Delaunay Graph Spanner Notes
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In these notes, we discuss the major results with respect to the Delaunay Graph as a spanner.

Delaunay Graph

P is a set of points in the plane, $DG(P)$ is a graph whose vertex set is P where u and v are connected by an edge only if the voronoi regions for u and v share an edge.

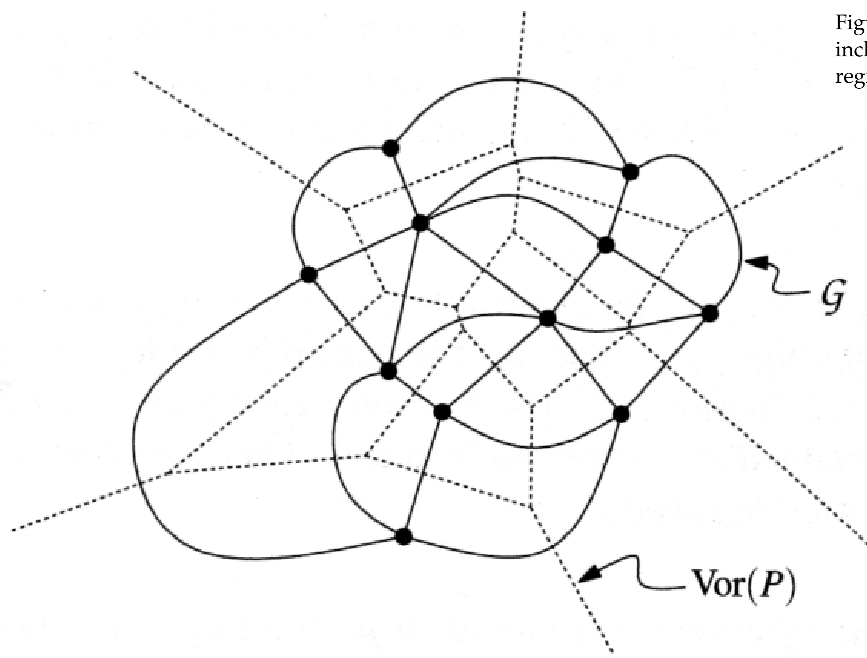

Figure 1: The Delaunay graph on P, including the boundaries of the Voronoi regions.

Dobkin's Results

The Delaunay triangulation of a set of points in the plane is a spanner with spanning ratio $c \leq ((1+\sqrt{5})/2)\pi \approx 5.08$. This was proven in the paper "Delaunay Graphs Are Almost as Good as Complete Graphs" by Dobkin, Friedman, and Supowit. [1] [2]

[1] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. In *Proceedings of the 28th Annual Symposium on Foundations of Computer Science*, SFCS '87, pages 20–26, Washington, DC, USA, 1987. IEEE Computer Society

[2] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete Comput. Geom.*, 5(4):399–407, May 1990

Introduction

We consider the path between two arbitray points $a, b \in P$. Let the line segment between a and b be the *direct line*. We construct *the direct DT path* by walking along the direct line, each time a new face of the Voronoi diagram is reached we add the corresponding edge in the Delaunay Graph.

One-Sided Path: The Easy Case

If all edges along the direct DT path between points $a, b \in P$ are either all above or all below the direct line, we say that this is a one-sided path.

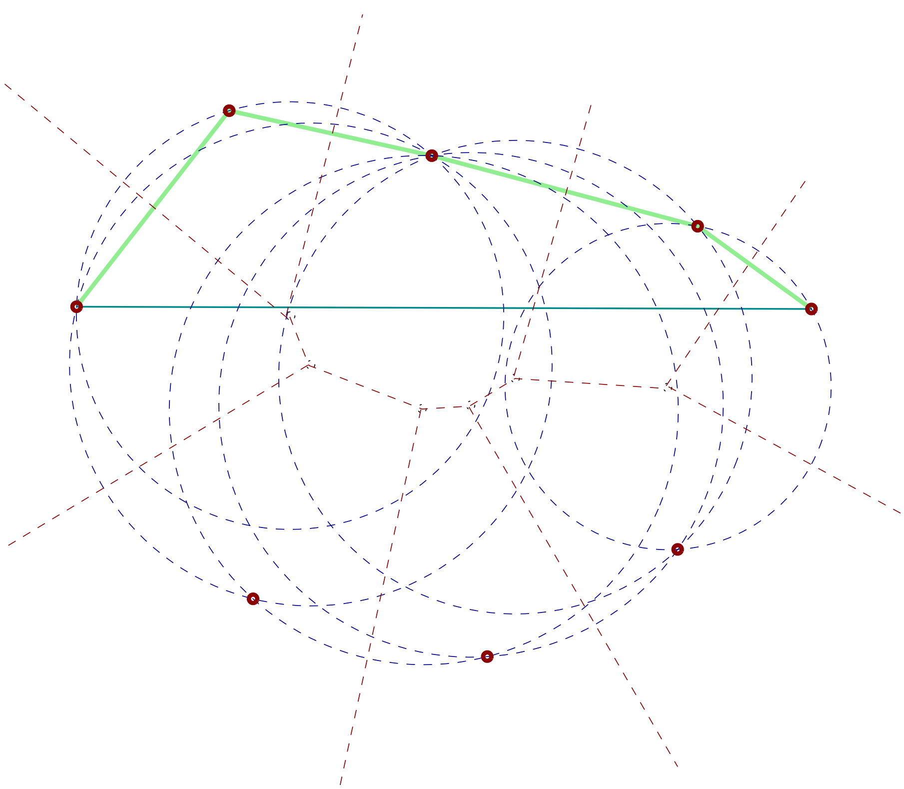

Figure 2: The cyan line shows the direct path, the green line shows the direct DT path, the dashed red lines show the boundaries of the Voronoi regions, and the circumcircles (also dashed) are blue.

Without loss of generality, we can say that the line segment between points a and b lies on the x-axis.

Lemma 1. *Points along a direct DT path are monotonic in x.*

Proof. All points along the perpendicular bisector of two adjacent points on the direct DT path, b_i and b_{i+1}, must be equidistant to each of these points. Since we know these points are adjacent along the direct path, they must share a Voronoi edge which must be a segment of this bisector. Since we know the direct line (which is the x-axis) crosses this Voronoi edge, we know b_i must lie to the left and b_{i+1} must lie to the right of the bisector. □

Lemma 2. *All points along the direct DT path from a to b are contained within or on the boundary of the circle with a and b diametrically opposed.* [3]

Proof. Let x be a point on the direct DT path outside of the circle with a and b diametrically opposed, whose centre is c. We know $d(x, c) > d(a, c) = d(b, c)$. Without loss of generality, assume $d(a, x) \geq d(b, x)$. Therefore, $d(a, x) > d(a, c)$. This means the Voronoi region of x could not intersect with the direct line between a and b. □

Lemma 3. *The boundary of a connected union of n circles centred on a line has length at most $\pi \cdot (x_r - x_l)$ where x_r and x_l are the extreme x coordinates of any of the circles.*

Proof. We prove by induction that the upper boundary of the circles has length at most $\frac{\pi}{2} \cdot (x_r - x_l)$, from which the lemma follows by symmetry.

In the case of a single circle, the upper boundary is half of the circumference of the circle:, $\frac{\pi}{2} \cdot (x_r - x_l)$. The lemma holds for $k \geq 1$ circles, we now show that it holds for $k + 1$ circles.

Without loss of generality, we say that the $k + 1$th circle is added at the right-most extremity of the k circles.

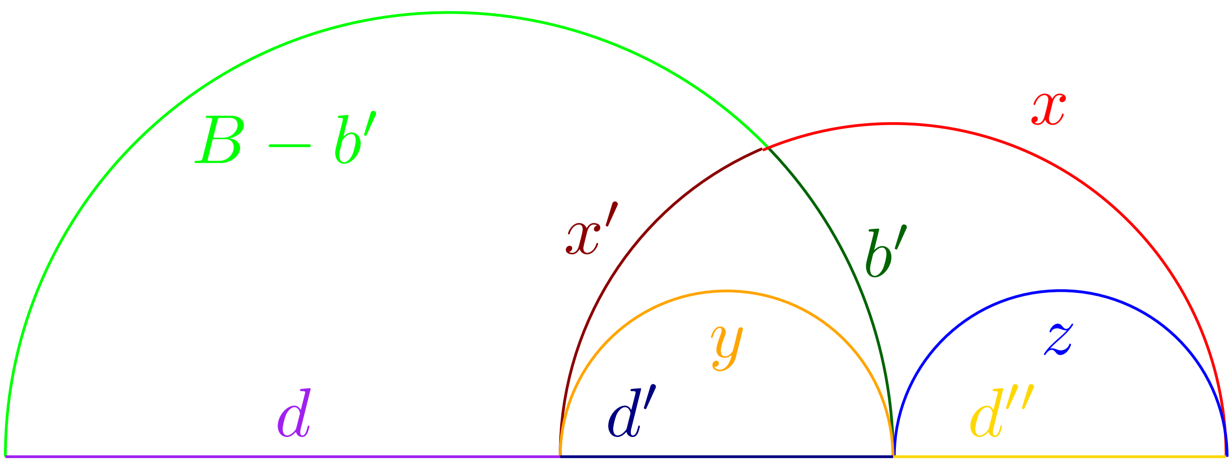

Figure 3: Let B be the upper boundary of the k circles. Let b' be the length of B contained within the $k + 1$th circle. Let x be the upper boundary of the new circle not contained within B, and x' be the rest. Let y be the circle whose left-most point is the left-most point of the $k + 1$th circle, and whose right-most point is the right-most point of B. Let z be the circle whose left-most point is the right-most point of B and whose right-most point is the right-most point of the $k + 1$th circle.

From the inductive hypothesis, we know

- $B \leq \frac{\pi}{2} \cdot (d + d')$
- $z \leq \frac{\pi}{2} \cdot d''$

Therefore $B - b' + x \leq \frac{\pi}{2} \cdot (d + d' + d'') = B + z$.
So we wish to show: $x - b' \leq z$.
We know:

- $y \leq x' + b' \leftrightarrow y - x' \leq b'$
- $x' + x = y + z$

So

$$\begin{aligned} x + x' &= y + z \\ x &= y + z - x' \\ x &\leq z + b' \\ x - b' &\leq z \end{aligned}$$

□

Let $a = b_0, \ldots, b_i, \ldots, b_n = b$ be the direct DT path from a to b. For each pair b_i, b_{i+1} create the circle on whose boundary these points lie, and whose centre is on the line segment between a and b. Let the union of these circles be C.

From lemma 3, we know that the length of the upper half of C is most $\pi/2$ times as long as the euclidean distance between a and b.

From lemma 1, we know that a one-sided direct DT path is an x-monotonic path along the upper half of C, and therefore the length of the upper half of C is an upper bound for the one-sided direct DT path, which is at most $\pi/2$.

The Harder Case

The direct DT path may cross the x-axis $\Omega(n)$ times, which can yield a much longer path.

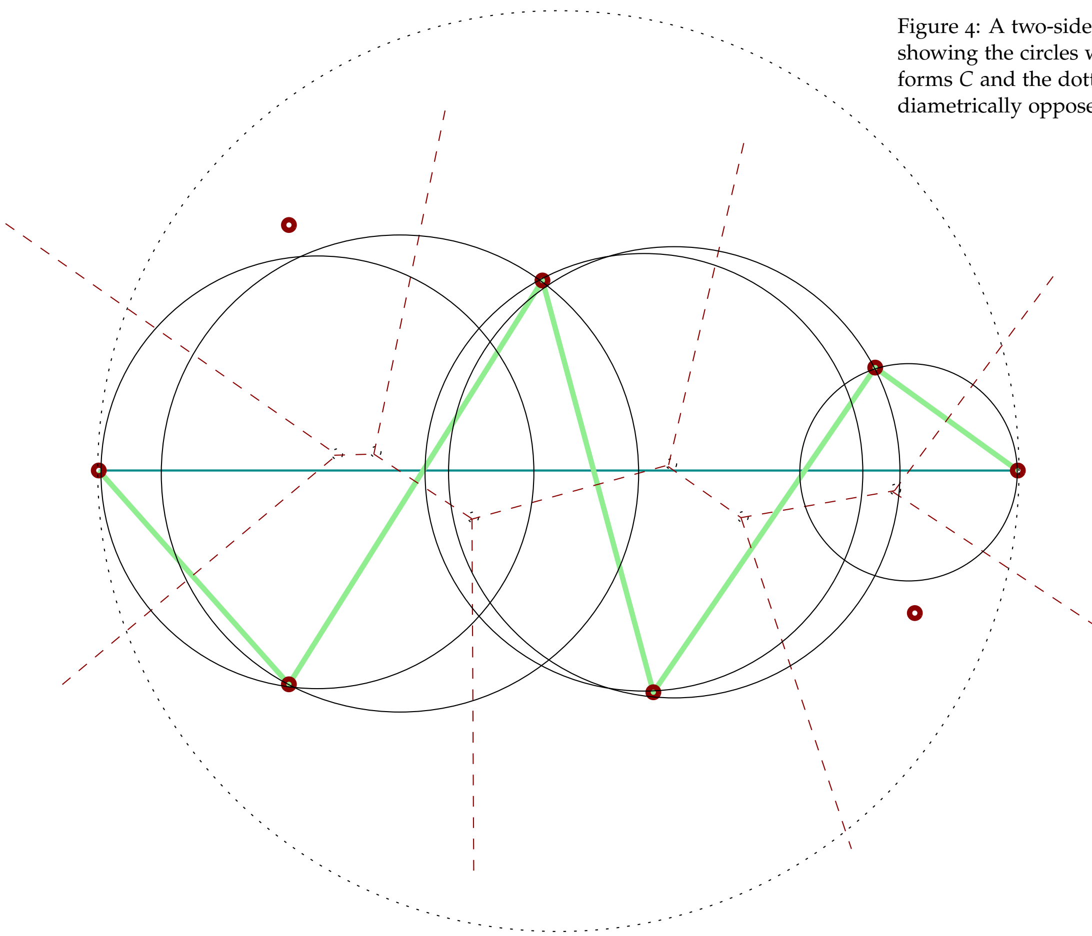

Figure 4: A two-sided direct DT path showing the circles whose union forms C and the dotted circle with a, b diametrically opposed.

The general idea is that we stick to the region above the x-axis as much as possible, and follow the path below the x-axis if the distance travelled across the axis is small compared to the distance travelled along the path before the next time the path crosses the x-axis.

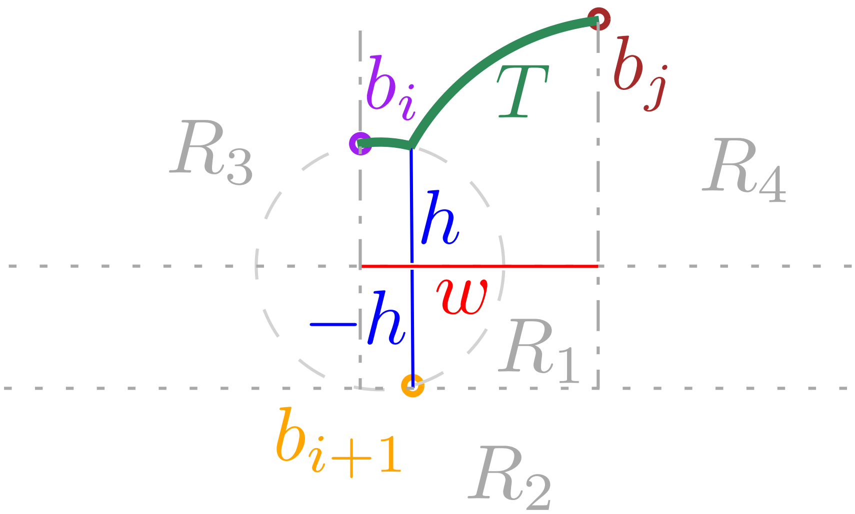

Figure 5:

Let b_i be the last point before the direct DT path dips below the x-axis, let b_j be the next point after b_i on or above the x-axis. Let T be the section of C between b_i and b_j. Let the length of T be t. Let $h = min\{y(q) : q \text{ lies on } T\}$, and $w = x(b_j) - x(b_i)$.

Let R_1 be the region between b_i and b_j above the line $y = -h$. Let R_2 be the region below the line $y = -h$. Let R_3 be the region above the line $y = -h$ and to the left of b_i. Let R_4 be the region above the line $y = -h$ and to the right of b_j.

To be specific, we take the direct DT path only if $h \leq w/4$.

Otherwise, we follow the lower convex hull of all points in P between b_i and b_j, who are above the x-axis and below the line segment between b_i and b_j.

The length of the path between b_i, b_j with no shortcuts is at most $t + 2(y(b_i) + y(b_j))$ and the length of the path between b_i, b_j using the lower hull is at most $t \cdot \pi/2$ (by Lemma 3).

(z_k, z_{k+1}) is an edge on the lower convex hull between b_i and b_j. Let L be the lower semi-circle of $circle(z_k, z_{k+1})$. Without loss of generality, assume $y(z_k) \leq y(z_{k+1})$.

Lemma 4. *The direct DT path from z_k to z_{k+1} is one-sided.*

Since we know the direct DT path from z_k to z_{k+1} is one-sided, we know its length is bounded by $\pi/2 \cdot d(z_k, z_{k+1})$.

Note 1. *A convex path P contained within a region R has length at most the boundary of R.*

From note 1, since the lower convex hull is a convex region within the region bounded by T and the line b_i, b_j, the sum of the length of all convex shortcut edges is at most $\pi/2 \cdot t$.

Note 2. *Where a, b, c are sides of a right triangle with c being the hypotenuse, then*

$$\frac{a}{2} + b \leq \frac{\sqrt{5}}{2} \cdot c$$

Theorem 1. *There exists a DT path from a to b of length*

$$\leq \frac{\pi}{2} \cdot (1 + \sqrt{5}) \cdot d(a, b)$$

Proof. We have shown the shortcut path has length at most $t \cdot \pi/2$.

In the case where $h \leq w/4$ and we don't take a shortcut, then let q be the lowest point on T, t be the length of T, t_i be the section of T between b_i and q, t_j be defined similarly for b_j, w_i be the projection of t_i on the x-axis, and w_j be defined similarly for t_j.

The length of the path is at most

$$
\begin{aligned}
t + 2(y(b_i) + y(b_j)) &= t + 2(2h + (y(b_i) - h) + (y(b_j) - h)) \\
&\leq t + 2(\frac{w}{2} + (y(b_i) - h) + (y(b_j) - h)) \\
&\leq t + 2(\frac{w_i}{2} + (y(b_i) - h) + \frac{w_j}{2} + (y(b_j) - h)) \\
&\leq t + 2(\frac{\sqrt{5}}{2} \cdot t_i + \frac{\sqrt{5}}{2} \cdot t_i) \qquad \text{(from Note 2)} \\
&\leq t + 2(\frac{\sqrt{5}}{2} \cdot (t_i + t_i)) \\
&\leq t + \sqrt{5} \cdot t \\
&\leq t(1 + \sqrt{5})
\end{aligned}
$$

From lemma 3, we know that half the boundary of the unioned circles is an upper bound on the length of the path, so it must be at most $\frac{\pi}{2} \cdot (1 + \sqrt{5}) \cdot d(a, b)$.

$\square$

Secondary Proofs

Note 2 states if a, b, c are sides of a right triangle with c being the hypotenuse, then

$$\frac{a}{2} + b \leq \frac{\sqrt{5}}{2} \cdot c$$

Proof of Note 2. Without loss of generality, assume $b \leq a$.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5a^2 + 5b^2 &= 5c^2 \\ a^2 + 4a^2 + 5b^2 &= 5c^2 \\ a^2 + 4a^2 + 4b^2 &\leq 5c^2 \\ a^2 + 4ab + 4b^2 &\leq 5c^2 \\ (a + 2b)^2 &\leq 5c^2 \\ a + 2b &\leq \sqrt{5} \cdot c \\ \frac{a}{2} + b &\leq \frac{\sqrt{5}}{2} \cdot c \end{aligned}$$

□

Proof of Lemma 4

Note 3. *No points of P are within L and R_3.*

Proof. From assumption, L and R_3 do not intersect.

□

Note 4. *No points of P are within L and R_2.*

Proof. Since $z_k \in P$, it must lie above T since it can't be within C, therefore $y(z_k) \geq h > w/4$ by the fact that we only build this lower hull in the case where $h > w/4$.

TODO: why is $y(q) \leq y(z_k)$?

□

Note 5. *No points of P are within L and R_4.*

Proof. Any point within L and R_4 must also be within C.

□

Lemma 4 states that the direct DT path from z_k to z_{k+1} is one-sided.

Proof of Lemma 4. We prove this by showing that L contains no points of P. From Lemmas 4, 5, and 6 it remains only to show that there are no points of P within L and R_1.

No points of P in R_1 exist between the lines $y = h$ and $y = -h$, since this area is strictly within C.

No points of P may exist in R_1 above h but below the line segment $z_k z_{k+1}$, otherwise it would be on the lower convex hull and z_k and z_{k+1} would not be adjacent.

$\square$

Keil's Results

TODO

References

[1] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. In *Proceedings of the 28th Annual Symposium on Foundations of Computer Science*, SFCS '87, pages 20–26, Washington, DC, USA, 1987. IEEE Computer Society.

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