A brief exposition of some applications of Linear Elastodynamics

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Abstract

The theories of Variational Calculus and Tensors are ubiquitious, useful and important in the studies of Mechanics and General Relativity. The aim of this short expository note is to include some examples from Mechanics in the aim of elucidating these areas.

1 Linear elastodynamics

On $\mathbb{R}^3$ consider the equations $\rho \mathbf{u}_{tt} = div(\mathbf{c} \cdot \nabla \mathbf{u})$ that is,

$$\rho u^i_{tt} = \frac{\partial}{\partial x^j}[c^{ijkl}\frac{\partial u^k}{\partial x^l}] \tag{1}$$

where ρ is a positive function, and $\mathbf{c}$ is a fourth-order tensor field (the **elasticity tensor**) on $\mathbb{R}^3$ with the symmetries $c^{ijkl} = c^{klij} = c^{jikl}$. On $\mathcal{F}(\mathbb{R}^3; \mathbb{R}^3) \times \mathcal{F}(\mathbb{R}^3; \mathbb{R}^3)$ (or more precisely on $H^1(\mathbb{R}^3 : \mathbb{R}^3) \times L^2(\mathbb{R}^3; \mathbb{R}^3)$ with suitable decay properties at infinity), define

$$\Omega((\mathbf{u}, \dot{\mathbf{u}}), (\mathbf{v}, \dot{\mathbf{v}}) = \int_{\mathbf{R}^3} \rho(\dot{\mathbf{v}} \cdot \mathbf{u} - \dot{\mathbf{u}} \cdot \mathbf{v}))d^3x \tag{2}$$

The form Ω is the canonical symplectic for $\mathbf{u}$ and their conjugate momenta $\pi = \rho\dot{\mathbf{u}}$ On the space of functions $\mathbf{u}$: $\mathbb{R}^3 \to \mathbb{R}^3$

$$\langle \mathbf{u}, \mathbf{v} \rangle_\rho = \int_{\mathbb{R}^3} \rho \mathbf{u} \cdot \mathbf{v} d^3x \tag{3}$$

There the operator $B\mathbf{u} = -(1/\rho)div(\mathbf{c} \cdot \nabla \mathbf{u})$ is symmetric with respect to this inner product and thus we can say that the operator $\mathrm{A}(\mathbf{u}, \dot{\mathbf{u}}) = (\dot{\mathbf{u}}, (1/\rho)div(\mathbf{c} \cdot \nabla \mathbf{u}))$ is Ω-skew. The equation (1) of linear elastodynamics are checked to be Hamiltonian with respect to Ω given by

$$H(\mathbf{u}, \dot{\mathbf{u}}) = \frac{1}{2}\int \rho \|\dot{\mathbf{u}}\|^2 d^3x + \frac{1}{2}\int c^{ijkl}e_{ij}e_{kl}d^3x \tag{4}$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u^i}{dx^j} + \frac{\partial u^j}{\partial x^i}\right)$$

1.1 Technical Supplement

Remark The symbols $\mathcal{F}$ and Den stand for function spaces included in the space of all functions and densities, chosen appropriate to the functional analysis needs of the particular problem. In practice this often means, among other things, that appropriate conditions at infinity are imposed to permit integration by parts. Also some particular spaces (such as spaces with compact support) are picked because then the integral is finite.

Remark Let $H^1(\mathbb{R}^3)$ denote the H^1 functions on $\mathbb{R}^3$, that is, functions which, along with their first derivatives are square integrable)

Spaces of functions Let $\mathcal{F}(\mathbb{R}^3)$ be the space of smooth functions $\phi : \mathbb{R}^3 \to \mathbb{R}$, and let $Den_C(\mathbb{R}^3)$ be the space of smooth densities on $\mathbb{R}^3$ with compact support. We will write a density $\pi \in Den_C(\mathbb{R}^3)$ as a function $\pi' \in \mathcal{F}(\mathbb{R}^3)$ with compact support times the volume element d^3x on $\mathbb{R}^3$ as $\pi = \pi' d^3x$. The spaces $\mathcal{F}$ and Den_C are in weak nondegenerate duality by the pairing $\langle \phi, \pi \rangle = \int \phi \pi' d^3x$ We can just think of densities as analogues of one-forms and we shall bypass the measure theoretic approach, although it is based on the more primitive physical concept of measuring the masses of portions of a body.