

Divergence-Free Shape Interpolation and Correspondence

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Shape Correspondence



Source: [1]

- Assign each point from one point cloud to another point cloud such that the point and its image correspond to each other
- Could also use mesh information (faces) for computation

What is a Divergence-Free Deformation Field?

- Given two point clouds $X = \{x_1, \dots, x_n\} \subset \Omega, Y = \{y_1, \dots, y_n\} \subset \Omega$, find a mapping $f : \Omega \rightarrow \Omega$ s.t. $f(X)$ fits the shape Y
- Idea: f should imitate real-world transformation from X to Y
- What is natural? Smoothness and continuity!

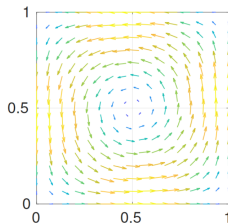
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⇒

$$\begin{cases} \dot{x}(t) = v(x(t)) \\ x(0) = x_n \end{cases}$$

- $v : \Omega \rightarrow \mathbb{R}^D$ is called deformation field!



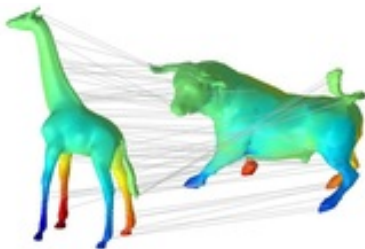
Source: [2]

What is a Divergence-Free Deformation Field?

- Two assumptions:
 - 1 Smoothness: $v \in C^\infty(\Omega, \mathbb{R}^D)$
 - 2 Divergence-free: $\nabla \cdot v = 0$
- Why? Smoothness guarantees unique solution of IVP by Picard-Lindelöf

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- Divergence-free vector fields conserve volume for any part $U \subset \Omega$ of the shape!



Source: [3]

Obtaining the Correspondence Mapping

- Evaluate solution operator at $t = 1$ to get correspondence mapping f :
 $f(x_n) := x(1)$
- $t=1$ is arbitrary and practical choice

Representing the Deformation Field

- We can now compute the correspondence given the deformation field
- But how to find the best deformation field? First, we need to represent the deformation field in an efficient way
- Compute a basis $\{v_1, v_2, \dots\}$ that spans the space of divergence-free deformation fields

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$$B_\phi = \left\{ \phi : [0, 1]^D \rightarrow \mathbb{R}, x \mapsto \prod_{d=1}^D \frac{1}{2} \sin(x_d \pi j_d) \mid j \in \mathbb{N}^D \right\}$$

- Sort the basis elements by eigenvalue

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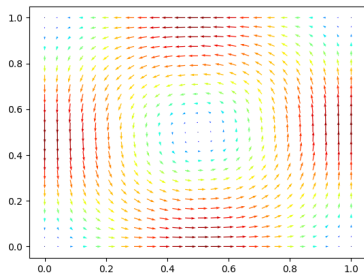
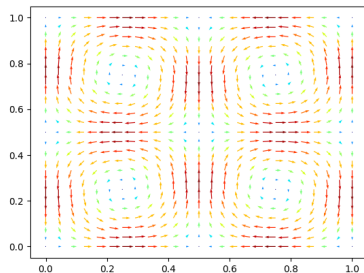
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- Sort the basis elements by eigenvalue

- Define $B_v = \bigcup_{k=1}^{\infty} \left\{ \nabla \times \begin{pmatrix} \phi_k \\ 0 \\ 0 \end{pmatrix}, \nabla \times \begin{pmatrix} 0 \\ \phi_k \\ 0 \end{pmatrix}, \nabla \times \begin{pmatrix} 0 \\ 0 \\ \phi_k \end{pmatrix} \right\}$

$$\Rightarrow v(x) = \sum_{k=1}^K v_k(x) a_k \text{ for a fixed } K.$$

Basis examples, sliced at $x_3 = 0.5$

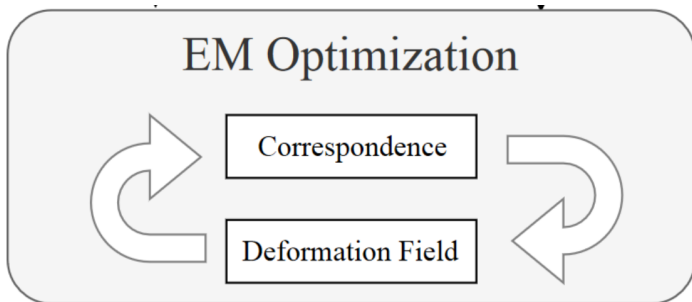
 v_1  v_{12}

Computing the Deformation Field

- Remember: $v(x) = \sum_{k=1}^K v_k(x) a_k$
- We only have to find the correct a_k
- If we had the a_k we could compute f which maps $\{x_1, \dots, x_n\}$ to be in the shape of Y
- We still need to assign the points $f(x_n)$ to points y_n

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- We still need to assign the points $f(x_n)$ to points y_n
- Idea: Do both at the same time with Expectation Maximization



Source: [2]

Shape Interpolation

- Remember the IVP:

$$\begin{cases} \dot{x}(t) = v(x(t)) \\ x(0) = x_n \end{cases}$$

- We defined $f := x(1)$
- What is $x(0.5)$?

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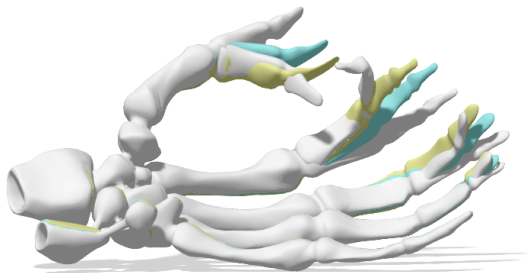
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Source: [2]

Project topic: Time-Varying Deformation Field

- There is only one time-independent deformation field



Source: [2]

- Possible Solution: Calculate multiple deformation fields for smaller timeframes
- Maybe even find a way to make deformation field vary over time

Sources

[1] https://geom1.github.io/WI18/LectureSlides/L17_Functional_Map.pdf

[2] Eisenberger, Marvin, Zorah Löhner, and Daniel Cremers.
"Divergence-free shape correspondence by deformation." Computer Graphics Forum. Vol. 38. No. 5. 2019.1

[3] https://www.researchgate.net/figure/Shape-correspondence-results-Notice-that-our-method-is-not-fig13_305750848