Taylor series

Given a function $f(x)$ with a in the domain of f, we might be able to write $f(a+x)$ as a power series

$$f(a+x) = f(a) + f'(a)x + \frac{f''(a)}{2}x^2 + \frac{f'''(a)}{3!}x^3 + \cdots + \frac{f^{(n)}(a)}{n!}x^n + \cdots$$

which is valid for all $|x| < R$ for some R (called the *radius of convergence*). Such a power series is called a *Taylor series* or *Taylor series for f about a*. In the case $a = 0$, it is also known as a *Maclaurin series* or *Maclaurin expansion*.

Such Taylor series exist for all "nice" functions, and there are theorems which specify the range of values of x for which this expansion is valid.

For example, the function $f(x) = 1/(1-x)$ has a Taylor series about $a = 0$, namely $f(x) = 1 + x + x^2 + x^3 + \cdots$, which is valid for $|x| < 1$, while $f(x) = e^x$ has a Taylor series about $a = 0$, which is $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, and this is valid for all x.

One of the basic requirements for the existence of a Taylor series at a point is that the function has to be "smooth", that is, you have to be able to differentiate the function infinitely many times at that point (as is clear from the power series expansion). For example, $f(x) = |x|$ has no Taylor series about $a = 0$, as it is not differentiable there, but it has a Taylor series about every other point.

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