ON ALGEBRAICALLY POSITIVE MATRICES WITH ASSOCIATED SIGN PATTERNS

AMITAKSHAR BISWAS AND SOUMYAJYOTI KUNDU

ABSTRACT. We study algebraically positive matrices and explore some of their interesting properties. We also explore the connections between algebraic positivity and irreducibility and how it is related to the fundamental theorem of Perron–Frobenius theory. Notably, we characterize all 3×3 symmetric sign pattern matrices that require algebraic positivity.

1. INTRODUCTION

Given any matrix $A \in GL_n(\mathbb{R})$, does there exist a polynomial $f \in \mathbb{R}[X]$ such that all the entries of $f(A)$ are positive? The answer, in general, is unfortunately no. As a matter of fact, if we consider the identity matrix I, then clearly there do not exist any real polynomial f such that $f(I)$ has all positive entries.

In [1], Kirkland, Qiao and Zhan (2016) introduced the concept of algebraically positive matrices, which although recent, has now found its use in various fields of study such as Matrix Analysis, Linear Algebra, Markov chains in probability theory, population models, iterative methods in numerical analysis, etc.

Definition 1.1. A *positive (nonnegative) matrix* is a matrix all of whose entries are positive (nonnegative) real numbers. The notation $A > 0$ $(A \geq 0)$ denotes that A is a positive (nonnegative) matrix.

Definition 1.2. A real square matrix A is said to be *algebraically positive* if there exists a real polynomial f such that $f(A)$ is a positive matrix. (Here $f(A) = a_n A^n + a_{n-1}A^{n-1} \cdots + a_1 A + a_0 I$ where $f(x) = a_n x^n + \cdots + a_1 x + a_0$ for some $a_0, \ldots a_n \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$.)

Recall that a *primitive matrix* is a square nonnegative matrix, some (integer) power of which is positive. Thus, all primitive matrices are examples of algebraically positive matrices. The primitive case is the heart of the Perron–Frobenius theory and its applications. Another example of algebraically positive matrices are eventually positive matrices. A real matrix A is called *eventually positive* if there exists a positive integer

s such that A^t is a positive matrix for all integers $t \geq s$. Hence, all eventually positive matrices are algebraically positive. The converse, however, is untrue. Throughout the text, we will denote I as the identity matrix whose order will be clear from the context, and A^T as the transpose of a matrix A. Also, whenever we consider matrices and polynomials, we assume them to be real. We will also express the elements of $\mathbb{R}^n$ as column vectors.

It is easy to see that a matrix A is algebraically positive if and only if $-A$, $P^T A P$ for any permutation matrix P and $A + dI$ for all $d \in \mathbb{R}$ are algebraically positive. This fact will serve as an useful tool in Section 4.

Now, suppose we are given a square matrix A, and we are asked if the given matrix is algebraically positive. Our immediate intuition will be to either search for a polynomial f such that $f(A)$ is a positive matrix or to prove the non-existence of any such polynomial. Again, suppose we are told that the given matrix A is algebraically positive and our task is to find the desired polynomial. Soon one will realise that it is next to impossible to find such a polynomial *unless* we are given some hint. The next lemma precisely serves that purpose.

Lemma 1.3. *Let A be a square matrix of order n. To check whether A is algebraically positive, we need only consider polynomials of degree at most $n - 1$.*

Proof. Let A be a square matrix of order n with characteristic polynomial $p(x)$ and let $f(x) \in \mathbb{R}[X]$. Then by the division algorithm, there exist real polynomials $q(x)$ and $r(x)$ with degree of $r(x)$ less than or equal to $n - 1$, such that $f(x) = p(x)q(x) + r(x)$. By the Cayley–Hamilton theorem, $p(A) = 0$. Thus $f(A) = r(A)$. $\qquad\square$

Thus, it is reassuring to know that in order to check if, say, a 3×3 matrix is algebraically positive, it is enough to look at all polynomials of degree less than or equal to 2. Recall that the *spectral radius* of a square matrix is the maximum of the moduli of the roots of its characteristic polynomial. A *simple eigenvalue* is an eigenvalue with algebraic multiplicity 1. A real vector is said to be *positive (nonnegative)* if all of its entries are positive (nonnegative). Before we proceed to the main content of this paper, we state the fundamental theorem of Perron–Frobenius theory without proof [3].

Theorem 1.4. *Suppose A is a primitive matrix, with spectral radius λ. Then λ is a simple root of the characteristic polynomial which is strictly greater than the modulus of any other root, and λ has positive eigenvectors.*

2. Recent Developments

Although Lemma 1.3 serves as a handy tool to check if a given matrix is algebraically positive, it is still not an efficient way to check, especially if we consider higher orders. In [1], Kirkland, Qiao and Zhan (2016) presented another alternative way to check if a given matrix is algebraically positive. We state the theorem without proof.

Theorem 2.1. *A real matrix is algebraically positive if and only if it has a simple real eigenvalue and corresponding left and right eigenvectors positive .*

Corollary 2.2. *A real symmetric matrix is algebraically positive if and only if it has a simple eigenvalue and corresponding right eigenvectors positive .*

Proof. Note that a real symmetric matrix always has real eigenvaules, and left and right eigenvectors of a symmetric matrix are just tranposes of each other. Thus, the result follows. □

Definition 2.3. A square $n \times n$ matrix $A = [a_{ij}]$ is called *reducible* if the indices $1, 2, \ldots, n$ can be partitioned into two disjoint nonempty sets $\{i_1, i_2, \ldots, i_\mu\}$ and $\{j_1, j_2, \ldots, j_\nu\}$ (with $\mu + \nu = n$) such that

$$a_{i_\alpha j_\beta} = 0$$

for $\alpha = 1, 2, \ldots, \mu$ and $\beta = 1, 2, \ldots, \nu$.

A square matrix that is not reducible is said to be *irreducible*. Recall that a matrix is reducible if and only if it can be placed into block lower-triangular form by simultaneous row/column permutations, i.e., if A is a reducible real matrix, then there is a permutation matrix P such that $A = P^T \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} P$, where A_1 and A_3 are square and nonvoid. For any real polynomial f, we have $f(A) = P^T \begin{bmatrix} f(A_1) & 0 \\ \star & f(A_3) \end{bmatrix} P$. Thus $f(A)$ always has zero entries and hence is not positive. Therefore, every algebraically positive matrix is irreducible.

Theorem 2.4. *Let $A \geq 0$ be of order $n \geq 2$ and let $\lambda^*(A)$ be its Perron root. Then the following are equivalent.*
   (i) A is irreducible.
   (ii) For any pair (i, j) of indices, $1 \leq i, j \leq n$, there is a positive integer $k = k(i, j)$, $k \leq n$, such that $A^k(i, j) > 0$.
   (iii) $(I + A)^{n-1} > 0$.
   (iv) $(I + A + A^2 + \cdots + A^{n-1}) > 0$.
   (v) $A + A^2 + \cdots + A^n > 0$.

Note that irreducibility and having a simple real eigenvalue are not sufficient for algebraic positivity. For example, $A = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -1 & 1 \\ 2 & -2 & -1 \end{bmatrix}$ is irreducible and has three simple real eigenvalues but is not algebraically positive.

Theorem 2.5. *If A is an irreducible real matrix all of whose off-diagonal entries are nonnegative (or nonpositive), then A is algebraically positive.*

Proof. It is enough to prove the nonnegative case by considering $-A$ if necessary. Choose a positive number d such that d is larger than the maximum absolute value of the diagonal entries of A. Then $A + dI$ is an irreducible nonnegative matrix. It follows from Theorem 2.4 that $(I + (dI + A))^{n-1} > 0$ where n is the order of A, i.e., $((d+1)I + A)^{n-1} > 0$. Thus, A is algebraically positive. $\square$

Corollary 2.6. *A nonnegative matrix is algebraically positive if and only if it is irreducible.*

Next, we provide yet another way to check if a given matrix is algebraically positive without proof [4].

Theorem 2.7. *A real square matrix A is algebraically positive if and only if it commutes with a unique (up to scalar multiplication) rank one positive matrix.*

3. Sign Patterns

Definition 3.1. A *sign pattern matrix* is a matrix whose entries belong to the set $\{+, -, 0\}$. For example, $\begin{bmatrix} + & - \\ - & 0 \end{bmatrix}, \begin{bmatrix} 0 & - \\ + & 0 \end{bmatrix}$, etc.

The sign pattern of a real matrix A is the matrix obtained from A by replacing all entries by their respective signs. Given a sign pattern A, the *pattern class* of A, denoted $Q(A)$, is the set of real matrices whose sign patterns are A.

Example 3.2. Let $A = \begin{bmatrix} + & - \\ - & 0 \end{bmatrix}$ be a sign pattern matrix and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$. Then B belongs to the pattern class of A, i.e., $B \in Q(A)$.

Let P be a property about real matrices. A sign pattern A is said to *require P* if every matrix in $Q(A)$ has property P and A is said to *allow P* if there exists a matrix in $Q(A)$ that has property P. If a sign pattern requires a property P, then it allows P. However, the converse is not true. Therefore, if a particular sign pattern require algebraic positivity, then all matrices belonging to its pattern class are algebraically positive.

A sign pattern A of order n is said to be *spectrally arbitrary* if given any monic real polynomial f of degree n, there is a matrix in $Q(A)$ whose characteristic polynomial is f.

Theorem 3.3. *No spectrally arbitrary sign pattern requires algebraic positivity.*

Proof. Clearly, any sign pattern of order 1 is not spectrally arbitrary. Now let A be a spectrally arbitrary sign pattern of order $n \geq 2$. Then there is a real matrix B in $Q(A)$ whose characteristic polynomial is $f(x) = (x - 1)^n$. Since B has no simple eigenvalue, by Theorem 2.1, B is not algebraically positive. This proves that A does not require algebraic positivity. $\square$

We state a necessary condition for a sign pattern to allow algebraic positivity.[1]

Theorem 3.4. *If a sign pattern allows algebraic positivity, then every row and column contains a $+$, or every row and column contains a $-$.*

Definition 3.5. Let A be a square sign pattern matrix. Then A is *symmetric* if $A = A^T$.

Note that matrices belonging to a symmetric sign pattern need not be symmetric. For example, $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ is a symmetric sign pattern and $\begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$ belongs to it but is not symmetric.

4. Main Results

In this section, we characterize all 3×3 symmetric sign pattern matrices that require algebraic positivity. However, to begin with, we characterize a smaller class of such matrices. We proceed with the following definition.

Definition 4.1. A matrix is called a *zero-diagonal* matrix if all of its diagonal entries are 0. For example, $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$, etc.

Before going to the main content, we state two lemmas.

Lemma 4.2. *A sign pattern A requires algebraic positivity if and only if the negated sign pattern, i.e., $-A$ does.*

Proof. Let A be a sign pattern matrix that requires algebraic positivity and let $B \in Q(-A)$. It follows from definition that $-B \in Q(A)$ is algebraically positive which implies B is algebraically positive. Thus, $-A$ too requires algebraic positivity. $\qquad\square$

Lemma 4.3. *A sign pattern requires algebraic positivity if and only if any 'permuted' sign pattern of it does. (The latter is simply a sign pattern obtained by simultaneously relabelling the rows and columns of the original sign pattern matrix.)*

Proof. The proof simply follows from the fact that A is algebraically positive if and only if $P^T A P$ is algebraically positive, where P is a permutation matrix. $\qquad\square$

Theorem 4.4. *Suppose A is a 3×3 symmetric zero-diagonal sign pattern matrix. Then A requires algebraic positivity if and only if A or $-A$ belongs to the following class,*

$$\mathcal{C} = \left\{ \begin{bmatrix} 0 & + & + \\ + & 0 & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & + \\ 0 & 0 & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix} \right\}.$$

Proof. Let A be a 3×3 symmetric zero-diagonal sign pattern matrix. Hence, A is of the form $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$, where $a, b, c \in \{+, -, 0\}$. By Lemma 4.3, $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$ is algebraically positive $\iff \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$ is algebraically positive $\iff \begin{bmatrix} 0 & b & a \\ b & 0 & c \\ a & c & 0 \end{bmatrix}$ is algebraically positive. We proceed with the proof by dividing it into various cases and in each case, either we prove that the given sign pattern requires algebraic positivity or give a counterexample, (i.e., a matrix belonging to that particular sign pattern which is not algebraically positive) or use Lemma 4.3. In all the counterexamples, we have used Theorem 2.1, where we have either shown the given matrix has no positive eigenvectors or even if it does, the corresponding eigenvalue is not simple.

We first characterize all 3×3 symmetric, zero-diagonal sign pattern matrices with non-zero off diagonal entries which require algebraic positivity and then proceed with the remaining cases. Let A be 3×3 symmetric, zero-diagonal sign pattern matrices with non-zero off diagonal entries. Note that the possible choices of the $(1, 2), (1, 3)$ and $(2, 3)$ entries of A are: $(+, +, +), (+, +, -), (+, -, +), (-, +, +), (-, -, -), (-, +, -), (-, -, +)$ and $(+, -, -)$.

Case 1. Let $A = \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$ and $B \in Q(A)$. Clearly, the entries in $(1, 2), (1, 3)$ and $(2, 3)$ of B are positive and it is easy to see that B^2 will have positive entries along the diagonal. Hence it is irreducible by Theorem 2.4 and thus, by Theorem 2.5, B is algebraically positive.

Case 2. Let $A = \begin{bmatrix} 0 & + & + \\ + & 0 & - \\ + & - & 0 \end{bmatrix}$. Consider $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$. Clearly, $B \in Q(A)$ but B has simple eigenvalue -2 with eigenvector $(-1, 1, 1)^T$, and eigenvalue 1 which is not simple. Therefore, by Theorem 2.1, B is not algebraically positive. Hence, A does not require algebraic positivity. Therefore, by Lemma 4.3, $\begin{bmatrix} 0 & + & - \\ + & 0 & + \\ - & + & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & - & + \\ - & 0 & + \\ + & + & 0 \end{bmatrix}$ also do not require algebraic positivity.

Finally, by Lemma 4.2, we can see that if A is a 3×3 symmetric zero-diagonal sign pattern matrix with non-zero off diagonal entries, then A requires algebraic positivity if and only if,

$$A \in \left\{ \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & - & - \\ - & 0 & - \\ - & - & 0 \end{bmatrix} \right\}.$$

Next we characterize all the remaining cases, i.e., 3×3 symmetric zero-diagonal sign pattern matrices with at least one zero off diagonal entry. Let A be such a matrix.

Case 1. Let exactly two among the $(1,2), (1,3)$ and $(2,3)$ entries of A be 0. Clearly,

$A \in \left\{ \begin{bmatrix} 0 & 0 & + \\ 0 & 0 & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ + & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$. By Theorem 3.4, A does not require

algebraic positivity.

Case 2. Let exactly one among the $(1,2), (1,3)$ and $(2,3)$ entries of A be 0.

Subcase (i): Let $A \in \left\{ \begin{bmatrix} 0 & + & + \\ + & 0 & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & + \\ 0 & 0 & + \\ + & + & 0 \end{bmatrix} \right\}$ and $B \in Q(A)$. It

is easy to see that in each case, B^2 will have positive entries in all the zero entries of B, proving that B is irreducible. Therefore, by Theorem 2.5, B is algebraically positive.

Subcase (ii): Let $A = \begin{bmatrix} 0 & + & - \\ + & 0 & 0 \\ - & 0 & 0 \end{bmatrix}$. Clearly by Theorem 3.4, A does not allow algebraic

positivity. Alternatively, consider $B = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$. Clearly, $B \in Q(A)$ but B has

simple eigenvalue $-\sqrt{2}$ with eigenvector $(\sqrt{2}, -1, 1)^T$, has simple eigenvalue $\sqrt{2}$ with eigenvector $(-\sqrt{2}, -1, 1)^T$ and simple eigenvalue 0 eigenvector $(0, 1, 1)^T$. Therefore, by Theorem 2.1, B is not algebraically positive. Hence, A does not require algebraic

positivity. Again by Lemma 4.3, $\begin{bmatrix} 0 & + & 0 \\ + & 0 & - \\ 0 & - & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & + \\ 0 & 0 & - \\ + & - & 0 \end{bmatrix}$ also do not require

algebraic positivity.

Case 3. Let $(1,2), (1,3)$ and $(2,3)$ entries of A be 0. Then A is the zero matrix which is clearly not algebraically positive.

Finally, using Lemma 4.2, we can conclude that if A is a 3×3 symmetric zero-diagonal sign pattern matrix. Then A requires algebraic positivity if and only if A or $-A$ belongs to the following class,

$$\mathcal{C} = \left\{ \begin{bmatrix} 0 & + & + \\ + & 0 & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & + \\ 0 & 0 & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix} \right\}.$$

$\square$

Definition 4.5. A matrix is called a *equidiagonal* matrix if all of its diagonal entries are equal. For example, $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 4 & 3 \end{bmatrix}$, etc. Clearly, all zero-diagonal matrices are equidiagonal. However, the converse is not true.

Corollary 4.6. *Let B be a real 3×3 symmetric equidiagonal matrix and $B \in Q(A)$ where A is a sign pattern matrix. Suppose either A or $-A$ belongs to $\mathcal{C}$, except for the diagonal entries. Then B is algebraically positive.*

Proof. It follows from Theorem 4.4 and the fact that B is algebraically positive if and only if $B + dI$ is algebraically positive for all real numbers d. $\square$

Corollary 4.7. *Suppose A is a real 3×3 symmetric Toeplitz matrix with its sign pattern matrix belonging to* $\left\{ \begin{bmatrix} 0 & + & 0 \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & - & 0 \\ - & 0 & - \\ 0 & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & - & - \\ - & 0 & - \\ - & - & 0 \end{bmatrix} \right\}$, *except for the diagonal entries. Then A is algebraically positive.*

We follow by characterizing another class of such matrices, however larger. We proceed with another definition.

Definition 4.8. A matrix of order n will be called a *lower zero-diagonal* matrix if its (n, n) entry is 0. For example, $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 4 & 0 \end{bmatrix}$, etc. Clearly, all zero-diagonal matrices are lower zero-diagonal. However, the converse is not true.

Theorem 4.9. *Suppose A is a 3×3 symmetric lower zero-diagonal sign pattern matrix. Then A requires algebraic positivity if and only if either A or $-A$ belongs to the following class,*

$$\hat{\mathcal{C}} = \mathcal{C} \; \bigcup \; \left\{ \begin{bmatrix} + & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & 0 & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & - & - \\ - & + & - \\ - & - & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & - & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} \star & + & + \\ + & \star & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & + & 0 \\ + & \star & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} \star & 0 & + \\ 0 & \star & + \\ + & + & 0 \end{bmatrix}, \right.$$

$$\left\{ \begin{bmatrix} \star & - & - \\ - & \star & 0 \\ - & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & - & 0 \\ - & \star & - \\ 0 & - & 0 \end{bmatrix}, \begin{bmatrix} \star & 0 & - \\ 0 & \star & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} + & + & - \\ + & - & 0 \\ - & 0 & 0 \end{bmatrix}, \begin{bmatrix} + & - & + \\ - & 0 & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} + & - & + \\ - & + & 0 \\ + & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} + & - & 0 \\ - & 0 & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & + & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & - & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & + \\ 0 & + & - \\ + & - & 0 \end{bmatrix}, \begin{bmatrix} + & 0 & - \\ 0 & 0 & + \\ - & + & 0 \end{bmatrix}, \begin{bmatrix} + & 0 & - \\ 0 & - & + \\ - & + & 0 \end{bmatrix} \right\},$$

where, $\star \in \{+, -, 0\}$

Proof. Again, as earlier, we use a similar approach, i.e., we divide the proof into various cases and sub cases and in each case, either we prove that the given sign pattern requires algebraic positivity or provide a counterexample, i.e., a matrix belonging to that particular sign pattern which is not algebraically positive. Note that the possible entries of the diagonal entries, excluding the negatives, are $(0,0,0)$, $(+,0,0)$, $(0,+,0)$, $(+,+,0)$ and $(+,-,0)$.

Case (i). All diagonal entries are 0. This case is already addressed in Theorem 4.4

Instead of studying the remaining cases separately, we study them altogether.

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} + & \star & \star \\ \star & 0 & \star \\ \star & \star & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \star & \star \\ \star & + & \star \\ \star & \star & 0 \end{bmatrix}$	$\begin{bmatrix} + & \star & \star \\ \star & + & \star \\ \star & \star & 0 \end{bmatrix}$	$\begin{bmatrix} + & \star & \star \\ \star & - & \star \\ \star & \star & 0 \end{bmatrix}$

Similar to the proof of Theorem 4.4, the possible non-zero choices of the $(1,2), (1,3)$ and $(2,3)$ entries of a sign pattern matrix are: $(+,+,+)$, $(+,+,-)$, $(+,-,+)$, $(-,+,+)$, $(-,-,-)$, $(-,+,-)$, $(-,-,+)$ and $(+,-,-)$.

Subcase (a) : $(+,+,-)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ 1 & -2 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & -2 \\ 1 & -2 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -2 \\ 1 & -2 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & -3 \\ 7 & -7 & 0 \end{bmatrix}$

Subcase (b) : $(+, -, +)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & -2 \\ 2 & 1 & 1 \\ -2 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix}$

Subcase (c) : $(-, +, +)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & 1 \\ -2 & 1 & 5 \\ 1 & 5 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 5 \\ 1 & 5 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 1 \\ -2 & -1 & 5 \\ 1 & 5 & 0 \end{bmatrix}$

Subcase (d) : $(-, -, +)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & -15 & -\frac{1}{10} \\ -15 & 0 & 1 \\ -\frac{1}{10} & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -2 & -1 \\ -2 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -15 & -\frac{1}{10} \\ -15 & 1 & 1 \\ -\frac{1}{10} & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -15 & -\frac{1}{10} \\ -15 & -1 & 1 \\ -\frac{1}{10} & 1 & 0 \end{bmatrix}$

Subcase (e) : $(-, +, -)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & -15 & 1 \\ -15 & 0 & -\frac{1}{10} \\ 1 & -\frac{1}{10} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -15 & 1 \\ -15 & 1 & -\frac{1}{10} \\ -1 & -\frac{1}{10} & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -15 & 1 \\ -15 & 1 & -\frac{1}{10} \\ 1 & -\frac{1}{10} & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & -15 & 1 \\ -15 & -1 & -\frac{1}{10} \\ 1 & -\frac{1}{10} & 0 \end{bmatrix}$

Subcase (f) : $(+, -, -)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & 1 & -15 \\ 1 & 0 & -\frac{1}{10} \\ -15 & -\frac{1}{10} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & -15 \\ 1 & 1 & -\frac{1}{10} \\ -15 & -\frac{1}{10} & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & -15 \\ 1 & 1 & -\frac{1}{10} \\ -15 & -\frac{1}{10} & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & -15 \\ 1 & -1 & -\frac{1}{10} \\ -15 & -\frac{1}{10} & 0 \end{bmatrix}$

Subcase (g) : $(+,+,+)$

Let $A \in \left\{ \begin{bmatrix} + & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix} \right\}$ and $B \in Q(A)$. It is easy to see that B^2 will have positive entries along the diagonal. Hence, B is irreducible (by Theorem 2.4) and thus, by Theorem 2.5, B is algebraically positive.

Now let $M \in Q\left(\begin{bmatrix} + & + & + \\ + & - & + \\ + & + & 0 \end{bmatrix} \right)$. Hence, M is of the form $\begin{bmatrix} x & + & + \\ + & -y & + \\ + & + & 0 \end{bmatrix}$, where x and y are positive real numbers. It is obvious that $M + 2yI$ is a positive matrix, which implies M is algebraically positive.

Subcase (h) : $(-,-,-)$

Let $-M \in Q\left(\begin{bmatrix} - & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix} \right)$. Suppose the negative entry of $-M$ be $-x$ where $x > 0$. Consider the matrix $-M + 2xI$. Clearly, in this matrix, the diagonal entries become x, $2x$ and $2x$ respectively. Thus $-M + 2xI$ is a positive matrix, which implies M is algebraically positive.

A similar argument holds for the cases, $\begin{bmatrix} 0 & - & - \\ - & + & - \\ - & - & 0 \end{bmatrix}$, $\begin{bmatrix} + & - & - \\ - & + & - \\ - & - & 0 \end{bmatrix}$ and $\begin{bmatrix} + & - & - \\ - & - & - \\ - & - & 0 \end{bmatrix}$.

Thus, if A is a 3×3 symmetric lower zero-diagonal sign pattern matrix with non-zero off diagonal entries, then A requires algebraic positivity if and only if A or $-A$ belongs to the following class,

$$\left\{ \begin{bmatrix} + & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & 0 & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & - & - \\ - & + & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & - & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix} \right\}.$$

Next we will characterize the remaining cases, i.e., matrices with at least one zero off diagonal entry. The cases in the previous part remain the same. Firstly, let us assume exactly two among the $(1,2), (1,3)$ and $(2,3)$ entries are 0.

Subcase (a) : $(+, 0, 0)$

Note that in all four cases, the third column is zero. Therefore, by Theorem 3.4, they do not require algebraic positivity.

Subcase (b) : $(0, +, 0)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Subcase (c) : $(0, 0, +)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Subcase (d) : $(-, 0, 0), (0, -, 0)$ and $(0, 0, -)$

It is easy to see that, by Theorem 3.4, all four cases do not require algebraic positivity.

Now, let us assume exactly one among the $(1, 2), (1, 3)$ and $(2, 3)$ entries is 0.

Subcase (e) : $(+, +, 0), (+, 0, +)$ and $(0, +, +)$

Let $B \in \left\{ \begin{bmatrix} \star & + & + \\ + & \star & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & + & 0 \\ + & \star & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} \star & 0 & + \\ 0 & \star & + \\ + & + & 0 \end{bmatrix} : \star \in \{+, -, 0\} \right\}$ and $A \in Q(B)$.

Assume that A is from either **Case (ii), Case (iii)** or **Case (iv)**. Then A^2 has positive entries in all the zero entries of A, proving that A is irreducible. Therefore, by Theorem 2.5, A is algebraically positive. Now if A is from **Case (v)**, then, let the negative entry of A be $-x$, where x is a positive real number. Then clearly, $M = A + 2xI$ will have positive diagonal entries, and M^2 will have positive entries in all the zero entries of M, proving that M is irreducible. Therefore, by Theorem 2.5, M is algebraically positive, which implies $M - 2xI = A$ is algebraically positive.

Subcase (f) : $(-, -, 0), (-, 0, -)$ and $(0, -, -)$

Again, let $B \in \left\{ \begin{bmatrix} \star & - & - \\ - & \star & 0 \\ - & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & - & 0 \\ - & \star & - \\ 0 & - & 0 \end{bmatrix}, \begin{bmatrix} \star & 0 & - \\ 0 & \star & - \\ - & - & 0 \end{bmatrix} : \star \in \{+, -, 0\} \right\}$ and $A \in Q(B)$. Assume that A is from either **Case (ii), Case (iii)** or **Case (iv)**. Now consider $-A$. Then the $(1, 2), (1, 3)$ and $(2, 3)$ entries of it are either $(+, +, 0), (+, 0, +)$ or $(0, +, +)$. Note that no matter in which among the four cases A belongs to, there always exist a positive real number ϵ such that $-A + \epsilon I$ has positive diagonal entries. Thus, $-A + \epsilon I$ is algebraically positive, which implies A is algebraically positive.

Subcase (g) : $(+, -, 0)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$	Requires

Subcase (h) : $(-, +, 0)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	Requires	Requires	Does not require by Theorem 3.4

Subcase (i) : $(+, 0, -)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

Subcase (j) : $(-, 0, +)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
Requires	$\begin{bmatrix} 0 & -15 & 0 \\ -15 & 1 & \frac{1}{10} \\ 0 & \frac{1}{10} & 0 \end{bmatrix}$	Requires	Requires

Subcase (k) : $(0, +, -)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
$\begin{bmatrix} 1 & 0 & \frac{1}{10} \\ 0 & 0 & -15 \\ \frac{1}{10} & -15 & 0 \end{bmatrix}$	Requires	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

Subcase (l) : $(0, -, +)$

Case (ii)	Case (iii)	Case (iv)	Case (v)
Requires	$\begin{bmatrix} 0 & 0 & -\frac{1}{10} \\ 0 & 1 & 15 \\ -\frac{1}{10} & 15 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$	Requires

Here are the proofs of those sign patterns requiring algebraic positivity:

Case (v) *Subcase (g)* : Let $A \in Q\left(\begin{bmatrix} + & + & - \\ + & - & 0 \\ - & 0 & 0 \end{bmatrix} \right)$ and $A = [a_{ij}]$. Consider $f(x) = c_2 x^2 + c_1 x + c_0$, where c_0, c_1 and c_2 are real numbers. Let $c_2 < -\frac{1}{a_{11}}$ and $c_2(a_{11} + a_{22}) + c_1 > 0$ and c_0 be greater than all the diagonal entries of $-c_1 A - c_2 A^2$. Then, $f(A) > 0$.

Case (iv) *Subcase (h)* : Let $A \in Q\left(\begin{bmatrix} + & - & + \\ - & + & 0 \\ + & 0 & 0 \end{bmatrix} \right)$ and $A = [a_{ij}]$. Consider $f(x) = c_2 x^2 + c_1 x + c_0$, where c_0, c_1 and c_2 are real numbers. Let $c_2 < 0$ and $c_2(a_{11} + a_{22}) + c_1 < 0$ and $c_2 a_{11} + c_1 > 0$ and c_0 be greater than all the diagonal entries of $-c_1 A - c_2 A^2$. Then, $f(A) > 0$.

Case (iii) *Subcase (h)* : Let $A \in Q\left(\begin{bmatrix} 0 & - & + \\ - & + & 0 \\ + & 0 & 0 \end{bmatrix} \right)$. Then, $A^2 \in Q\left(\begin{bmatrix} + & - & 0 \\ - & + & - \\ 0 & - & + \end{bmatrix} \right)$.

Now consider $-A^2$ whose sign pattern is $\begin{bmatrix} - & + & 0 \\ + & - & + \\ 0 & + & - \end{bmatrix}$. Clearly, there exists a positive real number d such that in $dI - A^2$, all its diagonal entries are positive, and is irreducible with non-negative off diagonal entries. Thus, by Theorem 2.5, $dI - A^2$ is algebraically positive which implies A^2 is algebraically positive. Again A^2 being algebraically positive implies A is algebraically positive (since A^2 is algebraically positive, there exists a real polynomial $f(x)$ such that $f(A^2)$ is a positive matrix and letting $g(x) = f(x^2)$ gives $g(A) > 0$. Thus, A is algebraically positive).

Case (ii) *Subcase (j)* : Similar argument as above.

Case (iii) *Subcase (k)* : Similar argument as above.

Case (ii) *Subcase (l)* : Similar argument as above.

Case (v) *Subcase (l)* : Similar argument as above.

Case (iv) & (v) *Subcase (j)* : Let $B \in \left\{ \begin{bmatrix} + & - & 0 \\ - & + & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & - & + \\ 0 & + & 0 \end{bmatrix} \right\}$ and $A = [a_{ij}] \in Q(B)$. Consider $f(x) = c_2 x^2 + c_1 x + c_0$, where c_0, c_1 and c_2 are real numbers. Let $c_2 < 0$ and $-(a_{11} + a_{22}) < \frac{c_1}{c_2} < -a_{22}$ and c_0 be greater than all the diagonal entries of $-c_1 A - c_2 A^2$. Then, $f(A) > 0$.

Finally, let us assume all $(1,2), (1,3)$ and $(2,3)$ entries are 0.

Subcase (m) : $(0,0,0)$
Clearly, by Theorem 3.4, no case requires algebraic positivity.

   Therefore, using Lemma 4.2, we can conclude that if A is a 3×3 symmetric lower zero-diagonal sign pattern matrix, then A requires algebraic positivity if and only if A or $-A$ belongs to the following class,

$$
\hat{\mathcal{C}} = \mathcal{C} \; \bigcup \; \left\{ \begin{bmatrix} + & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & 0 & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & - & - \\ - & + & - \\ - & - & 0 \end{bmatrix}, \right.
$$

$$
\begin{bmatrix} + & - & - \\ - & + & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & - & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & - & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} \star & + & + \\ + & \star & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & + & 0 \\ + & \star & + \\ 0 & + & 0 \end{bmatrix},
$$

$$
\begin{bmatrix} \star & 0 & + \\ 0 & \star & + \\ + & + & 0 \end{bmatrix}, \begin{bmatrix} \star & - & - \\ - & \star & 0 \\ - & 0 & 0 \end{bmatrix}, \begin{bmatrix} \star & - & 0 \\ - & \star & - \\ 0 & - & 0 \end{bmatrix}, \begin{bmatrix} \star & 0 & - \\ 0 & \star & - \\ - & - & 0 \end{bmatrix}, \begin{bmatrix} + & + & - \\ + & - & 0 \\ - & 0 & 0 \end{bmatrix}, \begin{bmatrix} + & - & + \\ - & 0 & 0 \\ + & 0 & 0 \end{bmatrix},
$$

$$
\begin{bmatrix} + & - & + \\ - & + & 0 \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & 0 & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & + & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & - & + \\ 0 & + & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & + \\ 0 & + & - \\ + & - & 0 \end{bmatrix}, \begin{bmatrix} + & 0 & - \\ 0 & 0 & + \\ - & + & 0 \end{bmatrix},
$$

$$
\left. \begin{bmatrix} + & 0 & - \\ 0 & - & + \\ - & + & 0 \end{bmatrix} \right\},
$$

where, $\star \in \{+, -, 0\}$

$\square$

Let A be a 3×3 symmetric lower zero-diagonal matrix. Hence, A is of the form $\begin{bmatrix} x & a & b \\ a & y & c \\ b & c & 0 \end{bmatrix}$ where a, b, c, x and y are real numbers. By Lemma 4.3, $\begin{bmatrix} x & a & b \\ a & y & c \\ b & c & 0 \end{bmatrix}$ is algebraically positive $\iff$ $\begin{bmatrix} 0 & c & b \\ c & y & a \\ b & a & x \end{bmatrix}$ is algebraically positive $\iff$ $\begin{bmatrix} x & b & a \\ b & 0 & c \\ a & c & y \end{bmatrix}$ is algebraically positive. Therefore, by characterizing all 3×3 symmetric lower zero-diagonal sign pattern matrices, we have managed to characterize all 3×3 symmetric sign pattern matrices with at least one zero diagonal entry. Note that although the positions of a, b and c gets interchanged among themselves, it still covers all the cases, since in the proof of Theorem 4.9, we have considered all possible choices for $(1, 2), (1, 3)$ and $(2, 3)$ entry. Hence, a permutation of a, b and c does not affect the total covering of cases.

Corollary 4.10. *Suppose A is a 3×3 symmetric sign pattern matrix with at least one zero diagonal entry. Then A requires algebraic positivity if and only if either A or $-A$ or any permuted sign pattern of A belongs to $\hat{\mathcal{C}}$.*

We now characterize the remaining cases, i.e., all 3×3 symmetric sign pattern matrices with non-zero diagonal entries.

Theorem 4.11. *Suppose A is a 3×3 symmetric sign pattern matrix with non-zero diagonal entries. Then A requires algebraic positivity if and only if either A or $-A$ or any permuted sign pattern of A belongs to the following class,*

$$
\hat{\mathcal{D}} = \left\{ \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & - \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & - \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & 0 & - \end{bmatrix}, \begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & + & - \end{bmatrix}, \begin{bmatrix} + & 0 & + \\ 0 & + & + \\ + & + & - \end{bmatrix}, \right.
$$

$$
\begin{bmatrix} + & - & 0 \\ - & + & - \\ 0 & - & - \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & + & 0 \\ - & 0 & - \end{bmatrix}, \begin{bmatrix} + & 0 & - \\ 0 & + & - \\ - & - & - \end{bmatrix}, \begin{bmatrix} + & - & + \\ - & + & 0 \\ + & 0 & - \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix},
$$

$$
\begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & + & + \end{bmatrix}, \begin{bmatrix} + & 0 & + \\ 0 & + & + \\ + & + & + \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & + & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & + & 0 \\ - & 0 & + \end{bmatrix},
$$

$$
\left. \begin{bmatrix} + & 0 & - \\ 0 & + & - \\ - & - & + \end{bmatrix} \right\}
$$

Proof. In Corollary 4.10, we have characterized all 3×3 symmetric sign pattern matrices with at least one zero diagonal entry. Therefore, we are left with all 3×3 symmetric

sign pattern matrices with non-zero diagonal entries. The possible choices of the diagonal entries, excluding the negatives, are : $(+, +, +)$, $(-, +, +)$, $(+, +, -)$ and $(+, -, +)$.

Case 1. Let A be a 3×3 symmetric sign pattern with $(+, +, -)$ as its diagonal entries respectively and let $B \in Q(A)$. Hence, B is of the form $\begin{bmatrix} x & \star & \star \\ \star & y & \star \\ \star & \star & -z \end{bmatrix}$ where x, y and z are positive real numbers. Note that B is algebraically positive if and only if $\begin{bmatrix} x + z & \star & \star \\ \star & y + z & \star \\ \star & \star & 0 \end{bmatrix}$ is algebraically positive. Therefore, the characterization of A, i.e., $\begin{bmatrix} + & \star & \star \\ \star & + & \star \\ \star & \star & - \end{bmatrix}$ is the same as the characterization of $\begin{bmatrix} + & \star & \star \\ \star & + & \star \\ \star & \star & 0 \end{bmatrix}$, if we base it on their off-diagonal sign choices. This is already addressed in the sub cases of Case (iv) from Theorem 4.9.

Case 2. Let A be a 3×3 symmetric sign pattern with $(+, +, +)$ as its diagonal entries respectively and let $B \in Q(A)$. Hence, B is of the form $\begin{bmatrix} x & \star & \star \\ \star & y & \star \\ \star & \star & z \end{bmatrix}$ where x, y and z are positive real numbers. In order to characterize this sign pattern, if we, say, divide the proof into cases and sub cases on the basis of their off-diagonal sign choices, it is easy to see that the cases and the sub cases which did not require algebraic positivity in Theorem 4.4 will be preserved in the corresponding cases and sub cases here, i.e., if C is a counterexample in some particular case in Theorem 4.4, then consider $C + I$ for the corresponding case here. Now for the remaining cases, it is clear that B is algebraically positive if and only if $\begin{bmatrix} x - z & \star & \star \\ \star & y - z & \star \\ \star & \star & 0 \end{bmatrix}$ is algebraically positive.

From Theorem 4.9, we know $\begin{bmatrix} x - z & \star & \star \\ \star & y - z & \star \\ \star & \star & 0 \end{bmatrix}$ is algebraically positive, provided the off-diagonal entries are $(+, +, 0)$, $(+, 0, +)$, $(0, +, +)$, $(+, +, +)$, $(-, -, 0)$, $(-, 0, -)$, $(0, -, -)$ and $(-, -, -)$, which implies B is algebraically positive (We need to consider the subcases which require algebraic positivity for all cases from (ii) to (iii) in the proof of Theorem 4.9). Note that these are the same combinations of the off-diagonal entries

for zero-diagonal sign pattern matrices that require algebraic positivity. Thus, the characterization of A, i.e., $\begin{bmatrix} + & \star & \star \\ \star & + & \star \\ \star & \star & + \end{bmatrix}$ is the same as the characterization of $\begin{bmatrix} 0 & \star & \star \\ \star & 0 & \star \\ \star & \star & 0 \end{bmatrix}$, if we base it on their off-diagonal sign choices.

Case 3. Let A be a 3×3 symmetric sign pattern with $(+, -, +)$ as its diagonal entries respectively and let $B \in Q(A)$. Hence, B is of the form $\begin{bmatrix} x & a & b \\ a & -y & c \\ b & c & z \end{bmatrix}$ where x, y and z are positive real numbers and a, b and c are real numbers. By Lemma 4.3, B is algebraically positive if and only if $\begin{bmatrix} x & b & a \\ b & z & c \\ a & c & -y \end{bmatrix}$ is algebraically positive. The latter is already addressed in Case 1.

Case 4. Similar argument as above.

Therefore, using Lemma 4.2, we can conclude that if A is a 3×3 symmetric sign pattern matrix with non-zero diagonal entries. Then A requires algebraic positivity if and only if either A or $-A$ or any permuted sign pattern of A belongs to the following class,

$$
\hat{\mathcal{D}} = \left\{ \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & - \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & + & - \\ - & - & - \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & 0 & - \end{bmatrix}, \begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & + & - \end{bmatrix}, \begin{bmatrix} + & 0 & + \\ 0 & + & + \\ + & + & - \end{bmatrix}, \right.
$$

$$
\begin{bmatrix} + & - & 0 \\ - & + & - \\ 0 & - & - \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & + & 0 \\ - & 0 & - \end{bmatrix}, \begin{bmatrix} + & 0 & - \\ 0 & + & - \\ - & - & - \end{bmatrix}, \begin{bmatrix} + & - & + \\ - & + & 0 \\ + & 0 & - \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix},
$$

$$
\begin{bmatrix} + & - & - \\ - & + & - \\ - & - & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ + & + & 0 \\ + & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & 0 \\ + & + & + \\ 0 & + & + \end{bmatrix}, \begin{bmatrix} + & 0 & + \\ 0 & + & + \\ + & + & + \end{bmatrix}, \begin{bmatrix} + & - & 0 \\ - & + & - \\ 0 & - & + \end{bmatrix}, \begin{bmatrix} + & - & - \\ - & + & 0 \\ - & 0 & + \end{bmatrix},
$$

$$
\left. \begin{bmatrix} + & 0 & - \\ 0 & + & - \\ - & - & + \end{bmatrix} \right\}
$$

$\square$

Finally, combining Corollary 4.10 and Theorem 4.11, we get our final result.

Theorem 4.12. *Suppose A is a 3×3 symmetric sign pattern matrix. Then A requires algebraic positivity if and only if either A or $-A$ or any permuted sign pattern of A belongs to $\hat{\mathcal{C}} \cup \hat{\mathcal{D}}$.*

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AMITAKSHAR BISWAS, STUDENT OF BACHELOR OF MATHEMATICS (B.MATH.), INDIAN STATISTICAL INSTITUTE, STATISTICS AND MATHEMATICS UNIT (SMU), 8TH MILE, MYSORE ROAD, BANGALORE, 560 059, INDIA.
Email address: bmat1904@isibang.ac.in, biswasamitakshar2013@gmail.com

SOUMYAJYOTI KUNDU, STUDENT OF BACHELOR OF MATHEMATICS (B.MATH.), INDIAN STATISTICAL INSTITUTE, STATISTICS AND MATHEMATICS UNIT (SMU), 8TH MILE, MYSORE ROAD, BANGALORE, 560 059, INDIA.
Email address: bmat1938@isibang.ac.in, soumyajyotik1729@gmail.com