Unit 3

KINEMATICS OF A RIGID BODY

RIGID BODY MOTION

- When all the particles of a rigid body move along paths which are equidistant from a fixed plane, the body is said to undergo planar motion.
-
- There are three types of planar motion:

  - Translation

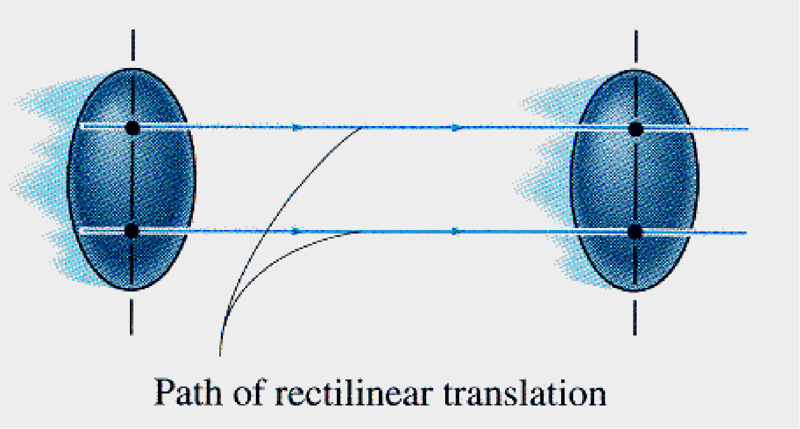
  - Rotation about a fixed axis

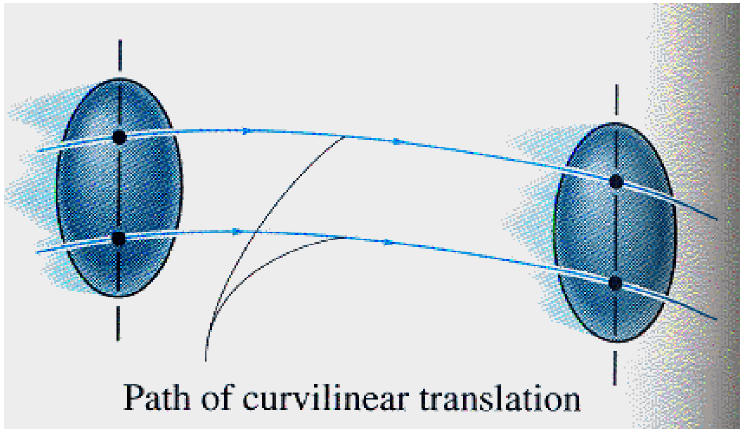
  - General Plane Motion

RIGID BODY MOTION

1. TRANSLATION:

- Translation occurs if any line segment on the body remains parallel to its original direction during the motion
- When the paths of motion for any two particles of the body are along equidistant straight lines, the motion is called rectilinear translation
- If the paths of motion are along curved lines which are equidistant the motion is called curvilinear translation

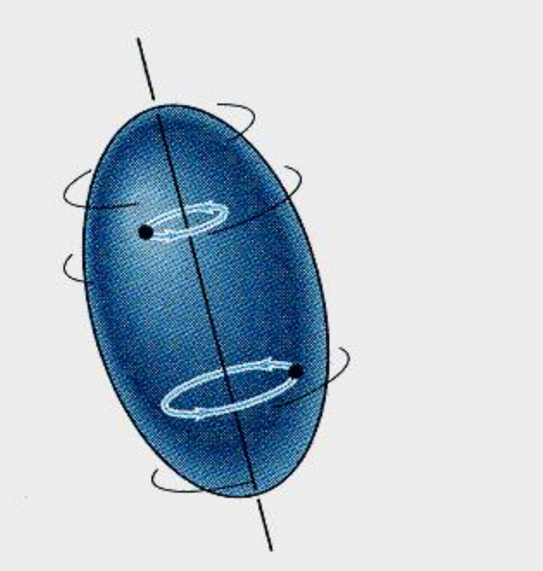




RIGID BODY MOTION

2. ROTATION ABOUT A FIXED AXIS:

- When a rigid body rotates about a fixed axis, all the particles of the body, except those which lie on the axis of rotation, moves along circular paths.

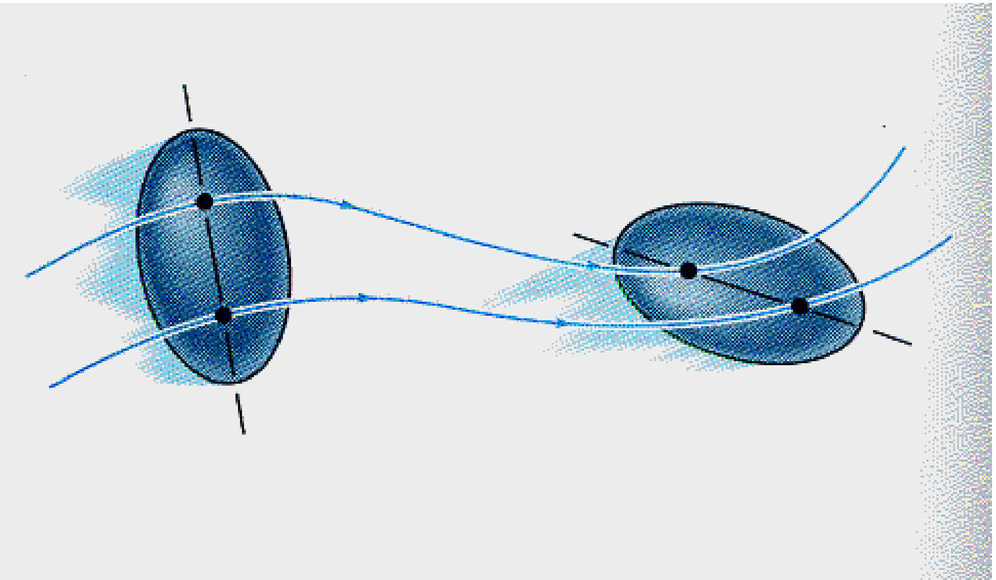


Rotation about a fixed axis

RIGID BODY MOTION

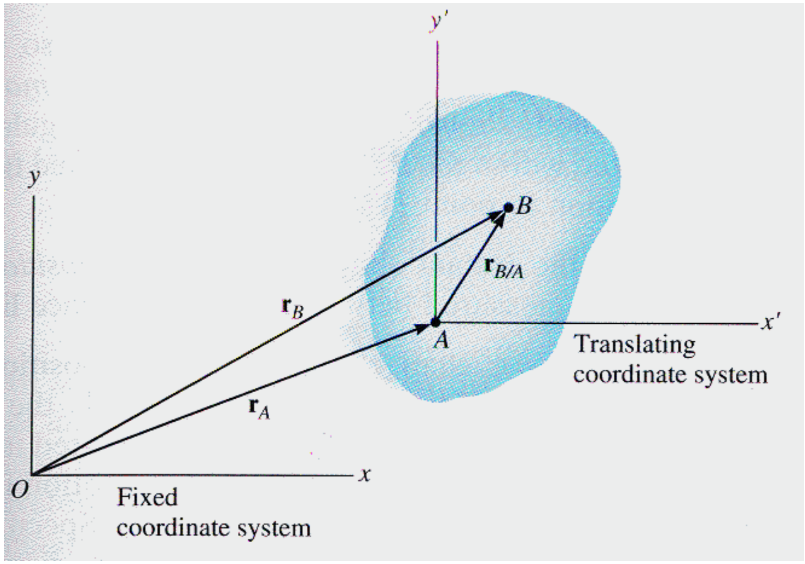
3. GENERAL PLANE MOTION (COMPLEX MOTION):

- When a body is subjected to general plane motion, it undergoes a combination of translation and rotation
- The translation occurs within a reference plane, and the rotation occurs about an axis perpendicular to the reference plane



General plane motion

TRANSLATION



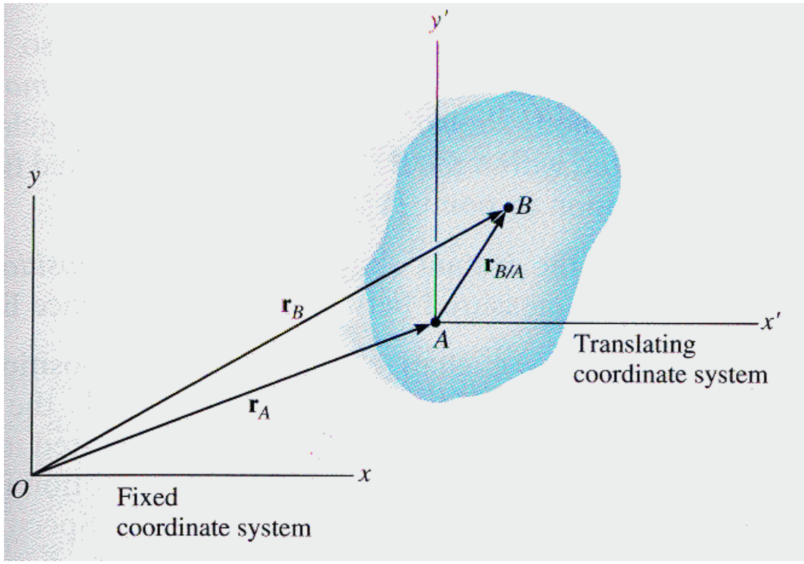
- Consider a rigid body which is subjected to either rectilinear or curvilinear translation in the xy plane

Position:

- The location of points A and B are defined from the fixed xy reference frame by using position vectors $\mathbf{r_A}$ and $\mathbf{r_B}$
- The position of B with respect to A is denoted by the relative position vector $\mathbf{r_{B/A}}$
- By vector addition:

$$\mathbf{r_B} = \mathbf{r_A} + \mathbf{r_{B/A}}$$

TRANSLATION



Velocity:

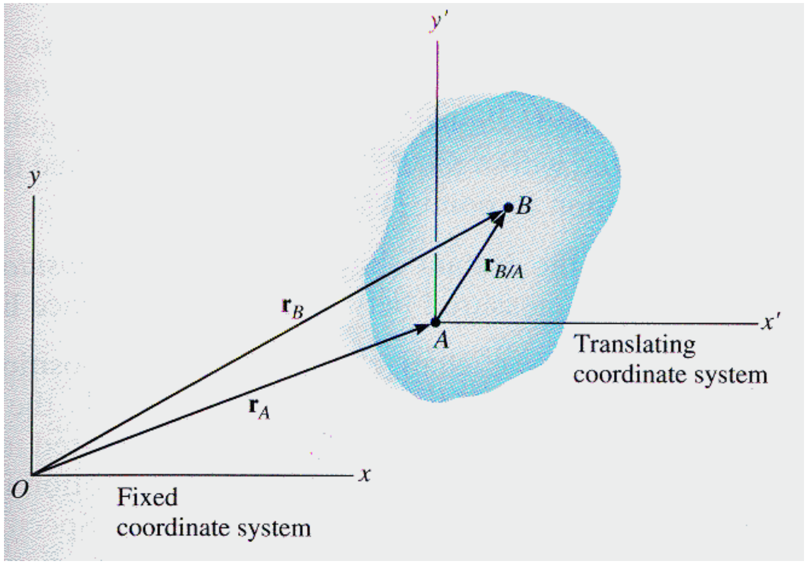
- A relationship between the instantaneous velocities of A and B is obtained by taking the time derivative of the position equation, which yields:

$$\mathbf{v_B} = \mathbf{v_A} + (\text{d}\, \mathbf{r_{B/A}} / \text{dt})$$

- Here $\mathbf{v_B}$ and $\mathbf{v_A}$ denote absolute velocities
- The term (d $\mathbf{r_{B/A}}$ / dt) = 0, since the magnitude of $\mathbf{r_{B/A}}$ is constant by definition of a rigid body, and because the body is translating the direction of $\mathbf{r_{B/A}}$ is constant.
- Therefore:

$$\mathbf{v_B} = \mathbf{v_A}$$

TRANSLATION



- **Acceleration:**
- Taking the time derivative of the acceleration equation yields:

$$\mathbf{a_B} = \mathbf{a_A}$$

- The above two equations indicate that all points in a rigid body subjected to either curvilinear or rectilinear translation move with the same velocity and acceleration
- So the kinematics of particle motion may be used to specify the kinematics of points located in a translating rigid body

ROTATION ABOUT A FIXED AXIS

- When a body is rotating about a fixed axis, any point P located in the body travels along a circular path.

Angular Motion:

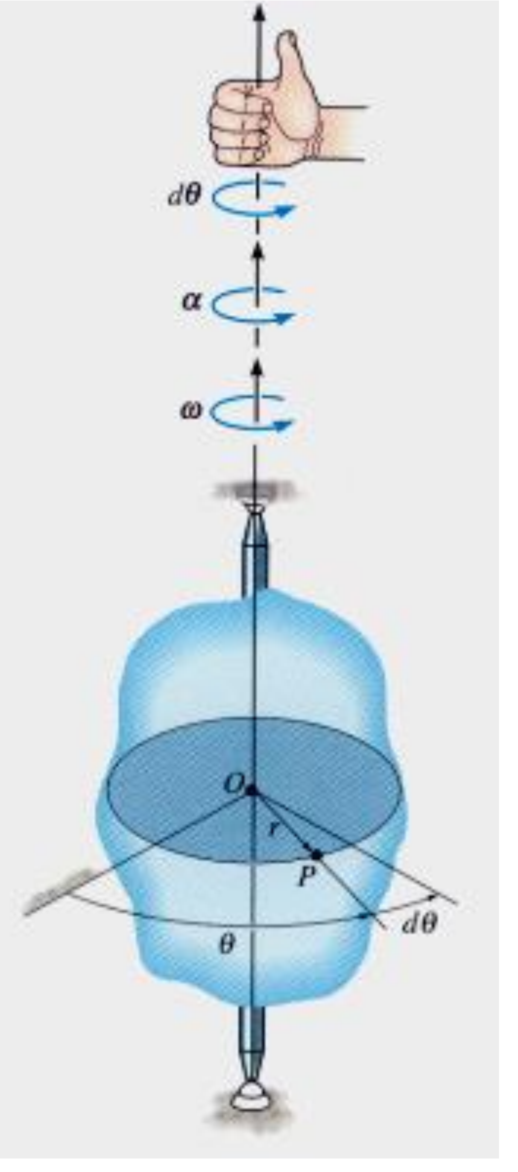
- Since a point is without dimension, it has no angular motion
- Only lines and bodies undergo angular motion
- Consider the body shown in the figure and the angular motion of a radial line r.

Angular Position:

- At any instant, the angular position of r is defined by the angle θ measured between a fixed reference line and r.

Angular Displacement:

- The change in angular position, which can be measured as a differential d**θ**, is called the angular displacement.



ROTATION ABOUT A FIXED AXIS

Angular Velocity:

- The time rate of change in the angular position is called the angular velocity ω
- Its magnitude is:

$$\omega = d\theta / dt \quad \text{-----------------------------}(1)$$

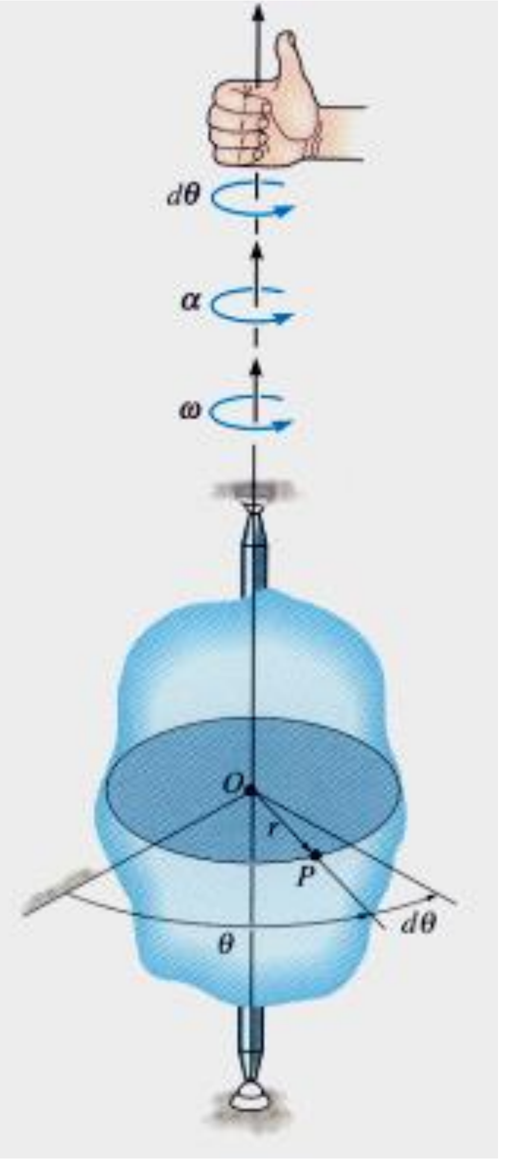
Angular Acceleration:

- The angular acceleration α measures the time rate of change of the angular velocity.
- Its magnitude is:

$$\alpha = d\omega / dt \quad \text{--------------------------------}(2)$$

or $$\alpha = d^2\theta / dt^2 \quad \text{-------------------------------}(3)$$

- The line of action of α is same as that for ω, however, its sense of direction depends on whether ω is increasing or decreasing with time.



ROTATION ABOUT A FIXED AXIS

- By eliminating dt from equations (1) and (2), we get:

$$\alpha \, d\theta = \omega \, d\omega \quad \text{---------------------------(4)}$$

- Similarity between the differential relations for angular motion and those developed for rectilinear motion of a particle

Constant angular acceleration:

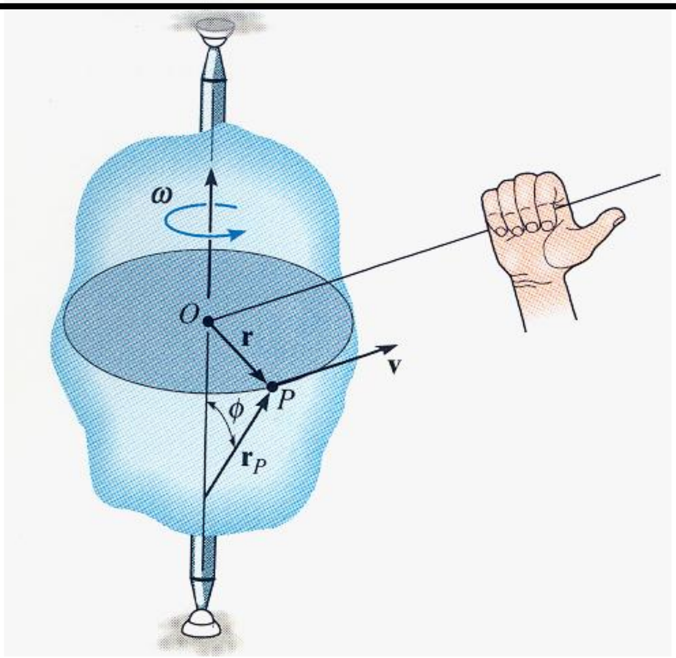
- If the angular acceleration of the body is constant, $\alpha = \alpha_c$, then eq (1), (2) and (4) when integrated, yield a set of formulas which relate the body's angular velocity, angular position and time. The results are:

$$\omega = \omega_0 + \alpha_c t \quad \text{------------------------------------(5)}$$

$$\theta = \theta_0 + \omega_0 t + (1/2)\, \alpha_c t^2 \quad \text{----------------------(6)}$$

$$\omega^2 = \omega_0^2 + 2\, \alpha_c\, (\theta - \theta_0) \quad \text{-----------------------(7)}$$

ROTATION ABOUT A FIXED AXIS



Motion of Point P:

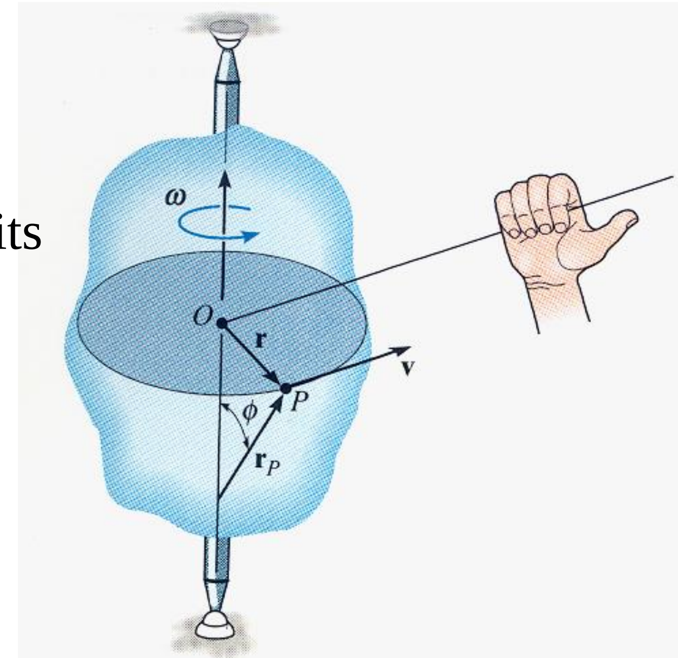
- As the rigid body rotates, point P travels along a circular path of radius r and center at point O.

Position:

- The position of P is defined by the position vector **r,** which extends from O to P.

ROTATION ABOUT A FIXED AXIS

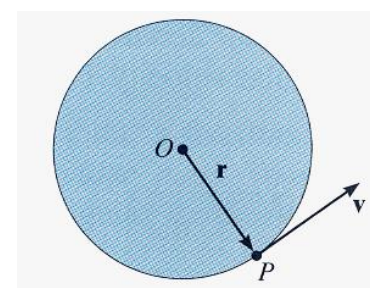
Velocity:



- The velocity of P has a magnitude which can be determined from the circular motion of P using its polar coordinate components $v_r = r^0$ and $v_\theta = r\theta^0$
- Since r is constant, the radial component $v_r = 0$, so that $v = v_\theta = r\theta^0$
- Because $\omega = \theta^0$, therefore:

$$v = \omega r$$

- The direction of **v** is tangent to the circular path.
- Both the magnitude and direction of **v** can also be found by using the cross product of w and r:



$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

ROTATION ABOUT A FIXED AXIS

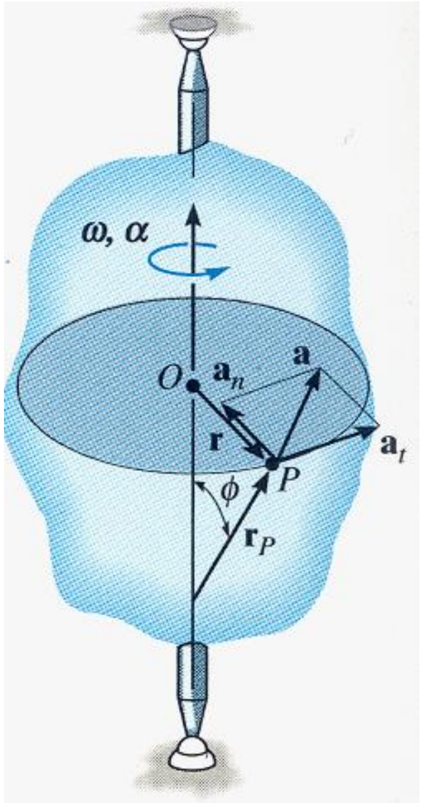
Acceleration:

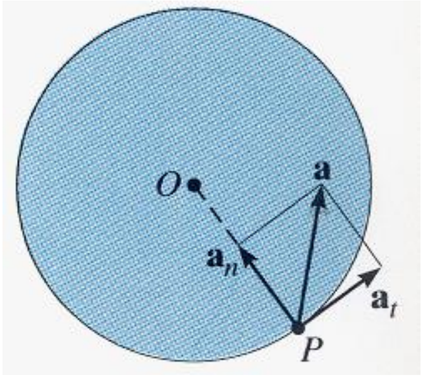
- The acceleration of P can be expressed in terms of its normal and tangential components
- Using $a_t = dv/dt$, $a_n = v^2 / \zeta$, noting that $\zeta = r$, $v = \omega r$, and $\alpha = d\omega/dt$, we have:

$$a_t = \alpha r$$

$$a_n = \omega^2 r$$

- The tangential component of acceleration represents the time rate of change in the velocity's magnitude. If the speed of P is increasing, then $\mathbf{a_t}$ acts in the same direction as **v**; if the speed is decreasing, $\mathbf{a_t}$ acts in the opposite direction of **v**; and if the speed is constant $\mathbf{a_t} = 0$
- The normal component of acceleration represents the time rate of change in the velocity's direction. The direction of $\mathbf{a_n}$ is always towards O, the center of the circular path





ROTATION ABOUT A FIXED AXIS

Acceleration:

- Like the velocity, the acceleration of point P may be expressed in terms of the vector cross product.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\boldsymbol{\omega} x \mathbf{r}) = \frac{d\boldsymbol{\omega}}{dt} \text{ x } \mathbf{r} + \boldsymbol{\omega} \text{ x } \frac{d\mathbf{r}}{dt}$$

- Since $d\boldsymbol{\omega}/dt = \boldsymbol{\alpha}$ and $d\mathbf{r}/dt = \mathbf{v} = \boldsymbol{\omega} \text{ x } \mathbf{r}$, so the above equation becomes:

$$\mathbf{a} = \boldsymbol{\alpha} \text{ x } \mathbf{r} + \boldsymbol{\omega} \text{ x } (\boldsymbol{\omega} \text{ x } \mathbf{r})$$

- In terms of the components, **a** can be expressed as:

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_n$$

$$\mathbf{a} = \boldsymbol{\alpha} \text{ x } \mathbf{r} - \omega^2 \mathbf{r}$$