Cross product

The *cross product* (or *vector product*) of the two three-dimensional real vectors $\mathbf{a}$ and $\mathbf{b}$, written $\mathbf{a} \times \mathbf{b}$ or $\mathbf{a} \wedge \mathbf{b}$, is another three-dimensional vector. It is defined as follows:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.$$

It can also be calculated by expanding the "determinant"

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

This is a vector (hence the name 'vector' product). The definition does not easily extend to vectors $\mathbf{a}$ and $\mathbf{b}$ with n components for $n \neq 3$. (Compare this to the dot product, which outputs a scalar and which does have a simple generalisation.)

The cross product combines information about the lengths of $\mathbf{a}$ and $\mathbf{b}$ and their directions. Specifically,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\, \hat{\mathbf{n}}$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the lengths (magnitudes) of $\mathbf{a}$ and $\mathbf{b}$, θ is the angle between the two vectors (with $0 \leq \theta \leq \pi$), and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. The vector $\hat{\mathbf{n}}$ is chosen so that $\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ satisfy the *right-hand rule*: if you use your right hand, point your first (index) finger in the direction of $\mathbf{a}$ and your second (middle) finger in the direction of $\mathbf{b}$, then your thumb, when lifted, will point in the direction of $\hat{\mathbf{n}}$ and hence in the direction of $\mathbf{a} \times \mathbf{b}$.

It follows that for non-zero vectors $\mathbf{a}$ and $\mathbf{b}$, the cross product is zero exactly when $\mathbf{a}$ and $\mathbf{b}$ are parallel to one another.

Any two vectors $\mathbf{a}$ and $\mathbf{b}$ define a parallelogram and a triangle. The area of the parallelogram is given by $|\mathbf{a} \times \mathbf{b}|$, and the area of the triangle is half of this.

For every choice of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ and every scalar λ, we have

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= -\mathbf{b} \times \mathbf{a} && \text{(anti-commutativity)} \\ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} && \text{(distributivity over addition)} \\ (\lambda \mathbf{a}) \times \mathbf{b} &= \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b}). \end{aligned}$$