5-Sided Orthogonal Point Enclosure

Notes on Rahul 2015, by Simon Pratt

The primary intention of these notes is to answer: how does [Rah15] answer 3D 5-sided point enclosure queries?

Orthogonal Point Enclosure Queries (OPEQ) Preprocess a set S of n axes-parallel rectangles in order to determine all rectangles containing a query point q. In particular, we wish to examine the case in which rectangles are 5-sided and in $\mathbb{R}^3$. To be precise, our rectangles are of the form $[x_1,x_2]\times[y_1,y_2]\times(-\infty,z]$. In these notes, we will prove the following:

Theorem 1 (5.1 in [Rah15])**.** *OPEQ on 5-sided rectangles can be answered using a structure of $O(n\lg^* n)$ size and $O(\lg n \lg\lg n + k)$ query time, where k is the size of the output.*

At a high level, we will use interval trees to solve 5-sided queries in $O(\lg^3 n+k)$ time, which is good for $k \geq \lg^3 n$. Otherwise, we'll use grids to break the structure into 4-sided queries which we solve with $O(n\lg^* n)$ space and $O(\lg n \lg^* n+k)$ query time using interval trees, shallow cuttings, and the following results:

Problem	Query	Space	Reference
2D 4-sided OPEQ	$O(\lg n+k)$	$O(n)$	[Cha86]
3D 3-sided OPEQ	$O(\lg n+k)$	$O(n)$	[Afs08]

1 5-Sided: Slow/Simple with Linear Space

An *interval tree* is a binary tree whose leaves store the values of the endpoints of a set of intervals [Ede83]. At each node v, store split(v) which is the largest value in the left subtree of v, and range(v) which is $(-\infty,\infty)$ at the root, and otherwise if range$(v)=[x_\ell,x_r]$, then v's left child will have range $[x_\ell,\text{split}(v)]$, and symmetrically for the right child. An interval is stored at the node v of minimal height such that the interval is contained within range(v). If we additionally maintain lists that store the left/right endpoints of all intervals stored at v in non-decreasing/non-increasing order, then space remains $O(n)$, but we can answer a 1D OPEQ in $O(\lg n+k)$ time.

Given a set of 4-sided rectangles of the form $[x_1,x_2]\times(-\infty,y]\times(-\infty,z]$, we can build an interval tree of all rectangles' projection onto the x-axis. Observe that at node v, if the query point q is to the left of split(v) then for each rectangle r stored at v, r contains q iff $q\in[x_1,\infty)\times(\infty,y]\times(\infty,z]$, and similarly (but symmetrically) if q is to the right of split(v). This

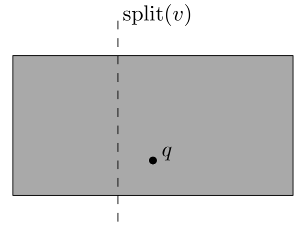

Figure 1: A 4-sided rectangle stored at v is reduced to 3-sided rectangles on either side of split(v) in an interval tree.

effectively reduces the problem to a 3-sided query, for which we can store [Afs08] at each node (see Figure 1). Since we perform at most $O(\lg n)$ of these queries (one at each level of the interval tree), the total query time is $O\left(\lg^2 n+k\right)$ and space is still $O(n)$. Call this structure D_4. We can use the same technique to solve 5-sided queries in $O\left(\lg^3 n+k\right)$ time by storing D_4 structures at the nodes of the interval tree obtained by y-projection instead. This proves the following:

Theorem 2 (2.1 in [Rah15])**.** *OPEQ on 5-sided rectangles can be answered using a structure of $O(n)$ size and $O\left(\lg^3 n+k\right)$ query time, where k is the size of the output.*

2 4-Sided: Faster with Near-Linear Space

Given two points $p,q\in\mathbb{R}^d$, then $p=(p_1,\ldots,p_d)$ *dominates* $q=(q_1,\ldots,q_d)$ if $p_i\geq q_i$ for all $i\in\{1,\ldots,d\}$. Given a set of points P, R is a *t-level shallow cutting* of P if (i) $|R|=O(n/t)$, (ii) any point p that is dominated by at most t points of P dominates a point in R, and (iii) each point in R is dominated by $O(t)$ points in P.

First, project the rectangles onto the plane. Then build an interval tree on the rectangles as we did in Section 1, such that at each node v, our 4-sided projections are stored as 3-sided rectangles either left or right of split(v).

At every node v of the interval tree, consider each 3-sided rectangle as a point, then we can take $\lg^{(i)} n$-level shallow cuttings R_i for all $0\leq i\leq \lg^* n$.

Question Why $\lg^* n$ levels of shallow cuttings?

Local Structure For each point $p\in R_i$, we build [Afs08] on all points that dominate p. Since p is in a $\lg^{(i)} n$-level shallow cutting of P, there are $O\left(\lg^{(i)} n\right)$ points that dominate p.

Lemma 1 (3.1 in [Rah15])**.** *Given a point set P in $\mathbb{R}^3$ and a strip $\mathcal{R}$ of the plane such that the projection of* *all points onto the plane falls within $\mathcal{R}$, we can compute a subdivision $\mathcal{A}$ of $\mathcal{R}$ with $|P|$ regions such that given a query point p in the plane, we can either find a point of P which dominates p, or no such point exists.*

Global Structure For every node v in the interval tree, for each R_i, compute the Lemma 1 subdivision $\mathcal{A}_i$, then store all the rectangles from these subdivisions in [Cha86].

To answer a query q, first query the global structure to find the largest i such that a point p corresponding to a rectangle from $\mathcal{A}_i$ encloses the query point. Then search the local structure of p, reporting all points (corresponding to rectangles of the input) that dominate q.

Space The local structure for each shallow cutting R_i takes $O(n)$ space, and there are $\lg^* n$ such cuttings.

Query Time Each of the $O(\lg n)$ nodes on the root to leaf path of the interval tree could return up to $O(\lg^* n)$ rectangles.

Remark You can instead build a data structure with $O(n)$ space and $O\left(\lg n \cdot \lg^{(c)} n + k\right)$ query time by only taking $c \geq 2$ shallow cuttings.

Theorem 3 (3.1 in [Rah15])**.** *OPEQ on 4-sided rectangles can be answered using a structure of $O(n \lg^* n)$ size and $O(\lg n \cdot \lg^* n + k)$ query time, where k is the size of the output.*

3 5-Sided: Putting it Together with Grids

We will use a grid technique adapted from [ABR00]. Let $t = \lg^4 m$, where m is the number of rectangles at the current level of recursion (initially n). Project S onto the plane, then draw an orthogonal $(2\sqrt{m/t}) \times (2\sqrt{m/t})$ grid over the resulting rectangles such that each horizontal and vertical slab contains the projections of $\sqrt{nt}$ sides. We create a tree by creating a node to store all rectangles which cross a grid line, then recurse on all other rectangles. Stop recursion when m reaches a constant. At each node of this tree: (i) *(slow)*: build the structure from Theorem 2, (ii) *(L_c)*: for each grid cell c, store at most $\lg^3 n$ of the rectangles which completely cover c in decreasing order of z span, and (iii) *(side)*: build the structure from Theorem 3 on the (at most 4) sides cut from each rectangle by the grid (see Figure 2).

To answer a query q, locate the grid cell c containing q and scan L_c, reporting rectangles until (a) we find a rectangle not containing q, or (b) we reach the end.

If (b), then $k \geq \lg^3 n$, thus querying the slow structure gives $O(k)$ query time. Otherwise, query the side structures then the recursive structures.

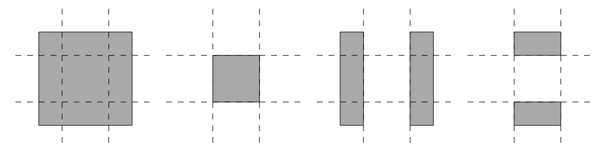

Figure 2: A rectangle (left) decomposes into an number of totally covered cells (middle-left), sides contained in adjacent vertical slabs (middle-right), and sides contained in adjacent horizontal slabs (right).

Space The bottleneck is the structure from Theorem 3 which occupies $O(n \lg^* n)$ space.

Query Time The bottleneck is the recursive case. Since we can find the grid cell containing q in $O(\lg n)$ time and the size of the subproblem is $\sqrt{nt}$, we get $Q(n) = Q(\sqrt{nt}) + O(\lg n)$. With $t = \lg^4 n$, this solves to $O(\lg n \lg\lg n + k)$ time, and this concludes the proof of Theorem 1.

Question Why doesn't it cost $\lg n \cdot \lg^* n$ at every level of this grid recursion?

Remark In the RAM model, we can find the grid cell containing q in $O(1)$ time. This suggests we can achieve $O(\lg\lg n + k)$ query time. Also, if Theorem 3 is not the time bottleneck, we can use the time-space trade-off to achieve $O(n)$ space.

References

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