Velocity and acceleration is Spherical Polar Coordinates

The Position Vector $\vec{r}$ in spherical polar coordinate is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

where

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\therefore \vec{r} = r\sin\theta\,\cancel{\cos\theta}\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k} \quad \longrightarrow (1)$$

The unit Vectors $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$ along r, θ and ϕ are

$$\hat{e}_r = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \hat{k}\cos\theta$$

as $\left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}$

$$= 1$$

$$\hat{e_\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{\sqrt{r^2(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta)}}$$

$$= \underline{\cos\theta\cos\phi\,\hat{i}} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e_\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = \frac{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}}{\sqrt{r^2\sin^2\theta(\sin^2\phi + \cos^2\phi)}}$$

$$= \underline{-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}}$$

Using value of $\hat{e_r}$, in equation (1) we get

$$\boxed{\vec{r} = r\hat{e_r}} \longrightarrow (2)$$

So the velocity is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e_r})$$

$$= \frac{dr}{dt}\hat{e_r} + r\frac{d}{dt}(\hat{e_r})$$

$$\vec{v} = \dot{r}\hat{e_r} + r\frac{d}{dt}(\hat{e_r}) \longrightarrow (3)$$

Now $\frac{d}{dt}(\hat{e}_r) = \frac{d}{dt}\left(\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \hat{k}\cos\theta\right)$

$= \cos\theta\dot{\theta}\cos\phi\,\hat{i} - \sin\theta\sin\phi\,\dot{\phi}\,\hat{i} + \cos\theta\dot{\theta}\sin\phi\,\hat{j} + \sin\theta\cos\phi\,\dot{\phi}\,\hat{j} + \hat{k}(-\sin\theta)\dot{\theta}$

$= \dot{\theta}\left[\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\right] + \dot{\phi}\left[\sin\theta\cos\phi\,\hat{j} - \sin\theta\sin\phi\,\hat{i}\right]$

$\frac{d}{dt}(\hat{e}_r) = \dot{\theta}\hat{e}_\theta + \dot{\phi}\sin\theta\,\hat{e}_\phi \longrightarrow (4)$

(using values of $\hat{e}_\theta$ and $\hat{e}_\phi$)

So using eqn (4) eqn (3) we get

$\vec{v} = \dot{r}\hat{e}_r + r\left(\dot{\theta}\hat{e}_\theta + \dot{\phi}\sin\theta\,\hat{e}_\phi\right)$

$$\boxed{\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\phi}\,\hat{e}_\phi} \longrightarrow (5)$$

which is the expression for velocity of Particle in Spherical Polar Coordinates

Expression for Acceleration

The acceleration is given by

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}\left[\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi\right)$$

$$\vec{a} = \frac{d}{dt}\left(\dot{r}\hat{e}_r\right) + \frac{d}{dt}\left(r\dot{\theta}\hat{e}_\theta\right) + \frac{d}{dt}\left(r\sin\theta\,\dot{\phi}\,\hat{e}_\phi\right)$$

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\hat{e}_r) + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}(\hat{e}_\theta) + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\sin\theta\,\dot{\phi}\,\hat{e}_\phi + r\cos\theta\,\dot{\theta}\dot{\phi}\,\hat{e}_\phi + r\sin\theta\,\dot{\phi}\frac{d}{dt}(\hat{e}_\phi) + r\sin\theta\,\hat{e}_\phi\,\ddot{\phi}$$

$$\longrightarrow (6)$$

Now $\frac{d}{dt}(\hat{e}_\theta) = \frac{d}{dt}\left(\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \hat{k}\sin\theta\right)$

$$= -\sin\theta\,\dot{\theta}\cos\phi\,\hat{i} - \cos\theta\sin\phi\,\dot{\phi}\,\hat{i} - \sin\theta\,\dot{\theta}\sin\phi\,\hat{j} + \dot{\phi}\cos\theta\cos\phi\,\hat{j} - \hat{k}\cos\theta\,\dot{\theta}$$

or

$$\frac{d}{dt}(\hat{e}_\theta) = -\dot{\theta}\left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\right.$$
$$\left. + \dot{\phi}\cos\theta(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)$$

$$= -\dot{\theta}\hat{e}_r + \dot{\phi}\cos\theta\,\hat{e}_\phi \longrightarrow (7)$$

(using values of $\hat{e}_r$ and $\hat{e}_\phi$)

Also

$$\frac{d}{dt}(\hat{e}_\phi) = \frac{d}{dt}(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

$$= -\cos\phi\,\dot{\phi}\,\hat{i} - \sin\phi\,\dot{\phi}\,\hat{j} \longrightarrow (8)$$

so using the values of $\frac{d}{dt}(\hat{e}_r)$, $\frac{d}{dt}(\hat{e}_\theta)$ and $\frac{d}{dt}(\hat{e}_\phi)$ in eqn (6) we get

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta + \dot{\phi}\sin\theta\,\hat{e}_\phi) + r\ddot{\theta}\hat{e}_\theta +$$
$$\dot{r}\dot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r + \dot{\phi}\cos\theta\,\hat{e}_\phi) +$$
$$\dot{r}\sin\theta\,\dot{\phi}\,\hat{e}_\phi + \cancel{r\cos\theta\dot{\theta}\hat{e}_\phi}\; r\cos\theta\,\dot{\theta}\dot{\phi}\,\hat{e}_\phi +$$
$$r\sin\theta\,\dot{\phi}(-\cos\phi\,\dot{\phi}\,\hat{i} - \sin\phi\,\dot{\phi}\,\hat{j}) +$$
$$r\sin\theta\,\hat{e}_\phi\,\ddot{\phi}$$

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\hat{e}_\phi + r\ddot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r + r\dot{\theta}\dot{\phi}\cos\theta\,\hat{e}_\phi + \dot{r}\sin\theta\,\dot{\phi}\hat{e}_\phi + r\cos\theta\,\dot{\theta}\dot{\phi}\hat{e}_\phi + r\sin\theta\,\hat{e}_\phi\ddot{\phi} - r\sin\theta\,\dot{\phi}^2(\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$$

$$\boxed{\cos\phi\,\hat{i} + \sin\phi\,\hat{j} = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta}$$

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\hat{e}_\phi + r\ddot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r + r\dot{\theta}\dot{\phi}\cos\theta\,\hat{e}_\phi + \dot{r}\sin\theta\,\dot{\phi}\hat{e}_\phi + r\cos\theta\,\dot{\theta}\dot{\phi}\hat{e}_\phi + r\sin\theta\,\hat{e}_\phi\ddot{\phi} - r\sin\theta\,\dot{\phi}^2(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta)$$

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\phi}\sin\theta\,\hat{e}_\phi + r\ddot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r + r\dot{\theta}\dot{\phi}\cos\theta\,\hat{e}_\phi + \dot{r}\sin\theta\,\dot{\phi}\hat{e}_\phi + r\cos\theta\,\dot{\theta}\dot{\phi}\hat{e}_\phi + r\sin\theta\,\hat{e}_\phi\ddot{\phi} - r\sin^2\theta\,\dot{\phi}^2\hat{e}_r - r\sin\theta\cos\theta\,\dot{\phi}^2\hat{e}_\theta$$

$$\boxed{\begin{aligned}\vec{a} = {} & \left(\ddot{r} - r\dot{\theta}^2 - \dot{\phi}^2 r\sin^2\theta\right)\hat{e}_r + \\ & \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\phi}^2\right)\hat{e}_\theta + \\ & \left(r\sin\theta\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\cos\dot{\theta}\dot{\phi}\right)\hat{e}_\phi\end{aligned}}$$

$\longrightarrow$ (9)

which is the expression for acceleration of a particle in spherical polar coordinates.