Interest Rate Models

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1 General class of models

Let us consider $(r, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $W_t \left\{W_t^1, \cdots, W_t^d\right\}$, $\mathcal{F}_t = \sigma\left\{W_u : u \leq t\right\}$

Remark 1.1. Where $\mathbb{P}$is what is termed *the historical measure*

- Short rate $\{r(t) : t \geq 0\}$ such that

$$r(t) = r(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dW_s \tag{1}$$

  Were both b and σ are $\mathcal{F}_t$ - progressive measurable processes. And dW_s is a d-dimension *Brownian Motion*(B.M.) We assume that $\int_0^T \left(|b(s)| + \|\|_{\mathbb{R}_d}^2\right) < +\infty$ a.s. $\forall$T

- $\exists \mathbb{Q} \cong \mathbb{P}$ such that $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{\int_0^\infty \gamma(s)dW_s - \dfrac{1}{2}\int_0^\infty \gamma^2(s)ds\right\}$

  Where $\gamma(s)$ is a $\mathcal{F}_t$ - progressive measurable process, and the above integrals are of Girasnov type and satisfy the Novikov condition is a $(\mathcal{F}_t, \mathbb{Q})$-martingale in $[0, T]$ $\forall T < +\infty$

$$\frac{P(t,T)}{S_0(t)} = \frac{P(t,T)}{\exp(\int_0^t r(s)ds} \tag{2}$$

  where $S_0(t)$ is the Money Market Account, and $P(t, T)$ price in t of a zero-coupon bond with maturity T

$\mathbb{Q}$ is an equivalent *Martingale Measure*(M.M) for the model, and this implies the **First Fundamental Theorem of Arbitrage**

Theorem 1.2. *Every restricted market ($P(\cdot, T_1); \cdots ; P(\cdot, T_m), S_0(t)$) m-finite is arbitrage free*

We can derive the fundamental relation $\dfrac{P(t,T)}{S_0(t)} = \mathbb{E}_{\mathbb{Q}}\left\{\dfrac{1}{S_0(T)}|\mathcal{F}_t\right\}$ [1]

$\Rightarrow P(t,T) = \mathbb{E}_{\mathbb{Q}}\left\{e^{-\int_0^T r(s)ds}|\mathcal{F}_t\right\}$

Definition 1.3. The short rate, r_t , is the (continuously compounded, annualized) interest rate at which an entity can borrow money for an infinitesimally short period of time from time t. Specifying the current short rate does not specify the entire yield curve. However arbitrage arguments show that, under some fairly relaxed technical conditions, if we model the evolution of r_t as a stochastic process under a risk-neutral measure $\mathbb{Q}$ then the price at time t of a zero-coupon bond maturing at time T is given by

:$P(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s\,ds\right)\middle|\,\mathcal{F}_t\right]$

where $\mathcal{F}$ is the natural filtration for the process.

[1] $P(T, T) = 1$

1.1 Dynamics of Short Rate Models

We will consider $t \Rightarrow P(t,T)$ (under $\mathbb{P}$, under $\mathbb{Q}$)

1. Due to Girasnov's Theorem $W_t - \int_0^t \gamma(s)ds =: W_t^*$ is a $(\mathbb{Q}, \mathcal{F}_t)$- d-dimensional Brownian Motion. So: $dr_t = r(0) + \int_0^t \sigma(s)dW_s^* + \int_0^t (b_s + \sigma(s)\gamma^T(s))ds$
   Note that $dW_s^* = dW_s - \gamma^T(s)ds$ and looking at $\int_0^t (b_s + \sigma(s)\gamma^T(s))ds$ we see that the drift has changed.

2. For every $T < +\infty$ $\exists \mathcal{F}_t$-progressive process $\{v(t,T) : t \leq T\}$ such that $\dfrac{d\mathbb{P}(t,T)}{P(t,T)} = r(t)dt + v(t,T)dW_t^*$
   Indeed P(t,T) $= e^{\int_0^t r(s)ds} \mathbb{E}_{\mathbb{Q}} \left\{ \dfrac{1}{S_0(T)} | \mathcal{F}_t \right\} = e^{\int_0^t r(s)ds} M_t^{(T)}$ since $M_t^{(T)}$ is a $(\mathbb{Q}, \mathcal{F}_t)$-martingale, $\exists \mathcal{F}_t$ -progressive process $(t \leq T)$ $\{h(t,T) : t \leq T\}$ such that $M_t^{(T)} = M_0^{(T)} + \int_0^t h(s,T)dW_s^*$ where $\{h(t,T) : t \leq T\}$ is a d-dimensional object.
   dℙ(t,T) = d $\left\{ e^{\int_0^T r(s)ds} \times M_t^{(T)} \right\} = M_t^{(T)} d \left\{ e^{\int_0^T r(s)ds} \right\} + e^{\int_0^T r(s)ds} dM_t^{(T)}$

   $= P(t,T)r(t)dt + e^{\int_0^T r(s)ds} h(t,T)dW_t^* = P(t,T)r(t)dt + P(t,T)v(t,T)dW_t^*$

   with $v(t,T) = \dfrac{e^{\int_0^T r(s)ds} h(t,T)}{P(t,T)}$
   As a consequence we can say $P(t,T) = P(0,T) \exp \int_0^t r(s)ds + \int_0^t v(s,T)dW_s^* - \frac{1}{2} \int_0^t v^2(s,T)$ $= P(0,T)S_0(t)Z_t^{(T)}$, where $Z_t^{(T)} := \exp \int_0^t r(s)ds + \int_0^t v(s,T)dW_s^* - \frac{1}{2} \int_0^t v^2(s,T)ds$ can be thought of as a 'pertubation' of a 'time series' and is also an $\mathcal{F}_t$-martingale under $\mathbb{Q}$Also $\dfrac{P(t,T)}{S_0(t)} = P(0,T)Z_t^{(T)} := \tilde{P}(t,T)$ $\dfrac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} = v(t,T)dW_t^*$

3. Under $\mathbb{P}$?
   $$\frac{d\mathbb{P}(t,T)}{\mathbb{P}(t,T)} = r(t)dt + v(t,T)dW_t^* \tag{3}$$
   Note: $dW_t^* = \left\{ dW_t - \gamma^T(t)dt \right\} = \underbrace{\left\{ r(t) - v(t,T)\gamma^T(t) \right\}}_{:=\alpha(t,T)} dt + v(t,T)dW_t$

P(t,T) $= P(0,T) \exp \int_0^t \alpha(t,T)ds + \int_0^t v(s,T)dW_s - \frac{1}{2} \int_0^t v(s,T)^2 ds$

$$\tilde{P}(t,T) = P(0,T) \exp - \int_0^t v(s,T)\gamma^T(s)ds + \int_0^t v(s,T)dW_s - \frac{1}{2} \int_0^t v(s,T)^2 ds$$

Remark 1.4. Recall $-\gamma =$ Market Price of Risk

1.2 Short rate diffusions

Let us consider Short rate diffusions $(\Omega, \mathcal{F}_t, \mathbb{P}); \underbrace{\mathbb{Q}}_{\gamma}, d = 1, W_t = \{W_t : t \geq 0\}$

Assume

$$\forall s \geq 0 \begin{cases} dr_t = b(t, r_t)dt + \sigma(t, r_t)dW_t^* \\ r_0 = \text{constant} \end{cases} \tag{4}$$

with values in Z $\mathbb{R}$ or $\mathbb{R}_+$. Take b, σ to be Lipschitz formulas Before proceeding let us recall what a Lipschitz Function is

Definition 1.5. Given two *metric spaces* (X, d_X) and (Y, d_Y), where d_X denotes the *metric* on the set X and d_Y is the metric on set Y (for example, Y might be the set of *real numbers* $\mathbb{R}$with the metric d_Y(x, y) = |x âĹŠ y|, and X might be a subset of $\mathbb{R}$), a function

$$f : X \to Y$$

is called 'Lipschitz continuous' if there exists a real constant K âĽě 0 such that, for all x_1 and x_2 in X,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

Any such K is referred to as 'a Lipschitz constant' for the function f. The smallest constant is sometimes called **the (best) Lipschitz constant**; however in most cases the latter notion is less relevant. If K = 1 the function is called a *short map*, and if $0 \leq K \leq 1$ the function is called a *contraction mapping*.

The inequality is (trivially) satisfied if $x_1 = x_2$. Otherwise, one can equivalently define a function to be Lipschitz continuous *if and only if* there exists a constant $K \geq 0$ such that, for all $x_1 \neq x_2$,

$$\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \leq K.$$

For real-valued functions of several real variables, this holds if and only if the absolute value of the slopes of all secant lines are bounded by K. The set of lines of slope K passing through a point on the graph of the function forms a circular cone, and a function is Lipschitz if and only if the graph of the function everywhere lies completely outside of this cone.

A function is called 'locally Lipschitz continuous' if for every x in X there exists a *neighborhood* U of x such that f restricted to U is Lipschitz continuous. Equivalently, if X is a *locally compact* metric space, then f is locally Lipschitz if and only if it is Lipschitz continuous on every compact subset of X. In spaces that are not locally compact, this is a necessary but not a sufficient condition.

More generally, a function f defined on X is said to be 'HÃűlder continuous' or to satisfy a *Hoelder condition* of order $\alpha > 0$on X if there exists a constant M $>$ 0 such that

$$d_Y(f(x), f(y)) \leq M d_X(x, y)^\alpha$$

for all x and y in X. Sometimes a HÃűlder condition of order Îś is also called a 'uniform Lipschitz condition of order' $\alpha > 0$.

If there exists a $K \geq 1$ with

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$$

then ÆŠ is called 'bilipschitz' (also written 'bi-Lipschitz'). A bilipschitz mapping is *injective function*, and is in fact a *homeomorphism* onto its image. A bilipschitz function is the same thing as an injective Lipschitz function whose *inverse function* is also Lipschitz. Surjective bilipschitz functions are exactly the isomorphisms of metric spaces.

1.3 How do we relate P(t,T) to a PDE

In Financial Modelling there are said to be two main approaches, the 'martingale' approach and the PDE approach. We know for example that the Black Scholes equation is a Parabolic PDE. Thanks to some of the theorems that one encounters in a good PDE Lecture Course or in a textbook, there is a range of techology to solve PDEs. This is not a 'PDE and analysis' course, so we won't venture too far into the study of PDE, but we will make some remarks about how these PDEs are solved.

$$\text{Let T be fixed} P(t,T) = \mathbb{E}_{\mathbb{Q}} \left[\exp\left(- \int_t^T r(s)\, ds \right) \middle| \mathcal{F}_t \right] \overset{=}{\text{Markov Property of SDE}} \tag{5}$$

$$\mathbb{E}_{\mathbb{Q}} \left[\exp\left(- \int_t^T r(s)\, ds \right) | r_t \right] = F(t, r_t \tag{6}$$

For some function $F(\cdot, \cdot)$ deterministic with random arguments, with

$$F(t, x) =: \mathbb{E}_{\mathbb{Q}} \left\{ \exp\left(- \int_t^T r(s)\, ds \right) | r_t = x \right\}$$

Theorem 1.6. *Fix $T > 0$, and adopt the previous notation. Assume that $F : [0,T] \times Z$ with $\Theta : Z \to \mathbb{R}_+$is a solution of*

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,r) + b(t,r)\dfrac{\partial}{\partial r}F(t,r) + \dfrac{1}{2}\sigma^2(t,r)\dfrac{\partial^2}{\partial r^2}F(t,r) = rF(t,r) \\ F(T,r) = \Theta(r) \end{cases} \tag{7}$$

Then: *$t \Rightarrow M_t := F(t,r_t)\exp - \int_0^t r(s)ds$ is a $(\mathcal{F}_t, \mathbb{Q})$- local martingale on [0,T]. If moreover: either*

1. *M_t is u.i. or*
2. *$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T (\partial_r F(t,r_t)\exp - \int_0^t r(s)ds\sigma(t,r_t))^2 dt\right]$*

Then $F(t,r_t) = \mathbb{E}_{\mathbb{Q}}(M_T|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}\left[F(T,r_t)\exp - \int_0^T r(s)ds|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\Theta(r_T)\exp - \int_0^T r(s)ds|\mathcal{F}\right.$ Therefore we can conclude:

$$F(t,r_t) = \mathbb{E}_{\mathbb{Q}}\left[\Theta(r_T)\exp - \int_0^T r(s)ds|r_t\right]$$

How to use this theorem:

1. T, b σ are given; we want to compute $P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\Theta(r_T)\exp - \int_0^T r(s)ds|r_t\right]$
2. Find a solution F(t,x) to 7 in the case $\Theta \equiv 1$
3. Check 1.6 (i) or (ii)
4. If true P(t,T) $=$ $F(t,r_t)$ and this solution is unique

Proof. We apply Ito's formulat to M_t

$$d(F(t,r_t)\exp - \int_0^t r(s)ds = \exp - \int_0^t r(u)du dF(t,r_t) - r(t)\exp - \int_0^t r(u)du F(t,r_t)dt$$

$$= -r(t)\exp - \int_0^t r_u du F(t,r_t)dt + \exp - \int_0^t r(u)du$$

$$\left\{\frac{\partial}{\partial t}F(t,r_t)dt + \frac{\partial}{\partial r}F(t,r_t)\left[b(t,r_t)dt + \sigma(t,r_t)dW_t^*\right] + \frac{1}{2}\frac{\partial^2}{\partial r^2}F(t,r_t)\underbrace{\sigma^2(t,r_t)}_{\text{Quadratic variation}}dt\right\}$$

$$= \frac{\partial}{\partial r}F(t,r_t)\sigma(t,r_t)\exp - \int_0^t r(s)ds dW_t^*$$

So $M_t = M_0 + \int_0^t \partial_r F(s,r_s)\exp - \int_0^s r(u)du\sigma(s,r_s)dW_s^* = (\mathcal{F},\mathbb{Q})$ Local Martingale. Moreover, either of the two 1.6 are sufficient conditions for M to be true for $(\mathcal{F}_t,\mathbb{Q})$-martingale □

1.4 Some final remarks on the PDE

These are generally solved by Finite difference methods, which are found in various 'Numerical Methods for solving PDE' books. Generally these methods came out of Physics or Mechanics, so there is a rich history and an extensive mathematical technology. Closed form solutions for option prices on zero-coupon bonds can also be found in this model. In general, derivatives prices can be estimated by either numerically solving the PDE in (7) with appropriate boundary conditions or by using Monte-Carlo methods. An example image of a simulation using Monte-Carlo Methods is provided in the next subsection.

1.5 Vasicek Model

The model specifies that the force of interest|instantaneous interest rate follows the stochastic differential equation:

$dr_t = a(b - r_t)\,dt + \sigma\,dW_t$

where W_t is a Wiener process under the risk neutral framework modelling the random market risk factor, in that it models the continuous inflow of randomness into the system. The standard deviation parameter, σ, determines the *volatility* of the interest rate and in a way characterizes the amplitude of the instantaneous randomness inflow. The typical parameters b, a and σ, together with the initial condition r_0, completely characterize the dynamics, and can be quickly characterized as follows, assuming a to be non-negative:

- b: "long term mean level". All future trajectories of r will evolve around a mean level b in the long run;
- a: "speed of reversion". a characterizes the velocity at which such trajectories will regroup around b in time;
- σ: "instantaneous volatility", measures instant by instant the amplitude of randomness entering the system. Higher σ implies more randomness

The following derived quantity is also of interest,

- $\sigma^2/(2a)$: "long term variance". All future trajectories of r will regroup around the long term mean with such variance after a long time.

a and σ tend to oppose each other: increasing σ increases the amount of randomness entering the system, but at the same time increasing a amounts to increasing the speed at which the system will stabilize statistically around the long term mean b with a corridor of variance determined also by a. This is clear when looking at the long term variance,

$\frac{\sigma^2}{2a}$

which increases with σ but decreases with a.

This model is an Ornstein Uhlenbeck stochastic process The following was a generation of 1000 Monte Carlo Simulations of the Vasicek Model using Python.