Achilles and the tortoise

The paradoxes of the ancient Greek philosopher Zeno, born approximately 490BC, have puzzled mathematicians and scientists for millennia. Perhaps the most famous paradox is that of the race between Achilles, the swift-footed warrior, and a lumbering tortoise. Sure of his superiority, Achilles gives the tortoise a head start. Zeno's argument "proves" that Achilles will never overtake the tortoise. Suppose the tortoise starts at point A. By the time Achilles gets there the tortoise has moved on to B. By the time Achilles gets to B, the tortoise will have moved on to C. And so on. Whenever Achilles gets to where the tortoise was a moment ago, the tortoise will have moved on, so Achilles never catches up.

To pick this apart, let's assume that Achilles and the tortoise both move at constant speed. Achilles' speed is 100 metres per minute and the tortoise's speed is 1 metre per minute (the actual numbers don't matter). Achilles is 100 times faster than the tortoise, so let's give the poor animal a very large head start: 100m. Now by the time Achilles has travelled the 100m to A the tortoise has moved 1m to point B (because it's 100 times slower than Achilles). When Achilles gets to B, the tortoise has moved 0.01m to C, etc.

Adding up the infinitely many distances Achilles has to traverse before he catches up with the tortoise, we get

$$100+1+\frac{1}{100}+\frac{1}{10000}+\frac{1}{1000000}+\cdots=100+1+\frac{1}{100}+\frac{1}{100^2}+\frac{1}{100^3}+\cdots.$$

This is a geometric series which sums to $101.010101\ldots$ metres.

Since he is travelling at constant speed Achilles can cover that finite distance in a finite amount of time (1 minute and 0.6 seconds)—after that time he will have caught up with the tortoise.

The flaw in Zeno's argument was the unstated assumption that the infinite sum of distances (or equivalently the infinite sum of time periods needed to traverse each distance) cannot be finite.

Below is a short animated video which tells this tale. Please use this video link if you require full screen.