Embracing infinity

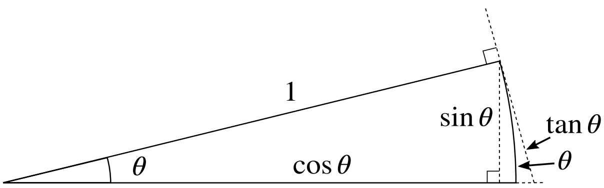
The notion of infinity intrigues philosophers, theologians, mathematicians, physicists, lay people, and children. Many people find talking about it irresistible, as any internet search engine will reveal. In mathematics we sometimes try to pin it down, sometimes to avoid it, sometimes to find ways to circumnavigate it, and sometimes to embrace it. The mathematics educator Caleb Gattegno famously said that mathematics is 'shot through with infinity', and again an internet search will show that this phrase has attracted many mathematicians.

In calculus, we embrace the infinite by understanding that when quantities become infinitesimally small (whatever that may mean), their relative behaviour is interesting. For example, when finding the gradient of a curve by taking a sequence of gradients of chords and making the chords shorter and shorter, eventually we get to the idea that the changes in the vertical and horizontal coordinates (the 'rise' and the 'run') are approaching zero. You might worry that this means that the gradient is approaching $0/0$ (whatever that might mean), but because we are looking at the relative behaviour rather than the absolute behaviour we are able to say (at least for a nice curve) that the gradient approaches some finite limit.

We also embrace the infinite in calculus by coming to the amazing conclusion that we can sum slices of areas (or volumes) on infinitesimally small intervals to make something finite, and we get 'true' finite results that we cannot calculate in the finite world.

Imagine slicing a circle into a very large number of sectors. The relative behaviour of very small quantities means that we can say that each of those sectors is (approximately) a very thin right-angled triangle, and, so long as we are using radians to measure angles, when the angle gets very small (the number of sectors is very large) we can say that

$$\sin\theta \approx \tan\theta \approx \theta.$$



From the same image you can also see that $\cos\theta \approx 1$ for very small θ, and this is a good enough approximation for many purposes, but with some more work you can also obtain the better approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$.

During your studies, you will find more and more often that you have to be careful about quantities that can become infinitely big or infinitely small.

Take care, however. Not everything changes just because it 'goes to infinity'. The sequence $1, 1, 1, \ldots$ does not change and that thought leads to a whole class of examples. It is also unwise to assume that mathematical objects that look superficially similar will behave in similar ways. For example, the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

converges to 2 (it is an example of a geometric progression), whereas the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges.

It is a useful exercise to take note of the appearance of the infinite throughout mathematics, because later on behaviour 'at infinity' might fit with some more advanced concepts.
For example, younger students are told to avoid dividing by zero. Later on, students sketch graphs with asymptotes to indicate similar behaviour in a continuous function. Undergraduate students studying complex analysis at university learn about singularities, and construct pathways that avoid the problematic points via tiny circular arcs.