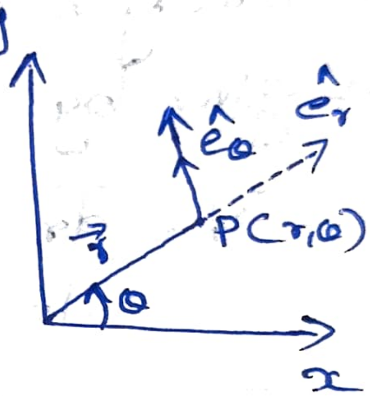
Velocity and acceleration in Plane Polar Coordinate system

Plane Polar Coordinates (r, θ)



The Position Vector $\vec{r}$ in Polar Coordinates is

$\vec{r} = x\hat{i} + y\hat{j}$ where $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} \longrightarrow (1)$$

The unit Vectors $\hat{e}_r$ and $\hat{e}_\theta$ are given by

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

(The unit Vectors are not Constant and changes with time)

So equation (1) becomes

$$\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$\boxed{\vec{r} = r\hat{e}_r} \longrightarrow (2)$$

So Velocity in ~~cylindrical~~ Polar Coordinates is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r)$$

$$\vec{v} = r\frac{d}{dt}(\hat{e}_r) + \hat{e}_r\frac{dr}{dt}$$

$$= r\dot{\hat{e}}_r + \hat{e}_r\dot{r} \longrightarrow (3)$$

$$\frac{dr}{dt} = \dot{r}$$

$$\frac{d(\hat{e}_r)}{dt} = \dot{\hat{e}}_r$$

But $\dot{\hat{e}}_r = \frac{d}{dt}(\hat{e}_r)$

$$\frac{d}{dt}(\hat{e}_r) = \frac{d}{dt}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$= -\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\cdot\frac{d\theta}{dt}\hat{j}$$

$$= -\sin\theta\,\dot{\theta}\,\hat{i} + \cos\theta\,\dot{\theta}\,\hat{j}$$

where $\dot{\theta} = \frac{d\theta}{dt}$

$$= \dot{\theta}[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}]$$

$$= \dot{\theta}\hat{e}_\theta$$

$$\dot{\hat{e}}_r = \frac{d}{dt}\left(\hat{e}_r\right) = \dot{\theta}\hat{e}_\theta$$

Equation 3 gives

$$\vec{V} = r\dot{\theta}\hat{e}_\theta + \dot{r}\hat{e}_r$$

$$\boxed{\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta} \longrightarrow (4)$$

which is expression for Velocity of Particles in Polar Coordinates.

In case of Polar Coordinate system

$\dot{r} = V_r$ = Radial Component of Velocity

$r\dot{\theta} = V_\theta$ = Transverse Component of Velocity

$$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta$$

Magnitude

$$V = \sqrt{V_r^2 + V_\theta^2}$$

$$\alpha = \tan^{-1}\left(\frac{V_\theta}{V_r}\right)$$

Acceleration

The acceleration is given by

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right]$$

$$\vec{a} = \frac{d}{dt}(\dot{r}\hat{e}_r) + \frac{d}{dt}(r\dot{\theta}\hat{e}_\theta)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\hat{e}_r) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta \longrightarrow (5)$$

But $\frac{d}{dt}(\hat{e}_\theta) = \dot{\hat{e}}_\theta$

$$= \frac{d}{dt}\left[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right]$$

$$= -\cos\theta\,\dot{\theta}\,\hat{i} + -\sin\theta\,\dot{\theta}\,\hat{j}$$

$$= -\cos\theta\,\dot{\theta}\,\hat{i} - \sin\theta\,\dot{\theta}\,\hat{j}$$

$$= -\dot{\theta}\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$= -\dot{\theta}\,\hat{e}_r$$

$$\frac{d}{dt}(\hat{e}_\theta) = -\dot{\theta}\hat{e}_r$$

$$\frac{d}{dt}(\hat{e}_r) = \dot{\theta}\hat{e}_\theta$$

$\therefore$ Eqn (5) becomes

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}(\dot{\theta}\hat{e}_r)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta} \longrightarrow (6)$$

which is expression for acceleration of Particle in Polar Coordinates.

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

where

$$a_r = (\ddot{r} - r\dot{\theta}^2)$$

$$a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$\therefore$ Eqn (6) becomes

$$\vec{a} = a_r\hat{e}_r + a_\theta \hat{e}_\theta$$

Magnitude

$$a = \sqrt{a_r^2 + a_\theta^2}$$

$$\phi = \tan^{-1}\left(\frac{a_\theta}{a_r}\right)$$