On Chern Simons Theory

Peadar Coyle

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Abstract

1 Introduction

One of the active areas of geometric topology at the border to mathematical physics is the theory of 'Chern-Simons' theory. The aim is to elucidate some of the techniques aimed at graduate students, and beginning researchers who know the basics of differential forms and differential manifolds. After an introduction to Lie groups and Lie algebras, principal bundles, connections and gauge transformations, we will carefully construct the Chern-Simons action and study the moduli space of its classical solutions. This yields Taubes' beautiful and influential description of Casson's invariant for homology 3-spheres via Chern-Simons Theory for which we need such concepts as differential operators and spectral flow. This naturally leads to subjects like the eta invariant and the rho invariant on the one hand as well as the quantization of the Chern-Simons action and Witten's invariants on the other. Furthermore some modern examples from Theoretical Condensed Matter physics will be included.

1.1 An overview

Let G be a semi-simple Lie group and $\mathfrak{g}$ its Lie algebra. A connection A on the trivial G-bundle over a closed oriented 3-manifold M is a $\mathfrak{g}$-valued 1-form, i.e. $A \in \Omega^1(M;\mathfrak{g})$. Chern-Simons theory is a quantum field theory in three dimensions, whose action is proportional to the Chern-Simons invariant given in [1] The celebrated work by Witten on the Jones Polynomial [7]. The other work [2, 6] are also good references for some of the material referenced.

$$cs(A{=}\frac{1}{8\pi^2}\int_M tr(A \wedge dA + \frac{1}{3}A \wedge [A \wedge A]).$$

Witten introduced Chern-Simons theory to knot theory in 1989 [7], when he described for each integer level $k \in \mathbb{Z}$ an invariant of a link $L = (L_j)$ in a 3-manifold M (and a list of finite-dimensional representations ρ_k of G associated to the link invariant L_j) as the (non-rigorous) Feynman path integral

$$Z_k(M, L) = \int_{\mathcal{A}/\mathcal{G}} \exp^{2\pi kics(A)} \prod_j tr_{\rho_j}(hol_A(L_j))dA,$$

where $\mathcal{A}$ is the space of G-connections, $\mathcal{G}$ is the space of gauge transformations. He interpreted these invariants using the axioms of Topological Quantum Field Theory (TQFT) as well as via an asymptotic expansion - the semiclassical approximation - by using the method of stationary phase. There have also been overlaps with knot Floer homology, and instantons, etc.

2 Lie groups

Lie groups are groups which are also differentiable manifolds, in which the group operations are smooth. Well-known examples are the general linear group, the unitary group, the orthogonal group and the special linear group.

Definition 2.1 *A Lie group G is a differential manifold with group structure such that the map $G \times G \to G$ given by $(g, h) \Rightarrow gh^{-1}$ is C^{∞}.*

Definition 2.2 *A Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ is a real vector space $\mathfrak{g}$ together with a bilinear operator $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (the bracket) such that for all $x, y, z \in \mathfrak{g}$,*

x,y = -[y,x] (anti-commutativity)

[x,y ,z] + [[y,z],x] + [[z,x],y] = 0 (Jacobi identity)

We will see that there is a Lie algebra associated to each Lie group, and that every connected, simply connected (i.e. the fundamental group $\pi_1(G)$ is trivial) Lie group is determined up to isomorphism by its Lie algebra. The study of such Lie groups then reduces to a study of their Lie algebras.

Definition 2.3 *Let $\varphi : M \to N$ be C^{∞}. Smooth vector fields X on M and Y on N are φ-related if $d\varphi \circ X = Y \circ \varphi$, which is short for $(d\varphi)_p(X_p) = Y_{\varphi(p)}$ for all $p \in M$.*

Exercise 2.4 *Let $\varphi : M \to N$ be C^{∞}. Let X_1 and X_2 be smooth vector fields on M, and let Y_1 and Y_2 be smooth vector fields on N. If X_i is $\varphi -$ related to Y_i, i =1,2, then $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$.*

Proof Let us consider a smooth $\varphi : M \to N$, and let X_1 and X_2 be smooth vector fields on M and let Y_1 and Y_2 be smooth vector fields on N. Let $d\varphi \circ X_1 = Y_1 \circ \varphi$ and $d\varphi \circ X_2 = Y_2 \circ \varphi$ Then $d\varphi \circ (X_1X_2 - X_2X_1)$
$\Rightarrow d\varphi \circ X_1X_2 - d\varphi X_2X_1$
$\Rightarrow Y_1 \circ \varphi(d\varphi \circ X_2) - Y_2 \circ \varphi(d\varphi \circ X_1)$
$\Rightarrow Y_1Y_2 \circ \varphi - Y_2Y_1 \circ \varphi$
$\Rightarrow (Y_1Y_2 - Y_2Y_1) \circ \varphi \Rightarrow [Y_1, Y_2] \circ \varphi$
So we have proved that X_i is φ-related to Y_i, i=1,2, then $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$

Definition 2.5 *Let G be a Lie group and $g \in G$. Left translation l_g and right translation r_g by g are diffeomorphism of G given by*

$$l_g h = gh \quad r_g = hg, \quad for\ all\ h \in G$$

A vector field X on G is called left-invariant if X is l_g-related to itself for all $g \in G$, that is

$$dl_g \circ X = X \circ l_g,$$

which is short for

$$dl_g(X_h) = X_{gh}\ for\ all\ h \in G.$$

Proposition 2.6 *Let G be a Lie group and $\mathfrak{g}$ is a set of left-invariant vector fields.*

1. *$\mathfrak{g}$ is a real vector space, and the map $\alpha : \mathfrak{g} \to T_eG$ defined by $\alpha(X) = X(e)$ is an isomorphism of $\mathfrak{g}$ with the tangent space T_eG of G at the identity. Consequently $dim(\mathfrak{g}) = dim(G)$.*
2. *Left invariant vector fields are smooth*
3. *The Lie bracket of two left-invariant vector fields is a left invariant vector field*
4. *$\mathfrak{g}$ is a Lie algebra under the Lie bracket operation of vector fields.*

Proof *It is not difficult to see that $\mathfrak{g}$ is a real vector space and that α is linear. Since $\alpha(X) = \alpha(Y)$ yields*

$$X_g = dl_g(X_e) = dl_g(\alpha(X)) = dl_g(\alpha(Y)) = dl_g(Y_e) = Y_g \quad for\ each\ g \in G,$$

α is injective. In order to see the surjectivity of α let $v \in T_eG$ and define

$$X_g = dl_g(v) \quad for\ all\ g, h \in G$$

This proves part (1).
For (2) let $X \in \mathfrak{g}$ and $f \in C^\infty(G)$. We need to show that $Xf \in C^\infty(G)$. Let $\varphi : G \times G \to G$ be a smooth map given by multiplication $\varphi(g, h) = gh$. Let i_e^1 and i_g^2 be the smooth maps $G \to G \times G$ given by $i_e^1(h) = (h, e)$ and $i_g^2(h) = (g, h)$ Let Y be any smooth vector field on G with $Y_e = X_e$. Then $[(0, Y)(f \circ \varphi)] \circ i_e^1$ is smooth and

$$[(0, Y)(f \circ \varphi)] \circ i_e^1(g) = (0, Y)_{(g,e)}(f \circ \varphi)$$

$$= 0_g(f \circ \varphi \circ i_e^1) + Y_e(f \circ \varphi \circ i_g^2)$$

$$= X_e(f \circ \varphi \circ i_g^2) = X_e(f \circ l_g)$$

$$= dl_g(X_e)f = X_g f = Xf(g)$$

which proves part (2). Since by (2), left-invariant vector fields are smooth, their Lie brackets are defined and by simple checking they are again left-invarint, which shows (3). (4) follows from (3) and by the fact that ***The vector space of all smooth vectors fields under the Lie bracket on vector fields is a Lie algebra***

Definition 2.7 *We define the Lie algebra of the Lie group G to be the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G*

Often it will be convenient to think of the T_eG as the Lie algerba of G with the Lie algebra stricture induced by the isomorphism α form Proposition 2.6(1).

Example 2.8 *The real line $\mathbb{R}$ is a Lie group under addition. The left-invariant fields are simply the constant vector fields* $\left\{\lambda(\frac{d}{dr})|\lambda \in \mathbb{R}\right\}$. *The bracket of any two such vector fields is 0.*

Example 2.9 *The Lie group* $GL(n, \mathbb{R}) = \det^{-1}(\mathbb{R}\ 0)$ *is a (differentiable) submanifold of* $\mathfrak{gl}(n\mathbb{R})$. $GL(n, \mathbb{R})$*has global coordinates* A_{ij} *which assigns to each matrix A the ij-th entry. Since* $\det(A^{-1}) = \det(A)^{-1}$ *and* $\det(AB) = \det(A)\det(B)$ *for any A,B* $\in GL(n, \mathbb{R})$, *the matrix* AB^{-1} *is invertible with the ij-th entry* $(AB^{-1})_{ij}$ *being a rational function in the entries of A and B with non-zero denominator. Therefore* $(A, B) \mapsto AB^{-1}$ *is* C^∞
Consider the isomorphisms $\alpha : \mathfrak{g} \to T_eG$ *from Proposition* 2.6*(1). Since* $T_eGL(n, \mathbb{R}) = T_e\mathfrak{gl}(n\mathbb{R})$ *and the map* $\beta : T_e\mathfrak{gl}(n\mathbb{R}) \to \mathfrak{gl}(n\mathbb{R})$ *given by* $\beta(v)_{ij} := v_{ij}$ *is a (canonical) isomorphism,* $\beta \circ \alpha$ *induces a Lie algebra structure on* $\mathfrak{gl}(n\mathbb{R})$. *We leave it as an exercise to show that the bracket agrees with the usual* $[A, B] = AB - BA$.

Taking the example above as a starting point, we can create other examples. The non-singular matrices $GL(n, \mathbb{C})$ of all n × n complex matrices $\mathfrak{gl}(n, \mathbb{C})$ is a Lie group with Lie algebra $\mathfrak{gl}(n, \mathbb{C})$. If V is an n-dimensional real vector space, a basis of V determines a diffeomorphism from End(V) to $\mathfrak{gl}(n\mathbb{R})$sending Aut(V) onto $GL(n, \mathbb{R})$. In this way End(V) is a Lie group from Lie algebra Aut(V). We get an analog example for complex case vector spaces.
The special linear group $SL(n, \mathbb{C})$, the Unitary group U(n), the special unitaroy group $SU(n, \mathbb{C})$are the most important examples for us.

2.1 Homomorphisms

Definition 2.10 *A map* $\varphi : G \to H$ *is a (Lie group) homomorphism if* φ *is both* C^∞ *and a homomorphism of groups. We call* φ *an isomorphism if* φ *is also a diffeomorphism. If* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are Lie algebras, a map* $\psi : \mathfrak{g} \to \mathfrak{h}$ *is a (Lie algebra) homomorphism if it is linear and it preserves brackets* ($\psi[X, Y] = [\psi(X), \psi(Y)]$ *for all* $X, Y \in \mathfrak{g}$). *If* ψ *is also a bijection, then* ψ *is an isomorphism.*

Let $\varphi : G \to H$ be a homomorphism. Then φ maps the identity of G to the identity of H, and the differential $d\varphi$ of φ is a linear transformation of $\mathfrak{g} = T_eG$ into $\mathfrak{h} = T_eH$. Notice that by the natural identification between Lie algebras and left-invariant vector fields $d\varphi(X)$ is the unique left-invariant vector field satisfying

$$(d\varphi(X))_e = d\varphi_e(X_e). \tag{1}$$

Theorem 2.11 *Let G and H Lie groups with Lie algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *respectively, and let* $\varphi : G \to H$ *be a homomorphism. Then*

1. *X and* $d\varphi(X)$ *are* φ*-related for each* $X \in \mathfrak{g}$.
2. $d\varphi$: $\mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra homomorphism.*

Proof The left-invariant vector fields $d\varphi(X)$ and X are φ-related to, because

$$(d\varphi(X))_{\varphi(g)} = dl)\varphi(g)d\varphi_e(X_e) = d(l_{\varphi(g)} \circ \varphi)X_e = d(\varphi \circ l_g)X_e = d\varphi(X_g).$$

To show part (2), let $X, Y \in \mathfrak{g}$. By an exercise result [X,Y] is φ-related to the left invariant vector field $[d\varphi(X), d\varphi(Y)]$, that is

$$(d\varphi([X,Y]))_e \stackrel{1}{=} d\varphi_e([X,Y]_e) = [d\varphi(X), d\varphi(Y)]_{\varphi(e)} = [d\varphi(X), d\varphi(Y)]_e$$

Therefore using the identification of left-invariant vector fields with tangent vectors at the identity, $d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)]$.[1]

2.2 Lie subgroups

Definition 2.12 *(H, ϕ) is a Lie subgroup of the Lie group G if*

1. *H is a Lie group;*
2. *(H, φ) is a submanifold of G;*
3. *$\varphi : H \to G$ is a Lie group homomorphism.*

(H, φ) is called a closed subgroup of G if $\phi(H)$ is also a closed subset of G.

Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{h} \subset \mathfrak{g}$ if [X,Y] $\in \mathfrak{h}$ for all X,Y $\in \mathfrak{h}$.

Let (H, ϕ) be a Lie subgroup of G, and let $\mathfrak{h}$ and $\mathfrak{g}$ be their respective Lie algebras. Then by Theorem (2.11) $d\varphi$ yields and isomorphism between $\mathfrak{h}$ and the Lie subalgebra $d\varphi(\mathfrak{h})$ of $\mathfrak{g}$. We will show in this section one of the fundamental theorems in Lie group theory, which asserts that there is a 1:1 correspondence between connected Lie subgroups of a Lie group and subalgebras of its Lie algebra. We need the following two propositions.

Proposition 2.13 *Let G be a connected Lie group, and let U be a neighbourhood of e. Then*

$$G = \bigcup_{n=1}^{\infty} U^n$$

where U^n consists of all n-fold products of elements of U.

Proof *Let us consider V:=$U \cup U^{-1}$ (where $U^-1 = \{g^{-1} \in U | g \in U\}$) and let*

$$H := \bigcup_{n=1}^{\infty} V^n \subset \bigcup_{n=1}^{\infty} U^n.$$

H is a subgroup of G and open in G, since $h \in H$ implies $hV \in H$. Therefore each coset of H is also open in G. Since

$$H = G \bigcup_{g \in G, gH \neq H}^{\infty} gH,$$

H is also closed. Since H is non-empty and G is connected we get $G = H$ ∎

[1] We had to be careful not to mix up notations, we had to take a detour via the tangent space at e, because of how φ-related was defined

$\mathfrak{g}$	B(X,Y)
$\mathfrak{gl}(n,\mathbb{R})$	$2\text{tr}(XY) - 2\text{tr}(X)tr(Y)$
$\mathfrak{sl}(n,\mathbb{R})$	$2\text{ntr}(XY)$
$\mathfrak{su}(n)$	$2\text{ntr}(XY)$
$\mathfrak{so}(n,\mathbb{C})$	(n-2)tr(XY)
$\mathfrak{sp}(n,\mathbb{R})$	(2n+2)tr(XY)
$\mathfrak{sp}(n,\mathbb{C})$	(2n+2)tr(XY)

Recall that a *k-dimensional distribution* $\mathcal{D}$ on a manifold M is a choice of a k-dimensional linear subspace $\mathcal{D}_p \subset T_pM$. A distribution is *smooth*, if it is locally spanned by smooth vector fields $X_1, \cdots, X_k$, or equivalently if $\mathcal{D}$ is a smooth subbundle of Tm. A smooth distribution is *involutive* if $[X, Y] \in \mathcal{D}$ for any smooth vector fields X,Y $\in \mathcal{D}$. It is *completely integrable*, if for each point $p \in M$ there is an integral manifold N of $\mathcal{D}$ passing through p, where N is intergral if

$$N \xrightarrow{i} M \; and \; di_p(T_pN) = D_p \; for \; each \; p \in N$$

A smooth coordinate chart (U, ϕ) is flat for $\mathcal{D}$, if $\phi(U) = U' \times U'' \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$, and at points of U, $\mathcal{D}$ is spanned by the first k coordinate vector fields $\dfrac{\partial}{\partial x^1}, \cdots, \dfrac{\partial}{\partial x^k}$. This implies that each slice of the form $x^{k+1} = c^{k+1}, \cdots, x^n = c^n$ for constants c^i is an integral manifold of $\mathcal{D}$. Beware that the terminology is inconsistent among various books. However, we will show the famous Frobenius' Theorem which states that terminology is of no consequence. Before that we need to recall a few definitions and prove a few lemmas.

Definition 2.14 . *Let $\mathcal{M}$ be a smooth manifold and let X be a vector field on $\mathcal{M}$. (1) An integral curve of X is a smooth curve $\gamma : J \to \mathcal{M}$ with $J \subset \mathbb{R}$ an open interval such that $\dot{\gamma}(t) = X_\gamma(t)$ for all $t \in J$. If $\theta \in J$, then $\gamma(\theta)$ is the starting point. (2) Let $\theta(p) : \mathcal{D}(p) \to \mathcal{M}$ be the maximal integral curve with starting point p. (3) The flow of X is the map $\theta : \mathcal{D} \to \mathcal{M}$ given by $\theta_t(p) = \gamma_p(t)$ for (t, p) in the domain $\mathcal{D} := \left\{(\mathcal{D}^{(p)}, p) | p \in M\right\}$. (4) X is called complete if the domain of its flow is $\mathbb{R} \times \mathcal{M}$.*

2.3 Killing forms

Consider a *Lie algebra* $\mathfrak{g}$over a field $\mathbb{K}$. Every element x of $\mathfrak{g}$defines the **adjoint endomorphism** ad(x) of $\mathfrak{g}$as

ad$(x)(y) = [x, y]$.

Now, supposing $\mathfrak{g}$is of finite dimension, the trace of the composition of two such endomorphisms defines a *symmetric bilinear form*

B(x, y) = trace(ad(x)∘ad(y))

with values in K, the **Killing form** on $\mathfrak{g}$

The Killing form for some Lie algebras $\mathfrak{g}$ are (for $X, Y \in \mathfrak{g}$):

See [3] for the Representation theory of Lie groups.

2.4 Classifying spaces

Classifying spaces are one of the more important notions of modern Algebraic Topology, particularly Homotopical Algebra, see [4] and references therein. We introduce the notion and an important theorem without proof.

Theorem 2.15 (Milnor) *Let G be a topological group. There exists a classifying space for G.*

BG is well-defined only up to 'homotopy equivalence'. In algebraic topology one considers the *classifying space* BG of a topological group G. By definition this is the quotient of a contractible space EG on which G acts freely. Thus there is a fibration $G \to EG \to BG$. For a Lie group G we can approximate EG by finite dimensional manifolds. Isomorphism classes of G bundles over a space X are in 1-1 correspondence with homotopy classes of maps from X to BG.

Example 2.16 *If $G = U(n)$ then BG is the Grassmanian of n-planes in $\mathbb{C}^\infty$, which for our purposes can be studied by taking n-planes in $\mathbb{C}^N$ for sufficiently lare N, in any given problem or calculation.*

Borel's theorem Suppose we have a spectral sequence (of vector spaces over a field k of characteristic zero) with

$$E_2^{p,q} = A^q \otimes B^p$$

where A is an exterior algebra on generators $e_i \in A^{2l_i}$, B is an algebra $B^0 = k$ and the sequence is compatible with products. Suppose $E_\infty^{p,q} = 0$ for p+q ¿ 0. Then B is a polynomial algebra on generators b_i in dim $2l_i$. The $b_i = d_{d_{l_i}} e_i$. It follows that for a compact Lie group G $H^*(BG)$ is a polynomial algebr, as above. For the classical groups G we see that $H^*(BG)$ has generators as follows:

- U(n): generators $c_i \in H^{2i}$ for i = 1,...,n
- Sp(n); generators $p_i \in H^{4i}$ for i = 1,...,n
- SO(2n+1) generators $p_i \in H^{4i}$ for i = 1,...,n

3 Topological Phases of Matter and Non-Abelian Anyons

A useful review article is [5] by Freedman, Simon et al. Let us consider some of the theory of Topological Quantum Computation. Topological quantum computation is predicated on the existence in nature of topological phases of matter. In this section, we will discuss the physics of topological phases from several different perspectives, using a variety of theoretical tools. Topological phases, the states of matter which support anyons, occur in many-particle physical systems. Therefore, we will be using field theory techniques to study these states. A canonical, but by no means unique, example of a field theory for a topological phase is Chern-Simons theory.

3.1 Chern-Simons Theory

Consider the simplest example of a TQFT, Abelian Chern-Simons theory, which is relevant to the Laughlin states at filling fractions of the form $\nu = \frac{1}{k}$, with k an odd interger. Although there are many ways to understand the Laughlin states, it is useful for us to take the viewpoint of a low-energy effective theory. Since quantum Hall systems are gapped, we should be able to describe the system by a field theory with very few degrees of freedom. Let us consider the action

$$S_{CS} = \frac{k}{4\pi} \int d^2\mathbf{r} dt \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \tag{2}$$

where k is an integer ϵ is the antisymmetric tensor. Here α is a U(1) gauge field and indices μ, ν, ρ atake the values 0 (for time direction), 1,2 (space-directions). The action represents the low-energy degrees of freedom of the system, which are purely topological.

The Chern-Simons gauge field α in (2) is an emergent degree of freedom which encodes the low-energy physics of a quantum Hall system. Although in this particular cases, it is simple-related to the electronic charge density, we will also be considering systems in which emergent Chern-Simons tauge fields cannot be related in a simple way to the underlying electronic degrees of freedom.

In the presence of an external electromagnetic field and quasiparticles, the action takes the form:

$$S = S_{CS} - \int d^2\mathbf{r} dt \left(\frac{1}{2\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho + j^{qp}_\mu a_\mu \right) \tag{3}$$

where j^{op}_μ is the quasiparticle current, $j^{qp}_0 = \rho^{qp}$ is the quasi-particle density $\mathbf{j}^{qp} = (j^{qp}_1, j^{qp}_2)$ is the quasiparticle spatial current, and A_μ is the external electromagnetic field. We will assume that the quasiparticles are not dynamical, but instead move along some classically-prescribed trajectories which determine j^{qp}_μ. The electircal current is:

$$j_\mu = \frac{\partial \mathcal{L}}{\partial A_\mu} = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \partial_\nu a_\rho \tag{4}$$

Since the action is quadratic, it is completely solvable, and one can integrate out the field a_μ to obtain the response of the current to the external electromagnetic field[2].The result of such a calcuation is precisely the quantized Hall conductivity $\sigma_{xx} = 0$ and $\sigma_{xy} = \frac{1}{k}e^2/h$.

The equation of motion obtained by varying a_0 is the Chern-Simons constraint:

$$\frac{k}{2\pi}\nabla \times \mathbf{a} = j_0^{qp} + \frac{1}{2\pi}B \tag{5}$$

According to this equation, each quasiparticle has Chern- Simons flux $2pi/k$ attached to it (the magnetic field is assumed fixed). Consequently, it has electrical charge 1/k, accord- ing to (4). As a result of the Chern-Simons flux, another quasiparticle moving in this Chern-Simons field picks up an Aharonov-Bohm phase. The action associated with taking one quasiparticle around another is, according to Eq.3, of the form

$$\frac{1}{2}k\int d\mathbf{r}dt\mathbf{j}\cdot\mathbf{a} = kQ\int_C d\mathbf{r}\cdot\mathbf{a} \tag{6}$$

where Q is the charge of the quasiparticle and the final integral is just the Chern-Simons flux enclosed in the path. (The factor of 1/2 on the left-hand side is due to the action of the Chern- Simons term itself which, according to the constraint (5) is -1/2 times the Aharanov-Bohm phase. This is cancelled by a factor of two coming from the fact that each particle sees the other's flux.) Thus the contribution to a path integral $e^{iS_{CS}}$ just gives an Aharonov-Bohm phase associated with moving a charge around the Chern-Simons flux attached to the other charges. The phases generated in this way give the quasipar- ticles of this Chern-Simons theory $\theta = \pi/k$ Abelian braiding statistics.[3] Therefore, an Abelian Chern-Simons term implements Abelian anyonic statistics. In fact, it does nothing else. An Abelian gauge field in 2 + 1 dimensions has only one trans- verse component; the other two components can be eliminated by fixing the gauge. This degree of freedom is fixed by the Chern-Simons constraint (5). Therefore, a Chern-Simons gauge field has no local degrees of freedom and no dynam- ics. We now turn to non-Abelian Chern-Simons theory. This TQFT describes non-Abelian anyons. It is analogous to the Abelian Chern-Simons described above, but different meth- ods are needed for its solution, as we describe in this section. The action can be written on an arbitrary manifold $\mathcal{M}$ in the form

$$S_{CS}[a] = \frac{k}{4\pi}\int_{\mathcal{M}} tr\left(a\wedge da + \frac{2}{3}a\wedge a\wedge a\right) \tag{7}$$

$$= \frac{k}{4\pi}\int_{\mathcal{M}} \epsilon^{\mu\nu\rho}\left(a_{\mu}^{\underline{a}}\partial_\nu a_{\rho}^{\underline{a}} + \frac{2}{3}f_{\underline{abc}}a_{\mu}^{\underline{a}}a_{\nu}^{\underline{b}}a_{\rho}^{\underline{b}}\right) \tag{8}$$

In this expression, the gauge field now takes values in the Lie algebra of the group G. $f_{\underline{abc}}$ are the structure constants of the Lie algebra which are simply $\epsilon_{\underline{abc}}$ for the case of SU(2). For the case of SU(2), we thus have a gauge field $a_{\mu}^{\underline{a}}$, where

[2]Why exactly does the action being quadratic mean that it is 'completely solvable'?

[3]The Chern-Simons effective action for a hierarchical state is equivalent to the action for the composite fermion state at the same filling fraction.

the underlined indices run from 1 to 3. A matter field transforming in the spin-j representation of the SU(2) gauge group a will couple to the combination $a_{\mu}^{\underline{a}} x_{\underline{a}}$, where $x_{\underline{a}}$ are the three generator matrices of su(2) in the spin-j representation. For gauge group G and coupling constant k (called the 'level'), we will denote such a theory by G_k . In this paper, we will be primarily concerned with $SU(2)_k$ Chern-Simons theory. To see that Chern-Simons theory is a TQFT, first note that the Chern-Simons action (24) is invariant under all diffeomorphisms of M to itself, $f : \mathcal{M} \to \mathcal{M}$. The differential form notation in (7) makes this manifest, but it can be checked in coordinate form for $x^{\mu} \to f^{\mu}(x)$. Diffeomorphism invariance stems from the absence of the metric tensor in the Chern-Simons action. Written out in component form, as in (7), indices are, instead, contracted with $\epsilon^{\mu\nu\lambda}$. Before analyzing the physics of this action (7), we will make two observations. First, as a result of the presence of $\epsilon^{\mu\nu\lambda}$, the action changes sign under parity or time-reversal transformations. In this paper, we will concentrate, for the most part, on topological phases which are chiral, i.e. which break parity and time-reversal symmetries. These are the phases which can appear in the fractional quantum Hall ef- fect, where the large magnetic field breaks P , T . Secondly, the Chern-Simons action is not quite fully invariant under gauge transformations $a_{\mu} \to g a_{\mu} g^{-1} + g\partial_{\mu} g^{-1}$, where $g : \mathcal{M} \to G$ is any function on the manifold taking values in the group G. On a closed manifold, it is only invariant under 'small' gauge transformations. Suppose that the manifold $\mathcal{M}$ is the 3-sphere, S^3. Then, gauge transformations are maps $S^3 \to G$, which can be classified topologically according to its homotopy $\pi_3(G)$. For any simple compact group G, $\pi_3(G) = \mathbb{Z}$, so gauge transformations can be classified according to their 'winding number'. Under a gauge transformation with winding m,

$$S_C S[a] \to S_C S[a] + 2\pi k m \tag{9}$$

While the action is invariant under 'small' gauge transformations, which are continuously connected to the identity and have m = 0, it is not invariant under 'large' gauge transformations ($m \neq 0$). However, it is sufficent for $\exp(iS)$ to be gague invariant, which will be the case so long as we require that the level k be an integer. The requirement that the level k be an integer is an example of the highly rigid structure of TQFTs. A small pertubation of the microscopic Hamiltonian cannot continuously change the value of k in the effective low energy theory; only a pertubation which is large enough to change k by an integer can do this.

The failure of gauge invariance under large gauge transformations is also reflected in the properties of Chern-Simons theory on a surface with boundary, where the Chern-Simons action is gauge invariant only up to a surface term. Consequently, there must be gapless degrees of freedom at the edge of the system whose dynamics is dictated by the requirement of gauge invariance of the combined bulk and edge.

To unravel the physics of Chern-Simons theory, it is useful to specialize to the case in which the spacetime manifold $\mathcal{M}$ can be decomposed into a product of a spatial surface and time $\mathcal{M} = \Sigma \times \mathbb{R}$. On such a manifold, Chern-Simons theory is a theory of the ground states of a topologically-ordered system on Σ. There are no excited states in Chern-Simons theory because the Hamiltonian vanishes.

This is seen most simply in $a_0 = 0$ gauge, where the momentum canonically conjugate to a_1 is $-\frac{k}{4\pi}a_2$, and the momentum canonically conjugate to a_2 is $\frac{k}{4\pi}a_1$ so that

$$\mathcal{H} = \frac{k}{4\pi} tr\left(a_2 \partial_0 a_1 - a_1 \partial_0 a_2\right) - \mathcal{L} = 0 \tag{10}$$

Note that this is a special feature of an ation with a Chern-Simons term alone. If the action had both a Chern-Simons and a Yang-Mills term, then the Hamiltonian would not vanish, and the theory would have both ground states and excited states with a finite gap. ince the Yang-Mills term is sublead- ing compared to the Chern-Simons term (i.e. irrelevant in a renormalization group (RG) sense), we can forget about it at energies smaller than the gap and consider the Chern-Simons term alone. Therefore, when Chern-Simons theory is viewed as an ef- fective field theory, it can only be valid at energies much smaller than the energy gap. As a result, it is unclear, at the moment, whether Chern-Simons theory has anything to say about the properties of quasiparticles – which are excitations above the gap – or, indeed, whether those properties are part of the universal low-energy physics of the system (i.e. are controlled by the infrared RG fixed point). Nevertheless, as we will see momentarily, it does and they are. Although the Hamiltonian vanishes, the theory is still not trivial because one must solve the constraint which follows by varying a_0 . For the sake of concreteness, we will specialize to the case G =SU(2). Then the constraint reads:

$$\epsilon_{ij} \partial_i a_j^{\underline{a}} + f^{\underline{abc}} a^{\underline{b}_1 a_2^{\underline{b}}} = 0 \tag{11}$$

where i,j = 1,2. The left-hand side of this equation is the field strength of the gauge field $a_i^{\underline{a}}$, $\underline{a} = 1, 2, 3$ is an $\mathfrak{su}(2)$ index. Since the field strength must vanish, we can always perform a gauge transformation so that $a_i^{\underline{a}} = 0$ locally. Therefore this theory has no local degrees of freedom. However, for some field configurations satisfying the constraint, there may be a global topological obstruction which prevents us from making the gauge field zero everywhere. Clearly, this can only happen if Σ is topologically non-trivial. The simplest non-trivial manifold is the annulus, which is topologically equivalent to the sphere with two punctures. Following Elitzur et al., 1989[4], let us take coordinates (r, ϕ) on the annulus, with $r_1 < r < r_2$, and let t be time. Then we can write $a_\mu = g\partial_\mu g^{-1}$, where

$$g(r, \phi, t) = e^{i\omega(r,\phi,t)} e^{i\frac{\phi}{k}\lambda(t)} \tag{12}$$

where $\omega(r, \phi, t)$ and $\lambda(t)$ take values in the Lie algebra $\mathfrak{su}(2)$ and $\omega(r, \phi, t)$ is a single-valued function of ϕ. The functions ω and ϕ are the dynamical variables of Chern-Simons theory on the annulus. Substituting (12) into the Chern-Simons action, we see that it now takes the form:

$$S = \frac{1}{2\pi} \int dttr(\lambda \partial_t \Omega) \tag{13}$$

where $\Omega(r, t) = \int_0^{2\pi} d\phi(\omega(r_1, \phi, t) - \omega(r_2, \phi, t))$. Therefore Ω is canonically conjugate to λ. By a gauge transformation, we can always rotate λ and Ω so that

[4] include this reference

they are along the 3 direction in $\mathfrak{su}(2)$, i.e. $\lambda = \lambda_3 T^3$, $\Omega = \Omega_3 T^3$. Since it is defined through the exponential in (12), Ω_3 takes values in $[0, 2\pi]$. Therefore, its canonical conjugate λ_3 is quantized to be an integer from the definition of λ in (12), we see that $\lambda_3 \equiv \lambda_3 + 2k$. However, by a gauge transformation given by a rotation around the 1-axis, we can transform $\lambda \to -\lambda$. Hence, the independent allowed values of λ are 0,1,$\cdots$,k.

On the two-punctured sphere, if one puncture is of type a, the other puncture must be of type $\bar{a}$. (If the topological charge at one puncture is measured along a loop around the puncture - e.g. by a Wilson loop then the lopp can be deformed so that it goes around the other puncture, but in the opposite direction. Therefore, the two punctures, necessarily have conjugate topological charges.) For SU(2), $a = \bar{a}$, so both punctures have the same topological charge. Therefore, the restriction to only k+1 different possible allowed boundary condition λ for the two-punctured sphere implies that there are k+1 different quasiparticle types in $SU(2)_k$ Chern-Simons theory. As we will describe in a later subsection, these allowed quasiparticle types can be identified with the $j = 0, \frac{1}{2}, \cdots, \frac{k}{2}$ representations of the $SU(2)_2$ Kac-Moody algebra.

4 Connections and the Cartan formalism

> On a manifold it is necessary to use covariant differentation, curvature measures its noncommutativity. Its combination as a characteristic form measures the nontriviality of the underlying bundle. This train of ideas is so simple, that their importance can not be exaggerated. - Shiing-shen Chern

For a smooth (for now, real) manifold M, we let $\mathcal{A}^1(M)$ denote smooth $\mathbb{C}$-valued differential forms and $\mathcal{A}$(M) denote smooth functions. Similarly for a complex bundle $E \to M$, we let E(M) denote (smooth, $\mathbb{C}$-valued) sections

4.1 Connections

Recall that a connection on a vector bundle E over a smooth manifold is a $\mathbb{C}$-homomorphism

$$\nabla : E(M) \to (\mathcal{A}^1 \otimes E)(M)$$

that maps global sections on M to global sections of $\mathcal{A}^1 \otimes E$, which satisfies the Leibniz rule $\nabla(fs) = (df)s + f\nabla s \;\; f \in \mathcal{A}(M) \;\; s \in E(M)$ This is essentially an way of differentiating sections of E, because for any vector field X on M, we can define

Definition 4.1 *The Covariant derivative w.r.t this connection of s in the direction of X*

$$\nabla_X s$$

. This satisfies

1. $\nabla_f X(s) \; = \; f\nabla_X s$
2. $\nabla_X(fs) \; = \; (Xf)_s \; + \; f\nabla_X s$

In fact these two properties **characterize** a connection.
We can describe a connection **locally** in terms of frames.

Definition 4.2 *Recall that a* **frame** *of an n-dimensional vector bundle E, over an open subset $U \subset M$, is a family of sections $(e_1, \cdots e_n) \; \in \; E(U)$ that form a basis at each point; thus $e_1, \cdots, e_n$ forms a vector bundle isomorphism between $E|_U$ and the trivial bundle.*

Then ∇ is **determined** over U by the elements $\nabla_{e_1}, \cdots \nabla_{e_n} \; \in \; (\mathcal{A}^1 \otimes E)(U)$. For any sections s of $E(U)$ can be written as $s \; = \; \sum_i f_i e_i$ for the f_i smooth functions, and consequently

$$\nabla s = \sum e_i(df_i) \; + \; \sum f_i \nabla e_i$$

In other words, if we use the fram e_i to identify each section of E(U) with the tuple f_i such that $s \; = \; \sum f_i e_i$ then ∇ acts by applying d and multiplying by suitable matrix corresponding to the ∇e_i. In view of this we make:

Definition 4.3 *Given a frame $\mathfrak{F} = e_1, \cdots, e_n$ over U and a connection ∇, we define the n-by-n matrix $\theta(\mathfrak{F})$ of 1-forms via*

$$\nabla \mathfrak{F} = \theta(\mathfrak{F})\mathfrak{F}$$

In other words, $\nabla e_i \; = \; \sum_j \theta(\mathfrak{F})_{ij} e_j$ for each j

Note that the θ itself makes no reference to the bundle: it is simply a matrix of 1-forms. Given a frame $\mathfrak{F}$, and given $g : U \to GL_n(\mathbb{C})$, we define a new frame $g\mathfrak{F}$ by multiplying on the left. We would like to determine how a connection **transforms** with respect to a change of frame, so we can think of a connection in a different way. Namely we have:

$$\nabla(g\mathfrak{F}) \;=\; (dg)\mathfrak{F} \;+\; g\nabla\mathfrak{F} \;=\; (dg)\mathfrak{F} \;+\; g\theta(\mathfrak{F})\mathfrak{F}$$

where df is considered as a matrix of 1-forms. As a result we get the **transformation law**

$$\theta(g\mathfrak{F}) = (dg)g^{-1} + g\theta(\mathfrak{F})g^{-1},\; g : U \to\; GL_n(\mathbb{C}) \tag{14}$$

Conversely, if we have for each local fram $\mathfrak{F}$ of a vector bundle $E \to M$ a matrix $\theta(\mathfrak{F})$ of 1-forms as above, which satisfy the transformation law 14 as above,then we get a connection on E.

Proposition 4.4 *Any vector bundle $E \to M$ admits a connection*

Proof It is easy to see that a convex combination of connections is a connection. Namely in each coordinate patch U over which E is trivial with a fixed frame, we choose the matrix θ arbitrarily and get some connection ∇'_U on $E|_U$. Let these various U'_s form an open cover $\mathfrak{A}$. Then we can find a partition of unity ϕ_U, $U \in \mathfrak{A}$ subordinate to $\mathfrak{A}$, and we can define our global connection va

$$\nabla \;=\; \sum_U \phi_U \nabla'_U$$

4.2 Curvature

We want to now describe the **curvature** of a connection. A connection is a means of differentiating sections; however, it may not satisfy the standard results for functions, that mixed partials are equal. The curvature will be the measure of how much that fails. Let M be a smooth manifold, $E \to M$ a smooth complex vector bundle. Given a connection ∇ on E ,the curvature is going to be a global section of $\mathcal{A}^2 \otimes \hom(E, E)$: in other words, the global differential 2-forms with coefficients in the vector bundle $\hom(E, E)$

Proposition 4.5 *Let s be a section of E, and X, Y vector fields. The map:*

$$s, X, Y \mapsto R(X, Y, s) = (\nabla_Y \nabla_X - \nabla_X \nabla_Y - \nabla_{[X,Y]})s$$

is a bundle map $E \to E$, and is $\mathcal{A}(M)$-linear in X, Y

Proof Calculation typically done to define the Riemannian curvature tensor in the case of the tangent bundle

Since the quantity $R(X, Y, s)$ is $\mathcal{A}(M)$-linear in all these quantities (X,Y,s),and clearly alternating in X,Y, we can think of it as a global section of the bundle $\mathcal{A}^2 \otimes \hom(E, E)$. Here recall that $\mathcal{A}^2$ is the bundle of 2-forms.

Definition 4.6 *The above elements of $\mathcal{A}^2 \otimes \hom(E, E)(M)$ is called the **curvature** of the connection ∇ and is denoted by Θ.*

We now wish to think of the curvature in another manner. To do this, we start by extending the connection ∇ to maps $\nabla : (E \otimes \mathcal{A}^p)(M) \to (E \otimes \mathcal{A}^{p+1})(M)$. The requirement is that the Leibnitz rule holds: that is,

$$\nabla(\omega s) = (d\omega)s + (-1)^p \omega \wedge \nabla s, \tag{15}$$

whenever ω is a p-form and s a global section. We can this locally and glue them. Thus:

Proposition 4.7 *One can extend ∇ to map s*

$$\nabla : (E \otimes \mathcal{A}^p)(M) \to (E \otimes \mathcal{A}^{p+1})(M)$$

satisfying 15

Given such an extension, we can consider the map

$$\nabla^2 : E(M) \to (E \otimes \mathcal{A}^2(M))$$

. This is $\mathcal{A}(M)$-linear. Indeed we can check this by computation

Example 4.8

$$\nabla^2(fs) = \nabla(\nabla(fs)) = \nabla(dfs + f\nabla s) = d^2fs + (-1)df\nabla s + df(\nabla s) + f\nabla^2 s = f\nabla^2 s$$

We now want to connect this $\mathcal{A}(M)$- linear map with the earlier curvature tensor

Proposition 4.9 *The vector bundle map ∇^2 is equal to the curvature tensor Θ*

Proof We can work in local coordinates, and assume that X,Y are the standard commuting vector fields ∂_i ,∂_j. We want to show that, given a section s, we have

$$\nabla^2(s)(\partial_i, \partial_j) = (\nabla_{\partial_i}\nabla_{\partial_i} - \nabla_{\partial_j}\nabla_{\partial_i})s \in E(M)$$

To do this we should check how ∇ was defined. Namely we have, by definition $\nabla s = \sum_i dx_i \nabla_{\partial_i} s$ and consequently

$$\nabla^2 s = \sum_{ij} dx_j \nabla_{\partial_j}(dx_i \nabla_{\partial_i} s)$$

This becomes, by the sign rules $\sum_{i<j}(\nabla_{\partial_j}\nabla_{\partial_i} - \nabla_{\partial_i}\nabla_{\partial_j} s dx_i \wedge dx_j$. It is easy to see that this, evaluated on on (∂_i, ∂_j), gives the desired quantity. It follows that ∇^2 is equal to the curvature tensor Θ

As a result, we may calculate **curvature** in a frame. Let $\mathfrak{F} = e_1, \cdots, e_n$ be a frame and let $\theta(\mathfrak{F})$ be the connection matrix. Then we can obtain an n-by-n **curvature matrix** $\Theta(\mathfrak{F})$ of 2-forms such that

$$\Theta(\mathfrak{F}) = \nabla^2(\mathfrak{F})$$

The follow result enables us to compute $\Theta(\mathfrak{F})$.

Proposition 4.10 (Cartan)

$$\Theta(\mathfrak{F}) = d\theta(\mathfrak{F}) - \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F}) \tag{16}$$

Note that $\theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})$ is not zero in general! The reason is that one is working with matrices of 1-forms, not just plain 1-forms. The wedge product is a matrix product in a sense.

Proof Indeed, we need to determine how ∇^2 acts on the fram e_i. Namely with an abuse of notation:

$$\nabla^2(\mathfrak{F}) = \nabla(\nabla\mathfrak{F}) = \nabla(\theta\mathfrak{F})(\mathfrak{F}) = d\theta(\mathfrak{F})\mathfrak{F} - \theta(\mathfrak{F}) \wedge (\theta(\mathfrak{F})\mathfrak{F})$$

We have used this formula that describes how ∇ acts on a product with a form. As a result the proof holds.

Finally, we shall need an expression for $d\Theta$. We state this in terms of a local frame.

Proposition 4.11 (Bianchi identity) *With respect to a frame $\mathfrak{F}$ $d\Theta(\mathfrak{F}) = [\theta(\mathfrak{F}), \Theta(\mathfrak{F})]$*

Here the right side consists of matrices, so we talk about the commutator. We shall use this identity at a crucial point in showing that the Chern-Weil homomorphism is even well-defined.

Proof This is a simple condition. For, by Cartan's equations,

$$d\Theta(\mathfrak{F}) = d(d\theta(\mathfrak{F}) - \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})$$

=

$$-d\theta(\mathfrak{F}) \wedge \theta(\mathfrak{F}) + \theta(\mathfrak{F}) \wedge d\theta(\mathfrak{F})$$

Similarly,

$$[\theta(\mathfrak{F}), \Theta(\mathfrak{F})] = [\theta(\mathfrak{F}), d\theta(\mathfrak{F}) + \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})]$$

$[\theta(\mathfrak{F}), d\theta(\mathfrak{F})]$ because $[\theta(\mathfrak{F}), \theta(\mathfrak{F}) \wedge \theta(\mathfrak{F})] = 0$

5 Connections on Principal bundles

5.1 Some equivalent notions

If $\pi : P \to M$ is a principal G-bundle, then for every $p \in P$ we have a map $i_p : G \to P$ given by the formula $i_p(g) = p \cdot g$. For $X \in \mathfrak{g}$ we let X^* be the *fundamental vector field corresponding to* X given by $X^*_p = di_p(X_e)$.

Definition 5.1 *A connection on a principal G-bundle* $P \to M$ *over a manifold* M *is a* $\mathfrak{g}$*-valued 1-form* A *on* P *satisfying*

1. $A(X^*) = X$ *for every* $X \in \mathfrak{g}$;
2. A *is G-equivariant, i.e.* $r_g^*(A) = Ad_{g^{-1}}A$ *for each* $g \in G$,

where $r_g : P \to P$ *denotes the right action by* $g \in G$. *We denote the set of connections on* P *by* $\mathcal{A}_p$.

G-equivariance says that if Y is a vector field on P and $g \in G$, then $A\left(dr_g\left(Y\right)\right) = Ad_{g^{-1}}(A(Y))$. Locally, a 1-form on an n-dimensional manifold $\mathcal{M}$with values in $\mathfrak{g}$has the expression

$$(dx_1 \otimes A_1) + \cdots + (dx_n \otimes A_n),$$

where the x_i are coordinates on M and the A_i are elements of $\mathfrak{g}$. Given a vector bundle $E \to \mathcal{M}$, $\Omega^p(E)$ will denote the p=forms with values in E, i.e. smooth sections of the vector bundle

$$\bigwedge^p T^*M \otimes E \to \mathcal{M},$$

where the symbols denote the wedge and tensor products of the bundles. A connection is therefor an element of $\Omega^1(P;\mathfrak{g})$, where

$$\Omega^k(P;\mathfrak{g}) := \Omega^k(P \times \mathfrak{g})$$

are the $\mathfrak{g}$-valued k-forms on P. Denote the subspace of G-equivariant $\mathfrak{g}$-valued k-forms on P by $\Omega^k(P;\mathfrak{g})^G$.

5.2 Chern-Weil Theory

Characteristic classes are mainly used in obstruction theory. For example, the *Euler class* $e(\pi) \in H^n(M;\mathbb{Z})$ is the primary obstruction to trivializing a real vector bundle $\pi : E \to M$ of rank r or a $GL(n,\mathbb{R})$-principal bundle $\pi : P \to M$. Chern-Weil theory is a way of describing characteristic classes of vector bundles or principal bundles using differential geometry, instead of the topological method of pulling back universal cohomology classes. Chern classes and Pontrjagin classes are represented in DeRham theory by differential forms which are functions of the curvature of a connection in the bundle. There are two approaches to defining characteristic forms, one uses invariant polynomials, the other formal power series.

5.2.1 Invariant polynomials

. Let V be a complex vector space. For k ≥ 1 let $S^k(V^*)$ be the vector space of linear maps

$$f : V \otimes \cdots \otimes V \to \mathbb{C}.$$

If we let $S^0(V^*) = \mathbb{C}$, then

$$S^*(V^*) = \oplus_{k=0}^{\infty} S^k(V^*)$$

is a commutative right with unit 1$\in S^0(V^*)$ and product

$$f \cdot g(v_1, \cdots, v_{k+1}) = \frac{1}{(k+1)!} \Sigma_\sigma f(v_{\sigma_1}, \cdots, v_{\sigma_k}) g(v_{\sigma_{k+1}}, \cdots, v_{\sigma_{k+1}})$$

for $f \in S^k(V^*)$ and $g \in S^l(V^*)$, where σ runs over all the the permutations of 1, ... , k+1

6 Quantization

Most visions of happiness blithely assume beating all the statistical odds. - Alain de Botton

Let us introduce the notions of Quantization and Geometric Quantization.

6.1 Line bundles and connections.

Suppose L is a complex line bundle over M. Let $\{U_\alpha\}$ be an open cover of M so that $L|_{U_\alpha}$ is trivial. Pick a connection ∇on L. On U_α let $\nabla = d - 2\pi i A_\alpha$[5] **Gauge change':** Suppose $g : U_\alpha \cap U_\beta \to S^1 = \{|z| = 1\} \subset \mathbb{C}$ is a gauge transformation, i.e., a change of trivialization. Then we write $g(x) = \exp(-2\pi i f(x))$. Under a gauge change,

$$2\pi i A_\alpha \mapsto dgg^{-1} + g(-2\pi i)A_\alpha g^{-1} = -2\pi i(A_\alpha + df).$$

Hence $A_\alpha \mapsto A_\alpha + df$. **Curvature:** The curvature is given by $-2\pi i dA_\alpha + (-2\pi i)^2 A_\alpha \wedge A_\alpha = -2\pi i dA_\alpha$, since we're dealing with 1 x 1 matrices (and they commute)! Moreover, dA_α transforms to $gda_\alpha g^{-1} = dA_\alpha$, i.e dA_α is invariant under gauge change. Therefore, dA_α can be patched into a closed 2-form on M. [6] The cohomology class of the closed 2-form m is called the *first Chern class* of L and is denoted $c_1(L) \in H^2_{dR}(M;\mathbb{R})$. Note that $c_1(L) = \frac{i}{2\pi}[F_A]$.

Remark 6.1 *$c_1(L)$ is actually an element of $H^2(M;\mathbb{Z}) \subset H^2(M;\mathbb{R})$.*

Theorem 6.2 *Let ω be a closed 2-form on M such that $[\omega] \in H^2(M;\mathbb{Z}) \subset H^2(M;\mathbb{R})$. Then there exists a complex line bundle $L \to M$ and a connection ∇such that $\omega = \frac{i}{2\pi}F_A$(In particular, this means that $c_1(L) = [\omega]$.)*

Proof *Chose a good cover U_α of M. A good cover is a cover for which $U_\alpha \simeq \mathbb{R}^n$, $U_\alpha \cap U_\beta \simeq \mathbb{R}^n$ or $\emptyset$, $U_\alpha \cap U_\beta \cap U_\gamma \simeq \mathbb{R}^n$, or $\emptyset$,etc Here $\simeq$ means "diffeomorphic to", and dim M = n. Such a good cover exists on any smooth manifold, and this can be constructed by using a Riemannian metric to construst geodesically convex neighbourhoods.*
Over U_α, construct the trivial bundle $U_\alpha \times \mathbb{C} \to U_\alpha$ with connection $-2\pi i A_\alpha$ so that $dA_\alpha = \omega$ on U_α. Here we are using the fact that $U_\alpha \simeq \mathbb{R}^n$ and the Poincare lemma to find a primitive for ω.
Next, on overlaps $U_\alpha \cap U_\beta \simeq \mathbb{R}^n$, $A_\alpha - A_\beta = df_{\alpha\beta}$ since $dA_\alpha = dA_\beta = \omega$. Again, we are using the Poincare lemma. Observe that the choice of $f_{\alpha\beta}$ is unique up to the choice of a constant function.
We now use $g_{\alpha\beta} = \exp(-2\pi i f_{\alpha\beta})$ to patch the $U_\alpha \times \mathbb{C}'s$. Namely , we glue $(U_\alpha \times U_\beta) \times \mathbb{C} \subset U_\beta \times \mathbb{C}$ to $(U_\alpha \times U_\beta) \times \mathbb{C} \subset U_\alpha \times \mathbb{C}$ by sending (x,z) to

[5]Note the minus sign - apparently this is used to make c_1(L) agree with the usual one

[6]Closed forms are elements of the set: $Z^r(M,\mathbb{R}) = \{\omega \in A^r(M)|d\omega = 0\}$ or the kernel of the homomorphism $d : A^r(M) \to A^{r-1}(M)$ Recall that a closed differential form is not necessarily exact, but that an exact differential form via Poincare's lemma is always closed. Recall also that elements of the set $B^r(M,\mathbb{R}) = \{\omega \in A^r(M)|\omega = d\beta \ for\ some\ \beta \in A^{r-1}(M)\}$. Clearly $A^r(M)$ is the cochain group of differential forms on a manifold M, also remember that differential forms are sections of the cotangent bundle, so it is an abuse of notation to call them 'on a manifold M'

(x,g(x)z).
In order to make sure that the gluing is consistent, we need to verify the following on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$:

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma},$$

or, equivalently,

$$f_{\alpha\beta} + f_{\beta\gamma} = f_{\alpha\gamma} mod \mathbb{Z}. \tag{17}$$

We ask whether it is possible to choose complex numbers $a_{\alpha\beta}$ so that $\bar{f}_{\alpha\beta} = f_{\alpha\beta} + a_{\alpha\beta}$ satisfies the above equation. To answer this, consider the simplical complex for M corresponding to the good cover $\{U_\alpha\}$: To each U_α, assign a vertex (0-simplex) v_α. To each nontrivial $U_\alpha \cap U_\beta$, assign an edge (1-simplex) between v_α and v_β. To each nontrivial $U_\alpha \cap U_\beta \cap U_\gamma$, place a 2-simplex with vertices $v_\alpha, v_\beta, v_\gamma$. With respect to this simplical decomposition of M, $\delta f = \{f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\beta}\}$ is a 2-cocycle with values in $\mathbb{C}$. Now the question can be rephrased as follows: Is there a 1-cochain $a = \{a_{\alpha\beta}\}$ with values in $\mathbb{C}$such that $f - \delta a$ has values in $\mathbb{Z}$? THis is precisely the same as asking for [ω] to be in $H^2(M;\mathbb{Z})$. ∎

6.2 Geometric quantization

Given a sympletic manifold (M, ω), construct a complex line bundle $L \to M$ and a connection ∇such that the curvature is $-2\pi i\omega$. Let $C^\infty(M)$ be the Poisson algebra of C^∞-functions on (M, ω), and let $\Gamma(L)$ be the smooth sections of L. By *(geometric) quantization* we mean a Lie algebra representation of $C^\infty(M)$ on $\Gamma(L)$, i.e., a Lie algebra homomorphism from $C^\infty(M)$ to $End(\Gamma(L))$. (Usually the operators are unbounded.) In the case at hand, assign:

$$f \mapsto \nabla_{X_f} - 2\pi i f.$$

The assignment is a Lie algebra homomorphism:

$$f, g \mapsto [\nabla_{X_f} - 2\pi i f, \nabla_{X_g} - 2\pi i g] \tag{18}$$
$$= (\nabla_{[X_f, X_g]} - 2\pi i\omega(X_f, X_g)) - 2\pi i X_f g - 2\pi i X_g f \tag{19}$$
$$= \nabla_{X_{\{f,g\}}} + 2\pi i\omega(X_f, X_g) \tag{20}$$
$$= \nabla_{X_{\{f,g\}}} + 2\pi i\{f, g\} \tag{21}$$

Primordial Example: Consider $(\mathbb{R}^{2n}, \omega = dpdq)$ with coordinates (p,q). (Here, p stands for momentum a covector, and q for position a vector.) Construct the trivial line bundle $\mathbb{R}^{2n} \times \mathbb{C}$ with connection $\nabla = d - \frac{i}{\hbar}A$. The quantize by sending

$$f \mapsto \nabla_{X_f} - \frac{i}{\hbar} f$$

We compute that $X_p = \frac{\partial}{\partial q}$ $X_q = \frac{\partial}{\partial p}$. Hence, upon quantizing:

$$p \mapsto \nabla_{-\frac{\partial}{\partial q}} - \frac{i}{\hbar} p = -\frac{\partial}{\partial q} - \frac{i}{\hbar} p dq(-\frac{\partial}{\partial q}) - \frac{i}{\hbar} p = \frac{\partial}{\partial q}$$

$$q \mapsto \nabla_{-\frac{\partial}{\partial p}} - \frac{i}{\hbar}q = -\frac{\partial}{\partial p} - \frac{i}{\hbar}pdq(-\frac{\partial}{\partial p}) - \frac{i}{\hbar}q = \frac{\partial}{\partial p} - \frac{i}{\hbar}q.$$

If we restrict to sections that are only functions in q, then the above more or less reduces to our quantum mechanics picture.

7 Path integrals

7.1 Sigma models.

Let us study maps u:$X \to M$ between Riemannian manifold. Let Map(X,M) be the set of smooth maps form X to M. Then given $u \in Map(X, M)$ we define the *energy functional:*

$$S_X : Map(X, M) \to \mathbb{R}$$
$$u \mapsto \int_X |du|^2 dvol_X$$

More precisely, at $x \in X$, we take an orthonormal basis $e_1, \cdots, e_n$ of T_xX (Here dim X = n). Then du^2 means $\Sigma_{i=1}^n \langle u_* e_i, u_* e_i \rangle$ $\langle \cdot, \cdot \rangle$ is with respect to the Riemannian metric for M. By 'functional' we mean a function on some space of functions. A critical point of the energy functional is called a 'harmonic map'.

Remark 7.1 *The energy functional (the generic term is 'action') has the following properties*

1. *If f: X' $\to$ X is an isometry, then* $S_{X'}(u \circ f) = S_X(u)$.
2. *If -X is X with reversed orientation, then* $S_{-X}(u) = -S_X(u)$.
3. *If* $X = X_1 \sqcup X_2$ *(disjoint union), then* $S_X(u) = S_{X_1}(u|_{X_1}) + S_{X_2}(u|_{X_2})$
4. *Suppose* $X = X_+ \cup X_-$, *where* $\partial X_+ = \partial X_- = Y$ *is a codimension 1 submanifold of X. If* $u_+ \in Map(X_+, M)$, $u_- \in Map(X_-, M)$, $u_+|Y = u_-|Y$, *then* $S_X(u) = S_{X_+}(u_+) + S_{X_-}(u_-)$. *Here u is defined to be* u_+ *on* X_+ *and* u_- *on* X_-.

7.2 Feynman path integral

. In *classical mechanics*, the trajectory of a particle between two points (say a and b) in configuration space minimizes the action $S(\gamma)$. In *quantum mechanics*, to each path γ you assign a "probability function“ $e^{iS(\gamma)/\hbar}$ and integrate over the space of all paths connection a and b:

$$\int e^{iS(\gamma)/\hbar} d\mu(y)$$

This is called the *Feynman path integral.* Here $d\mu$ is some measure on the paths connecting a and b.

Remark 7.2 *The Feynman path integral has been rigorously defined only in some cases, even defining a measure in the measure theoretic sense is difficult. When we go from quantum to classical (i.e. in the large* $\hbar$ *limit), we expect the rapid oscilations of* $e^{iS(\gamma)/\hbar}$ *to cancel each other, except near the critical points of* $S(\gamma)$. *Hence the main contributions are the classical trajectories.*

Sigma model: Let us consider the sigma model. Let $C_X = Map(X, M)$. If X does not have boundary, then the 'partition function'

$$Z(X) = \int_{C_X(\alpha)} e^{iS(\gamma)/\hbar} d\mu_X(u) >$$

Here we are integrating over $C_X(\alpha)$ which is the subset of C_X consisting of maps u: $X \to M$ which restrict to α on $\partial X = Y$. **Plan:** Although Z(X) may not be rigorously defined, we can write down expected properties of Z(X) and Z_Y, which is some vector subspace of functions on C_Y that Z(X) should live in.
Axioms:

1. (Orientation) $Z_{-Y} = Z_Y^*$ where Z_Y^* is the dual vector space of Z_Y.
2. (Multiplication) $Z_{Y_1 \sqcup Y_2} = Z_{Y_1} \otimes Z_{Y_2}$
3. (Gluing) $Z(X) = \langle Z(X_+), Z(X_-)\rangle$, where $X = X_+ \cup X_-$, $\partial X_+ = \partial X_- = Y$, and the pairing is between Z_Y and Z_Y^*

Explanation We explain the Gluing Axiom. Using the expected properties of the Feynman path integral (e.g., Fubini's theorem),

$$Z(X) = \int_{C_X} e^{iS_X(u)(\gamma)/\hbar} d\mu(u)$$

$= \int_{C_Y} \left(\int_{C_{X_+}(\alpha)} e^{iS_{X_+}(u_+)(\gamma)/\hbar} d\mu_{X_+}(u_+) \cdot \int_{C_{X_-}(\alpha)} e^{iS_{X_-}(u_-)(\gamma)/\hbar} d\mu_{X_-}(u_-) \right) d\mu_Y(\alpha)$
$= \int_{C_Y} Z(X_+)(\alpha) \cdot Z(X_-)(\alpha) d\mu_Y(\alpha) = \langle Z(X_+, Z(X_-)\rangle$ Here, $u_+ = u|_{X_+}$ $u_- = u|_{X_-}$, and $\alpha = u|_Y$. We are also using $S_X(u) = S_{X_+}(u_+) + S_{X_-}(u_-)$.

7.3 Topological Quantum Field Theory (TQFT) axioms.

We now formulate the TQFT in the sense of Atiyah. They are almost the same as the axioms derived for the sigma model above; the only major difference is that we ask the vector spaces to be finite-dimensional.

Consider a commutative ring Λ with 1, such as $\mathbb{C}$(A) A finitely generated Λ-module $Z(\Sigma)$ associated to each oriented closed smooth d-dimensional manifold Σ (corresponding to the *homotopy* axiom),

(B) An element $Z(M) \in Z(\partial M)$ associated to each oriented smooth (d+1)-dimensional manifold (with boundary) M (corresponding to an "additive" axiom).

These data are subject to the following axioms

(1) Z is "functorial" with respect to orientation preserving *diffeomorphisms* of Σ and M,

(2) Z is "involutory", i.e. $Z(\Sigma^*) = Z(\Sigma)^*$ where Σ^* is Σ with opposite orientation and $Z(\Sigma)^*$ denotes the dual module,

(3) Z is "multiplicative".

Furthermore, Atiyah adds two axiom to them. Namely, they are (4) and (5).

(4) $Z(\phi) = \Lambda$ for the d-dimensional empty manifold and $Z(\phi) = 1$ for the (d+1)-dimensional empty manifold. If we view $Z(M)$, for closed M, as a numerical invariant of M, then for a manifold with boundary we should think of

$Z(M) \in Z(\partial M)$ as a "relative" invariant. Let $f : \Sigma \times I \to \Sigma \times I$ be an orientation preserving diffeomorphism, and identify opposite ends of $\Sigma \times I$ by f. This gives a manifold Σ_f and our axioms imply $\Sigma_f = \text{Trace } \Sigma(f)$ where $\Sigma(f)$ is the induced automorphism of $Z(\Sigma)$.

(5) $Z(M^*) = \overline{Z(M)}$ (the *hermitian* axiom). Equivalently, $Z(M^*)$ is the disjoint of $Z(M)$

Note that for a manifold M with boundary Σ we can always form the double $M \cup_\Sigma M^*$ which is a closed manifold. (5) shows that $Z(M \cup_\Sigma M^*) = |Z(M)|^2$ where on the right we compute the norm in the hermitian (possibly indefinite) metric.

7.4 Loop groups

To further understand TQFT and the like, we need some further representation theory.

7.4.1 Maurer-Cartan form.

Definition 7.3 *Let G be a Lie group. Then the Maurer-Cartan form μ is a left-invariant 1-form on G wit values in the Lie group $\mathfrak{g}$which satisfies $\mu(e)(A) = A$, where $A \int T_e G = \mathfrak{g}$ (More generally, $\mu(g)(g(I + tA)) = I + tA$, if we write A as $I + tA$.)*

Notation: Often write A $\in T_e G$ as $I+tA$ or e^{tA} and think of A as an equivalence class of arcs. For matrix Lie groups (i.e., subgroups of $GL(n, \mathbb{C})$, the Maurer-Cartain form is $\mu = X^{-1}dX$, where X is an n x n matrix whose (ij)-th entry is the coordinate function x_{ij}. One verifies that $\mu(I)(I+tA) = dX(I+tA) = I+tA$ and $\mu(g)(g(I + tA)) = g^{-1}g(I + tA) = I + tA$. Now let G = SU(2). We recall SU(2) is diffeomorphic to S^3. If A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$, then a,b determine A and $\{|a|^2 + |b|^2 = 1\}$ is the unit sphere in $\mathbb{C}^2$. Consider the 3-form

$$\sigma = \frac{1}{24\pi^2} Tr(\mu \wedge \mu \wedge \mu),$$

where μ is the Maurer-Cartan form.

Lemma 7.4 *—σ— generates the integral cohomology group $H^3(SU(2); \mathbb{Z}) \simeq H^3(S^3; \mathbb{Z}) \simeq \mathbb{Z}$*

Proof *Let us perform the calculation at $e \in G$ and rely on the left-invariance. $\mathfrak{su}(n\mathbb{C})$(2) is the set of traceless skew-Hermitian matrices and has an $\mathbb{R}$-basis:*

$$\{A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & -0 \end{pmatrix}, C = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\}$$

We compute that:

$$\sigma(e)(A, B, C) = \frac{1}{24\pi^2}(3!)Tr(ABC) = \frac{1}{4\pi^2} Tr \begin{pmatrix} 0 & 1 \\ -1 & -0 \end{pmatrix} = -\frac{1}{2\pi^2}$$

Here 3! comes from observing that we are taking alternating sums when evaluating three tangent vectors A,B,C and each sum is the same. Since $2\pi^2$ is the volume of the unit 3-sphere in $\mathbb{R}^4$ $\int_{S^3} \sigma = -1$ and $|\sigma|$ generates $H^3(SU(2); \mathbb{Z})$. ∎

7.5 The loop group

Suppose G = SU(2). And let LG be the loop group, i.e., the group of smooth maps $S^1 \to G$

Lemma 7.5 $H^2(LG;\mathbb{Z}) = \mathbb{Z}$

Proof *First observe that $LG \simeq G \times \Omega G$, where ΩG is the set of based loops, namely smooth maps$S^1 \to G$ which map $1 \mapsto e$ (Here we view the 1-sphere as a subspace of $\mathbb{C}$and e the identity of G) In fact, we can send $\gamma \in LG$ to $(\gamma(1),(\gamma(1))^{-1}\gamma)$ Now, we have $\pi_i(\Omega G, e) \simeq \pi_{i+1}(G,e)$. (Here $e \in \Omega G$ refers to the map $S^1 \to G$ which maps to $e \in G$.) For the isomorphism, we can think of a map $(S^i, pt) \to (\Omega G, e)$ as a maps $(S^{i+1}, pt) \to (G, e)$. We then have:*

$$\pi_1(G) = \pi_1(S^3) = 0, \pi_2(G) = 0, \pi_3(G) = \mathbb{Z}$$

$$\pi_1(LG) - \pi_1(G) \times \pi_1(\Omega G) = \pi_1(G) \times \pi_2(G) = 0, \pi_2(LG) = \pi_2(G) \times \pi_3(G) = \mathbb{Z}$$

By the Hurewicz isomorphism theorem, the first nontrivial π_i and H_i agree, and we have

$$H_2(LG) \simeq \pi_2(G) \simeq \mathbb{Z}$$

▮

Lemma 7.6 *A generator for $H^2(LG;\mathbb{Z})$ is given by $\omega_0 = \int_{S^1} \phi^*\sigma$, where $\phi : LG \times S^1 \to G$ is the evaluation map $\phi(\gamma,\theta) = \gamma(\theta)$.*

We need to explain the integration operation. First,

$$\phi^*\sigma(\gamma,\theta)\left(\xi,\eta,\frac{\partial}{\partial\theta}\right) = \sigma(\gamma(\theta))(\xi(\theta),\eta(\theta),\gamma^{'}(\theta)),$$

where $\xi,\eta \in T_\gamma LG$. Then

$$\omega_0(\gamma)(\xi,\eta) = \left(\int_{S^1}\phi^*(\sigma)\right)(\gamma)(\epsilon,\eta)$$

$$= \int_0^{2\pi} \sigma(\gamma(\theta))(\epsilon(\theta),\eta(\theta),\gamma^{'}(\theta))d\theta \quad = \frac{1}{4\pi^2}\int_0^{2\pi} Tr(\gamma^{-1}\xi(\theta)\cdot\gamma^{-1}\eta(\theta)\cdot\gamma^{-1}\gamma^{'}(\theta))d\theta$$

The composition

$$H^3(G) \xrightarrow{\phi^*} H^3(LG \times S^1) \xrightarrow{\int_{S^1}} H^2(LG)$$

is called a *transgression.* It is not hard to see that the composition sends generators to generators. Now let ω be the left-invariant 2-form on LG given by the extending the Lie algebra 2-cocycle ω (with $[\omega] \in H^2(L\mathfrak{g};\mathbb{C})$), where

$$\omega(e)(\xi,\eta) = \frac{1}{4\pi^2}\int_0^{2\pi}\langle\xi^{'}(\theta),\eta(\theta)\rangle d\theta.$$

Here $\langle,\rangle$ is the Killing form for $\mathfrak{su}(n\mathbb{C})(2)$, which is a multiple of (A,B) $\mapsto$ $Tr(AB)$. If we set $\xi = X \otimes t^m$ and $\eta = Y \otimes t^n$, and if $t = e^{i\theta}$, then $\frac{d}{d\theta}e^{im\theta} = ime^{im\theta}$ and

$$\omega(e)(X \otimes t^m, Y \otimes t^n) = \frac{i}{2\pi}\langle X,Y\rangle m\delta_{m+n,0}. \tag{22}$$

Observe that $\langle X,Y\rangle m\delta_{m+n,0}$ is the Lie algebra 2-cocycle. The 2-cocycle property translates into $d\omega$ being closed.

Lemma 7.7 *$\omega = \omega_0 + d\beta$, where*

$$\beta(\gamma)(\xi) = \frac{1}{8\pi^2}\int_0^{2\pi} \mathrm{Tr}(\gamma^{-1}\gamma'(\theta)\cdot\gamma^{-1}\xi(\theta))d\theta$$

Hence ω is also a generator for $H^2(LG;\mathbb{Z})$.

Proof *We compare*

$$\omega_0(\gamma)(\xi,\eta) = \frac{1}{4\pi^2}\int_0^{2\pi} Tr(\gamma^{-1}\xi(\theta)\cdot\gamma^{-1}\eta(\theta)\cdot\gamma^{-1}\gamma'(\theta))d\theta \tag{23}$$

and

$$\omega(\gamma)(\xi,\eta) = \frac{1}{4\pi^2}\int_0^{2\pi} Tr((\gamma^{-1}\xi)'(\theta)\cdot\gamma^{-1}\eta(\theta))d\theta \tag{24}$$

Let $\tilde{\xi}$ and $\tilde{\eta}$ be left invariant extensions of ξ and η to all LG. Then we can use the Cartan formula:

$$d\beta(\gamma)(\xi,\eta) = \tilde{\xi}(\beta(\tilde{\eta})) - \tilde{\eta}(\beta(\tilde{\xi})) - \beta([\tilde{\xi},\tilde{\xi}]).$$

We have $\beta(\gamma)([\tilde{\xi},\tilde{\xi}]) = \omega_0(\gamma)(\xi,\eta)$, and since $\gamma^{-1}\tilde{\xi}(\theta), \gamma^{-1}\tilde{\eta}(\theta)$ are constant for all γ, we have

$$\tilde{\xi}(\beta(\tilde{\eta})) = \frac{1}{8\pi^2}\int_0^{2\pi} Tr((\gamma^{-1}\xi)'\cdot\gamma^{-1}\eta)d\theta,$$

$$-\tilde{\eta}(\beta(\tilde{\xi})) = -\frac{1}{8\pi^2}Tr((\gamma^{-1}\eta)'\cdot\gamma^{-1}\xi)d\theta$$

This proves $\omega = \omega_0 + d\beta$.

8 The Wess-Zumino-Witten Model

Let us discuss the Wess-Zumino-Witten (WZW) model.

Definition 8.1 *Let Σ be a compact Riemann surface, without boundary. Let G be the Lie group SU(2). Consider $Map(\Sigma, G)$, the space of smooth maps $f: \Sigma \to G$. We first define the energy functional*

$$E_\Sigma(f) = -i\int_\Sigma \mathrm{Tr}\left(f^{-1}\partial f \wedge f^{-1}\bar{\partial} f\right).$$

Interpretation: *First recall that the Killing form of G=SU(2) is a constant multiple of (X,Y) $\mapsto \mathrm{Tr}(X,Y)$. If we use the local holomorphic coordinate $z = x + iy$ for Σ, then*

$$\partial f = \frac{\partial f}{\partial z}dz = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)(dx + idy)$$

$$\bar{\partial} f = \frac{\partial f}{\partial z}dz = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)(dx - idy)$$

If f(z) = e (which we may assume since $f^{-1}\partial f$ and $f^{-1}\bar{\partial} f$ are left-invariant), then:

$$-i\mathrm{Tr}(f^{-1}\partial f \wedge f^{-1}\bar{\partial} f) = -\frac{1}{2}\mathrm{Tr}\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right) dxdy.$$

Hence, $E_\Sigma(f)$ is, up to constant multiple, equal to the energy $\int_\Sigma |df|^2 dvol$ defined previously. Note that there is no metric defined for Σ. A complex structure on Σ defines a metric up to a conformal factor, i.e. $g \cong fg$, where f is a positive function on Σ. Hence $E_\Sigma(f)$ only depends on the *conformal class* of the metric, corresponding to the complex structure on Σ. **WZW action** Let k be a nonnegaitve integer, called the level. Then define:

$$S_\Sigma(f) = \frac{k}{4\pi}E_\Sigma(f) - 2\pi ik \int_B \tilde{f}^*\sigma,$$

where B is a 3-manifold with $\partial B = \Sigma$, $\tilde{f} : B \to G$ is an extension of $f : \Sigma \to G$, and $\sigma = \dfrac{1}{24\pi^2}\mathrm{Tr}(\mu \wedge \mu \wedge \mu)$ where μ is the Maurer-Cartan form.

Remark 8.2 *$S_\Sigma(f)$ is, strictly speaking, $S_\Sigma(\tilde{f})$. To remove the dependence on the extension $\tilde{f}$ we exponentiate it.*

8.1 Polyakov-Wiegmann formula

Proposition 8.3 (Polakov-Wiegmann formaula) . *Let Σ be a closed Riemann surface. Given $f, g : \Sigma \to G$, we have:*

$$\exp(-S_\Sigma(fg)) = \exp(-S_\Sigma(f) - S_\Sigma(g) + \Gamma_\Sigma(f, g)),$$

where $\Gamma_\Sigma(f, g)) = -\dfrac{ik}{2\pi}\int_\Sigma \mathrm{Tr}(f^{-1}\bar{\partial} f \wedge \partial g g^{-1})$.

Proof We will often use the identity $\mathrm{Tr}(\omega \wedge \eta) = (1)^{pq}\mathrm{Tr}(\eta \wedge \omega)$, where ω is a p-form with values in $\mathfrak{g}$and η is a q-form with values in $\mathfrak{g}$. We compute

$$-\frac{k}{4\pi}E_\Sigma(f) = \frac{ik}{4\pi}\int_\Sigma \mathrm{Tr}\left((fg)^{-1}\partial(fg) \wedge (fg)^{-1}\bar{\partial}(fg)\right)$$

$$= \frac{ik}{4\pi}\int_\Sigma \mathrm{Tr}\left(g^{-1}f^{-1})(f\partial g + \partial f \cdot g) \wedge (g^{-1}f^{-1})(f\bar{\partial} g + \bar{\partial} g + \bar{\partial} f \cdot g)\right)$$

$$= \frac{ik}{4\pi}\int_\Sigma \mathrm{Tr}\left(g^{-1}\partial g + g^{-1}f^{-1}\partial f \cdot g) \wedge (g^{-1}\bar{\partial} g + g^{-1}f^{-1}\bar{\partial} f \cdot g)\right)$$

$$= \frac{ik}{4\pi}\int_\Sigma \mathrm{Tr}\left(g^{-1}\partial g \wedge g^{-1}\bar{\partial} g + f^{-1}\partial f \wedge f^{-1}\bar{\partial} f + \partial g g^{-1} \wedge f^{-1}\bar{\partial} f + f^{-1}\bar{\partial} f + f^{-1}\partial f \wedge \bar{\partial} g \cdot g^{-1}\right)$$

$$= -\frac{k}{4\pi}\left(E_\Sigma(f) + E_\Sigma(g)\right) + \frac{ik}{4\pi}\int_\Sigma \mathrm{Tr}\left(-f^{-1}\bar{\partial} f \wedge \partial g g^{-1} + f^{-1}\partial f \wedge \bar{\partial} g \cdot g^{-1}\right)$$

Next,

$$\mathrm{Tr}\left(f^{-1}df \wedge dg g^{-1}\right) = \mathrm{Tr}\left(f^{-1}(\partial f + \bar{\partial} f) \wedge (\partial g + \bar{\partial} g)g^{-1}\right)$$

$$= \mathrm{Tr}(f^{-1}\partial f \wedge \bar{\partial} g g^{-1}) + \mathrm{Tr}(f^{-1}\bar{\partial} f \wedge \partial g g^{-1}),$$

since terms of the form $\partial f \wedge \partial g$ and $\bar{\partial} f \wedge \bar{\partial} g$ are zero. Hence we conclude that:

$$-\frac{k}{4\pi}E_\Sigma(f) = \frac{ik}{4\pi}\left(-E_\Sigma(f) - E_\Sigma(g)) + \Gamma_\Sigma(f,g) + \frac{ik}{4\pi}\int_\Sigma \mathrm{Tr}(f^{-1}df \wedge dgg^{-1}\right). \tag{25}$$

Next consider the Wess-Zumino terms. Ommiting the tidles for convenience, we have:

$$2\pi ik \int_B (fg)^*\sigma = \frac{2\pi ik}{24\pi^2}\int_B \mathrm{Tr}\left((fg)^{-1}d(fg) \wedge (fg)^{-1}d(fg) \wedge (fg)^{-1}d(fg)\right)$$

$$= \frac{ik}{12\pi}\int_B (A_1 + 3A_2 + 3A_3 + A_4),$$

$$= 2\pi ik \int_B (f^*\sigma + g^*\sigma) + \frac{ik}{4\pi}\int_B (A_2 + A_3)$$

where

$$A_1 = \mathrm{Tr}(f^{-1}dff^{-1}dff^{-1}df) \;\; A_2 = \mathrm{Tr}\left(dgg^{-1}f^{-1}dff^{-1}df\right)$$
$$A_3 = \mathrm{Tr}(dgg^{-1}dgg^{-1}f^{-1}df) \;\; A_4 = \mathrm{Tr}(g^{-1}dgg^{-1}dgg^{-1}dg)$$

Now,

$$d(\mathrm{Tr}(dgg^{-1}f^{-1}df)) = dgg^{-1}dgg^{-1}df + dgg^{-1}f^{-1}dff^{-1}df = A_2 + A_3,$$

using $d(f^{-1}) = -f^{-1}dff^{-1}$. Finally,

$$\frac{ik}{4\pi}\int_B (A_2 + A_3) = \frac{ik}{4\pi}\int_\Sigma \mathrm{Tr}(dgg^{-1}f^{-1}df), \tag{26}$$

using Stoke's theorem. Combining equations (26 and ,25) gives the result. ∎

8.2 Line bundles over

$LG_\mathbb{C}$. Let us consider the complexification $G_\mathbb{C}$ instead of G. Consider $\mathbb{CP}^1 = \mathbb{C}\cup\{\infty\}$. Let $D_0 = \{|z| \leq 1\} \subset \mathbb{C}$, $D_\infty = \{|z| \geq 1\}\cup\{\infty\}$, and $S^1 = \{|z| = 1\} = \partial D_0 = -\partial D_\infty$. We define a complex line bundle $\mathcal{L}$ over $LG_\mathbb{C}$ as follows: Let $Map_0(D_\infty, G)$ be the set of smooth maps $f_\infty : D_\infty \to G_\mathbb{C}$ with $f_\infty(\infty) = e$. Then let $\mathcal{L} = Map_0(D_\infty, G_\mathbb{C})/\cong$, where $(f_\infty, u) \cong (g_\infty, v)$ if:

1. $f_\infty|_{S^1} = g_\infty|_{S^1}$.
2. If $g_\infty = f_\infty h_\infty$, then

   $$v = u \cdot \exp(-S_{\mathbb{CP}^1}(h) + \Gamma_{D_\infty}(f_\infty, h\infty)).$$

   Here h is an extension of h_∞ to D_0 by e. (Note that $h_\infty|_{S^1} = e$.) The equivalence class of (f_∞, u) will be denoted $[f_\infty, u]$. The projection $\mathcal{L} \to LG_\mathbb{C}$ is given by $[f_\infty, u] \to f_\infty|_{S^1}$ We can view $f_0 : D_0 \to G_\mathbb{C}$ as an element of $\mathcal{L}$ as follows: Let f_∞ be a smooth extension of f_0, i.e., $f_\infty|_{S^1} = f_0|_{S^1}$. Then assign $f_0 \mapsto [f_\infty, \exp(-S_{\mathbb{CP}^1}(f))]$.

Lemma 8.4 $[f_\infty, \exp(-S_{\mathbb{CP}^1}(f))]$. *does not depend on the extension* f_∞.

Proof *See* ∎

8.3 An Euler characteristic of the gauge equivalence classes of *SU(2)*-connections

In 1988 Taubes defined an invariant for homology 3-spheres M by defining an euler-characteristic on the space of gauge equivalence classes of SU(2)-connections. Then he proved that his invariant is actually the same as Casson's invariant for Homology 3-spheres. This is a little survey particularly of the gauge theoretic view on Casson's invariant for homology 3-spheres, which related it to Chern-Simons theory and leads to a refinement of the Casson invariant, Floer's instanton homology.

8.3.1 The Casson invariant.

Let M be a closed 3-manifold with $H^i(M) = H^i(S^3)$. Consider a Heegaard decomposition of M, i.e. let X_k, $k = 1, 2$

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