University Question Paper Solution

Unit 1: Introduction

1. Determine whether the following systems are: i) Memoryless, ii) Stable iii) Causal iv) Linear and v) Time-invariant.
   i) $y(n)= nx(n)$
   ii) $y(t)= e^{x(t)}$ **[Dec 12, 10 marks]**

   Solution:-

(i) $y[n] = n x[n]$

A. output depends only on the present value of the input
∴ Memoryless

B. $|x[n]| \le M_x < \infty$ Bounded input
$|y[n]| = |n||x[n]|$ as $n \to \infty$ output is unbounded
∴ Unstable

C. system o/p does not depend on future values
∴ Causal

D. system satisfies both superposition & homogeneity conditions. $H\{a_1 x_1[n] + a_2 x_2(n)\} = a_1 y_1[n] + a_2 y_2[n]$.
∴ Linear

E. $y[n-n_0] = (n-n_0)\, x[n-n_0] \neq H\{x[n-n_0]\}$
∴ Time-variant

ii) $y(t) = e^{x(t)}$
A. Memoryless, B. stable, C. Causal
D. Non-linear E. Time-invariant.

2. Distinguish between: i) Deterministic and random signals and ii) Energy and periodic signals. **[Dec 12, 6 marks]**

   Solution;-

(i) Deterministic signals
- no uncertainty wrt its value at any time
- energy signals
- can be modelled

Random signals.
- uncertainty before it occurs.
- noise, power signals

(ii) Energy signals
- finite energy & zero power
- Non-periodic, deterministic signals

$E = \int_{-\infty}^{\infty} x^2(t)\, dt$

Power signals
- infinite energy & finite power
- Periodic, random signal (∞ in length)

$P = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} x^2[n]$

3. For any arbitrary signal x(t) which is an even signal, show that [**Dec 12, 4 marks**]

$$\int_{-\infty}^{\infty} x(t)dt = 2\int_{0}^{+\infty} x(t)dt .$$

Solution:-

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(-t) = x_e(t) \ \underline{\text{Even.}} \qquad \therefore x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

$$\text{and } x_o(-t) = -x_o(t) \qquad \text{and } x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

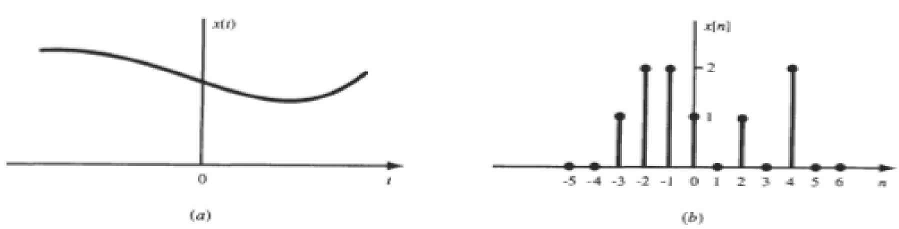
$$\int_{-\infty}^{+\infty} x(t)dt = \int_{-\infty}^{0} x(t)dt + \int_{0}^{+\infty} x(t)dt = \int_{0}^{+\infty} x(t)\,dt + \int_{0}^{+\infty} x(t)\,dt = 2\int_{0}^{+\infty} x(t)dt.$$

4. Distinguish between: [**Dec 09, 8 marks**]

i) Continuous time and discrete time signals
ii) Even and odd signals
iii) Periodic and non-periodic signals
iv) Energy and power signals.

i) Continuous-Time and Discrete-Time Signals

A signal x(t) is a continuous-time signal if t is a continuous variable. If t is a discrete variable, that is, x(t) is defined at discrete times, then x(t) is a discrete-time signal. Since a discrete-time signal is defined at discrete times, a discrete-time signal is often identified as a sequence of numbers, denoted by {x,) or x[n], where n = integer. Illustrations of a continuous-time signal x(t) and of a discrete-time signal x[n] are shown in Fig. 1-2.



1.2 Graphical representation of (a) continuous-time and (b) discrete-time signals

ii) Even and Odd Signals

A signal ***x (t)*** or ***x[n]*** is referred to as an ***even*** signal if

$$x(-t) = x(t)$$

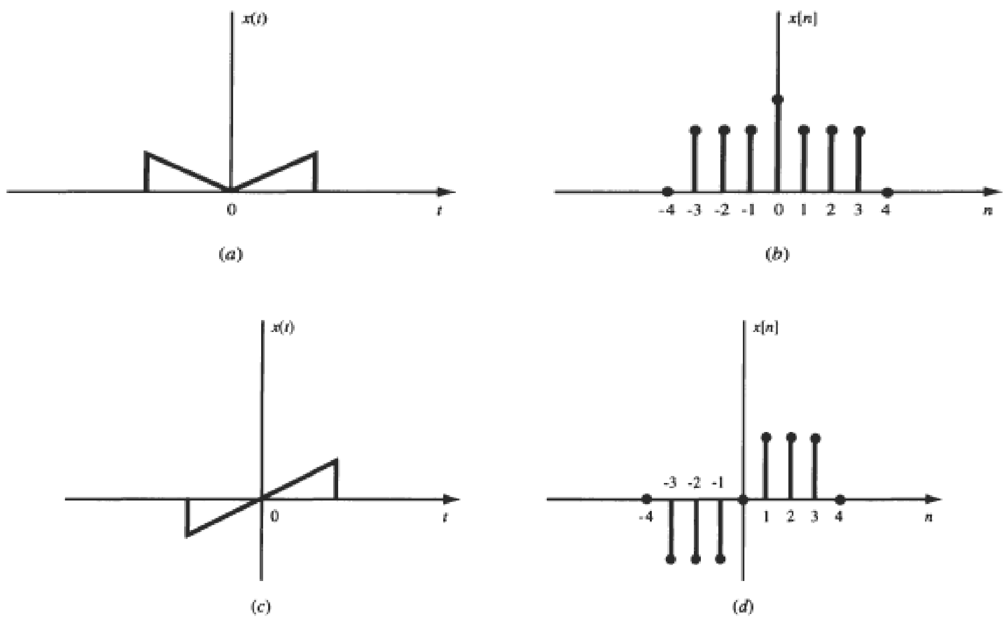
$$x[-n] = x[n] \quad \text{-------------(1.3)}$$

A signal ***x (t)*** or ***x[n]*** is referred to as an ***odd*** signal if

$$x(-t) = -x(t)$$

$$x[-n] = -x[n] \quad \text{--------------(1.4)}$$

Examples of even and odd signals are shown in Fig. 1.3.



1.3 Examples of even signals (a and b) and odd signals (c and d)**.**

Any signal x(t) or x[n] can be expressed as a sum of two signals, one of which is even and one of which is odd. That is,

$$x(t) = x_o(t) + x_e(t) \quad \text{-------(1.5)}$$

Where,

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right) \quad \text{-----(1.6)}$$

Similarly for x[n],

$$x[n] = x_o[n] + x_e[n] \quad \text{-------(1.7)}$$

Where,

$$x_e[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$$

$$x_o[n] = \frac{1}{2}\left(x[n] - x[-n]\right) \quad \text{--------(1.8)}$$

Note that the product of two even signals or of two odd signals is an even signal and that the product of an even signal and an odd signal is an odd signal.

ii) Periodic and Nonperiodic Signals

A continuous-time signal x (t) is said to be periodic with period T if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \qquad \text{all } t \qquad \text{............(1.9)}$$

An example of such a signal is given in Fig. 1-4(a). From Eq. (1.9) or Fig. 1-4(a) it follows that

$$x(t+mT) = x(t) \quad \text{---------------------------(1.10)}$$

for all t and any integer m. The fundamental period T, of x(t) is the smallest positive value of T for which Eq. (1.9) holds. Note that this definition does not work for a constant signal x(t) (known as a dc signal). For a constant signal x(t) the fundamental period is undefined since x(t) is periodic for any choice of T (and so there is no smallest positive value). Any continuous-time signal which is not periodic is called a nonperiodic (or aperiodic) signal.

Periodic discrete-time signals are defined analogously. A sequence (discrete-time signal) x[n] is periodic with period N if there is a positive integer N for which

$$x[n+N] = x[n] \qquad \text{all } n \qquad \text{..........(1.11)}$$

An example of such a sequence is given in Fig. 1-4(b). From Eq. (1.11) and Fig. 1-4(b) it follows that

$$x[n + mN] = x[n] \qquad \text{……………………..(1.12)}$$

for all n and any integer m. The fundamental period N_o of x[n] is the smallest positive integer N for which Eq.(1.11) holds. Any sequence which is not periodic is called a nonperiodic (or aperiodic sequence.

Note that a sequence obtained by uniform sampling of a periodic continuous-time signal may not be periodic. Note also that the sum of two continuous-time periodic signals may not be periodic but that the sum of two periodic sequences is always periodic.

iii) **Energy and Power Signals**

Consider v(t) to be the voltage across a resistor R producing a current i(t). The instantaneous power p(t) per ohm is defined as

$$p(t) = \frac{v(t)i(t)}{R} = i^2(t) \qquad \text{…………(1.13)}$$

Total energy E and average power P on a per-ohm basis are

$$E = \int_{-\infty}^{\infty} i^2(t)\,dt \quad \text{joules}$$

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)\,dt \quad \text{watts} \qquad \text{……(1.14)}$$

For an arbitrary continuous-time signal x(t), the normalized energy content E of x(t) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt \qquad \text{…………………(1.15)}$$

The normalized average power P of x(t) is defined as

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\,dt \qquad (1.16)$$

Similarly, for a discrete-time signal x[n], the normalized energy content ***E*** of x[n] is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad (1.17)$$

The normalized average power P of x[n] is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \qquad (1.18)$$

5. Write the formal definition of a signal and a system. With neat sketches for illustration, briefly describe the five commonly used methods of classifying signals based on different features. [**July10, 12 marks**]

Solution:-

Signal definition

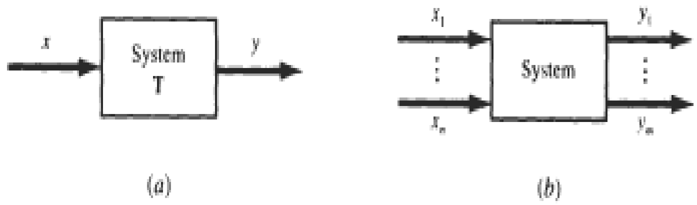
A **signal** is a function representing a physical quantity or variable, and typically it contains information about the behaviour or nature of the phenomenon. For instance, in a RC circuit the signal may represent the voltage across the capacitor or the current flowing in the resistor. Mathematically, a signal is represented as a function of an independent variable **'t'**. Usually **'t'** represents time. Thus, a signal is denoted by **x(t)**.

System definition

A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal.Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a transformation (or mapping) of **x** into y. This transformation is represented by the mathematical notation

y= Tx --(1.1)

where **T** is the operator representing some well-defined rule by which **x** is transformed into y.



1.1 System with single or multiple input and output signals

Classification of signals

Basically seven different classifications are there:

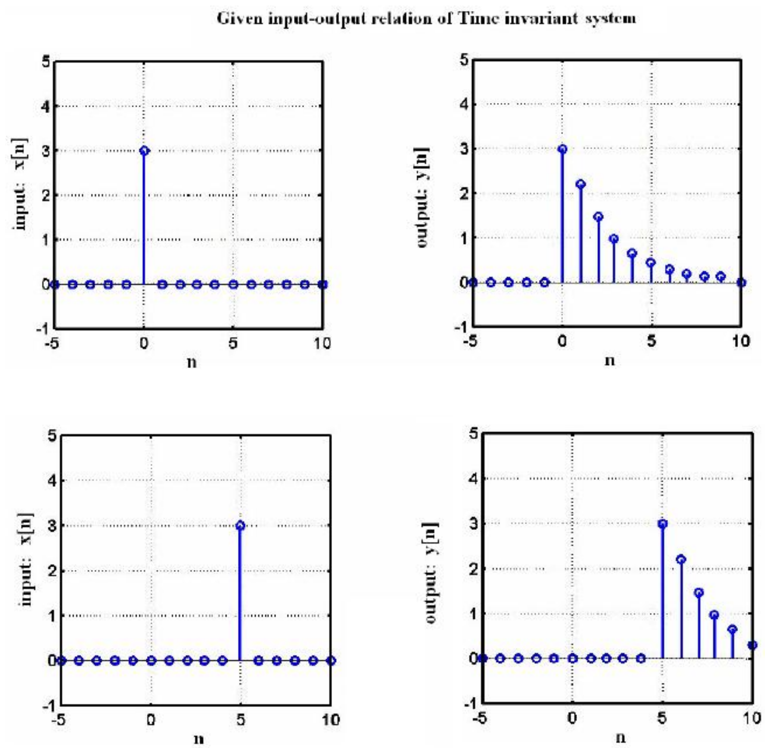
- Continuous-Time and Discrete-Time Signals
- Analog and Digital Signals
- Real and Complex Signals
- Deterministic and Random Signals
- Even and Odd Signals
- Periodic and Nonperiodic Signals
- Energy and Power Signals

6. Explain the following properties of systems with suitable example:
   i) Time invariance ii) Stability iii) Linearity. [**June/July09, 6 marks**]

   Solution:-

   Time invariance:

   A system is time invariant, if its output depends on the input applied, and not on the time of application of the input. Hence, time invariant systems, give delayed outputs for delayed inputs.



Stability

A system is stable if 'bounded input results in a bounded output'. This condition, denoted by BIBO, can be represented by:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad implies \quad \sum_{n=-\infty}^{\infty} |y[n]| < \infty \quad for \quad all \quad n \qquad ....(1.42)$$

Hence, a finite input should produce a finite output, if the system is stable. Some examples of stable and unstable systems are given in figure 1.21

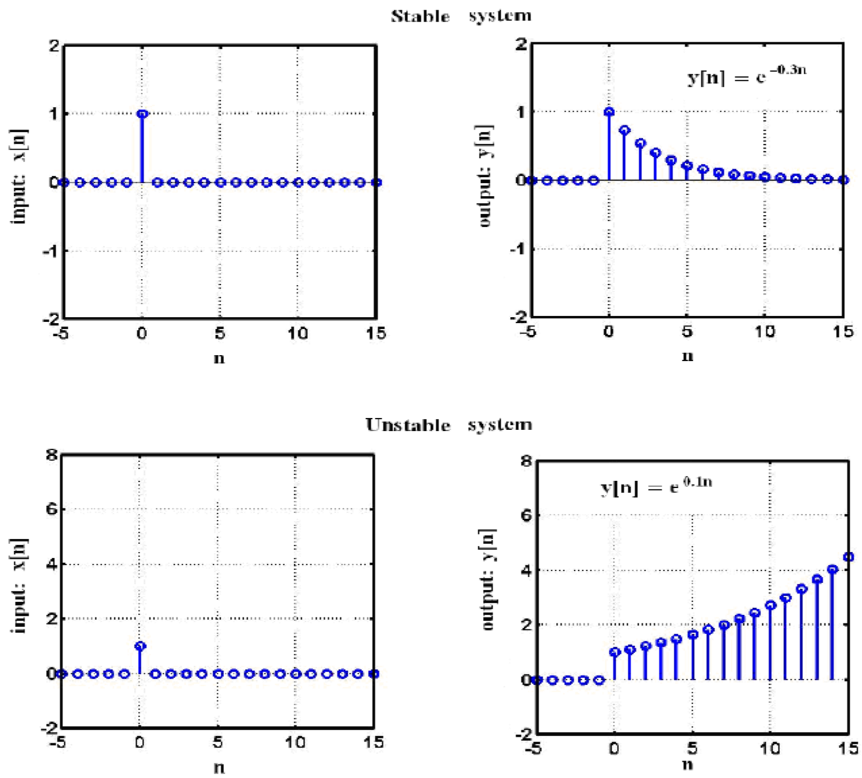
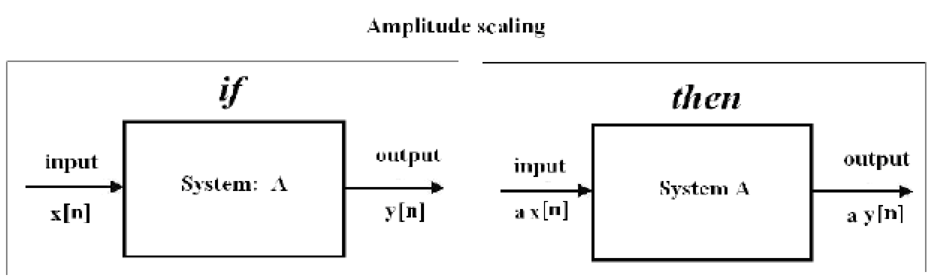


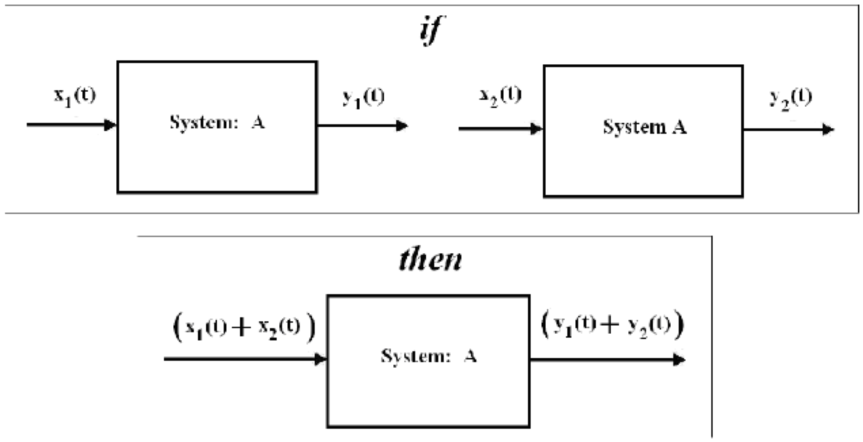
Fig 1.21 Examples for system stability

Linearity:

The system is a device which accepts a signal, transforms it to another desirable signal, and is available at its output. We give the signal to the system, because the output is s

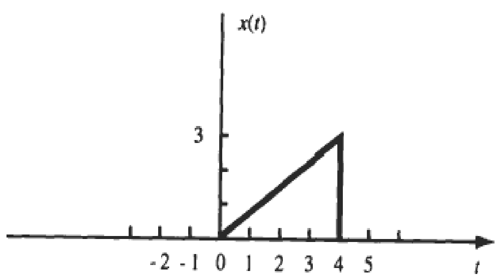


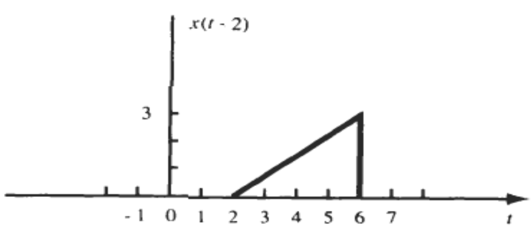
Superposition principle



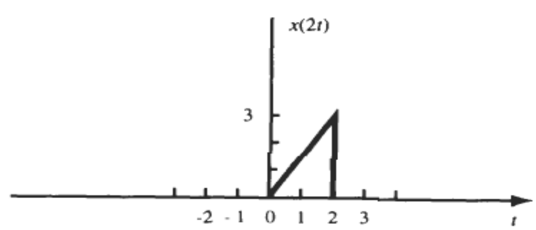
7. A continuous-time signal ***x (t)*** is shown in Fig. 1-17. Sketch and label each of the following signals.
   (i) x(t - 2); (ii) x(2t); (iii) x (t / 2) ; (iv)l x (- t) [**Jan 05 , 10 marks**]

Solution:



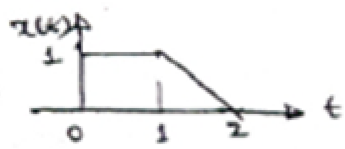


(a)

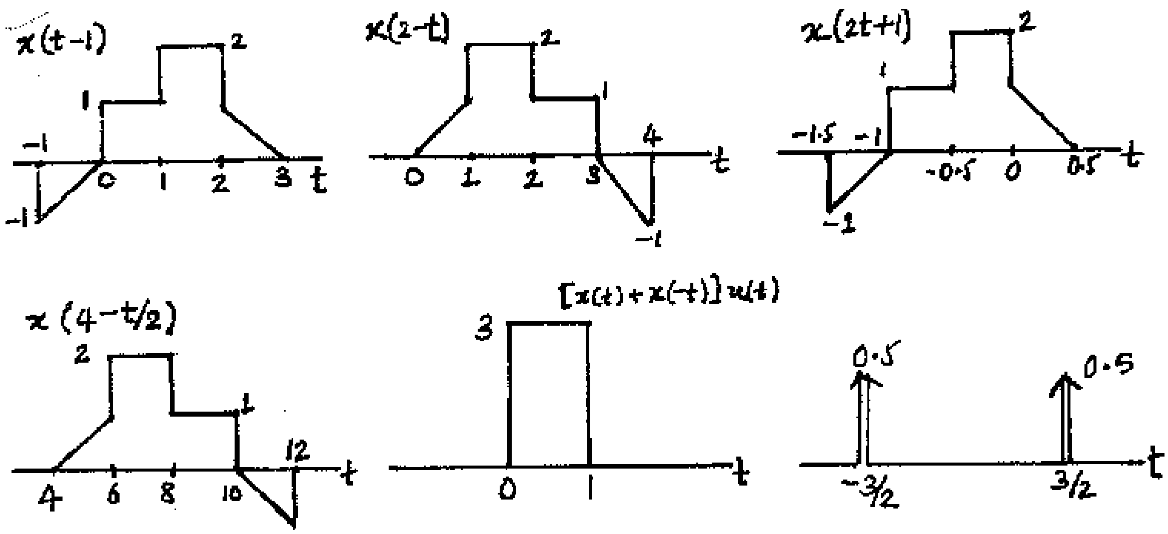


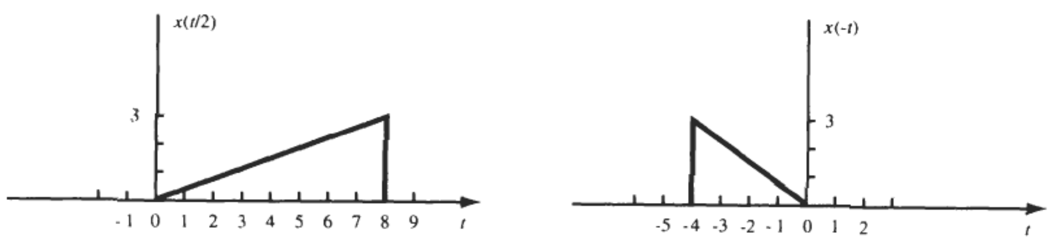
(b)

8. Given the signal *x(t)* as shown in fig 1.b, sketch the following: [**Jan/Feb 05, 4 marks**]
   i) X(2t+3)
   ii) X(t/2 -2)



Solution:-





(c) (d)

9. Find the even and odd components of x (t) = e^{jt}. [**Jan 05 , 10 marks**]

Solution:

Let $x_e(t)$ and $x_o(t)$ be the even and odd components of e^{jt}, respectively.

$$e^{jt} = x_e(t) + x_o(t)$$

From Eqs. (*1.5*) and (*1.6*) and using Euler's formula, we obtain

$$x_e(t) = \tfrac{1}{2}(e^{jt} + e^{-jt}) = \cos t$$

$$x_o(t) = \tfrac{1}{2}(e^{jt} - e^{-jt}) = j \sin t$$

10. Show that the product of two even signals or of two odd signals is an even signal and that the product of an even and an odd signaI is an odd signal.

Solution:

Let $x(t) = x_1(t)x_2(t)$. If $x_1(t)$ and $x_2(t)$ are both even, then

$$x(-t) = x_1(-t)x_2(-t) = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even. If $x_1(t)$ and $x_2(t)$ are both odd, then

$$x(-t) = x_1(-t)x_2(-t) = -x_1(t)[-x_2(t)] = x_1(t)x_2(t) = x(t)$$

and $x(t)$ is even. If $x_1(t)$ is even and $x_2(t)$ is odd, then

$$x(-t) = x_1(-t)x_2(-t) = x_1(t)[-x_2(t)] = -x_1(t)x_2(t) = -x(t)$$

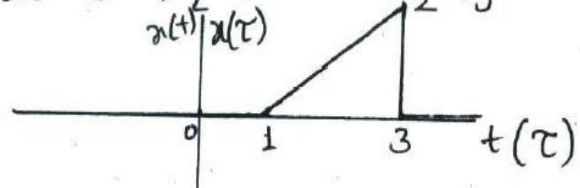
and x(t) is odd. Note that in the above proof, variable I represents either a continuous or a discrete variable.

2: Time-domain representations for LTI systems – 1

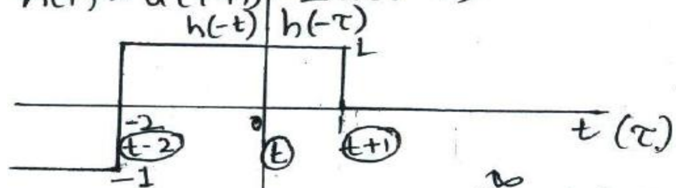
1. Find the convolution integral of x(t) and h(t), and sketch the convolved signal,
   x(t) = (t −1){u(t −1) +u(t −3)} and h(t)= [u(t +1) − 2u(t −2)]. [**Dec 12, 12marks**]

Solution:

$x(t) = (t-1)\{u(t-2) - u(t-3)\}$



$h(t) = u(t+1) - 2u(t-2)$



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

For $t < 0$, $w_t(\tau) = 0 \therefore y(t) = 0$

for $0 \le t \le 2$, $w_t(\tau) = (\tau - 1)\cdot 1$, $1 \le \tau \le t+1$

$$\therefore y(t) = \int_1^{t+1} (\tau - 1)\, d\tau = \frac{t^2}{2}$$

for $2 \le t \le 3$, $w_t(\tau) = (\tau - 1)\cdot 1$, $1 \le \tau \le 3$

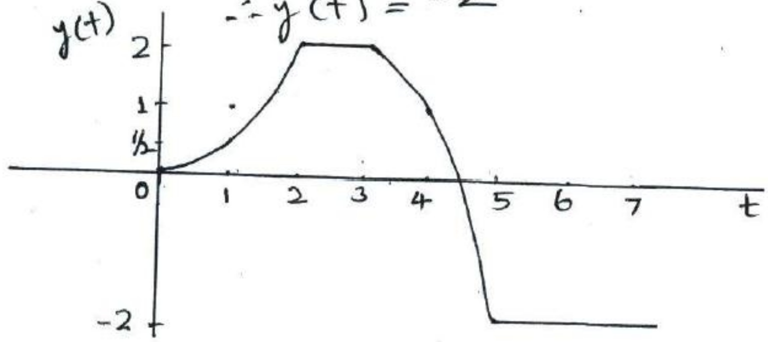
$$\therefore y(t) = 2$$

for $3 \le t < 5$, $w_t(\tau) = \begin{cases} (\tau-1)\cdot 1 & t-2 \le \tau \le 3 \\ (\tau-1)\cdot(-1) & 1 \le \tau \le t-2 \end{cases}$

$$\therefore y(t) = -t^2 + 6t - 7$$

for $t \ge 5$, $w_t(\tau) = (\tau - 1)(-1)$ $1 \le \tau \le 3$

$$\therefore y(t) = -2$$



2. Determine the discrete-time convolution sum of the given sequences.x(n) = {1, 2, 3, 4} and h(n)= {1, 5, 1} **[Dec 12, 8marks]**

Solution:-

$$x[n] = \{1, \underset{\uparrow}{2}, 3, 4\} \;\&\; h[n] = \{1, \underset{\uparrow}{5}, 1\}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$y[n] = \{1, 7, \underset{\uparrow}{14}, 21, 23, 4\}$$

3. Explain any four properties of continuous and / or discrete time systems. Illustrate with suitable examples. [**June 7, 8marks**]

Solution:-

Stability

A system is stable if 'bounded input results in a bounded output'. This condition, denoted by BIBO, can be represented by:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad implies \quad \sum_{n=-\infty}^{\infty} |y[n]| < \infty \quad for \quad all \quad n \qquad \text{.......(1.42)}$$

Hence, a finite input should produce a finite output, if the system is stable. Some examples of stable and unstable systems are given in figure 1.21

Memory

The system is memory-less if its instantaneous output depends only on the current input.

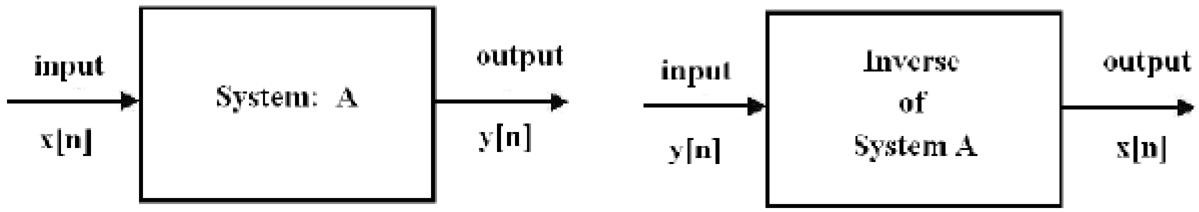
In memory-less systems, the output does not depend on the previous or the future input.

Examples of memory less systems:

$$y[n] = ax[n]$$
$$y[n] = ax^2[n]$$
$$i[n] = a_0 + a_1 v[n] + a_2 v^2[n] + a_3 v^3[n] + ...$$

Invertibility:

A system is invertible if,



A system is causal, if its output at any instant depends on the current and past values of input. The output of a causal system does not depend on the future values of input. This can be represented as:

$$y[n] = F\{x[m]\} \quad for\ m \le n$$

For a causal system, the output should occur only after the input is applied, hence,

$$x[n] = 0\ for\ n < 0 \text{ implies } \quad y[n] = 0\ for\ n < 0$$

All physical systems are causal (examples in figure 7.5). Non-causal systems do not exist. This classification of a system may seem redundant. But, it is not so. This is because, sometimes, it may be necessary to design systems for given specifications. When a system design problem is attempted, it becomes necessary to test the causality of the system, which if not satisfied, cannot be realized by any means. **Hypothetical examples** of non-causal systems are given in figure below.

4. What do you mean by impulse response of an LTI system? How can the above be interpreted? Starting from fundamentals, deduce the equation for the response of an LTI system, if the input sequence *x(n)* and the impulse response are given. [**June 7, 7marks**]

Solution:-
The impulse response of a continuous time system is defined as the output of the system when its input is an unit impulse, $\delta(t)$. Usually the impulse response is denoted by $h(t)$

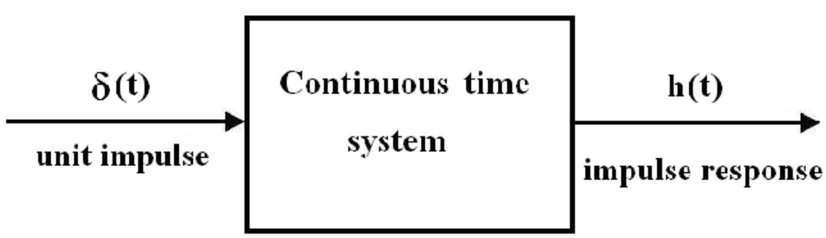
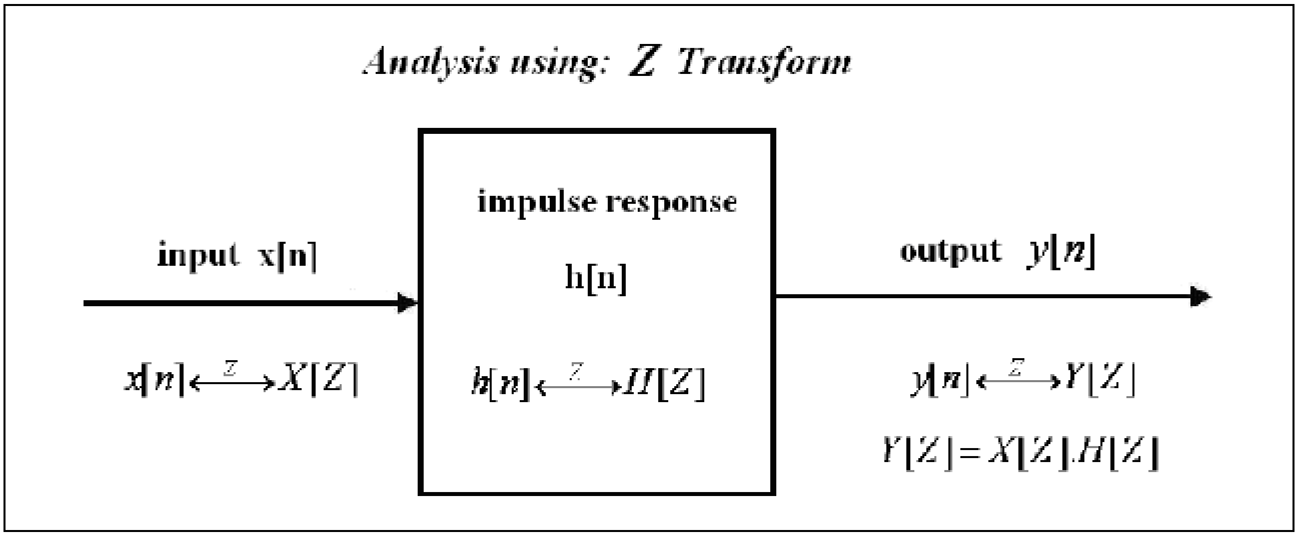


Figure 2: The impulse response of a continuous time system

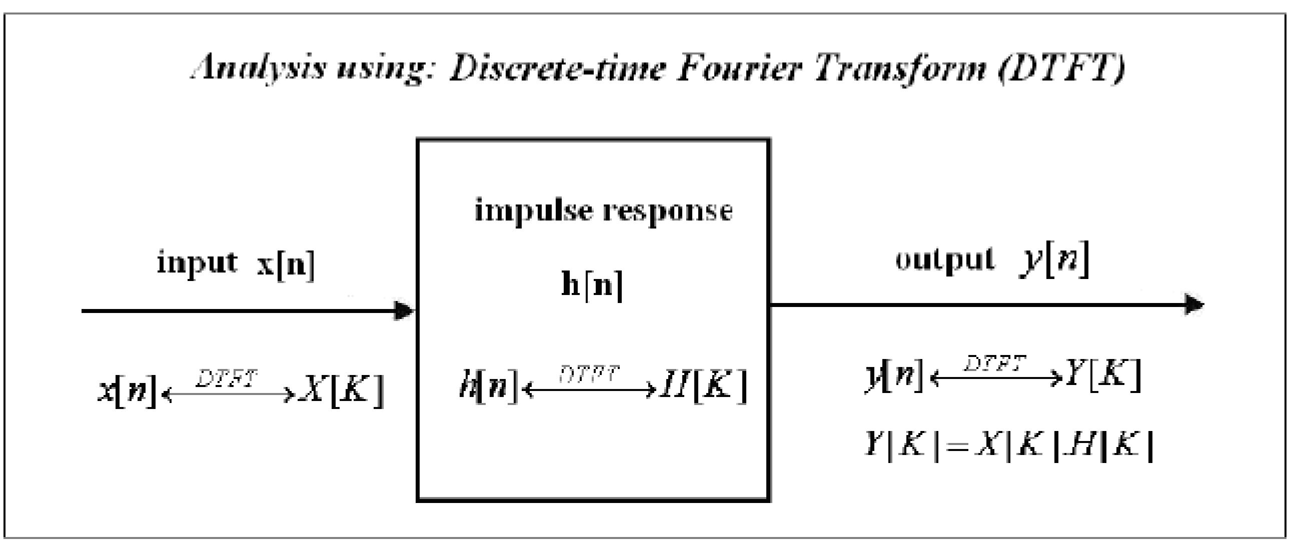
Evaluation from Z-transforms:

Another method of computing the convolution of two sequences is through use of Z-transforms. This method will be discussed later while doing Z-transforms. This approach converts convolution to multiplication in the transformed domain.



Evaluation from Discrete Time Fourier transform (DTFT):

It is possible to compute the convolution of two sequences by transforming them to the frequency domain through application of the Discrete Fourier Transform. This approach also converts the convolution operator to multiplication. Since efficient algorithms for DFT computation exist, this method is often used during software implementation of the convolution operator.



Evaluation from block diagram representation:

While small length, finite duration sequences can be convolved by any of the above three methods, when the sequences to be convolved are of infinite length, the convolution is easier performed by direct use of the 'convolution sum' .

5. The input ***x (t)*** and the impulse response ***h (t)*** of a continuous time LTI system are

given **by**

$$x(t) = u(t) \qquad h(t) = e^{-\alpha t}u(t),\ \alpha > 0$$

i)Compute the output y (t) by Eq
ii) Compute the output y (t)

Solution:-

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$y(t) = \int_0^t e^{-\alpha(t-\tau)}\,d\tau = e^{-\alpha t}\int_0^t e^{\alpha\tau}\,d\tau$$

$$= e^{-\alpha t}\frac{1}{\alpha}(e^{\alpha t}-1) = \frac{1}{\alpha}(1-e^{-\alpha t})$$

Thus, we can write the output $y(t)$ as

$$y(t) = \frac{1}{\alpha}(1-e^{-\alpha t})u(t)$$

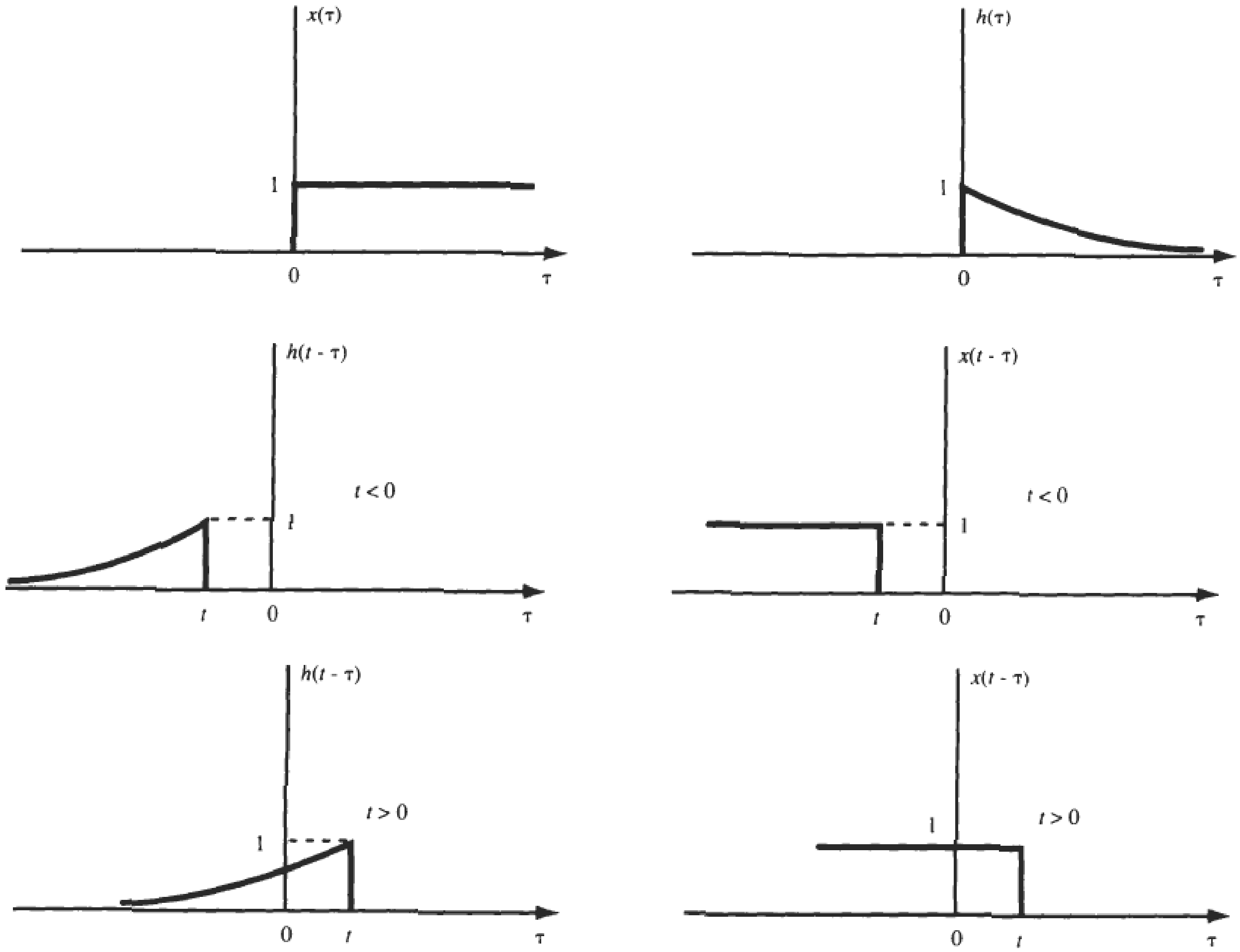
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

Functions $h(\tau)$ and $x(t-\tau)$ are shown in Fig. 2-4(b) for $t<0$ and $t>0$. Again from Fig. 2-4(b) we see that for $t<0$, $h(\tau)$ and $x(t-\tau)$ do not overlap, while for $t>0$, they overlap from $\tau=0$ to $\tau=t$. Hence, for $t<0$, $y(t)=0$. For $t>0$, we have

$$y(t) = \int_0^t e^{-\alpha\tau}\,d\tau = \frac{1}{\alpha}(1-e^{-\alpha t})$$

Thus, we can write the output $y(t)$ as

$$y(t) = \frac{1}{\alpha}(1-e^{-\alpha t})u(t)$$



6. Compute the output y(t for a continuous-time LTI system whose impulse response h (t) and the input x(t) are given by

$$h(t) = e^{-\alpha t}u(t) \qquad x(t) = e^{\alpha t}u(-t) \qquad \alpha > 0$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Solution:-

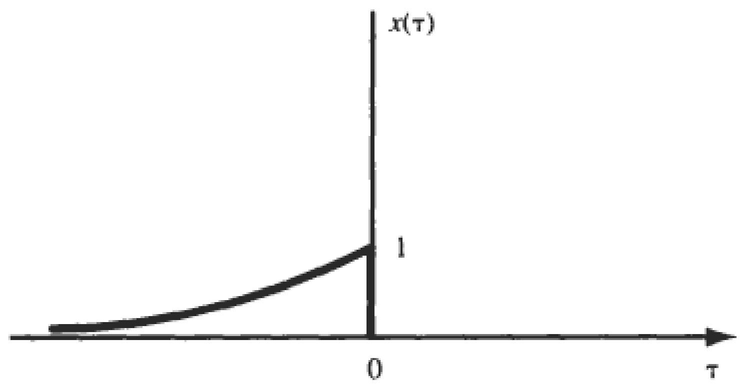
Functions $x(\tau)$ and $h(t-\tau)$ are shown in Fig. 2-5(a) for $t<0$ and $t>0$. From Fig. 2-5(a) we see that for $t<0$, $x(\tau)$ and $h(t-\tau)$ overlap from $\tau=-\infty$ to $\tau=t$, while for $t>0$, they overlap from $\tau=-\infty$ to $\tau=0$. Hence, for $t<0$, we have

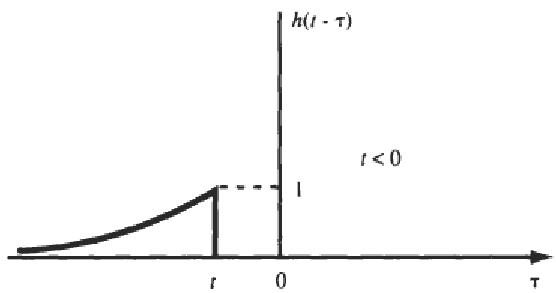
$$y(t) = \int_{-\infty}^{t} e^{\alpha\tau}e^{-\alpha(t-\tau)}\,d\tau = e^{-\alpha t}\int_{-\infty}^{t} e^{2\alpha\tau}\,d\tau = \frac{1}{2\alpha}e^{\alpha t} \qquad (2.66a)$$

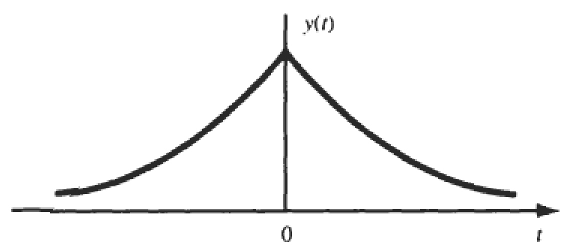
For $t > 0$, we have

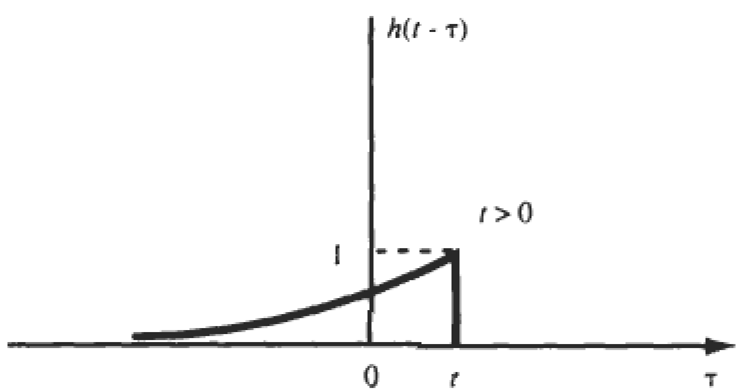
$$y(t) = \int_{-\infty}^{0} e^{\alpha\tau} e^{-\alpha(t-\tau)}\, d\tau = e^{-\alpha t}\int_{-\infty}^{0} e^{2\alpha\tau}\, d\tau = \frac{1}{2\alpha} e^{-\alpha t}$$

$$y(t) = \frac{1}{2\alpha} e^{-\alpha|t|} \qquad \alpha > 0$$









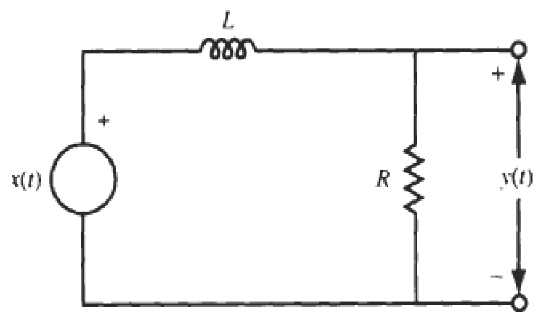
7. Consider the RL circuit shown in Fig. 2-33. Find the differential equation relating the output voltage y (t) across R and the input voltage x(t).
   i) Find the impulse response h (t) of the circuit.
   ii) Find the step response s(t) of the circuit.

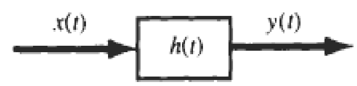
   Solution:-

   Ans. (a) $\dfrac{dy(t)}{dt} + \dfrac{R}{L}y(t) = \dfrac{R}{L}x(t)$

   (b) $h(t) = \dfrac{R}{L}e^{-(R/L)t}u(t)$

   (c) $s(t) = [1 - e^{-(R/L)t}]u(t)$

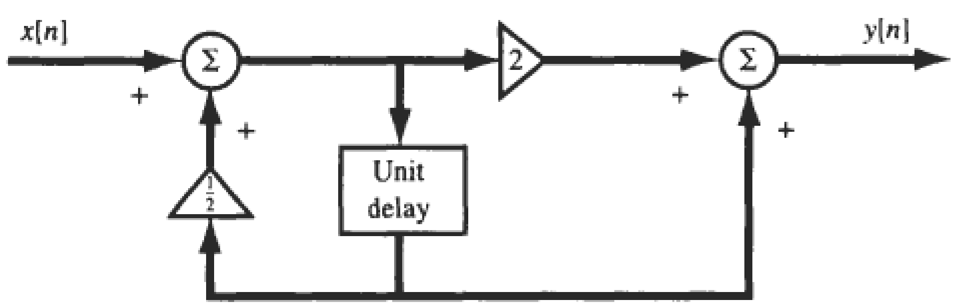




8. Is the system described by the differential equation

$$\frac{dy(t)}{dt} + 5y(t) + 2 = x(t)$$

   Solution:-
   No, it is nonlinear
9. Write the input-output equation for the system shown in



Solution:- 2y[n] - y[n – 1] = 4x[n] + 2x[n – 1]

10. Consider a continuous-time LTI system described by

$$y(t) = \mathbf{T}\{x(t)\} = \frac{1}{T}\int_{t-T/2}^{t+T/2} x(\tau)\,d\tau$$

(a) Find and sketch the impulse response *h (t)* of the system.
(b) Is this system causal?

Solution:-

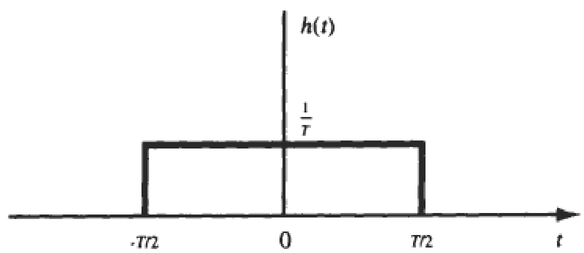
$$y(t) = \frac{1}{T}x(t) * u\left(t+\frac{T}{2}\right) - \frac{1}{T}x(t) * u\left(t-\frac{T}{2}\right)$$

$$= x(t) * \frac{1}{T}\left[u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right)\right] = x(t) * h(t)$$

Thus, we obtain

$$h(t) = \frac{1}{T}\left[u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right)\right] = \begin{cases} 1/T & -T/2 < t \le T/2 \\ 0 & \text{otherwise} \end{cases}$$

(*b*) From Fig. 2-16 or Eq. (2.75) we see that $h(t) \neq 0$ for $t < 0$. Hence, the system is not causal.



Unit 3: Time-domain representations for LTI systems – 2

1. Determine the condition of the impulse response of the system if system is, i) Memory less ii) Stable. **[Dec12, 6marks]**

Solution:-

Condition of the impulse response of the system if system is

(i) Memoryless

- Define & explain
- condition $h[k] = c\,\delta[k]$

or $h(\tau) = c\,\delta(\tau)$

(ii) stable

- Define & explain
- condition $\sum_{k=-\infty}^{+\infty} |h[k]| < \infty$

$\int_{-\infty}^{+\infty} |h(\tau)|\,d\tau < \infty$

2. Find the total response of the system given by,

$$\frac{d^2y(t)}{dt^2} + 3\frac{d}{dt}y(t) + 2y(t) = 2x(t) \text{ with } y(0) = -1;\ \left.\frac{d}{dt}y(t)\right|_{t=0} = 1 \text{ and } x(t) = \cos(t)u(t).$$

Solution:-

Given: $\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = 2x(t)$

with $y(0) = -1$; $\left.\frac{d}{dt}y(t)\right|_{t=0} = 1$ and $x(t) = \cos(t)\,u(t)$

$y(t) = y^{(n)}(t) + y^{(p)}(t)$

Characteristic equation: $r^2 + 3r + 2 = 0$

$\therefore r_1 = -2$ & $r_2 = -1$

$\therefore y^{(n)}(t) = C_1 e^{-2t} + C_2 e^{-t}$

Given: $x(t) = \cos(t)\, u(t)$, the particular solution is of the form,

$y^{(p)}(t) = K_1 \cos(t) + K_2 \sin(t)$

Substituting in the difference equation & simplifying we get, $(K_1 + 3K_2)\cos(t) + (K_2 - 3K_1)\sin(t) = 2\cos(t)$

Solving, $K_1 = 1/5$ & $K_2 = 3/5$.

$\therefore y(t) = C_1 e^{-2t} + C_2 e^{-t} + \frac{1}{5}\cos(t) + \frac{1}{5}\sin(t)$.

Using initial conditions in $y(t)$, we get, $C_1 = \frac{4}{5}$ & $C_2 = -2$

$\therefore y(t) = \frac{4}{5} e^{-2t} - 2e^{-t} + \frac{1}{5}\cos(t) + \frac{3}{5}\sin(t)$

3. The system shown in Fig. is formed by connecting two systems in ***cascade.*** The impulse responses of the systems are given by ***h , (t)*** and ***h 2 (0 ,*** respectively, and

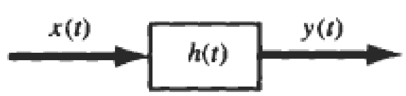
$$h_1(t) = e^{-2t}u(t) \qquad h_2(t) = 2e^{-t}u(t)$$

(a) Find the impulse response h (t) of the overall system shown in Fig.

(b) Determine if the overall system is BIBO stable. [**Dec 11, 10 marks**]

Solution:-

(a)



(a) Let $w(t)$ be the output of the first system. By Eq. (2.6)

$$w(t) = x(t) * h_1(t)$$

Then we have

$$y(t) = w(t) * h_2(t) = [x(t) * h_1(t)] * h_2(t)$$

But by the associativity property of convolution (2.8), Eq. (2.79) can be rewritten as

$$y(t) = x(t) * [h_1(t) * h_2(t)] = x(t) * h(t) \tag{2.80}$$

Therefore, the impulse response of the overall system is given by

$$h(t) = h_1(t) * h_2(t) \tag{2.81}$$

Thus, with the given $h_1(t)$ and $h_2(t)$, we have

$$h(t) = \int_{-\infty}^{\infty} h_1(\tau) h_2(t-\tau)\, d\tau = \int_{-\infty}^{\infty} e^{-2\tau} u(\tau) 2e^{-(t-\tau)} u(t-\tau)\, d\tau$$

$$= 2e^{-t} \int_{-\infty}^{\infty} e^{-\tau} u(\tau) u(t-\tau)\, d\tau = 2e^{-t}\left[\int_0^t e^{-\tau}\, d\tau\right] u(t)$$

$$= 2(e^{-t} - e^{-2t}) u(t)$$

(b) Using the above $h(t)$, we have

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = 2\int_0^{\infty} (e^{-\tau} - e^{-2\tau})\, d\tau = 2\left[\int_0^{\infty} e^{-\tau}\, d\tau - \int_0^{\infty} e^{-2\tau}\, d\tau\right]$$

$$= 2\left(1 - \tfrac{1}{2}\right) = 1 < \infty$$

Thus, the system is BIBO stable.

4. Verify the following

(a) $x[n] * h[n] = h[n] * x[n]$

(b) $\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$

[**Dec 11, 10 marks**]

Solution:-

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

By changing the variable $n - k = m$, we have

$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[n-m] h[m] = \sum_{m=-\infty}^{\infty} h[m] x[n-m] = h[n] * x[n]$$

(b) Let $x[n] * h_1[n] = f_1[n]$ and $h_1[n] * h_2[n] = f_2[n]$. Then

$$f_1[n] = \sum_{k=-\infty}^{\infty} x[k] h_1[n-k]$$

and

$$\{x[n] * h_1[n]\} * h_2[n] = f_1[n] * h_2[n] = \sum_{m=-\infty}^{\infty} f_1[m] h_2[n-m]$$

$$= \sum_{m=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x[k] h_1[m-k]\right] h_2[n-m]$$

Substituting $r = m - k$ and interchanging the order of summation, we have

$$\{x[n] * h_1[n]\} * h_2[n] = \sum_{k=-\infty}^{\infty} x[k]\left(\sum_{r=-\infty}^{\infty} h_1[r]h_2[n-k-r]\right)$$

Now, since

$$f_2[n] = \sum_{r=-\infty}^{\infty} h_1[r]h_2[n-r]$$

we have

$$f_2[n-k] = \sum_{r=-\infty}^{\infty} h_1[r]h_2[n-k-r]$$

Thus, $$\{x[n] * h_1[n]\} * h_2[n] = \sum_{k=-\infty}^{\infty} x[k]f_2[n-k]$$

$$= x[n] * f_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

5. Show that [**Jun 10 , 10 marks**]

Show that

(a) $x[n] * \delta[n] = x[n]$

(b) $x[n] * \delta[n - n_0] = x[n - n_0]$

Solution:-

$$x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = x[n]$$

(b) Similarly, we have

$$x[n] * \delta[n-n_0] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k-n_0] = x[n-n_0]$$

6.

Compute $y[n] = x[n] * h[n]$, where

(a) $x[n] = \alpha^n u[n]$, $h[n] = \beta^n u[n]$

(b) $x[n] = \alpha^n u[n]$, $h[n] = \alpha^{-n} u[-n]$, $0 < \alpha < 1$

[**Jan 10, 10 marks**]

Solution:-

(*a*) From Eq. (*2.35*) we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k u[k]\beta^{n-k}u[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \alpha^k \beta^{n-k} u[k]u[n-k]$$

Since $$u[k]u[n-k] = \begin{cases} 1 & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

we have

$$y[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} = \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k \qquad n \ge 0$$

Using Eq. (*1.90*), we obtain

$$y[n] = \begin{cases} \beta^n \dfrac{1-(\alpha/\beta)^{n+1}}{1-(\alpha/\beta)} u[n] & \alpha \ne \beta \\ \beta^n (n+1) u[n] & \alpha = \beta \end{cases}$$

or $$y[n] = \begin{cases} \dfrac{1}{\beta - \alpha} (\beta^{n+1} - \alpha^{n+1}) u[n] & \alpha \ne \beta \\ \beta^n (n+1) u[n] & \alpha = \beta \end{cases}$$

(*b*)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k u[k] \alpha^{-(n-k)} u[-(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} \alpha^{-n} \alpha^{2k} u[k]u[k-n]$$

For $n \le 0$, we have

$$u[k]u[k-n] = \begin{cases} 1 & 0 \le k \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = \alpha^{-n} \sum_{k=0}^{\infty} \alpha^{2k} = \alpha^{-n} \sum_{k=0}^{\infty} (\alpha^2)^k = \frac{\alpha^{-n}}{1-\alpha^2} \qquad n \le 0$$

For $n \ge 0$, we have

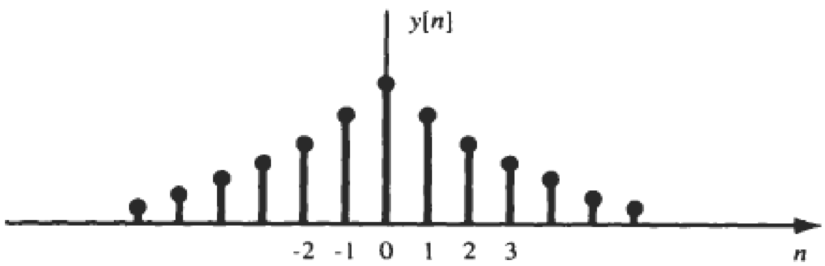
$$u[k]u[k-n] = \begin{cases} 1 & n \le k \\ 0 & \text{otherwise} \end{cases}$$

Thus, using Eq. (*1.92*), we have

$$y[n] = \alpha^{-n} \sum_{k=n}^{\infty} (\alpha^2)^k = \alpha^{-n} \frac{\alpha^{2n}}{1-\alpha^2} = \frac{\alpha^n}{1-\alpha^2} \qquad n \ge 0$$

Combining Eqs. (*2.136a*) and (*2.136b*), we obtain

$$y[n] = \frac{\alpha^{|n|}}{1-\alpha^2} \qquad \text{all } n$$



7. Find the impulse response h [n] for each of the causal LTI discrete-time systems satisfying the following difference equations and indicate whether each system is a FIR or an IIR system.
   (*a*) $y[n] = x[n] - 2x[n-2] + x[n-3]$
   (*b*) $y[n] + 2y[n-1] = x[n] + x[n-1]$
   (*c*) $y[n] - \frac{1}{2}y[n-2] = 2x[n] - x[n-2]$ [**July05, 10 marks**]

$$h[n] = \delta[n] - 2\delta[n-2] + \delta[n-3]$$

or

$$h[n] = \{1, 0, -2, 1\}$$

Since $h[n]$ has only four terms, the system is a FIR system.

(b) $h[n] = -2h[n-1] + \delta[n] + \delta[n-1]$
Since the system is causal, $h[-1] = 0$. Then

$$h[0] = -2h[-1] + \delta[0] + \delta[-1] = \delta[0] = 1$$
$$h[1] = -2h[0] + \delta[1] + \delta[0] = -2 + 1 = -1$$
$$h[2] = -2h[1] + \delta[2] + \delta[1] = -2(-1) = 2$$
$$h[3] = -2h[2] + \delta[3] + \delta[2] = -2(2) = -2^2$$
$$\vdots$$
$$h[n] = -2h[n-1] + \delta[n] + \delta[n-1] = (-1)^n 2^{n-1}$$

Hence, $$h[n] = \delta[n] + (-1)^n 2^{n-1} u[n-1]$$

Since $h[n]$ has infinite terms, the system is an IIR system.

(c) $h[n] = \frac{1}{2}h[n-2] + 2\delta[n] - \delta[n-2]$
Since the system is causal, $h[-2] = h[-1] = 0$. Then

$$h[0] = \tfrac{1}{2}h[-2] + 2\delta[0] - \delta[-2] = 2\delta[0] = 2$$
$$h[1] = \tfrac{1}{2}h[-1] + 2\delta[1] - \delta[-1] = 0$$
$$h[2] = \tfrac{1}{2}h[0] + 2\delta[2] - \delta[0] = \tfrac{1}{2}(2) - 1 = 0$$
$$h[3] = \tfrac{1}{2}h[1] + 2\delta[3] - \delta[1] = 0$$
$$\vdots$$

Hence, $$h[n] = 2\delta[n]$$

Since $h[n]$ has only one term, the system is a FIR system.

Unit 4: Fourier representation for signals – 1

1. One period of the DTFS coefficients of a signal is given by x(k)= $(1/2)^{k,}$ on $0 \leq K \leq 9$. Find the time-domain signal x(n) assuming N = 10. **[Dec 12, 6marks]**

Solution:

$N = 10, \quad X(k) = (1/2)^{k}, \quad 0 \leq k \leq 9$

$\Omega_0 = \frac{2\pi}{N} = \frac{\pi}{5}$ rad.

$\therefore x[n] = \sum_{k=0}^{N-1} X(k) e^{j\Omega_0 n k}, \quad n = 0, 1, \ldots 9$

substituting & expanding

$x[n] = \frac{1 - (1/2)}{1 - (1/2) e^{j\frac{\pi n}{5}}}$

2. Prove the following properties of DTFs: i) Convolution ii) Parseval relationship iii) Duality iv) Symmetry. **[Dec 12, 6marks]**

Solution:

Convolution:

$$x_1[n] * x_2[n] \leftrightarrow X_1(\Omega) X_2(\Omega)$$

As in the case of the z-transform, this convolution property plays an important role in the study of discrete-time LTI systems.

Duality:

The duality property of a continuous-time Fourier transform is expressed as

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

There is no discrete-time counterpart of this property. However, there is a duality between the discrete-time Fourier transform and the continuous-time Fourier series. Let

$$x[n] \leftrightarrow X(\Omega)$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

$$X(\Omega + 2\pi) = X(\Omega)$$

Since Ω is a continuous variable, letting $\Omega = t$ and $n = -k$

$$X(t) = \sum_{k=-\infty}^{\infty} x[-k]\, e^{jkt}$$

Since X(t) is periodic with period To = 2 π and the fundamental frequency $\omega_0 = 2\pi/T_0 = 1$,Equation indicates that the Fourier series coefficients of X(t) will be x [- k] . This duality relationship is denoted by

$$X(t) \overset{FS}{\longleftrightarrow} c_k = x[-k]$$

Where F_S denotes the Fourier series and c_k **,** are its Fourier coefficients.

Parseval's Relations:

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2[n] = \frac{1}{2\pi}\int_{2\pi} X_1(\Omega) X_2(-\Omega)\, d\Omega$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|^2\, d\Omega$$

3. State and prove tbe periodic time shift and periodic properties of DTFS**.[May/June 10 6 marks]**

Solution:

Periodicity

As a consequence of Eq. (6.41), in the discrete-time case we have to consider values of R(radians) only over the range0 < Ω < 2π or $\pi < \Omega < \pi$**,** while in the continuous-time case we have to consider values of ***0*** (radians/second) over the entire range $-\infty < \omega < \infty$.

$$X(\Omega + 2\pi) = X(\Omega)$$

Time Shifting:

$$x[n - n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega)$$

$$\mathscr{F}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0]\, e^{-j\Omega n}$$

By the change of variable $m = n - n_0$, we obtain

$$\mathscr{F}\{x[n-n_0]\} = \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega(m+n_0)}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega m} = e^{-j\Omega n_0} X(\Omega)$$

$$x[n-n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega)$$

4. Give the significance of time and frequency domain representation of signals.Give examples. **.[May/June 10, 4 marks]**

5. If FS representationof a signal x(t0) is X[k]. Derive the FS of a signal x(t-t_0)**.[Dec09/Jan 10, 6 marks]**

Solution:

<u>Time Shifting:</u>

$$x[n-n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega)$$

$$\mathscr{F}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0]\, e^{-j\Omega n}$$

By the change of variable $m = n - n_0$, we obtain

$$\mathscr{F}\{x[n-n_0]\} = \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega(m+n_0)}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega m} = e^{-j\Omega n_0} X(\Omega)$$

$$x[n-n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega)$$

6. Find the inverse Fourier transform x [n]

Solution:

$$X(\Omega) = 2\pi\delta(\Omega - \Omega_0) \qquad |\Omega|, |\Omega_0| \le \pi$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\Omega - \Omega_0)\, e^{j\Omega n}\, d\Omega = e^{j\Omega_0 n}$$

$$e^{j\Omega_0 n} \longleftrightarrow 2\pi\delta(\Omega - \Omega_0) \qquad |\Omega|, |\Omega_0| \le \pi$$

7. Determine the discrete Fourier series representation for each of the following sequences:

(a) $x[n] = \cos\frac{\pi}{4}n$

(b) $x[n] = \cos\frac{\pi}{3}n + \sin\frac{\pi}{4}n$

(c) $x[n] = \cos^2\left(\frac{\pi}{8}n\right)$

Solution:

(a) The fundamental period of $x[n]$ is $N_0 = 8$, and $\Omega_0 = 2\pi/N_0 = \pi/4$. Rather than using Eq. (6.8) to evaluate the Fourier coefficients c_k, we use Euler's formula and get

$$\cos\frac{\pi}{4}n = \frac{1}{2}\left(e^{j(\pi/4)n} + e^{-j(\pi/4)n}\right) = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{-j\Omega_0 n}$$

Thus, the Fourier coefficients for $x[n]$ are $c_1 = \frac{1}{2}$, $c_{-1} = c_{-1+8} = c_7 = \frac{1}{2}$, and all other $c_k = 0$. Hence, the discrete Fourier series of $x[n]$ is

$$x[n] = \cos\frac{\pi}{4}n = \frac{1}{2}e^{j\Omega_0 n} + \frac{1}{2}e^{j7\Omega_0 n} \qquad \Omega_0 = \frac{\pi}{4}$$

(b) From Prob. 1.16(*i*) the fundamental period of $x[n]$ is $N_0 = 24$, and $\Omega_0 = 2\pi/N_0 = \pi/12$. Again by Euler's formula we have

$$x[n] = \frac{1}{2}\left(e^{j(\pi/3)n} + e^{-j(\pi/3)n}\right) + \frac{1}{2j}\left(e^{j(\pi/4)n} - e^{-j(\pi/4)n}\right)$$

$$= \tfrac{1}{2}e^{-j4\Omega_0 n} + j\tfrac{1}{2}e^{-j3\Omega_0 n} - j\tfrac{1}{2}e^{j3\Omega_0 n} + \tfrac{1}{2}e^{j4\Omega_0 n}$$

Thus, $c_3 = -j(\frac{1}{2}), c_4 = \frac{1}{2}, c_{-4} = c_{-4+24} = c_{20} = \frac{1}{2}, c_{-3} = c_{-3+24} = c_{21} = j(\frac{1}{2})$, and all other $c_k = 0$. Hence, the discrete Fourier series of $x[n]$ is

$$x[n] = -j\frac{1}{2}e^{j3\Omega_0 n} + \frac{1}{2}e^{j4\Omega_0 n} + \frac{1}{2}e^{j20\Omega_0 n} + j\frac{1}{2}e^{j21\Omega_0 n} \qquad \Omega_0 = \frac{\pi}{12}$$

(c) From Prob. 1.16(j) the fundamental period of $x[n]$ is $N_0 = 8$, and $\Omega_0 = 2\pi/N_0 = \pi/4$. Again by Euler's formula we have

$$x[n] = \left(\tfrac{1}{2}e^{j(\pi/8)n} + \tfrac{1}{2}e^{-j(\pi/8)n}\right)^2 = \tfrac{1}{4}e^{j(\pi/4)n} + \tfrac{1}{2} + \tfrac{1}{4}e^{-j(\pi/4)n}$$
$$= \tfrac{1}{4}e^{j\Omega_0 n} + \tfrac{1}{2} + \tfrac{1}{4}e^{-j\Omega_0 n}$$

Thus, $c_0 = \frac{1}{2}, c_1 = \frac{1}{4}, c_{-1} = c_{-1+8} = c_7 = \frac{1}{4}$, and all other $c_k = 0$. Hence, the discrete Fourier series of $x[n]$ is

$$x[n] = \frac{1}{2} + \frac{1}{4}e^{j\Omega_0 n} + \frac{1}{4}e^{j7\Omega_0 n} \qquad \Omega_0 = \frac{\pi}{4}$$

8. Find the Fourier transform of x [n] = - a^n u[-n-1]

Solution:

$$X(z) = \frac{1}{1 - az^{-1}} \qquad |z| < |a|$$

Thus, $X(e^{j\Omega})$ exists for $|a| > 1$ because the ROC of $X(z)$ then contains the unit circle. Thus,

$$X(\Omega) = X(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}} \qquad |a| > 1$$

9. Verify the convolution theorem that is,

$$x_1[n] * x_2[n] \longleftrightarrow X_1(\Omega)X_2(\Omega)$$

Solution:

$$\mathscr{F}\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]\right)e^{-j\Omega n}$$

Changing the order of summation, we get

$$\mathscr{F}\{x_1[n] * x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k]\left(\sum_{n=-\infty}^{\infty} x_2[n-k]e^{-j\Omega n}\right)$$

By the time-shifting property Eq. (6.43)

$$\sum_{n=-\infty}^{\infty} x_2[n-k]e^{-j\Omega n} = e^{-j\Omega k}X_2(\Omega)$$

Thus, we have

$$\mathscr{F}\{x_1[n] * x_2[n]\} = \sum_{k=-\infty}^{\infty} x_1[k]e^{-j\Omega k}X_2(\Omega)$$
$$= \left(\sum_{k=-\infty}^{\infty} x_1[k]e^{-j\Omega k}\right)X_2(\Omega) = X_1(\Omega)X_2(\Omega)$$

$$x_1[n] * x_2[n] \longleftrightarrow X_1(\Omega)X_2(\Omega)$$

10. Using the convolution theorem, find the inverse Fourier transform x [n] .

$$X(\Omega) = \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} \qquad |a| < 1$$

Solution:

$$a^n u[n] \longleftrightarrow \frac{1}{1 - ae^{-j\Omega}} \qquad |a| < 1$$

$$X(\Omega) = \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} = \left(\frac{1}{1 - ae^{-j\Omega}}\right)\left(\frac{1}{1 - ae^{-j\Omega}}\right)$$

$$x[n] = a^n u[n] * a^n u[n] = \sum_{k=-\infty}^{\infty} a^k u[k] a^{n-k} u[n-k]$$

$$= a^n \sum_{k=0}^{n} 1 = (n+1) a^n u[n]$$

$$(n+1) a^n u[n] \longleftrightarrow \frac{1}{\left(1 - ae^{-j\Omega}\right)^2} \qquad |a| < 1$$

UNIT 5: Fourier representation for signals – 2

1. Define the DTFT of a signal. Establish the relation between DTFT and Z transform of a signal. **[Jan/Feb 04, 7marks]**

Solution:

If the sequence $x(n)$ has a finite duration of length N or less, the sequence can be recovered from its N-point DFT. Hence its z-transform is uniquely determined by its N-point DFT. Consequently, $X(z)$ can be expressed as a function of the DFT {X(k)} as follows

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \right] z^{-n}$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} z^{-1} \right)^n$$

Relationship to the *z*-transform. Let us consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

with a ROC that includes the unit circle. If $X(z)$ is sampled at the N equally spaced points on the unit circle $z_k = e^{j2\pi k/N}$, 0, 1, 2, ..., $N-1$, we obtain

$$X(k) \equiv X(z)|_{z=e^{j2\pi nk/N}} \qquad k = 0, 1, \ldots, N-1$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N}$$

$$X(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j2\pi k/N} z^{-1}}$$

When evaluated on the unit circle, (5.1.38) yields the Fourier transform of the finite-duration sequence in terms of its DFT, in the form

$$X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{-j(\omega - 2\pi k/N)}}$$

2. State and prove the following properties of Fourier transform. i) Time shifting property ii) Differentiation in time property iii) Frequency shifting property:

[July 09, 9marks]

Time Shifting:

$$x[n-n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega)$$

$$\mathscr{F}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0]\, e^{-j\Omega n}$$

By the change of variable $m = n - n_0$, we obtain

$$\mathscr{F}\{x[n-n_0]\} = \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega(m+n_0)}$$

$$= e^{-j\Omega n_0} \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega m} = e^{-j\Omega n_0} X(\Omega)$$

$$x[n-n_0] \longleftrightarrow e^{-j\Omega n_0} X(\Omega)$$

Frequency Shifting:

$$e^{j\Omega_0 n} x[n] \leftrightarrow X(\Omega - \Omega_0)$$

Differentiation in Frequency:

$$nx[n] \longleftrightarrow j\frac{dX(\Omega)}{d\Omega}$$

3. Show that the real and odd continuous time non periodic signal has purely imaginary Fourier transform. (4 Marks) **[Jan/Feb 05, 4marks]**

Solution:

Even and Odd Sequences:

When $x[n]$ is real, let

$$x[n] = x_e[n] + x_o[n]$$

where $x_e[n]$ and $x_o[n]$ are the even and odd components of $x[n]$, respectively. Let

$$x[n] \overset{\text{DFS}}{\longleftrightarrow} c_k$$

Then

$$x_e[n] \overset{\text{DFS}}{\longleftrightarrow} \text{Re}[c_k] \qquad (6.18a)$$

$$x_o[n] \overset{\text{DFS}}{\longleftrightarrow} j\,\text{Im}[c_k] \qquad (6.18b)$$

Thus, we see that if $x[n]$ is real and even, then its Fourier coefficients are real, while if $x[n]$ is real and odd, its Fourier coefficients are imaginary.

4. Use the equation describing the DTFT representation to determine the time-domain signals corresponding to the following DTFTs : **[July 06, 8marks]**
   i) $X(e^{j\Omega})= \text{Cos}(\Omega)+j\ \text{Sin}(\Omega)$
   ii) $X(e^{j\Omega})=\{1$, for $\pi/4<\Omega< 3\pi/4$; 0 otherwise and $X(e^{j\Omega})=-4\ \Omega$

Solution:

i)

(a) $X(e^{j\Omega}) = \cos(2\Omega) + j\sin(2\Omega)$

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(2+n)}\,d\Omega \\ &\quad \text{by orthogonality} \\ &= \delta[n+2] \end{aligned}$$

ii)

$$|X(e^{j\Omega})| = \begin{cases} 1, & \pi/4 < |\Omega| < 3\pi/4, \\ 0 & \text{otherwise} \end{cases} \qquad \arg\{X(e^{j\Omega})\} = -4\Omega$$

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{0.25\pi}^{0.75\pi} e^{j\Omega(n-4)}\,d\Omega + \frac{1}{2\pi}\int_{-0.75\pi}^{-0.25\pi} e^{j\Omega(n-4)}\,d\Omega \\ &= \frac{\sin(0.75\pi(n-4)) - \sin(0.25\pi(n-4))}{\pi(n-4)} \end{aligned}$$

5. Explain the reconstruction of CT signals implemented with zero-order device. **[Jan/Feb 05, 4marks]**

Solution:

6. Find the DTFT of the sequence $x(n) = \alpha^n u(n)$ and determine magnitude and phase spectrum.

Solution;

$$h[n] = \alpha^n u[n], \quad |\alpha| < 1$$

$$\begin{aligned} H(e^{j\Omega}) &= \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} \\ &= \frac{1}{1-\alpha e^{-j\Omega}} \end{aligned}$$

$$
\begin{aligned}
Y(e^{j\Omega})(1-\alpha e^{-j\Omega}) &= X(e^{j\Omega}) \\
y[n]-\alpha y[n-1] &= x[n]
\end{aligned}
$$

7. Plot the magnitude and phase spectrum of $x(t) = e^{-at}u(t)$

Solution;

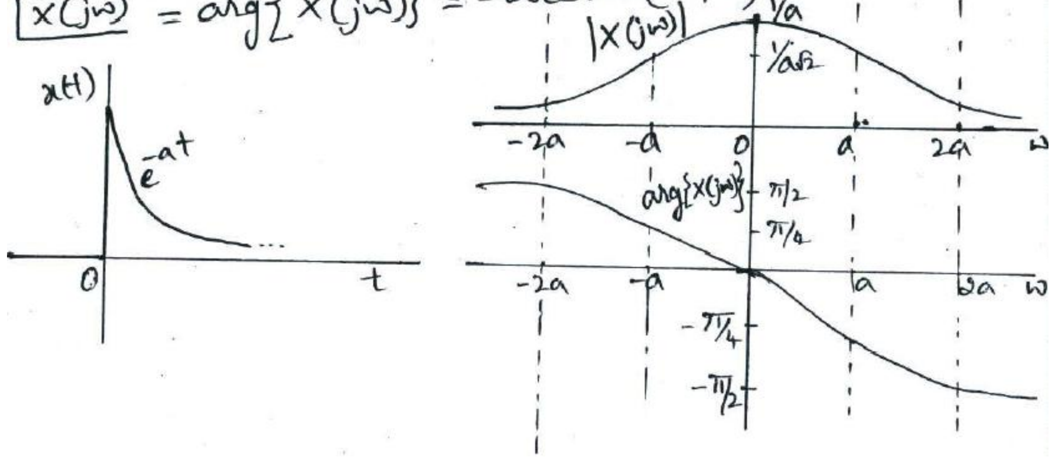
$x(t) = e^{-at}u(t)$ $a \leq 0$, FT does not converge

for $a > 0$, FT

$X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt = \left[\frac{1}{a+j\omega}\right]$

$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$

$\angle X(j\omega) = \arg\{X(j\omega)\} = -\arctan(\omega/a)$



8. Find the inverse Fourier transform of the spectra,

$$x(j\omega) = \begin{cases} 2\cos(\omega), & |\omega| < \pi \\ 0, & |\omega| > 0 \end{cases}$$

[July 07, 8marks]

Solution:

$X(j\omega) = \begin{cases} 2\cos(\omega), & |\omega| < \pi \\ 0 & |\omega| > \pi \end{cases}$

Inverse FT

$x(t) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} 2\cos(\omega)\, e^{j\omega t}\, d\omega$ solve

Simplification
$$x(t) = \frac{\sin(\pi(t+1))}{\pi(t+1)} + \frac{\sin(\pi(t-1))}{\pi(t-1)}$$
$$= \frac{2t \sin \pi t}{\pi(1-t^2)}$$

9. Find the DTFT of the sequence x(n) =(1/3)n u(n+2) and determine magnitude and phase spectrum. **[July 08, 6marks]**

Solution:

$$x[n] = \left(\tfrac{1}{3}\right)^n u[n+2]$$

$$x[n] = (\frac{1}{3})^n u[n+2]$$
$$= (\frac{1}{3})^{-2}(\frac{1}{3})^{n+2} u[n+2]$$
$$(\frac{1}{3})^n u[n] \xleftrightarrow{DTFT} \frac{1}{1-\frac{1}{3}e^{-j\Omega}}$$
$$s[n+2] \xleftrightarrow{DTFT} e^{j2\Omega} S(e^{j\Omega})$$
$$X(e^{j\Omega}) = \frac{9e^{j2\Omega}}{1-\frac{1}{3}e^{-j\Omega}}$$

10. Use the defining equation for the FT to evaluate the frequency-domain representations for the following signals: **[July 06, 6marks]**
   i) $X(t)= e^{-2t}u(t-3)$
   ii) $X(t)=e^{-4t}$ Sketch the magnitude and phase spectra.

Solution:

i)

$$x(t) = e^{-2t}u(t-3)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$
$$= \int_{3}^{\infty} e^{-2t}e^{-j\omega t}\,dt$$
$$= \frac{e^{-3(2+j\omega)}}{2+j\omega}$$

ii)

$$x(t) = e^{-4|t|}$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} e^{-4|t|} e^{-j\omega t}\,dt \\ &= \int_{0}^{\infty} e^{-4t} e^{-j\omega t}\,dt + \int_{-\infty}^{0} e^{4t} e^{-j\omega t}\,dt \\ &= \frac{8}{16+\omega^2} \end{aligned}$$

11. Use the de.ning equation for the DTFT to evaluate the frequency-domain representations or the following signals. Sketch the magnitude and phase spectra.

Solution: : **[July 07, 8marks]**

$x[n] = (\frac{3}{4})^n u[n-4]$

$$\begin{aligned} X(e^{j\Omega}) &= \sum_{m=-\infty}^{\infty} x[n] e^{-j\Omega n} \\ &= \sum_{m=4}^{\infty} (\frac{3}{4})^n e^{-j\Omega n} \\ &= \sum_{m=4}^{\infty} (\frac{3}{4} e^{-j\Omega})^n \\ &= \frac{(\frac{3}{4} e^{-j\Omega})^4}{1 - \frac{3}{4} e^{-j\Omega}} \end{aligned}$$

$$\begin{aligned} |X(e^{j\Omega})| &= \frac{(\frac{3}{4})^4}{(\frac{25}{16} - \frac{3}{2}\cos(\Omega))^{0.5}} \\ \angle X(e^{j\Omega}) &= -4\Omega + \arctan\left(\frac{3\sin(\Omega)}{4 - 3\cos(\Omega)}\right) \end{aligned}$$

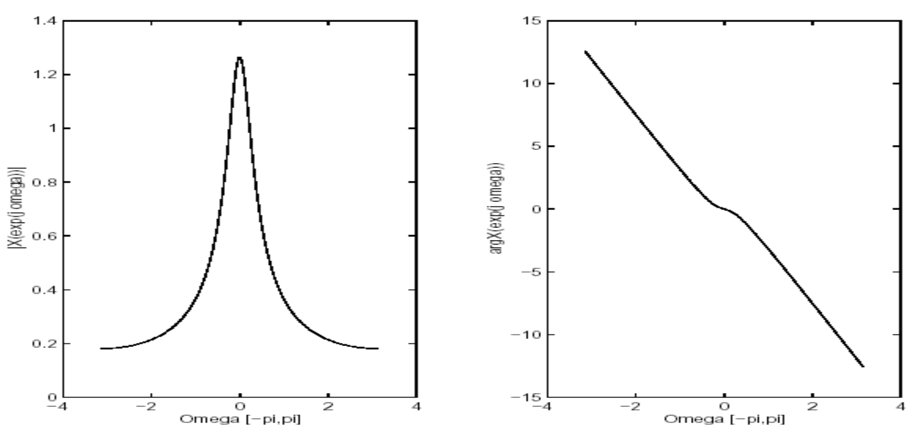


Figure P3.52. (a) Graph of the magnitude and phase

UNIT 6:Applications of Fourier representations

1. December 2011

Find the frequency response and impulse response of the system described by the differential equation.

$$\frac{d^2}{dt^2}y(t)+5\frac{d}{dt}y(t)+6y(t)=-\frac{d}{dt}x(t)$$ **(08 Marks)**

Solution :

Given : $\frac{d^2}{dt^2}y(t)+5\frac{d}{dt}y(t)+6y(t)=-\frac{dx(t)}{dt}$

Applying FT, $(j\omega)^2Y(j\omega)+5(j\omega)Y(j\omega)+6Y(j\omega)=-j\omega X(j\omega)$

Frequency Response $H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{-j\omega}{(j\omega)^2+5j\omega+6}$

Using partial fraction & taking inverse FT,

Impulse Response $h(t)=\left(2e^{-2t}-3e^{-3t}\right)u(t)$

2. December 2011

Find the Nyquist rate for each of the following signals:

i) $x_1(t)=\sin c(200t)$ ii) $x_2(t)=\sin c^2(500t)$ **(06 Marks)**

Solution :

(i) $x_1(t)=\operatorname{sinc}(200t)$

$=\frac{\sin(200\pi t)}{200\pi t}\Rightarrow\omega=200\pi$ rad/sec

$\therefore$ Nyquist rate $=2\times200\pi=400\pi$ rad/sec or 200Hz

(ii) $x_2(t)=\operatorname{sinc}^2(500t)$

$\therefore$ Nyquist rate $=2\times1000\pi$ rad/sec or 1000Hz

3. December 2012

Find the DTFT of the sequence $x(n)=\alpha^n u(n)$ and determine magnitude and phase spectrum. **(04 Marks)**

Solution :

Given: $x[n] = \alpha^n u[n]$

DTFT $X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\Omega})^n$

sum diverges for $|\alpha| \geq 1$

for $|\alpha| < 1$, sum converges

$$\therefore X(e^{j\Omega}) = \frac{1}{1-\alpha e^{-j\Omega}}$$

$$= \frac{1}{1-\alpha\cos\Omega + j\alpha\sin\Omega}, \text{ for } \alpha \text{ - real valued}$$

Magnitude spectrum $|X(e^{j\Omega})| = \frac{1}{(\alpha^2+1-2\alpha\cos\Omega)^{1/2}}$ even & 2π periodic

Phase spectrum $\arg\{X(e^{j\Omega})\} = -\arctan\left(\frac{\alpha\sin\Omega}{1-\alpha\cos\Omega}\right)$ odd & 2π periodic

4. December 2012

Plot the magnitude and phase spectrum of $x(t) = e^{-at}u(t)$. **(08 Marks)**

Solution :

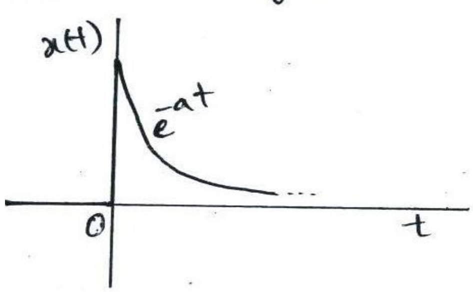
$x(t) = e^{-at}u(t)$ $\quad a \leq 0$, FT does not converge

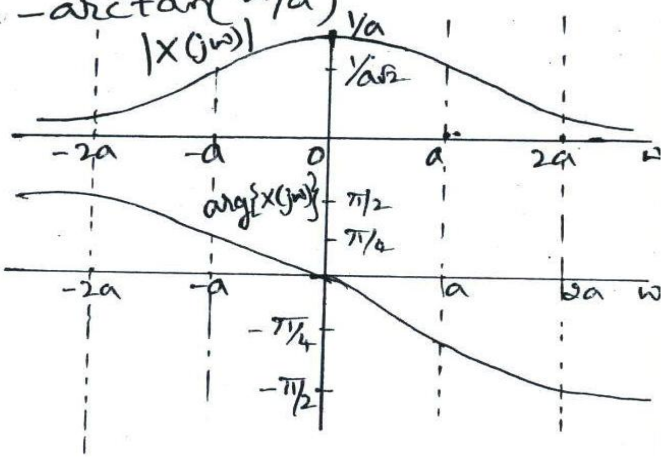
for $a > 0$, FT

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-at} u(t) e^{-j\omega t} dt = \left[\frac{1}{a+j\omega}\right]$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\underline{\angle X(j\omega)} = \arg\{X(j\omega)\} = -\arctan(\omega/a)$$





5. December 2012

Find the inverse Fourier transform of the spectra, $x(j\omega) = \begin{cases} 2\cos(\omega), & |\omega| < \pi \\ 0, & |\omega| > 0 \end{cases}$ **(08 Marks)**

Solution :

$$X(j\omega) = \begin{cases} 2\cos(\omega), & |\omega| < \pi \\ 0 & |\omega| > \pi \end{cases}$$

Inverse FT

$$x(t) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} 2\cos(\omega)\, e^{j\omega t}\, d\omega \quad \text{solve}$$

simplification

$$x(t) = \frac{\sin(\pi(t+1))}{\pi(t+1)} + \frac{\sin(\pi(t-1))}{\pi(t-1)}$$

$$= \frac{2t \sin \pi t}{\pi(1-t^2)}$$

6. Use the equation describing the DTFT representation to determine the time-domain signals corresponding to the following DTFT's. **[May/Jun 10, 8marks]**

Solution :

(a) $X(e^{j\Omega}) = \cos(2\Omega) + j\sin(2\Omega)$

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega(2+n)}\, d\Omega \\ &\quad \text{by orthogonality} \\ &= \delta[n+2] \end{aligned}$$

(b) $X(e^{j\Omega}) = \sin(\Omega) + \cos(\frac{\Omega}{2})$

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{e^{j\Omega} - e^{-j\Omega}}{2j} + \frac{e^{j\frac{\Omega}{2}} + e^{-j\frac{\Omega}{2}}}{2}\right)e^{j\Omega n}\, d\Omega \\ &= \frac{1}{2j}\delta[n+1] - \frac{1}{2j}\delta[n-1] + \frac{1}{2\pi}\frac{\cos(\pi n)}{n+0.5} - \frac{1}{2\pi}\frac{\cos(\pi n)}{n-0.5} \end{aligned}$$

7. Use the equation describing the FT representation to determine the time-domain signals correspondingto the following FT's. **[July 07, 8marks]**

Solution:

$$X(j\omega) = \begin{cases} \cos(2\omega), & |\omega| < \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-0.25\pi}^{0.25\pi} \frac{e^{j2\omega}+e^{-j2\omega}}{2}e^{j\omega t}\,d\omega \\ &= \frac{1}{2\pi}\int_{-0.25\pi}^{0.25\pi} \frac{1}{2}e^{j(t+2)\omega}\,dt + \frac{1}{2\pi}\int_{-0.25\pi}^{0.25\pi} \frac{1}{2}e^{j(t-2)\omega}\,d\omega \\ &= \frac{\sin(0.25\pi(t+2))}{2\pi(t+2)} + \frac{\sin(0.25\pi(t-2))}{2\pi(t-2)} \end{aligned}$$

$$x(t) = \begin{cases} \frac{\sin(0.25\pi(t+2))}{2\pi(t+2)} + \frac{\sin(0.25\pi(t-2))}{2\pi(t-2)} & t \neq 2, -2 \\ \frac{1}{8} & t = \pm 2 \end{cases}$$

8. Determine the appropriate Fourier representation for the following time-domain signals, using the defining equations. **[July 6, 10marks]**

Solution:

(a) $x(t) = e^{-t}\cos(2\pi t)u(t)$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= \frac{1}{2}\int_{0}^{\infty} e^{-t}(e^{j2\pi t}+e^{-j2\pi t})e^{-j\omega t}\,dt \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2}\int_{0}^{\infty} e^{-t(1-j2\pi+j\omega)}\,dt + \frac{1}{2}\int_{0}^{\infty} e^{-t(1+j2\pi+j\omega)}\,dt \\ &= \frac{1}{2}\left[\frac{1}{1-j(2\pi-\omega)} + \frac{1}{1+j(2\pi+\omega)}\right] \end{aligned}$$

UNIT 7:Z-Transforms – 1

1. **December 2011**

Find the z-transform of

(a) $x[n] = -a^n u[-n-1]$

(b) $x[n] = a^{-n} u[-n-1]$

(a) From Eq. (4.3)

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

By Eq. (1.91)

$$\sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1-a^{-1}z} \qquad \text{if } |a^{-1}z| < 1 \text{ or } |z| < |a|$$

Thus,

$$X(z) = 1 - \frac{1}{1-a^{-1}z} = \frac{-a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a} = \frac{1}{1-az^{-1}} \qquad |z| < |a|$$

2. **December 20112**

Prove the complex conjugation and time-advance properties. **(06 Marks)**

Solution:

Z-transform properties

(i) Conjugation

$x^*[n] \xleftrightarrow{ZT} X^*(z^*)$ ROC remain same

– Proof

ii) Time advance

$x[n]$ – right sided sequence

$Z\{x[n+1]\} = Zx(z) - zx[0]$ with ROC being entire z-plane except at $z=0$.

– Proof

3. **December 2012**

Find the z-transform of the signal along with ROC.

$x(n) = n\sin\left(\frac{\pi}{2}n\right)u(n)$ **(06 Marks)**

Solution:

$$x[n] = n\sin\left(\frac{\pi}{2}n\right)u[-n]$$

$$\Rightarrow x[n] = -n\sin\left(-\frac{\pi}{2}n\right)u[-n]$$

$$\sin\left(\frac{\pi}{2}n\right)u[n] \xleftrightarrow{ZT} \frac{\sin\left(\frac{\pi}{2}\right)z^{-1}}{1-2\cos\left(\frac{\pi}{2}\right)z^{-1}+z^{-2}} \quad ; |z|>1$$

$$\xleftrightarrow{ZT} \frac{z}{z^2+1}$$

Using differentiation in z-domain,

$$n\sin\left(\frac{\pi}{2}n\right)u[n] \xleftrightarrow{ZT} -z\frac{d}{dz}\left[\frac{z}{z^2+1}\right] \quad ; |z|>1$$

$$\xleftrightarrow{ZT} \frac{z^3-z}{(z^2+1)^2}$$

Using time-reversal property,

$$-n\sin\left(-\frac{\pi}{2}n\right)u[-n] \xleftrightarrow{ZT} \left.\frac{z^3-z}{(z^2+1)^2}\right|_{z=\frac{1}{z}} = \frac{z(1-z^2)}{(z^2+1)^2} \quad ; |z|<1$$

4. December 2012

Determine the inverse z-transform of the following x(z) by partial fraction expansion method,

$$x(z) = \frac{z+2}{2z^2-7z+3}$$

if the ROCs are i) $|z|>3$ ii) $|z|<\frac{1}{2}$ and iii) $\frac{1}{2}<|z|<3$. **(08 Marks)**

$$X(z) = \frac{z+2}{2z^2-7z+3}$$

$$F(z) = \frac{X(z)}{z} = \frac{z+2}{z(2z^2-7z+3)} = \frac{z+2}{2z\left(z-\frac{1}{2}\right)(z-3)}$$

$$= \frac{A_0}{z} + \frac{A_1}{z-\frac{1}{2}} + \frac{A_2}{z-3}$$

Solving $A_0 = \frac{2}{3}$, $A_1 = -1$, $A_2 = 1/3$

$$\therefore X(z) = \frac{2}{3} - \frac{z}{z-1/2} + \frac{z/3}{z-3}$$

(i) $|z| > 3$, all poles are interior, $x[n]$ causal.

$$\therefore x[n] = \frac{2}{3}\delta[n] - \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}(3)^n u[n]$$

(ii) $|z| < \frac{1}{2}$, both poles are exterior, $x[n]$ anticausal

$$\therefore x[n] = \frac{2}{3}\delta[n] - \left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}(3)^n u[-n-1]$$

iii) $\frac{1}{2} < |z| < 3$, $|z| < 3$ anticausal pole $p_2 = 3$ exterior
$|z| > \frac{1}{2}$ causal pole $p_1 = \frac{1}{2}$ interior

$$\therefore x[n] = \frac{2}{3}\delta[n] - \left(\frac{1}{2}\right)^n u[n] - \frac{1}{3}(3)^n u[-n-1]$$

7.31. Determine (i) transfer function and (ii) impulse response representations for the systems described by the following difference equations:

(a) $y[n] - \frac{1}{2}y[n-1] = 2x[n-1]$

$$Y(z)\left(1 - \frac{1}{2}z^{-1}\right) = 2z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

$$h[n] = 2\left(\frac{1}{2}\right)^{n-1}u[n-1]$$

(b) $y[n] = x[n] - x[n-2] + x[n-4] - x[n-6]$

$$\begin{aligned} Y(z) &= \left(1 - z^{-2} + z^{-4} - z^{-6}\right) X(z) \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= 1 - z^{-2} + z^{-4} - z^{-6} \\ h[n] &= \delta[n] - \delta[n-2] + \delta[n-4] - \delta[n-6] \end{aligned}$$

7.32. Determine (i) transfer function and (ii) difference-equation representations for the systems with the following impulse responses:
(a) $h[n] = 3\left(\frac{1}{4}\right)^n u[n-1]$

$$\begin{aligned} h[n] &= 3\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)^{n-1} u[n-1] \\ H(z) &= \frac{\frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} \\ &= \frac{Y(z)}{X(z)} \end{aligned}$$

Taking the inverse z-transform yields:

$$y[n] - \frac{1}{4}y[n-1] = \frac{3}{4}x[n-1]$$

(b) $h[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-2} u[n-1]$

$$\begin{aligned} h[n] &= \left(\frac{1}{3}\right)^n u[n] + 2\left(\frac{1}{2}\right)^{n-1} u[n-1] \\ H(z) &= \frac{1 + \frac{3}{2}z^{-1} - \frac{2}{3}z^{-2}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}} \\ &= \frac{Y(z)}{X(z)} \end{aligned}$$

Taking the inverse z-transform yields:

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] \quad = \quad x[n] + \frac{3}{2}x[n-1] - \frac{2}{3}x[n-2]$$

7.35. For each system described below, identify the transfer function of the inverse system, and determine whether it can be both causal and stable.

(a) $H(z) = \frac{1-8z^{-1}+16z^{-2}}{1-\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}}$

$$\begin{aligned} H(z) &= \frac{(z-4)^2}{(z-\frac{1}{2})^2} \\ H^{inv}(z) &= \frac{(z-\frac{1}{2})^2}{(z-4)^2} \\ \text{poles at:} \quad & z = 4 \text{ (double)} \end{aligned}$$

For the inverse system, not all poles are inside $|z| = 1$, so the system is not causal and stable.

(b) $H(z) = \frac{z^2 - \frac{81}{100}}{z^2-1}$

$$\begin{aligned} H^{inv}(z) &= \frac{z^2-1}{z^2-\frac{81}{100}} \\ \text{poles at:} \quad & z = \frac{9}{10} \text{ (double)} \end{aligned}$$

For the inverse system, all poles are inside $|z| = 1$, so the system can be causal and stable.

(c) $h[n] = 10\left(\frac{-1}{2}\right)^n u[n] - 9\left(\frac{-1}{4}\right)^n u[n]$

$$\begin{aligned} H(z) &= \frac{z(z-2)}{(z+\frac{1}{2})(z+\frac{1}{4})} \\ H^{inv}(z) &= \frac{(z+\frac{1}{2})(z+\frac{1}{4})}{z(z-2)} \end{aligned}$$

UNIT 8: Z-Transforms – II

1. December 2012

A system has impulse response $h(n)=\left(\frac{1}{2}\right)^n u(n)$, determine the input to the system if the output is given by,

$y(n)=\frac{1}{3}u(n)+\frac{2}{3}\left(-\frac{1}{2}\right)^n u(n)$. **(08 Marks)**

Solution :

$$h[n] = (1/2)^n u[n]$$

$$y[n] = \frac{1}{3}u[n] + \left(\frac{2}{3}\right)(-1/2)^n u[n]$$

$$H(z) = \frac{Y(z)}{X(z)} \Rightarrow X(z) = \frac{Y(z)}{H(z)} = \frac{(1-1/2\,z^{-1})^2}{(1-1/2\,z^{-1}-1/2\,z^{-2})}$$

$$= -\frac{1}{2} + \frac{1/6}{(1-z^{-1})} + \frac{4/3}{(1+1/2\,z^{-1})}$$

$$\therefore x[n] = -\frac{1}{2}\delta[n] + \frac{1}{6}u[n] + \left(\frac{4}{3}\right)(-1/2)^n u[n]$$

5. December 2012

Solve the following difference equation using unilateral z-transform,

$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=x(n)$, for $n\geq 0$, with initial conditions $y(-1)=4$, $y(-2)=10$, and $x(n)=\left(\frac{1}{4}\right)^n u(n)$. **(12 Marks)**

Solution:

$$y[n] \xrightarrow{ZT} Y(z) = \frac{2-\frac{9}{4}z^{-1}+\frac{1}{2}z^{-2}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

Partial fraction given $Y(z) = \frac{2/3}{(1-z^{-1})} + \frac{1}{(1-\frac{1}{2}z^{-1})} + \frac{1/3}{(1-\frac{1}{4}z^{-1})}$

Inverse z-transform,

$$y[n] = \frac{2}{3}u[n] + \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{4}\right)^n u[n];\ n\geq 0$$

7.24. Use the method of partial fractions to obtain the time-domain signals corresponding to the following z-transforms:

(a) $X(z) = \frac{1+\frac{7}{6}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \quad |z| > \frac{1}{2}$

$$\begin{aligned} X(z) &= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{3}z^{-1}} \\ 1 &= A + B \\ \frac{7}{6} &= \frac{1}{3}A - \frac{1}{2}B \\ X(z) &= \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1+\frac{1}{3}z^{-1}} \\ x[n] &= \left[2(\frac{1}{2})^n - (-\frac{1}{3})^n\right] u[n] \end{aligned}$$

(b) $X(z) = \frac{1+\frac{7}{6}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \quad |z| < \frac{1}{3}$

same as (a), but $x[n]$ is left-sided

$$x[n] = \left[-2(\frac{1}{2})^n + (-\frac{1}{3})^n\right] u[-n-1]$$

(c) $X(z) = \frac{1+\frac{7}{6}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \quad \frac{1}{3} < |z| < \frac{1}{2}$

same as (a), but $x[n]$ is two-sided

$$x[n] = -2(\frac{1}{2})^n u[-n-1] - (-\frac{1}{3})^n u[n]$$

(d) $X(z) = \frac{z^2-3z}{z^2+\frac{3}{2}z-1}, \quad \frac{1}{2} < |z| < 2$

$x[n]$ is two-sided

$$\begin{aligned} X(z) &= \frac{1-3z^{-1}}{1+\frac{3}{2}z^{-1}-z^{-2}} \\ &= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+2z^{-1}} \end{aligned}$$

(e) $X(z) = \frac{3z^2-\frac{1}{4}z}{z^2-16}, \quad |z| > 4$

$x[n]$ is right-sided

$$\begin{aligned} X(z) &= \frac{A}{1+4z^{-1}} + \frac{B}{1-4z^{-1}} \\ 3 &= A+B \\ -\frac{1}{4} &= -4A+4B \\ X(z) &= \frac{\frac{49}{32}}{1+4z^{-1}} + \frac{\frac{47}{32}}{1-4z^{-1}} \\ x[n] &= \left[\frac{49}{32}(-4)^n + \frac{47}{32}4^n\right]u[n] \end{aligned}$$

A causal system has input $x[n]$ and output $y[n]$. Use the transfer function to determine the impulse response of this system.

(a) $x[n] = \delta[n] + \frac{1}{4}\delta[n-1] - \frac{1}{8}\delta[n-2], \quad y[n] = \delta[n] - \frac{3}{4}\delta[n-1]$

$$\begin{aligned} X(z) &= 1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-1} \\ Y(z) &= 1 - \frac{3}{4}z^{-1} \end{aligned}$$

$$
\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{-\frac{2}{3}}{1-\frac{1}{4}z^{-1}} + \frac{\frac{5}{3}}{1+\frac{1}{2}z^{-1}} \\
h[n] &= \frac{1}{3}\left[5(-\frac{1}{2})^n - 2(\frac{1}{4})^n\right]u[n]
\end{aligned}
$$

(b) $x[n] = (-3)^n\, u[n], \quad y[n] = 4(2)^n u[n] - \left(\frac{1}{2}\right)^n u[n]$

$$
\begin{aligned}
X(z) &= \frac{1}{1+3z^{-1}} \\
Y(z) &= \frac{3}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} \\
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{10}{1-2z^{-1}} + \frac{-7}{1-\frac{1}{2}z^{-1}} \\
h[n] &= \left[10(2)^n - 7(\frac{1}{2})^n\right]u[n]
\end{aligned}
$$

7.30. A system has impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$. Determine the input to the system if the output is given by

$$
H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}
$$

(a) $y[n] = 2\delta[n-4]$

$$
\begin{aligned}
Y(z) &= 2z^{-4} \\
X(z) &= \frac{Y(z)}{H(z)} \\
&= 2z^{-4} - z^{-5} \\
x[n] &= 2\delta[n-4] - \delta[n-5]
\end{aligned}
$$

(b) $y[n] = \frac{1}{3}u[n] + \frac{2}{3}\left(\frac{-1}{2}\right)^n u[n]$

$$
\begin{aligned}
Y(z) &= \frac{\frac{1}{3}}{1 - z^{-1}} + \frac{\frac{2}{3}}{1 + \frac{1}{2}z^{-1}} \\
X(z) &= \frac{Y(z)}{H(z)} \\
&= -\frac{1}{2} + \frac{\frac{1}{6}}{1 - z^{-1}} + \frac{\frac{4}{3}}{1 + \frac{1}{2}z^{-1}} \\
x[n] &= -\frac{1}{2}\delta[n] + \frac{1}{6}u[n] + \frac{4}{3}(-\frac{1}{2})^n u[n]
\end{aligned}
$$