The Laplace Operator on Manifolds and Cauchy Problem

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Contents

1 The Laplace Operator

Idea 1. The aim is to introduce some notions associated with the Laplace Operator. And to enhance some of the remarks made in the PDE II problem class. Particularly on Riemannian Manifolds and volume forms. A good reference is [5], or any Differential Geometry book/ a good PDE book. The idea of the 'Hodge Star Operator' is introduced and defined.

Let V be a real vector space with scalar product $<\cdot,\cdot>$,and let $\bigwedge^p V$ be the p-fold exterior product of V. We then obtain a scalar product on $\bigwedge^p V$ by

$$< v_1 \wedge \cdots \wedge v_p, \omega_1 \wedge \cdots \wedge \omega_p >= det(< v_i, \omega_j >) \tag{1}$$

and bilinear extensions to $\bigwedge^p V$ if $e_1, \cdots, e_d$ is an orthonormal basis of V,

$$e_{i_1} \wedge \cdots \wedge_{i_p} with 1 \leq i_1 < i_2 \cdots < i_p \leq d \tag{2}$$

constitutes an orthonormal basis of $\bigwedge^p V$.

Orientation We've spoken of 'orientation' in some examples in PDE II so let us define what an ***orientation*** is. An orientation on V is obtained by distinguishing a basis of V as positive. Any other basis that is obtained from this basis by a base change with positive determinant is likewise called positive, and the remaining bases are called negative. Let now V carry and orientation. We define the linear star operator (or Hodge star operator[1])

$$* : \bigwedge^p (V) \to \bigwedge^{d-p} (V)$$

$(0 \leq p \leq d)$ by

$$* \, (e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{j_1} \wedge \cdots \wedge e_{j_{d-p}} \tag{3}$$

where $j_1, \cdots, j_{d-p}$ is selected such that $e_{i_1}, \cdots, e_{i_p}, e_{j_1}, \cdots, e_{j_{d-p}}$ is a positive basis of V. Since the star operator is supposed to be linear it is determined by its values on some basis (3). In particular

$$* \, (1) = e_{i_1} \wedge \cdots \wedge e_{i_p}, \tag{4}$$

$$* \, (e_{i_1} \wedge \cdots \wedge e_{i_p}) = 1, \tag{5}$$

if $e_1, \cdots, e_d$ is a positive basis. From the rules of multilinear algebra, it easily follows that if A is a d x d matrix, and if $f_1, \cdots, f_d \in V$, then

$$*(Af_1 \wedge \cdots \wedge Af_p = (detA) * (f_1 \wedge \cdots \wedge f_p)$$

[1]After the late great British Mathematician and Analyst

In particular, this implies that the star operator does not depend on the choise of positive orthonormal basis (O.N.B) in V, as any two such bases are related by a linear transformation with determinant 1. For a negative basis instead of a positive one, one gets a minus sign on the r.h.s of (3) (17) (5)

Lemma 1.1.

$$** = (-1)^{p(d-p)} : \bigwedge^{p}(V) \to \bigwedge^{p}(V)$$

Proof. $**$ maps $\bigwedge^p(V)$ onto itself. Suppose

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{j_1} \wedge \cdots \wedge e_{j_{d-p}}$$

(c.f (3))

Then

$$**(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{i_1} \wedge \cdots \wedge e_{i_p}$$

depending on positive or negative basis of V. The proof follows as $(-1)^{p(d-p)}$ is the determinant of the basis change from $e_{j_1} \wedge \cdots \wedge e_{j_{d-p}}$ to $e_{i_1} \wedge \cdots \wedge e_{i_p}$ $\square$

Lemma 1.2. *For v,w* $\in \bigwedge^p(V)$

$$< v, w >= *(w \wedge *v) = *(v \wedge *w) \tag{6}$$

Proof. It suffices to prove (6) for elements of the basis (2). For any two different of these base vectors

$$v \wedge *w = 0,$$

whereas $*(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge *(e_{i_1} \wedge \cdots \wedge e_{i_p})) = *(e_1 \wedge \cdots \wedge e_d)$, where $e_1, \cdots, e_d$ is an O.N.B ((3)) and this = 1 (5) $\square$

The claim clear follows.

Lemma 1.3. *Let* $v_1, \cdots v_d$ *be an arbitrary positive basis of V. Then*

$$*(1) = \frac{1}{\sqrt{det(< v_i, v_j >)}} v_1 \wedge \cdots \wedge v_d \tag{7}$$

Proof. Let $e_1, \cdots, e_d$ be a positive O.N.B as before. Then $v_1 \wedge \cdots \wedge v_d = (det(< v_i, v_j >))^{1/2} e_1 \wedge \cdots \wedge e_d$ and the claim follows from (17) $\square$

Let now M be an oriented Riemannian manifold of dimension d. Since M is oriented, we may select an orientation of all tangent spaces T_xM, hence also on all cotangent spaces T_x^*M in a consistent manner. We simply choose the Euclidean O.N.B $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}$ of $\mathbb{R}^d$ as being positive. Since all chart transitions of an oritented manifold have positive functional determinant, our basis will not depend on the choice of charts. So we have a basis for the tangent space,and we have a Riemannian structure, so we have a scalar product on each T_x^*M. We thus obtain a star operator (which preserves the base points)

$$* : \bigwedge^p (T_x^* M) \to \bigwedge^{d-p} (T_x^* M).$$

We recall that the metric on $T_x^*M$is given by $(g^{ij}(x)) = (g_{ij}(x))^{-1}$. Therefore by (3) we have in local coordinates

$$*\,(1) = \sqrt{g_{ij}} dx^1 \wedge \cdots \wedge dx^d \tag{8}$$

This expression is called the ***volume form***[2]. In particular we get this nice formula (provided the integral is finite)

$$vol(M) := \int_M *(1) \tag{9}$$

[2]This was mentioned in a PDE problem class

2 PDE Exercises

$$f : \mathbb{R}^n \to \mathbb{C}$$

given $\tilde{M}f$ is a solution of the homogeneous Cauchy problem, with $\tilde{M}f(0,x) = 0$, $\frac{\partial \tilde{M}f}{\partial t}(0,x) = f(x)$ Let $g : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}$ be sufficiently often continuously differentiable. Show without using any explicit formular for $\tilde{M}f$ that the function Qg defined by

$$Qg(t,x) := \int_0^t \tilde{M}g_s(t-s,x)ds,$$

where $g_s(x) := g(s,x)$, is the solution of the Cauchy problem for the inhomogeneous wave equation with right hand side g and zero initial conditions. We want to show that this implies $\Box Qg = g$ $Qg(0,x) = \frac{\partial Qg}{\partial t}(0,x) = 0$
For n = 1 this gives an alternative solution of Example 1.

$$\tilde{M}f(0,x) = \frac{1}{2c}\int_{x-t}^{x+t} f(\tau)d\tau$$

This implies

$$Qg(t,x) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} g(s,\tau)d\tau ds \tag{10}$$

We can then write Qg(0,x) = $\int_0^0 \cdots ds = 0$ $\frac{\partial Q}{\partial t}(t,x) = \tilde{M}g_s(t-s,x)$ We let s = t + $\int_0^t \frac{\partial}{\partial t}\tilde{M}g_s(t-s,x)ds$ We know $\tilde{M}g_t(0,x) = 0$ SO:

$$\frac{\partial Qg}{\partial t}(t,x) = \int_0^t \frac{\partial}{\partial t}\tilde{M}g_s(t-s,x)ds \tag{11}$$

In particular if t=0 We get $\frac{\partial Qg}{\partial t}(0,x) = \int_0^0 \cdots = 0$ We also have to look at the box operator $\Box = \frac{\partial^2}{\partial t^2} - \triangle_x$ We obtain by $\frac{\partial}{\partial t}$11 By parameter dependent integrals;

$$\frac{\partial^2 Qg}{\partial t^2}(t,x) = \int_0^t \frac{\partial^2}{\partial t^2}\tilde{M}g_s(t-s,x) + \frac{\partial}{\partial t}\tilde{M}g_s(t-s,x)|_{s=t}$$

$$\frac{\partial^2 Qg(t,x)}{\partial t^2} = g(t,x) + \int_0^t \cdots ds$$

By parameter dependent integrals;

$$\triangle_x Qg(t,x) = \int_0^t \triangle_x \tilde{M}g_s(t-s,x)ds$$

$$\Box Qg(t,x) = g(t,x) + \int_0^t \Box Mg(t-s,x)ds$$

The integral above goes to zero by the homogeneous Cauchy problem solution. Therefore we can write

$$\Box Qg(t,x) = g(t,x) \tag{12}$$

We only need the properties of $\tilde{M}g_s$. This should be seen as an exercise in parameter dependent integrals.

3 Examples of distributions and their order

Let us recall the criterion: T:$C_c^\infty(U) \to$ $\mathbb{C}$ linear T is a distribution $\iff$ $\forall K \subset U$ compact $\exists p(K) \in \mathbb{N}_0$, $C(K) \in [0,\infty)$ such that for $\phi \in C_c^\infty(U)$ with supp$\phi \subset K$

$$|T(\phi)| \leq C(K) \sum_{|\alpha| \leq p(K)} \|D^\alpha \phi\|_\infty$$

Where the last norm is the suprenum norm.

Example 3.1.

$$\delta_{x_0}$$

$$|\delta_{x_0}| = |\phi(x_0)| \leq \|\phi\|_\infty$$

(c=1, p = 0, indepedent of K.

Example 3.2.

$$f \in L^1_{loc}(U)$$

$T_f(\phi) = \int_U f(x)\phi(x)ds$ $|T_f(\phi)| = |\int_U f(x)\phi(x)dx| \leq \int_U |f(x)||\phi(x)|dx$ Assume $K \subset U$ compact $supp\phi \subset K = \int_K |f(x)||\phi(x)|dx \leq \|\phi\|_\infty \int_K |f(x)|dx$ the integral above can be considered as C(K) and it is less than infinity (p=0, independent of K).

Example 3.3. $T(\phi) = \int_M \phi|_M.\omega$ M $\subset U$ k-dim oriented subset and we can consider ω a k(top) form There is a **trick**: In local coordinates $Y_1, \cdots Y_k$ $V \subset M$ open $y_1 : V \to$ $\mathbb{R}$ $\omega = g(y_1, \cdots, y_k)dY_1 \wedge \cdots \wedge dY_k$ We use the pullback in fact, but we can sloppy write

$$\int_V \phi|_M \omega = \int_V \phi(Y_1 \cdots Y_k) g(Y_1, \cdots, Y_k) dy_1, \cdots, dy_k$$

$\int_M \phi_M \omega =$ sum over open sets, where we have coordinates using partition of unity. We need a volume form or take the modulus of $|g(y_1, ..., y_k)|$ If we define, in local coordinates of the correct orientation $|\omega(x)| = |g(y_1, \cdots, y_k)|dy_1\wedge$

$\cdots \wedge dy_k$ then $|\omega|$ is well-defined independent of co-ordinates. We can then write the following

$$|\int_M \phi|_M.\omega| \leq \int_M |\phi|_M.|\omega| = \int_{K\cap M} |\phi|_{|M}.|\omega| \leq \|\phi\|_\infty. \int_{K\cap M} |\omega|$$

Because $supp\phi \subset K$ we can do this trick. The integral above becomes C(K) and is finite, since ω has a finite number. This is indeed a distribution and clearly of order 0.

Example 3.4. Show that the sequence (T_n), where T_n is given by the locally integrable function $ne^{-\frac{n^2x^2}{2}}$, converges in $C^{-\infty}(\mathbb{R})$ and compute its limit.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R})$ (i.e., ϕ is a C^∞ function with compact suport). Then

$$\langle T_n, \phi\rangle = \int_{\mathbb{R}} n\, e^{-n^2x^2}\phi(x)\, dx = \int_{\mathbb{R}} e^{-x^2}\phi\big(\frac{x}{n}\big)\, dx.$$

We have

$$\lim_{n\to\infty} e^{-x^2}\phi\big(\frac{x}{n}\big) = e^{-x^2}\phi(0) \quad \forall x \in \mathbb{R}$$

and

$$\Big|e^{-x^2}\phi\big(\frac{x}{n}\big)\Big| \leq \|\phi\|_\infty e^{-x^2}.$$

The dominated convergence theorem implies that

$$\lim_{n\to\infty} \langle T_n, \phi\rangle = \phi(0),$$

that is, T_n converges in the distribution sense to Dirac's δ_0. □

Example 3.5. Formulate the homogeneous Cauchy problem for the heat equation on $\mathbb{R}^n$, and give uniqueness and existence results, including a solution formula, under a boundedness condition.

Assume now that the initial condition ϕ is real valued, nonnegative, compactly supported and not identically zero. Show that the solution $f(t,x)$ satisfies

- f(t,x) $>$ 0 for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$,
- $lim_{|x|\to\infty} f(t,x) = 0$ for any fixed t $>$ 0,
- $lim_{t\to\infty} f(t,x) = 0$ for any fixed $x \in \mathbb{R}^n$

Example 3.6. Let us consider what happens to a linear constant coefficient partial differential operator, P(D). *the fundamental solution of P can never be a distribution with compact support.*

Proof. In fact, assume we have P(D)u = f, where u is a distribution, then u has compact support $\iff \dfrac{f}{P(\xi)}$ is analytic. (This result can be found in Chapter 7 of Volume 1 of Hormanders treatise). Now, if we have

$$P(D)u = \delta \tag{13}$$

obviously $\dfrac{\delta}{P(\xi)}$ is never an analytic function for a polynomial P. So the fundamental solution of P can not be compactly supported. $\square$

Example 3.7. I have a sequence (T_n), where T_n is given by the locally integrable function $ne^{-\dfrac{n^2x^2}{2}}$, converges in $C^{-\infty}(\mathbb{R})$ and compute its limit. I suspect that the limit tends towards zero since the exponential tending towards infinity will become zero. Is this enough to prove this, in conjunction with the definition? I've already written the definition of a locally integrable function and I already understand the definition (sometimes called the 'weak-dual convergence') of the convergence of a sequence of distributions. I've not considered any topologies in these cases, as I don't understand Frechet spaces and the like.

Proof. Let $\phi \in \mathcal{D}(\mathbb{R})$ (i.e., ϕ is a C^∞ function with compact suport). Then

$$\langle T_n, \phi \rangle = \int_{\mathbb{R}} n\, e^{-n^2x^2} \phi(x)\, dx = \int_{\mathbb{R}} e^{-x^2} \phi\big(\frac{x}{n}\big)\, dx.$$

We have

$$\lim_{n \to \infty} e^{-x^2} \phi\big(\frac{x}{n}\big) = e^{-x^2}\phi(0) \quad \forall x \in \mathbb{R}$$

and

$$\Big| e^{-x^2} \phi\big(\frac{x}{n}\big) \Big| \leq \|\phi\|_\infty e^{-x^2}.$$

The dominated convergence theorem implies that

$$\lim_{n \to \infty} \langle T_n, \phi \rangle = \phi(0) \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}\, \phi(0),$$

that is, T_n converges in the distribution sense to Dirac's $\sqrt{\pi}\, \delta_0$. $\square$

Example 3.8. Show that he distribution given by the locally integrable function $\frac{1}{2}e^{|x|}$ is a fundamental solution of the differential operator $-\frac{\partial^2}{\partial x^2}+id$ on $\mathbb{R}^1$

Proof. You can check directly by noting the fact that

$$\frac{d}{dx}e^{|x|} = (H(x) + H(-x))e^{|x|}$$

where H is the Heaviside function. On the other hand,you can use fourier transform to get the desired result,in fact,let u satisfies

$$(1 - \frac{d}{dx^2})u = \delta$$

Take fourier transform on both sides,then get

$$\hat{u} = \frac{1}{1+x^2}$$

then the result follows easily. □

Example 3.9. Let us consider $\lim_{\epsilon\to 0} \frac{\epsilon}{x^2+\epsilon^2}$ in $C^{-\infty}(\mathbb{R})$

Proof. Firstly let us use $y = \frac{x}{\epsilon}$
So we have $\lim_{\epsilon\to 0} \frac{\epsilon}{\epsilon^2 y^2+\epsilon^2}$
We know that $\int_{\mathbb{R}} g(x)dx$ is Continuous and bounded.
$y_\epsilon(x) = 1/\epsilon(g(\frac{x}{\epsilon})dy$
Let us now form a distribution
$T_{y_\epsilon}(\phi) = \int_{\mathbb{R}} \epsilon^{-1}y(\frac{x}{\epsilon})\phi(x)$
$= \int_{\mathbb{R}} g(y)\phi(\epsilon y)dy$ $\lim_{\epsilon\to 0} T_{y_\epsilon}(\phi) = \lim_{\epsilon\to 0}$
$= \int_{\mathbb{R}} g(y) \lim_{\epsilon\to 0} \phi(\epsilon y)dy$
$= \int_{\mathbb{R}} g(y)\phi(0)dy$
$\phi(0) \int_{\mathbb{R}} g(y)dy$ by the assumption that this integral is x.
$= x \cdot \phi(0)$ □

Example 3.10. Question 4 on June 20th 2011 Suppose there exists a non-zero solution $f \in C^\infty(\mathbb{R}^n)$ of th equation $D'f = 0$

Proof. DT = S, where $S \in C_c^{-\infty}(U)$ By the a proposition $\exists!$ $\tilde{S} : C^\infty(U) \to \mathbb{C}$ such that $\tilde{S}(f) = S(f)$ for all $f \in C_c^\infty(U)$ but $f \in C^\infty$ therefore by a Proposition

$$\tilde{S}(f) = 0 \tag{14}$$

□

4 Lie Derivatives

The aim is to give a nice contained introduction to the properties of the Lie Derivative. This is covered in any good book on analysis on manifolds, or differential geometry, such as [5, 9] note that Warner[9] is better seen as a reference book rather than a book for learning from. Also one can see the 'Basic Riemannian Geometry' by FE Burstall as a chapter in [4]. A pdf is also available from [2] The Lie Derivative is a method of computing the 'directional derivative' of a vector field with respect to another vector field. We defined the ***Lie Derivative of a function*** $f \in C^\infty(M)$ in the direction of a vector field X $\in Vect(M)$ by

$$(L_X f)_m = X_m(df)_m \tag{15}$$

for all $m \in M$. We have an isomorphism between $X \in Vect(M)$ and $L_X \in Der(C^\infty(M))$ so we can write $L_X$as X. We would like to extend these ideas to vector fields. On a manifold it isn't as easy as in a Euclidean space. The problem is due to the fact different vectors live in different tangent spaces. So we replace the notion of vectors $X \in T_mM$ with that of a vector field.

Idea 2. We can now use the **flow** of the vector field to **push** values of Y back to m and differentiate.

We say the Lie derivative of Y with respect to the vector field X We have to compare the value $Y_m \in T_mM$ of Y at m with the value of Y at a point of M that lies close to m in the direction specified by X, i.e. with the value

$$Y_{\phi_t^X}(m) \in T_{\phi_t^X}(m)M$$

We want to use the flow of X to **push** the values of $Y_{\phi_t^X}(m)$ back to $Y_m$and then differentiate.

Definition 4.1.

$$(L_XY)_m = lim_{t\to\,0}\frac{(\phi^X_{-t,*}Y)_m - (\phi^X_{-0,*}Y)_m}{t}$$

$$= \frac{d}{dt}|_{t=0}(\phi^X_{-t,*}Y)_m$$

Where $(\phi^X_{-t,*}$ denotes the pushforware by $(\phi^X_{-t}$, this does allows us to cmpare two vectors that live in different tangent spaces.

5 Operators and Mathematical Physics

5.1 The Heat Kernel

The following section on the Heat Kernel and the following subsection on Magnetic Fields where both taken from the excellent and Mathematical Physics orientated textbook[7] The heat kernel 17 is the simplest example of a ***semigroup***. Clearly, equation 19 is a linear equation and $e^{t\triangle}$ is an *operator valued* solution of the heat equation, in the sense that for every initial condition f, the solution g_t is given by $e^t \triangle f$, i.e., the heat kernel applied to the function f. This relation can be written, in an admittedly formal way, as

$$\frac{d}{dt}e^{t\triangle} = \triangle e^{t\triangle}, \tag{16}$$

a notation that is familiar when dealint with finite systems of linear ODE[3], in which case $e^{t\triangle}$ is replaced by a t-dependent matrix P_t

$$t \rightarrow P_t$$

: one parameter group of matrices It is easy to see that the heat kernel 17 shares all these properties except for invertibility. Hence we call $e^{t\triangle}$ the ***heat semigroup*** and there is no solution for $t < 0$

Definition 5.1. Define the heat kernel on $\mathbb{R}^n \times \mathbb{R}^n$ to be

$$e^{t\triangle}(x,y) = (4\pi t)^{-n/2} exp \frac{-|x-y|^2}{4t} \tag{17}$$

THe action of the heat kernel on functions is, by definition, $e^{t\triangle}f(x) = \int_{\mathbb{R}^n} e^{t\triangle}(x,y)f(y)dy$ If $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq 2$, then, by Theorem 5.8 in [7]

$$e^{t\triangle}\hat{f}(h) = exp{-4\pi^2}|k|^2 t\hat{f}(k) \tag{18}$$

$$\triangle g_t = \frac{d}{dt}g_t \tag{19}$$

The heat equation is a model for heat conduction and g_t is the temperature distribution (as a function $x \in \mathbb{R}^n$) at time t. The kernel, given by 17 satisfies 19 for each $y \in \mathbb{R}^n$(as can be verified by explicit calculation) and satisfies the initial condition

$$\lim_{t\_0} e^{t\triangle}(\cdot\,, y) = \delta_y in \mathcal{D}^{'}(\mathbb{R}^n) \tag{20}$$

[3]Ordinary Differential Equations, as always is abbreviated

5.2 Magnetic fields

This may seem a bit of a hodge podge of ideas, but I was told that Magnetic fields were H^1_A spaces.

Remark 5.2. I'm not as of early 2012 familiar with Sobolev spaces but I thought this section was well written.

In differential geometry it is often necessary to consider **connections**, which are more complicated derivatives than ∇. The simplest example is a connection on a 'U(1)-bundle' over $\mathbb{R}^n$[9]. (Or for further explanations of the Geometry of the Aharanov-Bohm effect[**?**]). This bundle is not too complicated,it merely means acting on complex valued functions f by $(\nabla + iA(x))$, with $A(x) : \mathbb{R}^n \to \mathbb{R}^n$ being some pre-assigned, real vector field. The same operator occurs in quantum mechanics of particles in external magnetic fields (with n=3). The introduction of a magnetic field $B : \mathbb{R}^3 \to \mathbb{R}^3$ in quantum mechanics involes replacing ∇ by $\nabla + iA(x)$ (in appropriate units). Here A is called a ***vector potential*** and satisfies curl A = B.

In general A is not a bounded vector field, e.g., if B is a constant magnetic field (0,0,1), then a suitable vector potential A is given by A(x) = $(x_2, 0, 0)$. Unline in the differential geometric setting, A need not be smooth either, because we could add an arbitrary gradient to A, $A \to A + \nabla\Xi$, and still get the same magnetic field B. This is called ***gauge invariance***.[4] The problem is that Ξ (and hence A) could be a wild function - even if B is well behaved. For these reasons we want to find a large class of A's afor which we can make (distributional)sense of $(\nabla + iA(x))$ and $(\nabla + iA(x))^2$ when action on a suitable calss of $L^2(\mathbb{R}^3)$ functions. For general dimension n, the appropriate condition on A, which we assume henceforth, is

$$A_j \in L^1_{loc}(\mathbb{R}^n) \; for \; j = 1, \cdots, n \tag{21}$$

because of this condition the functions $A_j f$ are in $L^1_{loc}(\mathbb{R}^n)$ for every $f \in L^2_{loc}(\mathbb{R}^n)$. Therefore the expression

$$(\nabla + iA)f$$

called the ***covariant derivative***(with respect to A) of f, is a distribution ofr every $f \in L^2_{loc}(\mathbb{R}^n)$.

[4] A phrase which strikes fear into the hearts of Grad Students everywhere!

6 Yang Mills

The majority of this section is inspired by [1]. To define the Yang-Mills Lagrangian, we need to define the 'Trace' of an End(E) valued form. Recall that the Trace of a maTrix is the sum of its diagonal enTries. The Trace is independent of the choice of basis - an invariant notion that is independent of the choice of basis. A definition of the Trace that mkes this clear is as follows. Consider $End(V) \simeq V \otimes V^*$ - an isomorphism that does not depend on any choice of basis - so the pairing between V and V^* defines a linear map

$$Tr : End(V) \to \mathbb{R}$$

$$v \otimes f \mapsto f(v)$$

To see that this v is really a Trace, pick e_i of V and let ϵ^j be a dual basis of V^*. Writing $T \in End(V)$ as

$$T = T^i_j e_i \otimes \epsilon^j$$

We have

$$Tr(T) = T^i_j e_i(\epsilon^j) = T^i_j \delta^j_i = T^i_i$$

which is of course the sum of the diagonal enTries.
This implies that if we have a section T of $End(E)$, we can define a funciton Tr(T) on the base manifold M whose value at $p \in M$ is the Trace of the endomorphism T(p) of the fiber E_p:

$$Tr(T)(p) = Tr(T(p))$$

If $T \in \Gamma(End(E))$ and $\omega \in \Omega^p(M)$ we define

$$Tr(T \otimes \omega) = Tr(T)\omega$$

Now we can write down the **Yang-Mills Lagrangian**: If D is a connection on E, this is the n-form given by

$$\mathcal{L}_{YM} = \frac{1}{2} Tr(F \wedge *F) \tag{22}$$

where F is the curvature of D. Note that by the defintion of the hodge star operator (also in this collection of notes), we can write this in local coordinates as

$$\mathcal{L}_{YM} = \frac{1}{4} Tr(F_{\mu\nu} F^{\mu\nu}) vol \tag{23}$$

If we integrate $\mathcal{L}_{YM}$ over M we get the **Yang-Mills action**

$$S_Y M = \frac{1}{2}\int_M Tr(F \wedge *F) \tag{24}$$

This needs some elaboration. So let us explain these formulas better. We choose the physics convention $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ where the generators of the Lie algebra are Hermitian.

$$\mathcal{L}_{YM} = \mathcal{I} = -\int Tr(F \wedge *F) \tag{25}$$

by another convention (there is a lot of ambiguity of signs in this subject). The first thing to note is that F has vector **and** Lie algebra indices. The Trace is over the Lie algebra, **not** over the vector indices. The vector indices are just those of the field sTrength in QED. In Yang-Mills the curvature form is Lie Algebra valued.
In this case $F_{\mu\nu} = F^a_{\mu\nu}T^a$ where the summation convention is used,and where T^a are the generators of $\mathfrak{su}(n)$. To be explicit, F has not only tensor components but maTrix components

$$(F_{\mu\nu})_{ij} = F^a_{\mu\nu}T^a T^a_{ij}$$

The inner product of F with itself $< F, F >= \int F \wedge *F$ where $*$is the Hodge $*$- operator. Thus we are calculating $\mathcal{I} = -Tr < F, F >$. It is a standard exercise to find the exterior product of two r-forms. We find

$$F \wedge *F = \frac{1}{2!}F_{\mu\nu}F^{\mu\nu}dx^1 \wedge \cdots \wedge dx^4$$

Note that the differential forms don't 'know' the Lie algebra. The algebra hasn't come into the calcuation yet. $Tr(F_{\mu\nu}F^{\mu\nu}) = Tr(F^a_{\mu\nu}T^aF^{\mu\nu b}T^b$ $= Tr(T^aT^b)F^a_{\mu\nu}F^{\mu\nu b} = \frac{1}{2}\delta^{ab}F^a_{\mu\nu}F^{\mu\nu b} = \frac{1}{2}F^a_{\mu\nu}F^{\mu\nu a}$ where e have used the standard normalization convention for the T^a, $TrT^aT^b = \frac{1}{2}\delta^{ab}$. (This comes from the fact we want $\mathfrak{su}(2)$ to live in $\mathfrak{su}(n)$, and the generators of $\mathfrak{su}(2)$ are taken to be $T^a = \frac{\sigma^a}{2}$ where σ^a are the Pauli matrices.) Thus, we find

$$\mathcal{I} = -Tr(F \wedge *F)$$

which can be written as

$$\frac{1}{4}\int d^4x F^a_{\mu\nu}F^{a\mu\nu}$$

7 Metrics and Connections

When we think of Manifolds we often think in terms of a Riemannian Metric with a Euclidean Metric. Another important notion is that of a connection. We define a flat connection to be any submersion $\phi : M \to S$ between two manifolds to be a subbundle E $\subseteq$ TM such that

1. TM $= E \oplus kerT\phi$

2. $[E, E] \subseteq E$ (That is, sections of E are closed under the Lie bracket, and so by the Frobenius theorem E is integrable

3. Every path in S has a horizontal lift through each of its points

(1) and (3) imply that there is an Ehresman connection. (2) implies the flatness propert, which implies that the fibration has a discrete structure group. **Curvature** Let P $\in \Omega^1(M; TM)$ be a fiber projection, i.e $P \circ P = P$. This is the most general case of a (first order) connection. If P is of constant rank, then both are subvector bundles of TM. If im P is some primarily fixed sub vector bundle, P can be called a connection for it. Let (E, p, M, S) be a fiber bundle; we consider the fiber linear tangent mapping (pushforward) $T_p : TE \to TM$ and its $kerT_p =: VE$ whcih is called the vertical bundle of E.

Definition 7.1. A **connection** on the fiber bundle (E,p,M,S) is a vector bundle 1-form $\Phi \in \Omega^1(E; VE)$ with values in the vertical bundle VE such that $\Phi \circ \Phi = \Phi$ and $Im\Phi = VE$; so Φ is just a projection $TE \to VE$. Then ker Φ is of constant rnak, so ker Φ is a subvectorbundle of TE, it is called the space of horizontal vector bundles (denoted HE). Clearly $TE = HE \oplus VE$ and $T_uE = H_uE \oplus V_uE \; \forall u \in E$.

Definition 7.2. Let E be a vector bundle on the differentiable manifold M with bundle metric $< .,. >$. A connection D on E is called **metric** if $d < \mu, \nu > = < D\mu, \nu > + < \mu, D\nu >$ for all $\mu, \nu \in \Gamma(E)$

A metric condition respects the metric.

7.1 Curvature of a Connection

One of the most powerful and important ideas in Differential Geometry is that of the 'curvature' of a connection.

Definition 7.3. Let ∇ be a connection on a vector bundle $\pi : E \to M$. The bilinear map

$$R : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{L}(Sec(E), Sec(E)), \tag{26}$$

defined for $X, Y \in \mathcal{X}(M)$ and $s \in Sec(E)$ by

$$R(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s, \tag{27}$$

is called the curvature of the connection ∇

We can make some remarks, before considering the local form.

Remark 7.4.

1. Note that R is a trilinear map R:Sec(TM) $\times$ Sec(TM) $\times$ Sec(E) $\to$ Sec(E), which is also $C^\infty(M)$-linear in each argument and skew-symmetric in the first two arguments. Hence

$$R \in Sec(\bigwedge^2 T^*M \otimes E^* \otimes E), \tag{28}$$

i.e. the curvature of a connection on a vector bundle E with base manifold M is a differential 2-form on M valued in the endomorphism bundle of E.

2. The curvature tensor R of a connection ∇ is often denoted by R^∇.

3. We immediately see that curvature tensor of the trivial connection on a trivial bundle vanishes, since this covariant derivative with respect to a vector field X is just the Lie derivative with respect to X.

7.1.1 Local form and components

Remember that a covariant derivative is locally characterized by a connection 1-form

$$\mathcal{A} \in \Omega^1(U) \otimes \mathfrak{gl}(r, \mathbb{R}) \tag{29}$$

It is clearn that the de Rham differenial can be extended, for any manifold M, to $\Omega(M) \otimes \mathfrak{gl}(r, \mathbb{R})$. Just set

$$d(\alpha \otimes A) = (d\alpha) \otimes A \tag{30}$$

where $(\alpha \in \Omega(M), A \in \mathfrak{gl}(r, \mathbb{R}))$. Note here we apply the universal property of the tensor product.

8 Berry Phase

8.1 Introduction

The following is based on an amalgamation of notes made from [6] and the canonical textbook for learning about the Mathematics of the Geometric phase is ,[3].

We invite the reader to perform the following simple experiment. Put your arm out in front of you keeping your thumb pointing up perpendicular to your arm. Move your arm up over your head, then bring it down to your side, and at last bring the arm back in front of you again. In this experiment an object (your thumb) was taken along a closed path traced by another object (your arm) in a way that a simple local law of transport was applied. In our case the local law consisted of two ingredients: (1) preserve the orthogonality of your thumb with respect to your arm and (2) do not rotate the thumb about its instantaneous axis (i.e your arm). Performing the experiment, in this way you will manage to avoid rotations of your thumb locally, however in the end you will experience a rotation of 90°globally. The experiment above can be regarded as the archetypical example of the phenomenon called anholonomy by physicists and holonomy by mathematicians. In this paper we consider the manifestation of this phenomenon in the realm of quantum theory. The objects to be transported along closed paths in suitable manifolds will be wave functions representing quantum systems. After applying local laws dictated by inputs coming from physics, one ends up with a new wave function that has picked up a complex phase factor. Phases of this kind are called Geometric Phases with the famous Berry Phase being a special case.

8.2 Space of Rays

Let us consider a quantum system with physical states represented by elements $|\phi\rangle$ of some Hilbert space $\mathcal{H}$ with scalar product $\langle|\rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. For simplicity we assume that $\mathcal{H}$ is finite dimensional $\mathcal{H} \cong \mathbb{C}^{n+1}$ with $n > 1$. The infinite dimensional case can be studied by taking the inductive limit $n \to \infty$. Let us denote the complex amplitudes characterizing the state $|\psi\rangle$ by Z^α, $\alpha = 0, 1, \cdots, n$ For a normalized state we have

$$\|\psi\|^2 = \langle\psi|\psi\rangle \equiv \delta_{\alpha\beta}\bar{Z}^\alpha Z^\beta \equiv \bar{Z}_\alpha Z^\alpha = 1, \tag{31}$$

where summation over repreated indices is understood, indices are raised and lowered by $\delta^{\alpha\beta}$ and $\delta_{\alpha\beta}$ respectively, and the overbar refers to complex conjugation. A normalised state lies on the unit sphere $\mathcal{S} \cong S^{2n+1}$ in $\mathbb{C}^{n+1}$.

Two nonzero states $|\psi\rangle$ and $|\phi\rangle$ are equivalent $|\psi\rangle \sim |\phi\rangle$ iff they are related as $|\psi\rangle = \lambda|\phi\rangle$ for some nonzero complex number λ. For equivalent states quanities like

$$\frac{\langle\psi|\mathbf{A}|\psi\rangle}{\langle\psi|\psi\rangle}, \frac{|\langle\psi||\phi\rangle|^2}{\|\psi\|^2\|\phi\|^2} \tag{32}$$

having physical meaning (mean value of a physical quantity represented by a Hermitian operator $\mathbf{A}$, transition probability from a physical state represented by $|\psi\rangle$to one represented by $|\phi\rangle$) are invariant. Hence the real space of states representig the physical states of a quantum system unambigiously is the set of equivalence classes $\mathcal{P} \equiv \mathcal{H}/\sim \mathcal{P}$ is called the *space of rays.* For $\mathcal{H} \cong \mathbb{C}^{n+1}$ we have $\mathcal{P} \cong \mathbb{C}\mathcal{P}\ bb^n$, where $\mathbb{C}\mathcal{P}\ bb^n$ is the n-dimensional complex projective space. For normalised states $|\psi\rangle$ and $|\phi\rangle$ are equivalent iff $|\psi\rangle = \lambda|\phi\rangle$, where $|\lambda| = 1$ i.e. $\lambda \in U(1)$. In other words, two normalised states are equivalent iff they differ merely in a complex phase.

It is well known that $\mathcal{S}$ can be regarded as the total space of a principal bundle $\mathcal{P}$ with the structure group U(1). This meanas that we have the projection

$$\pi : |\psi\rangle \in \mathcal{S} \subset \mathcal{H} \to |\psi\rangle\langle\psi| \in \mathcal{P}\ , \tag{33}$$

where the rank one projector

$$|\psi\rangle\langle\psi|$$

represents the equivalence class of $|\psi\rangle$. Since we will use this bundle frequencly we will call it η_1. Then we have

$$\eta_1 : U(1) \hookrightarrow \mathcal{S} \xrightarrow{\pi} \mathcal{P} \tag{34}$$

For $Z^0 \neq 0$ our space of rays $\mathcal{P}$ can be given in local coordinates as

$$\omega^j \equiv \frac{Z^j}{Z^0},\ j = 1, \cdots, n \tag{35}$$

The ω^j are inhomogeneous co-ordinates for $\mathbb{CP}^n$ on the coordinate patch $\mathcal{U}_0$ defined by the condition $Z^0 \neq 0$.

$\mathcal{P}$ is a compact complex manifold with a natural Riemannian metric g. This metric g is induced from the scalar product on $\mathcal{H}$. Let us motivate the constructio of g by using the physical input provided by the invariance of the transition probability of 31. For this we define a **distance** between $|\psi\rangle\langle\psi|$ and $|\phi\rangle\langle\phi|$ in $\mathcal{P}$ as follows

$$\cos^2(\delta(\psi,\phi)/2) \equiv \frac{|\langle\psi||\phi\rangle|^2}{\|\phi\|^2\|\psi\|^2} \tag{36}$$

This defnition makes sense due to the Cauchy-Schwarz inequality applied to the R.H.S of 36 is non-negative and less than or equal to one. It is equal to one iff

$$|\psi\rangle$$

is a nonzero complex multiple of $|\phi\rangle$ i.e. iff they define the same point in $\mathcal{P}$. Hence in this case $\delta(\psi, \phi) = 0$ as we expected.

Suppose now that $|\psi\rangle$ and $|\phi\rangle$ are seperated by an infinitesimal distance $ds = \delta(\psi, \phi)$. Putting this into the definition 36, using the local coordinates ω^j for $|\psi\rangle$ and $\omega^j + d\omega^j$ for $|\phi\rangle$ after Taylor expanding both sides one gets[5]

$$ds^2 = 4g_{j\bar{k}} d\omega^j d\omega^{\bar{k}}, j, \bar{k} = 1, 2, \cdots, n \tag{37}$$

where

$$g_{j\bar{k}} \equiv \frac{(1 + \bar{\omega}_l \omega^l)\delta_{jk} - \bar{\omega}_j \omega_k}{(1 + \bar{\omega}_m \omega^m)^2} \tag{38}$$

with

$$d\omega^{\bar{k}} \equiv d\bar{\omega}^k$$

. The line element 37) defines the **Fubini-Study metric** for $\mathcal{P}$.

8.3 The Pancharatnam Connection

Having defined our basic entity the space of rays $\mathcal{P}$ and the principal U(1) bundle η_1 now we define a connection giving rise to a local law of parallel transport. In the mathematical literature the connection we are going to define is called the canonical connection on our principal bundle.

The information we need is an adaptation of Pancharatnam's study of polarized light to quantum mechanics. Let us consder two normalized states $|\psi\rangle$ and $|\phi\rangle$. When these states belong to the same ray then we have $|\psi\rangle = e^{i\theta}|\phi\rangle$ for some phase fact $e^{i\theta}$, hence the phase difference is θ How do we work out the phase difference between $|\psi\rangle$ and $|\phi\rangle$ (not orthogonal) when the states belong to *different rays?* To compare the phases of non-orthogonal states belonging to different rays Pancharatnam employed the following simple rule: two states are 'in phase' iff their inference is maximal. In order to find the state $|\phi\rangle \equiv e^{i\theta}|\phi'\rangle$ from the ray spanned by the representative $|\phi'\rangle$ which is in phase with $|\psi\rangle$we have to find a θ modulo 2π for which the interference term in

$$\|\psi + e^{i\theta}\phi'\|^2 = 2(1 + Re(e^{i\theta}\langle\psi||\phi'\rangle)) \tag{39}$$

[5]The author was unable to replicate this step

is maximal. Obviously the interference is maximal iff $(e^{i\theta}\langle\psi||\phi^{'}\rangle))$ is a real positive number i.e.

$$e^{i\theta} = \frac{\langle\phi^{'}|\psi\rangle}{|\langle\phi^{'}|\psi\rangle|}, |\phi\rangle = |\phi^{'} \frac{\rangle\phi^{'}|\psi\rangle}{|\langle\phi^{'}|\psi\rangle|} \tag{40}$$

Hence for the state $|\phi\rangle$ 'in phase' with $|\psi\rangle$ we have

$$\langle\psi|\phi\rangle = |\langle\psi|\phi^{'}\rangle| \in \mathbb{R}^{+} \tag{41}$$

9 Some Probabilistic formula for the solutions of PDEs

One of the more modern treatments for solving PDEs involves Stochastic Calculus. This can in turn be part of the larger field of 'Stochastic Geometry', however that would take us too far afield. Let us continue with the examples

9.1 Feynman Kac

Let L be a 2nd Order PDO on M, e.g. a) M differentiable manifold and L $= A_0 + \frac{1}{2}\sum_{i=1}^{r} A_i^2$ b) $M = \mathbb{R}^n$ and L $= \sum b_i\partial_i + \frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^*)_{ij}\partial_i\partial_j$ For x $\in$ M, let $X_t(x)$ be an L-diffusion, starting from x at time t=0, i.e. $X_0(x) = x$. Note that $X_t(x)$ can be constructed as a solution of the SDE

$$dX = A_0(X)dt + \sum_{i=1}^{r} A_i(X) \circ dW^i \tag{42}$$

$X_0 = x$

$$dX = b(X)dt + \sigma(X)dW \tag{43}$$

$X_0 = x$ where W is a Brownian Motion over $\mathbb{R}^r$ Suppose that the lifetime of $X_t(x)$ is finite a.s. $\forall x \in M$.

Proposition 9.1 (Feynman-Kac formula)**.** *Let $f : M \to \mathbb{R}$ be continuous and bounded $V : M \to \mathbb{R}$ be continuous and bounded above, i.e. $V(x) \leq K$ for $K \in \mathbb{R}$ Let u: $\mathbb{R}_+ \times M \to \mathbb{R}$ be a solution of the folllowing* ***initial value problem****(IVP) and note that u needs to be bounded.*

$$\frac{\partial}{\partial t}u = Lu + Vu \tag{44}$$

$$u|_{t=0} = f \tag{45}$$

i.e

$$(\frac{\partial}{\partial u})(t_i) = Lu(t_i) + V(\cdot)u(t_i) \tag{46}$$

$u(t,\cdot) = f$ *on M Then u is given by the following formula*

$$u(t,x) = \mathbb{E}[exp(\int_0^t V(X_s(x))ds) \cdot f(X_t(x))]$$

Remark 9.2. A remark which is unimportant for this course, is that there are substantial and deep links between $u(t,\cdot) = e^{tH}f$- and semigroup theory.

Proof. Fix t >0. Consider

$$Y_s := A_s \cdot Z_s \tag{47}$$

where $A_S = \exp(\int_0^s V(X_r(x))dr)$and $Z_s := u(t-s, X_s(x))$ for $0 \leq s \leq t$ Then $(Y_s)_{0\leq s\leq t}$ is a Martingale **Indeed** □

9.2 Schrodinger Operators

Remark 9.3. Operators of the form L+V = H where (V is the multiplication operator by V, or potential) and L is a Laplacian, and H is the Hamiltonian are called **Schrodinger Operators**.

We start with a definition

Definition 9.4. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. The *Schrödinger operator* **H** on the Hilbert space $L^2(\mathbb{R}^n)$ is given by the action

$$\psi \Rightarrow -\nabla^2\psi + V(x)\psi, \quad \psi \in L^2(\mathbb{R}^n).$$

This can be obviously re-written as:

$$\psi \Rightarrow [-\nabla^2 + V(x)]\psi, \quad \psi \in L^2(\mathbb{R}^n),$$

where $[-\nabla^2 + V(x)]$ is the *Schrödinger* operator, which is now called the Hamiltonian Operator, **H**.

For stationary quantum systems such as electrons in 'stable' atoms the *Schrödinger equation* takes the very simple form :

$$\mathbf{H}\psi = E\psi$$

, where E stands for energy eigenvalues of the stationary quantum states. Thus, in quantum mechanics of systems with finite degrees of freedom that are 'stationary', the Schrödinger operator is used to calculate the (time-independent) energy states of a quantum system with potential energy $V(x)$. Schrödinger called this operator the Hamilton operator, or the Hamiltonian, and the latter name is currently used in almost all of quantum physics publications, etc. The eigenvalues give the energy levels, and the wavefunctions are given by the eigenfunctions. In the more general, non-stationary, or 'dynamic' case, the Schrödinger equation takes the general form:

$$\mathbf{H}\psi = (-i)\frac{\partial\psi}{\partial t}$$

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10 Complex Analysis

When studying for a Riemann Surfaces class, I felt it was necessary to write down a few of the 'classical' Complex Analysis definitions. It is beneficial (for reference) to have these all in a short document.

10.1 Some definitions

Singularities are very interesting, the following largely comes from the excellent textbook [8]

A function can fail to be holomorphic in a variety of ways. A large class of interesting cases can be managed by exploring those singularities that are isolated in the same sense that zeroes are necessarily isolated. This excludes cases such as Log(z) or $\sqrt{z}$ near the origin, where we have to introduce branch cuts just to have a well-defined function, but does include a large and very important set of possibilities.

Definition 10.1 (Definition of an isolated singularity)**.** Suppose that we have an open U $\subset \mathbb{C}$ and that $z \in \mathbb{C}$. We define $U^* = U - z$. We say that a function f that is holomorphic on U^* has an *isolated singularity* at z. Since U is open, $N_r(z) \subset U$ for some r, and we define $N_r^*(z) = N_r(z) \cap U^*$. This is just a punctured disk, which is a special case of an annulus, so we can write down a Laurent expansion:

$$f(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{48}$$

Classification of isolated singularities The Laurent series allows us to classify isolated singularities into three types:

1. The point z is said to be a *removable singularity* if $a_{-n} = 0$ for all $n > 0$.
2. If $a_{-N} \neq 0$ but $a_{-n} = 0$ for all $n > N$ then z is a *pole of order* N.
3. If infinitely many negative terms are present, then z is an *isolated essential singularity.*

We also have a very important removable singularity theorem.

Theorem 10.2 (Riemann removable singularities theorem)**.** *If z is an isolated singularity of f(z) then it is a removable singularity if and only if there is an $r > 0$ such that $f(z)$ is bounded on $N_r^*(z)$.*

Remark 10.3. Note that one way is obvious - if the function indeed has a Taylor series it is bounded on a neighbourhood of z. For the other way, consider the negative terms in the Laurent series:

$$a_{-n} = \frac{1}{2\pi i}\int_{\phi_r} f(z)(z-z_0)^{n-1}dz \tag{49}$$

Now f is bounded on a circle centred on z_0 of radius s, for any s with $0 < s < r$, so suppose that $|f| < M$ for $|z| < r$.

$$|a_{-n}| \leq \frac{1}{2\pi} M s^{n-1} 2\pi s = M s^n \tag{50}$$

Since s can be as small as we will with $a_{-n} = 0$

Theorem 10.4 (Identity Theorem)**.** *Suppose X and Y are Riemann surfaces and $f_1, f_2 : X \to Y$ are two holomorphic mappings which coincide on a set $A \subset X$ having a limit point $a \in X$. Then f_1 and f_2 are identically equal.*

Proof. Let G be the set of all points $x \in X$ having an open neighbourhood W such that $f_1|W = f_2|W$. By definition G is open. We claim that G is also closed. For, suppose b is a boundary point of G. Then $f_1(b) = f_2(b)$ since f_1 and f_2 are continuous. Choose charts $\phi : U \to V$ on X and $\phi : U' \to V'$ on Y with $b \in U$ and $f_1(U) \subset U'$. We may also assume that U is connected. The mappings

$$g_i := \psi \circ f_i \circ \phi^{-1} : V \to V' \subset \mathbb{C} \tag{51}$$

are holomorphic. Since $U \cap G \neq \emptyset$, the Identity Theorem for holomorphic functions in $\mathbb{C}$ implies g_1 and g_2 are identically equal. Thus $f_1|U = f_2|U$. Hence $b \in G$ and thus G is closed. Now since X is connected either G $= \emptyset$ or G $=$ X. But the first case is exluded since $a \in G$ (using the Identity Theorem in the plane again). Hence f_1 and f_2 coincid on all of X. □ □

Definition 10.5. Let X be a Riemann surface and Y be an open subset of X. By a *meromorphic function* on Y we mean a holomorphic function $f : Y' \to \mathbb{C}$, where $Y' \subset Y$ is an open subset, such that the following hold:

- $Y\ Y'$ contain only isolated points
- For every point $p \in Y\ Y'$ one has

$$\lim_{x \to p} |f(x)| = \infty. \tag{52}$$

The points of $Y\ Y'$ are called the *poles* of f. The set of all meromorphic functions on Y is denoted by $\mathcal{M}(Y)$.

Remark 10.6. (a)Let (U,z) be a coordinate neighbourhood of a pole p of f with $z(p) = 0$. Then f may be expanded in a Laurent series

$$f = \sum_{v=-k}^{\infty} c_v z^v \tag{53}$$

in a neighbourhood of p. (b) $\mathcal{M}(Y)$ has the natural structure of a $\mathbb{C}$-algebra. First of all the sm and the product of two meromorphic functions $f, g \in \mathcal{M}(Y)$ are holomorphic functions at those points where both f and g are holomorphic. Then one holomorphically extends, using Riemann's Removable Singularities Theorem, f+g (resp. fg) across any singularities which are removable.

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