Re-discovering the logarithm—Mercator

One amazing thing about the natural logarithm is that it was discovered in many different contexts. Although we now know that it is intimately related to the number e, its discovery was independent of that.

One discovery goes back to the Dutchman Nicolaus Mercator (1620–1687), who, incidentally, also designed the fountains at Versailles! In his 1668 work *Logarithmotechnia* Mercator considered sums such as

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}=\frac{7}{12}.$$

There is an obvious pattern here, which you can continue. The next term to add would be $1/5$, then subtract $1/6$, then add $1/7$ and so on. As you continue this pattern, the result of the sum gets alternately smaller (if you are subtracting the last term) and larger (if you are adding). But the amount by which it changes decreases each time, in such a way that as you include more and more terms the results home in on a limit. And this limit is … $\ln 2$. More generally, if for any number x you compute the sums

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots-\frac{x^n}{n}$$

if n is even and

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{x^n}{n}$$

if n is odd, you will find that their limit is $\ln(1+x)$ as n tends to infinity.