Delaunay Graph Spanner Notes

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In these notes, we discuss the major results with respect to the Delaunay Graph as a spanner.

Delaunay Graph

P is a set of points in the plane, $DG(P)$ is a graph whose vertex set is P where u and v are connected by an edge only if the voronoi regions for u and v share an edge.

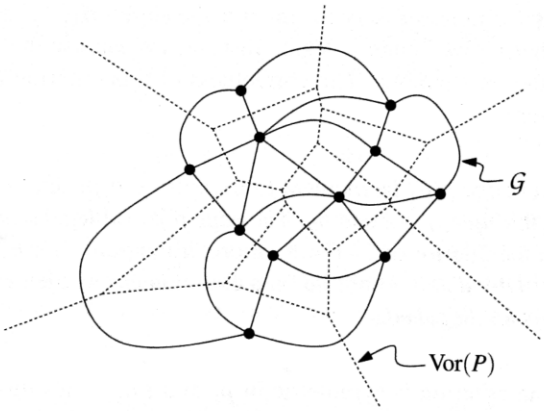

Figure 1: The Delaunay graph on P, including the boundaries of the Voronoi regions.

Dobkin's Results

The Delaunay triangulation of a set of points in the plane is a spanner with spanning ratio $c \leq ((1+\sqrt{5})/2)\pi \approx 5.08$. This was proven in the paper "Delaunay Graphs Are Almost as Good as Complete Graphs" by Dobkin, Friedman, and Supowit. [1] [2]

Introduction

We consider the path between two arbitray points $a, b \in P$. Let the line segment between a and b be the *direct line*. We construct *the direct DT path* by walking along the direct line, each time a new face of the Voronoi diagram is reached we add the corresponding edge in the

[1] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. In *Proceedings of the 28th Annual Symposium on Foundations of Computer Science*, SFCS '87, pages 20–26, Washington, DC, USA, 1987. IEEE Computer Society

[2] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete Comput. Geom.*, 5(4):399–407, May 1990

Delaunay Graph.

One-Sided Path: The Easy Case

If all edges along the direct DT path between points $a, b \in P$ are either all above or all below the direct line, we say that this is a one-sided path.

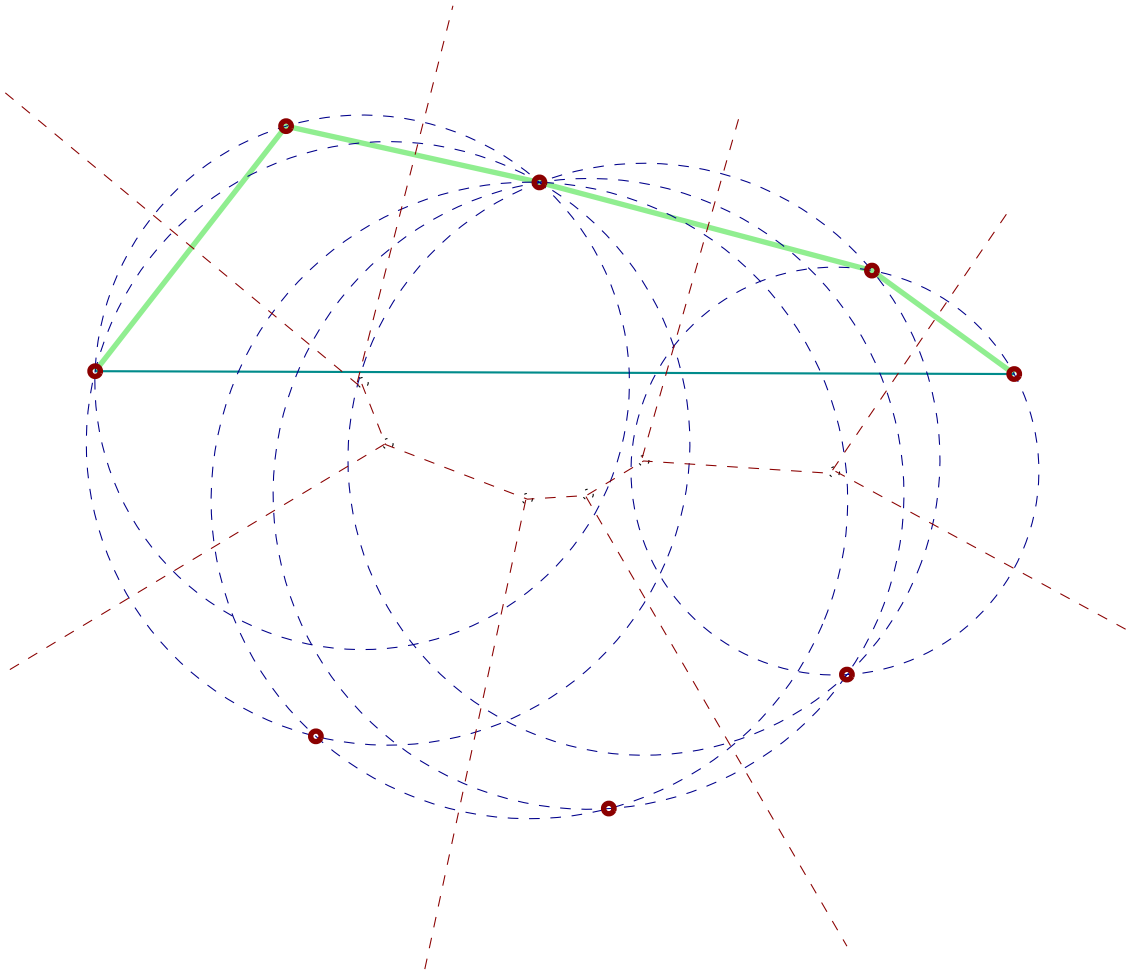

Figure 2: The cyan line shows the direct path, the green line shows the direct DT path, the dashed red lines show the boundaries of the Voronoi regions, and the circumcircles (also dashed) are blue.

Without loss of generality, we can say that the line segment between points a and b lies on the x-axis.

Lemma 1. *Points along a direct DT path are monotonic in x.*

Lemma 2. *All points along the direct DT path from a to b are contained within or on the boundary of the circle with a and b diametrically opposed.*

Lemma 3. *The boundary of a connected union of circles has boundary at most $\pi \cdot (x_r - x_l)$ where x_r and x_l are the extreme x coordinates of any of the circles.*

From these lemmas, it follows that the one-sided path is at most $\pi/2$ times as long as the euclidean distance between the endpoints.

The Harder Case

The direct DT path may cross the x-axis $\Omega(n)$ times, which can yield a much longer path.

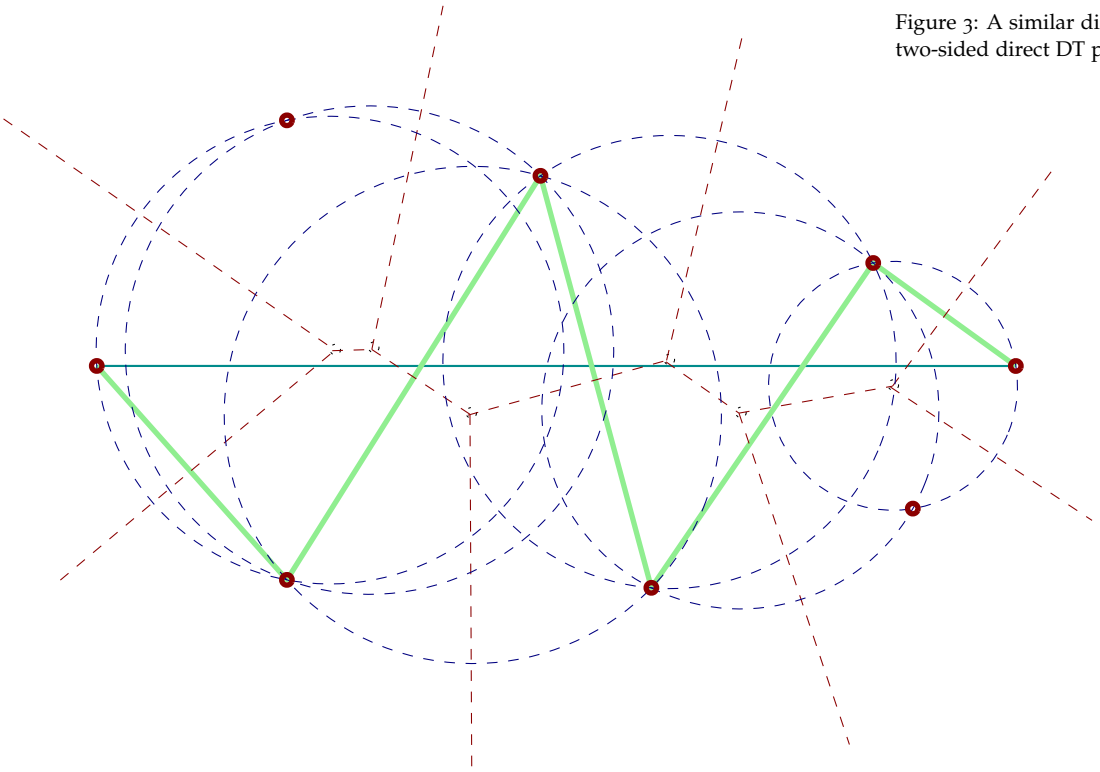

Figure 3: A similar diagram with a two-sided direct DT path.

The general idea is that we stick to the region above the x-axis as much as possible, and follow the path below the x-axis if it isn't too far from the next point above the x-axis. [3]

Otherwise, we follow the lower convex hull of all points in P between b_i and b_j, who are above the x-axis and below the line segment between b_i and b_j.

[3] Let $a = p_0, \ldots, p_i, \ldots, p_n = b$ be the direct DT path from a to b. For each pair p_i, p_{i+1} create the circle on whose boundary these points lie, and whose centre is on the line segment between a and b. Let the union of these circles be T.

Let $h = min\{y(q) : q \text{ lies on } T\}$, and $w = x(b_j) - x(b_i)$ where b_i is the last point before the direct DT path dips below the x-axis and b_j is the first point on or above the x-axis after b_i. We take the direct DT path only if $h \leq w/4$.

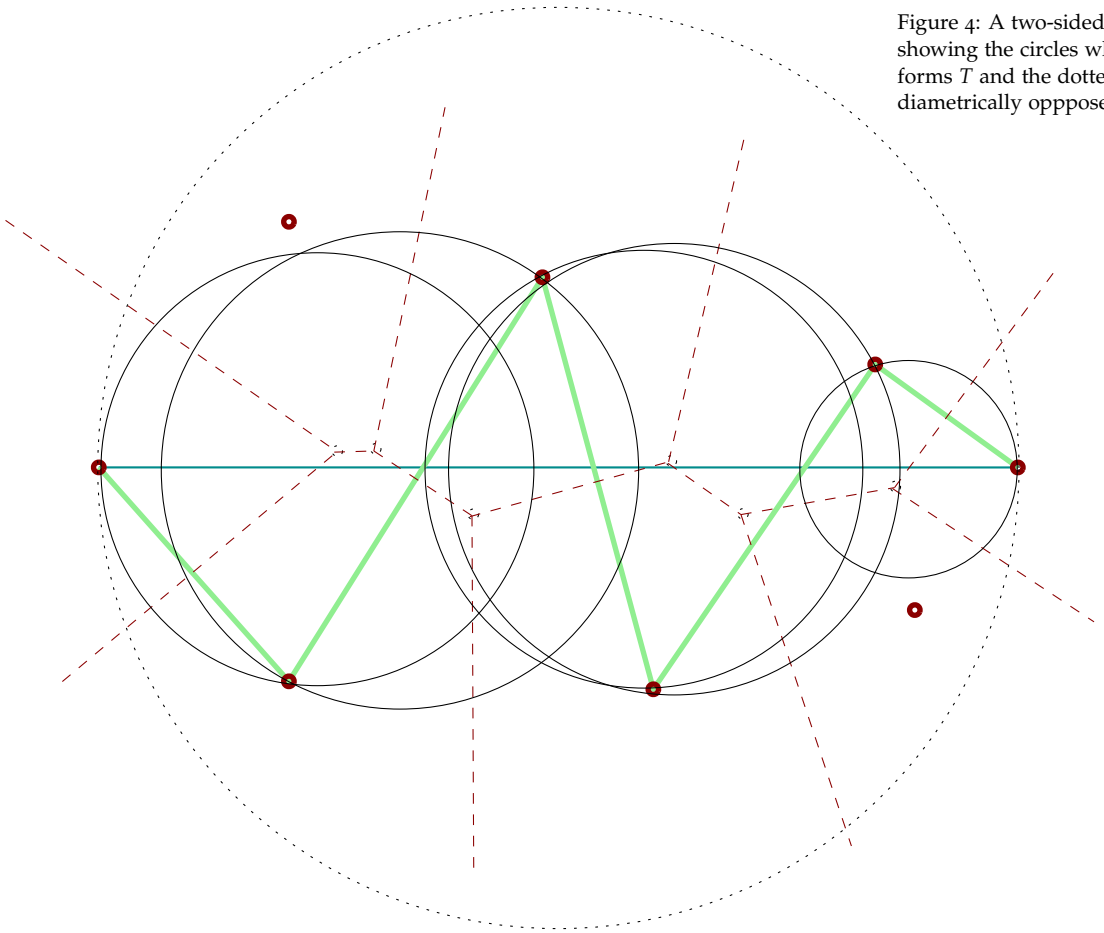

Figure 4: A two-sided direct DT path showing the circles whose union forms T and the dotted circle with a, b diametrically oppposed.

Keil's Results

TODO

References

[1] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. In *Proceedings of the 28th Annual Symposium on Foundations of Computer Science*, SFCS '87, pages 20–26, Washington, DC, USA, 1987. IEEE Computer Society.

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