1 Topology

The first three propositions are from early in my topology class. I chose this particular set of problems mostly because I thought Proposition 3 had an interesting proof. For context, the *section* S_α of a set A equipped with the order topology is the subset $\{x \in A \mid x < \alpha\}$. S_Ω is the so-called "minimal uncountable well-ordered set", which is a set with the property that any section of S_Ω is countable, but S_Ω is uncountable.

Proposition 1. S_Ω has no largest element.

Proof. Assume towards contradiction that S_Ω has a largest element, which we will call a. Then, S_a contains every element element in S_Ω smaller than a. By assumption, every element in S_Ω except a is smaller than a. Since S_Ω is uncountable, $S_a = S_\Omega - \{a\}$ is also uncountable, a contradiction to the assumption that S_x is countable for any $x \in S_\Omega$. Thus, a does not exist, and S_Ω has no largest element. $\square$

Proposition 2. For every element $a \in S_\Omega$, the subset $\{x \mid a < x\}$ is uncountable.

Proof. Let $a \in S_\Omega$ be an arbitrary element. Assume towards contradiction that the set $\{x \mid a < x\}$ is countable. Then $S_a = S_\Omega - \{x \mid a < x\} - \{a\}$ must be uncountable, since subtraction of an uncountable set by a countable set is uncountable. This is a contradiction, by our premise that S_a is countable. Thus $\{x \mid a < x\}$ is uncountable. $\square$

Proposition 3. Let $X_0 \subset S_\Omega$ consisting of all elements $x \in S_\Omega$ where x has no immediate predecessor. Then X_0 is uncountable.

Proof. To prove this claim we will construct a partition of S_Ω where each subset in the partition has exactly one element of X_0 and is countable. Then, we will argue that since S_Ω is uncountable, there must be an uncountable set of totally disjoint subsets of S_Ω which each contain one element of X_0, and thus X_0 is uncountable.

First, we construct a function $\phi : S_\Omega \to S_\Omega$ defined as follows:

$$\phi(x) = \begin{cases} x & \text{if } x \text{ has no immediate predecessor} \\ P(x) & \text{otherwise} \end{cases}$$

where if $\operatorname{Pred}(x)$ represents the predecessor of x then $P : S_\Omega \to S_\Omega$ maps x to the smallest element in the descending chain

$$x, \operatorname{Pred}(x), \operatorname{Pred}(\operatorname{Pred}(x)), \cdots,$$

where the chain terminates at length n when the nth element has no predecessor. Note that this is well defined because S_Ω is well-orderd, and thus every set has a smallest element. This element clearly does not have a predecessor.

Now, we construct a partition Q of S_Ω where $x \sim y$ are in the same partition when $\phi(x) = \phi(y)$. It is clear that this is a partition.

We can see that for any arbitrary x with a predecessor, $x \sim P(x)$ because $\phi(x) = \phi(P(x))$. Since $\phi(x)$ has no predecessor, the subset containing x in the partition has this as an element with no predecessor. Now, pick y, z with no predecessors where $y \neq z$. Since $\phi(y) = y \neq z = \phi(z)$, these elements are in different subsets in the partition. Thus, each subset of the partition has exactly one element with no predecessors.

Now, consider some arbitrary element x. If x has a successor $\operatorname{Succ}(x)$, then clearly $\operatorname{Succ}(x) \sim x$. Thus, for any subset S of a partition containing some element y, then let

$$U = \{\phi(y), \operatorname{Succ}(\phi(y)), \operatorname{Succ}(\operatorname{Succ}(\phi(y))), \cdots\}.$$

Then $U \subset S$. Now, assume towards contradiction that there is some element $z \in S$ such that no continued successor of $\phi(y)$ is z, i.e. $z \notin U$. Then $\phi(z) \neq \phi(y)$, and thus z is not in S. Thus, $S = U$, and is countable.

Now we have shown that every set in Q is countable and contains exactly one element with no immediate predecessor. Furthermore, since no countable union of countable sets can be uncountable, there must be an uncountable set of sets in Q, since Q is a partition of S_Ω, which is uncountable. Since all sets in Q are totally disjoint and contain one element in X_0, X_0 must be uncountable. $\square$

2 Complex Analysis

The next two propositions are from my complex analysis class, which is the math class that I took most recently, and thus I felt should be included in this document. By nature of the class, the proofs are more symbol-pushing and computing integrals, but are nonetheless interesting.

Proposition 4. If $f(z)$ is continuous on a domain D and $f(z)^8$ is analytic on D, then $f(z)$ is analytic on D.

Proof. First, note that if $f(z)^2$ is analytic implies that $f(z)$ is analytic, then $f(z)^8$ is analytic implies that $f(z)^4$ is analytic implies that $f(z)^2$ is analytic implies that $f(z)$ is analytic, and thus it suffices to show that $f(z)^2$ is analytic. (Since $f(z)$ continuous implies that higher powers of $f(z)$ are continuous.)

We will show that if $f(z)^2$ is analytic on D and $f(z)$ is continuous, then $f(z)$ is analytic on D. Consider the domain $D^* = \{z \in D \mid f(z) \neq 0\}$. We have that $f(z)^2$ is analytic on D^*. Now, since $f(z)^2$ is analytic we can compute the derivative

$$\begin{aligned}
\frac{df(z)^2}{dz} &= \lim_{h\to 0} \frac{f(z+h)^2 - f(z)^2}{h} \\
&= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \frac{f(z+h) + f(z)}{1} \\
&= \left(\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}\right)\left(\lim_{h\to 0} \frac{f(z+h) + f(z)}{1}\right) \\
&= \left(\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}\right) 2f(z).
\end{aligned}$$

Thus,

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \frac{df(z)^2/dz}{2f(z)}$$

which is defined when $f(z) \neq 0$. This means that $f'(z)$ is defined when $f(z) \neq 0$ and by Goursat's Theorem, we have that $f(z)$ is analytic on D^*. Now, note that $f(z) = 0$ exactly when $f(z)^2 = 0$. Since $f(z)^2$ is analytic, it has isolated zeros, and thus similarly does $f(z)$. By (a corollary to) Morera's Theorem, since $f(z)$ is continuous at its zeros on D, and since it is analytic everywhere except its zeros on D, then, in fact, it is analytic everywhere on D. $\square$

Proposition 5. We can check that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}.$$

using reside theory or by differentiating both sides of

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}.$$

Proof. Observe that

$$\int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} + \int_{\Gamma_R} \frac{dx}{(x^2+a^2)^2} = \int_{\partial D_R} \frac{dx}{(x^2+a^2)^2},$$

where D_R is a semicircle centered (i.e., the circle is centered) at zero with radius R and Γ_R is the non-real component of the boundary of the semicircle. By residue theory, this integral is the sum of the residues of the poles contained in the semicircle multiplied by $2\pi i$. Note that $(x^2+a^2)^2 = (x+ia)^2(x-ia)^2$ has a double zero at $x = ia$ in the positive half-plane, and no other zeros in the positive half-plane. Thus, $\frac{1}{(x^2+a^2)^2}$ has a double pole at $x = ia$, and this means that the residue of this function at $x = ia$ can be computed by

$$\lim_{x\to ia} \frac{d}{dx} \frac{dx}{(x^2+a^2)^2} = \lim_{x\to ia} -\frac{2}{(x+ia)^3} = -\frac{2}{(2ia)^3}.$$

Multiplying this by $2\pi i$, we get $\pi/2a^3$. Thus, for $R > a$, we get

$$\int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} + \int_{\Gamma_R} \frac{dx}{(x^2+a^2)^2} = \pi/2a^3.$$

Similarly, since the right hand side is invariant of R (other than requiring R to be sufficiently large),

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} + \int_{\Gamma_R} \frac{dx}{(x^2+a^2)^2} = \pi/2a^3.$$

We can use the ML-bound on the second integral in the left hand side of the equation, with

$$L = \pi R \qquad M = \left|\frac{1}{(a^2-R^2)^2}\right| \sim \left|\frac{1}{R^4}\right| \text{ as } R \to \infty.$$

Since

$$|ML| = \left|\frac{\pi R}{(a^2-R^2)^2}\right| \sim \frac{\pi}{R^3} \to 0 \text{ as } R \to \infty,$$

we have that

$$\int_{\Gamma_R} \frac{dx}{(x^2+a^2)^2} \to 0 \text{ as } R \to \infty,$$

and finally

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+a^2)^2} + \int_{\Gamma_R} \frac{dx}{(x^2+a^2)^2} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} + = \pi/2a^3.$$

We can also arrive at the same result by differentiating both sides of

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{\pi}{a}.$$

Take

$$\frac{d}{da} \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \frac{d}{da} \frac{\pi}{a}.$$

Let's start with the left hand side. By Leibnitz rule, we can switch the integral and the derivative

$$\frac{d}{da} \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = \int_{-\infty}^{\infty} \frac{\partial}{\partial a} \frac{dx}{x^2+a^2} = \int_{-\infty}^{\infty} \frac{2a}{(x^2+a^2)^2} dx = 2a \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

So by plugging in the right hand side of the first equation and evaluating the derivative we have

$$2a \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{a^2}$$

and thus

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}.$$

□