General tragedy

What's the general solution to an equation? To answer this question you first need to find a general form for the equation. For a linear equation it's

$$ax + b = 0$$

and for a quadratic it's

$$ax^2 + bx + c = 0,$$

where a, b and (for the quadratic) c are real numbers and a is not zero.

The formulae for the general solutions are well known. They are

$$x = -\frac{b}{a}$$

for the linear equation and

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the quadratic equation.

They work whatever the values of a, b and c (though in the case of the quadratic the resulting solutions may involve the square root of a negative number, that is, a complex number). Versions of both formulae were known to the Babylonians nearly 4000 years ago.

A natural question is whether there are similar general formulae for polynomial equations of higher degree, for example a cubic equation

$$ax^3 + bx^2 + cx + d = 0,$$

or a quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0,$$

or a quintic equation

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0.$$

It's not an easy question: it took mathematicians until the 16th century to show that the answer is "yes" for the cubic and the quartic equation and to provide those general formulae (which are too complicated to write down here). After that considerable achievement the hunt was on for the general solution of the quintic, but by 1800 it still hadn't been found. In 1824 the Norwegian mathematician Niels Henrik Abel explained why: he proved that there isn't one. This doesn't mean that a quintic equation has no solutions. In fact, the fundamental theorem of algebra says that every non-constant polynomial equation in one variable has solutions (which may be complex numbers). Neither does it mean that there *never* is a general formula: if the equation is particularly nice, for example,

$$ax^5 + b = 0,$$

then there may be one, in this case

$$x = \sqrt[5]{-b/a}.$$

What Abel proved is that there isn't a general formula which works for the most generic form of the equation when none of the coefficients are specified. Abel's result applies not just to the quintic, but to all polynomial equations in which the greatest power of x (the *degree* of the equation) is greater than 4. When you are faced with such an equation you either have to hope that it has a particularly nice form for which there is a formula, or use one of several techniques for finding or approximating solutions.

A few years after Abel delivered his result, the young mathematical prodigy Evariste Galois, from France, decided to find out exactly why the general quintic equation should be unsolvable. He discovered that, in a particular mathematical sense, the solutions of an equation could be swapped around, just like the corners of a square are swapped around by applying one of its symmetries, for example rotating it by 90 degrees or reflecting it in an axis. Galois developed a theory of symmetry for equations, which explained why there is not a general solution for a quintic: in the general case the symmetries just don't behave in the right way. In doing so Galois laid the foundations for what is now known as *group theory*.

Neither Abel nor Galois proved their results too soon. Abel died in 1829, at the tender age of 27, from tuberculosis and in abject poverty. Mathematicians of the day had recognised his genius, but they had never given him a job. Galois died in 1832. He had been challenged to a duel for reasons that are not entirely clear, but seem to be connected to a woman he was in love with, Stephanie-Felice du Motel. In the night before the fateful morning Galois jotted down his mathematical discoveries in a letter to a friend. The next morning he met his adversary and died from his wounds the following day. He was only 20 years old.