Computational Physics

Unit 3

Kinematics of particles, assumptions, Cartesian, Cylindrical and Spherical frames, and motion of particles in them. Translation and rotation of rigid bodies in 2D – Translation and rotation of rigid bodies in 3D.

Introduction

Kinematics: is the branch of dynamics which describes the motion of bodies *without* reference to the forces that either causes the motion or are generated as a result of the motion.

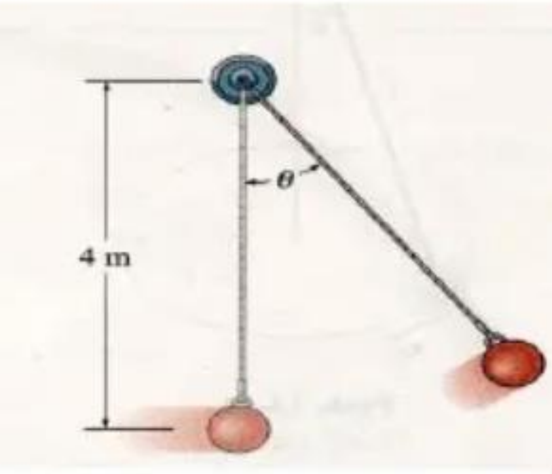
- Kinematics is often referred to as the "geometry of motion"

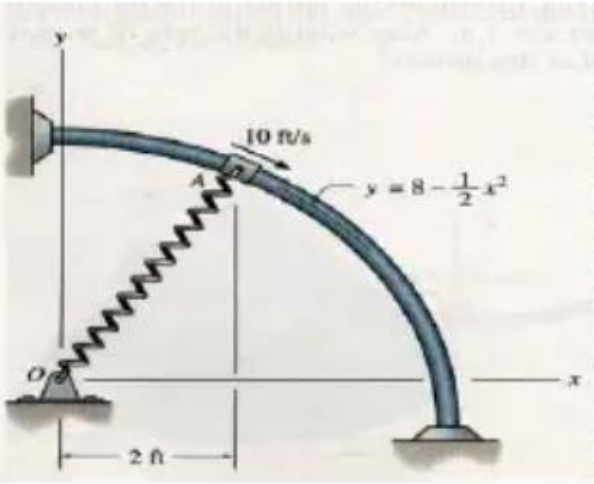
Examples of *kinematics problems* that engage the attention of engineers.

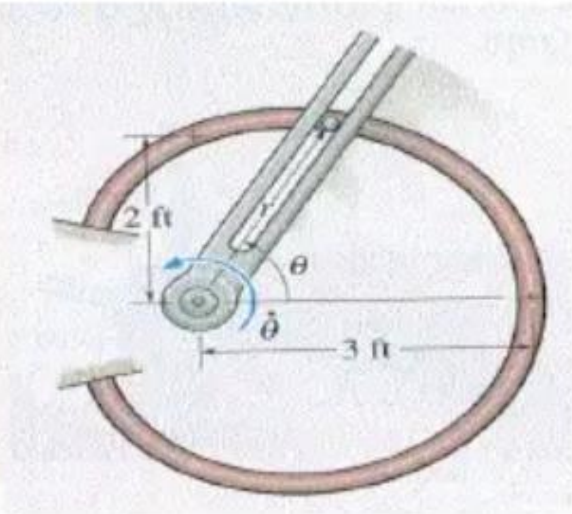
- The design of *cams*, *gears*, *linkages*, and other *machine elements* to control or produce certain desired motions, and
- The calculation of flight trajectory for aircraft, rockets and spacecraft.

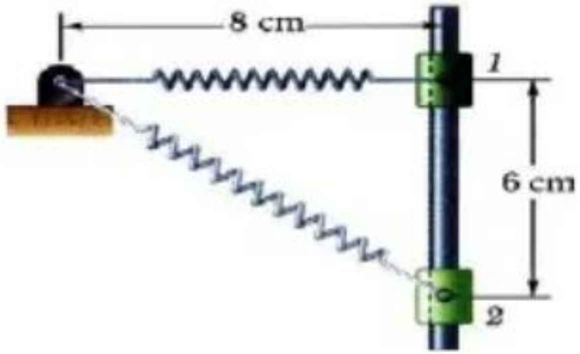
- If the particle is *confined* to a *specified path*, as with a bead sliding along a fixed wire, its motion is said to be *Constrained.*

Example 1. - A small rock tied to the end of a string and whirled in a circle undergoes constrained motion until the string breaks



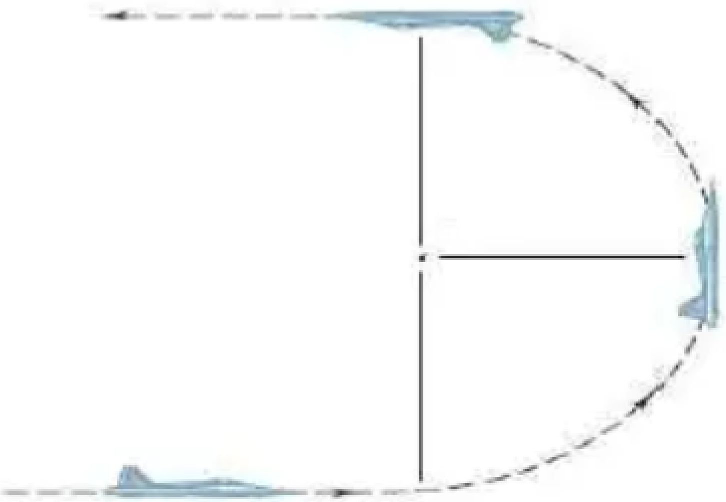






- If there are *no* **physical guides**, the motion is said to be *unconstrained*.

Example 2. - Airplane, rocket



- The position of particle P at any time t can be described by specifying its:
  - *Rectangular* coordinates; X,Y,Z
  - *Cylindrical* coordinates; r,θ,z
  - *Spherical* coordinates; R, θ,Φ
- Also described by measurements along the tangent *t* and normal *n* to the curve(path variable).

- The motion of particles(or rigid bodies) may be described by using coordinates measured from fixed reference axis *(absolute motion analysis)* or by using coordinates measured from moving reference axis *(relative motion analysis).*

Rectangular co-ordinates (x-y-z)

- This is particularly useful for describing motions where the *x,y and z-components* of acceleration are *independently generated.*

- When the position of a particle P is defined at any instant by its rectangular coordinate x,y and z, it is convenient to resolve the *velocity v* and the *acceleration a* of the particle into rectangular components.

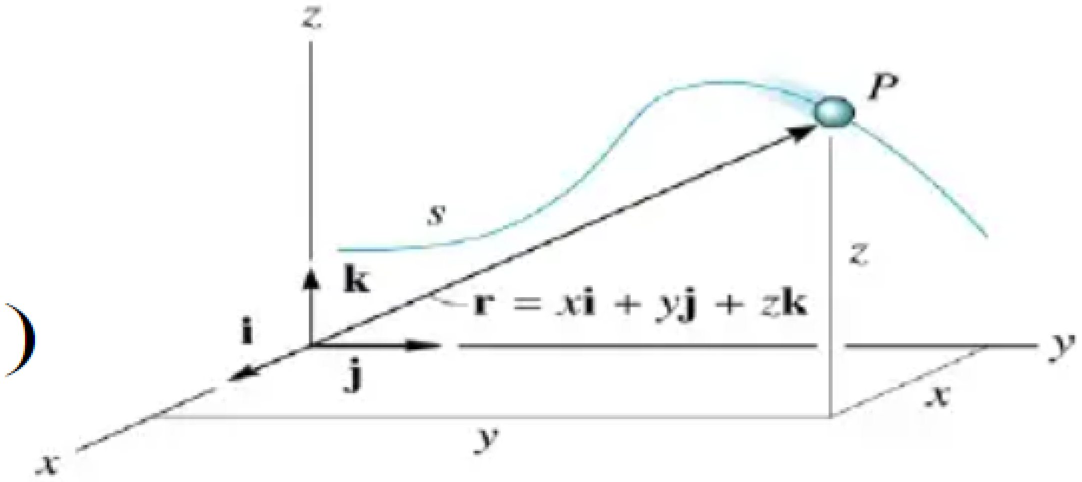
- Resolving the position vector r of the particle into rectangular components,

$$r = xi + yj + zk$$

- Differentiating

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

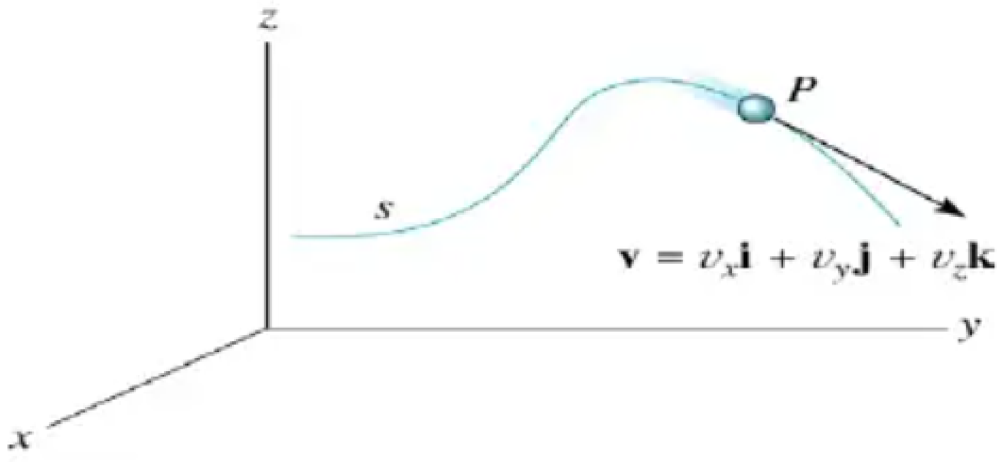
$$v = \dot{x}\, i + \dot{y}\, j + \dot{z}\, k$$



Position

- All of the following are equivalent:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$
$$= \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$
$$= v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$



Velocity

- Since the *speed* is defined as the magnitude of the velocity, we have:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

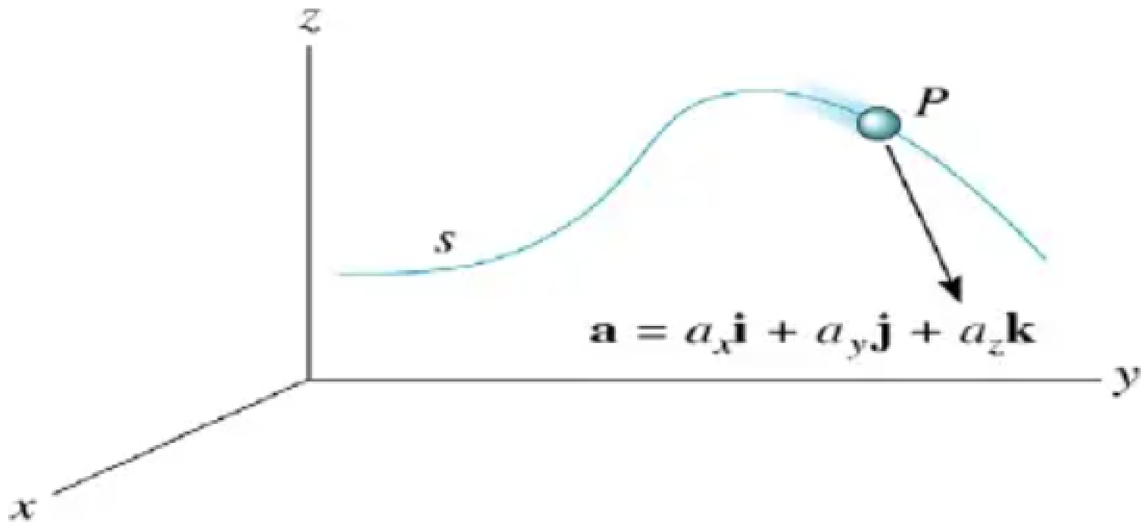
Similarly,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v_x\hat{i} + v_y\hat{j} + v_z\hat{k})$$

$$= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$= \dot{v}_x\hat{i} + \dot{v}_y\hat{j} + \dot{v}_z\hat{k}$$

$$= \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$



Acceleration

- The magnitude of the acceleration vector is:

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

- From the above equations that the scalar components of the velocity and acceleration are

$$v_x = \dot{x} \qquad v_y = \dot{y} \qquad v_z = \dot{z}$$

$$a_x = \ddot{x} \qquad a_y = \ddot{y} \qquad a_z = \ddot{z}$$

- The use of *rectangular components* to describe the *position*, the *velocity* and the *acceleration* of a particle is particularly *effective* when the component a_x of the acceleration depends only upon t, x and/or v_x, similarly for a_y and a_z.

- The motion of the particle in the *x direction*, its motion in the *y direction*, and its motion in the *z direction* can be considered *separately*.

- When a particle moves along a curved path, it is sometimes convenient to describe its motion using coordinates other than Cartesian.

Kinematics of Rigid Bodies

In physics, a **rigid body**, also known as a **rigid object**, is a solid body in which deformation is zero or negligible. The distance between any two given points on a rigid body remains constant in time regardless of external forces or moments exerted on it.

The body is in rest:When a body does not change its position with respect to time.
The body is in motion: if a body changes its position with respect to time.

There are different types of motion:

- Translational
- Rotational
- Periodic and
- Non-periodic

What is Translatory Motion?

The motion in which all points of a moving body move uniformly along a straight line is called translatory motion.

Translatory motion can be of two types:

- rectilinear and
- curvilinear.

Difference between rectilinear and curvilinear translatory motion

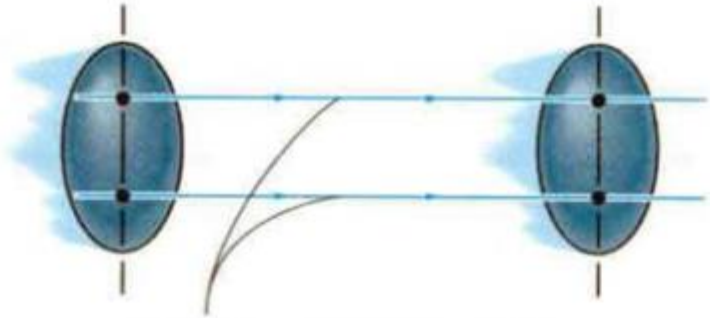
Rectilinear Motion	Curvilinear Motion
When an object in translatory motion moves along a straight line, it is said to be in rectilinear motion	When an object in translatory motion moves along a curved path, it is said to be in curvilinear motion
A car moving along a straight path and the train moving in a straight track are examples of rectilinear motion	A stone thrown up in the air at a certain angle and a car taking a turn are examples of curvilinear motion

Key Features of Translatory Motion:

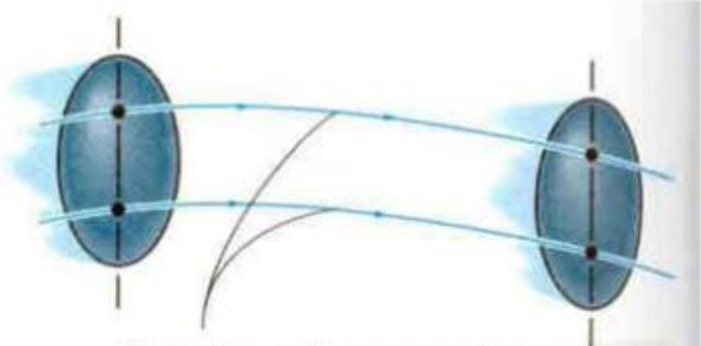
Translation:
- No rotation of any line in the body
- All points in body have same velocity and acceleration
- No relative motion between any two particles

- ❑ In translatory motion, every point on the object undergoes the same displacement over a given time interval.
- ❑ There is no relative motion between different parts of the object.

Translation



Path of rectilinear translation

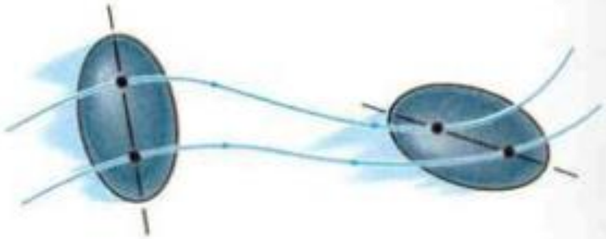


Path of curvilinear translation

Translation: Translation occurs if every line segment on the body remains parallel to its original direction during the motion. When all points move along straight lines, the motion is called rectilinear translation. When the paths of motion are curved lines, the motion is called curvilinear translation.

Rotation about a fixed axis

Rotation about a fixed axis. In this case, all the particles of the body, except those on the axis of rotation, move along circular paths in planes perpendicular to the axis of rotation.



General plane motion

General plane motion. In this case, the body undergoes both translation and rotation. Translation occurs within a plane and rotation occurs about an axis perpendicular to this plane.

Rigid-Body Motion

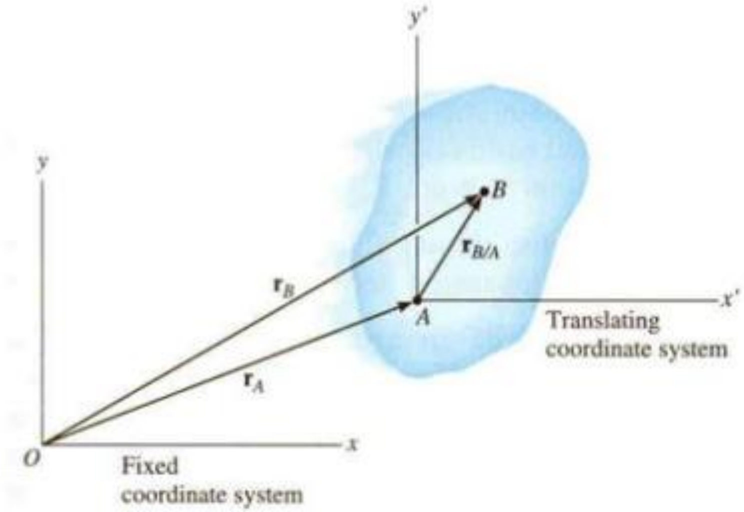
Types of rigid body planar motion

Translation – only linear direction

Rotational about fixed axis – rotational motion

General plane motion – consists of both linear and rotational motion

RIGID-BODY MOTION: TRANSLATION



The positions of two points A and B on a translating body can be related by

$$\boldsymbol{r}_B = \boldsymbol{r}_A + \boldsymbol{r}_{B/A}$$

where $\boldsymbol{r}_A$ & $\boldsymbol{r}_B$ are the absolute position vectors defined from the fixed x-y coordinate system, and $\boldsymbol{r}_{B/A}$ is the relative-position vector between B and A.

The velocity at B is $\boldsymbol{v}_B = \boldsymbol{v}_A + d\boldsymbol{r}_{B/A}/dt$.

Now $d\boldsymbol{r}_{B/A}/dt = \boldsymbol{0}$ since $\boldsymbol{r}_{B/A}$ is constant. So, $\boldsymbol{v}_B = \boldsymbol{v}_A$, and by following similar logic, $\boldsymbol{a}_B = \boldsymbol{a}_A$.

Note, all points in a rigid body subjected to translation move with the same velocity and acceleration.

Translation

Position

$$\mathrm{r}_B = \mathrm{r}_A + \mathrm{r}_{B/A}$$

Velocity

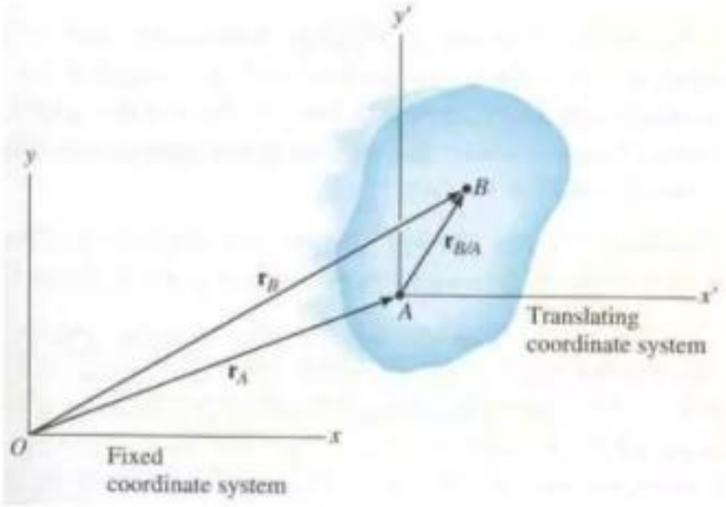
$$\mathrm{v}_B = \mathrm{v}_A + d\ \mathrm{r}_{B/A} / dt$$

$$\mathrm{r}_{B/A} = const.$$

$$\mathrm{v}_B = \mathrm{v}_A$$

Acceleration

$$\mathrm{a}_B = \mathrm{a}_A$$



All points move with same velocity and acceleration

If the linear acceleration of the body is constant, $a=a_c$ the equations for linear velocity and acceleration can be integrated to yield the set of algebraic equations below

- Constant acceleration

$$v = v_0 + a_c t$$

$$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$$

$$v^2 = v_0^2 + 2a_c(s - s_0)$$

Summary

- Time dependent acceleration

$$s(t)$$

$$v = \frac{ds}{dt}$$

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

$$a\,ds = v\,dv$$

- Constant acceleration

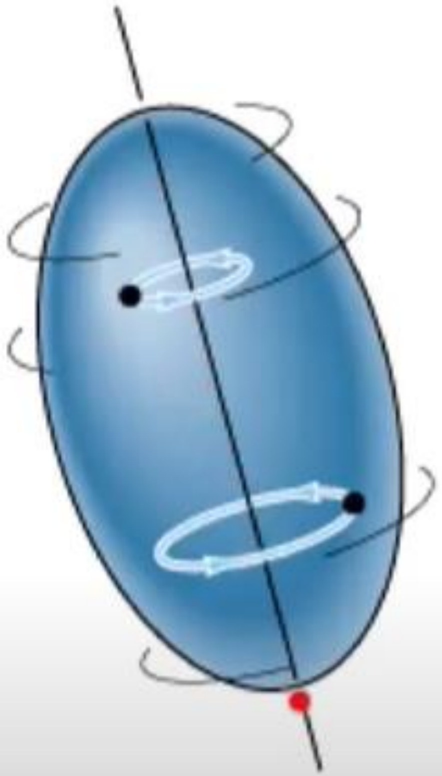
$$v = v_0 + a_c t$$

$$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$$

$$v^2 = v_0^2 + 2a_c(s - s_0)$$

Rotation about Fixed Axis

All particles of the body move along circular paths except those which lie on the axis of rotation



Rotation about a fixed axis

RIGID-BODY MOTION: ROTATION ABOUT A FIXED AXIS

When a body rotates about a fixed axis, any point P in the body travels along a circular path. The angular position of P is defined by θ.

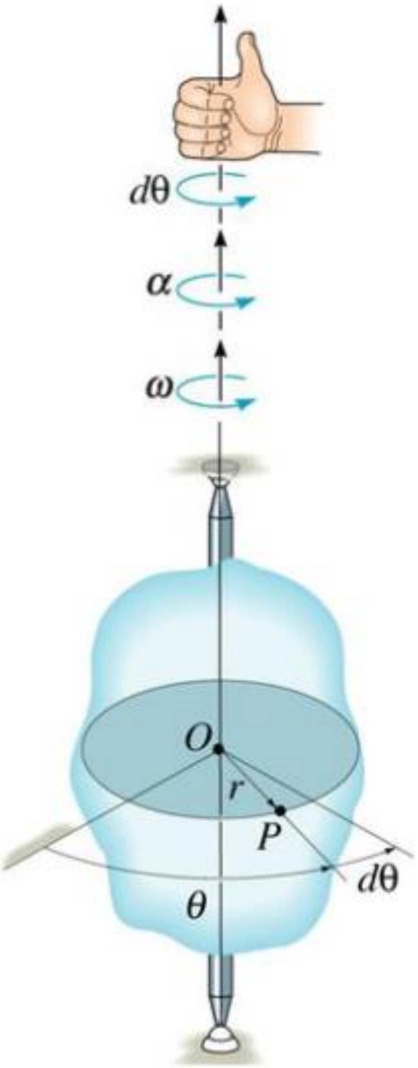
Angular Position (θ)

- Defined by the angle θ measured between a fixed reference line and *r*
- Measured in *rad*

Angular Displacement

- Measured as dθ
- Vector quantity
- Measured in radians or revolutions
- 1 rev = 2 π rad

RIGID-BODY MOTION: ROTATION ABOUT A FIXED AXIS



When a body rotates about a fixed axis, any point P in the body travels along a circular path. The angular position of P is defined by θ.

The change in angular position, $d\theta$, is called the angular displacement, with units of either radians or revolutions. They are related by

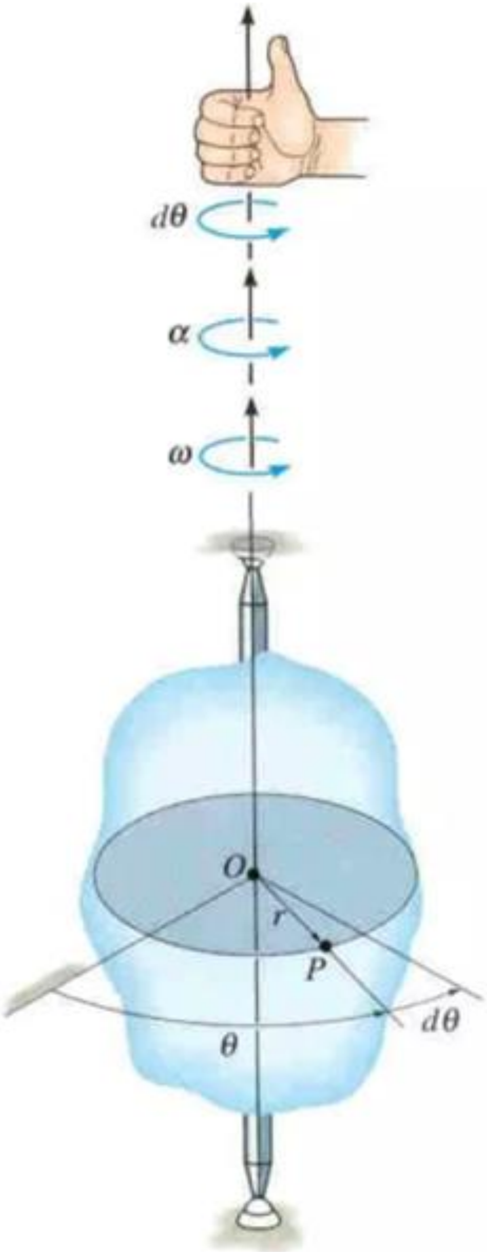
$$1 \text{ revolution} = 2\pi \text{ radians}$$

Angular velocity, ω, is obtained by taking the time derivative of angular displacement:

$\omega = d\theta/dt$ (rad/s) $+$

Similarly, angular acceleration is

$\alpha = d^2\theta/dt^2 = d\omega/dt$ or $\alpha = \omega(d\omega/d\theta)$ $+$ rad/s^2



Angular velocity (ω)

"the time rate of change in the angular position"

$$\boxed{\omega = \frac{d\theta}{dt} = \dot{\theta}}$$

Angular acceleration

"the time rate of change of the angular velocity"

$$\boxed{\alpha = \frac{d\omega}{dt} = \ddot{\theta}} \qquad \boxed{\alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}}$$

$\alpha = f(\theta)$

$$\boxed{\alpha \, d\theta = \omega \, d\omega}$$

RIGID-BODY MOTION: ROTATION ABOUT A FIXED AXIS

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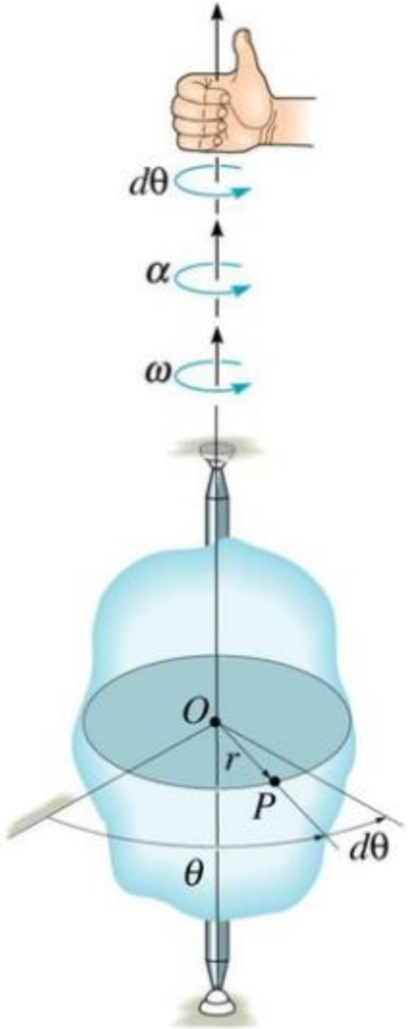


If the angular acceleration of the body is constant, $\boldsymbol{\alpha} = \boldsymbol{\alpha}_C$, the equations for angular velocity and acceleration can be integrated to yield the set of algebraic equations below.

$$\omega = \omega_O + \alpha_C t$$
$$\theta = \theta_O + \omega_O t + 0.5\alpha_C t^2$$
$$\omega^2 = (\omega_O)^2 + 2\alpha_C\,(\theta - \theta_O)$$

θ_O and ω_O are the initial values of the body's angular position and angular velocity. Note these equations are very similar to the constant acceleration relations developed for the rectilinear motion of a particle.

Comparison

The kinematics equations for rotational and translation motion:

$\boxed{s(t)}$ $\boxed{v = \frac{ds}{dt}}$ $\boxed{a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}}$	$\boxed{\theta(t)}$ $\boxed{\omega = \frac{d\theta}{dt}}$ $\boxed{\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}}$
$\boxed{a\,ds = v\,dv}$	$\boxed{\alpha\,d\theta = \omega\,d\omega}$
$v = v_0 + a_c t$	$\omega = \omega_0 + \alpha_c t$
$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$
$v^2 = v_0^2 + 2a_c(s - s_0)$	$\omega^2 = \omega_0{}^2 + 2\alpha_c(\theta - \theta_0)$

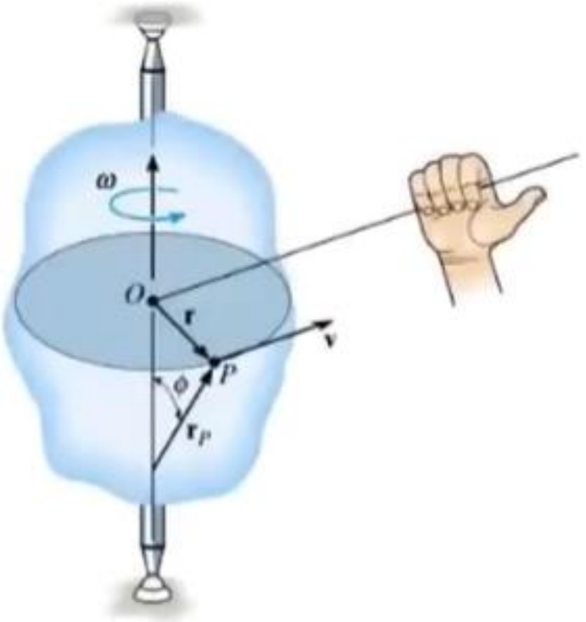
Rigid Body Under Constant Angular Acceleration

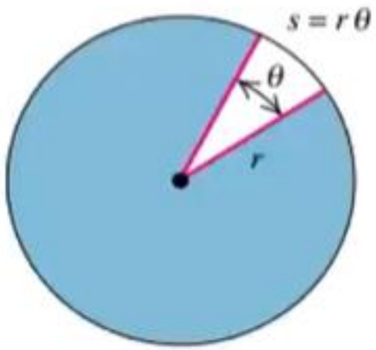
Motion of Point P

Position :

Is defined by the position vector *r*

The arc-length is $s = r\theta$



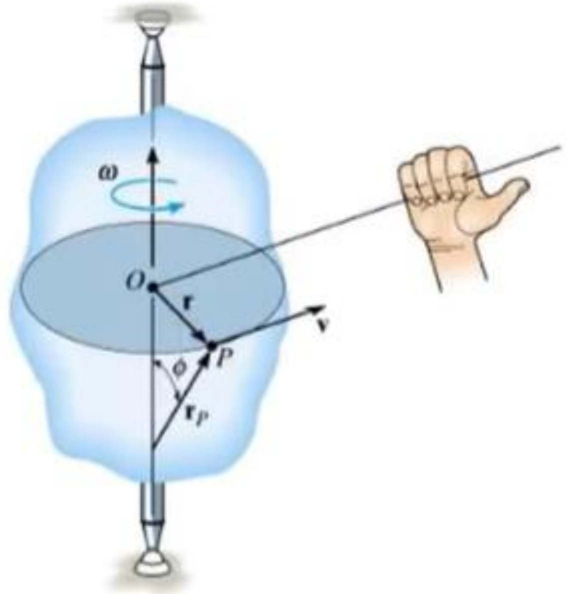


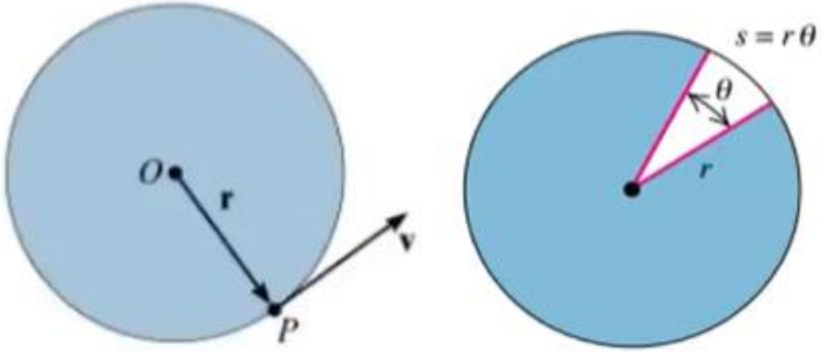
Motion of Point P

Velocity

$$v = \frac{ds}{dt} = \frac{d(r\theta)}{dt} = \frac{d\theta}{dt} r$$

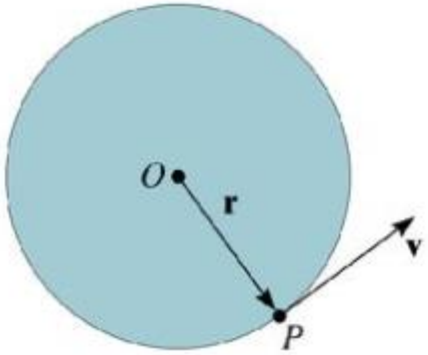
$$v = \omega r$$

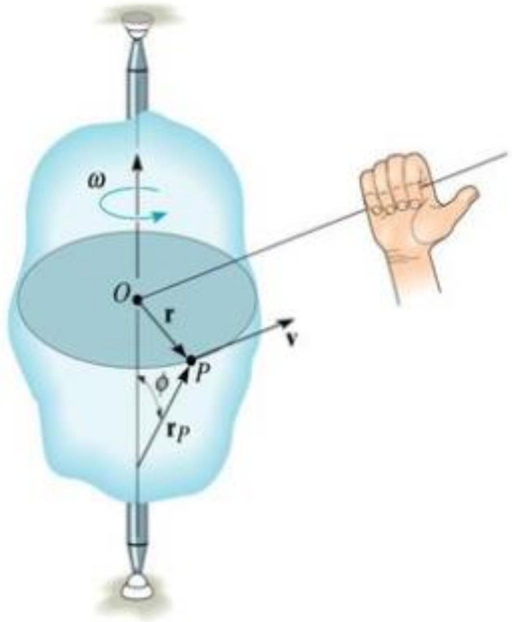




The magnitude of the velocity of P is equal to ωr. The velocity's direction is tangent to the circular path of P.

RIGID-BODY ROTATION: VELOCITY OF POINT P



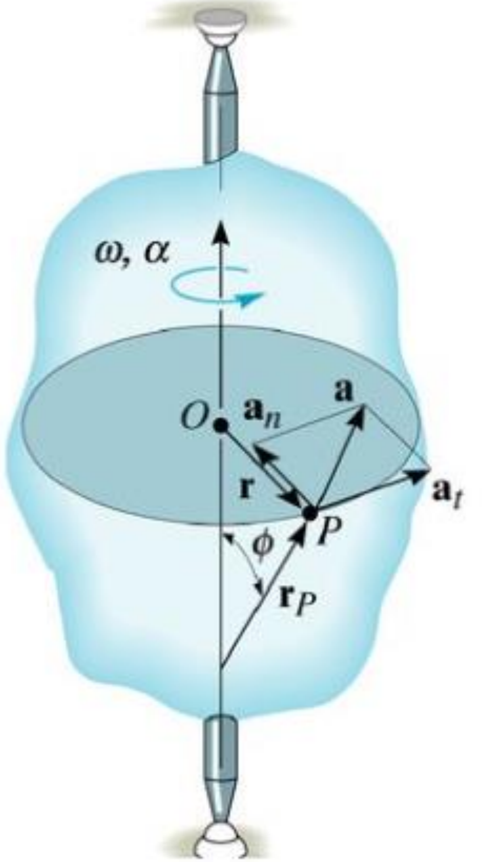


In the vector formulation, the magnitude and direction of $\boldsymbol{v}$ can be determined from the cross product of $\boldsymbol{\omega}$ and $\boldsymbol{r}_p$. Here $\boldsymbol{r}_p$ is a vector from any point on the axis of rotation to P.

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}_p = \boldsymbol{\omega} \times \boldsymbol{r}$$

The direction of $\boldsymbol{v}$ is determined by the right-hand rule.

RIGID-BODY ROTATION: ACCELERATION OF POINT P



The acceleration of P is expressed in terms of its normal ($\boldsymbol{a}_n$) and tangential ($\boldsymbol{a}_t$) components. In scalar form, these are $a_t = \alpha\, r$ and $a_n = \omega^2\, r$.

The tangential component, $\boldsymbol{a}_t$, represents the time rate of change in the velocity's magnitude. It is directed tangent to the path of motion.

The normal component, $\boldsymbol{a}_n$, represents the time rate of change in the velocity's direction. It is directed toward the center of the circular path.

RIGID-BODY ROTATION: ACCELERATION OF POINT P

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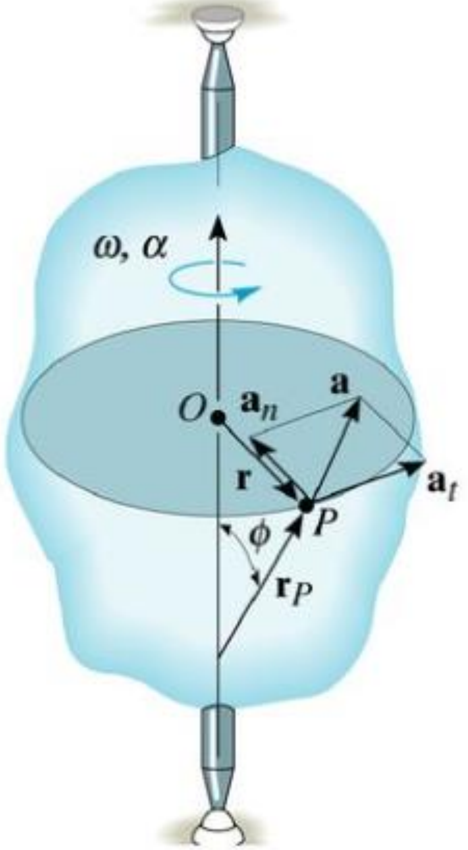


Using the vector formulation, the acceleration of P can also be defined by differentiating the velocity.

$$\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{dt} = \mathrm{d}\boldsymbol{\omega}/\mathrm{dt} \times \boldsymbol{r}_\mathrm{P} + \boldsymbol{\omega} \times \mathrm{d}\boldsymbol{r}_\mathrm{P}/\mathrm{dt}$$

$$= \boldsymbol{\alpha} \times \boldsymbol{r}_\mathrm{P} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_\mathrm{P})$$

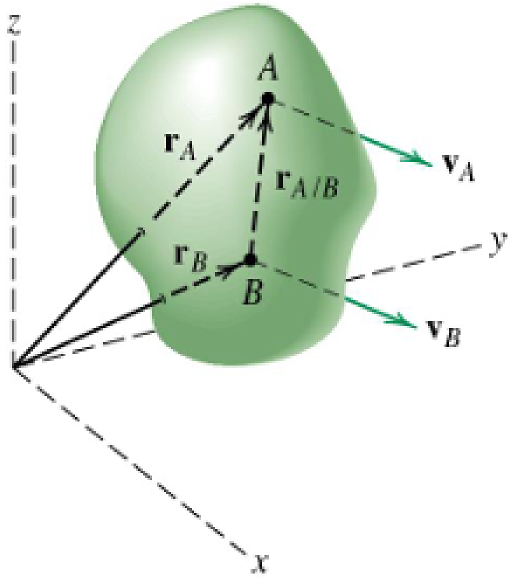
It can be shown that this equation reduces to

$$\boldsymbol{a} = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r} = \boldsymbol{a}_\mathrm{t} + \boldsymbol{a}_\mathrm{n}$$

The magnitude of the acceleration vector is $a = \sqrt{(a_t)^2 + (a_n)^2}$

THREE-DIMENSIONAL KINEMATICS OF RIGID BODIES

1. TRANSLATION



$$\vec{r}_A = \vec{r}_B + \vec{r}_{A/B}$$

$$\vec{v}_A = \vec{v}_B$$

$$\vec{a}_A = \vec{a}_B$$

$\vec{r}_{A/B}$ remains constant and therefore its time derivative is zero.

Thus, all points in the body have the same velocity and the same acceleration.

2. FIXED-AXIS ROTATION

Fixed Axis Rotation

:: ω does not change its direction since its lies along the fixed axis

:: Any point *A* in the body (not on the axis) moves in a circular arc in a plane normal to the axis; its velocity:

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}}$$

since **r** = **h** + **b** and ω x **h** = **0**

And acceleration of *A* is given by time derivative:

$$\boxed{\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}$$

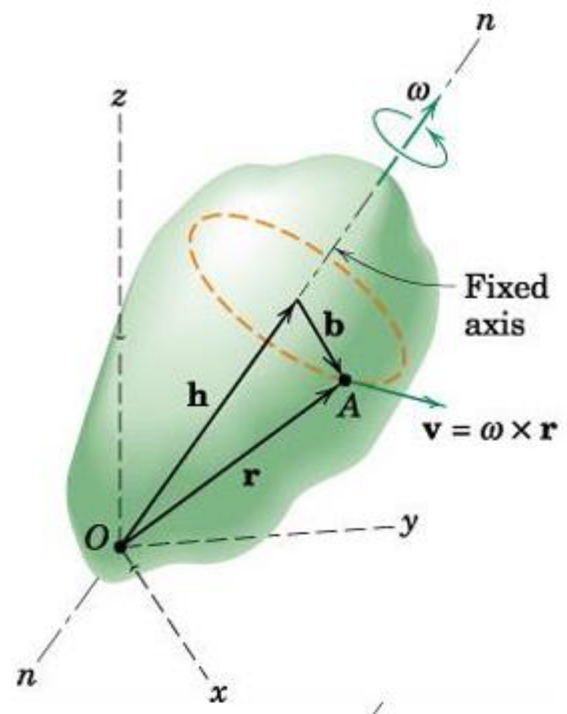
since $\dot{\mathbf{r}} = \mathbf{v} = \omega \times \mathbf{r}$

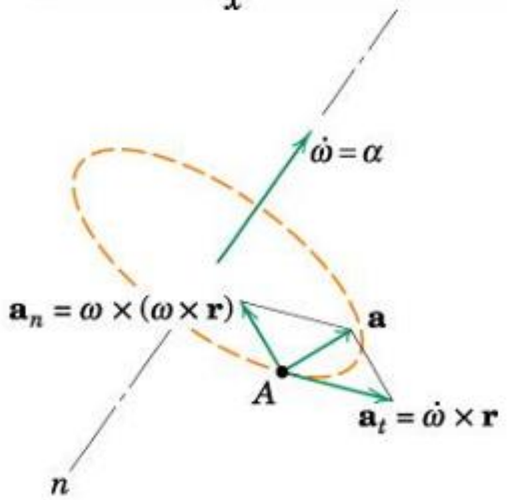
The normal and tangential components of **a** for the circular motion:

$$a_n = |\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})| = b\omega^2 \qquad a_t = |\dot{\boldsymbol{\omega}} \times \mathbf{r}| = b\alpha$$

Since **v** and **a** are perpendicular to ω and $\dot{\omega}$:

$$\mathbf{v}\cdot\boldsymbol{\omega} = 0, \mathbf{v}\cdot\dot{\boldsymbol{\omega}} = 0, \mathbf{a}\cdot\boldsymbol{\omega} = 0, \text{and } \mathbf{a}\cdot\dot{\boldsymbol{\omega}} = 0$$





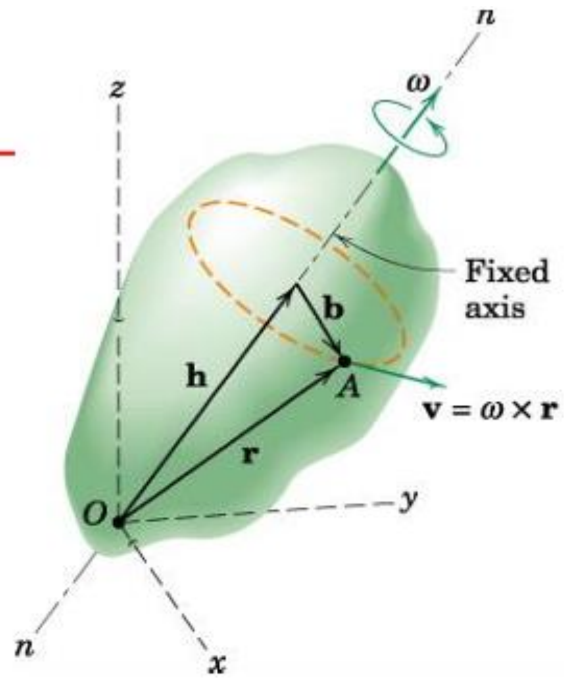
3-D Kinematics

Rotation about a Fixed Point

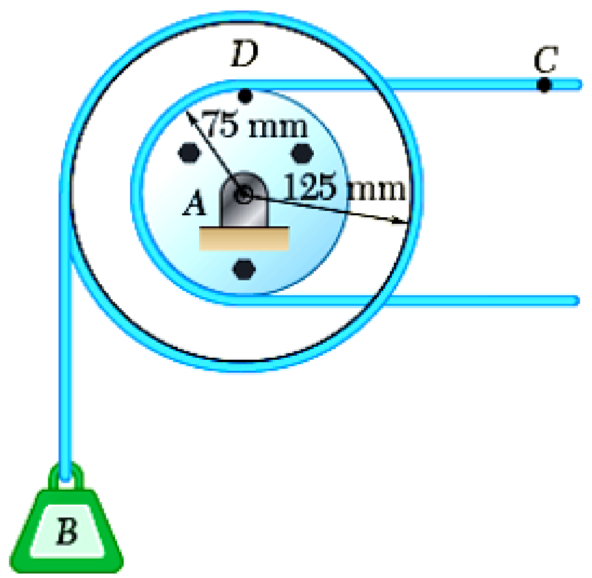
When a rigid body rotates @ a fixed point *O* with the instantaneous axis of rotation *n-n*, vel **v** and accln **a** = $\dot{\mathbf{v}}$ of any point *A* in the body are given by the same expressions derived when the axis was fixed:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a} = \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

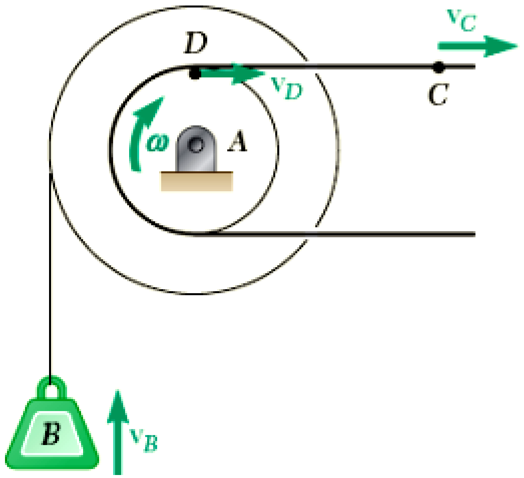


Sample Problem?



Cable C has a constant acceleration of 225 mm/s^2 and an initial velocity of 300 mm/s, both directed to the right.

Determine *(a)* the number of revolutions of the pulley in 2 s, *(b)* the velocity and change in position of the load B after 2 s, and *(c)* the acceleration of the point D on the rim of the inner pulley at $t = 0$.



SOLUTION:

- The tangential velocity and acceleration of D are equal to the velocity and acceleration of C.

$$(\vec{v}_D)_0 = (\vec{v}_C)_0 = 300\ \mathrm{mm/s} \rightarrow$$
$$(v_D)_0 = r\omega_0$$
$$\omega_0 = \frac{(v_D)_0}{r} = \frac{300}{75} = 4\ \mathrm{rad/s}$$

$$(\vec{a}_D)_t = \vec{a}_C = 9\ \mathrm{in./s} \rightarrow$$
$$(a_D)_t = r\alpha$$
$$\alpha = \frac{(a_D)_t}{r} = \frac{9}{3} = 3\ \mathrm{rad/s^2}$$

- Apply the relations for uniformly accelerated rotation to determine velocity and angular position of pulley after 2 s.

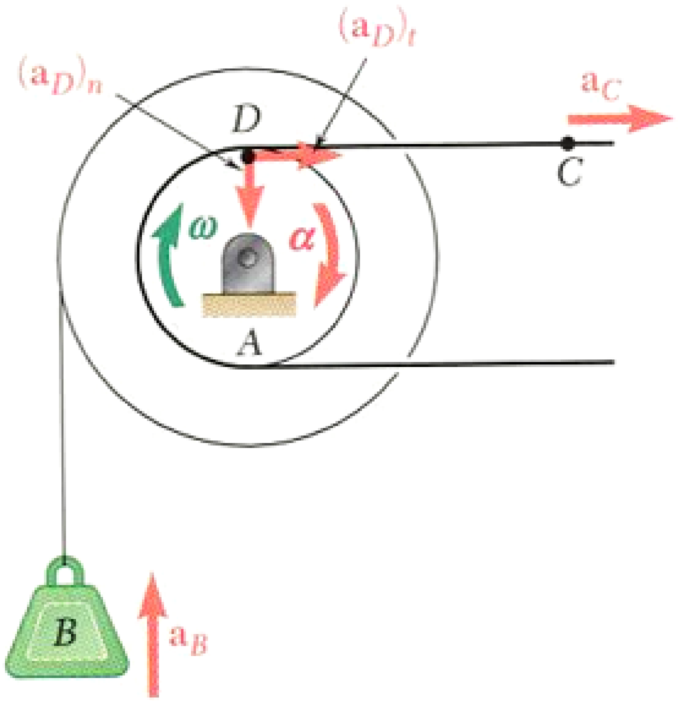
$$\omega = \omega_0 + \alpha t = 4\ \mathrm{rad/s} + \left(3\ \mathrm{rad/s^2}\right)(2\ \mathrm{s}) = 10\ \mathrm{rad/s}$$

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 = (4\ \mathrm{rad/s})(2\ \mathrm{s}) + \tfrac{1}{2}\left(3\ \mathrm{rad/s^2}\right)(2\ \mathrm{s})^2$$
$$= 14\ \mathrm{rad}$$

$$N = (14\ \mathrm{rad})\left(\frac{1\ \mathrm{rev}}{2\pi\ \mathrm{rad}}\right) = \text{number of revs} \qquad \boxed{N = 2.23\ \mathrm{rev}}$$

$$v_B = r\omega = (125\ \mathrm{mm})(10\ \mathrm{rad/s})$$
$$\Delta y_B = r\theta = (125\ \mathrm{mm})(14\ \mathrm{rad})$$

$$\boxed{\begin{array}{l} \vec{v}_B = 1.25\ \mathrm{m/s} \uparrow \\ \Delta y_B = 1.75\ \mathrm{m} \end{array}}$$



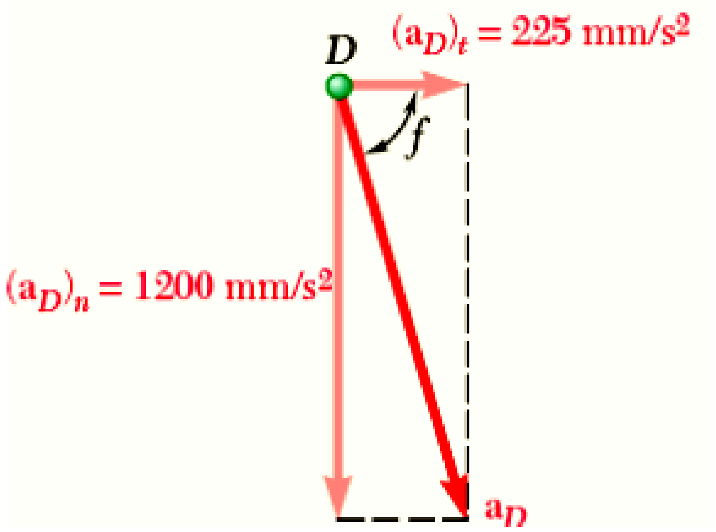
- Evaluate the initial tangential and normal acceleration components of D.

$$(\vec{a}_D)_t = \vec{a}_C = 225\ \text{mm/s}^2 \rightarrow$$

$$(a_D)_n = r_D\omega_0^2 = (75\ \text{mm})(4\ \text{rad/s})^2 = 1200\ \text{mm/s}^2$$

$$\boxed{(\vec{a}_D)_t = 225\ \text{mm/s}^2 \rightarrow \qquad (\vec{a}_D)_n = 1200\ \text{mm/s}^2 \downarrow}$$

Magnitude and direction of the total acceleration,



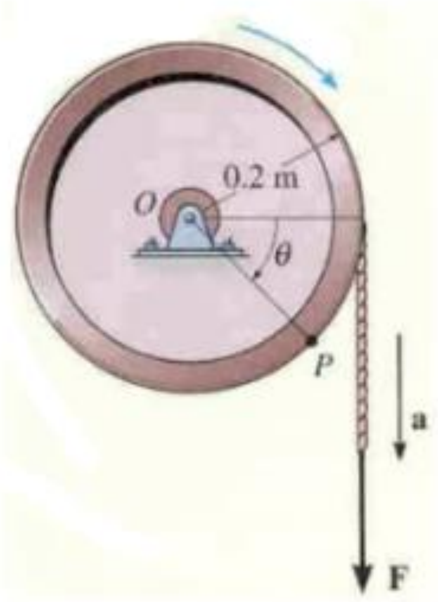
$$a_D = \sqrt{(a_D)_t^2 + (a_D)_n^2}$$
$$= \sqrt{225^2 + 1200^2}$$

$$\boxed{a_D = 1220\ \text{mm/s}^2}$$

$$\tan\phi = \frac{(a_D)_n}{(a_D)_t}$$
$$= \frac{1200}{225}$$

$$\boxed{\phi = 79.4°}$$

Problem



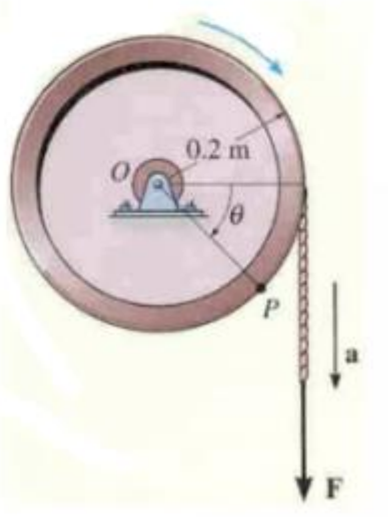
Rest

$a_t = 4t \text{ m/s}^2$

ω=?

θ=?

Problem



Rest

$a_t = 4t\ m/s^2$

$\omega = ?$

$\theta = ?$

$$(a_P)_t = \alpha\, r$$

$$(4t) = \alpha\,(0.2)$$

$$\alpha = 20t \ \ rad/s^2$$

$$\alpha = \frac{d\omega}{dt} = 20\,t$$

$$\int_0^{\omega} d\omega = \int_0^{t} 20t\ dt$$

$$\omega = 10t^2 \quad rad/s$$

$$\omega = \frac{d\theta}{dt} = 10\,t^2$$

$$\int_0^{\theta} d\theta = \int_0^{t} 10t^2\ dt$$

$$\theta = 3.33t^3 \quad rad$$

Thank You…