Riemann Surfaces

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Contents

1 Answers for Riemann Surfaces

Example 1.1. Proof of the $\mathbb{C}$-algebra structure of meromorphic functions

Proof. $\mathcal{M}(Y)$ has the natural structure of a $\mathbb{C}$-algebra. First of all the sm and the product of two meromorphic functions $f, g \in \mathcal{M}(Y)$ are holomorphic functions at those points where both f and g are holomorphic. Then one holomorphically extends, using Riemann's Removable Singularities Theorem, f+g (resp. fg) across any singularities which are removable. □

Example 1.2. Exercise 6 2) Let Γ be a lattice in C. Can you describe all holomorphic functions on the torus

$$\frac{\mathbb{C}}{\Gamma}$$

using a similar reasoning as in part 1) of this exercise?

Proof. Any holomorphic function on complex tori is constant, as it is a bounded holomorphic function on the universal covering space, and by Liouvilles theorem this is constant (and by the Maximum moduli theorem) □

Example 1.3. Exercise 9 Let us consider a point $p \in P$, where P is the set of poles. We've defined $\hat{f}(p) = f(p)$ if p $\notin$P

Alternatively the function $\hat{f}$ is infinity. But we know that p is definitely in the set of poles, and we know that $\hat{f}$ is mapping from the domain ∞ to the codomain ∞. On the Riemann sphere, by definition, this is therefore a continuous map.

Example 1.4. Exercise 12 1) Let

$$f : X \to Y$$

be a non-constant morphism of Riemann surfaces. Prove that if X is compact, then f is surjective. In particular, Y is compact as well in this case. 2) Use the first part of this exercise to conclude that $\mathcal{O}_X(X) = \mathbb{C}$ for every compact Riemann surface X, i. e., the only holomorphic functions on a compact Riemann surface are constants.

Let us introduce a corollary

Corollary 1.5. *Let $f : M \to N$ be a non-constant holomorphic mapping, then f is open, i.e. an image of any open set is open.*

Corollary 1.6. *Let $f : M \to N$ be a non-constant holomorphic mapping and M compact. Then f is surjective f(M) = N and N is also compact.*

Proof. The previous corollary implies that f(M) is open. On the other hand, f(M) is compact since it is a continuous image of a compact set. f(M) is open, closed and non-empty, therefore f(M) = N and N compact. □

For question 2, it is simply a case of Liouvilles theorem.

Theorem 1.7 (Liouville)**.** *There are no non-constant holomorphic functions on compact Riemann surfaces.*

Proof. An existence of a non-constant holomorphic mapping $f : M \to \mathbb{C}$ contradicts to the previous corollary since $\mathbb{C}$is not compact. □

Example 1.8. Exercise 14

Proposition 1.9. *Any doubly periodic holomorphic function on $\mathbb{C}$is constant.*

Proof. : Global holomorphic functions on $\dfrac{\mathbb{C}}{L}$ are constant. □

Proof. By Liouville's theorem, bounded holomorphic functions on $\mathbb{C}$are constant. □

To get any interesting functions, we must allow poles.

Definition 1.10. : An elliptic function is a doubly periodic meromorphic function on $\mathbb{C}$.

Elliptic functions are thus meromorphic functions on a torus $\dfrac{\mathbb{C}}{L}$. The reason for the name is lost in the dawn of time. (Really, elliptic functions can be used to express the arc-length on the ellipse.)

Example 1.11. Exercise 15 Prove that a Riemann surface X is path-connected.

Proof. Consider a point x_0 and consider the set of all points S which x_0 can be connected to by a path. This set S is an open subset of X, and all open subsets of X can be mapped to an open subset of $\mathbb{C}$which is open, closed and non-empty. □

Example 1.12. Prove that 0 is the only ramification point of the holomorphic map $\mathbb{C} \to \mathbb{C}$, $z \mapsto z^k$, $k \geq 2$