Some Remarks on the Fubini-Study Metric

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November 7, 2012

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1 A Little Complex Analysis

We want to introduce the notion of a 'Fubini-Study' metric which is important in Complex Manifold Theory and Differential Geometry (and the associated theories such as Mathematical Physics). But first we need to introduce a little Complex Analysis. The source is of course Griffiths and Harris. Let M be a complex manifold, $p \in M$ any point, and $z = (z_1, \cdots, z_n)$ a holomorphic co-ordinate system around p. There are three different notions of a tangent space to M at p,which we now describe:

- $T_{\mathbb{R},p}(M)$ is the usual **real tangent space** to M at p,when we consider M a real manifold of dimension 2n. $T_{\mathbb{R},p}(M)$ can be realized as the space of $\mathbb{R}-$linear derivations on the ring of real-valued C^∞-functions in a neighbourhood of p; if we write $z_i = x_i + iy_i$, $T_{\mathbb{R},p}(M) = \mathbb{R}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$.

- $T_{\mathbb{C},p}(M) = T_{\mathbb{R},p}(M) \otimes_{\mathbb{R}} \mathbb{C}$ is called the **complexified tangent space** to M at p. It can be realized as the space of $\mathbb{C}-$ linear derivations in the ring of complex valued C^∞-functions on M around p. We can write $T_{\mathbb{C},p}(M) = \mathbb{C}\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$
$=\mathbb{C}\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$

- $T'_p(M) = \mathbb{C}\frac{\partial}{\partial z_i} \subset T_{\mathbb{C},p}(M)$ is called the **holomorphic tangent space** to M at p. It can be realized as the subspace of $T_{\mathbb{C},p}(M)$ consisting of derivations that vanish on antiholomorphic functions (i.e. F such that T is holomorphic), and so is independent of the holomorphic co-ordinate system chosen. The subspace $T''_p(M) = \mathbb{C}\frac{\partial}{\partial \bar{z}_i}$ is called the **antiholomorphic tangent space** to M at p; clearly $T_{\mathbb{C},p}(M) = T'_p(M) \oplus T''_p(M)$

Now we consider some **Calculus on Complex Manifolds**. Let M be a complex manifold of dimension n. A **hermitian metric** on M is given by a positive definite hermitian inner product $(,)_z : T'_z(M) \otimes T'_z(M) \Rightarrow \mathbb{C}$ on the holomorphic tangent space at z for each $z \in M$,
depending smoothly on z - that is, such that for local co-ordinates z on M the function $h_ij(z) = (\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})_z$ are C^∞ Writing $(,)_z$ in terms of the basis $dz_i \otimes d\bar{z}_j$ for $(T'_z(M) \otimes T'_z(\bar{M})^* = T^{*'}_z(M) \otimes T^{*\text{J}}_z(M)$, the hermitian metric is given by $ds^2 = \sum_{i,j} h_{ij}(z) dz_i \otimes d\bar{z}_j$ So let us describe the **Fubini-Study Metric** Let $z_0, \cdots, z_n$ be co-ordinates on $\mathbb{C}^{n+1}$ and denote by $\pi : \mathbb{C}^{n+1} -$

$0 \Rightarrow \mathbb{P}^n$ the standard projection map. Let $U \subset \mathbb{P}^n$ be an open set and $Z : U \Rightarrow \mathbb{C}^{n-1} - 0$ a lifting of U, i.e. a holomorphic map with $\pi \circ z = id$; consider the differential form

$\omega = \dfrac{i}{2\pi}\partial\bar{\partial}log\|z\|^2$ If $Z' : U \Rightarrow \mathbb{C}^{n-1} - 0$ is another lifting, then $Z' = f.Z$ with f a nonzero holomorphic function, so that

$\dfrac{i}{2\pi}\partial\bar{\partial}log\|z\|^2 = \frac{i}{2\pi}\partial\bar{\partial}(log\|z\|^2 + logf + log\tilde{f})$

$= \omega + \dfrac{i}{2\pi}(\partial\bar{\partial}logf - \bar{\partial}\partial log\tilde{f})$

$= \omega$ Therefore $\omega$is independent of the lifting chosen; since liftings always exist locally, ω is a globally defined differential form in $\mathbb{P}^n$. (By the sheaf properties of differential forms) Clearly ω is of type (1,1). To see that ω is positive, first note that the unitary group $U(n+1)$ acts transitively on $\mathbb{P}^n$ and leaves the form ω positive everywhere if it is positive at one point. Now let $w_i = z_i/z_0$ be co-oridnates on the open set $U_0 = (z_0 \neq 0)$in $\mathbb{P}^n$ and use the lifting $Z = (1, w_1, \cdots, w_n)$ on U_0 ; we have (after some substitutions

$\omega = \dfrac{i}{2\pi}[\frac{\sum dw_i \wedge d\bar{w}_i}{1+\sum w_i\bar{w}_i} - \frac{(\sum \bar{w}_i dw_i \wedge \sum w_i d\bar{w}_i)}{(1+\sum w_i\bar{w}_i)^2}]$ At the point $[1, 0, \cdots, 0]$,

$\omega = \frac{i}{2\pi}\sum dw_i \wedge d\bar{w}_i > 0$ Thus ω defines a particular hermitian metric on the projective complex space called the **Fubini-Study metric**. That was the aim of the article!