Divergence-Free Shape Interpolation and Correspondence

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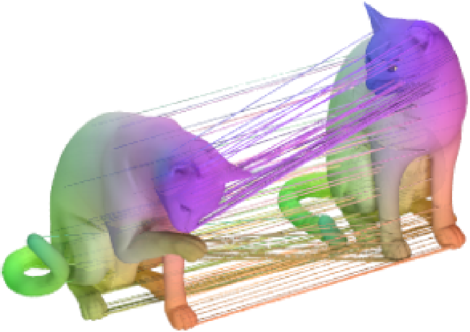
Shape Correspondence



Source: [1]

- Assign each point from one point cloud to another point cloud such that the point and its image correspond to each other
- Could also use mesh information (faces) for computation

What is a Divergence-Free Deformation Field?

- Given two point clouds $X = \{x_1, ..., x_n\} \subset \Omega, Y = \{y_1, ..., y_n\} \subset \Omega$, find a mapping $f : \Omega \to \Omega$ s.t. $f(X)$ fits the shape Y
- Idea: f should imitate real-world transformation from X to Y
- What is natural? Smoothness and continuity!

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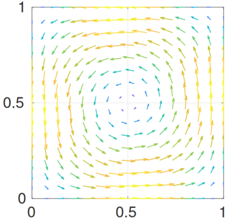
$\Rightarrow$

$$\begin{cases} \dot{x}(t) = v(x(t)) \\ x(0) = x_n \end{cases}$$

- $v : \Omega \to \mathbb{R}^D$ is called deformation field!



Source: [2]

What is a Divergence-Free Deformation Field?

- Two assumptions:
  1. Smoothness: $v \in C^{\infty}(\Omega, \mathbb{R}^D)$
  2. Divergence-free: $\nabla \cdot v = 0$
- Why? Smoothness guarantees unique solution of IVP by Picard-Lindelöf

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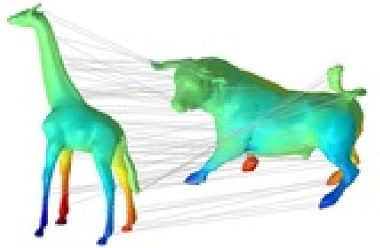
- Why? Smoothness guarantees unique solution of IVP by Picard-Lindelöf
- Divergence-free vector fields conserve volume for any part $U \subset \Omega$ of the shape!



Source: [3]

Obtaining the Correspondence Mapping

- Evaluate solution operator at $t = 1$ to get correspondence mapping f: $f(x_n) := x(1)$
- t=1 is arbitrary and practical choice

Representing the Deformation Field

- We can now compute the correspondence given the deformation field
- But how to find the best deformation field? First, we need to represent the deformation field in an efficient way
- Compute a basis $\{v_1, v_2, ...\}$ that spans the space of divergence-free deformation fields

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$$B_\phi = \left\{ \phi : [0,1]^D \to \mathbb{R}, x \mapsto \prod_{d=1}^{D} \frac{1}{2} \sin(x_d \pi j_d) \middle| j \in \mathbb{N}^D \right\}$$

- Sort the basis elements by eigenvalue

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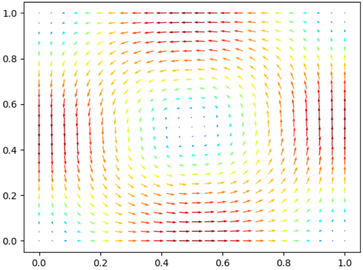
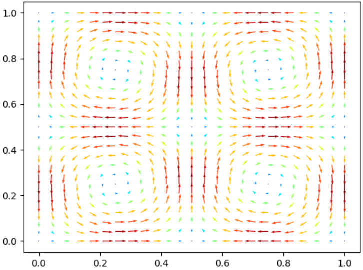
- Sort the basis elements by eigenvalue
- Define $B_v = \bigcup_{k=1}^{\infty} \left\{ \nabla \times \begin{pmatrix} \phi_k \\ 0 \\ 0 \end{pmatrix}, \nabla \times \begin{pmatrix} 0 \\ \phi_k \\ 0 \end{pmatrix}, \nabla \times \begin{pmatrix} 0 \\ 0 \\ \phi_k \end{pmatrix} \right\}$

$\Rightarrow$ $v(x) = \sum_{k=1}^{K} v_k(x) a_k$ for a fixed K.

Basis examples, sliced at $x_3 = 0.5$



v_1



v_{12}

Computing the Deformation Field

- Remember: $v(x) = \sum_{k=1}^{K} v_k(x) a_k$
- We only have to find the correct a_k
- If we had the a_k we could compute f which maps $\{x_1, ..., x_n\}$ to be in the shape of Y
- We still need to assign the points $f(x_n)$ to points y_n

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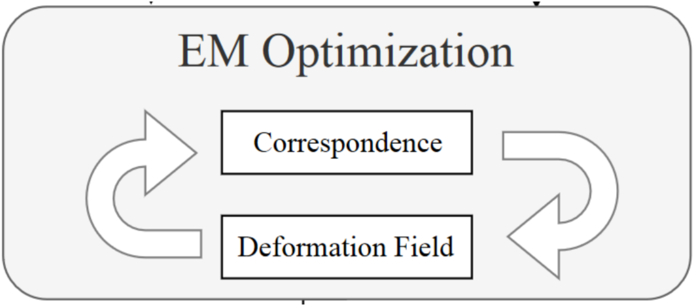
- Idea: Do both at the same time with Expectation Maximization



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Shape Interpolation

- Remember the IVP:

$$\begin{cases} \dot{x}(t) = v(x(t)) \\ x(0) = x_n \end{cases}$$

- We defined $f := x(1)$
- What is $x(0.5)$?

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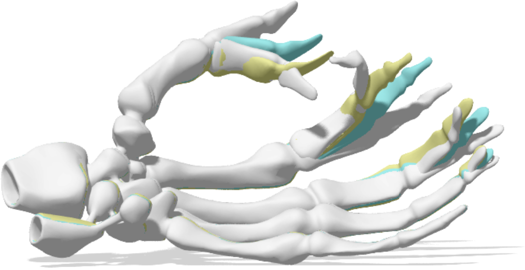
Project topic: Time-Varying Deformation Field

- There is only one time-independent deformation field



Source: [2]

- Possible Solution: Calculate multiple deformation fields for smaller timeframes
- Maybe even find a way to make deformation field vary over time

Sources

[1] https://geoml.github.io/WI18/LectureSlides/L17_Functional_Map.pdf

[2] Eisenberger, Marvin, Zorah Lähner, and Daniel Cremers. "Divergence-free shape correspondence by deformation." Computer Graphics Forum. Vol. 38. No. 5. 2019.1

[3] https://www.researchgate.net/figure/Shape-correspondence-results-Notice-that-our-method-is-no
fig13_305750848