Velocity and Acceleration in cylindrical Coordinates.

The Position Vector $\vec{r}$ in cylindrical Coordinates is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

where $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\therefore \vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\hat{k} \longrightarrow (1)$$

The unit Vectors $\hat{e}_r$, $\hat{e}_\theta$ are given by

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

So eqn (1) becomes

$$\vec{r} = r\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right) + z\hat{k}$$

$$\boxed{\vec{r} = r\hat{e}_r + z\hat{k}} \longrightarrow (2)$$

So velocity in cylindrical Coordinate is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(r\hat{e}_r + z\hat{k}\right)$$

ie, $$\vec{v} = \hat{e}_r\frac{dr}{dt} + r\frac{d}{dt}(\hat{e}_r) + \frac{dz}{dt}\hat{k}$$

$$= \hat{e}_r\dot{r} + r\dot{\hat{e}}_r + \dot{z}\hat{k} \longrightarrow (3)$$

But $$\frac{d}{dt}(\hat{e}_r) = \frac{d}{dt}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$$

$$= -\sin\theta\,\dot{\theta}\,\hat{i} + \cos\theta\,\dot{\theta}\,\hat{j}$$

$$= \dot{\theta}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$$

$$= \dot{\theta}\hat{e}_\theta$$

$$\frac{d}{dt}(\hat{e}_r) = \dot{\theta}\hat{e}_\theta$$

Eqn (3) becomes

$$\boxed{\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}} \longrightarrow (4)$$

which is expression for velocity of Particle in cylindrical Coordinate.

$$\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}k$$

where $V_r = \dot{r}$

$V_\theta = r\dot{\theta}$

$V_z = \dot{z}$

$$V = \sqrt{V_r^2 + V_\theta^2 + V_z^2}$$

Acceleration :

The acceleration is given by

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + \dot{z}k\right)$$

$$\vec{a} = \frac{d}{dt}\left(\dot{r}\hat{e}_r\right) + \frac{d}{dt}\left(r\dot{\theta}\hat{e}_\theta\right) + \frac{d}{dt}\left(\dot{z}\hat{k}\right)$$

$$= \dot{r}\dot{\hat{e}}_r + \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta + \ddot{z}\hat{k} \rightarrow (5)$$

But $\frac{d}{dt}\left(\hat{e}_\theta\right) = \dot{\hat{e}}_\theta$

$$= \frac{d}{dt}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right)$$

$$= -\cos\theta\,\dot{\theta}\,\hat{i} - \sin\theta\,\dot{\theta}\,\hat{j}$$

$$= -\dot{\theta}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)$$

$$= -\dot{\theta}\hat{e}_r$$

$$\dot{\hat{e}}_r = \dot{\theta}\hat{e}_\theta$$

$$\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_r$$

$\therefore$ equation (5) becomes

$$\vec{a} = \dot{r}\dot{\theta}\hat{e}_\theta + \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}(\dot{\theta}\hat{e}_r) + \ddot{z}\hat{k}$$

$$= \dot{r}\dot{\theta}\hat{e}_\theta + \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r + \ddot{z}\hat{k}$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}} \rightarrow (6)$$

which is expression for acceleration of a Particle in cylindrical Coordinates.

where

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$a_z = \ddot{z}$$

$$a = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$$