AdaBoost for Learning Binary and Multiclass Discriminations

(set to the "music" of Perl scripts)

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CONTENTS

1. Why You Should Learn AdaBoost

- Despite all belief to the contrary, most research contributions are merely incremental.

- However, every once in a while someone does something that just takes your breath away.

- The AdaBoost algorithm for machine learning by Yoav Freund and Robert Schapire is one such contribution.

- Until AdaBoost, the conventional wisdom in pattern classification was that your chosen features needed to be as class discriminatory as possible.

- AdaBoost turned that conventional wisdom on its head.

- With AdaBoost, it is possible to use even weak features for creating a pattern classifier, assuming you have a sufficient number of such features and assuming you just want to carry out binary classifications (or multiclass classifications in the manner described in Section 11 of this tutorial).

2. The Conventional Wisdom

- Revisiting the conventional wisdom, if you want to predict the class label for a new data element, you undertake the steps described below:

- You first get hold of as much training data as you can.

- You come up with a decent number of features. You try to select each feature so that it can discriminate well between the classes. You believe that the greater the power of a feature to discriminate between the classes at hand, the better your classifier.

- If you have too many features and you are not sure which ones might work the best, you carry out a feature selection step through either PCA (Principal Components Analysis), LDA (Linear Discriminant Analysis), a combination of PCA and LDA, or a greedy algorithm like the one that starts by choosing the most class-discriminatory feature and then adds additional features, one feature at a time, on the basis of the class discriminations achieved by the features chosen so far, etc. [See my "Constructing Optimal Subspaces Tutorial" for further information regarding this issue.]

- Once you have specified your feature space, you can use one of several approaches for predicting the class label for a new data element:
  - If your feature space is sparsely populated with the training samples and you have multiple classes to deal with, you are not likely to do much better than a Nearest Neighbor (NN) classifier.

– For NN based classification, you calculate the distance from your new data element to each of the training samples and you give the new data point the class label that corresponds to the nearest training sample. [As a variation on this, you find the k nearest training-data neighbors for your new data element and the class label you give your new data element is a majority vote (or a weighted majority vote) from those k training samples. This is known as the k-NN algorithm. In such algorithms, the distance calculations can be speeded up by using a k-d tree to represent the training samples.]

– For another variation on the NN idea, you might get better results by using NN to the class means as calculated from the training data as opposed to the training samples directly.

– If you don't want to use NN and if you are trying to solve a binary classification problem for the case of two linearly separable classes, you could try using linear SVM (Support Vector Machine) for your classifier. This will give you a maximum-margin decision boundary between the two classes. Or, if your classes are not linearly separable, you could construct a nonlinear SVM that uses a "kernel trick" to project the training data into a higher-dimensional space where they become linearly separable.

  - If you have multiple classes and you are comfortable using parametric models for the class distributions, you should be able to use model-based similarity criterion to predict the class label for your new data element. If you can fit Gaussians to your training data, you could, for example, calculate Mahalanobis distance between your data element and the means for each of the Gaussians to figure out as to which Gaussian provides the best class label for the new data element.

- Regardless of how you carry out classification, all the approaches listed above have one thing in common: The better the features are at discriminating between the classes, the better the performance of the classification algorithm.

3. Introduction to AdaBoost

- AdaBoost stands for Adaptive Boosting. [Literally, boosting here means to aggregate a set of weak classifiers into a strong classifier.]

- This approach is founded on the notion of using a set of weak classifiers and pooling the classification results of such classifiers to produce a provably strong classifier.

- In the sequence of weak classifiers used, each classifier focuses its discriminatory firepower on the training samples misclassified by the previous weak classifier. You could say that by just focusing on the training data samples misclassified by the previous weak classifier, each weak classifier contributes its bit — the best it can — to improving the overall classification rate.

- The AdaBoost approach comes with the theoretical guarantee that as you bring in more and more weak classifiers, your final misclassification rate for the training data can be made arbitrarily small.

- The AdaBoost approach also comes with a bound on the generalization error. This classification error includes the testing data that was NOT used for training, but that is assumed to be derived from the same data source as the training data.

- The main reference for the AdaBoost algorithm is the original paper by Freund and Schapire: "*A Decision-Theoretic Generalization of On-Line Learning and an Application to Boosting*," Proc. of the 2nd European Conf. on Computational Learning Theory, 1995.

- AdaBoost has become even more famous after it was shown by Viola and Jones how the algorithm could be used to create face detectors with false positive rates as low as 10^{-6}.

- Any algorithm that detects a face by scanning an image with a moving window (of different sizes to account for the fact that the size of a blob that is a face is not known in advance) and applying a classification rule to the pixels in the window must possess an extremely low false-positive rate for the detector to be effective. False positive here means declaring a blob of non-face pixels as a face.

4. The Notation

- We represent our labeled training data by the set

$$\{(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_m, y_m)\}$$

  with $x_i \in \mathcal{X}$ and $y_i \in \{-1, 1\}$. The set $\mathcal{X}$ represents our training data and the set $\{-1, 1\}$ the two class labels for the data elements. Note that each training data sample x_i can be a point in a multidimensional feature space. Obviously, m is the total number of training samples.

- The good thing is that we can be relaxed about the design of the feature space in which the training samples $x_i \in \mathcal{X}$ reside. As long as a feature is relevant to the classification problem at hand, it should work as a dimension of the feature space.

- For the purpose of drawing samples for training a classifier, AdaBoost maintains a probability distribution over all the training samples. This distribution is modified iteratively with each application of a new weak classifier to the data. We'll denote this probability distribution by $D_t(x_i)$. The subscript t refers to the different iterations of the AdaBoost algorithm.

- Initially, the probability distribution is uniform over the training samples. That is, $D_0(x_i)$ is considered to be uniform over x_i.

- The weak classifier chosen at iteration t of the AdaBoost algorithm will be denoted h_t. And the class label predicted by this weak classifier for the training data element x_i denoted $h_t(x_i)$. By comparing $h_t(x_i)$ with y_i for $i = 1, 2, \ldots, m$, we can assess the classification error rate for the classifier h_t.

- We use ϵ_t to denote the misclassification rate for the weak classifier h_t. [At iteration t, each candidate weak classifier is trained using a subsampling of all of the training data as made possible by the probability distribution $D_t(x)$. (The higher the probability $D_t(x)$ for a given training sample x, the greater the chance that it will be chosen for training the candidate classifier $h(t)$.) An important issue related to the selection of h_t: From all the different possibilities for h_t, we choose the one that minimizes the misclassification rate ϵ_t, *as measured over* **all** *of the training data*, using a formula shown in the next section.]

- In most practical implementation of AdaBoost, a weak classifier consists of a single feature that is thresholded appropriately. [Consider a computer vision application of AdaBoost and let's say you have a total of $F = \{f_1, f_2, ..., f_{|F|}\}$ features for characterizing a blob of pixels. At each iteration t, for the weak classifier h_t, you find the feature f_t that results in the smallest ϵ_t. This is best done by constructing an ordered list of **all** the training samples with respect to each candidate f_t. For any given f_t, you step through all its discrete values, from the lowest to the highest, and, considering each such value as a possible decision threshold θ, you update two quantities $error\_rate\_type\_1$ and $error\_rate\_type\_2$. The first, $error\_rate\_type\_1$, measures the misclassification rate if you classify all training samples that lie above θ as being of class $+1$ and all training sample that lie below θ as being of class -1. The second, $error\_rate\_type\_2$, measures the misclassification rate when you reverse the sense in which you use the decision threshold. That is, for $error\_rate\_type\_2$, you measure the misclassification rate when all training samples above θ are labeled -1 and all those below as $+1$. Finally, for a given f_t, you choose the threshold θ which corresponds to the smallest of the $error\_rate\_type\_1$ and $error\_rate\_type\_2$ values. If $error\_rate\_type\_1$ yielded the best value θ_t for θ, you say you will consider f_t with positive polarity and decision threshold θ_t as a candidate for the weak classifier h_t. On the other hand, if $error\_rate\_type\_2$ yielded the best θ_t, you change the polarity to negative. This logic when considered over all possible features in F returns the triple (f_t, p_t, θ_t), where p_t represents the polarity, as the best weak classifier at iteration t. When stepping through the discrete values for a given feature f_t, there is a little bit more to the calculation of $error\_rate\_type\_1$ and $error\_rate\_type\_2$ than just updating the two types of counts of the misclassified training samples:**the contribution made**

by each training sample to these two error rates must be weighted by the probability $D_t(x)$ associated with that sample. This fact will become clearer in the next section. When single features are used as weak classifiers, they are sometimes referred to as **decision stumps**. Note that factoring $D_t(x)$ into the calculation of the two types of misclassification rates is how sample probabilities $D_t(x)$ affect the design of h_t for creating decision stumps. That is, for the case of decision stumps, you do not directly subsample all of your training data for designing h_t.]

- We use α_t to denote how much trust we can place in the weak classifier h_t. Obviously, the larger the value of ϵ_t for a classifier, the lower our trust must be. We use the following relationship between α_t and ϵ_t:

$$\alpha_t = \frac{1}{2} \ln \frac{1 - \epsilon_t}{\epsilon_t}$$

- As ϵ_t varies from 1 to 0, α_t will vary from $-\infty$ to $+\infty$. ϵ_t being close 1 means that the weak classifier fails almost completely on the overall training dataset. And ϵ_t being close to 0 means that your weak classifier is actually a powerful classifier. (A weak classifier whose ϵ_t is close to 0 would negate all the reasons for using AdaBoost.)

- We will use H to denote the final classifier. This classifier carries out a weighted aggregation of the classifications produced by the individual weak classifiers to predict the class label for a new data sample. The weighting used for each weak classifier will be proportional to the degree of trust we can place in that classifier. [Weighting the individual weak classifiers h_t with the trust factor α_t makes it possible for a single strong weak classifier to dominate over several not-so-strong weak classifiers.]

5. The AdaBoost Algorithm

- Let’s say we can conjure up T weak classifiers. These classifiers can be as simple as what you get when you subject each feature of a multi-dimensional feature space to a suitable threshold. [For illustration, let’s say you are using the value of “yellow-ness” to weakly classify fruit in a supermarket. Such a weak classifier could work as follows: if the value of yellow color is below a decision threshold d_{th}, you predict the fruit’s label to be apple, otherwise you predict it to be an orange. As long as such a weak classifier does better than a monkey (meaning that as along as its classification error rate is less than 50%, it’s a good enough weak classifier. (Should it happen that this decision rule gives an error rate exceeding 50%, you can flip its “polarity” to yield a classification rule with an error rate of less than 50%.]

- What the note in blue on the previous slide implies is that *any* feature, if at all it is relevant to the objects of interest, can be used as a weak classifier.

- For $t = 1, 2, \ldots, T$, we do the following:

  1. Using the training samples thrown up by the probability distribution $D_t(x_i)$, we construct a weak classifier that is denoted h_t. We will make sure that D_t is such that h_t specifically targets the training samples misclassified by the previous weak classifier.

  2. We apply h_t to *all* of our training data (even though the classifier was constructed using just those training samples that were thrown up by the probability distribution D_t). The classifier h_t represents a mapping $h_t : \mathcal{X} \rightarrow \{-1, 1\}$.

3. Using *all* of the training data, we then estimate the misclassification rate $Prob\{h_t(x_i) \neq y_i\}$ for the h_t classifier:

$$\epsilon_t = \frac{1}{2}\sum_{i=1}^{m} D_t(x_i) \cdot \left|h_t(x_i) - y_i\right|$$

Note how the misclassifications are weighted by the probabilities associated with the samples. Now read again the long comment on Slides 15 and 16. [**As stated on pages 15 and 16, if you are using single features as weak classifiers (which is normally the case in computer vision applications of AdaBoost), you are NOT likely to use the formula shown above for estimating the misclassification rate for a single weak classifier. Instead, you are more likely to use the logic presented on pages 15 and 16.**]

4. Next, we calculate the trust factor for h_t by

$$\alpha_t = \frac{1}{2}\ln\left(\frac{1-\epsilon_t}{\epsilon_t}\right)$$

5. Finally, we update the probability distribution over the training data for the next iteration:

$$D_{t+1}(x_i) \;=\; \frac{D_t(x_i)e^{-\alpha_t y_i h_t(x_i)}}{Z_t}$$

where the role of Z_t is to serve as a normalizer. That is, we set a value for Z_t so that $\sum_{i=1}^{m} D_{t+1}(x_i) \;=\; 1$, where m is the total number of training samples. This implies

$$Z_t \;=\; \sum_{i=1}^{m} D_t(i)e^{-\alpha_t h_t(x_i)y_i}$$

Note the highly intuitive manner in which $D_{t+1}(x_i)$ acquires smaller probabilities at those samples that were correctly classified by h_t. The product $y_i h_t(x_i)$ will always be positive for such samples.

6. We then go back to Step 1 for the next iteration.

- At the end of T iterations, we construct the final classifier H as follows:

$$H(x) \;=\; sign\left(\sum_{t=1}^{T}\alpha_t h_t(x)\right)$$

where x is the new data element whose class label we wish to predict on the strength of the information in the training data. If for a test data sample x, $H(x)$ turns out to be positive, the predicted class label for x is 1. Otherwise, it is -1. [The above formula for the final classifier makes sense only if you use 1 and -1 as class labels. Note, however, it is not uncommon for an AdaBoost implementation to use 1 and 0 as the two class labels, with 1 for the object you are looking for in an image and 0 for the background. When using 1 and 0 for class labels, the formula for $H(x)$ is likely to compare the summation $\sum_{t=1}^{T} \alpha_t h_t(x)$ against $\frac{1}{2}\sum_{t=1}^{T} \alpha_t$. Using 1 and 0 for the class labels makes it easier to trade off the true-positive rate for the false-positive rate for final classification — an important issue in designing object detectors in computer vision applications of AdaBoost. For such tradeoffs, you are likely to compare $\sum_{t=1}^{T} \alpha_t h_t(x)$ with a decision threshold whose value would depend on what your desired true-positive rate is for the final classifier. For such logic to make sense, you'd keep on adding weak classifiers until the false-positive rate falls below a user-specified threshold.]

- Schapire and Singer have shown that the training error of the final classifier is bounded by

$$\frac{1}{m}\left|\left\{i : \; H(x_i) \; \neq \; y_i\right\}\right| \quad \leq \quad \prod_{t=1}^{T} Z_t$$

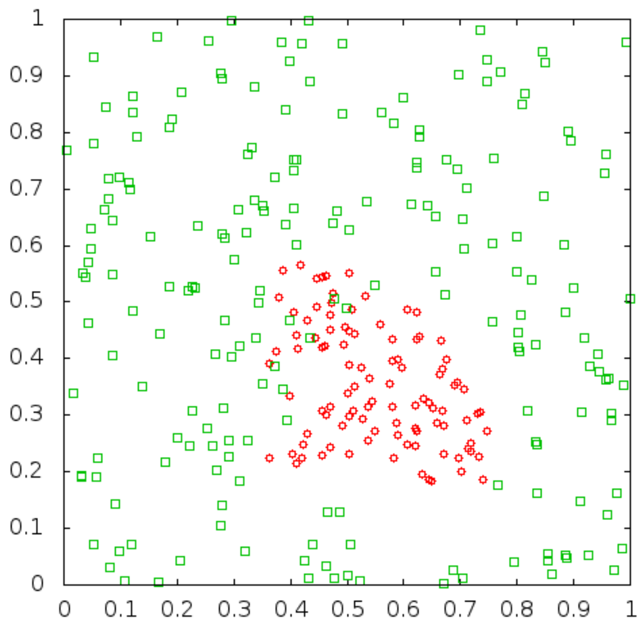
where Z_t is as shown on Slide 21.

- The bound shown above implies that for any value of T, which is the number of weak classifiers used, you want to use those that yield the smallest values for $\{Z_t | t = 1 \ldots T\}$. Generalizing this argument to treating T on a running basis, from amongst all the weak classifiers at your disposal, you want to pick that classifier for each iteration t that has the least misclassification rate.

- If you examine the formula for Z_t on Slide 21, the smaller the misclassification rate for a given weak classifier, the smaller the associated Z_t. Now you can see why in the comment block on Slide 15, at each iteration t, we wanted a feature for h_t that minimized ϵ_t as measured over all of the training data.

6. An Example for Illustrating AdaBoost

- We will use the two-class example shown below to illustrate AdaBoost.



- The red points were generated by the Perl script shown on the next slide. We will refer to the red points as the "red circles" or as just "circles".

- Here is the script for the red circle points:

```
sub gen_points_in_circle {
    my @points_in_circle;
    foreach my $i (0..$N1-1) {
        my ($x,$y) = (1,1);
        while ( ($x**2 + $y**2) >= 1 ) {
            $x = Math::Random::random_uniform();
            $y = Math::Random::random_uniform();
        }
        ($x,$y) = (0.4*($x-0.6), 0.4*($y-0.3));
        push @points_in_circle, ["circle_$i",$x+0.6,$y+0.3];
    }
    return \@points_in_circle;
}
```

where `$N1` specifies the number of points you want in a circle of radius ≈ 0.4 centered at (0.6,0.3).

- The green "square" points in the figure are generated by the following Perl script:

```
sub gen_points_in_rest_of_square {
    my @points_in_square;
    foreach my $i (0..$N2-1) {
        my ($x,$y) = (0.6,0.3);
        while ( (($x-0.6)**2 + ($y-0.3)**2) < 0.2**2 ) {
            $x = Math::Random::random_uniform();
            $y = Math::Random::random_uniform();
        }
        push @points_in_square, ["square_$i",$x,$y];
    }
    return \@points_in_square;
}
```

- In the second script on the previous slide, the variable `$N2` controls the number of green square points you want in the rest of the $[0, 1] \times [0, 1]$ box shown in the figure on Slide 25.

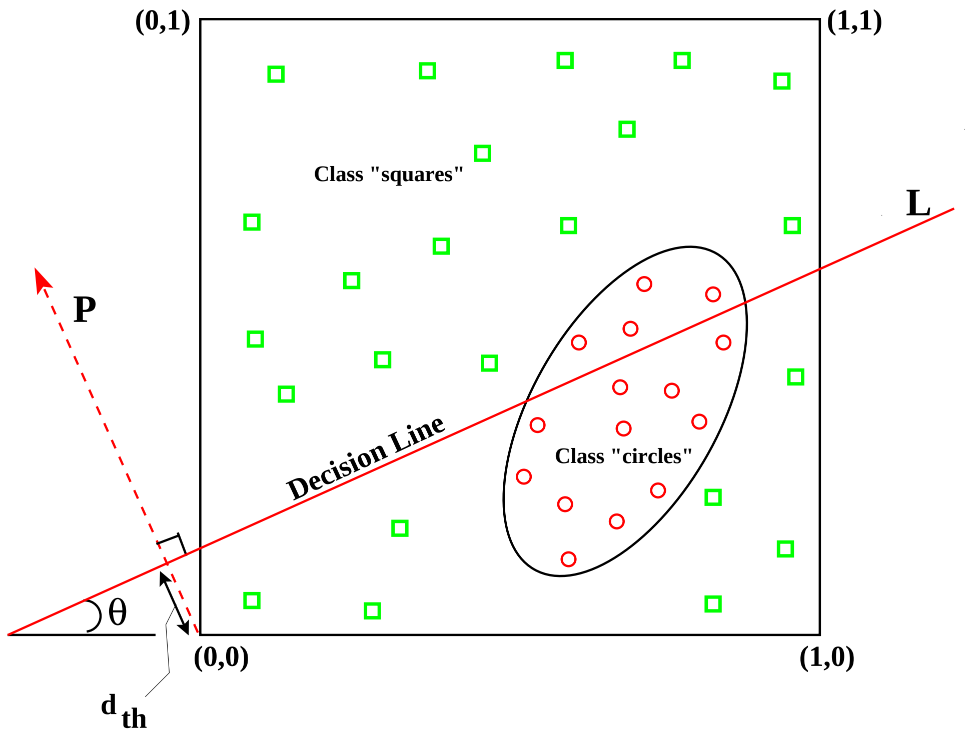
- Note that we make no attempt to generate the training data points according to any particular statistical distribution in the respective domains. For example, the red circle points are NOT uniformly distributed (despite the impression that may be created by calls to `Math::Random::random_uniform`) in the area where they predominate. Along the same lines, the green square points are NOT uniformly distributed in the rest of the $[0, 1] \times [0, 1]$ box. How the red and the green points are statistically distributed is NOT important to our illustration of AdaBoost.

- We can use any of a number of approaches for solving the classification problem depicted in the figure on Slide 25. For example, it would be trivial to construct an SVM classifier for this case. Simpler still, an NN classifier would in all likelihood work just as well (although it would be a bit slower for obvious reasons).

- The next section demonstrates how we can use AdaBoost to solve this problem.

7. Weak Classifiers in Our Example Study

- A weak classifier for our example will be a linear decision line as shown in the figure below. Such a classifier will declare all of the data points on one side of the line as being "red circles" and the data points on the other side as being "green squares".



- This sort of a classifier is characterized by the triple (θ, d_{th}, p), where θ is the orientation of the decision line L, and d_{th} the threshold on the projections of the data points on the perpendicular P to L (with the stipulation that the perpendicular pass through the origin). We say that the polarity p of the classifier is 1 if it classifies all of the data points whose projections are *less than or equal to* d_{th} as being "circle" and all the data points whose projections are *greater than* d_{th} as being "square". For the opposite case, we consider the classifier's polarity to be -1.

- Although, for a given orientation θ, the decision line L constitutes a weak classifier for almost every value of d_{th} (as long we choose a polarity that yields a classification error rate of less than 0.5), that will NOT be our approach to the construction of a set of weak classifiers.

- For the discussion that follows, for each orientation θ, we will choose that decision line L as our weak classifier which yields the smallest classification error.

- In our demonstration of AdaBoost, each iteration of the algorithm randomly chooses a value for the orientation θ of the decision line L. Subsequently we step along the perpendicular P to L to find the best value for the threshold d_{th} and the best polarity for the classifier. Best means yielding the least classification error.

- The script that is shown on Slides 30 and 31 demonstrates how we search for the best weak classifier for a given orientation θ for the decision line. In lines 7 and 8, the script first constructs a unit vector along the perpendicular P to the decision line L.

- At this point, it's good to recall that, in general, only a portion of the training data is used for the training of each weak classifier. See Slide 19.

- The training data for the current weak classifier is sorted in lines 10 through 21 in the ascending order of the data projections on P.

- Subsequently, in lines 25 through 36, the script steps through these projection points on P, from the smallest value to the largest, and at, each projection point on the perpendicular P, it calculates two types of classification errors that are described next. Let the projection point under consideration be denoted s.

- The first type of classification error corresponds to the case when the predicted labels for all the data points whose projections are less than or equal to s are considered to be "circle" and all the data points whose projections are greater than s as being "square".

- The second type of the classification error corresponds to the opposite case. That is, when all the data points whose projections are less than or equal to s are labeled "square", and all the data points whose projections are greater than s are labeled "circle".

- In the script, these two types of errors are stored in the hashes `%type_1_errors` and `%type_2_errors`.

- The polarity of the weak classifier chosen for a given θ is determined by which of the two minimum values for the two types of errors is the least. And d_{th} is the corresponding threshold.

- Lines 37 through 42 sort the two types of errors stored in the hashes `%type_1_errors` and `%type_2_errors` in order to determine which type of error is the least. The type that yields the smallest value determines the polarity of the weak classifier, as set in line 43.

- Subsequently, in lines 44 through 47, we find the decision threshold d_{th} that corresponds to the best weak classifier for the decision line orientation used.

```
sub find_best_weak_linear_classifier_at_given_orientation {
    my $orientation = shift;
    my $training_sample_array_ref = shift;
    my @training_samples = @{$training_sample_array_ref};
    my $polarity;
    my $PI = 3.14159265358979;
    my $orient_in_rad = $orientation * $PI / 180.0;
    my @projection_vec =  (-1.0 * sin($orient_in_rad),
                                        cos($orient_in_rad));
    my %projections;
    foreach my $label (@training_samples) {
        my $projection = $all_sample_labels_with_data{$label}->[0] *
                                             $projection_vec[0]    +
                         $all_sample_labels_with_data{$label}->[1] *
                                             $projection_vec[1];
        $projections{$label} = $projection;
    }
    # Create a sorted list of class labels along the perpendicular
    # line P with the sorting criterion being the location of the
    # projection point
    my @sorted_projections = sort {$projections{$a} <=>
                                $projections{$b}} keys %projections;
    my (%type_1_errors, %type_2_errors);
    my $how_many_circle_labels = 0;
    my $how_many_square_labels = 0;
    foreach my $i (0..@sorted_projections-1) {
        $how_many_circle_labels++ if $sorted_projections[$i] =~
                                                           /circle/;
        $how_many_square_labels++ if $sorted_projections[$i] =~
                                                           /square/;
        my $error1 = ($N1 - $how_many_circle_labels +
                     $how_many_square_labels) / (1.0 * ($N1 + $N2));
        $type_1_errors{$sorted_projections[$i]} = $error1;
        my $error2 = ($how_many_circle_labels + $N2 -
                     $how_many_square_labels) / (1.0 * ($N1 + $N2));
        $type_2_errors{$sorted_projections[$i]} = $error2;
    }
    my @sorted_type_1_errors = sort {$type_1_errors{$a} <=>
                            $type_1_errors{$b}} keys %type_1_errors;
    my @sorted_type_2_errors = sort {$type_2_errors{$a} <=>
                            $type_2_errors{$b}} keys %type_2_errors;
    my $least_type_1_error = $type_1_errors{$sorted_type_1_errors[0]};
    my $least_type_2_error = $type_2_errors{$sorted_type_2_errors[0]};
    $polarity = $least_type_1_error <= $least_type_2_error ? 1 : -1;
    my $error_for_polarity =$least_type_1_error <= $least_type_2_error ?
                              $least_type_1_error : $least_type_2_error;
    my $thresholding_label =$least_type_1_error <= $least_type_2_error ?
                    $sorted_type_1_errors[0] : $sorted_type_2_errors[0];
    return [$orientation, $projections{$thresholding_label}, $polarity,
                                                   $error_for_polarity];
}
```

- The next section presents the calling subroutine for the function shown above. That subroutine constitutes an implementation of the Steps 1 through 6 of Section 5.

8. An Implementation of the AdaBoost Algorithm

- We next present a Perl implementation of the Steps 1 through 6 of the AdaBoost algorithm as shown in Section 5.

- The script shown on Slides 35 and 36 starts out in lines 4 through 14 by selecting a subset of the training data on the basis of the current probability distribution over the data. The criterion used for choosing the training samples is very simple: We sort all the training samples in a descending order according to their probability values. We pick the top-ranked training samples by stepping through the sorted list until the accumulated probability mass is 0.5.

- A more sophisticated approach to the selection of training samples according to a given probability distribution over the data would consist of using an MCMC (Markov-Chain Monte-Carlo) sampler. [See Section 3.4 of my tutorial "Monte Carlo Integration in Bayesian Estimation" for an introduction to MCMC sampling.]

- For a Perl based implementation of MCMC sampling with the Metropolis-Hastings algorithm, see my "Random Point Generator" module available at:

  `http://search.cpan.org/~avikak/Algorithm-RandomPointGenerator-1.01/lib/Algorithm/RandomPointGenerator.pm`

  If copy-and-paste of this long URL is inconvenient, you can also access this module by Googling with a query string like "avi kak cpan random point generator".

- If you do not want to go the MCMC sampling route, but you would like to be a bit more sophisticated than the approach outlined in red in the second bullet on Slide 37, you could try the following method: You first create a fine grid in the $[0,1] \times [0,1]$ box, with its resolution set to the smallest of the intervals between the adjacent point coordinates along x and y. You would then allocate to each training sample a number of cells proportional to its probability. Subsequently, you would fire up a random-number generator for the two values needed for x and y. The two such random values obtained would determine the choice of the training sample for each such two calls to the random number generator.

- Going back to explaining the code on Slides 37 and 38, in lines 15 through 19, the script fires up the random number generator for an orientation for the weak classifier for the current iteration of the AdaBoost algorithm. It makes sure that the orientation selected is different from those used previously.

- The decision line orientation selected is stored in the array `@ORIENTATIONS_USED`.

- The decision-line orientation chosen and the training samples selected are shipped off in lines 21 through 24 to the subroutine `find_best_weak_linear_classifier_ at_given_orientation()` that was presented in the previous section.

- Next, in lines 33 through 40, the script applies the weak classifier returned by the call shown in the previous bullet to all of the training data in order to assess its classification error rate. This corresponds to Step 3 on Slide 20. This part of the code makes a call to the subroutine `weak_classify()` that is presented in the next section.

- Calculation of the classification error rate is followed in line 42 by a calculation of the trust factor α for the weak classifier in the current iteration of the AdaBoost algorithm.

- Subsequently, in lines 43 through 65, we update the probability distribution over all the training samples according to Step 5 on Slide 18.

- Starting on Slide 44, the next section presents the support routines needed by the Perl code discussed so far.

```
sub adaboost {
    my @weak_classifiers;
    my $decision_line_orientation = 0;
    foreach my $t (0..$NUMBER_OF_WEAK_CLASSIFIERS-1) {
        my @samples_to_be_used_for_training = ();
        # Select training samples for the current weak classifier:
        my $probability_mass_selected_samples = 0;
        foreach my $label (sort {$PROBABILITY_OVER_SAMPLES{$b} <=>
                        $PROBABILITY_OVER_SAMPLES{$a} }
                        keys %PROBABILITY_OVER_SAMPLES) {
            $probability_mass_selected_samples +=
                                    $PROBABILITY_OVER_SAMPLES{$label};
            last if $probability_mass_selected_samples > 0.5;
            push @samples_to_be_used_for_training, $label;
        }
        while (contained_in($decision_line_orientation,
                                               @ORIENTATIONS_USED)) {
            $decision_line_orientation = int(180 *
                                     Math::Random::random_uniform());
        }
        push @ORIENTATIONS_USED, $decision_line_orientation;
        my $learned_weak_classifier =
             find_best_weak_linear_classifier_at_given_orientation(
                                  $decision_line_orientation,
                                  \@samples_to_be_used_for_training);
        $LEARNED_WEAK_CLASSIFIERS{$t} = $learned_weak_classifier;
        my ($orientation, $threshold,
              $polarity, $error_over_training_samples) =
                                         @{$learned_weak_classifier};
        # Now find the overall classification error (meaning error
        # for all data points) for this weak classifier.  That will
        # allows us to calculate how much confidence we can place in
        # this weak classifier:
        my $error = 0;
        foreach my $label (@all_sample_labels) {
            my $data_point = $all_sample_labels_with_data{$label};
            my $predicted_label =  weak_classify($data_point,
                              $orientation, $threshold, $polarity);
            $error += $PROBABILITY_OVER_SAMPLES{$label}
                               unless $label =~ /$predicted_label/;
        }
        push @WEAK_CLASSIFIER_ERROR_RATES, $error;
        $ALL_ALPHAS[$t] =  0.5 * log((1 - $error) / $error);
        my %new_probability_over_samples;
        for my $label (keys %PROBABILITY_OVER_SAMPLES) {
            my $data_point_for_label =
                              $all_sample_labels_with_data{$label};
            my $predicted_label = weak_classify($data_point_for_label,
                              $orientation, $threshold, $polarity);
            my $exponent_for_prob_mod =$label =~ /$predicted_label/ ?
                                                          -1.0 : 1.0;
            $new_probability_over_samples{$label} =
                  $PROBABILITY_OVER_SAMPLES{$label} *
                       exp($exponent_for_prob_mod * $ALL_ALPHAS[$t]);
        }
        my @all_new_probabilities =
                                values %new_probability_over_samples;
        my $normalization = 0;
        foreach my $prob (@all_new_probabilities) {
            $normalization += $prob;
        }
        for my $label (keys %new_probability_over_samples) {
            $PROBABILITY_OVER_SAMPLES{$label} =
                    $new_probability_over_samples{$label}
                    / $normalization;
        }
    }
}
```

9. Some Support Routines

- Line 36 of the script shown in the previous section makes a call to the subroutine `weak_classify()` shown below:

```
sub weak_classify {
    my $data_point = shift;
    my $decision_line_orientation = shift;
    my $decision_threshold = shift;
    my $polarity = shift;
    my $PI = 3.14159265358979;
    my $orient_in_rad = $decision_line_orientation * $PI / 180.0;
    # The following defines the pass-through-origin perp to
    # the decision line:
    my @projection_vec =  (-1.0 * sin($orient_in_rad),
                                             cos($orient_in_rad));
    my $projection = $data_point->[0] * $projection_vec[0] +
                            $data_point->[1] * $projection_vec[1];
    return $projection <= $decision_threshold ?
                            "circle" : "square" if $polarity == 1;
    return $projection <= $decision_threshold ?
                           "square" : "circle" if $polarity == -1;
}
```

- The subroutine shown above takes four arguments that we describe in the next bullet.

- The first argument, stored in the local variable `$data_point` in line 1, is for the (x, y) coordinates of the point that needs to be classified. The second, stored in the local variable `$decision_line_orientation`, is for the orientation of the decision line for the weak classifier. The third, stored in `$decision_threshold`, is for supplying to the subroutine the value for d_{th}. And the last, stored in `$polarity`, is for the polarity of the weak classifier. The logic of how the weak classifier works should be obvious from the code in lines 9 through 16. We first construct a unit vector along the perpendicular to the decision lines 9 and 10. This is followed by projecting the training data points on the unit vector.

- Next let's consider the implementation of the final classifier $H(x)$ described on Slide 22. The subroutine `final_classify()` does this job and is presented on Slide 43.

- In the loop that starts in line 3 of the code shown on the next slide, we first extract in lines 4 and 5 one weak classifier at a time from all the weak classifiers stored in the hash `%LEARNED_WEAK_CLASSIFIERS`. In lines 6 and 7, we call on the `weak_classify()` presented on Slide 44 to classify our new data point.

- Subsequently, in lines 8 through 17, we format the numbers associated with the weak classifiers in order to produce an output that is easy to read and that allows for the different weak classifier performances to be compared easily. This formatting allows for the sort of a printout shown on Slide 54.

- We aggregate the individual weak classification results in lines 25 through 33 according to the formula shown on Slide 22.

```
sub final_classify {
    my $data_point = shift;
    my @classification_results;
    foreach my $t (0..$NUMBER_OF_WEAK_CLASSIFIERS-1) {
        my ($orientation, $threshold, $polarity, $error) =
                               @{$LEARNED_WEAK_CLASSIFIERS{$t}};
        my $result = weak_classify($data_point, $orientation,
                                         $threshold, $polarity);
        my $error_rate = $WEAK_CLASSIFIER_ERROR_RATES[$t];
        $error_rate =~ s/^(0\.\d\d)\d+$/$1/;
        $threshold =~ s/^(-?\d?\.\d\d)\d+$/$1/;
        $threshold = $threshold < 0 ? "$threshold" : " $threshold";
        if (length($orientation) == 1) {
            $orientation = "  $orientation";
        } elsif (length($orientation) == 2) {
            $orientation = " $orientation";
        }
        $polarity = $polarity > 0 ? " $polarity" : $polarity;
        print "Weak classifier $t (orientation: $orientation  " .
              "threshold: $threshold  polarity: $polarity  " .
              "error_rate: $error_rate):   $result\n";
        push @classification_results, $result;
    }
    #For weighted pooling of the results returned by the different
    #classifiers, treat "circle" as +1 and "square" as -1
    @classification_results = map {s/circle/1/;$_}
                                  @classification_results;
    @classification_results = map {s/square/-1/;$_}
                                  @classification_results;
    print "classifications: @classification_results\n";
    my $aggregate = 0;
    foreach my $i (0..@classification_results-1) {
        $aggregate += $classification_results[$i] * $ALL_ALPHAS[$i];
    }
    return $aggregate >= 0 ? "circle" : "square";
}
```

10. Using the Demonstration Code in an Interactive Session

- All of the code shown so far is in the script file `AdaBoost.pl` that you will find in the gzipped tar archive available from the URL:

https://engineering.purdue.edu/kak/distAdaBoost/AdaBoostScripts.tar.gz

- After you have unzipped and untarred the archive, executing the script `AdaBoost.pl` will place you in an interactive session in which you will be asked to enter the (x, y) coordinates of a point to classify in the $[0, 1] \times [0, 1]$ box. The script will then provide you with the "circle" versus "square" classification for the point you entered. Before going into the details of this interactive session, let's first see what all is done by the `AdaBoost.pl` script.

- The script **AdaBoost.pl** needs values for the following three user-defined global variables. The values currently set are shown below. But, obviously, you can change them as you wish.

```
# USER SPECIFIED GLOBAL VARIABLES:
my $N1 = 100;    # These are the "circle" points in Slide 25
my $N2 = 100;    # Points outside the circle but inside the
                 #  rest of the [0,1]x[0,1] box.  We refer to
                 #  to these points as "square" in Slide 25
my $NUMBER_OF_WEAK_CLASSIFIERS = 10;
```

- With regard to what it accomplishes, the **AdaBoost.pl** script first calls the two functions shown on Slide 26 for generating the number of points specified through the global variables `$N1` and `$N2`.

- Subsequently, it initializes the probability distribution over the training samples, as mentioned by the second bullet on Slide 13.

- Finally, `AdaBoost.pl` calls the following subroutines:

```
display_points_in_each_class( $outputstream );
visualize_data();
adaboost( $outputstream );
interactive_demo();
```

  where the first two calls are for the visualization of the training data. The call to `adaboost()` accomplishes what was explained in Section 8 of this tutorial. The last call above, `interactive_demo()` places the script in an interactive mode in which the user is prompted for the points (x, y) to classify and the script returns the final classifications for the points, while also displaying the results produced by each weak classifier.

- Note the argument `$outputstream` in the calls to `display_points_in_each_class()` and `adaboost()`. The role played by this argument is explained on the next slide.

- The value of the variable `$outputstream` is set at the very beginning when you fire up the script `AdaBoost.pl` depending on how you respond to the two prompts generated by the script. These prompts ask you whether you want the information related to the construction of the weak classifiers to be dumped in a file or to be displayed in your terminal window.

- When you are first becoming familiar with this Perl code, I recommend that you answer 'no' to the two prompts. With that answer, no information related to the weak classifiers will be put out.

- Shown on the next slide is the code for `interactive_demo()`:

```
sub interactive_demo {
1   # The following regex is from perl docs:
2   my $_num_regex = '^[+-]?\ *(\d+(\.\d*)?|\.\d+)([eE][+-]?\d+)?$';
3   print "\n\nThis AdaBoost demonstration is based on the " .
4         "following randomly selected decision line orientations " .
5         "for the weak classifiers: @ORIENTATIONS_USED\n\n";
6   for (;;) {
7       print "\n\nEnter the coordinates of the point you wish to " .
8             "classify: ";
9       my $answer = <STDIN>;
10      die "End of interactive demonstration" unless $answer;
11      next if $answer =~ /^\s*$/;
12      my @nums = split / /, $answer;
13      unless (($nums[0] =~ /$_num_regex/) &&
14                                   ($nums[1] =~ /$_num_regex/)) {
15          print "You entered an illegal character.  Try again " .
16                "or enter Contrl-D to exit\n";
17          next;
18      }
19      my $predicted_class = final_classify(\@nums);
20      print "The predicted class for the data point: " .
21                                                "$predicted_class\n";
22  }
}
```

- The user-interactive script shown above starts out in lines 3 through 5 by printing out the decision line orientations selected for the weak classifiers.

- The rest of the script on the previous slide is an infinite loop in which the user is asked for the (x, y) coordinates of a point in the $[0,1] \times [0,1]$ box that needs to be classified. In lines 13 through 17, the script makes sure that the information entered by the user is indeed a pair of floating point numbers.

- Finally, in line 19, it calls the `final_classify()` function of the previous section to classify the point.

- A command-line invocation such as

```
AdaBoost.pl
```

  automatically places the script in the interactive mode.

- In the interactive mode, after displaying the training data through a plot such as the one shown on Slide 25, and the information related to each weak classifier constructed, the script prints out the following message for the user:

```
    Enter the coordinates of the point you wish to classify:
```

- Let's say you enter the following coordinates: 0.2 0.7. Subsequently, the script will print out something like the following

```
Weak classifier 0 (or:   0  thr:  0.51  p:  1  err: 0.21): square
Weak classifier 1 (or: 151  thr: -0.10  p: -1  err: 0.35): square
Weak classifier 2 (or:  90  thr: -0.04  p: -1  err: 0.51): square
Weak classifier 3 (or:  25  thr:  0.30  p: -1  err: 0.57): circle
Weak classifier 4 (or:   3  thr:  0.41  p: -1  err: 0.46): circle
Weak classifier 5 (or: 162  thr: -0.07  p: -1  err: 0.45): square
Weak classifier 6 (or:  41  thr:  0.23  p: -1  err: 0.56): circle
Weak classifier 7 (or: 117  thr: -0.07  p: -1  err: 0.43): square
Weak classifier 8 (or: 135  thr: -0.09  p: -1  err: 0.50): square
Weak classifier 9 (or: 148  thr: -0.09  p: -1  err: 0.49): square

classifications: -1 -1 -1 1 1 -1 1 -1 -1 -1

The predicted class for the data point:    square

Enter the coordinates of the point you wish to classify:
```

- In the display you see on the previous slide, I have abbreviated some of the labels the script prints out so as not to overflow the page boundaries. The 'or' label is actually printed out as 'orientation', the 'thr' label as 'threshold', 'p' as 'polarity', and, finally, the 'err' label as 'error_rate'.

- You can exit the interactive session by entering $\langle Ctrl - d \rangle$ in response to the prompt.

11. Introduction to Codeword-Based Learning for Solving Multiclass Problems

- The rest of this tutorial is concerned with using AdaBoost when you have more than two classes to deal with.

- A not-so-uncommon way to use a binary classifier (such as, say, SVM or AdaBoost) for solving a multiclass classification problem is to devise a set of binary classifiers, with each binary classifier comparing one class against all the others. So if your data is modeled by N_c classes, you would need N_c one-versus-the-rest binary classifiers for solving the problem.

- In this tutorial, we will use a different strategy for solving a multiclass classification problem with AdaBoost. This strategy is based on codeword based learning.

- So what's codeword based learning of multiclass discriminations?

- The next several bullets explain this new idea that was first introduced by Dietterich and Bakiri in their seminal paper "*Solving Multiclass Learning Problems via Error-Correcting Output Codes*," Journal of Artificial Intelligence Research, 1995. The example that I present to explain codeword based learning is drawn from the introduction to the paper by Dietterich and Bakiri

- Consider the problem of digit recognition in handwritten material. We want to assign a digit to one of ten classes. A structural approach to solving this problem consists of identifying the presence or absence of six features in each digit and basing the final classification on which features are found to be present and which ones to be absent.

- The six structural features are:

```
vl:  contains vertical line
hl:  contains horizontal line
dl:  contains diagonal line
cc:  contains close curve
ol:  contains curve open to the left
or:  contains curve open to the right
```

- The presence or the absence of these features for each of the 10 digit classes may now be indicated by the following codeword matrix:

Class	vl	hl	dl	cc	ol	or
0	0	0	0	1	0	0
1	1	0	0	0	0	0
2	0	1	1	0	1	0
3	0	0	0	0	1	0
4	1	1	0	0	0	0
5	1	1	0	0	1	0
6	0	0	1	1	0	1
7	0	0	1	0	0	0
8	0	0	0	1	0	0
9	0	0	1	1	0	0

Note that each row in the table shown above is distinct so that each digit has a unique codeword.

- Let's now assume that we only have binary classifiers at our disposal. How can we use such classifiers to solve the multiclass recognition problem presented on the previous slide?

- If all we have are binary classifiers, each column of the matrix presented on the previous slide can be learned by a binary classifier from all of the training data in the following manner: Let's say we have 1000 training samples of handwritten digits available to us, distributed uniformly with respect to all the ten digits. For the learning of the first column, we divide the set of training samples into two halves, one containing the digits labeled 1, 4, and 5, and the other containing the digits labeled 0, 2, 3, 6, 7, 8, and 9. We now use an SVM or AdaBoost to create a binary classifier for making a distinction between these TWO categories, the first corresponding to the digits 1, 4, and 5 and the second category corresponding to the rest. We can refer to this binary classifier as f_vl.

- In this manner, we construct six binary classifiers, one for each column of the codeword matrix on Slide 58. We may labels the six binary classifiers as

```
  f_vl   f_hl   f_dl    f_cc    f_ol      f_or
```

- Now when we want to predict the class label of a new digit, we apply each of these six binary classifiers to the pixels for the digit. If the output of each binary classifier is thresholded to result in 0,1 classification, the output produced by the six binary classifiers will be a six-bit codeword like 110001. We assign to the new digit the class label of the row of the codeword matrix which is at the shortest Hamming distance from the codeword extracted for the new digit.

- For example, if the six binary classifiers yielded the codeword 110001 for the new data, you would give it the class label 4 since, of all the codewords shown in the matrix above, the Hamming distance from 110001 is the shortest to the codeword 110000 that corresponds to the digit 4. Recall that the Hamming distance measures the number of bit positions in which two codewords differ.

- Does the basic idea of codeword based learning as presented above suffer from any shortcomings? We examine this issue next.

- If the six features that correspond to the six columns of the codeword matrix shown above could be measured precisely, then the basic approach outlined above should work perfectly.

- In reality, unfortunately, there will always be errors associated with the extraction of those features from a blob of pixels. Yes, the basic approach does give us a little bit of wiggle room for dealing with such errors since we only use 10 out of 32 ($= 2^6$) different possible codewords and since Hamming distance is used to map the measured codeword to the nearest codeword in the matrix for classification. However, the degree of protection against errors is rather limited.

- If d is the shortest Hamming distance between any pair of the codewords in a codeword matrix, the basic approach protects us against making at most $\lfloor \frac{d-1}{2} \rfloor$ errors in the extraction of the codeword for a new object to be classified.

- For the codeword matrix shown on Slide 58, the smallest distance is only 1 between the codewords for the digit 4 and 5. The distance is also just 1 for the digit 7 and 8 and for 8 and 9. What that means is that the codeword matrix of Slide 58 cannot tolerate any errors at all in the output of the six binary classifiers. [Even if we did not use a set of binary classifiers to learn that codeword matrix and relied on directly extracting the six structural features listed at the top of Slide 58, representing the classes in the manner shown in the matrix of Slide 58 does not allow for any errors in the extraction of the six features.]

- Let's now see how the basic idea of a codeword matrix can be generalized to give us greater protection against measurement errors.

- The idea that is used for the generalization we need is based on what's known as error correction coding (ECC) that's used for reliable communications over noisy channels and for increasing the reliability of data storage in modern computers.

- In the ECC based approach to codeword matrix design, we are allowed to assign codewords of arbitrary length (subject to the constraint that for k classes, you will have a maximum of 2^k columns in the codeword matrix if you want the columns to be distinct) to the classes.

- For the 10 classes of digit recognition, here is an example drawn from the paper by Dietterich and Bakiri in which we assign 15-bit codewords to the classes:

Class	f0	f1	f2	f3	f4	f5	f6	f7	f8	f9	f10	f11	f12	f13	f14
0	1	1	0	0	0	0	1	0	1	0	0	1	1	0	1
1	0	0	1	1	1	1	0	1	0	1	1	0	0	1	0
2	1	0	0	1	0	0	0	1	1	1	1	0	1	0	1
3	0	0	1	1	0	1	1	1	0	0	0	0	1	0	1
4	1	1	1	0	1	0	1	1	0	0	1	0	0	0	1
5	0	1	0	0	1	1	0	1	1	1	0	0	0	0	1
6	1	0	1	1	1	0	0	0	0	1	0	1	0	0	1
7	0	0	0	1	1	1	1	0	1	0	1	1	0	0	1
8	1	1	0	1	0	1	1	0	0	1	0	0	0	1	1
9	0	1	1	1	0	0	0	0	1	0	1	0	0	1	1

- Note that the columns in the codeword matrix shown above carry no meaning — unlike the columns in the codeword matrix of Slide 58 where each column stood for a visual feature.

- So you can think of the codewords shown above as having been assigned more or less arbitrarily to the 10 different classes of interest to us. [As we will soon see, there do exist certain constraints on how the different codewords are laid out. Nonetheless, these constraints do not arise from a need to make the columns of the matrix meaningful in the sense they are on Slide 58.]

- One thing we can be certain about is that with so many more codewords possible with the 15 bits we now use ($2^{15} = 32768$), we can make sure that we have a large minimum Hamming distance between any pair of codewords in the matrix.

- As before, we can use either SVM or AdaBoost classifier for the learning required for each column of the codeword matrix of the previous slide. [Let's denote the binary classifier for learning the first column by f_0. We can acquire f_0 by dividing our training samples into two categories, one with the training samples for the classes $\{0, 2, 4, 6, 8\}$, and the other for the classes $\{1, 3, 5, 7, 9\}$. The purpose of f_0 will be to discriminate between these two categories. Similarly, to learn f_1, the binary classifier for the second column of the codeword matrix, we would need to divide the training data between the positive examples consisting of the samples for the classes $\{0, 4, 5, 8, 9\}$ and the negative examples consisting of the samples for the classes $\{1, 2, 3, 6, 7\}$. And so on.]

- Dietterich and Bakiri list the following two criteria for choosing the codewords for the different classes for solving the multiclass problem:

  1. Row Separation: The minimum Hamming distance between any two rows of the codeword matrix should be as large as possible.

  2. Column Separation: The minimum Hamming distance between any two columns should be large. The minimum Hamming distance between any column and the complement of every other column should also be large.

- The first criterion follows directly from how the "decoding" step — meaning predicting a class label for a binary code word as extracted from a test sample — is supposed to work (see the last bullet on Slide 60).

- Regarding the second criterion on the previous slide, it is based on the fact that the logic of error correction coding (assigning to a codeword extracted for a new object the class label for the Hamming-nearest codeword in the codeword matrix) works only when the bit-wise errors committed for the different columns of the codeword matrix are uncorrelated.

- If two columns are nearly the same, or if one column is close to being the same as the complement of another column, the bit-wise errors for two different positions in the codeword will be correlated (in the sense that one error will be predictable from the other).

- As I said earlier, if you have k classes, the largest number of columns you can have in your codeword matrix is 2^k. To all these columns, you must apply the Column Separation criterion mentioned on the previous slide.

- The Column Separation criterion makes it difficult to create codeword matrices for cases when the number of classes is less than 5. [For example, when you have only 3 classes, you have $k = 3$. In this case, you can have a maximum of 8 $(= 2^3)$ columns. If you eliminate from these the all-zeros and the all-ones columns, you are left with only 6 columns. Now if you eliminate the complements of the columns, you will be left with only 3 columns for the three classes. That does not give you much error protection in a codeword based approach to multiclass learning.]

- The smallest number of classes in which the codeword based approach to learning works is $k = 5$. Dietterich and Bakiri have provided the following codeword matrix for this number of classes:

Class	f0	f1	f2	f3	f4	f5	f6	f7	f8	f9	f10	f11	f12	f13	f14
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
3	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1
4	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
5	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

- The above codeword matrix for a 5-class problem was obtained by exhaustively searching through the different possible 15-bit codewords for the best set that satisfied the two criteria on Slide 67.

- In the next section, we will use the codeword matrix shown above to solve a contrived 5-class problem.

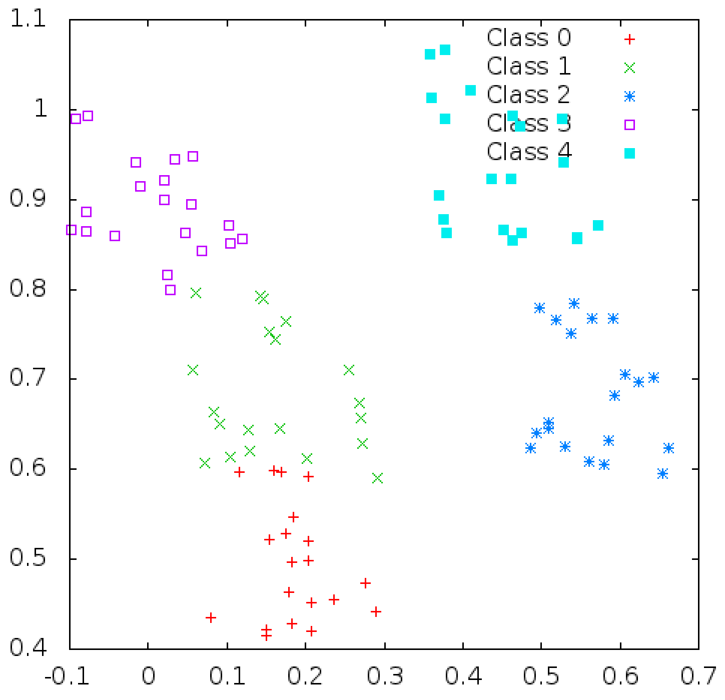
12. AdaBoost for Codeword-Based Learning of Multiclass Discriminations

- In the code that you can download from the URL:

`https://engineering.purdue.edu/kak/distAdaBoost/AdaBoostScripts.tar.gz`

  in addition to the file `AdaBoost.pl` that I have already talked about, you will also find another file named `MulticlassAdaBoost.pl` that is meant for showing how AdaBoost can be used to learn multiclass discriminations. [By "multiclass", I mean more than two classes.]

- Multiclass application of AdaBoost will be demonstrated on the sort of randomly generated class distributions shown in the figure on the next slide.



- When I say that the five distributions in the above figure are "randomly generated," what I mean is that when you work with the code in an interactive mode, for each interactive session you will see a different distribution for the five classes in the figure. Of course, for each session you can feed in as many *test* data points as you wish for classification and all of those classifications will be carried out with the same training data.

- Since you already know how AdaBoost works, I'll present this part of the tutorial in a top-down fashion with regard to the content of the file `MulticlassAdaBoost.pl`.

- The file `MulticlassAdaBoost.pl` starts out with the declarations:

```
my $NUMBER_OF_CLASSES = 5;
my $N = 20;
my $NUMBER_OF_WEAK_CLASSIFIERS = 10;
my %CODEWORD_MATRIX;
$CODEWORD_MATRIX{0} = [ qw/ 1 1 1 1 1 1 1 1 1 1 1 1 1 1 / ];
$CODEWORD_MATRIX{1} = [ qw/ 0 0 0 0 0 0 0 1 1 1 1 1 1 1 / ];
$CODEWORD_MATRIX{2} = [ qw/ 0 0 0 1 1 1 1 0 0 0 0 1 1 1 / ];
$CODEWORD_MATRIX{3} = [ qw/ 0 0 1 1 0 0 1 1 0 0 1 1 0 1 / ];
$CODEWORD_MATRIX{4} = [ qw/ 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 / ];
```

where the variable `$N` holds the number of data samples per class, and the variable `$NUMBER_OF_WEAK_CLASSIFIERS` specifies how many weak classifiers to construct for each column of the codeword matrix. The codeword matrix shown above is the one recommended by Dietterich and Bakiri.

- Next, the file `MulticlassAdaBoost.pl` declares the following global variables:

```
my %TRAINING_DATA;
my %ALL_SAMPLE_LABELS_WITH_DATA;
my %COLUMNS;
my $NUMBER_OF_COLUMNS;
my %POSITIVE_CLASSES_FOR_ALL_COLUMNS;
my %NEGATIVE_CLASSES_FOR_ALL_COLUMNS;
my %ORIENTATIONS_USED_FOR_ALL_COLUMNS;
my %WEAK_CLASSIFIERS_FOR_ALL_COLUMNS;
my %WEAK_CLASSIFIER_ERROR_RATES_FOR_ALL_COLUMNS;
my %ALPHAS_FOR_ALL_COLUMNS;
```

Several of these variables, whose names should convey the purpose they serve, are meant for convenience. [While all of the training data for all the classes is held in the hash `%TRAINING_DATA` where the keys are the class indexes, we also store the same data in the hash `%ALL_SAMPLE_LABELS_WITH_DATA` where the keys are the "class-x-sample-i" tags associated with the different data points. The hash `%COLUMNS` holds the individual columns of the `%CODEWORD_MATRIX` hash introduced earlier. The keys of the `%COLUMNS` hash are the column indexes and the values the columns of the codeword matrix. The variable `$NUMBER_OF_COLUMNS` is set to the number of columns of the codeword matrix.]

- The variables `%POSITIVE_CLASSES_FOR_ALL_COLUMNS` and `%CLASSES_CLASSES_FOR_ALL_COLUMNS` shown on the previous slide store for each column the positive classes and negative classes, respectively. [To explain what is meant by positive and negative classes for a column, let's look at the column indexed 0 in the codeword matrix shown on Slide 70. For this column, there exists just a single positive class, which is the class indexed 0. On the other hand, this column has 4 negative classes, these being classes index 1, 2, 3, 4. Along the same lines, for the column indexed 1 on Slide 70, the positive classes are indexed 0 and 4, and the negative classes indexed 1, 2, and 3.] The positive and the negative classes for the different columns of the codeword matrix are set by:

```
sub set_positive_and_negative_classes_for_columns {
    $NUMBER_OF_COLUMNS = @{$CODEWORD_MATRIX{0}};
    foreach my $col_index (0..$NUMBER_OF_COLUMNS-1) {
        my @column_bits = @{$COLUMNS{$col_index}};
        my (@positive_classes, @negative_classes);
        foreach my $bit_index (0..@column_bits-1) {
            push @{$POSITIVE_CLASSES_FOR_ALL_COLUMNS{$col_index}},
                           $bit_index if $column_bits[$bit_index];
            push @{$NEGATIVE_CLASSES_FOR_ALL_COLUMNS{$col_index}},
                       $bit_index unless $column_bits[$bit_index];
        }
    }
}
```

- Starting with Slide 79, we show the workhorse of `MulticlassBadaBoost.pl` script. It's this script's job to create all of the weak classifiers for the binary classification for the positive and the negative examples designated by the 1's and the 0's of a given column of the codeword matrix. The index of the column that this script is supposed to work on is supplied as its argument. In the script itself, the column index becomes the value of the local variable `$col_index` in line 1.

- Lines 2 through 14 of the script on Slide 79 are primarily for initializing the probability distribution of the training samples in accordance with the second bullet on Slide 13.

- The search for the weak classifiers begins in line 20. In lines 21 through 31, we use the current probability distribution over the training samples to choose a subset of the training data for the next weak classifier.

- The logic used in lines 32 through 63 for the construction of a weak classifier is exactly the same as described in Section 7 of this tutorial. [For a weak classifier, we randomly choose an orientation for the decision line, an example of which was shown in the figure on Slide 29. We then move this decision along its perpendicular until we find a position where the classification error rate over the training samples being used is the least. When we find the best position of the decision line, we take into account both polarities for the classification rule expressed by the decision line. The best location of the line on the perpendicular gives us the decision threshold d_{th} to use for this weak classifier.]

- In lines 65 through 70, we set the value of α_t as required by Step 4 on Slide 20. Note that the implementation you see for the calculation of α_t differs from its analytical form on Slide 20. The reason is that the analytical form shown on Slide 20 becomes problematic when the error rate ϵ_t is zero. As shown in line 69, when the error rate is less than 0.00001, we clamp α_t at a high value of 5.

- Lines 71 through 103 are devoted to the calculation of the new probability distribution over the training samples according to the classification errors made by the weak classifier just constructed. This calculation uses the formula shown in Step 5 on Slide 20.

```
sub generate_weak_classifiers_for_one_column_of_codeword_matrix {
    my $col_index = shift;
    my @positive_classes =
                  @{$POSITIVE_CLASSES_FOR_ALL_COLUMNS{$col_index}};
    my @negative_classes =
                  @{$NEGATIVE_CLASSES_FOR_ALL_COLUMNS{$col_index}};
    my $N_positives = 0;
    foreach my $class_index (@positive_classes) {
        $N_positives += @{$TRAINING_DATA{$class_index}};
    }
    my $N_negatives = 0;
    foreach my $class_index (@negative_classes) {
        $N_negatives += @{$TRAINING_DATA{$class_index}};
    }
    my $N_total = $N_positives + $N_negatives;
    my %probability_over_samples;
    foreach my $label (keys %ALL_SAMPLE_LABELS_WITH_DATA) {
        $probability_over_samples{$label} = 1.0 / $N_total;
    }
    my $decision_line_orientation = 0;
    foreach my $t (0..$NUMBER_OF_WEAK_CLASSIFIERS-1) {
        my @samples_to_be_used_for_training = ();
        # Select training samples for the current weak classifier:
        my $probability_mass_selected_samples = 0;
        foreach my $label (sort {$probability_over_samples{$b} <=>
                 $probability_over_samples{$a} }
                                keys %probability_over_samples) {
            $probability_mass_selected_samples +=
                               $probability_over_samples{$label};
            last if $probability_mass_selected_samples > 0.8;
            push @samples_to_be_used_for_training, $label;
        }
        while (contained_in($decision_line_orientation,
             @{$ORIENTATIONS_USED_FOR_ALL_COLUMNS{$col_index}})) {
            $decision_line_orientation =
                         int(180 * Math::Random::random_uniform());
        }
          push @{$ORIENTATIONS_USED_FOR_ALL_COLUMNS{$col_index}},
                                        $decision_line_orientation;
        my $learned_weak_classifier =
            find_best_weak_linear_classifier_at_given_orientation(
                 $decision_line_orientation, \@positive_classes,
                                              \@negative_classes );
        push @{$WEAK_CLASSIFIERS_FOR_ALL_COLUMNS{$col_index}},
                                           $learned_weak_classifier;
        my ($orientation, $threshold, $polarity,
            $error_over_training_samples) = @{$learned_weak_classifier};
        my $error = 0;
        foreach my $label (keys %ALL_SAMPLE_LABELS_WITH_DATA) {
            my $data_point = $ALL_SAMPLE_LABELS_WITH_DATA{$label};
            my $predicted_label = weak_classify($data_point,
                                   $orientation, $threshold, $polarity);
            $label =~ /class_(\d+)_sample/;
            my $class_index_for_label = $1;
            next if (contained_in($class_index_for_label,
                                                @positive_classes) &&
                                 ($predicted_label eq 'positive')) ||
                    (contained_in($class_index_for_label,
                                                @negative_classes) &&
                                    ($predicted_label eq 'negative'));

            $error += $probability_over_samples{$label};
        }
        push @{$WEAK_CLASSIFIER_ERROR_RATES_FOR_ALL_COLUMNS{$col_index}},
                                                                 $error;
        if ($error > 0.00001) {
            $ALPHAS_FOR_ALL_COLUMNS{$col_index}->[$t] =
                                        0.5 * log((1 - $error) / $error);
        } else {
            $ALPHAS_FOR_ALL_COLUMNS{$col_index}->[$t] = 5;
        }
        my %new_probability_over_samples;
```

```
        for my $label (keys %probability_over_samples) {
            my $data_point_for_label =
                                 $ALL_SAMPLE_LABELS_WITH_DATA{$label};
            my $predicted_label = weak_classify($data_point_for_label,
                                 $orientation, $threshold, $polarity);
            my $exponent_for_prob_mod;
            $label =~ /class_(\d+)_sample/;
            my $class_index_for_label = $1;
            if ( (contained_in($class_index_for_label,
                                                @positive_classes) &&
                                 ($predicted_label eq 'positive')) ||
                 (contained_in($class_index_for_label,
                                                @negative_classes) &&
                                ($predicted_label eq 'negative')) ) {
                $exponent_for_prob_mod = -1.0;
            } else {
                $exponent_for_prob_mod = 1.0;
            }
            $new_probability_over_samples{$label} =
                                  $probability_over_samples{$label} *
                                  exp($exponent_for_prob_mod *
                           $ALPHAS_FOR_ALL_COLUMNS{$col_index}->[$t]);
        }
        my @all_new_probabilities = values %new_probability_over_samples;
        my $normalization = 0;
        foreach my $prob (@all_new_probabilities) {
            $normalization += $prob;
        }
        for my $label (keys %new_probability_over_samples) {
            $probability_over_samples{$label} =
                 $new_probability_over_samples{$label} / $normalization;
        }
    }
}
```

- Another subroutine that plays a key role in the codeword based multiclass discriminations is `final_classify_for_one_column_of_codeword_matrix()` whose job is to aggregate the classifications returned by each of the weak classifiers for a given column and return the final classifications of a new data point. Shown on the next slide is the implementation of this subroutine. [This subroutine takes two arguments, the column index and the data point you want to classify. These become the values of local variables in lines 1 and 2 on the next slide.]

- After fetching the weak classifiers for the designated column in lines 3 and 4, the loop in lines 8 through 26 queries each of the weak classifiers and formats their output for the presentation of the results in lines 25 through 28.

- For aggregating the results returned by the individual weak classifiers, we use the formula on Slide 22. This aggregation is implemented in lines 33 through 44.

- Note that the call to `weak_classify()` in line 11 has the same implementation as shown previously on Slide 43.

```
sub final_classify_for_one_column_of_codeword_matrix {
1   my $col_index = shift;
2   my $data_point = shift;
3   my $number_of_weak_classifiers_for_this_column =
4       scalar( @{$WEAK_CLASSIFIERS_FOR_ALL_COLUMNS{$col_index}} );
5   my @classification_results;
6   my @weak_classifiers_for_this_column =
7                 @{$WEAK_CLASSIFIERS_FOR_ALL_COLUMNS{$col_index}};
8   foreach my $t (0..$number_of_weak_classifiers_for_this_column-1) {
9       my ($orientation, $threshold, $polarity, $error) =
10                           @{$weak_classifiers_for_this_column[$t]};
11      my $result = weak_classify($data_point, $orientation,
12                                                 $threshold, $polarity);
13      my $error_rate =
14       $WEAK_CLASSIFIER_ERROR_RATES_FOR_ALL_COLUMNS{$col_index}->[$t];
```

```
        $error_rate =~ s/^(0\.\d\d)\d+$/$1/;
        $error_rate = "   $error_rate" if length($error_rate) == 1;
        $threshold =~ s/^(-?\d?\.\d\d)\d+$/$1/;
        $threshold = $threshold < 0 ? "$threshold" : " $threshold";
        if (length($orientation) == 1) {
            $orientation = "  $orientation";
        } elsif (length($orientation) == 2) {
            $orientation = " $orientation";
        }
        $polarity = $polarity > 0 ? " $polarity" : $polarity;
        print "[Column $col_index] Weak classifier $t " .
              "(orientation: $orientation  threshold: $threshold  " .
              "polarity: $polarity  error_rate: $error_rate):   "   .
              "$result\n";
        push @classification_results, $result;
    }
    #For weighted pooling of the results returned by the different
    #classifiers, treat "positive" as +1 and "negative" as -1
    @classification_results =
                       map {s/positive/1/;$_} @classification_results;
    @classification_results =
                       map {s/negative/-1/;$_} @classification_results;
    print "[Column $col_index] weak classifications:
                                            @classification_results\n";
    my $aggregate = 0;
    foreach my $i (0..@classification_results-1) {
        $aggregate += $classification_results[$i] *
                             $ALPHAS_FOR_ALL_COLUMNS{$col_index}->[$i];
    }
    return $aggregate >= 0 ? "positive" : "negative";
}
```

13. An Interactive Session That Shows the Power of a Single Column of Codeword Matrix as a Binary Classifier

- When you first fire up `MulticlassAdaBoost.pl`, it will prompt you for whether you want the multiclass classifier as learned from the entire codeword matrix, or a single binary classifier for a single column of the matrix. If you opt for the latter, the script will enter the column-interactive mode and it will ask you to enter the column index for the column you are interested in.

- When you choose the single-column option in your response to the prompts, the script invokes the `overall_interactive()` subroutine.

- Shown on Slide 87 is the implementation of the subroutine `column_interactive()`.

- The interactive part of the script is the infinite loop in lines 12 through 36. The code in lines 13 through 26 first prompts the user for data entry and then checks that what the user entered makes sense. Line 17 is to allow for a user to terminate the interactive session at any time by entering $\langle Ctrl - d \rangle$.

- The result returned by calling `final_classify_for_one_column_of_codeword_matrix()` in line 27 is either 'positive' or 'negative'. The former means that the data point supplied by the user belongs to one the positive classes for this class as fetched in line 2 of the script. And the latter means that the result belongs to one of the negative classes fetched in line 4.

```
sub column_interactive {
    my $col_index = shift;
    my @positive_classes =
                  @{$POSITIVE_CLASSES_FOR_ALL_COLUMNS{$col_index}};
    my @negative_classes =
                  @{$NEGATIVE_CLASSES_FOR_ALL_COLUMNS{$col_index}};
    generate_weak_classifiers_for_one_column_of_codeword_matrix(
                                                       $col_index);
    print "\n\nThis column-specific demonstration is based on " .
          "the following randomly selected decision line " .
          "orientations for the weak classifiers: " .
          " @{$ORIENTATIONS_USED_FOR_ALL_COLUMNS{$col_index}}\n\n’’;
    for (;;) {
        print "\n\nEnter the coordinates of the point you wish to " .
              "classify: ‘‘;
        my $answer = <STDIN>;
        chomp $answer;
        die ‘‘End of interactive demonstration’’ unless $answer;
        next if $answer =~ /^\s*$/;
        my @nums = split / /, $answer;
        unless (($nums[0] =~ /$_num_regex/) &&
                                  ($nums[1] =~ /$_num_regex/)) {
            print "You entered an illegal character.  " .
                  "Try again or enter Contrl-D to exit\n’’;
            next;
        }
        my $predicted_class =
           final_classify_for_one_column_of_codeword_matrix($col_index,
                                                            \@nums);
        if ($predicted_class eq ’positive’) {
            print "The data point (@nums) is predicted to be in " .
                  "one of these classes: @positive_classes\n’’;
        } elsif ($predicted_class eq ’negative’) {
            print "The data point (@nums) is predicted to be in " .
                  "one of these classes: @negative_classes\n’’;
        }
    }
}
```

- When you invoke `MulticlassAdaBoost.pl` in the column-interactive mode, you will see a popup window with a plot such as the one shown on Slide 72 and the following sort of output in your terminal window:

```
=> MulticlassAdaBoost.pl
Prining out the points in class 0:
class_0_sample_0 0.0388130199253125 0.306701274378974
class_0_sample_1 -0.0483082434657033 0.333167487289401
::::
Prining out the points in class 1:
class_1_sample_0 0.439181892150296 0.15767914403357
class_1_sample_1 0.341741264689717 0.290947662466462
::::
Prining out the points in class 2:
class_2_sample_0 0.62339139639785 0.233743754191426
class_2_sample_1 0.663987994007291 0.185020510091793
::::
Prining out the points in class 3:
class_3_sample_0 0.0119872353686556 0.505153721523502
class_3_sample_1 0.192827443382858 0.416779605628115
::::
::::
Column 0: 1 0 0 0 0
Column 1: 1 0 0 0 1
Column 2: 1 0 0 1 0
Column 3: 1 0 0 1 1
::::
This column-specific demonstration is based on the following
 randomly selected decision line orientations for the weak
 classifiers: 0 173 55 37 96 105 174 113 166 30
Enter the coordinates of the point you wish to classify:
```

You enter the coordinates of the point you wish to classify in response to the prompt you see at the bottom.

- Let’s say you enter something like “0.2 0.6” in response to the prompt. Next, the system will come back with the following sort of classification results for your point:

```
[Column 5] Weak classifier 0 (.........):   positive
[Column 5] Weak classifier 1 (.........):   positive
[Column 5] Weak classifier 2 (.........):   negative
[Column 5] Weak classifier 3 (.........):   negative
[Column 5] Weak classifier 4 (.........):   negative
[Column 5] Weak classifier 5 (.........):   negative
[Column 5] Weak classifier 6 (.........):   positive
[Column 5] Weak classifier 7 (.........):   negative
[Column 5] Weak classifier 8 (.........):   positive
[Column 5] Weak classifier 9 (.........):   negative
[Column 5] weak classifications: 1 1 -1 -1 -1 -1 1 -1 1 -1

The data point (0.2 0.6) is predicted to be in one of these classes:

Enter the coordinates of the point you wish to classify:
```

where I have suppressed a lot of information regarding each weak classifier that is printed out inside the parentheses that currently just have dots.

- Each row of the output shown on the previous slide starts with the column of the codeword matrix whose discriminatory power you are investigating. Recall that any single column can only carry out a binary classification into two groups of classes, those corresponding to the 1's in the column and those corresponding to the 0's in the column.

- When a weak classifier reports 'positive' for the classification of the point you supplied, it means that, according to the weak classifier, the point belongs to the classes corresponding to the 1's in the column. Similarly, for the 'negative' classification.

14. An Interactive Session for Demonstrating Multiclass Classification with AdaBoost

- If you entered "1" in response to the prompt when you first fired up `MulticlassAdaBoost.pl`, the demo will work in the overall-interactive mode. In this mode, you will again see the output that was shown in a highly abbreviated form on Slide 88 for the column-interactive mode of operation of the script. As mentioned there, the output ends in the following prompt:

```
Enter the coordinates of the point you wish to classify:
```

- You are also shown a plot of the sort you saw on Slide 72 to help you decide what coordinates you should supply for classification. Let's say you entered "0.2 0.3" in response to the question shown above, you will see the kind of output that is displayed on the next slide.

```
[Column 0] Weak classifier 0 (..........):   negative
[Column 0] Weak classifier 1 (..........):   negative
::::
[Column 0] weak classifications: -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
[Column 1] Weak classifier 0 (..........):   negative
[Column 1] Weak classifier 1 (..........):   negative
::::
[Column 1] weak classifications: -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
[Column 2] Weak classifier 0 (..........):   negative
[Column 2] Weak classifier 1 (..........):   negative
::::
[Column 2] weak classifications: -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
[Column 3] Weak classifier 0 (..........):   negative
[Column 3] Weak classifier 1 (..........):   negative
::::
[Column 3] weak classifications: -1 -1 -1 -1 -1 -1 -1 1 -1 -1
[Column 4] Weak classifier 0 (..........):   negative
[Column 4] Weak classifier 1 (..........):   positive
::::
[Column 4] weak classifications: -1 1 -1 1 1 -1 -1 -1 -1 -1
[Column 5] Weak classifier 0 (..........):   positive
[Column 5] Weak classifier 1 (..........):   positive
::::
[Column 5] weak classifications: 1 1 -1 -1 -1 -1 -1 -1 1 -1
[Column 6] Weak classifier 0 (..........):   positive
[Column 6] Weak classifier 1 (..........):   positive
::::
[Column 6] weak classifications: 1 1 -1 -1 -1 -1 -1 -1 -1 -1
[Column 7] Weak classifier 0 (..........):   positive
[Column 7] Weak classifier 1 (..........):   positive
::::
[Column 7] weak classifications: 1 1 -1 1 1 1 -1 1 -1 -1
[Column 8] Weak classifier 0 (..........):   negative
[Column 8] Weak classifier 1 (..........):   negative
::::
::::
[Column 14] weak classifications: 1 1 1 1 1 1 1 1 1 1
Predicted codeword for point (.2 .3) is: 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1
Predicted class label for (0.2 0.3): 1    [with Hamming Distance: 0]


Enter the coordinates of the point you wish to classify:
```

- Note the final classification result near the bottom of the display on the previous slide. It says that the predicted class label is '1'. It also says that the Hamming distance between the binary code word obtained from the 15 columns of the codeword matrix and the codeword corresponding to class 1 is zero. In this case, we happen to be right on target. A cool thing about this user interaction is that you can verify the accuracy of the predicted class label by examining your point on the plot (like the one shown on Slide 72) that stays on your screen.

- In the overall-interactive, the data point you want to classify is handed over to the function `final_classify_with_all_columns()` whose implementation is shown on the next slide.

```
sub final_classify_with_all_columns {
    my $data_point = shift;
    my @prediction_codeword;
    foreach my $col_index (0..$NUMBER_OF_COLUMNS-1) {
        generate_weak_classifiers_for_one_column_of_codeword_matrix(
              $col_index) unless $WEAK_CLASSIFIERS_ALREADY_GENERATED;
        my $predicted_class =
          final_classify_for_one_column_of_codeword_matrix(
                                            $col_index, $data_point);
        push @prediction_codeword, 1 if $predicted_class eq ’positive’;
        push @prediction_codeword, 0 if $predicted_class eq ’negative’;
    }
    print "\nThe predicted codeword for the data point " .
                        "(@{$data_point}) is: @prediction_codeword\n’’;
    my $number_of_rows = scalar(keys %CODEWORD_MATRIX);
    my %hamming_distance;
    foreach my $row_index (0..$number_of_rows-1) {
        my @codeword = @{$CODEWORD_MATRIX{$row_index}};
        $hamming_distance{$row_index} = 0;
        foreach my $bit_index (0..$NUMBER_OF_COLUMNS-1) {
            $hamming_distance{$row_index}++
              if $prediction_codeword[$bit_index] !=
                                                 $codeword[$bit_index];
        }
    }
    my @sorted_rows = sort {$hamming_distance{$a} <=>
                         $hamming_distance{$b}} keys %hamming_distance;
    print "\n\nPredicted class label for (@{$data_point}): " . "
            "$sorted_rows[0]    [with Hamming Distance: " .
                               "$hamming_distance{$sorted_rows[0]}]\n’’;
}
```

- As it must, the subroutine on the previous slide first calls on `generate_weak_classifiers_for_one_` `_column_of_codeword_matrix()` in line 4 to generate the weak classifiers for each column of the codeword matrix.

- Subsequently, in lines 6 and 7, it calls on `final_classify_for_one_column_of_codeword_matrix()` to use weak classifiers for each column to carry out a binary classification of the new data point. When the class returned is 'positive', the new data point belongs to one of the classes whose bit is 1 in the column. Otherwise, its classification would be 'negative'.

- From the 'positive' or 'negative' labels returned for each of the column binary classifiers, we construct a codeword for the new data point in lines 9 and 10.

- In the rest of the script, we compare the codeword thus formed for the new data point against the codewords for each of the classes in the codeword matrix. This gives us the Hamming distance between the codeword for the new data point and the class codewords. In lines (25) through (30), we declare for the final classification that class label which is at the shortest Hamming distance to the codeword for the new data point.

15. Acknowledgments

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