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A potentially stable and potentially nilpotent tree sign pattern that is not spectrally arbitrary

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Abstract. It has been shown that a 4×4 tree sign pattern is spectrally arbitrary if and only if it is potentially stable and potentially nilpotent in [M. Arav, F. Hall, Z. Li, K. Kaphle, N. Manzagol, spectrally arbitrary tree sign patterns of order 4, Electron. J. Linear Algebra, 20 (2010), 180-197], where the authors asked whether the above important theorem can be extended to the 5×5 case. In this paper, we negatively answer this question by constructing a 5×5 potentially stable and potentially nilpotent tree sign pattern which is shown to be not spectrally arbitrary.

1. Introduction

A *sign pattern* is a matrix with entries in $\{+, -, 0\}$. For a real matrix A , $\text{sgn}(A)$ is the sign pattern whose entries are the signs of the corresponding entries of A . If A is an $n \times n$ sign pattern, the *qualitative class* of A , denoted by $Q(A)$, is the set of all $n \times n$ real matrices A with $\text{sgn}(A) = A$, and we call A is a *realization* of sign pattern A . A *subpattern* of A is a $n \times n$ sign pattern obtained from A by replacing some (possibly, none) nonzero entries of A with zeros. If B is a subpattern of A , then A is a *superpattern* of B . A diagonal sign pattern with nonzero diagonal entries is called a *signature pattern*; see e.g. [1].

An $n \times n$ sign pattern A is *potentially nilpotent* if there exists a nilpotent matrix $A \in Q(A)$, i.e., $A^k = 0$ for some positive integers k ; see e.g., [2-4] and references therein. An $n \times n$ sign pattern A is *potentially stable* (or, A *allows stability*) if there exists a real matrix $A \in Q(A)$ such that the number of eigenvalues of A with negative real parts is n ; see [5] for more details.

An $n \times n$ sign pattern A is a *spectrally arbitrary pattern* (*SAP*, for short) if, every real monic polynomial $p(x)$ of degree n , there exists some real matrix $A \in Q(A)$ such that the characteristic polynomial of A is $p(x)$; see, for example, [3], where this definition was introduced and the well-known Nilpotent-Jacobian method was developed to show that a sign pattern is spectrally arbitrary. There are considerable literatures about the spectral properties on the potential problems of sign pattern matrices; see e.g., [4] for a survey.

It's well known that the properties of being potentially nilpotent, potentially stable and spectrally arbitrary are preserved under transposition, permutation similarity and signature similarity. Two sign patterns are *equivalent* if one can be obtained from the other by any combination of these three operations.



An $n \times n$ sign pattern $A = (\alpha_{ij})$ has *signed digraph* $\Gamma(A)$ with vertex set $\{1, 2, \dots, n\}$ and for all i and j , a positive (negative) arc from i to j if and only if α_{ij} is positive (negative). A (directed) *simple cycle* of length k (or a k -cycle) is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \dots, (i_{k-1}, i_k), (i_k, i_1)$ such that the vertices i_1, i_2, \dots, i_k are distinct. A *composite cycle* γ in $\Gamma(A)$ is a product of simple cycles, say $\gamma = \gamma_1 \gamma_2 \cdots \gamma_m$, where the vertices sets of the simple cycle γ_i are mutually disjoint; see e.g., [1]. If the length of γ_i is l_i , then the length of γ is $\sum_{i=1}^m l_i$.

In [2], all 4×4 spectrally arbitrary tree sign patterns were characterized by an innovative application of Gröbner bases. It was shown that a 4×4 tree sign pattern is spectrally arbitrary if and only if it is potentially nilpotent and potentially stable; see [2, Theorem 3.10]. The authors asked whether we can extend the above result to the 5×5 case. That is, for the class of 5×5 tree sign patterns, is being potentially nilpotent and potentially stable equivalent to being spectrally arbitrary. In this article, we construct a 5×5 tree sign pattern that is potentially stable and potentially nilpotent, but not spectrally arbitrary. It indicates that the Theorem 3.10 in [2] cannot be extended to the 5×5 case in general.

2. Preliminaries and main result

We begin this section by constructing the following 5×5 sign pattern

$$A = \begin{pmatrix} 0 & + & + & 0 & 0 \\ - & 0 & 0 & 0 & 0 \\ + & 0 & 0 & + & + \\ 0 & 0 & - & - & 0 \\ 0 & 0 & - & 0 & + \end{pmatrix} \quad (1)$$

Following [6], it is clear that the associated signed graph of A is the signed tree as is shown in Figure 1.

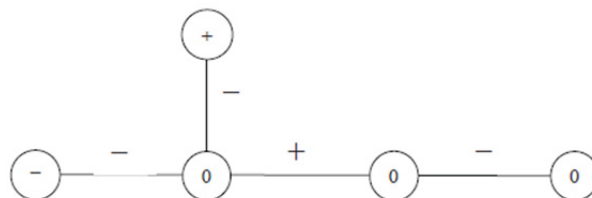


Figure 1. Signed tree graph of order 5.

To show that the sign pattern A described by (1) is potentially stable and potentially nilpotent, but not spectrally arbitrary, we need the following preliminaries. Note that although the specification of necessary and sufficient conditions for potential stability remains unsolved and it seems probable that the problem of recognizing potential stability is "NP-complete", there exist some constructions of potentially stable sign patterns presented by Bone in [3], one of which is the following:

Given an $(n-2) \times (n-2)$ matrix A^- , we construct an $n \times n$ matrix A as follows:

- (i) let A^- be a principal submatrix of A ;
- (ii) the 2×2 principal submatrix complementary to A^- contains a negative 2-cycle;
- (iii) A contains a positive n -cycle.

Lemma 1 [3, Theorem 3]. Suppose A is a sign pattern produced by the above construction. If A^- is potentially stable, then A is also potentially stable.

Now, we consider the potential stability of sign pattern A described by (1).

Lemma 2. If A is a 5×5 sign pattern described by (1), then A is potentially stable.

Proof. Assume that the principal submatrix

$$A^- = \begin{pmatrix} 0 & + & + \\ - & - & 0 \\ - & 0 & + \end{pmatrix}.$$

Obviously, the associated signed graph of $\Gamma(A^-)$ is the last graph in Line 4 in Figure 2 described in [6]. It follows that A^- is a potentially stable sign pattern. It is clear that the 2×2 principal submatrix

$$\begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}$$

contains a negative 2-cycle and A has a positive 5-cycle: (12)(21)(34)(43)(55). The statement follows readily from Lemma 1.

Example 3. Let $B = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ -\frac{16}{143} & 0 & 0 & 0 & 0 \\ \frac{1224}{715} & 0 & 0 & 1 & 1 \\ 0 & 0 & -\frac{107}{5} & -8 & 0 \\ 0 & 0 & -\frac{36}{5} & 0 & 1 \end{pmatrix}$. Then $B \in Q(A)$ and the eigenvalues of B are

approximately $-0.80151808 \pm 0.25419788i$, $-1.6105027 \pm 0.19730995i$, and -2.17595849 . It follows that B is stable, and thus A is potentially stable.

Next, we consider the potentially nilpotent property of sign pattern A .

Lemma 4. If A is a 5×5 sign pattern described by (1), then A is potentially nilpotent.

Proof. Without loss of generality, up to equivalence, assume that a realization A of A has the form

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 1 & 1 \\ 0 & 0 & -c & -d & 0 \\ 0 & 0 & -e & 0 & f \end{pmatrix} \quad (2)$$

where variables a, b, c, d, e and f are positive. Calculating the characteristic polynomial $p_A(x)$ of A , we have

$$\begin{aligned} p_A(x) = & x^5 - (f-d)x^4 + (a-b+c+e-df)x^3 \\ & - [f(a-b+c) - d(a-b+e)]x^2 \\ & + [(e+c)a - df(a-b)]x - (caf - aed). \end{aligned} \quad (3)$$

To make real matrix A to be nilpotent, let

$$f-d=0, \quad (4)$$

$$a-b+c+e-df=0, \quad (5)$$

$$f(a-b+c) - d(a-b+e)=0, \quad (6)$$

$$(e+c)a - df(a-b)=0, \quad (7)$$

and

$$caf - aed=0. \quad (8)$$

By (4) and (8), we get

$$e = c. \quad (9)$$

Solving the variables b, c, f from equalities (4), (5), (7) and (9) in terms of variables a and d , we have

$$e = c = \frac{d^4}{2(d^2 + a)} \quad (10)$$

and

$$b = \frac{a^2}{d^2 + a}. \quad (11)$$

It follows that A is potentially nilpotent.

Example 5. For the real matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{4} & -1 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & 1 \end{pmatrix}$$

with its sign pattern A described by (1), all the eigenvalues of A are 0s. So A is nilpotent and the sign pattern A is potentially nilpotent.

Now, we turn our attention to the main result.

Theorem 6. There exists a 5×5 potentially stable and potentially nilpotent tree sign pattern that is not spectrally arbitrary.

Proof. Consider the sign pattern A described by (1). By Lemmas 2 and 4, A is a potentially stable and potentially nilpotent tree sign pattern. So, it suffices to show that A is not a spectrally arbitrary pattern. We claim that sign pattern A cannot allow the polynomial $x^5 + \alpha x^3 + \beta x + \gamma$ for some variables α, β and nonzero variable γ . We verify this claim by a way of contradiction.

Suppose the sign pattern A allows the polynomial $x^5 + \alpha x^3 + \beta x + \gamma$. Then without loss of generality, assume that the realization A of A has the form (2) such that

$$p_A(x) = x^5 + \alpha x^3 + \beta x + \gamma.$$

Then by (3), we have

$$a - b + c + e - df = \alpha, \quad (12)$$

$$a(c + e) - df(a - b) = \beta, \quad (13)$$

and

$$acf - aed = \gamma. \quad (14)$$

By Equalities (4) and (6), we obtain that

$$e = c. \quad (15)$$

It follows readily that $\gamma = 0$. It is contradiction.

3. Concluding remarks

We have constructed a 5×5 tree sign pattern A that has been shown to be potentially stable and potentially nilpotent, but not spectrally arbitrary. Consequently, the above open question is answered negatively. Up to equivalence, it is well known that a 5×5 tree sign pattern is a star pattern, or a tridiagonal pattern or a double star pattern. Clearly, tree sign pattern A in this article is a double star sign pattern. It has been known that an $n \times n$ star sign pattern is spectrally arbitrary if and only if it is

potentially stable and potentially nilpotent. Therefore, to address on the open question that a 5×5 tree sign pattern is spectrally arbitrary if and only if it is potentially stable and potentially nilpotent, it suffices to consider 5×5 potentially nilpotent tridiagonal sign patterns. Since characterizing the 4×4 potentially nilpotent tridiagonal sign patterns (together with positive and negative diagonal entries) was posed as an another open question posed in [5], identifying the sufficient and necessary conditions for a 5×5 tridiagonal sign pattern to be potentially nilpotent remain also open.

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References

- [1] Hall F J and Li Z 2007 Sign pattern matrices, in: L. Hogben (Ed.), Handbook of Linear Algebra. Boca Raton: Chapman \& CRC Press
- [2] Arav M, Hall F, Li Z, Kaphle K and Manzagol N 2010 Spectrally arbitrary tree sign patterns of order 4 *Electron. J. Linear Algebra* **20** 180-197. A reference
- [3] Drew J H, Johnson C R, Olesky D D, van den Driessche P 2000 Spectrally arbitrary patterns *Linear Algebra Appl.* **308** 121-137
- [4] Catral M, Olesky D D and van den Driessche P 2009 Allow problems concerning spectral properties of sign patter matrices: A survey *Linear Algebra Appl.* **430** 3080-3094. More references
- [5] Bone T 1983 Positive feedback may sometimes promotes stability *Linear Algebra Appl.* **51** 143-151.
- [6] Johnson C R and Summers T S 1989 The potentially stable tree sign patterns for dimensions less than five *Linear Algebra Appl.* **126** 1-13