Curvature of space as a time-independent perturbation

Joshua Lin
Enrichment activity for Physics 137B, Fall 2017, Professor A.Charman

December 6, 2017

1 Overview

In our quantum physics lectures, both 137A and B, we were introduced to the Hamiltonian operator, which takes the form:

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V$$

where V is our potential. In class, we have only dealt with the case where we are interested in solutions to Schrodinger's equation:

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle$$

in flat space (usually one, two or three dimensional Euclidean space, and occassionally compact spaces such as a closed interval (infinite square well) or a circle (question in homework on calculating energy levels of a fidget spinner)). In flat space, we can always find a cartesian set of coordinates to describe the space, at least locally, which leads to a laplacian of the form:

$$\nabla^2_{\text{flat}} = \partial_i\partial_i$$

using the Einstein summation notation, and taking i to range over a cartesian basis. But what if we wanted to solve Schrodinger's equation on a more general manifold? If this manifold has geometry described by a metric g_{ij}, then our laplacian is generalised to the Laplace-Beltrami operator, and is given by:

$$\nabla^2_{\text{curved}} = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j)$$

(Note that in the case of Euclidean space, our metric g_{ij} is the delta function, and our operator reduces to $\partial_i\partial_i$ as expected). Now, assuming that the deviation of our new ∇^2 operator from the old ∇^2 operator in flat space is small in some sense, then it might be interesting to consider this as a time independent perturbation problem, namely:

$$H = -\frac{\hbar^2}{2m}\nabla^2_{\text{curved}} + V = \left(-\frac{\hbar^2}{2m}\nabla^2_{\text{flat}} + V\right) - \frac{\hbar^2}{2m}(\nabla^2_{\text{curved}} - \nabla^2_{\text{flat}})$$

where our perturbation is given by:

$$H_1 = -\frac{\hbar^2}{2m}(\nabla^2_{\text{curved}} - \nabla^2_{\text{flat}})$$

2 Hyperbolic Geometry

One of the most studied non-euclidean geometries is that of Hyperbolic Space, whose defining feature is a constant negative sectional curvature. As a model for the hyperbolic plane, consider a two dimensional space described by "geodesic-polar" coordinates r, θ, which are the analogue to polar coordinates in the Euclidean plane. The metric will be given by:

$$g_{rr} = 1, g_{\theta\theta} = S_k^2(r), g_{r\theta} = g_{\theta r} = 0$$

where

$$S_k(r) = \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}r)$$

and k is a negative real number. Notice firstly that in the limit as $k \to 0^-$, we have $g_{\theta\theta} \to r^2$, which is the usual metric for polar coordinates of the Euclidean plane. Notice also that we have sectional curvature: