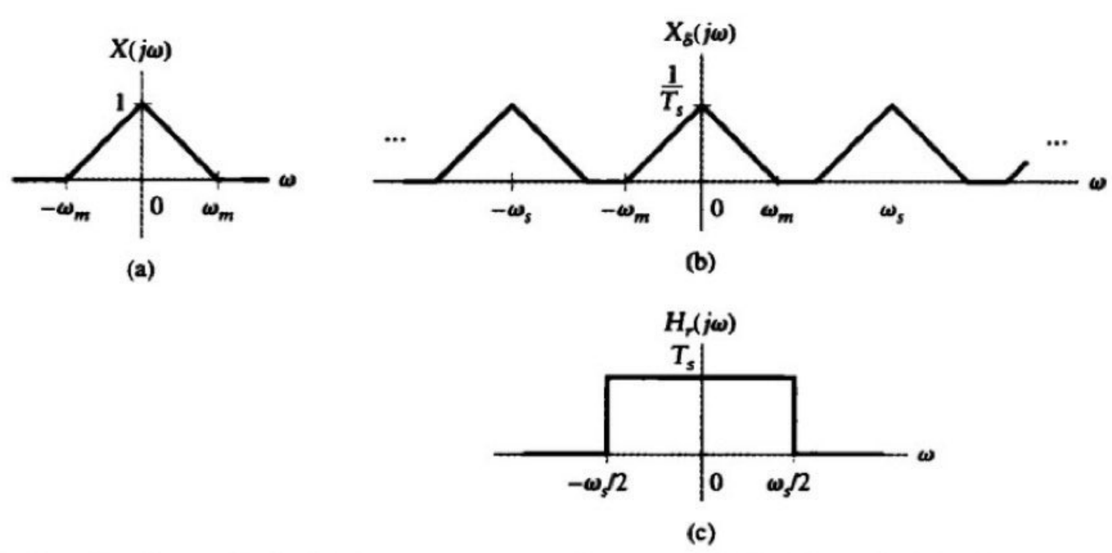


***Fig*: (a) Spectrum of original signal, (b) Spectrum of sampled signal (c) frequency response of reconstruction filter.**

- We then have $X(j\omega) = X_\delta(j\omega)\, H_r(j\omega)$
- Note that, multiplication by $H_r(j\omega)$ will not recover $X(j\omega)$ from $X_\delta(j\omega)$ if the conditions of the sampling theorem are not met and aliasing occur.
- Multiplication in the frequency domain transforms to convolution in the time domain.
- Hence, $x(t) = x_\delta(t) * h_r(t)$

  Where, $h_r(t) \xleftrightarrow{FT} H_r(j\omega)$.

  Substituting $x_\delta(t)$ in the above equation, we get

$$x(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT_s).$$

  Next we use

$$h_r(t) = \frac{T_s \sin\left(\frac{\omega_s}{2}t\right)}{\pi t}$$

$$= \mathrm{Ts}\left(\frac{\omega_s}{2\pi}\, sinc\left(\frac{\omega_s t}{2\pi}\right)\right)$$

$$= sinc\left(\frac{\omega_s t}{2\pi}\right)$$

Scanned with CamScanner

So,

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\mathrm{sinc}(\omega_s(t - nT_s)/(2\pi))$$

- In the time domain, we construct $x(t)$ as a weighted sum of sinc functions shifted by the sampling interval. The weights correspond to the values of the discrete-tine sequence.
- The value of the $x(t)$ at $t = nT_s$ is given by $x[n]$, because all of the shifted sinc functions are zero at nT_s, except the nth one and its value is unity.
- The operation described by the above equation is referred to as ideal *band limited interpolation*, since it indicates how to interpolate in between the samples of a band-limited signal.

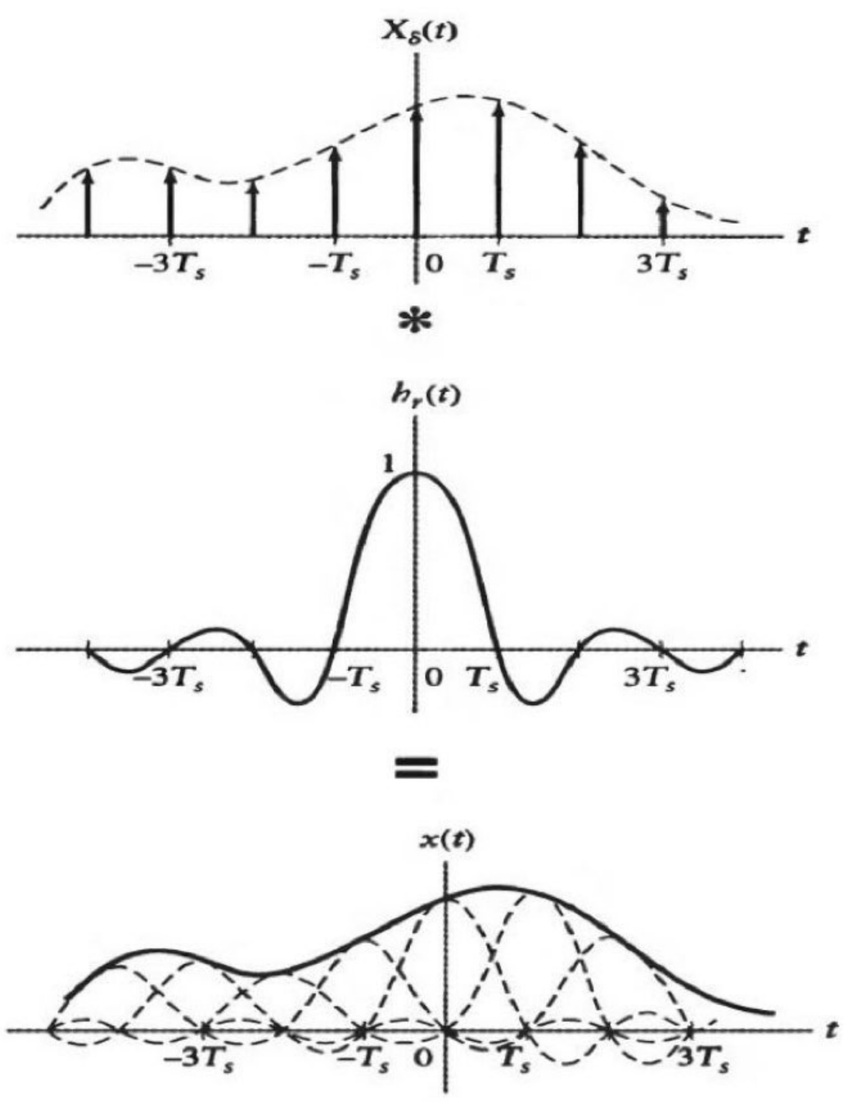


Fig: Ideal reconstruction in the time domain

Fourier series representation of finite duration on periodic signal:

- As DTFS is the only Fourier representation that can be evaluated numerically, so we apply DTFS and FS to signals that are not periodic to facilitate numerical computation of Fourier representations.
- Another advantage of this representation is understanding of relationship between the FT and corresponding FS representation.

Scanned with CamScanner

Relating the DTFS to DTFT:

- Let $x[n]$ be a finite duration aperiodic signal of length M i.e.;

$$x[n] = 0 \qquad \text{for } n < 0 \text{ and } n \geq M$$

- DTFT of this signal is

$$X(e^{j\Omega}) = \sum_{n=0}^{M-1} x[n]e^{-j\Omega n}.$$

- Let $\tilde{x}[n]$ be a periodic discrete-time signal with period $N \geq M$ such that one period of $\tilde{x}[n]$ is given by $x[n]$.
- The DTFS coefficients of $\tilde{x}\,[n]$ are given by

$$\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n} \qquad \text{(as } \tilde{x}\,[n] = x[n] \text{ within one period)}$$

Where $\Omega_0 = 2\pi / N$

$$\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{M-1} x[n]e^{-jk\Omega_o n} \qquad \text{(as } x[n] = 0 \text{ for } n \geq M\text{)}$$

A comparison of $\tilde{X}[K]$ and $X(e^{j\omega})$ reveals that

$$\tilde{X}[k] = \frac{1}{N}X(e^{j\Omega})\Big|_{\Omega=k\Omega_o}$$

- The DTFS coefficient of $\tilde{x}[n]$ are samples of the DTFT of $x[n]$, divided by N and evaluated by at intervals of 2π / N.
- Although $x[n]$ is not periodic, we define DTFS coefficients using $x[n]$, n = 0,1,….N-1 according to

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_o n}$$

So $X[K] = \tilde{X}[K] = (1/\text{N})\, X\,(e^{jk\omega_0})$ *(from above two equations)*

- DTFS coefficients of $x[n]$ correspond to the DTFS coefficients of periodically extended signal $\tilde{x}[n]$.
- The effect of sampling the DTFT of a finite-duration nonperiodic is to periodically extend the signal in the time domain. i.e.;

$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n + mN] \xleftrightarrow{DTFS;\Omega_o} \tilde{X}[k] = \frac{1}{N}X(e^{jk\Omega_o})$$

Scanned with CamScanner

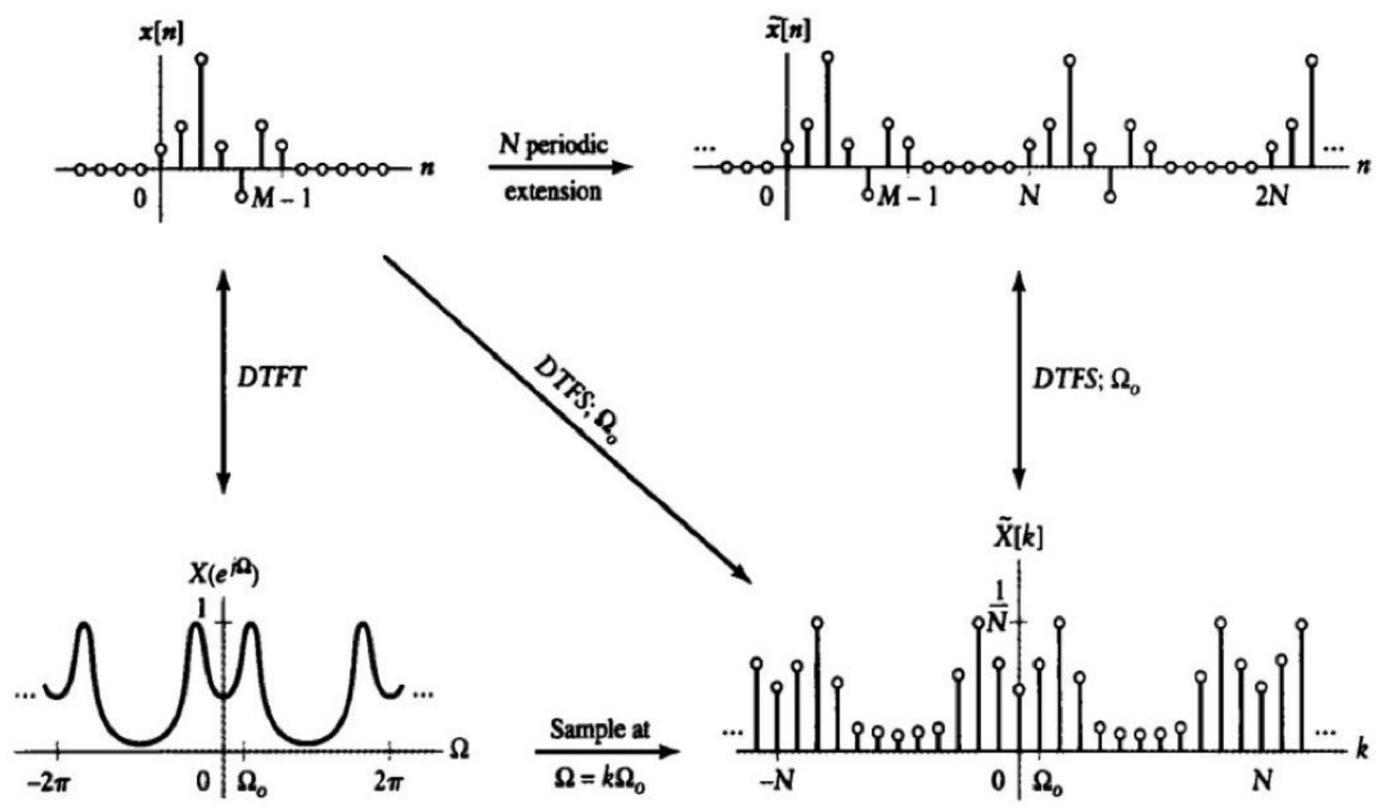


Fig: The DTFS of a finite duration nonperiodic signal

- The above relationship is dual to sampling frequency.
- Sampling a signal in time generates shifted replicas of the original signal in the frequency domain.

Dual:

- Sampling a signal in frequency generates shifted replicas of the original signal in the time domain.
- In order to prevent overlap or aliasing, of those shifted replicas in time, we require the frequency sampling interval Ω_0 to be less than or equal to $2\pi/ M$ ($\Omega_0 \leq 2\pi/ M \Rightarrow N \geq M$)

Relating the FS to the FT:

Let $x(t)$, an aperiodic signal have finite duration T_0,

i.e.; $x(t) = 0, \qquad t < 0 \ \& \ t \geq T_0$

Construct a periodic signal with period t

$$\tilde{x}(t) = \sum_{m=-\infty}^{\infty} x(t + mT)$$

With $T \geq T_0$ by periodically extending $x(t)$, the FS coefficients of $\tilde{x}(t)$ are

$$\tilde{X}[k] = \frac{1}{T}\int_0^T \tilde{x}(t)e^{-jk\omega_o t}\,dt$$

$$= \frac{1}{T}\int_0^{T_o} x(t)e^{-jk\omega_o t}\,dt$$

Scanned with CamScanner

Where we have used the relationship:

$$\tilde{x}(t) = x(t) \text{ for } 0 \le t \le T_0 \text{ and } \tilde{x}(t) = 0 \text{ for } T_0 < t < T$$

The FT of $x(t)$ is defined by

$$\mathbf{X(j\omega)} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{0}^{T_o} x(t)e^{-j\omega t}\,dt. \quad \text{(as } x(t) \text{ is finite duration)}$$

Hence, comparing $\tilde{X}(k)$ with $X(j\omega)$

$$\tilde{X}[k] = \frac{1}{T}X(j\omega)\Big|_{\omega=k\omega_o}$$

The FS coefficients are the samples the FT, normalized by T.

Scanned with CamScanner