

5-Sided Orthogonal Point Enclosure

Notes on Rahul 2015, by Simon Pratt

The primary intention of these notes is to answer: how does [Rah15] answer 3D 5-sided point enclosure queries?

Orthogonal Point Enclosure Queries (OPEQ)

Preprocess a set S of n axes-parallel rectangles in order to determine all rectangles containing a query point q . In particular, we wish to examine the case in which rectangles are 5-sided and in \mathbb{R}^3 . To be precise, our rectangles are of the form $[x_1, x_2] \times [y_1, y_2] \times (-\infty, z]$. In these notes, we will prove the following:

Theorem 1 (5.1 in [Rah15]). *OPEQ on 5-sided rectangles can be answered using a structure of $O(n \lg^* n)$ size and $O(\lg n \lg \lg n + k)$ query time, where k is the size of the output.*

At a high level, we will use interval trees to solve 5-sided queries in $O(\lg^3 n + k)$ time, which is good for $k \geq \lg^3 n$. Otherwise, we'll use grids to break the structure into 4-sided queries which we solve with $O(n \lg^* n)$ space and $O(\lg n \lg^* n + k)$ query time using interval trees, shallow cuttings, and the following results:

Problem	Query	Space	Reference
2D 4-sided OPEQ	$O(\lg n + k)$	$O(n)$	[Cha86]
3D 3-sided OPEQ	$O(\lg n + k)$	$O(n)$	[Afs08]

1 5-Sided: Slow/Simple with Linear Space

An *interval tree* is a binary tree whose leaves store the values of the endpoints of a set of intervals [Ede83]. At each node v , store $\text{split}(v)$ which is the largest value in the left subtree of v , and $\text{range}(v)$ which is $(-\infty, \infty)$ at the root, and otherwise if $\text{range}(v) = [x_\ell, x_r]$, then v 's left child will have range $[x_\ell, \text{split}(v)]$, and symmetrically for the right child. An interval is stored at the node v of minimal height such that the interval is contained within $\text{range}(v)$. If we additionally maintain lists that store the left/right endpoints of all intervals stored at v in non-decreasing/non-increasing order, then space remains $O(n)$, but we can answer a 1D OPEQ in $O(\lg n + k)$ time.

Given a set of 4-sided rectangles of the form $[x_1, x_2] \times (-\infty, y] \times (-\infty, z]$, we can build an interval tree of all rectangles' projection onto the x -axis. Observe that at node v , if the query point q is to the left of $\text{split}(v)$ then for each rectangle r stored at v , r contains q iff $q \in [x_1, \infty) \times (\infty, y] \times (\infty, z]$, and similarly (but symmetrically) if q is to the right of $\text{split}(v)$. This

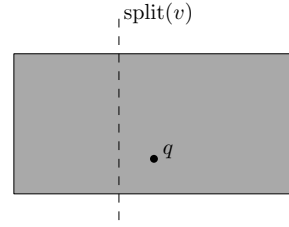


Figure 1: A 4-sided rectangle stored at v is reduced to 3-sided rectangles on either side of $\text{split}(v)$ in an interval tree.

effectively reduces the problem to a 3-sided query, for which we can store [Afs08] at each node (see Figure 1). Since we perform at most $O(\lg n)$ of these queries (one at each level of the interval tree), the total query time is $O(\lg^2 n + k)$ and space is still $O(n)$. Call this structure D_4 . We can use the same technique to solve 5-sided queries in $O(\lg^3 n + k)$ time by storing D_4 structures at the nodes of the interval tree obtained by y -projection instead. This proves the following:

Theorem 2 (2.1 in [Rah15]). *OPEQ on 5-sided rectangles can be answered using a structure of $O(n)$ size and $O(\lg^3 n + k)$ query time, where k is the size of the output.*

2 4-Sided: Faster with Near-Linear Space

Given two points $p, q \in \mathbb{R}^d$, then $p = (p_1, \dots, p_d)$ *dominates* $q = (q_1, \dots, q_d)$ if $p_i > q_i$ for all $i \in \{1, \dots, d\}$. Given a set of points P , R is a t -level shallow cutting of P if (i) $|R| = O(n/t)$, (ii) any point p that is dominated by at most t points of P dominates a point in R , and (iii) each point in R is dominated by $O(t)$ points in P . First, project the rectangles onto the plane. Then build an interval tree on the rectangles as we did in Section 1, such that at each node v , our 4-sided projections are stored as 3-sided rectangles either left or right of $\text{split}(v)$.

At every node v of the interval tree, consider each 3-sided rectangle as a point, then we can take $\lg^{(i)} n$ -level shallow cuttings R_i for all $0 \leq i \leq \lg^* n$.

Local Structure For each point $p \in R_i$, we build [Afs08] on all points that dominate p . Since p is in a $\lg^{(i)} n$ -level shallow cutting of P , there are $O(\lg^{(i)} n)$ points that dominate p .

Lemma 1 (3.1 in [Rah15]). *Given a point set P in \mathbb{R}^3 and a strip \mathcal{R} of the plane such that the projection of all points onto the plane falls within \mathcal{R} , we can compute a subdivision \mathcal{A} of \mathcal{R} with $|R|$ regions such that given*

a query point p in the plane, we can either find a point of P which dominates p , or no such point exists.

Global Structure For every node v in the interval tree, for each R_i , compute the Lemma 1 subdivision \mathcal{A}_i , then store all the rectangles from these subdivisions in [Cha86].

To answer a query q , first query the global structure to find the largest i such that a point p corresponding to a rectangle from \mathcal{A}_i encloses the query point. Then search the local structure of p , reporting all points (corresponding to rectangles of the input) that dominate q .

Space The local structure for each shallow cutting R_i takes $O(n)$ space, and there are $\lg^* n$ such cuttings.

Query Time Each of the $O(\lg n)$ nodes on the root to leaf path of the interval tree could return up to $O(\lg^* n)$ rectangles.

Remark You can instead build a data structure with $O(n)$ space and $O(\lg^{(c+1)} n + k)$ query time by only taking $c \geq 2$ shallow cuttings.

Theorem 3 (3.1 in [Rah15]). *OPEQ on 4-sided rectangles can be answered using a structure of $O(n \lg^* n)$ size and $O(\lg n \cdot \lg^* n + k)$ query time, where k is the size of the output.*

3 5-Sided: Putting it Together with Grids

We will use a grid technique adapted from [ABR00]. Let $t = \lg^4 m$, where m is the number of rectangles at the current level of recursion (initially n). Project S onto the plane, then draw an orthogonal $(2\sqrt{m/t}) \times (2\sqrt{m/t})$ grid over the resulting rectangles such that each horizontal and vertical slab contains the projections of \sqrt{nt} sides. We create a tree by creating a node to store all rectangles which cross a grid line, then recurse on all other rectangles. Stop recursion when m reaches a constant. At each node of this tree: (i) (*slow*): build the structure from Theorem 2, (ii) (L_c): for each grid cell c , store at most $\lg^3 n$ of the rectangles which completely cover c in decreasing order of z span, and (iii) (*side*): build the structure from Theorem 3 on the (at most 4) sides cut from each rectangle by the grid (see Figure 2).

To answer a query q , locate the grid cell c containing q and scan L_c , reporting rectangles until (a) we find a rectangle not containing q , or (b) we reach the end.

If (b), then $k \geq \lg^3 n$, thus querying the slow structure gives $O(k)$ query time. Otherwise, query the side structures then the recursive structures.

Space The bottleneck is the structure from Theorem 3 which occupies $O(n \lg^* n)$ space.

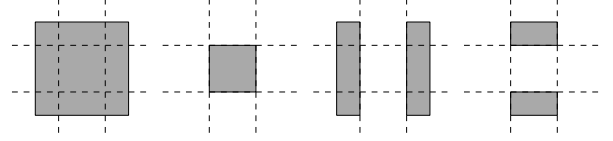


Figure 2: A rectangle (left) decomposes into an number of totally covered cells (middle-left), sides contained in adjacent vertical slabs (middle-right), and sides contained in adjacent horizontal slabs (right).

Query Time The bottleneck is the recursive case. Since we can find the grid cell containing q in $O(\lg n)$ time and the size of the subproblem is \sqrt{nt} , we get $Q(n) = Q(\sqrt{nt}) + O(\lg n)$. With $t = \lg^4 n$, this solves to $O(\lg n \lg \lg n + k)$ time, and this concludes the proof of Theorem 1.

Remark In the RAM model, we can find the grid cell containing q in $O(1)$ time. This suggests we can achieve $O(\lg \lg n + k)$ query time. Also, if Theorem 3 is not the time bottleneck, we can use the time-space trade-off to achieve $O(n)$ space.

References

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