Interest Rate Modelling

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CHAPTER 1

Preliminaries

1. Introduction to Interest Rate Modelling

There is a one-to-one correspondence between the class Q of all probability measures equivalent to $\mathbb{P}$ and the class Λ of all $\mathbb{F}$-adapted (or $\mathbb{F}$-predictable) process λ_t satisfying

$$\mathbb{P}\left(\int_0^{T^\star} |\lambda_u|^2 \, du < \infty\right) = 1$$

and

$$\mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_{T^\star}\left(\int_0^{\cdot} \lambda_u \, dW_u\right)\right) = 1$$

Thus our correspondence is

$$Q \ni \mathbb{P}^\lambda \iff \lambda \in \Lambda.$$

Consequently,

(i) $\frac{d\mathbb{Q}}{d\mathbb{P}} = \eta_{T^\star}$
(ii)

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} \mid \mathcal{F}_t &= \eta_t^{\mathbb{Q}} \\ &= \mathbb{E}_{\mathbb{P}}\left(\eta_{T^\star} \mid \mathcal{F}_t\right) \\ &= \mathcal{E}_t\left(\int_0^{\cdot} \lambda_u \, dW_u\right)\end{aligned}$$

Theorem 1.1 (Abstract Bayes formula). *Let $\mathbb{Q} \sim \mathbb{P}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} = \eta$. Suppose that $\mathcal{G} \subset \mathcal{F}$. We then have*

$$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{G}) = \frac{\mathbb{E}_{\mathbb{P}}(\eta X \mid \mathcal{G})}{\mathbb{E}_{\mathbb{P}}(\eta \mid \mathcal{G})}.$$

Note that is $\mathcal{G} = \{\emptyset, \Omega\}$ then the formula reduces to

$$\mathbb{E}_{\mathbb{Q}}(X) = \mathbb{E}_{\mathbb{P}}(\eta X).$$

If $Q \sim \mathbb{P}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \eta_t$, for all $t \in [0, T^\star]$, then

$$\mathbb{E}_{\mathbb{Q}}(X \mid \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(\eta_{T^\star} X \mid \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_{T^\star} \mid \mathcal{F}_t)}.$$

Hence if X is $\mathcal{F}_t$ measurable for some $T \in [0, T^\star]$ then

$$\mathbb{E}_\mathbb{Q}(X \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_\mathbb{P}(\eta_T X \,|\, \mathcal{F}_t)}{\eta_t} = \mathbb{E}_\mathbb{P}(\eta_t^{-1} \eta_T X \,|\, \mathcal{F}_t)$$

EXAMPLE 1.2. If $\eta_t = \mathcal{E}_t \left(\int_0^\cdot \lambda_u \, dW_u \right)$, then

$$\mathbb{E}_\mathbb{Q}(X \,|\, \mathcal{F}_t) = \mathbb{E}_\mathbb{P}(e^{\int_t^T \lambda_u \, dW_u - \frac{1}{2} \int_0^t |\lambda_u|^2 \, du} X \,|\, \mathcal{F}_t)$$

LEMMA 1.3. *A $\mathbb{F}$-adapted and $\mathbb{Q}$-integrable process M is a $(\mathbb{Q}, \mathbb{F})$-martingale if and only if the product $M\eta$ is a $(\mathbb{P}, \mathbb{F})$-martingale.*

PROOF. $\mathbb{E}_\mathbb{Q}(M_t \,|\, \mathcal{F}_s) = M_s$, $s \le t$, so

$$M_s = \mathbb{E}_\mathbb{Q}(M_t \,|\, \mathcal{F}_s) = \frac{\mathbb{E}_\mathbb{P}(\eta_t M_t \,|\, \mathcal{F}_s)}{\eta_s}$$

□

LEMMA 1.4. *If X and Y are two processes of the form*

$$dX_t = \alpha_t \, dt + \beta_t \, dW_t$$
$$dY_t = \tilde{\alpha}_t \, dt + \tilde{\beta}_t \, dW_t$$

then the product satisfies the Itô product formula

$$d(X_t Y_t) = X_t \, dY_t + Y_t \, dX_t + d\langle X, Y \rangle_t$$

If X is of the form $dX_t = \alpha_t \, dt + \beta_t \, dW_t$ and f is of class $C^2(\mathbb{R})$, then the continuos martingale part of $Y_t = f(X_t)$ is given as

$$\int_0^t f'(X_u) \beta_u \, dW_u$$

PROPOSITION 1.5.

PROOF OF PROPOSITION 1.1. Let $\mathbb{P}^\lambda$ be equivalent to $\mathbb{P}$, so that

$$d\eta_t = \eta_t \lambda_t \, dW_t$$

and

$$\frac{d\mathbb{P}^\lambda}{d\mathbb{P}} = \eta_t$$

on $(\Omega, \mathcal{F}_t)$, $t \in [0, T^\star]$.

Define $B(t, T)$ as follows, for all $t \in [0, T]$,

$$B(t,T) = B_t \mathbb{E}_{\mathbb{P}^\lambda} \left(\frac{1}{B_T} \,|\, \mathcal{F}_t \right)$$
$$= \mathbb{E}_{\mathbb{P}^\lambda} \left(e^{-\int_t^T r_u \, du} \,|\, \mathcal{F}_t \right)$$

For i), we simply apply Girsanov's theorem, replacing dW_t by $dW_t = dW_t^\lambda - \lambda_t \, dt$ in the dynamics of r under $\mathbb{P}$.

For ii), we first recall that $Z(t,T) = \frac{B(t,T)}{B_t}$ is given by

$$Z(t,T) = \mathbb{E}_{\mathbb{P}^\lambda}\left(\frac{1}{B_T} \mid \mathcal{F}_t\right)$$

is a $(\mathbb{P}^\lambda, \mathbb{F})$-martingale.

Note that $\mathbb{F}^\lambda \neq \mathbb{F}$ in general. From Lemma 1.3, we know that $\eta_t Z(t,T)$ is a $(\mathbb{P}, \mathbb{F})$-martingale. Thus applying the predictable representation property, there exists an $\mathbb{F}$-adapted process γ_t such that

$$M_t \equiv \eta_t Z(t,T) = Z(0,T) + \int_0^t \gamma_u \, dW_u$$

for all $t \in [0,T]$. Consequently, $dM_t = \gamma_t \, dW_t$ and hence

$$dZ(t,T) = d(\eta_t^{-1} M_t) = M_t \, d\eta_t^{-1} + \eta_t^{-1} \, dM_t + d\langle \eta^{-1}, M\rangle_t$$

where

$$d\eta_t^{-1} = -\eta_t^{-1} \lambda_t \, dW_t^\lambda.$$

We obtain

$$\begin{aligned} dZ(t,T) &= \eta_t Z(t,T)\left(-\eta_t^{-1}\lambda_t \, dW_t^\lambda\right) + \eta_t^{-1}\gamma_t\left(dW_t^\lambda + \lambda_t \, dt\right) + \left(-\eta_t^{-1}\lambda_t\gamma_t\right) \, dt \\ &= \eta_t^{-1}\left(\gamma_t - M_t\lambda_t\right) \, dW_t^\lambda \end{aligned}$$

so that

$$dZ(t,T) = \tilde{b}^\lambda(t,T) \, dW_t^\lambda$$

Since $B(t,T) = B_t Z(t,T)$, using again the Itô formula we have

$$\begin{aligned} dB(t,T) &= B_t \, dZ(t,T) + Z(t,T) \, dB_t \\ &= \frac{B(t,T)}{B_t} r_t B_t \, dt + B_t \tilde{b}^\lambda(t,T) \, dW_t^\lambda \\ &= r_t B(t,T) \, dt + B(t,T) \underbrace{\frac{B_t \tilde{b}^\lambda(t,T)}{B(t,T)}}_{b^\lambda(t,T)} \, dW_t^\lambda. \end{aligned}$$

We conclude that for all $T \in [0, T^\star]$, there exists an $\mathbb{F}$-adapted process $b^\lambda(t,T)$, $t \in [0,T]$ called the volatility of the bond , such that

$$dB(t,T) = B(t,T)(r_t \, dt + b^\lambda(t,T) \, dW_t^\lambda).$$

In fact, it does not depend on the choice of λ. For simplicity, we can write $b(t,T) \equiv b^\lambda(t,T)$.

The final formula is a special case of the well known result:

$$
\begin{aligned}
dX_t &= X_t(\alpha_t\,dt + \beta_t\,dW_t) \\
&\Updownarrow \\
X_t &= X_0 e^{\int_0^t \alpha_u\,du}\mathcal{E}_t\left(\int_0^{\cdot} \beta_u\,dW_u\right) \\
&= X_0 e^{\int_0^t \alpha_u\,du} e^{\int_0^t \beta_u\,dW_u - \frac{1}{2}\int_0^t |\beta_u|^2\,du}
\end{aligned}
$$

This completes our proof of Proposition 1.1, under the assumption that $\frac{1}{B_T}$ is $\mathbb{P}^\lambda$-integrable. $\square$

There are still several issues given this pricing formula.

(i) How to compute $b(t,T)$ explicitly in terms of μ and σ under the assumptions that

$$dr_t = \mu(r_t, t)\,dt + \sigma(r_t, t)\,dW_t$$

and $\lambda_t = \lambda(r_t, t)$ is the risk premium.

(ii) How can we calibrate our short-term rate model, meaning that

$$\mathbb{E}_{\mathbb{P}^\lambda}\left(\frac{1}{B_T}\right) = B(0,T) = P(0,T).$$

The issue of pricing bonds is related to solving a backward stochastic differential equation (BSDE). The general form is

$$X_t = X_0 + \int_0^t \mu(X_u, u)\,du + \int_0^t \xi_u\,dW_u \tag{$\star$}$$

where $\mu : \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}$ is some function and ξ is some $\mathbb{F}$-adapted process. We also fix $T > 0$ and postulate that X_T is a **known** $\mathcal{F}_T$-measurable random variable.

Definition 1.6. We say that (X, ξ) solves the BSDE with terminal condition with terminal condition Y ($\mathcal{F}_T$-measurable) if:

(i) (X, ξ) satisfies $(\star)$,

(ii) $X_T = Y$.

This can be extended to cases where $\mu : \mathbb{R}\times\mathbb{R}^+\times\Omega \to \mathbb{R}$ is $\mathbb{F}$-adapted.

CHAPTER 2

Markovian Models of the Short Rate

Let $\mathbb{P}^\star$ be a martingale measure in the sense that

$$B(t,T) = \mathbb{E}_{\mathbb{P}^\star}\left(e^{-\int_t^T r_u\,du} \,|\, \mathcal{F}_t\right).$$

In particular,

$$B(0,T) = \mathbb{E}_{\mathbb{P}^\star}\left(e^{-\int_0^T r_u\,du}\right).$$

We postulate that

$$dr_t = \mu(r_t,t)\,dt + \sigma(r_t,t)\,dW_t^\star, \tag{2.1}$$

where $W^\star$ is a Brownian motion under $\mathbb{P}^\star$. The filtration $\mathbb{F}$ is any filtration such that $W^\star$ is a BM with respect to $\mathbb{F}$. We assume that (2.1) has a unique (strong) solution.

Then it known that r_t has the Markov property with respect to $\mathbb{F}$, meaning that for any bounded continuous function $h : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_{\mathbb{P}^\star}\left(h(r_t) \,|\, \mathcal{F}_s\right) = \mathbb{E}_{\mathbb{P}^\star}\left(h(r_t) \,|\, r_s\right)$$

for all $s \leq t$.

Hence

$$\mathbb{E}_{\mathbb{P}^\star}\left(e^{-\int_t^T r_u\,du} \,|\, \mathcal{F}_t\right) = v(r_t,t,T) = \tilde{v}(r_t,t)$$

suppressing the dependence on T.

Goals:

(i) Compute explicitly $v(r_t,t,T)$ for some classical models
   - (a) Merton's model
   - (b) Vasicek's model
   - (c) CIR model (Bessel process)

   using either the probabilistic approach (martingale measure) or the analytic approach (PDEs).

(ii) Represent the price of the bond as follows

$$B(t,T) = \exp\left(m(t,T) - n(t,T)r_t\right)$$

For a fixed maturity T,

$$m(\cdot,T), n(\cdot,T) : [0,T] \to \mathbb{R}$$

can also be computed using the second method by separating variables in the PDE. Note that $m(T,T), n(T,T) = 0$.

(iii) Compute explicitly the volatility $b(t,T)$ of the bond by applying the Itô formula to the function $v(r_t,t,T)$.

(iv) Extend the model to the time-inhomogenous case in order to ensure that $B(0,T) = P(0,T)$ for all $T \in [0,T^\star]$.

1. Merton's model

Assure

$$r_t = r_0 + at + \sigma W_t^\star$$

where $W^\star = W^\lambda$ for some λ. Hence

$$dr_t = a\,dt + \sigma\,dW_t^\star, \quad r_0 > 0. \tag{2.2}$$

NOTE. *The generator of the time homogenous Markov diffusion can be represented as*

$$A_r = a\frac{\partial}{\partial r} + \frac{1}{2}r^2\frac{\partial^2}{\partial r^2}.$$

PROPOSITION 2.1. *The price $B(t,T)$ is given by*

$$B(t,T) = e^{-r_t(T-t)-\frac{1}{2}a(T-t)^2+\frac{1}{6}\sigma^2(T-t)^3}. \tag{2.3}$$

Hence

$$dB(t,T) = B(t,T)\left(r_t\,dt - \sigma(T-t)dW_t^\star\right).$$

Thus we have the volatility of the bond $b(t,T) = -\sigma(T-t)$.

PROOF. It is enough to calculate $B(0,T)$ and then establish the general formula for $B(t,T)$ using the property that r_t is a time-homogenous Markov process, thus

$$B(0,T) = v(r_0,T) \Rightarrow B(t,T) = v(r_t, T-t)$$

Computation of $B(0,T)$ is done as follows:

$$B(0,T) = \mathbb{E}_{\mathbb{P}^\star}\left(e^{-\int_0^T r_u\,du}\right) = \mathbb{E}_{\mathbb{P}^\star}\left(e^{-\xi_T}\right)$$

where the distribution of ξ_T can be found explicitly. We argue that

$$\xi_T \sim N\left(r_0T + \frac{1}{2}aT^2, \frac{1}{3}\sigma^2T^3\right)$$

We have

$$\begin{aligned}
\xi_T &= \int_0^T r_u \, du \\
&= \int_0^T \left(r_0 + au + \sigma W_u^\star\right) \, du \\
&= \int_0^T (r_u + au) \, du + \sigma \int_0^T W_u^\star \, du
\end{aligned}$$

Th rest proceeds quite simply.

We then derive the dynamics of $B(t,T)$. By the Itô formula, we have that since $B(t,T) = v(r_t, t, T)$, we must have

$$dB(t,T) = r_t B(t,T) \, dt + b(t,T) B(t,T) \, dW_t^\star.$$

Note that the martingale component comes from

$$\frac{\partial v}{\partial r} dr_t$$

and

$$\frac{\partial}{\partial r} v(r_t, t, T) = -(T-t) v(r_t, t, T)$$

so that

$$\begin{aligned}
\frac{\partial}{\partial r} v(r_t, t, T) \, dr_t &= -(T-t) v(r_t t, T)(a \, dt + \sigma \, dW_t^\star) \\
&\sim -\sigma(T-t) B(t,T) \, dW_t^\star
\end{aligned}$$

We then obtain the equality $B(t,T) = -\sigma(T-t)$. In particular, $B(t,T) = 0$. $\square$

Exercise 2.2. *Apply the PDE approach to obtain* (2.3).

2. Vasicek's Model

Consider the dynamics

$$dr_t = (a - br_t) \, dt + \sigma \, dW_t^\star. \tag{2.4}$$

Lemma 2.3. *The unique solution to Vasicek's equation is*

$$r_t = r_0 e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right) + \sigma \int_0^t e^{-b(t-u)} \, dW_u^\star. \tag{2.5}$$

Proposition 2.4. *The bond price in the Vasicek model is given by*

$$\begin{aligned}
B(t,T) &= \exp(m(t,T) - n(t,T) r_t) \\
n(t,T) &= \frac{1}{b}\left(1 - e^{-b(T-t)}\right)
\end{aligned}$$

and $m(t,T)$ is also known explicitly.

The volatility of the bond satisfies

$$b(t,T) = -\sigma n(t,T) = -\frac{\sigma}{b}\left(1 - e^{-b(T-t)}\right)$$

and

$$dB(t,T) = B(t,T)\left(r_t\,dt - \sigma n(t,T)\,dW_t^\star\right).$$

THEOREM 2.5 (Stochastic Fubini's theorem). *In the computation above, we obtain the following double integral*

$$\int_0^T \int_0^t e^{-b(t-u)}\,dW_u^\star\,dt = \frac{1}{b}\int_0^T \left(1 - e^{-b(T-u)}\right) dW_u^\star.$$

To obtain this result, we must use the stochastic Fubini theorem

$$\int_0^T \int_0^t f(t,u)\,dW_u^\star\,dt = \int_0^T \int_u^T f(t,u)\,dt\,dW_u^\star$$

where f is a continuous function.

2.1. PDE Approach to Vasicek's model. We can either use some known results or provide some simple arguments.

We start by postulating that $B(t,T) = v(r_t,t,T)$ where $v \in C^{2,1}(\mathbb{R}\times[0,T\star],\mathbb{R})$. On the other hand, we may apply the Itô formula and obtain

$$dv(r_t,t,T) = \left(\frac{\partial r}{\partial t} + \mu(r_t,t)\frac{\partial v}{\partial r} + \frac{1}{2}\sigma^2(r_t,t)\frac{\partial^2 v}{\partial r^2}\right)dt + \sigma(r_t,t)\frac{\partial v}{\partial r}dW_t^\star.$$

On the other hand, from Proposition 1.5 we have

$$dB(t,T) = dv(r_t,t,T) = r_t v(r_t,t,T)\,dt + b(t,T)v(r_t,t,T)\,dW_t^\star.$$

This means that

$$\underbrace{\left(\frac{\partial r}{\partial t} + \mu\frac{\partial v}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial r^2} - r_t v\right)dt}_{A_t} = \underbrace{\left(b(t,T)v - \sigma\frac{\partial v}{\partial r}\right)dW_t^\star}_{M_t}.$$

LEMMA 2.6. *If $(M_t)_{t\in[0,T^\star]}$ is a continuous local martingale and a process of finite variation then $M_t = M_0$ for $t \in [0,T^\star]$.*

Since r_t is a Gaussian process, we note that the unknown function should necessarily satisfy the following pricing PDE for $v = v(r_t,t,T)$,

$$\begin{cases} \frac{\partial r}{\partial t} + \mu\frac{\partial v}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial r^2} - r_t v = 0 \\ v(r_t,T,T) = h(r_t). \end{cases}$$

For the bond maturing at T, we set $h(r) = 1$.

To solve this PDE in the Vasicek case, we postulate that

$$v(r_t,t,T) = e^{m(t,T) - n(t,T)r_t}$$

and derive a system of two ODEs satisfied by the function m and n.

3. Valuation of Bond Options

Consider a European call option on a U-maturity zero-coupon bond with expiry T and strike K where $t \leq T < U$ and $K > 0$. The payoff at time T equals

$$C_T = (B(T,U) - K)^+ = (B(T,U) - KB(T,T))^+$$

We postulate that

$$\begin{aligned} C_t &= B_t \mathbb{E}_{\mathbb{P}^\star}\left(B_T^{-1} C_T \,|\, \mathcal{F}_t\right) \\ &= \mathbb{E}_{\mathbb{P}^\star}\left(e^{-\int_t^T r_u\, du}\left(v\left(r_T, T, U\right) - K\right)^+\right) \end{aligned}$$

The idea is to change the martingale measure $\mathbb{P}^\star$ to another probability measure $\mathbb{Q}$ such that

$$\begin{aligned} C_t &= B(t,T)\mathbb{E}_{\mathbb{Q}}\left(C_T \,|\, \mathcal{F}_t\right) \\ &= B(t,T)\mathbb{E}_{\mathbb{Q}}\left(\left(F_t\xi - K\right)^+ \,|\, \mathcal{F}_t\right) \end{aligned}$$

where $F_t = \frac{B(t,U)}{B(t,T)}$. The measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ and it is chosen in such a way such that $(F_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale.

Alternatively, consider a claim $X = C_T$ maturing at time T. Then

$$\begin{aligned} C_t &= B_t \mathbb{E}_{\mathbb{P}^\star}\left(\frac{C_T}{B_T} \,|\, \mathcal{F}_t\right) \\ \Phi_t(X) &= B_t \mathbb{E}_{\mathbb{P}^\star}\left(\frac{X}{B_T} \,|\, \mathcal{F}_t\right) \end{aligned}$$

EXAMPLE 2.7. In the context of equity options this approach yields the following representation of the price of a call option:

$$C_t = S_t \hat{P}\left(S_T > K \,|\, \mathcal{F}_t\right) - KB(t,T)\mathbb{P}^\star\left(S_T > K \,|\, \mathcal{F}_t\right)$$

where

$$\begin{aligned} &\frac{B_t}{S_t} \text{ is a } \hat{P}\text{-martingale} \\ &\frac{S_t}{B_t} \text{ is a } P^\star\text{-martingale} \end{aligned}$$

If $B_t = e^{rt}$ (deterministic) then $\hat{P} = P^\star$.

4. The CIR Model

We postulate that

$$dr_t = (a - br_t)\, dt + \sigma\sqrt{r_t}\, dW_t^\star,$$

where $a, b\sigma$ are positive constants. Using Yamada-Watanabe theorem, we obtain uniqueness and existence of solutions. A suitable comparison theorem tells us that if $r_0 > 0$ then $r_t \geq 0$ for $t \in [0, T]$. It is known that the solution r to the CIR equation is related to the Bessel process. It is known that

(i) $B(t,T) = e^{m(t,T)-n(t,T)r_t}$ where m and n can be computed explicitly using the PDE approach.
(ii) The price of a call option can be computed explicitly using the probabilistic approach.

One can prove that

$$C_t = B(t,U)\Phi_1(B(t,U), B(t,T), t, T, U) - KB(t,T)\Phi_2(B(t,U), B(t,T), t, T, U)$$

where Φ_1, Φ_2 are given explicitly in terms of the distribution of a Bessel process.

5. Calibration

We denote by $\hat{B}(0,T)$ the market price of a zero coupon bond with maturity T. We assume that

$$\hat{B}(0,T) = e^{-\int_0^T \hat{f}(0,u)\,du}$$

where the instantaneous forward rate is a differentiable function such that

$$\hat{f}_T(0,t)$$

exists for $t \in 0, T$. In general, we can fit to market data a model of the form

$$dr_t = (a(t) - br_t)\,dt + \sigma r_t^{\beta}\,W_t^{\star}$$

for $\beta \in [0,1]$.

Proposition 2.8. *Let $\beta = 0$. Then the model fits the market data if and only if $a(t) = \hat{f}_T(0,t) + h'(t) + b(\hat{f}(0,t) + h(t))$ where*

$$h(t) = \frac{\sigma^2\left(1 - e^{-bt}\right)^2}{2b^2}.$$

It is essential here to assume that the function $\hat{f}(0,T)$ is differentiable with respect to T. If we wish to produce a model such that $f(0,T) = \hat{f}(0,T)$.

CHAPTER 3

Lecture 1: Pecatti

1. Introduction

The final exam is a 2 hour written exam which involves a combination of computation and the construction of models. In this course we will describe some of the main developments in interest-rate modelling since Black and Scholes (1973) and Merton's (1973) original articles on the pricing of equity derivatives. In particular, we will focus on continuous- time, arbitrage-free models for the full term structure of interest rates. Other models which model a limited number of key interest rates or which operate in discrete time (for example, the Wilkie (1995) model) will be considered elsewhere. Here we will describe the basic principles of arbitrage-free pricing and cover var- ious frameworks for modelling: short-rate models (for example, Vasicek, Cox- Ingersoll-Ross, Hull-White); the Heath-Jarrow-Morton approach for modelling the forward-rate curve; and finally market models. The course works through various approaches and models in a historical sequence. Partly this is for history's sake, but, more importantly, the older models are simpler and easier to understand. This will allow us to build up gradually to the more up to date, but more complex, modelling techniques. The definitive graduate textbook for this course is [**?**] There are plenty of other references including the books by the following academics and academic-neophytes[**?**, **?**, **?**, **?**, **?**] or for a more 'practical' approach [**?**] and an extensive review of Term Structure Models is [**?**] or the book chapter [**?**]

2. Preliminary remarks

Definition 3.1. Interest: Fee paid by the borrower to the lender, for having borrowed money over a certain period of time. (One says the lender invests the money) Interest rates are computed by means of:

(1) A nominal value M, expressed by ₤,£ , $
(2) A time length Δ expressed in some time unit: days, years, etc
(3) A positive parameter R $>$ 0, the 'interest rates'

Remark. *TFAE:*

- *A future value M, invested over some time unit M'£ $>$ M*

- *The interest rate generated by M invested over Δ is (M'-M) $>$ 0 at rate R.*

DEFINITION 3.2 (Different Systems).
- The ratio R is *simple*, if $\Delta \leq 1$ and the future value of M is invested over Δ is FV = M(1 + R Δ), where the interest is MRΔ
- The rate R is 'complicated', if Δ = 1,2,3,..., hen the future value of M£ over Δ is $FV = M(1+R)^{\Delta}$

REMARK. *The **actual value** of M£ at t is T (t <T), associated with the short rate r is present value $PV = Me^{-r(T-t)}$*

3. Building models

We want to build a model of a financial market in continuous time, with emphasis on fixed-income products (Financial products providing deterministic cashflow).
Assumption H:

(1) No transaction cost
(2) Financial products are infinitely divisible
(3) Short selling is available
(4) No liquidity risk
(5) Continuous time
(6) Prices are linear

DEFINITION 3.3.
(1) A portfolio π is a combination of financial assets held by a single investor.
(2) The value of a portfolio π at time t is $\pi(t)$ = the sum of the value of components in t
(3) We say that the market satisfies a static **No Arbitrage** (N.A.) property, if $\forall \pi, \pi^{'}$, the following are held

$$\pi(T) \leq \pi^{'}(T) \to \forall t \leq T, \pi(t) \leq \pi^{'}(t) \tag{3.1}$$

We call 3.1 the *condition to verify N.A.*

3.1. Zero Coupon bonds and Associated Interest rates.

Before we begin with definitions of Zero coupon bonds, it is worth us including some of the nomeclature from the financial industry. Since some of these definitions are easily forgotten!

DEFINITION 3.4.
- spot market: *immediately* exercised trades (notice value 'date').
- fixed income: interest rate (IR) trading.
- money market: IR products with maturity in £ 1y.

- bond market: maturity >1y (government bonds, corporate bonds).
- swap market: interbank IR trading >1y (has *spot market qualities*).
- discount product: IR instrument, traded at a rebate w.r.t. its nominal value (discount bond, zero coupon bond, zero bond).
- coupon bond: instrument with periodic interest payments (coupons).
- unconditional exerc.: the trade must be completed by both counterparties in any case (forwards, futures).
- conditional exercise: the option holder has the right to chose whether (or, when) to exercise the trade, the option writer must comply (option contracts; but notice generalised contingent claims).
- derivative: general term for financial instruments which are derived from underlying (usually simpler) financial instruments.
- underlying: short for underlying instrument of a derivative instrument.
- maturity: calendar date on which a forward (or futures, or optional) trade must (or can) be exercised, or a coupon (or the principal) of a bond must be paid.
- spot price: current market value of an asset (underlying instrument).
- forward price: currently determined price to be paid (by the long c/party) in a forward contract for delivery (by the short counterparty) of the underlying asset at maturity (known today).
- futures price: dto. for a futures contract (known today). (Usually very close or equal to forward price.)
- future price: actual spot price of the asset in the future (unknown today).
- payoff: value of a derivative contract at maturity.
- Forward contracts: difference between forward price at conclusion of contract and spot price at maturity

There are two major types of traded financial products, those over the counter and those traded on the various financial exchanges around the world.

exchange traded product: financial instrument offered by a futures or options exchange which acts as intermediary to all counterparties

- standardised contract specifications no individual negotiation of product features
- margin account system, low counterparty risk
- usually highly liquid trading
- innovation possible
- example: futures contract

OTC product: a trade where two c/parties deal directly without intermediary.

- no strict standardised features, but standard conventions exist (for some markets)
- all contract specifications negotiated individually
- counterparty risk depends on (the credit rating of) your counterparty
- liquidity ranges from very high (FX spot and derivatives trading) to very low (structured/complex deals)
- highly innovative products
- example: forward contract

Definition 3.5. A **forward contract** is an agreement between two entities (counterparties) which commits both buyer (long position) and seller (short position) at time t ('today'), to buy resp. to sell a particular object S (underlying) at a certain future time (maturity) T > t at a predetermined price K (forward price).

Remark.
- *The Gain for the long position in the contract equals the loss for the short position and vice versa (zero sum game).*
- *The deal must be 'fair' to both parties. This is achieved by choosing the forward price K such that the contract has no value at its conclusion. Our task is the determination of the 'fair' forward price K.*

Definition 3.6. A zero bond is a financial instrument without running interest payments and only a single cash flow at maturity. Zero bonds are issued (and traded) below their nominal value (redemption value).

Or alternatively, and equivalently

Definition 3.7. Let T >0, a zero coupon bond with the maturity is a contract guarranteeing the owner 1*$* at time T.

Definition 3.8. A coupon bond is a financial instrument which pays periodic interest, plus once at maturity the principal amount (nominal value).

Let us introduce some notation to handle Zero-coupon bounds

Definition 3.9. Let P be a zero-coupon bound with the maturity T. $P(t, T)$ = the price of P at time $t < T$ value. Plainly $P(T, T) = 1$

We have some assumptions

Remark.
1. *There exists a market which satisfies assumption H, where bonds of any maturity are sold and bought*
2. $0 \leq P(t, T) \leq 1$
3. *$T \to P(t,T)$ the term structure of zero coupon bond prices. Smooth ($T \to P(t,T)$) - for every fixed T. Differentiable and continuous*

3.2. Relation with Forward Rate Agreement (FRA).

Definition 3.10. A Forward Rate Agreement is an over-the-counter contract between two parties specificying an interest rate (fixed in a future period) to be applied at a future date.

Remark. *We can use Zero-coupon bonds to simulate a FRA For instance, consider dates* $t < T < S$ *We have the following strategy at t:sell one T-bond and buy* $\frac{P(t,T)}{P(t,S)}$ *s-bond At T: pay 1\$ At S: Touch* $\frac{P(t,T)}{P(t,S)} \cdot P(s,S) = \frac{P(t,T)}{P(t,S)}$ *in* $S \geq$ *1. Since* $T \to P(t,T)$ *is non-increasing*

CHAPTER 4

Lecture 2: Pecatti

1. Lecture 2 on Interest Rate Models

Let us outline some assumptions, let us assume that the market is *smooth* (meaning that it follows assumption H). We can sell and buy zero-coupon bonds for all values up to T. $\forall t < T$; $P(t,T)$ = price in t of a zero-coupon bond with maturity T.

Remark. • $P(t,T) \leq 1$ $T \Rightarrow P(t,T)$ *non-increasing;* $\forall t < T$
• $P(T,T) = 1$

Remark. *We can use Zero-coupon bonds to simulate a FRA For instance, consider dates $t < T < S$ We have the following strategy at t:sell one T-bond and buy $\frac{P(t,T)}{P(t,S)}$ s-bond At T: pay 1\$ At S: Touch $\frac{P(t,T)}{P(t,S)} \cdot P(s,S) = \frac{P(t,T)}{P(t,S)}$ in $S \geq 1$. Since $T \rightarrow P(t,T)$ is non-increasing*

This operation is equivalent to *decide in t to invest 1\$ in T, and receive* $\dfrac{P(t,T)}{P(t,S)}$£in S We have the following equation

$$\begin{vmatrix} T & \rightarrow S \\ 1\$ & \dfrac{P(t,T)}{P(t,S)}\$ \end{vmatrix} \tag{4.1}$$

Problem: How do we describe the above 4.1 in terms of interest rates? $t < T < S$

(1) The **simple forward rate** associated with $[T,S]$, prevailing at t, is the quantity $F(t;T,S)$ satisfying $\frac{P(t,T)}{P(t,S)} = 1 + F(t;T,S)(S-T)$
$F(tlT,S) = \dfrac{1}{S-T}\left\{\frac{P(t,T)}{P(t,S)}\right\}$
(2) The *simple spot rate* for [t,T] is given by $F(t;t,T) = \dfrac{1}{T-t}\left\{\dfrac{1}{P(t,T)} - 1\right\}$
$1 = P(t,T)\left\{1 + F(t,t,T)(T-t)\right\}$
(3) t < T < S
The (forward) **continuous compounded** interest rate for [T,S] prevailing in t, noted $R(t;T,S)$
$\dfrac{P(t,T)}{P(t,S)} = \exp R(t;T,S)(S-T)$
$R(t;T,S) = \dfrac{-logP(t,S) - logP(t,T)}{S-T}$

(4) **Continuously compounded interest rate** for [t,T]
R(t,T) = R(t;t, T) = $\dfrac{-log(P(t,T))}{T-t}$
$(1 = P(t,T)e^{R(t,T)(T-t)})$

(5) **Instantaneous forward rate** with maturity T prevailing in t < T
$f(t,T) = \lim_{S \downarrow T} R(t;T,S) = \dfrac{-\partial}{\partial T} logP(t,T)$ This simplifies to

$$P(t,T) = \exp \int_t^T f(t,u)du \tag{4.2}$$

REMARK. *In fact 4.2 is the starting point of the Heath-Jarrow-Morton models which are very important in interest rate models.*

(6) **Short Rate** (Short time frame)
$r(t) = \lim_{T \downarrow t} R(t,T)$
$= \lim_{T \downarrow t} R(t;t,T) = -\dfrac{\partial}{\partial T} logP(t,T)|_{T=t}$

REMARK. *One can model* $\{r(t) : t \geq 0\}$ *a.s as a stochastic process (diffusion). These are the so-called 'short-rate models'*

(1) Vasicek
(2) Cox-Ingensell-Ross

2. Coupons; floating rate notes; Swaps

REMARK. *Let us recall the No Arbitrage(NA) property* $\pi, \pi^{'}, \ \pi(t), \pi^{'}(t), \ t >0$ $\forall t < T$
$(\pi(T) \leq \pi^{'}(T)) \longrightarrow \pi(t) \leq \pi^{'}(t)$ *(a)* ***Coupon bonds*** *Financial products composed of*

(1) $T_0 < T_1 < \cdots < T_n$ *= maturity (dates)*
(2) sequence of deterministic cashflows $c_1\$, c_2\$, \cdots, c_n\$$
(3) A 'nominal value' N£
Description: *The holder pays some price at* $t < T_0$ *and receives* $c_i\$ \ \forall T_i$, $i = 1, \cdots, n-1$ *and* $(c_n + N)\$$ *in* T_n

CHAPTER 5

The General Framework of Short Rate Models

1. General class of models

Let us consider $(r, \mathbb{F}, \mathbb{F}_t, \mathbb{P})$, $W_t \left\{W_t^1, \cdots, W_t^d\right\}$, $\mathbb{F}_t = \sigma\left\{W_u : u \leq t\right\}$

Remark. *Where* $\mathbb{P}$*is what is termed the historical measure*

- Short rate $\{r(t) : t \geq 0\}$ such that

$$r(t) = r(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dW_s \tag{5.1}$$

Were both b and σ are $\mathbb{F}_t$ - progressive measurable processes. And dW_s is a d-dimension *Brownian Motion*(B.M.) We assume that $\int_0^T \left(|b(s)| + \|\|\|_{\mathbb{R}_d}^2\right) < +\infty$ a.s. ∀T

- $\exists \mathbb{Q} \cong \mathbb{P}$ such that $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{\int_0^\infty \gamma(s)dW_s - \dfrac{1}{2}\int_0^\infty \gamma^2(s)ds\right\}$

Where $\gamma(s)$ is a $\mathbb{F}_t$ - progressive measurable process, and the above integrals are of Girasnov type and satisfy the Novikov condition is a $(\mathbb{F}_t, \mathbb{Q})$-martingale in $[0, T]$ $\forall T < +\infty$

$$\frac{P(t,T)}{S_0(t)} = \frac{P(t,T)}{\exp(\int_0^t r(s)ds} \tag{5.2}$$

where $S_0(t)$ is the Money Market Account, and $P(t, T)$ price in t of a zero-coupon bond with maturity T

$\mathbb{Q}$ is an equivalent *Martingale Measure*(M.M) for the model, and this implies the **First Fundamental Theorem of Arbitrage**

Theorem 5.1. *Every restricted market* $(P(\cdot, T_1); \cdots; P(\cdot, T_m), S_0(t))$ *m-finite is arbitrage free*

We can derive the fundamental relation $\dfrac{P(t,T)}{S_0(t)} = \mathbb{E}_{\mathbb{Q}}\left\{\dfrac{1}{S_0(T)}|\mathbb{F}_t\right\}$ [1] $\Rightarrow P(t,T) = \mathbb{E}_{\mathbb{Q}}\left\{e^{-\int_0^T r(s)ds}|\mathbb{F}_t\right\}$

Definition 5.2. The short rate, r_t , is the (continuously compounded, annualized) interest rate at which an entity can borrow money for an infinitesimally short period of time from time t. Specifying the current short rate does not specify

[1] $P(T,T) = 1$

the entire yield curve. However arbitrage arguments show that, under some fairly relaxed technical conditions, if we model the evolution of r_t as a stochastic process under a risk-neutral measure $\mathbb{Q}$ then the price at time t of a zero-coupon bond maturing at time T is given by

:$P(t,T) = \mathbb{E}\left[\exp\left(-\int_t^T r_s\,ds\right)\middle|\mathcal{F}_t\right]$

where $\mathcal{F}$ is the natural filtration for the process.

1.1. Dynamics of Short Rate Models. We will consider $t \Rightarrow P(t,T)$ (under $\mathbb{P}$, under $\mathbb{Q}$)

(1) Due to Girasnov's Theorem $W_t - \int_0^t \gamma(s)ds =: W_t^*$ is a $(\mathbb{Q}, \mathbb{F}_t)$- d-dimensional Brownian Motion. So: $dr_t = r(0) + \int_0^t \sigma(s)dW_s^* + \int_0^t (b_s + \sigma(s)\gamma^T(s))ds$
Note that $dW_s^* = dW_s - \gamma^T(s)ds$ and looking at $\int_0^t (b_s + \sigma(s)\gamma^T(s))ds$ we see that the drift has changed.

(2) For every $T < +\infty$ $\exists\mathbb{F}_t$-progressive process $\{v(t,T) : t \leq T\}$ such that $\dfrac{d\mathbb{P}(t,T)}{P(t,T)} = r(t)dt + v(t,T)dW_t^*$
Indeed P(t,T) $= e^{\int_)^t r(s)ds}\mathbb{E}_{\mathbb{Q}}\left\{\dfrac{1}{S_0(T)}|\mathbb{F}_t\right\} = e^{\int_0^t r(s)ds}M_t^{(T)}$ since $M_t^{(T)}$ is a $(\mathbb{Q}, \mathbb{F}_t)$-martingale, $\exists\mathbb{F}_t$ -progressive process $(t \leq T)$ $\{h(t,T) : t \leq T\}$ such that $M_t^{(T)} = M_0^{(T)} + \int_0^t h(s,T)dW_s^*$ where $\{h(t,T) : t \leq T\}$ is a d-dimensional object.
d$\mathbb{P}$(t,T) = d $\left\{e^{\int_0^T r(s)ds} \times M_t^{(T)}\right\} = M_t^{(T)}d\left\{e^{\int_0^T r(s)ds}\right\} + e^{\int_0^T r(s)ds}dM_t^{(T)}$

$$= P(t,T)r(t)dt + e^{\int_0^T r(s)ds}h(t,T)dW_t^* = P(t,T)r(t)dt + P(t,T)v(t,T)dW_t^*$$

with $v(t,T) = \dfrac{e^{\int_0^T r(s)ds}h(t,T)}{P(t,T)}$
As a consequence we can say $P(t,T) = P(0,T)\exp\int_0^t r(s)ds + \int_0^t v(s,T)dW_s^* - \frac{1}{2}\int_0^t v^2(s,T)ds$ $= P(0,T)S_0(t)Z_t^{(T)}$, where $Z_t^{(T)} := \exp\int_0^t r(s)ds + \int_0^t v(s,T)dW_s^* - \frac{1}{2}\int_0^t v^2(s,T)ds$ can be thought of as a 'pertubation' of a 'time series' and is also an $\mathbb{F}_t$-martingale under $\mathbb{Q}$Also $\dfrac{P(t,T)}{S_0(t)} = P(0,T)Z_t^{(T)} := \tilde{P}(t,T)$ $\dfrac{d\tilde{P}(t,T)}{\tilde{P}(t,T)} = v(t,T)dW_t^*$

(3) Under $\mathbb{P}$?

$$\frac{d\mathbb{P}(t,T)}{\mathbb{P}(t,T)} = r(t)dt + v(t,T)dW_t^* \tag{5.3}$$

Note: $dW_t^* = \left\{dW_t - \gamma^T(t)dt\right\} = \underbrace{\left\{r(t) - v(t,T)\gamma^T(t)\right\}}_{:=\alpha(t,T)}dt + v(t,T)dW_t$

P(t,T) $= P(0,T)\exp\int_0^t \alpha(t,T)ds + \int_0^t v(s,T)dW_s - \frac{1}{2}\int_0^t v(s,T)^2ds$

$$\tilde{P}(t,T) = P(0,T)\exp - \int_0^t v(s,T)\gamma^T(s)ds + \int_0^t v(s,T)dW_s - \frac{1}{2}\int_0^t v(s,T)^2ds$$

Remark. *Recall* $-\gamma =$ *Market Price of Risk*

1.2. Short rate diffusions. Let us consider Short rate diffusions $(\Omega, \mathbb{F}_t, \mathbb{P}); \underbrace{\mathbb{Q}}_{\gamma}, d = 1, W_t = \{W_t : t \geq 0\}$ **Assume**

$$\forall s \geq 0 \begin{cases} dr_t = b(t, r_t)dt + \sigma(t, r_t)dW_t^* \\ r_0 = \text{constant} \end{cases} \tag{5.4}$$

with values in Z $\mathbb{R}$ or $\mathbb{R}_+$. Take b, σ to be Lipschitz formulas Before proceeding let us recall what a Lipschitz Function is

DEFINITION 5.3. Given two *metric space*s (X, d_X) and (Y, d_Y), where d_X denotes the *metric* on the set X and d_Y is the metric on set Y (for example, Y might be the set of *real numbers* $\mathbb{R}$with the metric d_Y(x, y) = |x − y|, and X might be a subset of $\mathbb{R}$), a function

$$f : X \to Y$$

is called 'Lipschitz continuous' if there exists a real constant K 0 such that, for all x_1 and x_2 in X,

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2).$$

Any such K is referred to as 'a Lipschitz constant' for the function f. The smallest constant is sometimes called **the (best) Lipschitz constant**; however in most cases the latter notion is less relevant. If K = 1 the function is called a *short map*, and if $0 \leq K \leq 1$ the function is called a *contraction mapping*.

The inequality is (trivially) satisfied if $x_1 = x_2$. Otherwise, one can equivalently define a function to be Lipschitz continuous *if and only if* there exists a constant K ≥ 0 such that, for all $x_1 \neq x_2$,

$$\frac{d_Y(f(x_1), f(x_2))}{d_X(x_1, x_2)} \leq K.$$

For real-valued functions of several real variables, this holds if and only if the absolute value of the slopes of all secant lines are bounded by K. The set of lines of slope K passing through a point on the graph of the function forms a circular cone, and a function is Lipschitz if and only if the graph of the function everywhere lies completely outside of this cone.

A function is called 'locally Lipschitz continuous' if for every x in X there exists a *neighborhood* U of x such that f restricted to U is Lipschitz continuous. Equivalently, if X is a *locally compact* metric space, then f is locally Lipschitz if and only if it is Lipschitz continuous on every compact subset of X. In spaces that are not locally compact, this is a necessary but not a sufficient condition.

More generally, a function f defined on X is said to be 'Hölder continuous' or to satisfy a *Hoelder condition* of order $\alpha > 0$on X if there exists a constant M > 0 such that

$$d_Y(f(x), f(y)) \leq M d_X(x, y)^\alpha$$

for all x and y in X. Sometimes a Hölder condition of order is also called a 'uniform Lipschitz condition of order' $\alpha > 0$.

If there exists a $K \geq 1$ with

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$$

then f is called 'bilipschitz' (also written 'bi-Lipschitz'). A bilipschitz mapping is *injective function*, and is in fact a *homeomorphism* onto its image. A bilipschitz function is the same thing as an injective Lipschitz function whose *inverse function* is also Lipschitz. Surjective bilipschitz functions are exactly the isomorphisms of metric spaces.

1.3. How do we relate P(t,T) to a PDE. In Financial Modelling there are said to be two main approaches, the 'martingale' approach and the PDE approach. We know for example that the Black Scholes equation is a Parabolic PDE. Thanks to some of the theorems that one encounters in a good PDE Lecture Course or in a textbook, there is a range of techology to solve PDEs. This is not a 'PDE and analysis' course, so we won't venture too far into the study of PDE, but we will make some remarks about how these PDEs are solved.

$$\text{Let T be fixed} P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)\middle|\mathcal{F}_t\right] \overset{=}{\text{Markov Property of SDE}} \tag{5.5}$$

$$\mathbb{E}_{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\,ds\right)|r_t\right] = F(t, r_t \tag{5.6}$$

For some function $F(\cdot, \cdot)$ deterministic with random arguments, with

$$F(t, x) =: \mathbb{E}_{\mathbb{Q}}\left\{\exp\left(-\int_t^T r(s)\,ds\right)|r_t = x\right\}$$

Theorem 5.4. *Fix $T > 0$, and adopt the previous notation. Assume that $F : [0, T] \times Z$ with $\Theta : Z \to \mathbb{R}_+$is a solution of*

$$\begin{cases} \dfrac{\partial F}{\partial t}(t, r) + b(t, r)\dfrac{\partial}{\partial r}F(t, r) + \dfrac{1}{2}\sigma^2(t, r)\dfrac{\partial^2}{\partial r^2}F(t, r) = rF(t, r) \\ F(T, r) = \Theta(r) \end{cases} \tag{5.7}$$

Then: *$t \Rightarrow M_t := F(t, r_t)\exp - \int_0^t r(s)ds$ is a $(\mathbb{F}_t, \mathbb{Q})$- local martingale on [0,T]. If moreover: either*

(1) M_t is u.i. or

(2) $\mathbb{E}_{\mathbb{Q}}\left[\int_0^T (\partial_r F(t,r_t)\exp - \int_0^t r(s)ds\sigma(t,r_t))^2 dt\right]$

Then $F(t,r_t) = \mathbb{E}_{\mathbb{Q}}(M_T|\mathbb{F}_t) = \mathbb{E}_{\mathbb{Q}}\left[F(T,r_t)\exp - \int_0^T r(s)ds|\mathbb{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\Theta(r_T)\exp - \int_0^T r(s)ds|\mathbb{F}_t\right]$
Therefore we can conclude:

$$F(t,r_t) = \mathbb{E}_{\mathbb{Q}}\left[\Theta(r_T)\exp - \int_0^T r(s)ds|r_t\right]$$

How to use this theorem:

(1) T, b σ are given; we want to compute $P(t,T) = \mathbb{E}_{\mathbb{Q}}\left[\Theta(r_T)\exp - \int_0^T r(s)ds|r_t\right]$
(2) Find a solution F(t,x) to 5.7 in the case $\Theta \equiv 1$
(3) Check 5.4 (i) or (ii)
(4) If true P(t,T) = $F(t,r_t)$ and this solution is unique

PROOF. We apply Ito's formulat to M_t

$$d(F(t,r_t)\exp - \int_0^t r(s)ds = \exp - \int_0^t r(u)du dF(t,r_t) - r(t)\exp - \int_0^t r(u)du F(t,r_t)dt$$

$$= -r(t)\exp - \int_0^t r_u du F(t,r_t)dt + \exp - \int_0^t r(u)du$$

$$\left\{\frac{\partial}{\partial t}F(t,r_t)dt + \frac{\partial}{\partial r}F(t,r_t)\left[b(t,r_t)dt + \sigma(t,r_t)dW_t^*\right] + \frac{1}{2}\frac{\partial^2}{\partial r^2}F(t,r_t) \underbrace{\sigma^2(t,r_t)}_{\text{Quadratic variation}} dt\right\}$$

$$= \frac{\partial}{\partial r}F(t,r_t)\sigma(t,r_t)\exp - \int_0^t r(s)ds dW_t^*$$

So $M_t = M_0 + \int_0^t \partial_r F(s,r_s)\exp - \int_0^s r(u)du\sigma(s,r_s)dW_s^* = (\mathbb{F},\mathbb{Q})$ Local Martingale. Moreover, either of the two 5.4 are sufficient conditions for M to be true for $(\mathbb{F}_t,\mathbb{Q})$-martingale □

1.4. Some final remarks on the PDE. These are generally solved by Finite difference methods, which are found in various 'Numerical Methods for solving PDE' books. Generally these methods came out of Physics or Mechanics, so there is a rich history and an extensive mathematical technology. Closed form solutions for option prices on zero-coupon bonds can also be found in this model. In general, derivatives prices can be estimated by either numerically solving the PDE in (5.7) with appropriate boundary conditions or by using Monte-Carlo methods. An example image of a simulation using Monte-Carlo Methods is provided in the next subsection.

1.5. Vasicek Model. The model specifies that the force of interest|instantaneous interest rate follows the stochastic differential equation:

$dr_t = a(b - r_t)\,dt + \sigma\,dW_t$

where W_t is a Wiener process under the risk neutral framework modelling the random market risk factor, in that it models the continuous inflow of randomness into the system. The standard deviation parameter, σ, determines the *volatility* of the interest rate and in a way characterizes the amplitude of the instantaneous randomness inflow. The typical parameters b, a and σ, together with the initial condition r_0, completely characterize the dynamics, and can be quickly characterized as follows, assuming a to be non-negative:

- b: "long term mean level". All future trajectories of r will evolve around a mean level b in the long run;
- a: "speed of reversion". a characterizes the velocity at which such trajectories will regroup around b in time;
- σ: "instantaneous volatility", measures instant by instant the amplitude of randomness entering the system. Higher σ implies more randomness

The following derived quantity is also of interest,

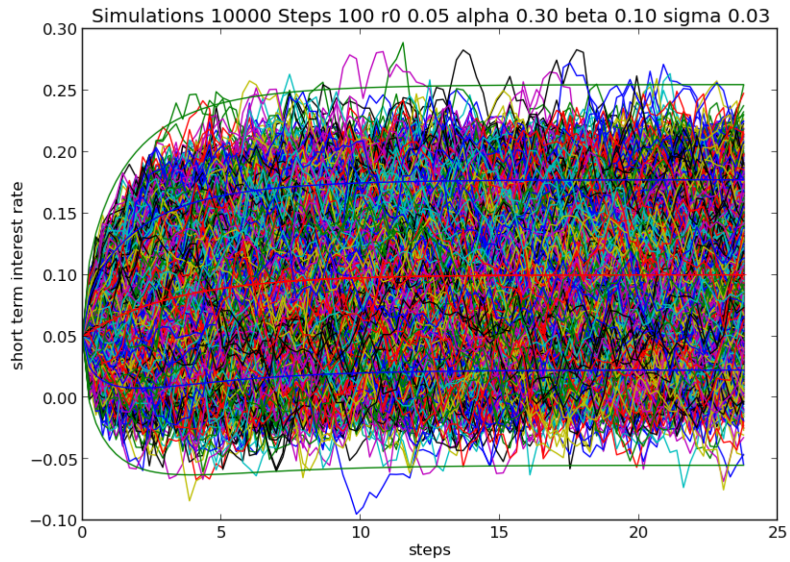
- $\sigma^2/(2a)$: "long term variance". All future trajectories of r will regroup around the long term mean with such variance after a long time.

a and σ tend to oppose each other: increasing σ increases the amount of randomness entering the system, but at the same time increasing a amounts to increasing the speed at which the system will stabilize statistically around the long term mean b with a corridor of variance determined also by a. This is clear when looking at the long term variance,

$\frac{\sigma^2}{2a}$

which increases with σ but decreases with a.

This model is an Ornstein Uhlenbeck stochastic process The following was a generation of 1000 Monte Carlo Simulations of the Vasicek Model using Python.



Simulations 10000 Steps 100 r0 0.05 alpha 0.30 beta 0.10 sigma 0.03
short term interest rate
steps
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−0.05
−0.10
0
5
10
15
20
25

CHAPTER 6

Examples of Interest Rate Models

We already introduced the Vasicek model [**?**], the dynamics of the Vasicek model [**?**]and the PDE approach to the Vasicek model[**?**] It is commonplace when studying Short rate models, to have to solve a PDE. Let us reintroduce in a slightly different notation, the Vasicek model.

$$dr(t) = d(t, r(t))dt + \sigma(t, r_t) \underbrace{BrownianMotion} dW_t^*, \ \forall s \geq 0$$
$$r(s) = x\text{deterministic initial condition}\forall t \geq s$$

This equation takes values in $Z = \mathbb{R}$ or $Z = \mathbb{R}_+$.

NOTE. *You may note that we will run into a problem here, in our previous course on Stochastic Calculus, we didn't introduce a Stochastic Differential Equation Framework, something to handle* $\mathbb{R}_+$

$$r(t) = r(s) + \int_s^t (b(u), r(u))du + \int_s^t \sigma(u, r_u)dW_u^*$$

$$\text{Assumptions}\forall s \geq 0 \begin{cases} dr_t = b(t, r_t)dt + \sigma(t, r_t) \underbrace{\text{Brownian Motion w.r.t. EMM}\mathbb{Q}} dW_t^* \\ r_s = x(\in Z); \end{cases} \qquad t \geq s$$

has a unique solution $\forall x \in Z$, with values in Z.