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Algebraic conditions and the sparsity of spectrally arbitrary patterns

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Abstract: Given a square matrix A, replacing each of its nonzero entries with the symbol * gives its *zero-nonzero pattern*. Such a pattern is said to be *spectrally arbitrary* when it carries essentially no information about the eigenvalues of A. A longstanding open question concerns the smallest possible number of nonzero entries in an $n \times n$ spectrally arbitrary pattern. The Generalized 2n Conjecture states that, for a pattern that meets an appropriate irreducibility condition, this number is 2n. An example of Shitov shows that this irreducibility is essential; following his technique, we construct a smaller such example. We then develop an appropriate algebraic condition and apply it computationally to show that, for $n \le 7$, the conjecture does hold for \mathbb{R} , and that there are essentially only two possible counterexamples over \mathbb{C} . Examining these two patterns, we highlight the problem of determining whether or not either is in fact spectrally arbitrary over \mathbb{C} . A general method for making this determination for a pattern remains a major goal; we introduce an algebraic tool that may be helpful.

Keywords: 2*n* conjecture, spectrally arbitrary patterns, zero-nonzero patterns, sign patterns, eigenvalues

MSC: 15A18, 15B35, 15B99

1 Introduction

A problem of active interest in combinatorial matrix theory is to relate combinatorial properties of a matrix to properties of the linear operator it represents. The combinatorial properties in question are often formulated in terms of some description of the matrix that depends only the signs or on the zero-nonzero character of its entries. A simple example is the *zero-nonzero pattern* of the matrix, which simply specifies which entries of the matrix are nonzero. This information can itself be represented as a matrix, defined formally as follows.

Definition 1.1. A zero-nonzero pattern matrix is a matrix with entries from the set $\{*, 0\}$.

We associate a zero-nonzero pattern to each matrix *A* in the following natural way.

Definition 1.2. Let A be an $m \times n$ matrix over some field. The *zero-nonzero pattern of* A is the $m \times n$ zero-nonzero pattern matrix A such that the (i, j)-entry of A is * if and only if the (i, j)-entry of A is nonzero. In this case, we call A a *realization* of the pattern A.

For matrices over an ordered field, such as \mathbb{R} , a refinement of this definition is given by the notion of a *sign pattern*. This is a matrix with entries from the set $\{+, -, 0\}$ that is taken to describe the signs of the entries of

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a matrix in the obvious way. Here we focus on zero-nonzero patterns, but many of the questions we consider have natural analogs for sign patterns as well.

The zero-nonzero pattern of A gives a combinatorial description of A, leading to the question of what information this description conveys about the operator-theoretic properties of A. For example, in the case where A is a square matrix, what information does it carry about the spectrum of A?

This question is evidently not vacuous. For example, if A has only zero entries, then we certainly know everything about its eigenvalues. On the other hand, for $n \ge 2$, it follows from [13, Theorem 1.2] that any conjugate-closed multiset of n complex numbers is the spectrum of some $n \times n$ matrix over \mathbb{R} that has only nonzero entries. Hence, a natural subquestion of the above is that of when the zero-nonzero pattern of a matrix carries any information about its eigenvalues. In particular, a zero-nonzero pattern that is consistent with any appropriate choice of eigenvalues—or, equivalently, any choice of characteristic polynomial—over a certain field is said to be spectrally arbitrary over that field. This notion is given a precise definition as follows.

Definition 1.3. Let \mathbb{F} be a field. An $n \times n$ zero-nonzero pattern \mathcal{A} is said to be *spectrally arbitrary over* \mathbb{F} if for each monic polynomial $p(x) \in \mathbb{F}[x]$ of degree n, there exists a realization of \mathcal{A} over \mathbb{F} with characteristic polynomial p(x).

Hence, to say that the zero-nonzero pattern of a matrix carries some information about its eigenvalues is to say that the pattern is *not* spectrally arbitrary. We thus arrive at a central question, one which has received a great deal of attention.

Question 1.4. What combinatorial properties characterize those patterns that are spectrally arbitrary over a particular field?

A natural subquestion is the following.

Question 1.5. What is the smallest number of nonzero entries possible for an $n \times n$ zero-nonzero pattern that is spectrally arbitrary over a particular field?

This question was originally raised in [3] for sign patterns over \mathbb{R} . A partial answer was given there in the form of a lower bound of 2n-1 on the number of nonzero entries in a spectrally arbitrary sign pattern that is *irreducible*. The condition of being irreducible is an important one, and so we give its precise definition.

Definition 1.6. Let *A* be an $n \times n$ matrix. We say that *A* is *irreducible* if there is no permutation of $\{1, 2, ..., n\}$ that, when applied simultaneously to the rows and columns of *A*, results in a block upper triangular matrix.

The authors of [3] reported knowing of no spectrally arbitrary sign pattern achieving the lower bound of 2n-1. They did, on the other hand, give explicit families of $n \times n$ spectrally arbitrary sign patterns with 2n nonzero entries. In light of this, they put forward the following conjecture.

Conjecture 1.7 (The 2*n* Conjecture [3]). *Every* $n \times n$ *sign pattern that is spectrally arbitrary over* \mathbb{R} *has at least 2n nonzero entries.*

The method used in [3] to prove that the aforementioned examples were spectrally arbitrary has become known as the Nilpotent-Jacobian Method. It was shown in [2] that this method cannot successfully be applied to any pattern with fewer than 2*n* nonzero entries.

When considering the situation for zero-nonzero patterns, it is useful to observe that the set of matrices over \mathbb{R} with a given sign pattern is contained in the set of matrices that have the corresponding zero-nonzero pattern. It follows that a sign pattern cannot be spectrally arbitrary if its corresponding zero-nonzero pattern is not spectrally arbitrary. Hence, while Conjecture 1.7 asserts that no sign pattern with fewer than 2n nonzero entries is spectrally arbitrary, a stronger statement would be this same assertion for zero-nonzero patterns.

And the question of whether a particular zero-nonzero pattern is spectrally arbitrary may be considered over any field. Thus, the following could be considered the ultimate strengthening of Conjecture 1.7.

Conjecture 1.8 (The Generalized 2n Conjecture). Let \mathbb{F} be any field. Every irreducible $n \times n$ zero-nonzero pattern that is spectrally arbitrary over \mathbb{F} has at least 2n nonzero entries.

Conjecture 1.8 above, unlike Conjecture 1.7, includes the additional condition that the pattern is irreducible. This is necessary in the more general case where an arbitrary field is considered; Shitov demonstrated this in [15] by showing that a certain reducible $n \times n$ zero-nonzero pattern with 2n-1 nonzero entries is spectrally arbitrary over C. (In Section 3, we revisit the argument given there and present a smaller such example.)

Moreover, a theorem of [16] shows that the lower bound of 2n - 1 still holds over every field for zerononzero patterns that are irreducible, and the irreducibility is essential to the proof. For reference, we state this result below.

Theorem 1.9 ([16, Theorem 3]). Let \mathbb{F} be any field. Every irreducible $n \times n$ zero-nonzero pattern that is spectrally arbitrary over \mathbb{F} has at least 2n-1 nonzero entries.

Hence, to resolve Conjecture 1.8 it suffices to determine whether or not there exists an $n \times n$ irreducible zerononzero pattern with exactly 2n-1 nonzero entries that is spectrally arbitrary over some field. A counting argument is enough to establish that this cannot occur over any *finite* field, giving the following result.

Theorem 1.10 ([14]). Let \mathbb{F} be a finite field. Every irreducible $n \times n$ zero-nonzero pattern that is spectrally arbitrary over \mathbb{F} has at least 2n nonzero entries.

As noted in [16], the existence of irreducible polynomials of arbitrary degree over every finite field implies that the term "irreducible" can be removed from Theorem 1.10, since a reducible pattern cannot realize any irreducible polynomial, and therefore is already ruled out by the hypothesis that the pattern is spectrally arbitrary.

Returning to the case of \mathbb{R} , we briefly survey the current state of knowledge. For all spectrally arbitrary patterns, the lower bound of 2n nonzero entries was proved in [4] for $1 \le n \le 4$, and extended in [5] to n = 5. Because this work proved the bound for both reducible and irreducible zero-nonzero patterns, it established both Conjecture 1.7 and Conjecture 1.8 over \mathbb{R} for $1 \le n \le 5$. In addition, [5] established the bound for reducible zero-nonzero patterns (and hence also for reducible sign patterns) for n = 6 and n = 7. For these values of n, we show that Conjecture 1.8 does hold, which also implies that Conjecture 1.7 holds in the irreducible case; since that was the only remaining case for n = 6 and n = 7, our work here completes the proof of both conjectures for those values of n. Hence, what we prove here is sufficient to establish that Conjecture 1.7 holds, and that Conjecture 1.8 holds over \mathbb{R} , for $1 \le n \le 7$.

In fact, the results of the present paper subsume the case of \mathbb{R} , as our work applies over all fields with either characteristic 0 or characteristic p, except for some finitely many nonzero values of p. Conjecture 1.8 does hold over all such fields when $1 \le n \le 6$, and in the case of n = 7 there are (up to some natural notions of equivalence) only two patterns that could be counterexamples. We examine these patterns in Section 7, where we show that they are in fact not spectrally arbitrary over \mathbb{R} . Notably, however, the conjecture remains open over \mathbb{C} for n=7; essentially the only patterns of this size that could be counterexamples are the two mentioned above. Determining whether or not either of these is spectrally arbitrary over $\mathbb C$ seems to be a difficult problem. In Section 5 we highlight an algebraic tool that may be useful.

Our confirmation of the 2*n* Conjecture over \mathbb{R} for $1 \le n \le 7$ is achieved via a computer-assisted verification outlined in Section 6. This verification exploits an algebraic condition, introduced in Section 4, that can be checked easily by most computer algebra systems to rule out the possibility that a particular zero-nonzero pattern is spectrally arbitrary. Beyond enabling computational results, such algebraic conditions may also be useful for proving general results about spectrally arbitrary patterns. In Section 5 we prove a lemma that may

be useful in formulating further such conditions; in Section 7 we use this lemma to show that a particular pattern not ruled out by the computation is nevertheless not spectrally arbitrary over \mathbb{R} .

2 Preliminaries

Let \mathcal{A} be an $n \times n$ zero-nonzero pattern with m nonzero entries. Let A be the unique matrix with zero-nonzero pattern \mathcal{A} in which each nonzero entry is one of the algebraic indeterminates x_1, x_2, \ldots, x_m , with the indices of these indeterminates occurring in increasing order with respect to the usual row-major ordering of the matrix entries. Letting p(z) be the characteristic polynomial of A, we have

$$p(z) = z^{n} + \alpha_1 z^{n-1} + \alpha_2 z^{n-2} + \dots + \alpha_{n-1} z + \alpha_n, \tag{1}$$

where each α_i is a polynomial in the variables x_1, x_2, \ldots, x_m . Each α_i is homogeneous, as each of its terms is the product of i distinct variables. Moreover, its coefficients are from the set $\{1, -1\}$, so that we may, for any field \mathbb{F} desired, consider this family of polynomials as belonging to $\mathbb{F}[x_1, x_2, \ldots, x_m]$. When considering a pattern \mathcal{A} , we refer to these as its *associated polynomials*.

Example 2.1. The zero-nonzero pattern

That is, A is the matrix of indeterminates associated with the pattern \mathcal{A} in the manner described above. Hence, calculation of the coefficients of the characteristic polynomial of A shows that for this pattern the associated polynomials are:

$$\alpha_{1} = -x_{1} - x_{12}$$

$$\alpha_{2} = -x_{7}x_{8} - x_{2}x_{10} - x_{6}x_{11} + x_{1}x_{12} - x_{3}x_{13}$$

$$\alpha_{3} = x_{1}x_{7}x_{8} + x_{1}x_{6}x_{11} + x_{7}x_{8}x_{12} + x_{3}x_{12}x_{13} - x_{4}x_{13}x_{15}$$

$$\alpha_{4} = x_{2}x_{7}x_{8}x_{10} + x_{6}x_{7}x_{8}x_{11} - x_{5}x_{7}x_{9}x_{11} - x_{1}x_{7}x_{8}x_{12} - x_{3}x_{6}x_{10}x_{14} + x_{3}x_{7}x_{8}x_{13} + x_{3}x_{6}x_{11}x_{13}$$

$$+ x_{4}x_{12}x_{13}x_{15}$$

$$\alpha_{5} = -x_{1}x_{6}x_{7}x_{8}x_{11} + x_{1}x_{5}x_{7}x_{9}x_{11} - x_{3}x_{7}x_{8}x_{12}x_{13} - x_{4}x_{6}x_{10}x_{14}x_{15} + x_{4}x_{7}x_{8}x_{13}x_{15}$$

$$+ x_{4}x_{6}x_{11}x_{13}x_{15}$$

$$\alpha_{6} = x_{3}x_{6}x_{7}x_{8}x_{10}x_{14} - x_{3}x_{5}x_{7}x_{9}x_{10}x_{14} - x_{3}x_{6}x_{7}x_{8}x_{11}x_{13} + x_{3}x_{5}x_{7}x_{9}x_{11}x_{13} - x_{4}x_{7}x_{8}x_{12}x_{13}x_{15}$$

$$\alpha_{7} = x_{4}x_{6}x_{7}x_{8}x_{10}x_{14}x_{15} - x_{4}x_{5}x_{7}x_{9}x_{10}x_{14}x_{15} - x_{4}x_{6}x_{7}x_{8}x_{11}x_{13}x_{15} + x_{4}x_{5}x_{7}x_{9}x_{11}x_{13}x_{15}$$

A cursory glance at the polynomials above may reveal little reason to suspect that \mathcal{A} cannot be spectrally arbitrary. But a deeper investigation reveals that

$$x_4x_{15}\alpha_6-x_3\alpha_7=-x_4^2x_7x_8x_{12}x_{13}x_{15}^2.$$

In particular, this shows that α_6 and α_7 cannot be simultaneously zero while every x_i is nonzero. Hence, no realization of \mathcal{A} can have $\alpha_6 = \alpha_7 = 0$, and so \mathcal{A} is not spectrally arbitrary.

In the above example, the pattern \mathcal{A} is seen not to be spectrally arbitrary because it is not possible for all of its associated polynomials to vanish simultaneously without some x_i being zero. That is, it fails the following necessary (but not sufficient) condition for a pattern to be spectrally arbitrary.

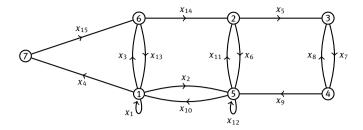


Figure 1: Weighted digraph for the pattern \mathcal{A} of Example 2.1.

Definition 2.2. Let \mathbb{F} be a field. An $n \times n$ zero-nonzero pattern \mathcal{A} is said to be *potentially nilpotent* over \mathbb{F} if \mathcal{A} has some realization A with entries in \mathbb{F} and the characteristic polynomial x^n .

We make use of the following standard connection between a square zero-nonzero pattern and a directed graph.

Definition 2.3. Let \mathcal{A} be an $n \times n$ zero-nonzero pattern. The *digraph* of \mathcal{A} is the directed graph on vertices 1, 2, . . . , n in which there is an arc from vertex i to vertex j precisely when the (i, j)-entry of \mathcal{A} is *. (Note that this digraph may have loops.)

A realization of a zero-nonzero pattern \mathcal{A} can be thought of as a way of assigning a nonzero weight to each edge of the digraph of \mathcal{A} ; the weight assigned to each arc is the value of the corresponding nonzero entry in the realization. One case of particular importance is where the nonzero entries are distinct algebraic indeterminates. The next example illustrates how these become weights for the arcs of the digraph.

Example 2.4. Figure 1 shows the digraph of the pattern \mathcal{A} of Example 2.1 with weights given as distinct variables; this corresponds to the realization A given in (2).

The condition of a pattern being irreducible (see Definition 1.6) has a simple interpretation in terms of its digraph.

Theorem 2.5 ([11, Theorem 6.2.24]). A pattern is irreducible if and only if its digraph is strongly connected.

When considering a realization of an irreducible zero-nonzero pattern, the following theorem allows us to apply a helpful normalization.

Theorem 2.6 ([7, Theorem 2.3]). Let A be an irreducible $n \times n$ matrix, and $S = \{(i_1, j_1), \ldots, (i_{n-1}, j_{n-1})\}$ be a set of positions of A whose entries are nonzero such that the corresponding arcs give a spanning tree in the underlying graph of the digraph of A. Then there exist nonzero values d_1, d_2, \ldots, d_n such that for $D = \text{diag}(d_1, \ldots, d_n)$ the (i_k, j_k) -entry of DAD^{-1} is equal to 1 for all $k \in \{1, \ldots, n-1\}$.

By the above theorem, when considering the question of which spectra are realized by some particular irreducible pattern, there exist n-1 entries which we may assume without loss of generality are equal to 1 in an arbitrary realization of the pattern. This can be of tremendous advantage to the efficiency of computations. To illustrate such a normalization, we revisit the pattern from Example 2.1.

Example 2.7. For the pattern \mathcal{A} of Example 2.1, the arcs corresponding to x_2 , x_4 , x_5 , x_7 , x_{14} , x_{15} form a spanning tree as required by Theorem 2.6. Thus, we can normalize these entries in the realization A from (2.1)

so as to give

To see the effect on the associated polynomials (3), we show these polynomials below, with each normalized variable replaced with a 1. (We display the 1s to highlight precisely where the normalization has an effect.)

$$\begin{split} &\alpha_{1} = -x_{1} - x_{12} \\ &\alpha_{2} = -1 \cdot x_{8} - 1 \cdot x_{10} - x_{6} x_{11} + x_{1} x_{12} - x_{3} x_{13} \\ &\alpha_{3} = x_{1} \cdot 1 \cdot x_{8} + x_{1} x_{6} x_{11} + 1 \cdot x_{8} x_{12} + x_{3} x_{12} x_{13} - 1 \cdot x_{13} \cdot 1 \\ &\alpha_{4} = 1 \cdot 1 \cdot x_{8} x_{10} + x_{6} 1 \cdot x_{8} x_{11} - 1 \cdot 1 \cdot x_{9} x_{11} - x_{1} 1 \cdot x_{8} x_{12} - x_{3} x_{6} x_{10} 1 \cdot + x_{3} 1 \cdot x_{8} x_{13} \\ &\quad + x_{3} x_{6} x_{11} x_{13} + 1 \cdot x_{12} x_{13} \cdot 1 \\ &\alpha_{5} = -x_{1} \cdot 1 \cdot 1 \cdot x_{8} x_{11} + x_{1} \cdot 1 \cdot 1 \cdot x_{9} x_{11} - x_{3} \cdot 1 \cdot x_{8} x_{12} x_{13} - 1 \cdot x_{6} x_{10} \cdot 1 \cdot 1 \\ &\quad + 1 \cdot 1 \cdot x_{8} x_{13} \cdot 1 + 1 \cdot x_{6} x_{11} x_{13} \cdot 1 \\ &\alpha_{6} = x_{3} x_{6} \cdot 1 \cdot x_{8} x_{10} \cdot 1 - x_{3} \cdot 1 \cdot 1 \cdot x_{9} x_{10} \cdot 1 - x_{3} x_{6} \cdot 1 \cdot x_{8} x_{11} x_{13} + x_{3} \cdot 1 \cdot 1 \cdot x_{9} x_{11} x_{13} - 1 \cdot 1 \cdot x_{8} x_{12} x_{13} \cdot 1 \\ &= x_{3} x_{6} x_{8} x_{10} - x_{3} x_{9} x_{10} - x_{3} x_{6} x_{8} x_{11} x_{13} + x_{3} x_{9} x_{11} x_{13} \cdot 1 + 1 \cdot 1 \cdot 1 \cdot x_{9} x_{11} x_{13} \cdot 1 \\ &= x_{6} x_{8} x_{10} - x_{9} x_{10} - x_{6} x_{8} x_{11} x_{13} + x_{9} x_{11} x_{13} \end{split}$$

Comparing the expressions for α_6 and α_7 above, we see that, once we neglect the term of $-x_8x_{12}x_{13}$ that appears in the expression for α_6 , the two differ by a factor of x_3 . In particular, then,

$$\alpha_6 - x_3 \alpha_7 = -x_8 x_{12} x_{13}$$
.

This example illustrates how the normalization made the relevant algebraic relationship between α_6 and α_7 easier to detect as compared with (3). What we gain from this is the knowledge that α_6 and α_7 cannot be simultaneously zero while all of the x_i are nonzero. This implies that \mathcal{A} is not potentially nilpotent, as no realization of \mathcal{A} can have $\alpha_6 = \alpha_7 = 0$. Hence, \mathcal{A} is not spectrally arbitrary.

The associated polynomials of \mathcal{A} are determined (up to permutation of the x_i) by the digraph of \mathcal{A} . This observation allows the computer search outlined in Section 6 to consider what happens algebraically for all patterns by conducting a systematic search of the strongly connected digraphs with 2n - 1 edges.

3 Reducibility and the generalized 2n Conjecture

As mentioned in Section 1, the irreducibility of the pattern is a necessary condition in Conjecture 1.8. This was first demonstrated in [15], by the construction of a reducible $n \times n$ zero-nonzero pattern that is spectrally arbitrary over \mathbb{C} despite having only 2n-1 nonzero entries, for n=708. Here we show that the same idea can be used to construct such an example with n=49.

The following fact is used explicitly in [15] and is a key to the argument there. Although the proof is basic, we include it here for the sake of completeness.

Lemma 3.1. Let \mathbb{F} be a field, let $p \in \mathbb{F}[x_1, \dots, x_n]$ be a polynomial of total degree k, and let $S \subseteq \mathbb{F}$ with |S| = n + k. Then there exists a subset $\{a_1, \dots, a_n\}$ of S such that $p(a_1, \dots, a_n) \neq 0$.

Proof. The proof is by induction on n. For n = 1, the conclusion follows since p is univariate of degree k, and hence cannot have more than k roots in \mathbb{F} . For n > 1, consider p as a polynomial in x_n with coefficients in $\mathbb{F}[x_1, \ldots, x_{n-1}]$. That is, let

$$p(x_1,\ldots,x_n) = \sum_{i=0}^t q_i(x_1,\ldots,x_{n-1})x_n^i,$$
 (4)

where t is the degree of p as a polynomial in x_n (i.e., t is the highest power to which x_n appears in any term of p). In particular, the total degree of q_t is at most k-t. Hence, by the inductive hypothesis, any subset of t of size t

We now follow the argument from [15] to give a smaller example showing the necessity of the irreducibility condition in Conjecture 1.8.

Theorem 3.2. For every $n \ge 49$, there exists an $n \times n$ reducible zero-nonzero pattern with 2n-1 nonzero entries that is spectrally arbitrary over \mathbb{C} . Moreover, when n=45+4k for some $k \ge 1$, there is an $n \times n$ reducible zero-nonzero pattern with 2n-k nonzero entries that is spectrally arbitrary over \mathbb{C} .

Proof. We begin with the 4×4 pattern

$$\mathcal{A} = \begin{bmatrix} 0 & * & 0 & * \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & 0 \end{bmatrix} \quad \text{and take} \quad A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & x_1 & 1 & 0 \\ 0 & 0 & x_2 & x_3 \\ x_4 & 0 & 0 & 0 \end{bmatrix}$$

as its suitably-normalized generic realization. Then taking $x_1 = x_4 = -1$ and $x_2 = x_3 = 1$ gives a nilpotent realization, while $x_1 = 4 - 2i$, $x_2 = 4 + 2i$, $x_3 = -16$, and $x_4 = -4$ give a realization with characteristic polynomial $(z-2)^4$. Note that this nilpotent realization, together with scalar multiples of the latter realization, show that \mathcal{A} realizes any characteristic polynomial with four identical roots.

Meanwhile, one may verify that A has the characteristic polynomial $z^4 + t_1z^3 + t_2z^2 + t_3z + t_4$ when its entries are chosen according to the equations

$$x_{4} = -\frac{t_{3}}{t_{1}},$$

$$x_{3} = -\frac{t_{1}t_{2}t_{3} - t_{1}^{2}t_{4} - t_{3}^{2}}{t_{1}t_{3}},$$

$$x_{2}^{2} + t_{1}x_{2} + \frac{t_{1}t_{2} - t_{3}}{t_{1}} = 0, \text{ and}$$

$$x_{1} = -x_{2} - t_{1}.$$
(5)

Hence, a realization of \mathcal{A} with this characteristic polynomial exists when all of the denominators in these equations are nonzero and the values for x_1 through x_4 given by the equations are nonzero. Those two conditions will be met provided

$$t_1t_3(t_1t_2-t_3)(t_1t_2t_3-t_1^2t_4-t_3^2)\neq 0,$$
 (6)

since, in particular, the above condition ensures that the constant term of the quadratic (5) is nonzero, so that neither 0 nor $-t_1$ can be among its roots, ensuring that it is possible to choose a nonzero value of x_2 according to (5) that still results in a nonzero value for x_1 .

We now proceed as in [15]. First, let $\lambda_1, \ldots, \lambda_4$ be the eigenvalues of A. Then each t_i is given by $(-1)^i$ times the ith elementary symmetric polynomial in $\lambda_1, \ldots, \lambda_4$. Hence, we may consider (6) as a polynomial in $\lambda_1, \ldots, \lambda_4$. As in [15], we call this polynomial ψ . We see from (6) that the total degree of ψ (in terms of

the variables λ_j) is 13. Therefore, by Lemma 3.1, given any 17 distinct complex numbers, some subset of 4 of them can be assigned as values to the λ_j in such a way that ψ will not vanish, so that these give a spectrum realizable by \mathcal{A} .

Therefore, if a set of complex numbers does not contain any subset of 4 values giving a realizable spectrum of A, then it can contain at most 16 distinct values, and (by the initial paragraph of the proof) each of those can occur at most 3 times. Hence, such a set can have size at most 48. Thus, any set of at least 49 complex numbers contains a spectrum realizable by A.

It follows from the above that the direct sum of \mathcal{A} with any spectrally arbitrary pattern of order at least 45 yields a (reducible) spectrally arbitrary pattern. In particular, results of [6, 8] show that a particular well-studied $k \times k$ zero-nonzero pattern denoted by \mathfrak{T}_k is spectrally arbitrary for every $k \ge 3$. Thus, for each $n \ge 49$ we have that $\mathcal{A} \oplus \mathfrak{T}_{n-4}$ is a reducible zero-nonzero pattern of order n with 2n-1 nonzero entries that is spectrally arbitrary over \mathbb{C} .

Building on the above, the direct sum of \mathcal{A} with the spectrally arbitrary pattern $\mathcal{A} \oplus \mathcal{T}_{45}$ yields a spectrally arbitrary pattern of order n=53 with 2n-2 nonzero entries. Repeating this argument shows that the direct sum of \mathcal{T}_{45} with k copies of \mathcal{A} gives an $n \times n$ zero-nonzero pattern that is spectrally arbitrary over \mathbb{C} and has 2n-k nonzero entries, with n=45+4k.

4 Algebraic conditions

In order for a pattern \mathcal{A} to be spectrally arbitrary, it must allow a zero trace, which implies that the associated polynomial α_1 cannot be a monomial. Of course, more generally, in order for \mathcal{A} to be potentially nilpotent, *none* of the polynomials α_i can be a monomial. This condition can be generalized even further; not only may none of the polynomials be a monomial, but in fact there cannot be any monomial in the ideal they generate. That is, we have the following result of [18].

Theorem 4.1 ([18, Lemma 3.3]). Let \mathbb{F} be a field and let \mathcal{A} be an $n \times n$ zero-nonzero pattern with m nonzero entries and associated polynomials $\alpha_1, \ldots, \alpha_n$. If

$$x_1x_2\cdots x_m\in\sqrt{\langle\alpha_1,\ldots,\alpha_n\rangle}\subseteq\mathbb{F}[x_1,x_2,\ldots,x_m],$$
 (7)

then A is not potentially nilpotent over \mathbb{F} .

The notation used in (7) serves to emphasize that the ideal $\sqrt{\langle \alpha_1, \ldots, \alpha_n \rangle}$ is considered as generated over \mathbb{F} . In the case where $\mathbb{F} = \mathbb{Q}$, we can check condition (7) efficiently using any standard computer algebra package. The next result shows that this check over \mathbb{Q} alone is sufficient to rule out \mathcal{A} being potentially nilpotent over all but a very limited class of fields.

Theorem 4.2. Let A be an $n \times n$ zero-nonzero pattern with m nonzero entries and associated polynomials $\alpha_1, \ldots, \alpha_n$. If

$$x_1x_2\cdots x_m\in\sqrt{\langle\alpha_1,\ldots,\alpha_n\rangle}\subseteq\mathbb{Q}[x_1,x_2,\ldots,x_m],$$
 (8)

then A is not potentially nilpotent, except possibly over some fields of characteristic $q \neq 0$, for finitely many values of q.

Proof. Suppose equation (8) holds. This means that some $k \in \mathbb{Z}^+$ gives $(x_1x_2 \cdots x_m)^k \in \langle \alpha_1, \dots, \alpha_n \rangle$, meaning that

$$(x_1 x_2 \cdots x_m)^k = \sum_{i=1}^n p_i(x_1, \dots, x_m) \alpha_i(x_1, \dots, x_m)$$
 (9)

for some $p_1, \ldots, p_n \in \mathbb{Q}[x_1, \ldots, x_m]$.

We first consider the case where $\mathbb F$ is a field of characteristic 0. In this case, we can rely on the standard homomorphism from $\mathbb Q$ to $\mathbb F$. Since this homomorphism extends to the respective polynomial rings, it follows

that (9) holds over $\mathbb{F}[x_1,\ldots,x_m]$ when each polynomial on the right-hand side is replaced with its homomorphic image. (The images of the α_i simply give the associated polynomials as we would consider them over \mathbb{F} .) Hence, any assignment of a value in \mathbb{F} to each of x_1,\ldots,x_m that causes every α_i to vanish necessarily causes $x_1x_2\cdots x_m$ to vanish as well, implying that at least one x_i must have been assigned a value of 0. Since each realization A of the zero-nonzero pattern A corresponds to an assignment of a *nonzero* value to each of x_1,\ldots,x_m , it follows that no such realization can cause every α_i to vanish, meaning that no such realization can have the characteristic polynomial x^n .

Now consider the case where \mathbb{F} is a field of characteristic $q \neq 0$. Let M be the least common multiple of all the denominators of coefficients appearing in the polynomials p_i . Then we have

$$M \cdot (x_1 x_2 \cdots x_m)^k = \sum_{i=1}^n \hat{p}_i(x_1, \dots, x_m) \alpha_i(x_1, \dots, x_m),$$
 (10)

where $\hat{p}_1, \ldots, \hat{p}_n \in \mathbb{Z}[x_1, \ldots, x_m]$. Let D be the finite set of prime divisors of M. When $q \notin D$, we have that q is relatively prime to M. In this case, applying the standard homomorphism from \mathbb{Z} to \mathbb{F} to (10) shows that a nonzero multiple of $x_1x_2\cdots x_m$ is contained in the ideal generated by $\alpha_1, \ldots, \alpha_n$ in $\mathbb{F}[x_1, \ldots, x_m]$. The remainder of the proof proceeds as in the characteristic 0 case.

To say that a zero-nonzero pattern \mathcal{A} is potentially nilpotent is to say that it allows all of the coefficients of the characteristic polynomial (other than the leading coefficient) to vanish simultaneously. It will be useful to consider a stronger condition that includes this as a special case.

Definition 4.3. Let \mathcal{A} be an $n \times n$ zero-nonzero pattern. We say that \mathcal{A} is *coefficient support arbitrary* if for every $S \subseteq \{1, \ldots, n\}$, there is a matrix A with zero-nonzero pattern \mathcal{A} such that $\alpha_i \neq 0$ if and only if $i \in S$.

That is, to say that \mathcal{A} is coefficient support arbitrary is to say that \mathcal{A} allows the vector of coefficients $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{F}^n$ to take on any support desired. In particular, then, its support can be empty, and so a zero-nonzero pattern that is coefficient support arbitrary is necessarily potentially nilpotent. But it is easy to see that the converse is not true; that is, a pattern can be potentially nilpotent without being coefficient support arbitrary.

Of course, a pattern that fails to be coefficient support arbitrary cannot be spectrally arbitrary. Just as with the condition of being potentially nilpotent, there is an algebraic condition which, if satisfied, implies that the pattern is not coefficient support arbitrary. This condition is given by the next theorem, whose proof follows closely that of Theorem 4.2.

Theorem 4.4. Let A be an $n \times n$ zero-nonzero pattern with m nonzero entries and associated polynomials $\alpha_1, \ldots, \alpha_n$. If there exists an $S \subseteq \{1, \ldots, n\}$ such that

$$x_1x_2\cdots x_m\prod_{i\in S}\alpha_i\in\sqrt{\langle\alpha_i:i\notin S\rangle}\subseteq\mathbb{Q}[x_1,\ldots,x_m],$$
(11)

then A is not coefficient support arbitrary, except possibly over some fields of characteristic $q \neq 0$, for finitely many values of q.

Proof. Suppose $S \subseteq \{1, ..., n\}$ satisfies condition (11). This means that there exists some $k \in \mathbb{Z}^+$ such that

$$\left(x_1 x_2 \cdots x_m \prod_{j \in S} \alpha_j\right)^k = \sum_{i \notin S} p_i(x_1, \dots, x_m) \alpha_i(x_1, \dots, x_m), \tag{12}$$

for some polynomials $p_i \in \mathbb{Q}[x_1, \dots, x_m]$. Let M be the least common multiple of the denominators of the coefficients appearing in the polynomials p_i on the right-hand side of (12). Then we have

$$M \cdot \left(x_1 x_2 \cdots x_m \prod_{j \in S} \alpha_j\right)^k = \sum_{i \notin S} \hat{p}_i(x_1, \dots, x_m) \alpha_i(x_1, \dots, x_m)$$
(13)

for some polynomials \hat{p}_i with coefficients in \mathbb{Z} . Let D be the finite set of prime divisors of M.

Let A be a realization of A over some field \mathbb{F} . Suppose for the sake of contradiction that when the variables x_1, \ldots, x_m are assigned values according to the corresponding nonzero entries of A, the support of the vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$ is equal to S. This means that α_j is zero when $j \notin S$ and nonzero when $j \in S$.

When \mathbb{F} has either characteristic 0, or has characteristic $q \notin D$ (so that q is relatively prime to M), the image of (13) under the standard homomorphism from \mathbb{Z} to \mathbb{F} gives an equation over $\mathbb{F}[x_1, \ldots, x_m]$ in which the left-hand side does not vanish, but the right-hand side does, when the variables take on those values as given by the entries of A. This contradiction shows that A cannot be coefficient support arbitrary over \mathbb{F} . \square

A converse for Theorem 4.4 holds over \mathbb{C} , due to this field being algebraically closed.

Theorem 4.5. Let A be a zero-nonzero pattern. If A is not coefficient support arbitrary over \mathbb{C} , then A satisfies (11) for some $S \subseteq \{1, \ldots, n\}$.

Proof. Let \mathcal{A} be a pattern that is not coefficient support arbitrary over \mathbb{C} ; that is, suppose there is a set $S \subseteq \{1, \ldots, n\}$ such that there is no realization of \mathcal{A} for which the vector $(\alpha_1, \ldots, \alpha_n)$ of associated polynomials has support S.

This means that there are no nonzero values for x_1, \ldots, x_m for which $\alpha_i \neq 0$ for $i \in S$ while $\alpha_i = 0$ for $i \notin S$. In other words, any common zero of the polynomials in the set $\{\alpha_i : i \notin S\}$ must also be a zero of the polynomial $x_1x_2 \cdots x_m \prod_{j \in S} \alpha_j$. Because $\mathbb C$ is algebraically closed, it follows from Hilbert's Nullstellensatz that

$$x_1x_2\cdots x_m\prod_{j\in S}\alpha_j\in\sqrt{\langle\alpha_i:i\notin S\rangle}\subseteq\mathbb{C}[x_1,\ldots,x_m].$$
 (14)

This means that there exists some $k \in \mathbb{Z}^+$ such that

$$\left(x_1x_2\cdots x_m\prod_{j\in S}\alpha_j\right)^k\in\langle\alpha_i:i\not\in S\rangle. \tag{15}$$

Note that the ideals in equations (14) and (15) are ideals of $\mathbb{C}[x_1, \ldots, x_m]$. However, since the polynomial on the left-hand side of equation (15) is in $\mathbb{Q}[x_1, \ldots, x_m]$, it is contained in

$$\mathbb{O}[x_1,\ldots,x_m]\cap\langle\alpha_i:i\notin S\rangle$$
,

which is an ideal of $\mathbb{Q}[x_1,\ldots,x_m]$. Moreover, since the polynomials α_i are themselves contained in $\mathbb{Q}[x_1,\ldots,x_m]$, this ideal is identical with the ideal generated by those polynomials over $\mathbb{Q}[x_1,\ldots,x_m]$. It follows that \mathcal{A} satisfies condition (11) for the same set \mathcal{S} .

For a given pattern \mathcal{A} , the result of Theorem 4.5 also holds over any algebraically closed field of characteristic $p \neq 0$, except possibly for finitely many values of p. This can be shown by adapting the proof of Theorem 4.5 by arguing as in the proof of Theorem 4.4.

5 Further algebraic conditions

Theorem 4.4 gives an algebraic condition, that of (11), whose utility seems somewhat one-sided. If the condition is satisfied for a particular pattern, then the pattern is not spectrally arbitrary. However, when the condition is not satisfied, this does not give any immediate conclusion. In contrast, it would be desirable to supply a purely algebraic condition that is *equivalent* to a pattern being spectrally arbitrary. (The analogous goal for sign patterns over $\mathbb R$ was stated as [14, Problem 14] in the unpublished notes of Shader.) In this section, we show that the question can be reframed in terms of the surjectivity of a related map that is simpler to study from an algebraic standpoint.

Definition 5.1. Let \mathcal{A} be an $n \times n$ zero-nonzero pattern over \mathbb{F} , and let $\alpha_1, \ldots, \alpha_n$ be the associated polynomials for \mathcal{A} as defined in (1). We define the *polynomial map associated with* \mathcal{A} to be the map $F_{\alpha} : \mathbb{F}^m \to \mathbb{F}^n$ defined by

$$F_{\alpha}(x_1,\ldots,x_m)=(\alpha_1(x_1,\ldots,x_m),\ldots,\alpha_n(x_1,\ldots,x_m)). \tag{16}$$

In terms of Definition 5.1, the pattern \mathcal{A} is spectrally arbitrary if and only if F_{α} maps $(\mathbb{F} \setminus \{0\})^m$ surjectively onto \mathbb{F}^n . Considering $(\mathbb{F} \setminus \{0\})^m$ as the domain of the function (as opposed to simply \mathbb{F}^m) may complicate the application of algebraic (and, when \mathbb{F} is taken to be \mathbb{R} or \mathbb{C} , analytic or topological) tools. However, the following lemma shows that it is possible to instead consider a related map that avoids this complication.

Lemma 5.2. Let A be an $n \times n$ zero-nonzero pattern over \mathbb{F} , and let $F_{\alpha} : \mathbb{F}^m \to \mathbb{F}^n$ be the polynomial map associated with A. Let $\hat{F}_{\alpha} : \mathbb{F}^{m+1} \to \mathbb{F}^{n+1}$ be defined by

$$(x_1, x_2, \ldots, x_m, y) \mapsto (f_1, f_2, \ldots, f_n, x_1 x_2 \cdots x_m y),$$

where $f_i = \alpha_i(x_1, x_2, ..., x_m)$. Then \mathcal{A} is spectrally arbitrary over \mathbb{F} if and only if \hat{F}_{α} is surjective.

Proof. Suppose \mathcal{A} is spectrally arbitrary. Then for each $(a_1, a_2, \ldots, a_n, a_{n+1}) \in \mathbb{F}^{n+1}$ there is a choice of values for x_1, \ldots, x_m such that $f_1 = a_1, f_2 = a_2, \ldots$, and $f_n = a_n$ such that each x_i is nonzero, allowing y to be chosen so that $x_1 \cdots x_m y = a_{n+1}$. Therefore, \hat{F}_{α} is surjective.

Now assume \hat{F}_{α} is surjective. Then, for any $a_1, a_2, \ldots, a_n \in \mathbb{F}$, there exist $x_1, \ldots, x_m, y \in \mathbb{F}$ so that $\hat{F}_{\alpha}(x_1, x_2, \ldots, x_m, y) = (a_1, a_2, a_3, \ldots, a_n, 1)$. In particular, $x_1x_2 \cdots x_my = 1$, implying that the x_i are all nonzero. This shows that for each $(a_1, a_2, \ldots, a_n) \in \mathbb{F}^n$ there is a choice of nonzero x_i at which the polynomial map F_{α} associated with A evaluates to (a_1, a_2, \ldots, a_n) . Therefore, A is spectrally arbitrary over A.

The question of whether a polynomial map such as the \hat{F}_{α} of Lemma 5.2 is surjective is one that can be studied from a variety of perspectives. For example, the tools of elimination theory from algebraic geometry allow one to gain information about this question by considering the polynomial ideal

$$I = \langle f_1 - c_1, \dots, f_n - c_n, x_1 x_2 \cdots x_m y - c_{n+1} \rangle \subseteq \mathbb{F}[x_1, \dots, x_m, y, c_1, \dots, c_n, c_{n+1}].$$

One may ask whether the variety V(I) associated with this ideal, when projected onto its final n+1 coordinates, gives all of \mathbb{F}^{n+1} . This question is equivalent to that of whether \hat{F}_{α} is surjective, which by Lemma 5.2 is equivalent to the question of whether \mathcal{A} is spectrally arbitrary. This reformulation of the question can be studied by considering the appropriate elimination ideal, many properties of which can be computed efficiently using Gröbner bases.

6 Verification of the 2n Conjecture for $n \le 7$

Let \mathcal{A} be an $n \times n$ zero-nonzero pattern, and recall that if \mathcal{A} satisfies condition (11) of Theorem 4.4 for some $S \subseteq \{1, \ldots, n\}$, then \mathcal{A} cannot be coefficient support arbitrary, and hence cannot be spectrally arbitrary. For a fixed such S, it is computationally feasible to use a computer algebra system to check this condition, at least when the number of nonzero entries of \mathcal{A} is not large. Thus, in light of Conjecture 1.8, it is a natural idea to use such software to check, for a fixed n and each possible S, whether this condition is satisfied for each pattern \mathcal{A} with 2n-1 nonzero entries.

Of course, there is no reason to expect that every pattern that fails to be spectrally arbitrary in fact fails the condition given in (11). In particular, it might happen that a pattern is coefficient support arbitrary without being spectrally arbitrary. At the same time, it seems plausible that a pattern might have to satisfy (11) if it fails to be spectrally arbitrary *for the reason that it is too sparse*.

In any case, an exhaustive computation was performed using the SageMath software package [17]. To reduce the number of patterns to a feasible number, some avoidance of equivalent patterns was necessary. For zero-nonzero patterns, the appropriate notion of equivalence is provided by two observations.

- 1. Taking the transpose of a matrix does not affect its characteristic polynomial. So two patterns may be considered equivalent when they are transposes of one another.
- 2. Applying a permutation similarity to a matrix does not affect its characteristic polynomial, and two patterns are permutation similar if and only if they have the same digraph. So two patterns may be considered equivalent when their digraphs are isomorphic.

The algorithm used in the computation was designed to exploit observation (2). Thus, an important feature provided by the SageMath software package was isomorph-free iteration over all digraphs of a fixed order. In particular, the software includes an implementation of the "orderly generation" scheme of McKay [12] that allows the generation of every digraph on a fixed number of vertices that has some property P, as long as property P is preserved by deleting edges. Of course, this does not hold for the property of having exactly 2n-1 edges. But the property of having at most 2n-1 edges is a suitable property with which to apply the generation scheme, and that is how the computer search proceeds.

The version of SageMath used did not allow the orderly generation of digraphs with loops. Therefore, the approach adopted was first to generate every digraph on n vertices with $m \le 2n - 3$ non-loop edges (since, for a pattern to be spectrally arbitrary, its digraph must have at least 2 loops) and then, for each such digraph that was strongly connected (so as to give an irreducible pattern), to consider every way of assigning loops to some 2n - 1 - m of the vertices.

For each digraph with loops so generated, a spanning tree was chosen (essentially at random) and the entries corresponding to the edges of this spanning tree were normalized to 1. Then the associated polynomials were computed and the condition given in (11) was checked. The complete search algorithm as implemented is outlined in Algorithm 1, while the actual source code used appears in the Appendix.

for each loopless digraph D with n vertices and $m \le 2n - 3$ edges **do**

```
if D is strongly connected then
    \ell \leftarrow (2n-1)-m;
    for each L \subseteq \{1, 2, ..., n\} with |L| = \ell do
         D' \leftarrow the result of adding a loop to D at vertex i for every i \in L;
         T \leftarrow a spanning tree of D':
         \mathcal{A} \leftarrow the n \times n zero-nonzero pattern corresponding to D';
         A \leftarrow realization of A with those entries corresponding to edges of T equal to 1 and the
          other n entries equal to the indeterminates x_1, x_2, \ldots, x_n;
         for each S \subseteq \{1, \ldots, n\} do
             record whether or not (11) is satisfied
         end
        if there is no S for which (11) is satisfied then
             record that A is 'exceptional' in that it may be spectrally arbitrary
         else
             record that A is not spectrally arbitrary (by Theorem 4.4)
         end
    end
end
```

Algorithm 1: Procedure used for the exhaustive consideration of sparse $n \times n$ patterns using SageMath (see the Appendix for the source code)

The results of the computer search are summarized in Table 1. Note that the number of patterns counted in the second column of the table is not, as would be ideal, equal to the number of inequivalent $n \times n$ patterns having 2n-1 nonzero entries and a digraph with at least two loops. In particular, although every such pattern is included in the search, some patterns are generated more than once. This is the case for two reasons. First,

Table 1: Results of exhaustive search of irreducible patterns with 2n-1 nonzero entries using Algorithm 1. Note that the number of patterns generated is not equal to the number of inequivalent such patterns (see discussion).

n	Number of patterns generated	Number satisfying (11) for some S
3	3	3
4	28	28
5	655	655
6	22,671	22,671
7	991,242	991,237
8	53,132,934	unknown

the algorithm is not able to avoid considering a pattern and also its transpose, which corresponds to the same digraph but with every edge reversed. In addition, it may happen that different ways of adding loops to the "base" digraph *D* in Algorithm 1 result in isomorphic digraphs; in this case, each one will be considered separately and contribute separately to the count given in the table.

As the table shows, for $n \le 6$, every pattern generated was found to satisfy condition (11) for some set S. By Theorem 4.4, this implies that when $n \le 6$, every irreducible $n \times n$ zero-nonzero pattern with 2n-1 nonzero entries fails to be coefficient support arbitrary over every field of characteristic 0. (The theorem also implies that no such pattern is coefficient support arbitrary over any field of characteristic $p \ne 0$, except possibly for finitely many values of p.) Combined with Theorem 1.10, this shows that the Generalized 2n Conjecture (i.e., Conjecture 1.8) holds for $n \le 6$, except possibly over some infinite fields of characteristic p > 0, for finitely many values of p.

The above observations would apply for n=7 as well, except that for exactly 5 of the 991,242 patterns considered by the search algorithm, the condition given in (11) is not satisfied for any $S\subseteq\{1,2,\ldots,7\}$. Among these patterns we find two distinct pairs of a pattern and its transpose. Hence, after taking into account equivalence, there are only 3 different patterns to consider. Moreover, 2 of these 3 differ only in the placement of one of the loops in such a way that happens not to affect the associated polynomials. Hence, there are in effect only two different 7×7 patterns that could be counterexamples to Conjecture 1.8 over a field of characteristic 0 or $q\neq0$, except possibly for finitely many values of q. It follows that for n=7 it suffices to consider only signings of these as potential counterexamples to the original Conjecture 1.7. We present these patterns in detail in the next section, where we show that neither one is spectrally arbitrary over \mathbb{R} . Hence, in fact no signing of either one can give a counterexample to Conjecture 1.7. Therefore, the original 2n Conjecture does in fact hold for $n\leq7$. Meanwhile, the above observations show that the stronger Generalized 2n Conjecture holds for $n\leq6$, except possibly over some infinite fields of characteristic $p\neq0$, for finitely many values of p.

We may also consider the computational feasibility of performing the same verification for n = 8. For example, running the iteration without the expensive check of (11) reveals that the process would require checking 53,132,934 digraphs. Extrapolation based on the length of time the process took to check the first 5,000 of these suggests that this would require perhaps 6 months of computing time. (The search for n = 7 required a little over one day.)

In any case, based on the outcome for n = 7, it seems reasonable to suspect that for n = 8 a much larger number of $n \times n$ patterns exist for which the condition given in (11) is insufficient to resolve the question of whether or not they are spectrally arbitrary.

When considering a zero-nonzero pattern for which the condition given in (11) fails to give a conclusion (i.e., one for which the condition is not satisfied for any subset S), there seems to be an absence of effective tools for determining whether or not the pattern is spectrally arbitrary. This situation is already in evidence for the very small number of 7×7 such patterns, as we will see in the next section.

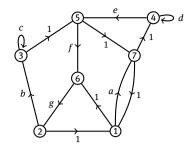


Figure 2: Digraph for the first exceptional pattern found by Algorithm 1

7 The exceptional patterns for n = 7

Section 6 outlined an exhaustive search that found first that for $n \le 6$ no $n \times n$ zero-nonzero pattern is coefficient support arbitrary over any field of characteristic 0 or of characteristic $p \ne 0$, except possibly for finitely many values of p, and also that for n = 7 this is true for all but a very small number of patterns. In fact, in order to understand every exceptional case found by the search, it suffices to consider only two patterns. We now analyze these and present what we know and do not know about their properties.

Example 7.1. The first pattern found by Algorithm 1 not to satisfy (11) for any $S \subseteq \{1, ..., 7\}$ is that with the digraph shown in Figure 2.

Following Theorem 2.6, the edges in Figure 2 are labeled such that those belonging to a particular spanning tree are normalized to have weight 1, while the weights of the remaining edges are a, b, \ldots, g . This allows us to express the associated polynomials more succinctly; they are as follows.

$$\alpha_{1} = -c - d$$

$$\alpha_{2} = cd - a$$

$$\alpha_{3} = a(c + d) - g - e$$

$$\alpha_{4} = -gbf - cda + ce + g(c + d)$$

$$\alpha_{5} = gd(bf - c)$$

$$\alpha_{6} = g(af - 1)(b - e)$$

$$\alpha_{7} = g(af - 1)(ce - bd)$$

We now consider the question of which values $k \in \mathbb{R}$ have the property that some matrix with this pattern has the characteristic polynomial $x^7 + kx$. For such a matrix, we have $\alpha_i = 0$ for all $i \neq 6$. Then, to begin with, $\alpha_1 = 0$ implies that d = -c. Substituting for d in the expression for α_2 , we find that $\alpha_2 = 0$ implies $0 = -c^2 - a$, so that then $a = -c^2$. Similarly, substituting d = -c in the expression for α_3 shows that $\alpha_3 = 0$ requires 0 = -g - e, so that g = -e.

We now consider $\alpha_4 = 0$ and use the conditions derived above that d = -c, $a = -c^2$, and g = -e to obtain

$$0 = \alpha_4 = -gbf - cda + ce + g(c + d)$$

$$= -gbf - cda + ce$$

$$= -gc - c(-c)(-c^2) + ce$$

$$= -gc - c^4 + ce$$

$$= ec - c^4 + ce$$

$$= 2ec - c^4$$

$$= c(2e - c^3).$$

Hence, due to the condition that $c \neq 0$, we have $e = \frac{1}{2}c^3$.

Figure 3: Digraph for the second exceptional pattern found by Algorithm 1

Finally, $\alpha_5 = 0$ gives 0 = dg(bf - c). Under the condition that $dg \neq 0$, this gives c = bf. In fact, we can now express all variables in terms of b and f. In particular, we have:

$$a = -b^{2}f^{2}$$
 $c = bf$
 $d = -bf$ $e = \frac{1}{2}c^{3} = \frac{1}{2}b^{3}f^{3}$
 $g = -e = -\frac{1}{2}b^{3}f^{3}$

Substituting all of the above into α_7 , and letting $y = b^2 f^3$, we can write

$$\begin{split} \alpha_7 &= g(af-1)(ce-bd) \\ &= -\frac{1}{2}b^3f^3(-b^2f^3-1)(\frac{1}{2}b^4f^4+b^2f) \\ &= \frac{1}{2}b^5f^4(b^2f^3+1)(\frac{1}{2}b^2f^3+1) \\ &= \frac{1}{2}b^3fy(y+1)(\frac{1}{2}y+1). \end{split}$$

In particular, given that $b^3 f y = b^5 f^4 \neq 0$, the assumption that $\alpha_7 = 0$ implies that y = -1 or y = -2.

Finally, we have not assumed any conditions on α_6 , but we can still write it in terms of b, f, and y, as we did for α_7 above. In particular, we have

$$\alpha_6 = g(af - 1)(b - e)$$

$$= -\frac{1}{2}b^3f^3(-b^2f^3 - 1)(b - \frac{1}{2}b^3f^3)$$

$$= \frac{1}{4}b^4f^3(b^2f^3 + 1)(2 - b^2f^3)$$

$$= \frac{1}{2}b^2y(y + 1)(2 - y).$$

Since we have either y = -1 or y = -2, the value of y(y + 1)(2 - y) is either 0 or 8. Hence, by the above, we have either $\alpha_6 = 0$ or $\alpha_6 = 4b^2$. In either case, we must have $\alpha_6 \ge 0$.

Hence, the polynomial $x^7 + kx$ is not realized by any matrix over \mathbb{R} with this pattern when k < 0. In particular, the pattern is not spectrally arbitrary over \mathbb{R} .

Example 7.2. The other pattern found by Algorithm 1 that did not satisfy (11) for any $S \subseteq \{1, ..., 7\}$ is that with the digraph shown in Figure 3. The edges in Figure 3 are labeled with those belonging to a particular spanning tree normalized to have weight 1, while the weights of the remaining edges are labeled, in row-major order, as a, b, ..., g. This allows us to write the associated polynomials as follows.

$$\alpha_1 = -a - b$$
 $\alpha_2 = ab - c - d$
 $\alpha_3 = ac - d(-a - b) - eg - f$
 $\alpha_4 = -d(ab - c) + egb - f(-a - b)$
 $\alpha_5 = -fab + eg(c + d) - ed - acd$
 $\alpha_6 = e(f - bd)(g - 1)$
 $\alpha_7 = -(g - 1)(bf + cd)e$

To show that A is not spectrally arbitrary over \mathbb{R} , we can apply Lemma 5.2 by showing that the map \hat{F}_{α} given by

$$\hat{F}_{\alpha}(a, b, c, d, e, f, g, y) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, abcdefgy)$$
(17)

does not map \mathbb{R}^8 surjectively onto itself.

By experimental computation, H. Tracy Hall [10] discovered a particular polynomial not realized by \mathcal{A} , one with integer coefficients and 7 terms. One of the referees was able to find a simpler polynomial that \mathcal{A} also fails to realize, namely

$$x^7 + 2x^2 - 1$$
.

Accordingly, we claim that (0, 0, 0, 0, 2, 0, -1, 1) is not in $\hat{F}_{\alpha}[\mathbb{R}^8]$. This can be shown by considering the ideal

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 - 2, \alpha_6, \alpha_7 - (-1), abcdefgy - 1 \rangle.$$
 (18)

In particular, using a Gröbner basis calculation to compute the intersection of this ideal with $\mathbb{C}[a]$ shows that the ideal given by this intersection (i.e., the corresponding elimination ideal) is generated by

$$a^6 + a^5 + a^4 + a^3 + a^2 + \frac{1}{3}a + \frac{1}{3}$$
. (19)

In any case, it is even more straightforward to use software to check that the above polynomial is contained in the ideal (18). This alone is enough to imply that if *a* through *g* and *y* are given values such that

$$(\alpha_1, \ldots, \alpha_7, abcdefgy) = (0, 0, 0, 0, 2, 0, -1, 1),$$

then the polynomial (19) must vanish at these values. But this is not possible when $a \in \mathbb{R}$, as can be verified by a simple computation via an application of Sturm's Theorem; see [1, Theorem 2.50]. Hence, the pattern corresponding to this digraph is not spectrally arbitrary over \mathbb{R} .

8 Conclusion

Building on the results of [4] and [5], our work here shows that Conjecture 1.7, the original 2n Conjecture, holds for $n \le 7$. We also have established that Conjecture 1.8, the Generalized 2n Conjecture, holds for $n \le 6$ over every field, except possibly for infinite fields of nonzero characteristic p, for finitely many values of p. Moreover, for such fields we know that there are only two patterns, given in Examples 7.1 and 7.2, that could be counterexamples for n = 7, and that these are not counterexamples over \mathbb{R} .

For n = 8, our algorithm would need to check 53,132,934 patterns, which is computationally not feasible at this time. It is possible that a partial run of our algorithm with n = 8 will yield a pattern that is spectrally arbitrary, but further techniques would be needed to show that it is spectrally arbitrary.

Our algorithm relies on the notion of a coefficient support arbitrary pattern. The two patterns explored in Section 7 are coefficient support arbitrary over \mathbb{C} , but we do not know if these are in fact spectrally arbitrary over \mathbb{C} . Meanwhile, the authors were able to use Mathematica to check specific polynomials to verify that each of the two patterns is coefficient support arbitrary over \mathbb{R} as well, although in Section 7 we saw that these patterns are not spectrally arbitrary over that field. Hence, being coefficient support arbitrary is in general not sufficient to imply that a pattern is spectrally arbitrary.

Over \mathbb{R} , it is possible to define *coefficient sign arbitrary* in a natural way analogous to the notion of coefficient support arbitrary. While the conclusion of Example 7.1 shows that the pattern considered there is not coefficient sign arbitrary, computations similar to those mentioned above were able to show that the pattern of Example 7.2 in fact is. Hence, over \mathbb{R} , even being coefficient sign arbitrary is not sufficient to imply that a pattern is spectrally arbitrary.

Ultimately, new tools are needed in order to show that a pattern with 2n-1 nonzero entries is spectrally arbitrary. The Nilpotent Jacobian Method laid out in [14] and the Nilpotent Centralizer Method laid out in [9] both require the pattern to have at least 2n nonzero entries, as shown in [2]. Given appropriate new tools or methods, it may be possible to show that, as we suspect, the Generalized 2n conjecture is not true over \mathbb{C} , and it even seems plausible that the two patterns presented in Section 7 are counterexamples.

Appendix: Code for computation

```
Q_poly_ring = QQ['x0,x1,x2,x3,x4,x5,x6,x7,x8']
x_variable = list(Q_poly_ring.gens())
x0, x1, x2, x3, x4, x5, x6, x7, x8 = Q_poly_ring.gens()
def check_ideal_condition_from_digraph(D, spanning_tree_edges=[]):
    n = D.num_verts();
    A = D.adjacency matrix();
    if spanning_tree_edges == []:
        spanning tree edges = D.min spanning tree();
    # Edges returned by min_spanning_tree() might not respect direction
    # in the digraph, so this must be corrected for
    normalized_arcs = []
    for e in spanning_tree_edges:
        if A[e[0],e[1]] == 1:
            normalized_arcs.append( (e[0],e[1]) )
        else:
            normalized arcs.append((e[1],e[0]))
    current_var_index = 0;
B = matrix(Q_poly_ring, A)
    for r in range(n):
        for c in range(n):
            if B[r,c] == 1 and (r,c) not in normalized_arcs:
                B[r,c] = x_variable[current_var_index]
current_var_index += 1
    coeffs = B.characteristic_polynomial().coefficients(sparse=False)
    coeffs.reverse()
    alpha = coeffs[1:]
    # If one of the coefficients is 0, then certainly it is in the ideal
    # generated by the others, so we can short-circuit the test
    if 0 in alpha:
        return set([alpha.index(0)])
    monomial = 1
    for i in range(n):
        monomial = monomial * x_variable[i]
    for S in Subsets(range(n)):
        if len(S) < n:
            product_of_subset_of_alphas = 1
            for j in S:
                 product_of_subset_of_alphas *= alpha[j]
            other_alphas = [alpha[k] for k in range(n) if k not in S]
            I = ideal(other alphas)
            if monomial * product_of_subset_of_alphas in I.radical():
                 return set(S)
    return None
def search_all_sparse_patterns(n):
    num_digraphs_considered = 0;
    indices_of_exceptional_digraphs = [];
max_num_nonloop_edges = 2*n-1-2;
    for D in digraphs(n, lambda G: G.size() <= max_num_nonloop_edges):</pre>
        if D.is_strongly_connected():
            \# calculate number of loops required to get up to 2n-1 edges
            number_of_nonloop_edges = D.size();
            number of loops required = 2*n-1 - number of nonloop edges;
            if number_of_loops_required <= n:</pre>
                 # make a copy of the digraph that allows loops
                 D_plus_loops = DiGraph(D, loops=True);
```

```
# consider every possible assignment of loops
            for looped vertices in Subsets(range(n), number of loops required):
                # remove loops remaining from previous iteration
                D_plus_loops.remove_loops();
                # add the next set of loops to the digraph
                loop_list = [[i,i] for i in looped_vertices.list()];
                D plus loops.add edges(loop list);
                # analyze the pattern that results from this looped digraph
                dependent_set_of_alphas = check_ideal_condition_from_digraph(
                    D_plus_loops)
                if dependent_set_of_alphas == None:
                    indices_of_exceptional_digraphs.append(
                        num digraphs considered);
                    print D_plus_loops.adjacency_matrix().list()
                num_digraphs_considered += 1;
print "Number of digraphs considered:", num_digraphs_considered
num_exceptional_digraphs = len(indices_of_exceptional_digraphs);
print "Number of exceptional digraphs:", num_exceptional_digraphs
return indices_of_exceptional_digraphs
```

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