Differential Geometry and Information Geometry

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Abstract

1   Connections

Firstly, let us solidify and recall some notation The set of all vector fields $\Gamma(TM)$ takes on the structure of a module over the commutative algebra of smooth functions on M, denoted $C^\infty(M)$.

Definition An affine connection on M is the operation ∇ which assigns to every vector field $\mathbf{X}$ a linear map but not necessarily a $C^\infty(M)$- linear map[1] $\nabla_{\mathbf{X}}$ on the space $\mathcal{O}(M)$ of vector fields - we can say that the is a module of vector fields on the sheaf of rings.

$$\nabla_{\mathbf{X}}(\lambda\mathbf{Y} + \mu\mathbf{Z}) \tag{1}$$
$$= \lambda\nabla_{\mathbf{X}}\mathbf{Y} + \mu\nabla_{\mathbf{X}}\mathbf{Z} \tag{2}$$

for every $\lambda, \mu \in \mathbb{R}$ such that the following conditions are satisfied

- for arbitrary (smooth) functions f,g on M

$$\nabla_{f\mathbf{X}+g\mathbf{Y}}(\mathbf{Z}) = f\nabla_{\mathbf{X}}(\mathbf{Z}) + g\nabla_{\mathbf{Y}}(\mathbf{Z}) \tag{3}$$

  this is called C^∞-linearity

- for arbitrary function f

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = (\nabla_{\mathbf{X}}f)\mathbf{Y} + f(\nabla_{\mathbf{X}}(\mathbf{Y}) \tag{4}$$

  recall that $\nabla_{\mathbf{X}}f$ is just the usual directional derivative of a function f along a vector field so we'll say $\nabla_{\mathbf{X}} = df$

Let us write down explicit formulae in a given local co-ordinates $\{x_i\}$(i=1,$\cdots$,n) on a manifold M. Let $\mathbf{X} = X^i\mathbf{e}_i = X^i\dfrac{\partial}{\partial x^i}$ $\mathbf{Y} = Y^i\mathbf{e}_i = Y^i\dfrac{\partial}{\partial x^i}$ So lets write

$$\nabla_{\mathbf{X}}\mathbf{Y} = \nabla_{X^i\partial_i}Y^k\partial_k = X^i\left(\nabla_i\left(Y^k\partial_k\right)\right) \tag{5}$$

where ∂_i is merely the base vector field, and $\nabla_i = \nabla_{\partial_i}$, the X^i comes up after the second equal sign in (5) because it is merely a function. According to (3)

$$\nabla_i(Y^k\partial_k) = \nabla_i(Y^k)\partial_k + Y^k(\nabla_i\partial_k) \tag{6}$$

Let us now decompose the vector field $\nabla_i\partial_k$ over the basis ∂_i:

$$\nabla_i\partial_k = \Gamma_{ik}^m\partial_m \tag{7}$$

(take the two subscripts on the l.h.s and put them at the bottom on the Christoffel symbol (Γ) and include a dummy index) and

$$\nabla_i(Y^k\partial_k) = \frac{\partial Y^k(x)}{\partial x^i}\partial_k + Y^k\Gamma_{im}^k\partial_m \tag{8}$$

And to return to where we began

$$\nabla_{\mathbf{X}}(\mathbf{Y}) = X^i\frac{\partial Y^m}{\partial x^i}\partial_m + X^iY^k\Gamma_{im}^k\partial_m \tag{9}$$

The coefficients $\left\{\Gamma_{im}^k\right\}$ are called *Christoffel symbols* in coordinates $\{x_i\}$. We can say that these coefficients define a covariant derivative or a **connection**

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[1] There is a map which is linear over the reals but not necessarily over the ring of functions

1.1 Transformation of Christoffel symbols for an arbitrary connection

Let ∇ be a connection on a manifold M. Let Γ_{ik}^m be Christoffel symbols of this connection given in local coordinates $\{x_i\}$. Then according to (7) and (8) we have

$$\nabla_{\mathbf{X}}\mathbf{Y} = X^m \frac{\partial Y^i}{\partial x^m}\frac{\partial}{\partial x^i} + X^m \Gamma_{im}^k Y^k \frac{\partial}{\partial x^i} \tag{10}$$

and in particular $\Gamma_{im}^k \partial_i = \nabla_{\partial_m}\partial_k$ Let us use this relation to calculate Christoffel symbols in a new coordinates $\left\{x^{i'}\right\}$

$$\Gamma_{m'k'}^{i'}\partial_{i'} = \nabla_{\partial_{m'}}\partial_{k'}$$

We have that $\partial_{m'} = \frac{\partial}{\partial x^{m'}} = \frac{\partial x^m}{\partial x^{m'}}\frac{\partial}{\partial x^m} = \frac{\partial x^m}{\partial x^{m'}}\partial_m$ Hence due to properties (3) and (4) we have

$$\Gamma_{m'k'}^{i'}\partial_{i'} = \nabla_{\partial_{m'}}\partial_{k'} = \nabla_{\partial_{m'}}\left(\frac{\partial x^k}{\partial x^{k'}}\partial_k\right) = \left(\frac{\partial x^k}{\partial x^{k'}}\right)\nabla_{\partial_{m'}}\partial_k + \frac{\partial}{\partial x^{m'}}\left(\frac{\partial x^k}{\partial x^{k'}}\right)\partial_k$$

$$= \left(\frac{\partial x^k}{\partial x^{k'}}\right)_{\nabla_{\frac{\partial x^m}{\partial x^{m'}}\partial_m}}\partial_k + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma_{mk}^i\partial_i + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\partial_k$$

$$= \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\Gamma_{mk}^i\frac{\partial x^{i'}}{\partial x^i}\partial_{i'} + \frac{\partial^2 x^k}{\partial x^{m'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^k}\partial_{i'}$$

We get the transformation law: If $\left\{\Gamma_{km}^i\right\}$ are Christoffel symbols fo the connection ∇ in local coordinates $\left\{x^i\right\}$ and $\left\{\Gamma_{k'm'}^{i'}\right\}$ are Christoffel syombols of this connection in new local coordinates $\left\{x^{i'}\right\}$ then

$$\Gamma_{k'm'}^{i'} = \frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^m}{\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^i}\Gamma_{km}^i + \frac{\partial^2 x^r}{\partial x^{k'}\partial x^{m'}}\frac{\partial x^{i'}}{\partial x^r} \tag{11}$$

Remark Christoffel symbols do not transform as a tensor. If the second term is euqal to zero, i.e. transformation of coordinates are linear then the transformation rule above is the same as a transformation rule for tensors of the type $\binom{1}{2}$. In general this ins note true. Christoffel symbols do not transform as a tensor under arbitrary non-linear coordinate transformations.

2 What is a Geodesic

Let us recall the definition of a geodesic. A **geodesic** on a smooth manifold M with an **affine connection** ∇ is defined as a curve $\gamma(t)$ such that **parallel transport** along the curve preserves the tangent vector to the curve, so

$$\nabla_{\dot\gamma}\dot\gamma = 0 \tag{12}$$

at each point along the curve, where $\dot\gamma$ is the derivative w.r.t. t. More precisely, in order to define the covariant derivative of $\dot\gamma$ to a continuously differentiable

vector field on an open set. However, the resulting value of 12 is independent of the choice of extension. We can however write this in local form,

$$\frac{d^2 x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0,$$

where $x^\mu(t)$ are the coordinates of the curve $\gamma(t)$ and $\Gamma^\lambda_{\mu\nu}$ are the Christoffel symbols of the connection ∇. This is just an ODE for the coordinates. It has a unique solution, given an initial position and an initial velocity. Therefore, fromt he point of view of classical mechainics, geodesics can be thought of as trajectories of **free particles** in a manifold. Indeed, the equation $\nabla_{\dot\gamma} \dot\gamma = 0$ means that the acceleration of the curve has no components in the direction of the surface (the direction of the surface is the same as the direction of the affine connection). We can also say that the acceleration of the curve is perpendicular to the tangent plane of the surface at each point of the curve. So the motion is completely determined by the bending of the surface. This also the idea of General Relativity where particles move on geodesics and the bending is caused by gravity.

3   Bundles and Connections

Let us review some of the fundamental and classical notions of Principal Bundles and Connections. Let G be a Lie group. A principal G-bundle P over a smooth manifold X is a manifold with a (smooth) right G action and orbit space $P/G = X$ We demand that the action admits local product structures, i.e. it is locally equivalent to the product space U x G, where U is an open set in X. Then we have a fibration $\pi : P \to X$. We say that P has structure group G. Three useful ways of defining a connection on such a bundle are:

1. As a field of 'horizontal subspaces' $H \subset TP$ transverse to the fibers of π That is, for each $p \in P$ we have the decomposition

   $$TP_p = H_p \oplus T(\pi^{-1}(x))$$

   where $\pi(p) = x$. The field of subspaces is required to be preserved by the action of G on P.

2. As a 1-form A on P with values in the Lie algebra $\mathfrak{g}$ of G i.e. a section of the vector bundle $T^*P \otimes \mathfrak{g}$ over P. Again we require this to be invariant under G, acting by a combination of the given action on P and the adjoing action on $\mathfrak{g}$. Also, A should be restricted to the canonical right-invariant form on the fibres.

3. For any linear representation of G, on $\mathbb{C}^n$ or $\mathbb{R}^n$, we get a vector bundle E over X associated with P and the representation. The fibres of E are copies of $\mathbb{C}^n$ or $\mathbb{R}^n$ respectively. Conversely, if given a vector bundle E we can make a principal bundle. For example, if E is a complex n-plane bundle we get a principal bundle P with the structure group $GL(n, \mathbb{C})$ by taking the set of all 'frames' in E. A point in the fibre of P over x $\in$ X is a set of basis vectors for E_x. Additional algebraic structure on E yields a principal bundle with a smaller structure group. For example is E is a complex vector bundle with a Hermitian metric we get a principal bundle U(n) bundle of orthonormal frams.e

For classical groups, (automorphisms of a vector space preserving some linear algebraic structure) the concepts of principal and vector bundles are completely equivalent.

Now given a vector bundle E (real or complex) a connection on the frame bundle can be defined by a **covariant derivative** on E, that is a linear map:

$$\nabla : \Omega^0_X(E) \to \Omega^1_X(E) \tag{13}$$

Here we introduce the notation $\Omega^p_X(E)$ to denote sections of $\bigwedge^p T^*X \otimes E$ - the p-forms with values in E. The map ∇ is required to satisfy the Leibniz rule $\nabla(f.s) = df.s + f\nabla s$ for any f a real or complex function on X and $s \in \Gamma(E)$, where $\Gamma(E)$ denotes sections on E. To understand this definition, consider a tangent vector v to X at a point x. If s is a section of E we can make the contraction $\langle \nabla s, v \rangle \in E_x$ which is to be thought of as the derivative of s in the direction v at x. If E has additional algebraic structure we can require the covariant derivative to be compatible with this; for example if E has a metric, a compatible connection is one for which $\nabla(s,t) = (\nabla s, t) + (s, \nabla t)$, for all sections s,t. One see the equivalence of these viewpoints as follows to go from (2) to (1) we regard H_p (the horizontal space) to be the kernel of $A_p : TP_p \to \mathfrak{g}$. To go from (3) to (1) we first observe that ∇ is a local operator; that is, if two sections s_1, s_2 agree on an open set U in X, so also do ∇s_1 ∇s_2. This follows from Leibnitz rule by considering ϕs_i, where ϕ is a cutoff function. Then we say that a local section δ of the frame bundle (i.e. a collection of sections $s_1, \cdots, s_n$ of E) is horizontal at x in X if all the ∇s_i vanish at x. Finally we define H_p to be the tangent space to a horizontal section δ through p, regarded as a submanifold of P.