Connections and Fiber Bundles

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1 Connections and the Cartan formalism

> On a manifold it is necessary to use covariant differentation, curvature measures its noncommutativity. Its combination as a characteristic form measures the nontriviality of the underlying bundle. This train of ideas is so simple, that their importance can not be exaggerated. - Shiing-shen Chern

For a smooth (for now, real) manifold M, we let $\mathcal{A}^1(M)$ denote smooth $\mathbb{C}$-valued differential forms and $\mathcal{A}$(M) denote smooth functions. Similarly for a complex bundle $E \to M$, we let E(M) denote (smooth, $\mathbb{C}$-valued) sections

1.1 Connections

Recall that a connection on a vector bundle E over a smooth manifold is a $\mathbb{C}$-homomorphism

$$\nabla : E(M) \to (\mathcal{A}^1 \otimes E)(M)$$

that maps global sections on M to global sections of $\mathcal{A}^1 \otimes E$, which satisfies the Leibniz rule $\nabla(fs) = (df)s+f\nabla s$ $f \in \mathcal{A}(M)$ $s \in E(M)$ This is essentially an way of differentiating sections of E, because for any vector field X on M, we can define

Definition 1.1. The Covariant derivative w.r.t this connection of s in the direction of X

$$\nabla_X s$$

. This satisfies

1. $\nabla_f X(s) \ = \ f\nabla_X s$
2. $\nabla_X(fs) \ = \ (Xf)_s \ + \ f\nabla_X s$

In fact these two properties **characterize** a connection.
We can describe a connection **locally** in terms of frames.

Definition 1.2. Recall that a **frame** of an n-dimensional vector bundle E, over an open subset $U \subset M$, is a family of sections $(e_1, \cdots e_n) \ \in \ E(U)$ that form a basis at each point; thus $e_1, \cdots, e_n$ forms a vector bundle isomorphism between $E|_U$ and the trivial bundle.

Then ∇ is **determined** over U by the elements $\nabla_{e_1}, \cdots \nabla_{e_n} \ \in \ (\mathcal{A}^1 \otimes E)(U)$. For any sections s of $E(U)$ can be written as $s \ = \ \sum_i f_i e_i$ for the f_i smooth functions, and consequently

$$\nabla s = \sum e_i(df_i) \ + \ \sum f_i \nabla e_i$$

In other words, if we use the fram e_i to identify each section of E(U) with the tuple f_i such that $s = \sum f_i e_i$ then ∇ acts by applying d and multiplying by suitable matrix corresponding to the ∇e_i. In view of this we make:

Definition 1.3. Given a frame $\mathfrak{F} = e_1, \cdots, e_n$ over U and a connection ∇, we define the n-by-n matrix $\theta(\mathfrak{F})$ of 1-forms via

$$\nabla \mathfrak{F} = \theta(\mathfrak{F})\mathfrak{F}$$

In other words, $\nabla e_i = \sum_j \theta(\mathfrak{F})_{ij} e_j$ for each j

Note that the θ itself makes no reference to the bundle: it is simply a matrix of 1-forms. Given a frame $\mathfrak{F}$, and given $g : U \to GL_n(\mathbb{C})$, we define a new frame $g\mathfrak{F}$ by multiplying on the left. We would like to determine how a connection **transforms** with respect to a change of frame, so we can think of a connection in a different way. Namely we have:

$$\nabla(g\mathfrak{F}) = (dg)\mathfrak{F} + g\nabla\mathfrak{F} = (dg)\mathfrak{F} + g\theta(\mathfrak{F})\mathfrak{F}$$

where df is considered as a matrix of 1-forms. As a result we get the **transformation law**

$$\theta(g\mathfrak{F}) = (dg)g^{-1} + g\theta(\mathfrak{F})g^{-1},\ g : U \to\ GL_n(\mathbb{C}) \tag{1}$$

Conversely, if we have for each local fram $\mathfrak{F}$ of a vector bundle $E \to M$ a matrix $\theta(\mathfrak{F})$ of 1-forms as above, which satisfy the transformation law 1 as above,then we get a connection on E.

Proposition 1.4. *Any vector bundle $E \to M$ admits a connection*

Proof. It is easy to see that a convex combination of connections is a connection. Namely in each coordinate patch U over which E is trivial with a fixed frame, we choose the matrix θ arbitrarily and get some connection ∇'_U on $E|_U$. Let these various U'_s form an open cover $\mathfrak{A}$. Then we can find a partition of unity ϕ_U, $U \in \mathfrak{A}$ subordinate to $\mathfrak{A}$, and we can define our global connection va

$$\nabla = \sum_U \phi_U \nabla'_U$$

□

1.2 Curvature

We want to now describe the **curvature** of a connection. A connection is a means of differentiating sections; however, it may not satisfy the standard results for functions, that mixed partials are equal. The curvature will be the measure of how much that fails. Let M be a smooth manifold, $E \to M$ a smooth complex vector bundle. Given a connection ∇ on E ,the curvature is going to be a global section of $\mathcal{A}^2 \otimes \hom(E, E)$: in other words, the global differential 2-forms with coefficients in the vector bundle $\hom(E, E)$

Proposition 1.5. *Let s be a section of E, and X, Y vector fields. The map:*

$$s, X, Y \mapsto R(X, Y, s) = (\nabla_Y \nabla_X - \nabla_X \nabla_Y - \nabla_{[X,Y]})s$$

is a bundle map $E \to E$, and is $\mathcal{A}(M)$-linear in X, Y

Proof. Calculation typically done to define the Riemannian curvature tensor in the case of the tangent bundle □

Since the quantity $R(X, Y, s)$ is $\mathcal{A}(M)$-linear in all these quantities (X,Y,s),and clearly alternating in X,Y, we can think of it as a global section of the bundle $\mathcal{A}^2 \otimes \hom(E, E)$. Here recall that $\mathcal{A}^2$ is the bundle of 2-forms.

Definition 1.6. The above elements of $\mathcal{A}^2 \otimes \hom(E, E)(M)$ is called the **curvature** of the connection ∇ and is denoted by Θ.

We now wish to think of the curvature in another manner. To do this, we start by extending the connection ∇ to maps $\nabla : (E \otimes \mathcal{A}^p)(M) \to (E \otimes \mathcal{A}^{p+1})(M)$. The requirement is that the Leibnitz rule holds: that is,

$$\nabla(\omega s) = (d\omega)s + (-1)^p \omega \wedge \nabla s, \tag{2}$$

whenever ω is a p-form and s a global section. We can this locally and glue them. Thus:

Proposition 1.7. *One can extend ∇ to map s*

$$\nabla : (E \otimes \mathcal{A}^p)(M) \to (E \otimes \mathcal{A}^{p+1})(M)$$

satisfying 2

Given such an extension, we can consider the map

$$\nabla^2 : E(M) \to (E \otimes \mathcal{A}^2(M))$$

. This is $\mathcal{A}(M)$-linear. Indeed we can check this by computation

Example 1.8.

$\nabla^2(fs) = \nabla(\nabla(fs)) = \nabla(dfs+f\nabla s) = d^2fs+(-1)df\nabla s+df(\nabla s)+f\nabla^2 s = f\nabla^2 s$

We now want to connect this $\mathcal{A}(M)$- linear map with the earlier curvature tensor

Proposition 1.9. *The vector bundle map ∇^2 is equal to the curvature tensor Θ*

Proof. We can work in local coordinates, and assume that X,Y are the standard commuting vector fields ∂_i ,∂_j. We want to show that, given a section s, we have

$$\nabla^2(s)(\partial_i, \partial_j) = (\nabla_{\partial_i}\nabla_{\partial_i} - \nabla_{\partial_j}\nabla_{\partial_i})s \in E(M)$$

To do this we should check how ∇ was defined. Namely we have, by definition $\nabla s = \sum_i dx_i \nabla_{\partial_i} s$ and consequently

$$\nabla^2 s = \sum_{ij} dx_j \nabla_{\partial_j}(dx_i \nabla_{\partial_i} s)$$

This becomes, by the sign rules $\sum_{i<j}(\nabla_{\partial_j \nabla_{\partial_i} - \nabla_{\partial_i}\nabla_{\partial_j}} s dx_i \wedge dx_j$. It is easy to see that this, evaluated on on (∂_i, ∂_j), gives the desired quantity. It follows that ∇^2 is equal to the curvature tensor Θ □

As a result, we may calculate **curvature** in a frame. Let $\mathfrak{F} = e_1, \cdots, e_n$ be a frame and let $\theta(\mathfrak{F})$ be the connection matrix. Then we can obtain an n-by-n **curvature matrix** $\Theta(\mathfrak{F})$ of 2-forms such that

$$\Theta(\mathfrak{F}) = \nabla^2(\mathfrak{F})$$

The follow result enables us to compute $\Theta(\mathfrak{F})$.

Proposition 1.10 (Cartan)**.**

$$\Theta(\mathfrak{F}) = d\theta(\mathfrak{F}) - \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F}) \tag{3}$$

Note that $\theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})$ is not zero in general! The reason is that one is working with matrices of 1-forms, not just plain 1-forms. The wedge product is a matrix product in a sense.

Proof. Indeed, we need to determine how ∇^2 acts on the fram e_i. Namely with an abuse of notation:

$$\nabla^2(\mathfrak{F}) = \nabla(\nabla\mathfrak{F}) = \nabla(\theta\mathfrak{F})(\mathfrak{F}) = d\theta(\mathfrak{F})\mathfrak{F} - \theta(\mathfrak{F}) \wedge (\theta(\mathfrak{F})\mathfrak{F})$$

We have used this formula that describes how ∇ acts on a product with a form. As a result the proof holds. □

Finally, we shall need an expression for $d\Theta$. We state this in terms of a local frame.

Proposition 1.11 (Bianchi identity)**.** *With respect to a frame* $\mathfrak{F}$ $d\Theta(\mathfrak{F}) = [\theta(\mathfrak{F}), \Theta(\mathfrak{F})]$

Here the right side consists of matrices, so we talk about the commutator. We shall use this identity at a crucial point in showing that the Chern-Weil homomorphism is even well-defined.

Proof. This is a simple condition. For, by Cartan's equations,

$$d\Theta(\mathfrak{F}) = d(d\theta(\mathfrak{F}) - \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})$$

$=$

$$-d\theta(\mathfrak{F}) \wedge \theta(\mathfrak{F}) + \theta(\mathfrak{F}) \wedge d\theta(\mathfrak{F})$$

Similarly,

$$[\theta(\mathfrak{F}), \Theta(\mathfrak{F})] = [\theta(\mathfrak{F}), d\theta(\mathfrak{F}) + \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})]$$

$[\theta(\mathfrak{F}), d\theta(\mathfrak{F})]$ because $[\theta(\mathfrak{F}), \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})] = 0$ $\square$

2 Questions for Fiber Bundles and Connections

Idea 1. The idea behind these questions is to elaborate on some of the technicalities in the course. They are based upon Questions from the Part III of the Cambridge Mathematical Tripos 2010. And the questions were written by Professor A.G. Kovalev - Answered provided by Peadar Coyle

Showthat every (real) vector bundle can be given a positive definite inner product, varying smoothly in the fibers, i.e given a local trivialization (U_α, Φ_α) by a smooth math $g_\alpha : x \in U_\alpha \to g_\alpha(x) \in Sym_+(k, \mathbb{R})$ denotes the set of all real-positive definite k x k symmetric matrices. [Hint: you might like to use a partion of unity] Deduce that any vector bundle admits an O(n) - structure. Discuss geometric interpretation of the associated principal O(n)-bundle. Are there analogous results for complex vector bundles?

Proof. A: We know that every vector space can be given an inner product. Let us recall some facts

Recall 1. If V is a vector space over $\mathbb{R}$, a positive-definite inner product on V is a symmetric bilinear map

$$<\cdot,\cdot> : V \times V \to \mathbb{R}, (v, w) \to < v, w >$$

such that $< v, v >> 0 \ \forall v \in V - 0$ If $<\cdot,\cdot>$ and $<\cdot,\cdot>'$ are positive definite inner products on V and $a, a' \in \mathbb{R}^0$ are both non zero then

$$a<\cdot,\cdot> + a'<\cdot,\cdot>' : V \times V \to \mathbb{R},$$

$$a<\cdot,\cdot> + a'<\cdot,\cdot>'(v, w) = a < v, w > +a' < v, w >'$$

is also a positive-definite inner product.

If W is a subspace of V and $<\cdot,\cdot>$is a positive definite inner product on V, let

$$W^\perp = v \in V :< v, w >= 0 \forall w \in W$$

be the orthogonal complement of W in V. In particular

$$V = W \oplus W^\perp$$

Furthermore, the quotient project map

$$\pi : V \to V/W$$

induces an isomorphism from $W^{\perp} \to V/W$ so that

$$V \cong W \oplus (V/W)$$

If M is a smooth manifold and $V \to M$ is a smooth real vector bundle of rank k, a ***Riemannian metric*** on V is a positive-definite inner product in each fiber $V_x \cong \mathbb{R}^k$ of V that varies smoothly with $x \in M$. The smoothness requirement is one of the following equivalent conditions:

- The map $<\cdot,\cdot>$: $V \times_M V \to \mathbb{R}$ is smooth;
- the section $<\cdot,\cdot>$of the vector bundle $(V \otimes V^*) \to M$ is smooth;
- if s_1 and s_2 are smooth sections of the vector bundle $V \to M$, then the map $< s_1, s_2 >$: $M \to \mathbb{R}, m \to < s_1(m), s_2(m) >$, is smooth;
- if h: $V|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{R}^k$ is a trivialization of tV, the the matrix valued function

  $$B : \mathcal{U} \to Sym_k(\mathbb{R}) s.t. < h^-1(m, v), h^-1(m, w) >= v^t B(m) w$$

  $\forall m \in \mathcal{U}, v, w \in \mathbb{R}^k$, is smooth.

Every real vector bundle admits a Riemannian metric (by a theorem). Such a metric can be constructed by covering M by a locally finite collection of trivializations for V and patching together positive definite inner-products on each trivialization with a partition of unity. If W is a subspace of V and $<\cdot,\cdot>$ is a Riemannian metric on V, let

$$W^{\perp} = v \in V :< v, w >= 0 \forall w \in W$$

be the orthogonal complement of W in V. In particular

$$V = W \oplus W^{\perp}$$

Furthermore, the quotient project map

$$\pi : V \to V/W$$

induces a vector bundle isomorphism from $W^{\perp} \to V/W$ so that

$$V \cong W \oplus (V/W)$$

So what about the complex case? If V is a vector space over $\mathbb{C}$, a nondegenerate Hermitian inner product on V is a map

$$<\cdot,\cdot>: V \times V \to \mathbb{C}, (v,w) \to < v, w >$$

which is $\mathbb{C}$-antilinear in the first imput, $\mathbb{C}$linear in the second input $< w, v >= \overline{< v, w >}$ and $< v, v >> 0$ $\forall v \in V - 0$ If $<\cdot,\cdot>$ and $<\cdot,\cdot>'$ are nondegenerate Hermitian inner products on V and $a, a' \in \mathbb{R}^0$ are both non zero then

$$a<\cdot,\cdot> + a'<\cdot,\cdot>' : V \times V \to \mathbb{R},$$

$$a<\cdot,\cdot> + a'<\cdot,\cdot>'(v,w) = a < v, w > +a' < v, w >'$$

is also a nondegenerate Hermitian inner product on V. If W is a complex subspace of V and $<\cdot,\cdot>$is a nondegenerate Hermitian inner product on V, let

$$W^{\perp} = v \in V :< v, w >= 0 \forall w \in W$$

be the orthogonal complement of W in V. In particular

$$V = W \oplus W^{\perp}$$

Furthermore, the quotient project map

$$\pi : V \to V/W$$

induces an isomorphism from $W^{\perp} \to V/W$ so that

$$V \cong W \oplus (V/W)$$

If M is a smooth manifold and $V \to M$ is a smooth complex vector bundle of rank k, a ***Hermitian metric*** on V is a positive-definite inner product in each fiber $V_x \cong \mathbb{C}^k$ of V that varies smoothly with $x \in M$. The smoothness requirement is one of the following equivalent conditions:

- The map $<\cdot,\cdot>$: $V \times_M V \to \mathbb{C}$ is smooth;
- the section $<\cdot,\cdot>$of the vector bundle $(V \otimes V^*) \to M$ is smooth;
- if s_1 and s_2 are smooth sections of the vector bundle $V \to M$, then the map $< s_1, s_2 >$ on M is smooth;
- if h: $V|_{\mathcal{U}} \to \mathcal{U} \times \mathbb{C}^k$ is a trivialization of tV, the the matrix valued function

  $$B : \mathcal{U} \to Sym_k(\mathbb{C}) s.t. < h^-1(m, v), h^-1(m, w) >= \overline{v}^t B(m) w$$

  $\forall m \in \mathcal{U}, v, w \in \mathbb{C}^k$, is smooth.

Similarly to the real case, every complex vector bundle admits a Hermitian metric. If W is a subspace of V and $<\cdot,\cdot>$is a Hermitian inner product on V, let

$$W^{\perp} = v \in V :< v, w >= 0 \forall w \in W$$

be the orthogonal complement of W in V. In particular

$$V = W \oplus W^{\perp}$$

Furthermore, the quotient project map

$$\pi : V \to V/W$$

induces an isomorphism of complex vector bundles over M so that

$$V \cong W \oplus (V/W)$$

□

It follows from the definition of a vector bundle E that one can define over the intersection of two trivializing neighbourhoods U_β, U_α a compositye map

$$\Phi_\beta \circ \Phi_\alpha^{-1}(b, v) = (b, \phi_{\beta\alpha}(b)v)$$

where $(b, v) \in (U_\beta, U_\alpha \times \mathbb{R}^k$ We know from a definition (in any textbook that contains differential geometry and connections) that there is an equivalence between a well-defined positive-definite inner product on the fibers E_p and the existence of an O(n)-structure on a rank n vector bundle. We can say that the system oflocal trivializations naturally define a G-structure on vector bundle E. All vector bundles admit such a structure.

Remark 2.1. We can make a remark about the geometric structure of the vector bundle. We can define a vector bundle with inner product by modifying the formal definition given in the lecture notes, we replace vector space by 'inner product space' and isomorphism by 'isometry' (scalar product preserving). This will by definition force all the transition functions to take values in O(n).

We can play the same game with rank k complex vector bundles, we know that these have Hermitian inner products 'varying smoothly with the fibre' $\iff$ there is a U(k)-structure on this vector bundle. These are equivalent by definition. So there is an analogous structure on a complex vector bundle which is a unitary structure, (unitary is complex orthogonal) - this is a good metaphor to have in mind.

3 The Laplace Operator

Idea 2. The aim is to introduce some notions associated with the Laplace Operator. And to enhance some of the remarks made in the PDE II problem class. Particularly on Riemannian Manifolds and volume forms. A good reference is [2], or any Differential Geometry book/ a good PDE book. The idea of the 'Hodge Star Operator' is introduced and defined.

Let V be a real vector space with scalar product $<\cdot,\cdot>$,and let $\bigwedge^p V$ be the p-fold exterior product of V. We then obtain a scalar product on $\bigwedge^p V$ by

$$< v_1 \wedge \cdots \wedge v_p, \omega_1 \wedge \cdots \wedge \omega_p >= det(< v_i, \omega_j >) \tag{4}$$

and bilinear extensions to $\bigwedge^p V$ if $e_1, \cdots, e_d$ is an orthonormal basis of V,

$$e_{i_1} \wedge \cdots \wedge_{i_p} with 1 \leq i_1 < i_2 \cdots < i_p \leq d \tag{5}$$

constitutes an orthonormal basis of $\bigwedge^p V$.

Orientation We've spoken of 'orientation' in some examples in PDE II so let us define what an ***orientation*** is. An orientation on V is obtained by distinguishing a basis of V as positive. Any other basis that is obtained from this basis by a base change with positive determinant is likewise called positive, and the remaining bases are called negative. Let now V carry and orientation. We define the linear star operator (or Hodge star operator[1])

$$* : \bigwedge^{p}(V) \to \bigwedge^{d-p}(V)$$

$(0 \leq p \leq d)$ by

$$*\,(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{j_1} \wedge \cdots \wedge e_{j_{d-p}} \tag{6}$$

where $j_1, \cdots, j_{d-p}$ is selected such that $e_{i_1}, \cdots, e_{i_p}, e_{j_1}, \cdots, e_{j_{d-p}}$ is a positive basis of V. Since the star operator is supposed to be linear it is determined by its values on some basis (6). In particular

$$*\,(1) = e_{i_1} \wedge \cdots \wedge e_{i_p}, \tag{7}$$

$$*\,(e_{i_1} \wedge \cdots \wedge e_{i_p}) = 1, \tag{8}$$

if $e_1, \cdots, e_d$ is a positive basis. From the rules of multilinear algebra, it easily follows that if A is a d x d matrix, and if $f_1, \cdots, f_d \in V$, then

$$*(Af_1 \wedge \cdots \wedge Af_p = (detA) * (f_1 \wedge \cdots \wedge f_p)$$

[1]After the late great British Mathematician and Analyst

In particular, this implies that the star operator does not depend on the choise of positive orthonormal basis (O.N.B) in V, as any two such bases are related by a linear transformation with determinant 1. For a negative basis instead of a positive one, one gets a minus sign on the r.h.s of (6) (7) (8)

Lemma 3.1.

$$** = (-1)^{p(d-p)} : \bigwedge^p(V) \to \bigwedge^p(V)$$

Proof. $**$ maps $\bigwedge^p(V)$ onto itself. Suppose

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{j_1} \wedge \cdots \wedge e_{j_{d-p}}$$

(c.f (6))

Then

$$* * (e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{i_1} \wedge \cdots \wedge e_{i_p}$$

depending on positive or negative basis of V. The proof follows as $(-1)^{p(d-p)}$ is the determinant of the basis change from $e_{j_1} \wedge \cdots \wedge e_{j_{d-p}}$ to $e_{i_1} \wedge \cdots \wedge e_{i_p}$ □

Lemma 3.2. *For v,w* $\in \bigwedge^p(V)$

$$< v, w >= *(w \wedge *v) = *(v \wedge *w) \tag{9}$$

Proof. It suffices to prove (9) for elements of the basis (5). For any two different of these base vectors

$$v \wedge *w = 0,$$

whereas $*(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \cdots \wedge e_{i_p})) = *(e_1 \wedge \cdots \wedge e_d)$, where $e_1, \cdots, e_d$ is an O.N.B ((6)) and this $=$ 1 (8) □

The claim clear follows.

Lemma 3.3. *Let* $v_1, \cdots v_d$ *be an arbitrary positive basis of V. Then*

$$* (1) = \frac{1}{\sqrt{det(< v_i, v_j >)}} v_1 \wedge \cdots \wedge v_d \tag{10}$$

Proof. Let $e_1, \cdots, e_d$ be a positive O.N.B as before. Then $v_1 \wedge \cdots \wedge v_d = (det(< v_i, v_j >))^{1/2} e_1 \wedge \cdots \wedge e_d$ and the claim follows from (7) □

Let now M be an oriented Riemannian manifold of dimension d. Since M is oriented, we may select an orientation of all tangent spaces $T_x M$, hence also on all cotangent spaces $T_x^* M$ in a consistent manner. We simply choose the Euclidean O.N.B $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}$ of $\mathbb{R}^d$ as being positive. Since all chart transitions of an oritented manifold have positive functional determinant, our basis will not depend on the choice of charts. So we have a basis for the tangent space,and we have a Riemannian structure, so we have a scalar product on each T_x^*M. We thus obtain a star operator (which preserves the base points)

$$* : \bigwedge^p (T_x^*M) \to \bigwedge^{d-p} (T_x^*M).$$

We recall that the metric on $T_x^*M$is given by $(g^{ij}(x)) = (g_{ij}(x))^{-1}$. Therefore by (6) we have in local coordinates

$$*\,(1) = \sqrt{g_{ij}}dx^1 \wedge \cdots \wedge dx^d \tag{11}$$

This expression is called the ***volume form***[2]. In particular we get this nice formula (provided the integral is finite)

$$vol(M) := \int_M *(1) \tag{12}$$

[2]This was mentioned in a PDE problem class

4 Yang Mills

The majority of this section is inspired by [1]. To define the Yang-Mills Lagrangian, we need to define the 'Trace' of an End(E) valued form. Recall that the Trace of a maTrix is the sum of its diagonal enTries. The Trace is independent of the choice of basis - an invariant notion that is independent of the choice of basis. A definition of the Trace that mkes this clear is as follows. Consider $End(V) \simeq V \otimes V^*$ - an isomorphism that does not depend on any choice of basis - so the pairing between V and V^* defines a linear map

$$Tr : End(V) \Rightarrow \mathbb{R}$$

$$v \otimes f \mapsto f(v)$$

To see that this v is really a Trace, pick e_i of V and let ϵ^j be a dual basis of V^*. Writing $T \in End(V)$ as

$$T = T^i_j e_i \otimes \epsilon^j$$

We have

$$Tr(T) = T^i_j e_i(\epsilon^j) = T^i_j \delta^j_i = T^i_i$$

which is of course the sum of the diagonal enTries.
This implies that if we have a section T of $End(E)$, we can define a funciton Tr(T) on the base manifold M whose value at $p \in M$ is the Trace of the endomorphism T(p) of the fiber E_p:

$$Tr(T)(p) = Tr(T(p))$$

If $T \in \Gamma(End(E))$ and $\omega \in \Omega^p(M)$ we define

$$Tr(T \otimes \omega) = Tr(T)\omega$$

Now we can write down the **Yang-Mills Lagrangian**: If D is a connection on E, this is the n-form given by

$$\mathcal{L}_{YM} = \frac{1}{2} Tr(F \wedge *F) \tag{13}$$

where F is the curvature of D. Note that by the defintion of the hodge star operator (also in this collection of notes), we can write this in local co-ordinates as

$$\mathcal{L}_{YM} = \frac{1}{4} Tr(F_{\mu\nu} F^{\mu\nu}) vol \tag{14}$$

If we integrate $\mathcal{L}_{YM}$ over M we get the **Yang-Mills action**

$$S_Y M = \frac{1}{2} \int_M Tr(F \wedge *F) \tag{15}$$

This needs some elaboration. So let us explain these formulas better. We choose the physics convention $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ where the generators of the Lie algebra are Hermitian.

$$\mathcal{L}_{YM} = \mathcal{I} = -\int Tr(F \wedge *F) \tag{16}$$

by another convention (there is a lot of ambiguity of signs in this subject). The first thing to note is that F has vector **and** Lie algebra indices. The Trace is over the Lie algebra, **not** over the vector indices. The vector indices are just those of the field sTrength in QED. In Yang-Mills the curvature form is Lie Algebra valued.
In this case $F_{\mu\nu} = F^a_{\mu\nu} T^a$ where the summation convention is used,and where T^a are the generators of $\mathfrak{su}(n)$. To be explicit, F has not only tensor components but maTrix components

$$(F_{\mu\nu})_{ij} = F^a_{\mu\nu} T^a T^a_{ij}$$

The inner product of F with itself $< F, F >= \int F \wedge *F$ where $*$is the Hodge $*$- operator. Thus we are calculating $\mathcal{I} = -Tr < F, F >$. It is a standard exercise to find the exterior product of two r-forms. We find

$$F \wedge *F = \frac{1}{2!} F_{\mu\nu} F^{\mu\nu} dx^1 \wedge \cdots \wedge dx^4$$

Note that the differential forms don't 'know' the Lie algebra. The algebra hasn't come into the calcuation yet. $Tr(F_{\mu\nu}F^{\mu\nu}) = Tr(F^a_{\mu\nu} T^a F^{\mu\nu b} T^b$ $= Tr(T^a T^b) F^a_{\mu\nu} F^{\mu\nu b} = \frac{1}{2}\delta^{ab} F^a_{\mu\nu} F^{\mu\nu b} = \frac{1}{2} F^a_{\mu\nu} F^{\mu\nu a}$ where e have used the standard normalization convention for the T^a, $Tr T^a T^b = \frac{1}{2}\delta^{ab}$. (This comes from the fact we want $\mathfrak{su}(2)$ to live in $\mathfrak{su}(n)$, and the generators of $\mathfrak{su}(2)$ are taken to be $T^a = \frac{\sigma^a}{2}$ where σ^a are the Pauli matrices.) Thus, we find

$$\mathcal{I} = -Tr(F \wedge *F)$$

which can be written as

$$\frac{1}{4} \int d^4x F^a_{\mu\nu} F^{a\mu\nu}$$

References

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