AP Calculus BC Concepts

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1 Euler's Method

The key to Euler's method is understanding the concept of estimating the next value of a function f by scaling its derivative from an initial point $f(x_i)$ by a constant distance $(x_{i+1} - x_i)$ defined on the x-axis. In other words, given an initial point $f(x_i)$, you can approximate the next point $f(x_{i+1})$ using

$$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

Additionally, we can rewrite the above approximation as

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

This form of the approximation helps us estimate the derivative of f on the interval $[x_i, x_{i+1}]$. We can also extend this approximation of $f'(x_i)$ such that we are estimating the derivative at an intermediate point. In other words, for points x_{i-1}, x_i, x_{i+1}, we can estimate $f'(x_i)$ via

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}$$

Euler's method is simply using this approximation iteratively so that each point estimate is dependent on the estimated point that was last computed, all the way down to the initial point $f(x_i)$.

2 Mean Value Theorem

Given a function f defined on an interval $[a, b]$, there is at least one c such that $a < c < b$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is pretty easy to understand once you look at a graph visualizing what it says. If there is a continuous, differentiable function going from one point to another, then no matter what path it takes between a and b, at some point, the function must take a value $(f(c))$ such that it's derivative at that value $(f'(c))$ is equal to the slope defined between points a and b. The easiest way is if the function is identical to the slope, then the derivative is constant over the interval $[a, b]$ and the theorem obviously checks out. Now, let's consider that the function's derivative right beyond point a is much higher; then, to adjust its trajectory to get to point b, it must counteract the initial higher derivative with a derivative lower than $f'(c)$ before the function gets to b. Hence, to transition from a derivative higher than $f'(c)$ to a derivative lower then $f'(c)$, the derivative must at some point take on value equal to $f'(c)$.

3 Intermediate Value Theorem

This theorem is probably the most simple theorem to understand in BC calculus, though very important. Intermediate Value theorem simply says that if a function f defined on an interval $[a, b]$ takes values $f(a)$ and $f(b)$, then it also must also take on every value between $f(a)$ and $f(b)$ (Note: it can take on more values than every one between $f(a)$ and $f(b)$, though it must at least take on every value between).

4 L'Hopital's Rule

You don't need to be able to prove L'Hôpital's rule fortunately, you only need to know how to use it. Whenever you're asked to find the limit of a ratio of functions, meaning you are asked to find

$$\lim_{x\to c} \frac{f(x)}{g(x)}$$

where c is a constant and f and g are functions, but you run into an indeterminate form (i.e. $\lim_{x\to c} f(x)/g(x) = 0/0$), then you know to use L'Hôpital's rule. Simply note that the rule implies

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = \lim_{x\to c} \frac{f''(x)}{g''(x)} = \dots$$

so long as the function in the denominator is differentiable. Hence, all you need to do is differentiate the functions in the fraction iteratively until you get a limit that you can actually compute.

5 Riemann Sums

Riemann Sums are very easy to handle. All you need to remember is that a **Left-hand Riemann sum** means that your function values $f(x_i), f(x_{i+1}), f(x_{i+2}), ...$ will intersect with the top left-hand corner of the rectangles drawn over the graph used for estimation, while a **Right-hand Riemann sum** means that the values $f(x_i), f(x_{i+1}), f(x_{i+2}), ...$ will intersect with the top right-hand corner of the rectangles drawn over the graph used for estimation. The estimate then takes the form

$$\sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i)$$

for left-hand sums and

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

for right-hand sums

6 Polar Coordinates

Polar coordinates can sometimes ease the process of integrating in the (x, y) plane. Consider a circle of radius r defined by

$$x^2 + y^2 = r^2$$

From trigonometry, we know that we can define our cartesian coordinates (x, y) in terms of the radius r and angle θ via

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Substituting back into the original equation, we see that it checks out.

$$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

$$r^2(\cos^2\theta + \sin^2\theta) = r^2$$

This leads us to definitions of our polar coordinates (r, θ) in terms of our cartesian coordinates (x, y).

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

7 Implicit Differentiation

Consider the circle defined in the last section with a radius r

$$x^2 + y^2 = r^2$$

Suppose that you want to find the slope of a line tangent to the circle. Differentiating this equation seems confusing at first, but recall the chain rule. If we have a function nested within a function, (i.e. the square of our function y), then we know its derivative can be written as $2yy'$. Applying this to the whole equation gives

$$d(x^2) + d(y^2) = d(r^2)$$
$$2x + 2yy' = 0$$
$$y' = \frac{-x}{y}$$

Hence we have differentiated implicitly as our derivative y' is defined in terms of y without having to deal with a more complex expression only in terms of x.

8 Integration by Parts

We will approach integration by parts via derivation. Given a function $h(x)$ which can be written as the product of two functions $f(x)$ and $g(x)$, let us differentiate said function. This gives us

$$h'(x) = d(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

via the product rule. Now, let us integrate both sides of our function in terms of f and g.

$$\int d(f(x)g(x)) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Rewriting the above equation gives us the form of the 'integration by parts' equation that is most familiar,

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

Hence, when you see a function that is a product of two functions, look at the constituent parts of the function and examining which antiderivative will be the easiest to work with when solving with integration by parts.

9 Partial Fraction Decomposition

Working with partial fractions theoretically can get very messy and daunting, so being comfortable with examples of how to use partial fractions is enough for BC calculus. Consider the function

$$f(x) = \frac{1}{x^2 + 2x - 3}$$

We can see that the quadratic in the denominator can be rewritten as

$$(x + 3)(x - 1)$$

Because of this, we can actually rewrite the function in terms of a sum of fractions defined, in a sense, by their base roots. So, here, we know that we have 2 roots to our quadratic and thus we rewrite as

$$\frac{1}{x^2 + 2x - 3} = \frac{A}{x + 3} + \frac{B}{x - 1}$$

where A and B are constants. Now, multiplying through by our denominator on the left in "root-product" form (i.e. $(x-3)(x-1)$) leads us to

$$1 = A(x-1) + B(x+3)$$

From here, we can use our roots to solve easily for values of A and B. Setting $x=-3$ gives us that $A=-\frac{1}{4}$, and setting $x=1$ gives us that $B=\frac{1}{4}$. This gives us our final form of rewriting the function as

$$f(x) = \frac{1}{4}\left(\frac{1}{x-1} - \frac{1}{x+3}\right)$$

The key thing to remember about partial fractions is that the order of the polynomial in the denominator represents how many terms you will be rewriting the function in.

10 Improper Integrals

An improper integral is an integral with one or both limits infinite or an integrand which approaches infinity on either end of the integration bounds. Consider the integral

$$\int_1^\infty \frac{1}{x^2}dx$$

We can't integrate this normally, so we must do some tricks. Substituting in a variable t for our upper bound, we can integrate the expression as follows.

$$\int_1^t \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t}$$

Now, we simply take the limit as $t \to \infty$ to get an equivalent result to computing the original integral.

$$\lim_{t\to\infty} 1 - \frac{1}{t} = 1$$

Hence, you can replace infinite bounds with arbitrary variables to ease the process of integrating, and then take the limit afterwards to get the same result.

11 Convergence Tests for Series

11.1 Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

11.2 Integral Test

Suppose that f is a positive, decreasing function on $[k,\infty)$ and that $f(n) = a_n$. Then, if

$$\int_k^\infty f(x)dx$$

is convergent (or divergent), so is

$$\sum_{n=k}^\infty a_n$$

11.3 Comparison Test

Suppose that we have two series $\sum a_n$, $\sum b_n$ with $a_n, b_n \geq 0$ for all n and $a_n \leq b_n$ for all n. Then, if

$$\sum b_n$$

is convergent, then so is $\sum a_n$, and if

$$\sum a_n$$

is divergent, then so is $\sum b_n$ (think about what the inequality $a_n \leq b_n$ for all n implies!).

11.4 Limit Comparison Test

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n. Define

$$c = \lim_{n\infty} \frac{a_n}{b_n}$$

If $0 < c < \infty$, then either both series converge or both series diverge.

11.5 Alternating Series Test

Suppose that we have a series $\sum a_n$ and either

$$a_n = (-1)^n b_n$$

or

$$a_n = (-1)^{n+1} b_n, \;\; b_n \geq 0 \; \forall n$$

Then, if $\lim_{n\to\infty} b_n = 0$ and $\{b_n\}_{n\geq 0}$ is eventually a decreasing sequence, then the series $\sum a_n$ is convergent.

11.6 Ratio Test

Suppose we have series $\sum a_n$. Define limit L as

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

1. if $L < 1$, the series is convergent.
2. if $L > 1$, the series is divergent.
3. if $L = 1$, the series could be either.

11.7 Root Test

Suppose we have series $\sum a_n$. Define limit L as

$$L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

1. if $L < 1$, the series is convergent.
2. if $L > 1$, the series is divergent.
3. if $L = 1$, the series could be either.

11.8 Series Testing Guidelines

1. Does the series look like its terms won't converge to zero? If so, use the divergence test.
2. Is the series geometric? If so, use the comparison test.
3. Is the series a rational expression involving only polynomials or polynomials under radicals? If so, use the comparison test or the limit comparison test (remember series terms must all be positive for these two tests!).
4. Does the series contain factorials or powers? If so, use the ratio test.
5. Can the series be written in the form $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$? If so, use the alternating series test.
6. Can the series terms be written in the form $a_n = (b_n)^n$? If so, the root test may work.
7. If $a_n = f(n)$ for some positive, decreasing function and $\int_a^\infty f(x)dx$ is easy to evaluate, then use the integral test.