Delaunay Graph Spanner Notes
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In these notes, we discuss the major results with respect to the Delaunay Graph as a spanner.

Delaunay Graph

P is a set of points in the plane, $DG(P)$ is a graph whose vertex set is P where u and v are connected by an edge only if the voronoi regions for u and v share an edge.

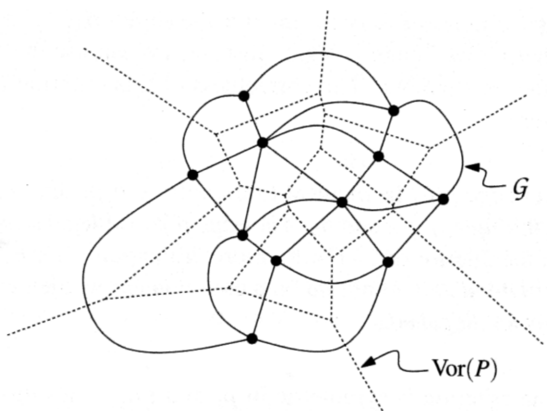

Figure 1: The Delaunay graph on P, including the boundaries of the Voronoi regions.

Dobkin's Results

The Delaunay triangulation of a set of points in the plane is a spanner with spanning ratio $c \leq ((1+\sqrt{5})/2)\pi \approx 5.08$. This was proven in the paper "Delaunay Graphs Are Almost as Good as Complete Graphs" by Dobkin, Friedman, and Supowit. [1] [2]

Introduction

We consider the path between two arbitray points $a, b \in P$. Let the line segment between a and b be the *direct line*. We construct *the direct DT path* by walking along the direct line, each time a new face of the Voronoi diagram is reached we add the corresponding edge in the

[1] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. In *Proceedings of the 28th Annual Symposium on Foundations of Computer Science*, SFCS '87, pages 20–26, Washington, DC, USA, 1987. IEEE Computer Society

[2] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete Comput. Geom.*, 5(4):399–407, May 1990

Delaunay Graph.

One-Sided Path: The Easy Case

If all edges along the direct DT path between points $a, b \in P$ are either all above or all below the direct line, we say that this is a one-sided path.

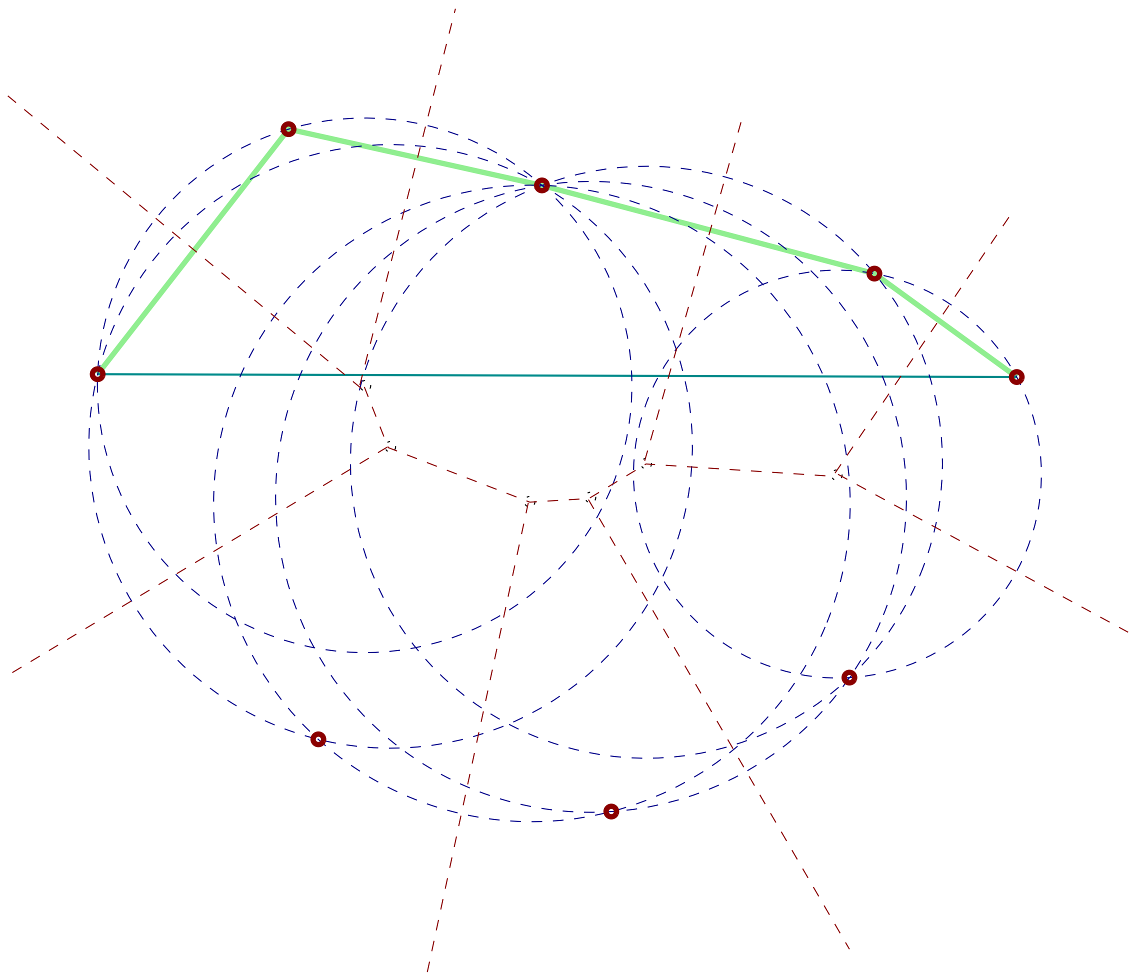

Figure 2: The cyan line shows the direct path, the green line shows the direct DT path, the dashed red lines show the boundaries of the Voronoi regions, and the circumcircles (also dashed) are blue.

Without loss of generality, we can say that the line segment between points a and b lies on the x-axis.

Lemma 1. *Points along a direct DT path are monotonic in* x.

Lemma 2. *All points along the direct DT path from* a *to* b *are contained within or on the boundary of the circle with* a *and* b *diametrically opposed.*

Lemma 3. *The boundary of a connected union of* n *circles has length at most* $\pi \cdot (x_r - x_l)$ *where* x_r *and* x_l *are the extreme* x *coordinates of any of the circles.*

Proof. We prove by induction that the upper boundary of the circles has length at most $\frac{\pi}{2} \cdot (x_r - x_l)$, from which the lemma follows by symmetry.

In the case of a single circle, the upper boundary is half of the circumference of the circle:, $\frac{\pi}{2} \cdot (x_r - x_l)$. The lemma holds for $k \geq 1$ circles, we now show that it holds for $k+1$ circles.

Without loss of generality, we say that the $k+1$th circle is added at the right-most extremity of the k circles.

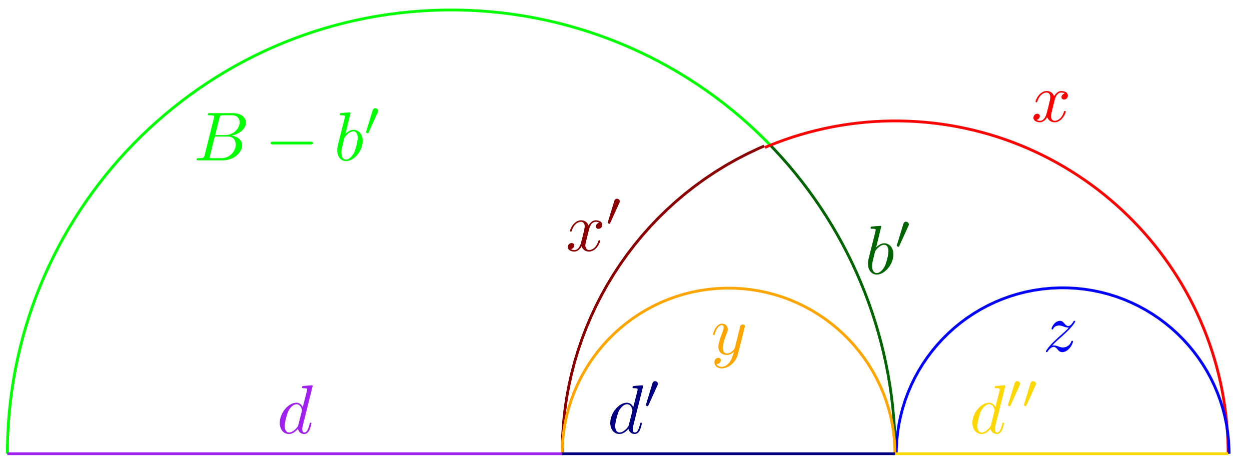

Figure 3: Let B be the upper boundary of the k circles. Let b' be the length of B contained within the $k+1$th circle. Let x be the upper boundary of the new circle not contained within B, and x' be the rest. Let y be the circle whose left-most point is the left-most point of the $k+1$th circle, and whose right-most point is the right-most point of B. Let z be the circle whose left-most point is the right-most point of B and whose right-most point is the right-most point of the $k+1$th circle.

From the inductive hypothesis, we know

- $B \leq \frac{\pi}{2} \cdot (d + d')$
- $z \leq \frac{\pi}{2} \cdot d''$

Therefore $B - b' + x \leq \frac{\pi}{2} \cdot (d + d' + d'') = B + z$.
So we wish to show: $x - b' \leq z$.
We know:

- $y \leq x' + b' \leftrightarrow y - x' \leq b'$
- $x' + x = y + z$

So

$$\begin{aligned} x + x' &= y + z \\ x &= y + z - x' \\ x &\leq z + b' \\ x - b' &\leq z \end{aligned}$$

$\square$

From lemmas 1 and 3, it follows that the one-sided path is at most $\pi/2$ times as long as the euclidean distance between the endpoints.

The properties of the delaunay triangulation don't seem to allow for any kind of zig-zag one-sided path.

The Harder Case

The direct DT path may cross the x-axis $\Omega(n)$ times, which can yield a much longer path.

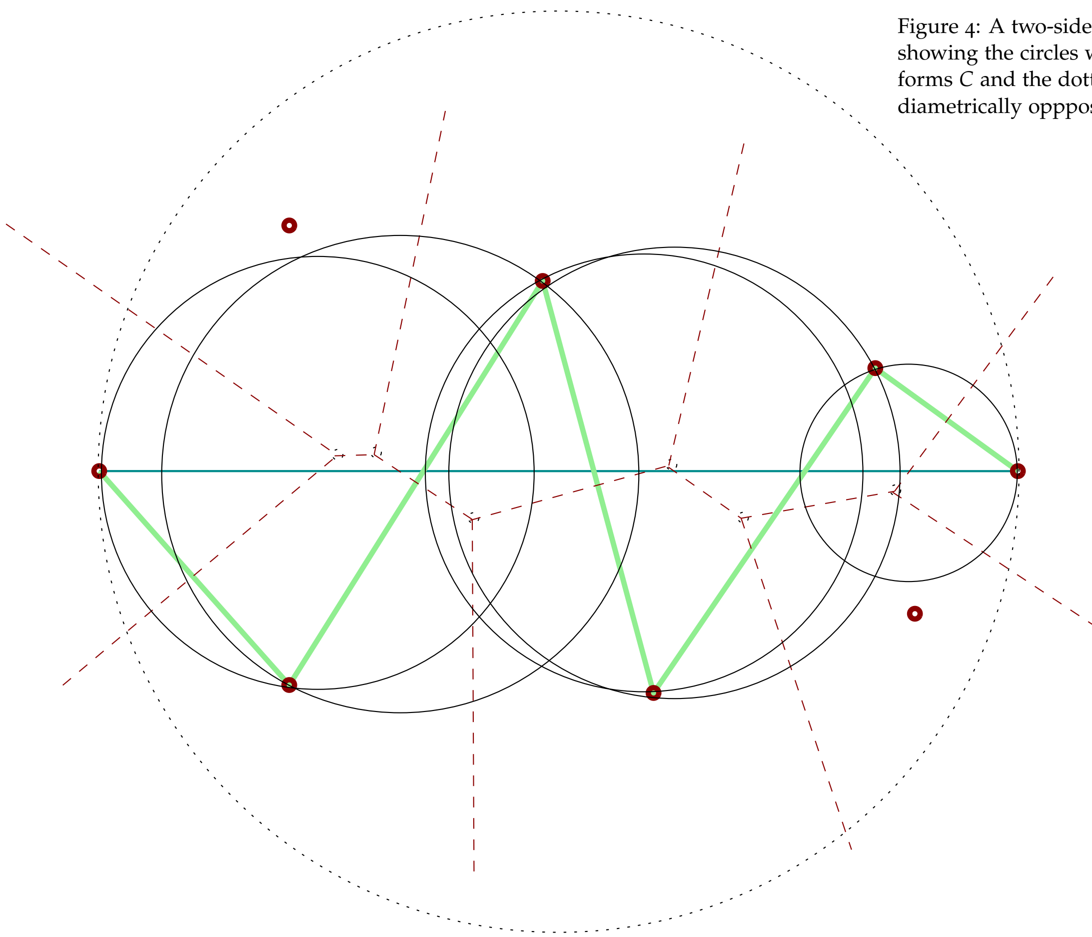

Figure 4: A two-sided direct DT path showing the circles whose union forms C and the dotted circle with a, b diametrically oppposed.

The general idea is that we stick to the region above the x-axis as much as possible, and follow the path below the x-axis if it isn't too far from the next point above the x-axis.

Otherwise, we follow the lower convex hull of all points in P between b_i and b_j, who are above the x-axis and below the line segment between b_i and b_j.

To be specific, we take the direct DT path only if $h \leq w/4$.

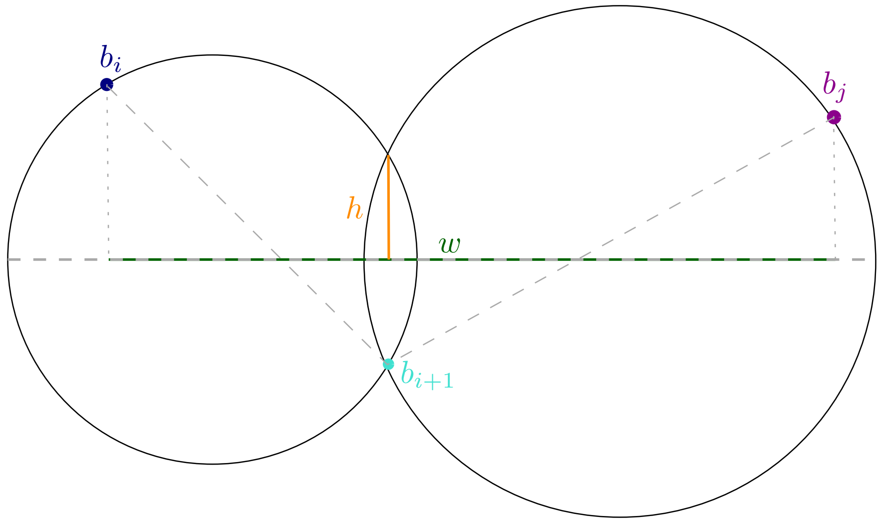

Figure 5: Let $a = p_0, \ldots, p_i, \ldots, p_n = b$ be the direct DT path from a to b. For each pair p_i, p_{i+1} create the circle on whose boundary these points lie, and whose centre is on the line segment between a and b. Let the union of these circles be C. Let b_i be the last point before the direct DT path dips below the x-axis, let b_j be the next point after b_i on or above the x-axis. Let T be the section of C between b_i and b_j. Let $h = min\{y(q) : q \text{ lies on } T\}$, and $w = x(b_j) - x(b_i)$.

This is still within the spanning ratio because...?

Lemma 4. *If $e = (u, v)$ is an edge on the lower convex hull between b_i and b_j, then the direct DT path from u to v is one-sided.*

Keil's Results

TODO

References

[1] David P. Dobkin, Steven J. Friedman, and Kenneth J. Supowit. Delaunay graphs are almost as good as complete graphs. In *Proceedings of the 28th Annual Symposium on Foundations of Computer Science*, SFCS '87, pages 20–26, Washington, DC, USA, 1987. IEEE Computer Society.

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